Chapter 4

Moments and Moment Generating Functions

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4.1 Introduction

- In this chapter we define a third special mathematical expectation, called the *moment generating function* (abbreviated mgf) of a random variable.
- The obvious purpose of the moment-generating function is in determining moments of random variables.
- However, the most important contribution is to establish distributions of functions of random variables.

2.2 Moments

- The following Definition 1 yields an expected value called the rth *moment about the mean* of the random variable X, which we denote by μ_r.

- Definition 1 – Moment about the Mean

  The *rth moment of a random variable X about the mean μ*, also called the *rth central moment*, is defined as

  $$\mu_r = E[(X-\mu)^r],$$

  where $r = 0,1,2,\cdots$.

- We have,

$$\mu_r = E[(X-\mu)^r] = \begin{cases} \sum_x (x-\mu)^r f(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx, & \text{if } X \text{ is continuous.} \end{cases}$$

- It follows that $\mu_0 = 1, \mu_1 = 0$ and $\mu_2 = \sigma^2$, i.e., the second central moment or second moment about the mean is the variance, $Var(X)$ of X.
- The following Definition 2 yields an expected value called the rth *moment about the origin* of the random variable X, which we denote by μ_r'.

- Definition 2 – Moment about the Origin

  The *rth moment of a random variable X about the origin* is defined as

  $$\mu_r' = E[X^r],$$

  where $r = 0,1,2,\cdots$.
- We have,

$$\mu_r' = E[X^r] = \begin{cases} \sum_x x^r f(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} x^r f(x)\, dx, & \text{if } X \text{ is continuous.} \end{cases}$$

- Note that, the expected value of X, $E(X)$ is the first moment about the origin, i.e., $\mu_1' = \mu$.

- It can be shown that, the relationship between these moments is given by

$$\mu_r = \mu'_r - \binom{r}{1}\mu'_{r-1}\,\mu + \cdots + (-1)^i\binom{r}{i}\mu'_{r-i}\,\mu^i + \cdots + (-1)^r\mu'_0\,\mu^r.$$

As special cases we have, using $\mu'_1 = \mu$ and $\mu'_0 = 1$,

$$\mu_1 = \mu'_1 - \mu'_0\,\mu = \mu - \mu = 0,$$

$$\mu_2 = \mu'_2 - \mu^2,$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3,$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4.$$

4.2 Moment Generating Functions

4.2.1 Defining Moment Generating Functions

- Although the moments of a random variable can be determined directly from Definition 1 and 2, an alternative procedure exists.
- This procedure requires us to utilize a *moment-generating function*.

- **Definition 3 – Moment Generating Function (MGF)**

  The *moment-generating function* of the random variable X is denoted as $M_X(t)$ and given by $M_X(t) = E(e^{tX})$, that is,

$$M_X(t) = \begin{cases} \sum_x e^{tx} f(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & \text{if } X \text{ is continuous.} \end{cases}$$

  Where no confusion can result we often write $M(t)$ instead of $M_X(t)$.

- Moment-generating functions will exist only if the sum or integral of Definition 3 converges, i.e., as long as the sum or integral is finite for some interval of t around 0.

  That is, $M_X(t)$ is the MGF of X if there is a positive number h such that the above sum or integral exists and is finite for $-h < t < h$.

- When a moment-generating function $M_X(t)$ of a random variable X does exist, we can derive the Taylor series expansion of the moment generating function $M_X(t)$ as follows:

$$M_X(t) = E(e^{tX}) = E\left(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots + + \frac{(tX)^r}{r!} + \cdots\right)$$

$$= 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + \mu t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \cdots + \mu_r' \frac{t^r}{r!} + \cdots.$$

4.2.2 Deriving Moments from Moment Generating Function

- Since each moment is an expected value, and the definition of expected value involves either a sum (in the discrete case) or an integral (in the continuous case), it seems that the computation of moments could be tedious.
- If a moment-generating function of a random variable X does exist, it can be used to generate all the moments of that variable.

  The method is described in Theorem 1 without proof.

- Theorem 1

  Let X be a random variable with moment generating function $M_X(t)$.

