Probabilistic programming exercises 4. Jan Boerman & Jasper van Rooijen

1 a

$$\begin{aligned}
&wp(\text{if } (x=0)\ \{\text{skip}\}\ \text{else}\{x := 1+x\ [\tfrac{1}{3}]\ x:=6\},\ x) \\
&= [x=0]\cdot wp(\text{skip}, x) + [\neg x=0]\cdot wp(x:=1+x\ [\tfrac{1}{3}]\ x:=6, x) \\
&= [x=0]\cdot x + [\neg x=0]\cdot\left(\tfrac{1}{3}\cdot wp(x:=1+x, x) + \tfrac{2}{3}\cdot wp(x:=6, x)\right) \\
&= 0 + [x\neq 0]\cdot\left(\tfrac{1}{3}\cdot[1+x] + \tfrac{2}{3}\cdot 6\right) \\
&= [x\neq 0]\cdot\left(\tfrac{1}{3}[1+x] + 4\right)
\end{aligned}$$

b

$$\begin{aligned}
&wp(\{\text{while}(\text{true})\{x:=1\}\}\ [\tfrac{1}{2}]\ \{\text{diverge}\}, 1) \\
&= \tfrac{1}{2}\cdot wp(\text{while}(\text{true})\{x:=1\}, 1) + \tfrac{1}{2}\cdot wp(\text{diverge}, 1) \\
&= \tfrac{1}{2}\cdot \text{lfp}\ X.\ ([\text{true}]\cdot wp(x:=1, X) + [\neg\text{true}]\cdot 1) + \tfrac{1}{2}\cdot 0 \\
&= \tfrac{1}{2}\cdot \text{lfp}\ X.\ wp(x:=1, X) \\
&= \tfrac{1}{2}\cdot 0 \\
&= 0
\end{aligned}$$

Ex 2

Prove that $wp(P, [F]) = [wp(P, F)]$

Proof by structural induction

base cases

(1) "skip" $wp(skip, [F]) = [wp(skip, F)]$

$$[F] = [F]$$

(2) "diverge" $wp(diverge, [F]) = [wp(diverge, F)]$

$$\lambda s.\, 0 = [false]$$

$$\lambda s.\, 0 = \lambda s.\, 0$$

(3) "assignment" $wp(x := E, [F]) = [wp(x := E, F)]$

$$\lambda s. [F][x := E] = \lambda s. [F[x := E]]$$

case $F(s) = true$:

$$1 \cdot [x := E] = [true[x := E]]$$

$$[x := E] = [x := E]$$

case $F(s) = false$:

$$0 \cdot x := E = [false[x := E]]$$

$$0 = [false]$$

$$0 = 0$$

Induction Hypothesis:

Suppose we know that for any arbitrary programs P and Q

we know that $wp(P, [F]) = [wp(P, F)]$

and $wp(Q, [F]) = [wp(Q, F)]$ hold.

Induction cases

(4) "sequential composition"

$$wp(P;Q,[F]) = [wp(P;Q,F)]$$
$$wp(P, wp(Q,[F])) = [wp(P, wp(Q,F))]$$
$$wp(P,[wp(Q,F)]) = [wp(P, wp(Q,F))]$$
$$[wp(P, wp(Q,F))] = [wp(P, wp(Q,F))]$$

(apply IH Q)

(apply IH P)

(5) "if then else"

$$wp(\text{if } (G)\, P \text{ else } Q, [F]) = [wp(\text{if}(G)\, P \text{ else } Q, F)]$$
$$[G]\cdot wp(P,[F]) + [\neg G]\cdot wp(Q,[F]) = [(G \wedge wp(P,F)) \vee (\neg G \wedge wp(Q,F))]$$
$$[G]\, wp(P,[F]) + [\neg G]\, wp(Q,[F]) = [G]\cdot[wp(P,F)] + [\neg G]\cdot[wp(Q,F)]$$
$$[G]\cdot wp(P,[F]) + [\neg G]\cdot wp(Q,[F]) = [G]\cdot wp(P,[F]) + [\neg G]\cdot wp(Q,[F])$$

(apply IH twice for the right hand side)

