Chapter 3

Mathematical Expectation

Prof. N. G. J. Dias

Senior Professor and Head

Department of Computer Systems Engineering

Faculty of Computing and Technology

University of Kelaniya

3.1 Mean of a Random Variables

3.1.1 Introduction

- A very important concept in probability and statistics is that of the *mathematical expectation*, *expected value*, or briefly the *expectation*, of a random variable.

3.1.2 Mathematical Expectation of a Discrete Random variable

- Definition 1 – Expectation of a Discrete Random variable

  Let X be a discrete random variable with possible distinct values $x_1, x_2, \cdots, x_n, \cdots$.

  Then the *mathematical expectation,* or the *expected value* of X is denoted by $E(X)$ and defined by

  $$E(X) = \sum_i x_i \, P(X = x_i).$$

  Using probability mass function $f(x)$, we can write $E(X) = \sum_i x_i \, f(x_i)$.

- The expectation of X is very often called the *mean* of X and is denoted by μ_X, or simply μ when the particular random variable is understood.

Example 3.1

Consider the experiment of tossing of a pair of fair dice.

The 36 outcomes in the sample space S is given by

$$S = \{(a, b) | a, b = 1,2,3,4,5,6\}..$$

(i) Let X be the random variable which gives the maximum of the numbers appear on the two dice.

Then the range space of X is $R_X = \{1,2,3,4,5,6\}$.

There is only 1 outcome $(1,1)$ whose maximum is 1; hence $P(X = 1) = \frac{1}{36}$.

There are 3 outcomes, $(1,2), (2,1)$ and $(2,2)$ whose maximum is 2; hence $P(X = 2) = \frac{3}{36}$.

There are 5 outcomes, $(1,3), (2,3), (3,3), (3,1)$ and $(3,2)$, whose maximum is 3; hence $P(X = 3) = \frac{5}{36}$.

Similarly, $P(X=4)=\frac{7}{36}, P(X=5)=\frac{9}{36}, P(X=6)=\frac{11}{36}$.

Thus, the probability distribution of X is given by

x_i	1	2	3	4	5	6
$P(X=x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Then the mean, expectation or expected value of X is

$$\mu = E(X) = 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right) = \frac{161}{36}$$

$$= 4.47.$$

(ii) Let Y be the random variable which gives the sum of the numbers appear on the two dice.

Then the range space of Y is $R_Y = \{2,3,\cdots,12\}$.

Thus, the probability distribution of Y is given by

y_i	2	3	4	5	6	7	8	9	10	11	12
$P(Y = y_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Then the mean, expectation or expected value of Y is

$$\begin{aligned}\mu_Y = E(Y) &= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) \\ &\quad +8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) \\ &= \frac{252}{36} = 7.\end{aligned}$$

Example 3.2

Consider the experiment of selecting a sample of 3 items from a box containing 12 items of which 3 are defective.

The sample space S consists of $\binom{12}{3} = 220$ different samples of size 3.

Let X denote the number of defective items in the sample; then X is a random variable with range space $R_X = \{0,1,2,3\}$.

From Example 2.9 of Chapter 2, we know that the probability distribution of X is given by:

x_i	0	1	2	3
$P(X = x_i)$	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

Thus, $\mu = E(X) = 0\left(\frac{84}{220}\right) + 1\left(\frac{108}{220}\right) + 2\left(\frac{27}{220}\right) + 3\left(\frac{1}{220}\right) = 0.75.$

Example 3.3

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Solution: Let X represent the number of good components in the sample.

As an exercise, show that the probability distribution of X is given by

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x = 0,1,2,3.$$

Simple calculation yields that $f(0) = \frac{1}{35}, f(1) = \frac{12}{35}, f(2) = \frac{18}{35}$, and $f(3) = \frac{4}{35}$.

Therefore, $\mu = E(X) = 0\left(\frac{1}{35}\right) + 1\left(\frac{12}{35}\right) + 2\left(\frac{18}{35}\right) + 3\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Example 3.4

Three horses a, b and c are in a race; suppose their respective probabilities of winning are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$.

