Fluid Thesis

Andrew Martin

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Abstract

Just so I don’t forget that there is an abstract environment...

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0.1 Introduction

0.2 Derivation of the Squire-Long equation

Squire-long / Bragg-Hawthorne equation for the stream function of axisymmetric inviscid fluid, using cylindrical coordinates

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$

radial component u, azimuthal (swirl) is v, axial component w

stream function satisfies

$\nabla \cdot u = 0 \longrightarrow$ streamfunction exists

Remember for cylindrical coordinates:

$$u = \frac{1}{r}\frac{\partial\Psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial\Psi}{\partial r}$$

Ψ is the stream function
r is the radius
$C = rv$
$H = \frac{p}{\rho} + \frac{1}{2}(u^2 + v^2 + w^2)$
H is conserved on stream surfaces
C is conserved on stream surfaces

vorticity

$$w = w_r e_r + w_\theta e_\theta + w_z e_z$$

where w_r, w_θ, w_z can be written in terms of the velocity

Considering cylindrical coordinates (z,r,θ) with corresponding velocity (u,v,w), vorticity components (w_z, w_r, w_θ). Axisymmetric flow as:

$$\omega_z = \frac{1}{r}\frac{\partial rv}{\partial r}, \quad \omega_r = -\frac{\partial rv}{\partial z}, \quad \omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

The continuity equation (conservation of mass) is satisfied by setting

$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad u = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$$

Where Ψ is the stream function This gives the azimuthal component for w_θ:

$$\begin{aligned}\omega_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\\ &= -\frac{1}{r}\frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \Psi}{\partial r}\\ &= -\frac{1}{r}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r}\right)\end{aligned}$$

Use the vorticity equation

$$w \times v - \frac{\partial w}{\partial t} = \nabla H$$

Where

$$H = \frac{1}{2}(w^2 + u^2 + v^2) + \frac{p}{\rho}$$

This gives:

$$\begin{aligned}u\omega_\theta - v\omega_r - \frac{\partial w}{\partial t} &= \frac{\partial H}{\partial x}\\ v\omega_z - w\omega_\theta - \frac{\partial u}{\partial t} &= \frac{\partial H}{\partial r}\\ w\omega_r - u\omega_z - \frac{\partial v}{\partial t} &= 0\end{aligned}$$

The last one is equivalent to the material derivative of rw set to 0:

$$\frac{D(rv)}{Dt} = 0$$

From the Bernoulli equation:

$$rv = C(\Psi)$$

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}|\mathbf{w}|^2 + \frac{p}{\rho} = H(\Psi)$$

Where $H(\Psi)$ and $C(\Psi)$ are arbitrary functions.

Rewriting ω:

$$\omega_z = w\frac{dC}{d\Psi}, \quad \omega_r = u\frac{dC}{d\Psi}$$

Giving

$$\frac{\omega_\theta}{r} = \frac{v\omega_r}{ru} + \frac{1}{ru}\frac{dH}{d\Psi}\frac{\partial \Psi}{\partial z} = \frac{C}{r^2}\frac{dC}{d\Psi} - \frac{dH}{d\Psi}$$

Which is the form taken by the second of the dynamic equations. Now, combining this last statement with the equation for ω_θ:

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$

Taken from Batchelor's An Introduction to Fluid Dynamics

Considering the flow far upstream where there is constant uniform axial velocity and rotates with angular velocity Ω

$$\Psi_{\text{upstream}} = \frac{1}{2}Wr^2$$
$$v = \Omega r, w = W$$

And

$$C = rv = \frac{v^2}{\Omega} = \Omega r^2 = 2\Omega\Psi/W$$
$$\frac{dC}{d\Psi} = 2\Omega/W$$

Since the flow is steady, the radial equation of motion yields:

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{w^2}{r} = \frac{C^2}{r^3}$$

$$\begin{aligned}
H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\
&= \frac{1}{2}(\Omega^2 r^2 + W^2) + \frac{p}{\rho} \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \frac{p}{\rho} \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \frac{1}{\rho}\frac{dp}{dr}dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \frac{C^2}{r^3}dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \frac{\Omega^2 r^4}{r^3}dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \Omega^2 r dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \frac{1}{2}\Omega^2 r^2 \\
&= \frac{2\Omega^2\Psi}{W} + \frac{1}{2}W^2
\end{aligned}$$

$$\begin{aligned}
\frac{dH}{d\Psi} &= \frac{\partial \frac{2\Omega^2\Psi}{W}}{\partial\Psi} \\
&= \frac{2\Omega^2}{W}
\end{aligned}$$

