МАТ3 (Оба колоквијума у збиру 25п за пролаз)

- Збирка више математике Миличић Ушћумлић

I колоквијум: Интеграли (одређени, неодређени)
Редови, диференцијалне једначине

Неодређени интеграли 11.10.2024.

Дефиниција: За ф-ју $F: I \to R$ где је I-интервал, кажемо да је примитивна ф-ја ф-је $f: I \to R$ ако је $F'(x) = f(x), x \in I$

Теорема: Ако је $F: I \to R$ примитивна ф-ја ф-је $f: I \to R$, тада је и $F(x)+c, c \in R$ такође примитивна ф-ја ф-је f.

Деф 2: Скуп свих примитивних ф-ја ф-је f називамо неодређени интеграл ф-је f и означавамо интеграл $\int f(x)dx$

Особине неодређеног интервала: $\int$- интеграл

1) $\left(\int f(x)\,dx\right)' = f(x)$

2) $\int F'(x)dx = F(x)+c$

3) $\int \lambda f(x)dx = \lambda \int f(x)dx$

4) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

$\int f(x)dx$	$F(x)+c$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1}+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^x	e^x+c
a^x	$\frac{a^x}{\ln a}+c$
$\sin x$	$-\cos x+c$
$\cos x$	$\sin x+c$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x+c$ или $-\arccos x+c$
$\frac{1}{1+x^2}$	$\operatorname{arctg} x+c$ или $-\operatorname{arcctg} x+c$
$\frac{1}{\cos^2 x}$	$\operatorname{tg} x+c$
$\frac{1}{\sin^2 x}$	$-\operatorname{ctg} x+c$

Задаци:

ш

−Рачунање интеграла за сваки задатак−

1. Свуда стављамо интеграл:

1. $\int \left(x^4 - \sqrt{x} + x\sqrt[3]{x} + \frac{1}{x^2}\right)dx = \int x^4 dx - \int \sqrt{x}\,dx + \int x\sqrt[3]{x}\,dx + \int \frac{1}{x^2}dx =$

$= \frac{x^{4+1}}{4+1} - \int x^{\frac{1}{2}}dx + \int x \cdot x^{\frac{1}{3}}dx + \int x^{-2}dx = \frac{x^5}{5} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \int x^{1+\frac{1}{3}}dx + \frac{x^{-2+1}}{-2+1} =$

$= \frac{x^5}{5} - \frac{x^{\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}+\frac{2}{2}} + \int x^{\frac{3}{3}+\frac{1}{3}}dx + \frac{x^{-1}}{-1} = \frac{x^5}{5} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \int x^{\frac{4}{3}}dx - \frac{1}{x} =$

2. Пишемо +C када у запису немамо више интеграла.

$= \frac{x^5}{5} - \frac{2}{3}x^{\frac{3}{2}} + \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} - \frac{1}{x} + C$

$= \frac{x^5}{5} - \frac{2}{3}\sqrt{x^3} + \frac{x^{\frac{7}{3}}}{\frac{7}{3}} - \frac{1}{x} + C = \frac{x^5}{5} - \frac{2}{3}\sqrt{x^3} + \frac{3}{7}\sqrt[3]{x^7} - \frac{1}{x} + C$

Погрешно записано треба обрнуто

2. $\int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx = \int \left(1 - \frac{1}{x^2}\right)\sqrt{x \cdot x^{\frac{1}{2}}}\,dx = \int \left(1 - \frac{1}{x^2}\right)\sqrt{x^{\frac{3}{2}}\cancel{dx}} = \int \left(1 - \frac{1}{x^2}\right)\left(x^{\frac{1}{2}}\right)^{\frac{3}{2}}dx =$

$= \int \left(1 - \frac{1}{x}\right)x^{\frac{3}{4}}dx = \int \left(x^{\frac{3}{4}} - \frac{1}{x^2}\cdot x^{\frac{3}{4}}\right)dx = \int \left(x^{\frac{3}{4}} - x^{-2}\cdot x^{\frac{3}{4}}\right)dx = \int \left(x^{\frac{3}{4}} - x^{-2+\frac{3}{4}}\right)dx =$

$= \int \left(x^{\frac{3}{4}} - x^{-\frac{5}{4}}\right)dx = \int x^{\frac{3}{4}}dx - \int x^{-\frac{5}{4}}dx = \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} - \frac{x^{-\frac{5}{4}+1}}{-\frac{5}{4}+1} + C = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} - \frac{x^{-\frac{1}{4}}}{-\frac{1}{4}} + C = \frac{4}{7}\sqrt[4]{x^7} + \frac{4}{\sqrt[4]{x}} + C$

3. $\int \frac{x^2 + x\sqrt[3]{x} + \sqrt{x}}{x\sqrt{x}}dx = \int \frac{x^2 + x\cdot x^{\frac{1}{3}} + x^{\frac{1}{2}}}{x \cdot x^{\frac{1}{2}}}dx = \int \frac{x^2 + x^{\frac{4}{3}} + x^{\frac{1}{2}}}{x^{\frac{3}{2}}}dx =$