Let X denote the payoff function for the winning horse, and suppose X pays \$2, \$6, or \$9 according as a, b or c wins the race.

The expected payoff for the race is

$$\begin{aligned} E(X) = X(a)P(a) + X(b)P(b) + X(c)P(c) &= 2\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right) + 9\left(\frac{1}{6}\right) \\ &= 4.5. \end{aligned}$$

Example 3.5

Suppose that a game is to be played with a single die assumed fair. In this game a player wins \$20 if a 2 turns up, \$40 if a 4 turns up; loses \$30 if a 6 turns up; while the player neither wins nor loses if any other face turns up.

Find the expected sum of money to be won.

Solution: Let X be the random variable giving the amount of money won on any toss.

The possible amounts won when the die turns up $1,2,\cdots,6$ are $x_1 = 0, x_2 = +20, x_3 = 0, x_4 = +40, x_5 = 0$ and $x_6 = -30$, respectively, while the probabilities of these are $f(x_1), f(x_2), \cdots, f(x_6)$.

Since, the die assumed to be fair, the probability distribution for X is displayed in the following table.

x_i	0	+20	0	+40	0	-30
$f(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Therefore, the expected value or expectation is

$$E(X) = 0\left(\frac{1}{6}\right) + 20\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) + 40\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) + (-30)\left(\frac{1}{6}\right) = 5.$$

It follows that the player can expect to win \$5. In a fair game, therefore, the player should be expected to pay \$5 in order to play the game.

- Now consider the following special case:

  Let X is a discrete random variable with possible n distinct values $x_1, x_2, \cdots, x_n$.

  If the probabilities are all equal, then the probability distribution of X can be expressed as $f(x) = P(X = x) = \begin{cases} \frac{1}{n}, & x = x_i \text{ for } i = 1,2,\cdots,n \\ 0, & otherwise. \end{cases}$.

Thus, the expected value or expectation of X is

$$E(X) = x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

which is called the *arithmetic mean*, or simply the *mean*, of $x_1, x_2, \cdots, x_n$.

Exercises

1. The probability distribution of the discrete random variable X is

$$f(x) = \binom{3}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{3-x}, x =, 1,2,3.$$

   Find the mean of X.

2. An attendant at a car wash is paid according to the number of cars that pass through. Suppose the probabilities are 1/12, 1/12, 1/4, 1/4, 1/6, and 1/6, respectively, that the attendant receives $7, $9, $11, $13, $15, or $17 between 4:00 pm and 5:00 pm on any sunny Friday. Find the attendant's expected earnings for this particular period.

3.1.3 Mathematical Expectation of a Continuous Random variable

- Definition 2 – Expectation of a Continuous Random variable

  Let X be a continuous random variable with probability density function $f(x)$.

  Then the *mathematical expectation*, or the *expected value* of X is denoted by $E(X)$ and defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \,.$$

- In this case also, the expectation of X is very often called the *mean* of X and is denoted by μ_X, or simply μ when the particular random variable is understood.

Example 3.6

The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2, \\ 0, & otherwise. \end{cases}$$

Using Definition 2, the expected value of X is then

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 x\left(\frac{1}{2}x\right)dx = \int_0^2 \frac{x^2}{2}\,dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}.$$

Example 3.7

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20000}{x^3}, & x > 100, \\ 0, & elsewhere. \end{cases}.$$

Find the expected life of this type of device.

Solution: Using Definition 2, we have

$$\mu = E(X) = \int_{-\infty}^{\infty} x \left(\frac{20000}{x^3}\right) dx = \int_{100}^{\infty} \frac{20000}{x^2} dx = 20000 \left[\frac{x^{-1}}{-1}\right]_{100}^{\infty}$$

$$= 20000 \left[-\frac{1}{x}\right]_{100}^{\infty} = 200.$$

Therefore, we can expect this type of device to last, *on average*, 200 hours.