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$
$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = \frac{2r^2\Omega^2}{W} - \frac{4\Omega^2}{W^2}\Psi$$

Or in a more 'standard' form

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{4\Omega^2}{W^2}\Psi = \frac{2r^2\Omega^2}{W}$$

0.2.1 Homogeneous ODE

Considering the case where Ψ is just a function of the radius, r. So Ψ does not depend on z, and $\frac{\partial^2 \Psi}{\partial z^2} = 0$

To simplify it into a homogeneous ODE, a change of variables is used:

$$\Psi = \frac{1}{2}Wr^2 + \psi = \frac{1}{2}Wr^2 + rF$$

$$\frac{\partial \Psi}{\partial r} = Wr + F + r\frac{\partial F}{\partial r}$$
$$\frac{\partial^2 \Psi}{\partial r^2} = W + 2\frac{\partial F}{\partial r} + r\frac{\partial^2 F}{\partial r^2}$$

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \Psi(\frac{4\Omega^2}{W^2} - \frac{1}{r^2}) = 0$$

$$r^2\frac{d^2 F}{dr^2} - r\frac{dF}{dr} + F(r^2k^2 - 1) = 0$$

Letting $k = \frac{2\Omega}{W}$ If we take $x = kr$, $\frac{dF}{dr} = \frac{dF}{dx}\frac{dx}{dr} = k$ and $\frac{d^2F}{dr^2} = k^2\frac{d^2F}{dx^2}$

$$\frac{x^2}{k^2}k^2\frac{d^2 F}{dx^2} - \frac{x}{k}k\frac{dF}{dx} + F(\frac{x^2}{k^2}k^2 - 1) = 0$$
$$x^2\frac{d^2 F}{dx^2} - x\frac{dF}{dx} + F(x^2 - 1) = 0$$

Which is the form of a bessel differential equation of order $\nu = 1$, giving solutions

$$F = AJ_1(kr) + BY_1(kr)$$

Returning to the streamfunction:

$$\Psi = \frac{1}{2}Wr^2 + r\left(AJ_1(kr) + BY_1(kr)\right)$$

And hence

$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r} = W + AkJ_0(kr) + BkY_0(kr)$$

A, and B rely on boundary conditions. In this case, it is necessary forthe streamlines to be the same as at the inlet along the boundary. Also introduce a vortex breakdown condition in the core of the stream, i.e. a region $0 < r < r_*$ where the streamfunction becomes zero:

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

Consider it as a matrix system

$$\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ RJ_1(kR) & RY_1(kR) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ 0 \end{pmatrix}$$

Giving

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{r_* R \left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \begin{pmatrix} RY_1(kR) & -r_* Y_1(kr_*) \\ -RJ_1(kR) & r_* J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ 0 \end{pmatrix}$$

$$\begin{aligned} A &= \frac{-\frac{1}{2}RWr_*^2 Y_1(kR)}{r_* R \left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \\ B &= \frac{\frac{1}{2}RWr_*^2 J_1(kR)}{r_* R \left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \end{aligned}$$

And hence

$$\begin{aligned} A &= \frac{-\frac{1}{2}Wr_* Y_1(kR)}{\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \\ B &= \frac{\frac{1}{2}Wr_* J_1(kR)}{\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \end{aligned}$$

With the requirement that $r_* \neq R$ so as to not divide by zero.

Using

$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r}$$

Gives

$$w = W + k(AJ_0(kr) + BY_0(kr))$$

Solving this for a given k (or alternatively a desired r_*) is done numerically using `MATLAB`.

The set of valid solutions to this problem are those which satisfy the constraint

$$w(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

The plot figure 0.2.1 shows the k, r_* combinations which satisfy the constraint.

Clearly this can only occur for values of $kR > 3.8$.

The first branch of this (extending from $kR \approx 3.8$) corresponds to natural solutions, whereas further branches give unwanted behaviour, which introduce reversed flow.