  Then the rth moment of the random variable X about the origin will be obtained by

$$\mu_r' = E(X^r) = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0},$$

  i.e., μ_r' is the rth derivative of $M_X(t)$ with respect to t, evaluated at the point $t = 0$.

- Note: If a MGF exists for a random variable X, then

  (1) The mean of X can be found by evaluating the first derivative of the MGF at $t = 0$, i.e., $\mu = E(X) = M_X'(0)$.

  (2) The variance of X can be found by evaluating the first and second derivatives of the MGF at $t = 0$, i.e.,

$$\sigma^2 = E(X^2) - [E(X)]^2 = M_X''(0) - [M_X'(0)]^2.$$

Example 1

The random variable X can assume the values 1 and -1 with probability $\frac{1}{2}$ each.

Find (a) the moment generating function,

(b) the first four moments about the origin.

Solution:

(a) $M_X(t) = E(e^{tX}) = e^{t(1)}\left(\frac{1}{2}\right) + e^{t(-1)}\left(\frac{1}{2}\right) = \frac{1}{2}(e^t + e^{-t})$.

(b) We have, $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^r}{r!} + \cdots$, and

$$e^{-t} = 1 - \frac{t}{1!} + \frac{t^2}{2!} + \cdots + (-1)^r\frac{t^r}{r!} + \cdots.$$

Then $M_X(t) = \frac{1}{2}(e^t + e^{-t}) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots.$ --- (1)

We also know that, $M_X(t) = 1 + \mu t + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \cdots + \mu_r'\frac{t^r}{r!} + \cdots.$ --- (2)

Then, comparing (1) and (2), we have

$$\mu = 0, \mu_2' = 1, \mu_3' = 0, \mu_4' = 1, \cdots.$$

The odd moments are all zero, and the even moments are all one.

Example 2

A random variable X has density function given by

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Find (a) the moment generating function,

(b) the first four moments about the origin.

Solution:

(a) $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} e^{tx}(2e^{-2x})\, dx = 2\int_0^{\infty} e^{(t-2)x}\, dx$

$$= \left.\frac{2e^{(t-2)x}}{t-2}\right|_0^{\infty} = \frac{2}{2-t},$$ assuming $t < 2$.

(b) If $|t| < 2$ we have

$$\frac{2}{2-t} = \frac{1}{1-{}^{t}/_{2}} = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots$$

$$= 1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \cdots. \qquad \text{--- (1)}$$

We also know that, $M_X(t) = 1 + \mu t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \cdots + \mu_r' \frac{t^r}{r!} + \cdots$. --- (2)

Therefore, on comparing terms of (1) and (2), we get

$$\mu = \frac{1}{2}, \mu_2' = \frac{1}{2}, \mu_3' = \frac{3}{4}, \mu_4' = \frac{3}{2}.$$

Example 3

Find the first four moments

(a) about the origin, (b) about the mean,

for a random variable X having density function $f(x) = \begin{cases} \frac{4x(9-x^2)}{81}, 0 \le x \le 3, \\ 0, \qquad otherwise. \end{cases}$

Solution:

(a) $\mu_1' = E(\mathrm{X}) = \int_0^3 \frac{4}{81} x \cdot x(9-x^2)dx = \frac{4}{81}\int_0^3 x^2(9-x^2)dx = \frac{8}{5} = \mu,$

$\mu_2' = E(X^2) = \int_0^3 \frac{4}{81} x^2 \cdot x(9-x^2)dx = \frac{4}{81}\int_0^3 x^3(9-x^2)dx = 3,$

$\mu_3' = E(X^3) = \int_0^3 \frac{4}{81} x^3 \cdot x(9-x^2)dx = \frac{4}{81}\int_0^3 x^4(9-x^2)dx = \frac{216}{35}$, and

$\mu_4' = E(X^4) = \int_0^3 \frac{4}{81} x^4 \cdot x(9-x^2)dx = \frac{4}{81}\int_0^3 x^5(9-x^2)dx = \frac{27}{2}.$

(b) Using the results on slide 5,

$$\mu_1 = 0,$$

$$\mu_2 = \mu_2' - \mu^2 = 3 - \left(\frac{8}{5}\right)^2 = \frac{11}{25} = \sigma^2,$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 = \frac{216}{35} - 3 \cdot (3) \cdot \left(\frac{8}{5}\right) + 2 \cdot \left(\frac{8}{5}\right)^3 = -\frac{32}{875},$$