Example 3.8

The density function of a random variable X is given by

$$f(x) = \begin{cases} 2(1+x)^{-3}, & 0 \le x, \\ 0, & otherwise. \end{cases}.$$

The expected value of X is then

$$\begin{aligned} E(X) = \int_{-\infty}^{\infty} xf(x)\,dx &= \int_{-\infty}^{0} xf(x)\,dx + \int_{0}^{\infty} xf(x)\,dx \\ &= \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x\{2(1+x)^{-3}\}\,dx \\ &= 0 + \int_{0}^{\infty} \frac{2x}{(1+x)^3}\,dx. \end{aligned}$$

Using the substitution $1 + x = u$, we get $u = 1$ when $x = 0$, $u = \infty$ when $x = \infty$, and $du = dx$.

Thus, we have $E(X) = \int_{1}^{\infty} \frac{2(u-1)}{u^3}\,du = \int_{1}^{\infty} \left\{\frac{2}{u^2} - \frac{2}{u^3}\right\} du = \left[-\frac{2}{u} + \frac{1}{u^2}\right]_1^{\infty} = 1.$

Example 3.9

The density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & otherwise. \end{cases}.$$

The expected value of X is then

$$\begin{aligned} E(X) = \int_{-\infty}^{\infty} xf(x)\,dx &= \int_{-\infty}^{a} xf(x)\,dx + \int_{a}^{b} xf(x)\,dx + \int_{b}^{\infty} xf(x)\,dx \\ &= \int_{-\infty}^{a} x \cdot 0\,dx + \int_{a}^{b} x\left(\frac{1}{b-a}\right) dx + \int_{b}^{\infty} x \cdot 0\,dx \\ &= 0 + \left(\frac{1}{b-a}\right)\int_{a}^{b} x\,dx + 0 \\ &= \left(\frac{1}{b-a}\right)\left[\frac{x^2}{2}\right]_a^b \\ &= \left(\frac{1}{b-a}\right)\left(\frac{b^2-a^2}{2}\right) = \frac{a+b}{2}. \end{aligned}$$

Exercises

1. If a dealer's profit, in units of \$5000, on a new automobile can be looked upon as a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 2, \\ 0, & elsewhere.' \end{cases}$$

   find the average profit per automobile.

2. The density function of the continuous random variable X, the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, is given as a random variable with probability density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2-x, & 1 \le x < 2,. \\ 0, & elsewhere. \end{cases}$$

   Find the average number of hours per year that families run their vacuum cleaners.

3.1.4 Functions of Random Variables

- Let X be a random variable.

  Now let us consider a new random variable $g(X)$, which depends on X; that is, each value of $g(X)$ is determined by the value of X.

  For instance, $g(X)$ might $3X-4$ or X^2+2, and whenever X assumes the value x, $g(X)$ assumes the value $g(x)$.

- **Definition 3 - Mathematical Expectation of a Function of a Random variable**

  Let X be a random variable.

  (i) If X is discrete random variable with probability mass function $f(x)$, then the *mathematical expectation*, or the *expected value* of the random variable $g(X)$ is denoted by $E[g(X)]$ and defined by

$$E[g(X)] = \sum_i g(x_i)\, P(X = x_i) = \sum_i g(x_i)\, f(x_i).$$

(ii) If X is continuous random variable with probability density function $f(x)$, then the *mathematical expectation*, or the *expected value* of the random variable $g(X)$ is denoted by $E[g(X)]$ and defined by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \,.$$

Example 3.10

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution: By Definition 3, the attendant can expect to receive

$$E[g(X)] = E[2X - 1] = \sum_4^9 (2x - 1)f(x)$$

$$= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + +(11)\left(\frac{1}{4}\right) + +(13)\left(\frac{1}{4}\right) + (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right)$$

$$= \$12.67.$$

Example 3.11

Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & otherwise. \end{cases}.$$

Find the expected value of $g(X) = 4X + 3$.