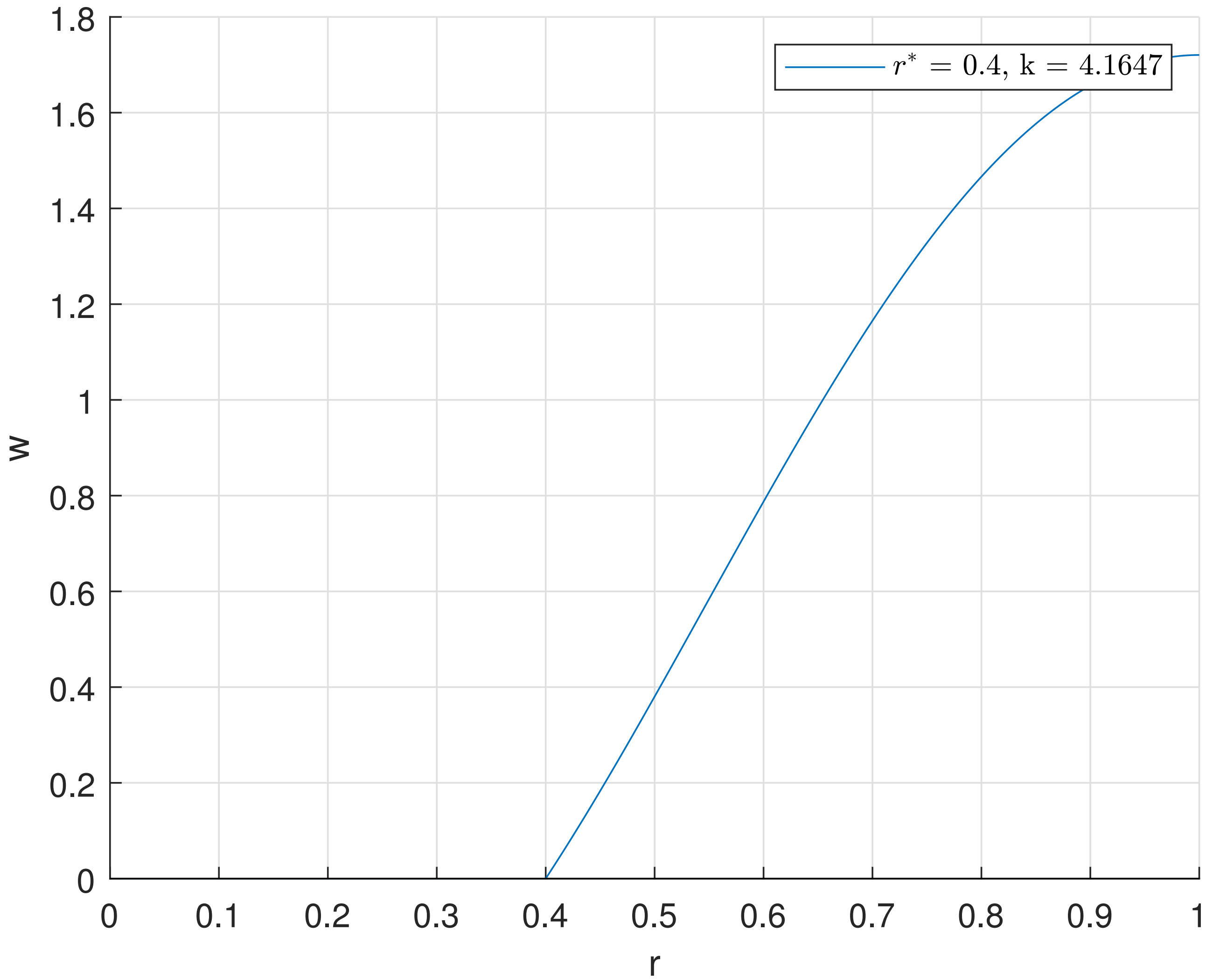

Figure 1: An example solution plot

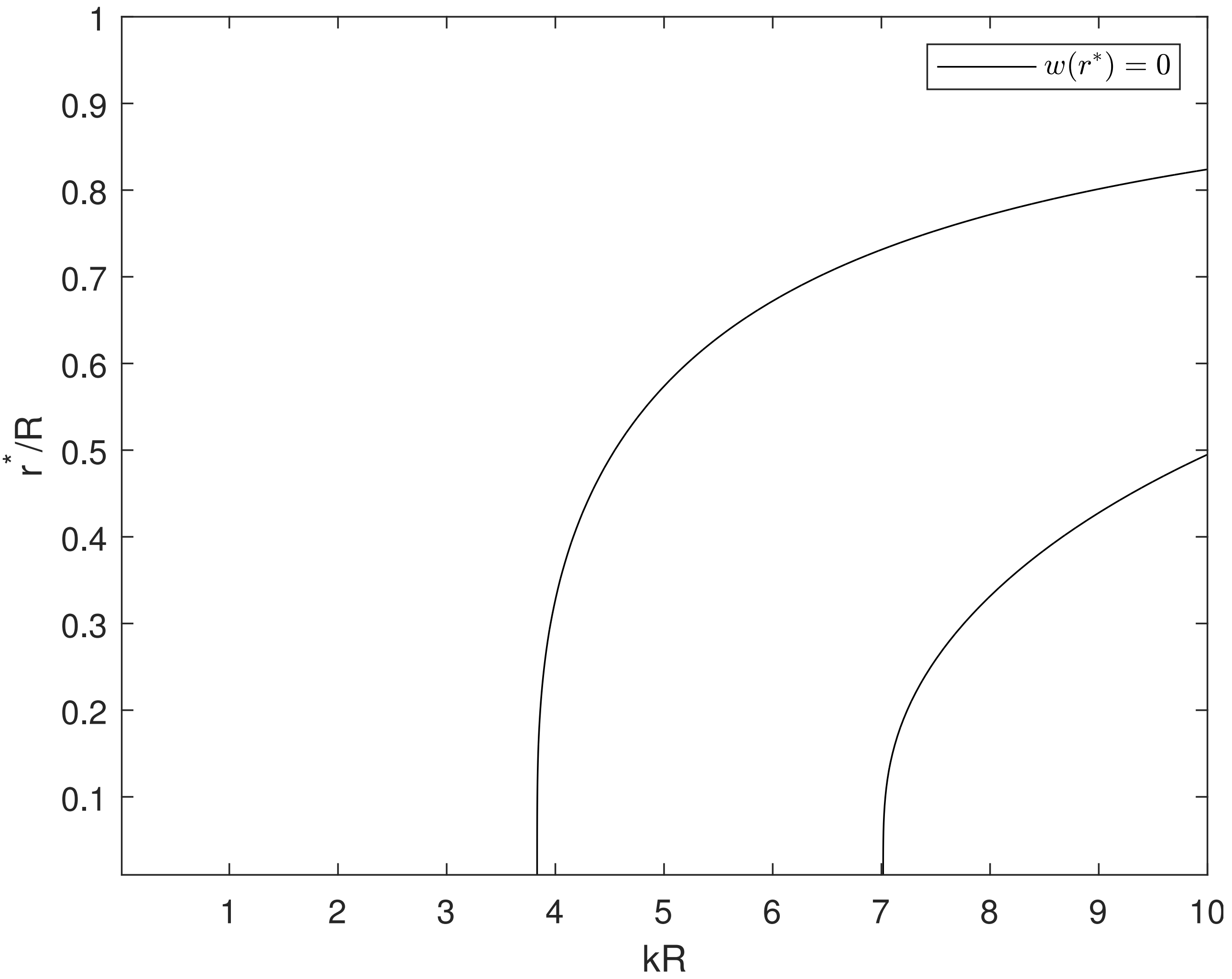

Figure 2: Solution set for the simplified problem

have to assume things for outside of the region for Ψ. I.e. if we go above the maximum input value then some assumption, and if we go below the minimum then it is a stagnation point

see if we can do it for the wall stagnation zones (i.e. psi goes to 0 near R) so when $\Psi > \frac{1}{2}WR^2$ Plug it into H and C

$$H = (\Omega R)^2 + \frac{1}{2}W^2$$

$$\frac{\partial H}{\partial \psi} = 0$$
$$C = \Omega R^2$$
$$\frac{\partial C}{\partial \Psi} = 0$$

Which then yields the separable first order ODE

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = 0$$

And hence

$$\frac{\partial \Psi}{\partial r} = Ar$$
$$\Psi = \frac{1}{2}Ar^2 + B$$

our left hand side could be written as

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)$$

using staggered grid

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{r}\frac{\partial \Psi}{\partial r}$$

at the boundary r=0

0.2.2 Numerics

Solving the ODE numerically:

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = r^2\frac{\partial H}{\partial \Psi} + C\frac{\partial C}{\partial \Psi}$$

finite difference - divide r as a grid of N intervals. So our grid spaces over R,

$$r_i = \Delta r_i, \quad \Delta = \frac{R}{N}$$

So (check this)

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}}{\Delta^2}$$
$$\frac{\partial \Psi}{\partial r} = \frac{\Psi_{i+1} - \Psi_{i-1}}{2\Delta}$$
$$\Psi_0 = 0, \quad \Psi_N = \frac{1}{2}WR^2$$

