Chapter 1: Tutorial

1. Let $X_1, X_2, \dots, X_n \sim Poisson(\lambda)$. Derive a method of moment estimators for λ.
2. Let $X_1, X_2, \dots, X_n \sim iid\ \ N(\mu, \sigma^2)$, both μ and σ^2 unknown. Derive a method of moment estimators for μ and σ.
3. Let $X_1, X_2, \dots, X_m \sim iid\ \ Bin(n, \theta)$, both n and θ unknown. Derive a method of moment estimators for n and θ.
4. Let $X_1, X_2, \dots, X_n \sim iid\ \ Unif(\theta_1, \theta_2)$, where $\theta_1 < \theta_2$, both unknown. Derive a method of moment estimators for θ_1 and θ_2.
5. Let $X_1, X_2, \dots, X_n \sim iid\ \ Gamma(\alpha, \beta)$, both α and β unknown. Derive a method of moment estimators for α and β.

   The survival time (in weeks) of 20 randomly selected male mouse exposed to 240 units of certain type of radiation are given below.

   $$152, 115, 109, 94, 88, 137, 152, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83, 69$$

   It is believed that the survival times have a gamma distribution. Estimate the corresponding parameters.
6. Let $x_1, x_2, \dots, x_n$ be n random measurements of random variable X with the density function
   $$f_X(x; \lambda) = \lambda x^{\lambda-1}, \ \ 0 < x < 1, \ \ \lambda > 0$$
   Derive a method of moment estimator for λ.
7. Let $x_1, x_2, \dots, x_n$ be a random sample of size n from a Poisson distribution with parameter λ. Derive the maximum likelihood estimator of λ.
8. Let $x_1, x_2, \dots, x_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Derive the maximum likelihood estimators of μ and σ^2.
9. Let $X_1, X_2, \dots, X_n \sim iid\ Poisson(\lambda)$. Find the MLE of $P(X \leq 1)$
10. Let $X_1, X_2, \dots, X_n \sim iid\ N(\mu, \sigma^2)$. Find the MLE of μ/σ.
11. Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with the density function
    $$f_X(x; \lambda) = \lambda e^{-\lambda x}, \ \ x > 0.$$
    Is the maximum likelihood estimators of λ unbiased?

Hint

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (i.i.d.) exponential random variables with rate parameter λ, i.e.,

$$X_i \sim \text{Exp}(\lambda) \quad i = 1, 2, \dots, n.$$

Then, the sum of these n independent exponential random variables

$$S_n = X_1 + X_2 + \cdots + X_n$$

follows a gamma distribution with shape parameter n and rate parameter λ:

$$S_n \sim \text{Gamma}(n, \lambda).$$

Further, for any positive integer $n(> 0)$, Gamma fucntion, $\Gamma(n) = (n-1)!$.

12. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a distribution with mean μ and variance σ^2. Show that $\bar{X}$ and $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$ are unbiased estimators of μ and σ^2, respectively.

13. Let $x_1, x_2, \dots, x_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Consider the maximum likelihood estimators of σ^2. Show the estimator $\hat{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$ is biased for σ^2, but it has a smaller MSE than $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$.

14. Let $X_1, X_2, \dots, X_n$ be a random sample from some distribution and $E(X) = \mu$ and $V(X) = \sigma^2$. Show that $\bar{X}$ is a better estimator than X_1 and $\frac{X_1 + X_2}{2}$ for μ in terms of MSE.

    Hint

    Distribution of sample variance of a normal distribution

    Suppose that $X_1, X_2, \dots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2. Let $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$ be the sample mean and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ be the sample variance. Then,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

15. Let $X_1, X_2, \dots, X_n$ be a random sample from a $N(\mu, \sigma^2)$. Let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ and $T = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$. Show that $\bar{X}$ is consistent for μ and T is consistent for σ^2.

16. Let $X_1, X_2, \dots, X_n$ be a random sample from a normal distribution

$$X_i \sim N(\mu, \sigma^2)$$

where μ is the unknown mean parameter and σ^2 is a known variance. Consider the following two estimators for μ:

- Estimator 1: The sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
- Estimator 2: The first observation X_1.

i. Show that both $\bar{X}$ and X_1 are unbiased estimators of μ.

ii. Compute the variances of both estimators and determine which is more efficient.

iii. Using the Factorization Theorem, determine which of the two estimators is a sufficient statistic for μ.

17. Let $X_1, X_2, \dots, X_n$ be a random sample from a Poisson distribution

$$X_i \sim \text{Poisson}(\lambda), \quad i = 1, 2, \dots, n.$$

Consider the following two estimators for λ:

- Estimator 1: The sample mean $\hat{\lambda_1} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
- Estimator 2: The mean of the first two observations $\hat{\lambda_2} = \frac{X_1 + X_2}{2}$

i. Show that both $\hat{\lambda_1}$ and $\hat{\lambda_2}$ are unbiased estimators of λ.

ii. Compute the variances of both estimators and determine which is more efficient.

iii. Use the Factorization Theorem to determine whether $\bar{X}$ or $\hat{\lambda_2}$ is a sufficient statistic for λ.