Solution: From Definition 3, we have

$$E[4X + 3] = \int_{-\infty}^{\infty} (4x + 3)\frac{x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2) dx = \frac{1}{3}[x^4 + x^3]_{-1}^{2} = 8.$$

Example 3.12

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2, \\ 0, & otherwise. \end{cases}$$

Find the expected value of $g(x) = 3X^2 - 2X$.

Solution: $E[3X^2 - 2X] = \int_{-\infty}^{\infty}(3x^2 - 2x)\frac{x}{2}dx = \frac{1}{2}\int_0^2(3x^3 - 2x^2)dx = \frac{10}{3}$.

Exercises

1. Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x) = P(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

   Find $\mu_{g(X)}$, where $g(X) = (2X + 1)^2$.

2. Find the expected value of the random variable $g(X) = x^2$, where X has the probability distribution of Exercise 1 in Section 3.1.2.

3. What is the dealer's average profit per automobile if the profit on each automobile is given by $g(X) = x^2$, where X is a random variable having the density function of Exercise 1 of Section 3.1.3?

4. The probability distribution of X, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

x	0	1	2	3	4
$f(x) = P(X = x)$	0.41	0.37	0.16	0.05	0.01

(i) Construct the cumulative distribution function of X.

(ii) Plot the probability function and the cumulative distribution function of X.

(iii) Find the expected number of imperfections, $E(X) = \mu$.

(iv) Find $E(X^2)$.

3.1.5 Some Theorems on Expectation

- Theorem 1: Let a and b any two constants and let X be a random variable.

  Then $E(aX + b) = aE(X) + b$.

Proof:

Case (i) Let X be discrete random variable with probability mass function $f(x) = P(X = x)$ and let $g(X) = aX + b$.

Then by Definition 3,

$$
\begin{aligned}
E(aX + b) &= \textstyle\sum_i (ax_i + b)\, P(X = x_i) \\
&= \textstyle\sum_i ax_i P(X = x_i) + \sum_i bP(X = x_i) \\
&= a \textstyle\sum_i x_i P(X = x_i) + b \sum_i P(X = x_i) \\
&= aE(X) + b,
\end{aligned}
$$

since $\sum_i x_i P(X = x_i) = E(X)$ and $\sum_i P(X = x_i) = 1$.

Case (ii) Let X be continuous random variable with probability density function $f(x)$ and let $g(X) = aX + b$.

Then by Definition 3,

$$\begin{aligned} E(aX + b) &= \int_{-\infty}^{\infty} (ax + b) f(x) dx \\ &= \int_{-\infty}^{\infty} ax f(x) dx + \int_{-\infty}^{\infty} b f(x) dx \\ &= a \int_{-\infty}^{\infty} x f(x) dx + b \int_{-\infty}^{\infty} f(x) dx \\ &= aE(X) + b, \end{aligned}$$

since for a continuous random variable $\int_{-\infty}^{\infty} x f(x) dx = E(X)$ and $\int_{-\infty}^{\infty} f(x) dx = 1$.

- **Corollary 1:** Setting $a = 0$, in the previous theorem we get $E(b) = b$, i.e.,

  if b is a constant, then $E(b) = b$.
- **Corollary 2:** Setting $b = 0$, in the previous theorem we get $E(aX) = aE(X)$,i.e.,

  if a is a constant, then $E(aX) = aE(X)$.

Example 3.13

Again consider the discrete random variable X with probability distribution given in Example 3.10.

Applying Theorem 1 to the discrete random variable $g(X) = 2X - 1$, we can write $E(2X - 1) = 2E(X) - 1$.

Now, $\mu = E(X) = \sum_4^9 xf(x) = 4\left(\frac{1}{12}\right) + 5\left(\frac{1}{12}\right) + 6\left(\frac{1}{4}\right) + 7\left(\frac{1}{4}\right) + 8\left(\frac{1}{6}\right) + 9\left(\frac{1}{6}\right)$

$$= \frac{41}{6}.$$

Therefore, $\mu_{2X-1} = E(2X - 1) = 2E(X) - 1 = 2 \cdot \frac{41}{6} - 1 = \$12.67$.

Example 3.14

Again consider the continuous random variable X with probability density function given in Example 3.11.