(6) "while" $\cancel{wp(P,[F]) = [wp(P,F}$

$$wp(\text{while } (G)\, P, [F]) = [wp(\text{while } (G)\, P, F)]$$
$$\text{lfp } X.\, ([G]\cdot wp(P,X) + [\neg G]\cdot[F]) = [\text{lfp } X.\, ((G \wedge wp(P,X)) \vee (\neg G \wedge F))]$$
$$\cancel{\text{lfp } X([G]\, wp(P,X) + [\neg G]\, [F]} =$$
$$\sup_{n\in\mathbb{N}} ([G]\cdot wp(P,X) + [\neg G]\cdot[F])^n(0) = [\sup_{n\in\mathbb{N}} ((G \wedge wp(P,X)) \vee (\neg G \wedge F))^n(\text{false})]$$
$$\sup_{n\in\mathbb{N}} ([G]\cdot wp(P,X) + [\neg G]\cdot[F])^n(0) = \sup_{n\in\mathbb{N}} [(G \wedge wp(P,X)) \vee (\neg G \wedge F)]^n(0)$$
$$\sup_{n\in\mathbb{N}} ([G]\cdot wp(P,X) + [\neg G]\cdot[F])^n(0) = \sup_{n\in\mathbb{N}} ([G]\cdot[wp(P,X)] + [\neg G]\cdot[F])^n(0)$$

now apply IH then the formulas are the same

Ex 3

$$wp(P,x) = wp(x:=1, wp(y:=0, \underbrace{wp(\text{while} \ldots, x)}_{\psi(x)}))$$

$$\psi(x) = [y=0] \cdot wp(x:=2x; \{y:=0\}[\tfrac{1}{4}]\{y:=1\}, x) + [y \neq 0] \cdot x$$

$$\psi(x) = [y=0] \cdot wp(x:=2x, \tfrac{1}{4} wp(y:=0, x) + \tfrac{3}{4} wp(y:=1, x)) + [y \neq 0] \cdot x$$

$$\psi(x) = [y=0] \cdot wp(x:=2 \cdot x, \tfrac{1}{4} x [y:=0] + \tfrac{3}{4} x [y:=1]) + [y \neq 0] \cdot x$$

$$\psi(x) = [y=0] \cdot \left(\tfrac{2}{4} x [y:=0] + \tfrac{6}{4} x [y:=1]\right) + [y \neq 0] \cdot x$$

$$\psi^0(x) = 1$$

$$\psi^1(x) = 2x$$

$$\psi^2(x) = 2x + \tfrac{1}{4} 2^2 x$$

$$\psi^3(x) = 2x + \tfrac{1}{4} 2^2 + \tfrac{1}{16} 2^3 x$$

$$\vdots$$

$$\psi^n(x) = \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{k-1} \cdot 2^k \cdot x$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2k-2} \cdot \left(\frac{1}{2}\right)^{-k} \cdot x = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k-2} x = 4 \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} x$$

$$= 4 \cdot \left(\frac{1}{1-\frac{1}{2}}\right) x = 4 \cdot 2x = 8x$$

$$wp(P,x) = wp(x:=1, wp(y:=0, \tfrac{1}{2}x)) = 8$$

~~welp something went wrong~~

welp something went wrong.....

Program transformation is not compositional. Assume program P to be a trivial, terminating program.
Assume that program Q is:

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1. Observe (x == 0)
2. if (x) {
3.    diverge;
4 }
5. x = 1;
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When considering wp (Q, f), as should be done in the case wp ($\widehat{P;Q}$, f), the wp will consider the diverge-statement on line 3.
However, when considering wp ($\hat{Q}$, f), the diverge-statement will not be considered, as ~~the~~ the if-statement can safely be ignored. This can be seen when constructing the weakest pre-expectations:

$$\begin{aligned} wp(Q, f) &= wp(\text{if } x \text{ then diverge else skip; end}, f) \\ &= [x] \cdot wp(\text{diverge}, f) + [\neg x] \cdot wp(\text{skip}, f) \\ &= [x] \cdot 0 + [\neg x] \cdot f = [\neg x] \cdot f \end{aligned}$$

$$\begin{aligned} wp(\hat{Q}, f) &= wp(\text{if } x \text{ then diverge else skip; } x := 0\text{; end}, f) \\ &= wp(\text{if } x \text{ then diverge else skip; }, f[x := 0]) \\ &= [x] \cdot wp(\text{diverge}, f[x:=0]) + [\neg x] \cdot wp(\text{skip}, f[x:=0]) \\ &= 0 \cdot 0 + 1 \cdot f[x:=0] = f[x:=0] \end{aligned}$$

Therefore we can conclude that the moment of transformation is significant and not compositional, therefore:

$$wp(\widehat{P;Q}, f) \neq wp(\hat{P};\hat{Q}, f)$$