Which should work for the index i until we reach the bifurcations/stagnations

Should end up with a matrix equation

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ & & \mathbf{A} & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \mathbf{\Psi} \\ \Psi_N \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ \frac{1}{2}WR^2 \end{pmatrix}$$

A should be the finite difference version of

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = 0$$

I.e. for the i^{th} row of $\mathbf{A}$

$$A(i) = \frac{A(i+1) - 2 * A(i) + A(i-1)}{\Delta^2} - \frac{A(i+1) - A(i-1)}{2r(i)\Delta}$$

$$A_{ij} = \begin{cases} 1 & j = i = 1 \\ 1/\Delta^2 + 1/(2r_i\Delta) & j = i-1 \\ 2/\Delta^2 & j = i \\ 1/\Delta^2 - 1/(2r_i\Delta) & j = i+1 \\ 1 & j = i = N \\ 0 & otherwise \end{cases}$$

For the full equation

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \Psi(\frac{4\Omega^2}{W^2} - \frac{1}{r^2}) = 0$$

$$\Psi = \frac{1}{2}Wr^2 + rF$$

$$F = \frac{\Psi}{r} - \frac{1}{2}Wr$$

Boundary conditions for F relate to those for Ψ.

$$\Psi(R) = \frac{1}{2}WR^2 \implies F(R) = 0$$

$$\Psi(r_*) = 0 \implies F(r_*) = \frac{1}{2}Wr_*^2$$

when we look at the vortex breakdown problem, introduce a coordinate transformation

$$\eta = \frac{r - r_*}{R - r_*}$$

$$\eta = 0, r = r_*, \eta = 1 r = R$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{1}{R - r_*}\frac{\partial \Psi}{\partial \eta}$$

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{(R - r_*)^2}\frac{\partial^2 \Psi}{\partial \eta^2}$$

use the same conditions we have used anyway where $\Psi(r_*) = w(r_*) = 0$ Rankine body problem: At some point on the radius r_0, we get $v = K/r_0$ for some constant K find $K = \Omega r_0^2$?

0.2.3 Rankine Body

$w = W$,

$$v = \begin{cases} \frac{\Gamma}{2\pi r}, & r > r_0 \\ \Omega r, & r \leq r_0 \end{cases}$$

Where the second condition was the previous solution. Since the velocity profile is now piecewise defined, the streamfunction must also be, i.e. it is necessary to split the streamfunction into 2 regions to solve this problem. The upstream regions:

$$\begin{cases} \Psi_{inner}, & 0 \leq r \leq r_0 \\ \Psi_{outer}, & r_0 \leq r \leq R \end{cases}$$

Note that r_0 is defined upstream, so the position of the region may have moved downstream to a new radius, $\hat{r}$, and hence, downstream, these regions will become around $\hat{r}$ instead of r_0. We enforce some similar conditions as to the normal problem:

$$\begin{aligned} \Psi(r_*) &= 0, \\ \Psi(R) &= \frac{1}{2}WR^2, \\ w(r_*) &= 0 \end{aligned}$$

With the added condition that Ψ must remain continuous around $\hat{r}$ I.e.

$$\lim_{r^- \to \hat{r}} \Psi(r^-) = \lim_{r^+ \to \hat{r}} \Psi(r^+)$$

And

$$\lim_{r^- \to \hat{r}} v(r^-) = \lim_{r^+ \to \hat{r}} v(r^+)$$

Where $\Psi(r^-)$ is Ψ defined for $r \leq \hat{r}$ and $\Psi(r^+)$ is defined in the region $r \geq \hat{r}$.

The region for $\Psi(r)$ with $r \in [0, r_0]$ will be the same as before, i.e.

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

For the region $r_0 < r < R$ the problem must be resolved from the SL equation

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C\frac{dC}{d\Psi} \\ C = rv &= \frac{\Gamma}{2\pi} \\ \frac{dC}{d\Psi} &= 0 \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2}(0 + \frac{\Gamma^2}{4\pi^2 r^2} + W^2) + \int \frac{C^2}{r^3} dr \\ &= \frac{1}{2}(\frac{\Gamma^2}{4\pi^2 r^2} + W^2) + \int \frac{\Gamma^2}{4\pi^2 r^3} dr \\ &= \frac{1}{2}(\frac{\Gamma^2}{4\pi^2 r^2} + W^2) - \frac{\Gamma^2}{8\pi^2 r^2} \\ &= \frac{W^2}{2} \\ \frac{dH}{d\Psi} &= 0 \end{aligned}$$