Applying Theorem 1 to the continuous random variable $g(X) = 4X + 3$, we can write $E(4X + 3) = 4E(X) + 3$.

Now, $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-1}^{2} x\left(\frac{x^2}{3}\right)dx = \int_{-1}^{2}\left(\frac{x^3}{3}\right)dx = \left[\frac{x^4}{12}\right]_{-1}^{2} = \frac{5}{4}$.

Therefore, $E(4X + 3) = 4E(X) + 3 = 4 \cdot \frac{5}{4} + 3 = 8$.

- Theorem 2: Let a and b any two constants and let X be a random variable.

  If $g(X)$ and $h(X)$ are functions of the random variable X, then

$$E[ag(X) + bh(X)] = aE[g(X)] + bE[h(X)].$$

Proof: Try this as an exercise.

- Note that, we can extend this theorem to more than two functions:

  If $c_1, c_2, \cdots, c_n$ are constants and $g_1(X), g_2(X), \cdots, g_n(X)$ are functions of the random variable X, then

$$E[c_1 g_1(X) + c_2 g_2(X) + \cdots + c_n g_n(X)]$$
$$= c_1 E[g_1(X)] + c_2 E[g_2(X)] + \cdots + c_n E[g_n(X)].$$

Example 3.15

Let X be a random variable with probability distribution as follows:

x	0	1	2	3
$f(x) = P(X = x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Find the expected value of $Y = (X-1)^2$.

Solution: Applying Theorem 2 to the function $Y = (X-1)^2$, we can write

$$E(Y) = E[(X-1)^2] = E(X^2 - 2X + 1) = E(X^2) - 2E(X) + E(1).$$

From Corollary 1, $E(1) = 1$, and by direct computation,

$$E(X) = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (2)(0) + (3)\left(\frac{1}{6}\right) = 1 \text{ and}$$

$$E(X^2) = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (4)(0) + (9)\left(\frac{1}{6}\right) = 2.$$

Hence, $E[(X-1)^2] = E(X^2) - 2E(X) + E(1) = 2 - 2 \cdot 1 + 1 = 1.$

Example 3.16

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & otherwise. \end{cases}$$

Find the expected values of $h(X) = 3X + 4$ and $g(x) = 6X + 3X^2$.

Solution:

$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 xf(x)\,dx = \int_0^1 x \cdot 2(1-x)\,dx = \frac{1}{3}$, and

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2(1-x)\,dx = \frac{1}{6}$.

Using Theorem 2 and 4,

$E[h(X)] = E[3X + 4] = 3E(X) + 4 = 3 \cdot \frac{1}{3} + 4 = 5$, and

$E[g(x)] = E[6X + 3X^2] = 6E(X) + 3E(X^2) = 6 \cdot \frac{1}{3} + 3 \cdot \frac{1}{6} = \frac{5}{2}$.

Example 3.17

Let X be a discrete random variable with probability mass function

$$f(x) = \begin{cases} \frac{x}{6}, & x = 1,2,3, \\ 0, & otherwise. \end{cases}.$$

Then

$$E(3X^3) = 3E(X^3) = 3\sum_x x^3 f(x) = 3\sum_x x^3\left(\frac{x}{6}\right) = 3\left\{\frac{1}{6} + \frac{16}{6} + \frac{81}{6}\right\} = 3 \cdot \frac{98}{6}$$

$$= 49.$$

Exercises

1. In a lottery there are 200 prizes of \$5, 20 prizes of \$25, and 5 prizes of \$100. Assuming that 10,000 tickets are to be issued and sold, what is a fair price to pay for a ticket? (Ans: 0.2 or 20 cents)

2. Let X have the probability density function

$$f(x) = \begin{cases} \frac{x+2}{18}, & -2 < x < 4, \\ 0, & elsewhere. \end{cases}.$$

   Find $E(X), E[(X+2)^3]$, and $E[6X - 2(X+2)^3]$.