$\int \left(\frac{x^2}{x^{\frac{3}{2}}} + \frac{x^{\frac{4}{3}}}{x^{\frac{3}{2}}} + \frac{x^{\frac{1}{2}}}{x^{\frac{3}{2}}}\right)dx = \int \left(x^{2-\frac{3}{2}} + x^{\frac{4}{3}-\frac{3}{2}} + x^{\frac{1}{2}-\frac{3}{2}}\right)dx = \int \left(x^{\frac{1}{2}} + x^{\frac{8}{6}-\frac{9}{6}} + x^{-1}\right)dx =$

$= \int x^{\frac{1}{2}}dx + \int \frac{x^{-\frac{1}{6}}}{1}dx + \int x^{-1}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1} + \ln|x| + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{5}{6}}}{\frac{5}{6}} + \ln|x| + C = \frac{2}{3}\sqrt{x^5} + \frac{6}{5}\sqrt[6]{x^2} + \ln|x| + C$

4. $\int \frac{x^4 + \sqrt{x^3} + \sqrt{x} + 3}{x\sqrt{x}}dx \leftarrow$ За домаћи

5. $\int \frac{(x^2+1)(x^2-2)}{\sqrt[3]{x^2}}dx = \int \frac{x^4 - 2x^2 + x^2 - 2}{x^{\frac{2}{3}}}dx = \int \frac{x^4 - x^2 - 2}{x^{\frac{2}{3}}}dx = \int \frac{x^4}{x^{\frac{2}{3}}} - \frac{x^2}{x^{\frac{2}{3}}} - \frac{2}{x^{\frac{2}{3}}}\,dx =$

$= \int \left(x^{4-\frac{2}{3}} - x^{2-\frac{2}{3}} - 2x^{-\frac{2}{3}}\right)dx = \int x^{\frac{10}{3}}dx - \int x^{\frac{4}{3}}dx - \int 2x^{-\frac{2}{3}}dx = \frac{x^{\frac{10}{3}+1}}{\frac{10}{3}+1} - \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} - 2\int x^{-\frac{2}{3}}dx =$

$= \frac{x^{\frac{13}{3}}}{\frac{13}{3}} - \frac{x^{\frac{7}{3}}}{\frac{7}{3}} - 2\cdot\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C = \frac{3}{13}\sqrt[3]{x^{13}} - \frac{3}{7}\sqrt[3]{x^7} - 2\cdot\frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C = \frac{3}{13}\sqrt[3]{2^{13}} - \frac{3}{7}\sqrt[3]{x^7} - 6\sqrt[3]{x} + C$

6. $\int \frac{18x^2 - 2}{3x-1}dx = \int \frac{2(9x^2-1)}{3x-1}dx = 2\int \frac{9x^2-1}{3x-1} = 2\int \frac{(3x-1)(3x+1)}{3x-1}dx = 2\int (3x+1)dx =$

$= 2\left(\int 3x\,dx + \int dx\right) = 2\left(3\int x\,dx + \int x^0 dx\right) = 2\cdot\left(3\frac{x^2}{2} + x\right) + C = 6\frac{x^2}{2} + 2x + C = 3x^2 + 2x + C$

7. $\int \frac{x^4}{1+x^2}dx = \int \frac{(x^4-1)+1}{1+x^2}dx = \int \left(\frac{x^4-1}{1+x^2} + \frac{1}{1+x^2}\right)dx = \int \frac{x^4-1}{1+x^2}dx + \int \frac{1}{1+x^2}dx =$

$= \int \frac{(x^2-1)\cancel{(x^2+1)}}{\cancel{1+x^2}}dx + \operatorname{arctg} x = \int (x^2-1)dx + \operatorname{arctg} x = \frac{x^3}{3} - x + \operatorname{arctg} x + C$

8. $\int \frac{4-x}{2+\sqrt{x}}dx \leftarrow$ За домаћи

Задачи: 11.10.2024.

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\sin^2 x + \cos^2 x = 1$

$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

– Рачунање интеграла за сваки задатак –
(тригонометрија)

9. $\int tg^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x} = \int\left(\frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}\right)dx = \int\frac{1}{\cos^2 x}dx - \int dx = tg\,x - x + C$

10. $\int \frac{\cos 2x}{\sin^2 x\cos^2 x}dx = \int\frac{\cos^2 x - \sin^2 x}{\sin^2 x\cos^2 x}dx = \int\frac{\cancel{\cos^2 x}}{\sin^2 x\cancel{\cos^2 x}}dx - \int\frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x}dx =$

$= \int\frac{1}{\sin^2 x}dx - \int\frac{1}{\cos^2 x}dx = -ctg\,x - tg\,x + C$

11. $\int\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2 dx = \int\left(\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}\right)dx = \int(1-\sin x)dx =$

$= \int dx - \int\sin x\,dx = x - (-\cos x) + C = x + \cos x + C$

12. $\int\sin^2\frac{x}{2}dx = \int\frac{1-\cos x}{2}dx = \frac{1}{2}\int(1-\cos x)dx = \frac{1}{2}(x - \sin x) + C$