And hence the SL equation gives

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$
$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$
$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = 0$$

Which results in:

$$\Psi = Cr^2 + D, \quad r \geq \hat{r}$$
$$w = \frac{1}{r}\frac{\partial\Psi}{\partial r} = 2C$$

With the requirement that there is no discontinuity at $\hat{r}$, i.e.

$$\Psi = \frac{1}{2}W\hat{r}^2 + \hat{r}\left(AJ_1(k\hat{r}) + BY_1(k\hat{r})\right) = C\hat{r}^2 + D$$

And using the same for w

$$w(\hat{r}) = W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) = 2C$$

And lastly the wall condition

$$\Psi(R) = \frac{1}{2}WR^2 = C\hat{r}^2 + D$$

With

$$w(r_*) = 0$$
$$\frac{\Gamma}{2\pi r_0} = \Omega r_0 \implies \Omega = \frac{\Gamma}{2\pi r_0^2}$$
$$k_{outer} = \frac{2\Gamma}{2\pi W r_0^2} = \frac{\Gamma}{\pi W r_0^2}$$

Noting that the values for A and B are obtained from the r_* condition.

The coefficients for Ψ have to be resolved, since the condition $\Psi_{inner}(R) = \frac{1}{2}WR^2$ cannot be imposed.

Parameters

$$r_0, \hat{r}, r_*, R, k, \Gamma, W, A, B, C, D$$

We can fix r_0, R, k, W and Γ. This is 11 parameters, where 5 are fixed. Require 6 conditions. Impose:

1). $w(r_*) = 0$ (as before)

2). $\Psi_{inner}(r_*) = 0$ (as before)

3). Since at the wall Ψ must remain the same, this applies to where v is changed, i.e. $\Psi_{inner}(\hat{r}) = \frac{1}{2}Wr_0^2$

4). For continuity, $\Psi_{outer}(\hat{r}) = \frac{1}{2}Wr_0^2$

5). $w_{outer}(\hat{r}) = w_{inner}(\hat{r})$

6). $\Psi_{outer}(R) = \frac{1}{2}WR^2$

Redo the problem instead getting A, B from 2) and 3)

$$\Psi_{inner}(r_*) = 0$$
$$\Psi_{inner}(\hat{r}) = \frac{1}{2}Wr_0^2$$

Use this for A, B

$$\begin{aligned}
\Psi_{inner}(r_*) &= \frac{1}{2}Wr_*^2 + r_*(AJ_1(kr_*) + BY_1(kr_*)) = 0 \\
&= r_*(AJ_1(kr_*) + BY_1(kr_*)) = -\frac{1}{2}Wr_*^2 \\
\Psi_{inner}(\hat{r}) &= \frac{1}{2}W\hat{r}^2 + \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}Wr_0^2 \\
&= \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}W(r_0^2 - \hat{r}^2)
\end{aligned}$$

This gives the matrix system for A, B below. Note that the system relies on the unknowns r_* and $\hat{r}$.

$$\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ \hat{r}J_1(k\hat{r}) & \hat{r}Y_1(k\hat{r}) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{det} \begin{pmatrix} \hat{r}Y_1(k\hat{r}) & -r_*Y_1(kr_*) \\ -\hat{r}J_1(k\hat{r}) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix}$$

$$\begin{aligned}
A &= \frac{1}{\det}\left(\hat{r}Y_1(k\hat{r})\left(-\frac{1}{2}Wr_*^2\right) - r_*Y_1(kr_*)\left(\frac{1}{2}W(r_0^2 - \hat{r}^2)\right)\right) \\
B &= \frac{1}{\det}\left(-\hat{r}J_1\left(-\frac{1}{2}Wr_*^2\right) + r_*J_1(kr_*)\left(\frac{1}{2}W(r_0^2 - \hat{r}^2)\right)\right)
\end{aligned}$$

Where

$$\begin{aligned}
\det &= \hat{r}r_*Y_1(k\hat{r})J_1(kr_*) - \hat{r}r_*J_1(k\hat{r})Y_1(kr_*) \\
&= \hat{r}r_*\left(Y_1(k\hat{r})J_1(kr_*) - J_1(k\hat{r})Y_1(kr_*)\right)
\end{aligned}$$

This for r_*

$$w_{inner}(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

Get C from:

$$\begin{aligned}
w_{outer}(\hat{r}) &= w_{inner}(\hat{r}) \\
2C &= W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) \\
C &= \frac{1}{2}\left(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r}))\right)
\end{aligned}$$