   (Ans: 2, 86.4, -160.8)

3. A continuous random variable X has probability density given by

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

   Find (a) $E(X)$ (b) $E(X^2)$. (Ans: ½, ½)

4. Suppose that

$$f(x) = \begin{cases} \frac{1}{5}, & x = 1,2,3,4,5, \\ 0, & elsewhere. \end{cases}$$

is the probability mass function of a discrete random variable X.

Compute $E(X)$ and $E(X^2)$.

Use these two results to find $E[(X+2)^2]$ by writing $(X+2)^2 = X^2 + 4X + 4$.

(Ans: 3, 11, 27)

3.2 Variance of a Random Variables

3.2.1 Introduction

- The mean, or expected value, of a random variable X is of special importance in statistics because it describes where the probability distribution is centered.
- By itself, however, the mean does not give an adequate description of the shape of the distribution.
- The most important measure of variability of a random variable X is obtained by applying Definition 3 with $g(X) = (X - \mu)^2$, where $\mu = E(X)$.
- The expectation of $g(X)$ is referred to as the *variance of the random variable* X or the *variance of the probability distribution* of X and is denoted by $Var(X)$ or the symbol σ_X^2 , or simply by σ^2 when it is clear to which random variable we refer.

3.2.1 Variance and Standard Deviation of a Random Variable

- **Definition 4 - Variance**

  Let X be a random variable with mean μ.

  Then the *variance* of the random variable X is denoted by $Var(X)$ or σ_X^2, and defined by

$$\sigma_X^2 = Var(X) = E[(X - \mu_X)^2].$$

- If X is a discrete random variable taking the values $x_1, x_2, \cdots, x_n$ and having probability mass function $f(x)$, then the variance of X can be expressed as

$$\sigma_X^2 = Var(X) = E[(X - \mu_X)^2] = \sum_{i=1}^{n}(x_i - \mu_X)^2 f(x_i).$$

  If X is a continuous random variable having probability density function $f(x)$, then the variance of X can be expressed as

$$\sigma_X^2 = Var(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty}(x - \mu_X)^2 f(x)\, dx.$$

- Note that, for any random variable X, $Var(X) \geq 0$.

- Definition 5 - Standard Deviation

  The positive square root of the variance, $\sqrt{Var(X)}$ or σ_X, is called the *standard deviation* of X.

- An alternative and preferred formula for finding $Var(X)$ or σ^2, which often simplifies the calculations, is stated in the following theorem.

- Theorem 3

  Let X be a random variable.

  Then $Var(X) = E(X^2) - [E(X)]^2$ or $Var(X) = E(X^2) - \mu^2$.

Proof: This can be easily proved using the definition of expectation and its properties.

Example 3.18

Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X.

x	0	1	2	3
$f(x) = P(X = x)$	0.51	0.38	0.10	0.01

Calculate σ^2.

Solution: First we compute,

$E(X) = 0 \cdot (0.51) + 1 \cdot (0.38) + 2 \cdot (0.10) + 3 \cdot (0.01) = 0.61$, and

$E(X^2) = 0^2.\,(0.51) + 1^2 \cdot (0.38) + 2^2 \cdot (0.10) + 3^2 \cdot (0.01) = 0.87.$

Using Theorem 3,

$$Var(X) = E(X^2) - [E(X)]^2 = 0.87 - (0.61)^2 = 0.4979.$$

Example 3.19

The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & elsewhere. \end{cases}$$

Find the mean, variance and standard deviation of X.

Solution: Let us first calculate $E(X)$ and $E(X^2)$.

$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_1^2 x \cdot 2(x-1)\, dx = 2\int_1^2 x(x-1)\, dx = \frac{5}{3}$, and

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_1^2 x^2 \cdot 2(x-1)\, dx = 2\int_1^2 x^2(x-1)\, dx = \frac{17}{6}$.

Therefore, by Theorem 3, $\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}$.

Standard deviation of $X = \sigma = \sqrt{1/18} = 0.23570$.

- Definition 6 - Variance of a Function of a Random variable

Let X be a random variable.