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4 \\ &= \frac{27}{2} - 4 \cdot \left(\frac{216}{35}\right) \cdot \left(\frac{8}{5}\right) + 6 \cdot (3) \cdot \left(\frac{8}{5}\right)^2 - 3 \cdot \left(\frac{8}{5}\right)^4 \\ &= \frac{3693}{8750}.\end{aligned}$$

Example 4

Find the moment generating function of the binomial random variable X and then use it to verify that $\mu = np$ and $\sigma^2 = npq$, where $q = 1 - p$.

Solution: We know that the probability distribution of the binomial random variable X is given by $f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$.

Thus, $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}.$$

Recognizing this last sum as the binomial expansion of $(pe^t + q)^n$, we obtain

$$M_X(t) = (pe^t + q)^n.$$

Now, $\dfrac{dM_X(t)}{dt} = n(pe^t + q)^{n-1} pe^t$, and

$$\frac{d^2 M_X(t)}{dt^2} = np[e^t(n-1)(pe^t + q)^{n-2} pe^t + (pe^t + q)^{n-1} e^t].$$

Setting $t = 0$, we get

$$\mu_1' = np \text{ and } \mu_2' = np[(n-1)p + 1].$$

Therefore, $\mu = \mu_1' = np$ and

$$\sigma^2 = \mu_2' - \mu^2 = np[(n-1)p + 1] - (np)^2 = np(1-p) = npq.$$

Exercises

1. The random variable X can assume the values $\frac{1}{2}$ and $-\frac{1}{2}$ with probability $\frac{1}{2}$ each. Find (a) the moment generating function,

   (b) the first four moments about the origin.

2. Find the moment generating function of a random variable having density function

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2, \\ 0, & otherwise. \end{cases}$$

   Determine the first four moments about the origin.

3. Find the moment generating function of a random variable X having density function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & otherwise. \end{cases}$$

Determine the moment generating function of X and the first four moments about the origin.

4. Let X have density function $f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & otherwise. \end{cases}$ Find the kth moment about (a) the origin, (b) the mean.

5. Let $f(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1,2,\cdots, \\ 0 & , elsewhere \end{cases}$, be the pdf of the random variable X. Find the MGF, mean and the variance of X.

6. For each of the following pdfs, compute $P(\mu - 2\sigma < X < \mu + 2\sigma)$.

(a) $f(x) = \begin{cases} 6x(1-x), & 0 < x < 1, \\ 0, & elsewhere. \end{cases}$ (b) $g(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1,2,\cdots, \\ 0 & , elsewhere. \end{cases}$

4.2.3 Some Theorems on Moment Generating Functions

- **Theorem 2 (Uniqueness Theorem)**

  Let X and Y be two random variables with moment generating functions $M_X(t)$ and $M_Y(t)$, respectively.

  If $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same probability distribution.

- **Theorem 3**

  Let X be a random variable with moment generating functions $M_X(t)$ and let a be a constant. Then $M_{X+a}(t) = e^{at}M_X(t)$.

Proof: $M_{X+a}(t) = E\left(e^{t(X+a)}\right) = e^{at}E(e^{tX}) = e^{at}M_X(t)$.

- Theorem 4

  Let X be a random variable with moment generating functions $M_X(t)$ and let a be a constant. Then $M_{aX}(t) = M_X(at)$.

Proof: $M_{aX}(t) = E\left(e^{t(aX)}\right) = E\left(e^{(at)X}\right) = M_X(at)$.

- Theorem 5

  Let X be a random variable with moment generating functions $M_X(t)$ and let a and $b (b \neq 0)$ be constants.

  Then $M_{(X+a)/b}(t) = e^{\frac{at}{b}} M_X(\frac{t}{b})$.

Proof: $M_{(X+a)/b}(t) = E\left(e^{t\left(\frac{X+a}{b}\right)}\right) = e^{\frac{at}{b}} E\left(e^{\left(\frac{t}{b}\right)X}\right) = e^{\frac{at}{b}} M_X(\frac{t}{b})$.