Get D here:

$$\Psi_{outer}(R) = CR^2 + D = \frac{1}{2}WR^2$$
$$D = \frac{1}{2}WR^2 - C$$

Hence get $\hat{r}$ from

$$\Psi_{outer}(\hat{r}) = C\hat{r}^2 + D = \frac{1}{2}Wr_0^2$$
$$C\hat{r}^2 + \frac{1}{2}WR^2 - C = \frac{1}{2}Wr_0^2$$
$$\left(\frac{1}{2}\left(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r}))\right)\right)(\hat{r}^2 - 1) = \frac{1}{2}W(r_0^2 - R^2)$$
$$\left(AJ_0(k\hat{r}) + BY_0(k\hat{r})\right)(\hat{r}^2 - 1) = \frac{1}{k}W(r_0^2 - R^2 - 1)$$

For physically valid solutions, we must impose the condition of no net change on the momentum from upstream to downstream on the momentum (Escudier, Keller). The momentum is defined as

$$s = 2\pi \int_0^{r_t} \left(\rho w^2 + p\right) r dr$$

Which comes to:

$$\Delta s = \frac{\pi}{4}\rho U^2 k^2 r_c^2 \left[-r_b^2 + \frac{1}{4}\left(\frac{r_b^4 - r_a^4}{r_c^2}\right) + \frac{3}{4}r_c^2 + \frac{1}{2}r_c^2 \log\left(\frac{r_b^2}{r_c^2}\right)\right] = 0$$

0.3 Burger's Vortex

Q-vortex without a Jet. Start with

$$w = W$$
$$v = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/\delta^2}\right)$$
$$\frac{d^2\Psi}{dr^2} - \frac{1}{r}\frac{d\Psi}{dr} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$

Solve from r_* to R numerically.

Generate grid from r_* to R.

Boundary conditions as normal

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

$$w(r_*) = 0$$

Non-dimensional parameter may be something like $\frac{\Gamma}{WR}$ (we can probably relate this to kr_0 for the rankine problem)

Eventually do the same thing as before with s and Δs.

$$s = \int_0^R \left(\rho w^2 + p\right) r dr = \int_0^{r_*} p(r_*) r dr + \int_{r_*}^R \left(\rho w^2 + p\right) r dr$$

Use

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v^2}{r} = \frac{\Gamma^2}{4\pi^2 r^3 +}\left(1 - e^{-r^2/\delta^2}\right)^2$$

$$\Psi = \frac{1}{2}Wr^2 \implies r = \sqrt{\frac{2\Psi}{W}}$$

$$\begin{aligned}
C = rv &= \frac{\Gamma}{2\pi}\left(1 - e^{-r^2/\delta^2}\right) \\
\frac{\partial C}{\partial \Psi} &= \frac{\Gamma}{2\pi}\frac{\partial}{\partial \Psi}\left(1 - e^{-r^2/\delta^2}\right) \\
&= \frac{-\Gamma}{2\pi}\frac{\partial}{\partial \Psi}\left(e^{-r^2/\delta^2}\right) \\
&= \frac{-\Gamma}{2\pi}\frac{\partial}{\partial \Psi}\left(e^{-2\Psi/W\delta^2}\right) \\
&= \frac{\Gamma}{W\delta^2\pi}\left(e^{-2\Psi/W\delta^2}\right) \\
&= \frac{\Gamma}{W\pi\delta^2}e^{-r^2/\delta^2}
\end{aligned}$$

$$\begin{aligned}
\frac{dH}{d\Psi} &= \frac{dH}{dr}\frac{dr}{d\Psi} \\
&= \frac{dr}{d\Psi}\frac{d}{dr}\left(\frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho}\right) \\
&= \frac{1}{\sqrt{2W\psi}}\frac{d}{dr}\left(\frac{1}{2}v^2 + \int \frac{C^2}{r^3}dr\right) \\
&= \frac{1}{Wr}\left(\frac{1}{2}\frac{dv^2}{dr} + \frac{v^2}{r}\right) \\
&= \frac{1}{Wr}\left(\frac{\Gamma^2}{4r\pi^2}\left(\frac{-1}{r^2} + 2e^{-r^2/\delta^2}\left(\frac{1}{r^2} + \frac{1}{\delta^2}\right) - e^{-2r^2/\delta^2}\left(\frac{1}{r^2} + \frac{2}{\delta^2}\right)\right) + \frac{\Gamma^2}{4r\pi^2}\left(\frac{1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}}{r^2}\right)\right) \\
&= \frac{\Gamma^2}{4Wr^2\pi^2}\left(2e^{-r^2/\delta^2}\left(\frac{1}{r^2} + \frac{1}{\delta^2}\right) - e^{-2r^2/\delta^2}\left(\frac{1}{r^2} + \frac{2}{\delta^2}\right) + \frac{-2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}}{r^2}\right) \\
&= \frac{\Gamma^2}{2Wr^2\delta^2\pi^2}\left(e^{-r^2/\delta^2} - e^{-2r^2/\delta^2}\right)
\end{aligned}$$