Then the variance of the random variable $g(X)$ is defined by

$$\sigma^2_{g(X)} = Var\big(g(X)\big) = E\left[\left(g(X) - \mu^2_{g(X)}\right)^2\right],$$

where $\mu_{g(X)} = E[g(X)]$.

Example 3.20

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution:

x	0	1	2	3
$f(x) = P(X = x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Solution: First, we find the mean μ of the random variable X.

$E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{8} = \frac{3}{2}.$

According to Theorem 2, $\mu_{2X+3} = E(2X + 3) = 2E(X) + 3 = 2 \cdot \frac{3}{2} + 3 = 6.$

Now using Definition 3, we have

$$\sigma^2_{2X+3} = E\left[\left((2X + 3) - \mu_{2X+3}\right)^2\right] = E[(2X - 3)^2] = \sum_{x=0}^{3}(2x - 3)^2 f(x)$$
$$= 4.$$

Example 3.21

Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & otherwise. \end{cases}.$$

Find the variance of the random variable $g(X) = 4X + 3$.

Solution: From Example 3.11 and Example 3.14, we have $E[4X+3] = 8$.

$$\sigma_{4X+3}^2 = E\left[\left((4X+3) - \mu_{4X+3}\right)^2\right]$$

$$= E[(4X-5)^2]$$

$$= \int_{-1}^{2} (4x-5)^2 \frac{x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (16x^4 - 40x^3 + 25)\, dx = \frac{51}{5}.$$

3.2.3 Properties of Variance

- Theorem 4: Let X be a random variable.

  If a and b are constants, then $Var(aX + b) = a^2 Var(X)$.

Proof: Try this as an exercise.

- Corollary 3: Setting $a = 0$, in the previous theorem we get $Var(b) = 0$, i.e.,

  if b is a constant, then $Var(b) = 0$.

- Corollary 4: Setting $b = 0$, in the previous theorem we get $Var(aX) = a^2 Var(X)$, i.e.,

  if a is a constant, then $Var(aX) = a^2 Var(X)$.

Exercises

1. Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that 0, 1, 2, or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures striking this subdivision.

2. The random variable X, representing the number of errors per 100 lines of software code, has the following probability distribution:

x	2	3	4	5	6
$f(x) = P(X = x)$	0.01	0.25	0.4	0.3	0.04

   Find the variance and standard deviation of X.

3. A dealer's profit, in units of $5000, on a new automobile is a random variable X having the density function given in Exercise 1 in Section 3.1.3. Find the variance of X.

4. Referring to Exercise 1 in Section 3.1.4, find $\sigma^2_{g(X)}$ for the function $g(X) = 3X^2 + 4$.
5. Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x) = P(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

   Find $E(X)$ and $E(X^2)$ and then, using these values, evaluate $E[(2X + 1)^2]$.
6. If a random variable X is defined such that $E[(X - 1)^2] = 10$ and $E[(X - 2)^2] = 6$, find μ and σ^2.

7. The total time, measured in units of 100 hours, that a teenager runs her hair dryer over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \leq x < 2, \\ 0, & elsewhere. \end{cases}$$

Evaluate the mean of the random variable $Y = 60X^2 + 39X$, where Y is equal to the number of kilowatt hours expended annually.

8. A manufacturing company has developed a machine for cleaning carpet that is fuel-efficient because it delivers carpet cleaner so rapidly. Of interest is a random variable Y , the amount in gallons per minute delivered. It is known that the density function is given by