13. $\int\left(4\cos x - \frac{5}{\sqrt{9-9x^2}}\right)dx = 4\int\cos x\,dx - \int\frac{5}{\sqrt{9(1-x^2)}}dx = 4\sin x - \frac{5}{3}\int\frac{1}{\sqrt{1-x^2}}dx = 4\sin x - \frac{5}{3}\arcsin x + C$

14. $\int\frac{e^{2x} + e^x\sin x}{e^x}dx = \int\left(\frac{e^{2x}}{e^x} + \frac{e^x\sin x}{e^x}\right)dx = \int(e^x + \sin x)dx = e^x - \cos x + C$

15. $\int\frac{2^{x+1} - 5^{x-1}}{10^x}dx = \int\left(\frac{2^{x+1}}{10^x} - \frac{5^{x-1}}{10^x}\right)dx = \int\left(\frac{2\cdot\cancel{2^x}}{\cancel{2^x}\cdot 5^x} - \frac{\cancel{5^x}\cdot 5^{-1}}{2^x\cdot\cancel{5^x}}\right)dx = \int\left(\frac{2}{5^x} - \frac{5^{-1}}{2^x}\right)dx =$

$= 2\int\frac{1}{5^x}dx - 5^{-1}\int\frac{1}{2^x}dx = 2\int\left(\frac{1}{5}\right)^x dx - \frac{1}{5}\int\left(\frac{1}{2}\right)^x dx = 2\cdot\frac{\left(\frac{1}{5}\right)^x}{\ln\left(\frac{1}{5}\right)} - \frac{1}{5}\cdot\frac{\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}} + C$

16. Одредити једначину криве $y = f(x)$ ако је познат извод ф-је $y' = 2\left(x - \frac{1}{x^3}\right)$ и тачка $M(1,2)$ која припада графику ф-је.

$y = \int y'dx = \int 2\left(x - \frac{1}{x^3}\right)dx = 2\int\left(x - \frac{1}{x^3}\right)dx = 2\left(\frac{x^2}{2} - \int x^{-3}dx\right) = 2\left(\frac{x^2}{2} - \frac{x^{-2}}{-2}\right) + C = x^2 + x^{-2} + C = x^2 + \frac{1}{x^2} + C$

$2 = 1^2 + \frac{1}{1^2} + C \qquad 2 = 1 + 1 + C \Rightarrow C = 0$

$y = x^2 + \frac{1}{x^2}$

f – ф $\quad \varphi$ – фи

– Смена променљивих –

$\int f(\varphi(x))\varphi'(x)dx = \begin{vmatrix}\varphi(x) = t\\ \varphi'(x)dx = dt\end{vmatrix} = \int f(t)dt$

17. $\int(x+1)^2dx = \begin{vmatrix}x+1 = t\\ 1dx = dt\end{vmatrix} = \int t^2dt = \frac{t^3}{3} + C = \frac{(x+1)^3}{3} + C$

18. $\int\frac{dx}{\sqrt[5]{(2x-3)^3}} = \int(2x-3)^{-\frac{3}{5}}dx = \begin{vmatrix}2x-3 = t\\ 2dx = dt\\ dx = \frac{dt}{2}\end{vmatrix} = \int t^{-\frac{3}{5}}\cdot\frac{dt}{2} = \frac{1}{2}\cdot\frac{t^{-\frac{3}{5}+1}}{-\frac{3}{5}+1} + C = \frac{1}{2}\cdot\frac{t^{\frac{2}{5}}}{\frac{2}{5}} + C =$

$= \frac{1}{2}\cdot\frac{5}{2}\cdot\sqrt[5]{t^2} + C = \frac{5}{4}\sqrt[5]{(2x-3)^2} + C$

19. $\int\sin 2x\,dx = \begin{vmatrix}2x = t\\ 2dx = dt\\ dx = \frac{dt}{2}\end{vmatrix} = \int\sin t\frac{dt}{2} = \frac{1}{2}(-\cos t) + C = -\frac{\cos 2x}{2} + C$

20. $\int \frac{dx}{\cos^2(4x-3)} = \left| \begin{array}{l} 4x-3=t \\ 4dx=dt \\ dx=\frac{dt}{4} \end{array} \right| = \int \frac{\frac{dt}{4}}{\cos^2(t)} = \frac{1}{4}\int \frac{dt}{\cos^2 t} = \frac{1}{4}\, tg\, t + C = \frac{1}{4}\, tg\,(4x-3) + C$

21. $\int e^{3x-4}dx = \left| \begin{array}{l} 3x-4=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{array} \right| = \int e^t \cdot \frac{dt}{3} = \frac{1}{3}\int e^t dt = \frac{1}{3} e^t + C = \frac{1}{3} e^{3x-4} + C$

22. $\int \frac{dx}{5-x} = \left| \begin{array}{l} 5-x=t \\ -dx=dt \\ dx=-dt \end{array} \right| = \int \frac{-dt}{t} = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|5-x| + C$

23. $\int \frac{x\,dx}{1+x^2} = \left| \begin{array}{l} 1+x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{dt}{2} \end{array} \right| = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|1+x^2| + C$

24. $\int \frac{x\,dx}{(1+x^2)^2} \leftarrow$ За домаћу