$$f(y) = \begin{cases} 1, & 7 < x < 8, \\ 0, & elsewhere. \end{cases}$$

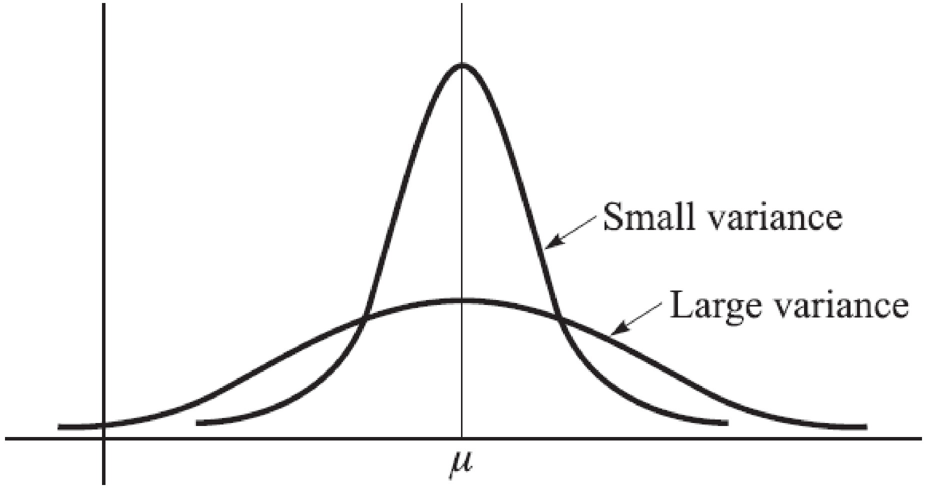
Find: $E(Y), E(Y^2)$, and $Var(Y)$.

- Note that if X has certain *dimensions* or *units*, such as *centimeters* (cm), then the variance of X has units cm^2, while the standard deviation has the same unit as X, i.e., cm.

  It is for this reason that the standard deviation is often used.

3.2.5 Chebyshev's Theorem

- The variance (or the standard deviation) is a measure of the *dispersion*, or *scatter*, of the values of the random variable about the mean .
- If the values tend to be concentrated near the mean, the variance is small; while if the values tend to be distributed far from the mean, the variance is large.
- Therefore, the probability that the random variable assumes a value within a certain interval about the mean is greater than for a similar random variable with a larger standard deviation.
- The situation is indicated graphically in the following figure for the case of two continuous distributions having the same mean μ.



- If we think of probability in terms of area, we would expect a continuous distribution with a large value of σ to indicate a greater variability, and therefore we should expect the area to be more spread out, as in the above figure.
- We can argue the same way for a discrete distribution as well.
- The following theorem, due to Chebyshev, gives a conservative estimate of the probability that a random variable assumes a value within k standard deviations of its mean for any real number k.

- Theorem 5 - Chebyshev's Theorem

  Let X be any random variable (discrete or continuous) with finite mean μ and variance σ^2.

  The probability that the random variable X will assume a value within k standard deviations of the mean μ is at least $1 - \frac{1}{k^2}$.

  That is, $P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$.

- Note that, since $P(\mu - k\sigma < X < \mu + k\sigma) = P(|X - \mu| \leq k\sigma)$, we can write the above inequality in the following form:

$$P(|X - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}.$$

  Further note that, since $P(|X - \mu| > k\sigma) = 1 - P(|X - \mu| \leq k\sigma)$, we can write the above inequality in the following equivalent form:

$$P(|X - \mu| > k\sigma) \leq \frac{1}{k^2}.$$

Example 3.22

A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, of an unknown probability distribution. Find bounds for

(a) $P(-4 < X < 20)$,

(b) $P(|X - 8| \geq 6)$.

Solution:

(a) $P(-4 < X < 20) = P[8 - 4(3) < X < 8 + 4(3)] \geq 1 - \frac{1}{4^2} = \frac{15}{16}$.

(b)
$$\begin{aligned} P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6) &= 1 - P(-6 < X - 8 < 6) \\ &= 1 - P[8 - 2(3) < X < 8 + 2(3)] \\ &\leq \frac{1}{4}, \end{aligned}$$

since $P[8 - 2(3) < X < 8 + 2(3)] \geq 1 - \frac{1}{2^2} = \frac{3}{4}$.

Exercise

1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{2\sqrt{3}}, & -\sqrt{3} < x < \sqrt{3}, \\ 0, & elsewhere. \end{cases}.$$

Find μ and σ^2. Hence, obtain a bound for $P\left(|X| \geq \frac{3}{2}\right)$.