How I got the middle term:

$$\begin{aligned}
\frac{dv^2}{dr} &= \frac{d}{dr}\left(\frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/\delta^2}\right)\right)^2 \\
&= \frac{\Gamma^2}{4\pi^2}\frac{d}{dr}\left(\frac{1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}}{r^2}\right) \\
&= \frac{\Gamma^2}{4\pi^2}\left(\frac{-2}{r^3} - 2\left(-\frac{2e^{-r^2/\delta^2}}{r^3} - \frac{2e^{-r^2/\delta^2}}{r\delta^2}\right) + \left(-\frac{2e^{-2r^2/\delta^2}}{r^3} - \frac{4e^{-2r^2/\delta^2}}{r\delta^2}\right)\right) \\
&= \frac{\Gamma^2}{2r\pi^2}\left(\frac{-1}{r^2} + 2e^{-r^2/\delta^2}\left(\frac{1}{r^2} + \frac{1}{\delta^2}\right) - e^{-2r^2/\delta^2}\left(\frac{1}{r^2} + \frac{2}{\delta^2}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} &= r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi} \\
&= \frac{\Gamma^2}{2W\delta^2\pi^2}\left(e^{-r^2/\delta^2} - e^{-2r^2/\delta^2}\right) - \frac{\Gamma}{2\pi}\left(1 - e^{-r^2/\delta^2}\right)\frac{\Gamma}{W\pi\delta^2}e^{-r^2/\delta^2} \\
&= \frac{\Gamma^2}{2W\delta^2\pi^2}\left(e^{-r^2/\delta^2} - e^{-2r^2/\delta^2} - \left(e^{-r^2/\delta^2} - e^{-2r^2/\delta^2}\right)\right) \\
&= 0
\end{aligned}
$$

$$
\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}
$$

$$
\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = \frac{\Gamma^2}{2\delta^2W\pi^2}\left(e^{-\frac{r^2}{\delta^2}}\left(e^{-\frac{r^2}{\delta^2}} - 1\right) + e^{-\frac{2r^2}{\delta^2}}\left(e^{\frac{r^2}{\delta^2}} - 1\right)\right)
$$

`dsolve()` gives

$$
\Psi(r) = ar^2 + b
$$

When we do finite differences, grab a computational variable (call it η for now)

$$
\eta = \frac{r - r_*}{R - r_*}
$$

So now we are computing in $0 < \eta < 1$. So try plugging it into the ODE:

$$
d\eta = \frac{dr}{R - r_*}
$$

$$
\frac{\partial\Psi}{\partial r} = \frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial r} = \frac{1}{R - r_*}\frac{\partial\Psi}{\partial\eta}
$$

If we don't know r_* the η vector becomes

$$
\begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{N-1} \\ r_* \end{pmatrix}
$$

This will be a non-linear problem so we will need to solve using `fsolve`.

To get guesses could use rankine vortex stuff -

For a linear flow we could use We know that $\psi(r_*) = 0$ and $\psi(R) = \frac{1}{2}WR^2$. We could guess that ψ is constant, $\psi = Ar^2 + B$.

Try plotting $w(r_*)$ for various r_* to help find guesses (require it to be 0)

If using η, the ODE becomes $\psi(\eta = 0) = 0$ and $\psi(\eta = 1) = \frac{1}{2}wR^2$

$$
\frac{\partial\Psi}{\partial r} = \frac{1}{R - r_*}\frac{\partial\Psi}{\partial\eta}
$$

$$
\frac{\partial^2\Psi}{\partial r^2} = \frac{1}{(R - r_*)^2}\frac{\partial^2\Psi}{\partial\eta^2}
$$

$$
r = \eta(R - r_*)) + r_*
$$

DE becomes

$$
\frac{1}{(R - r_*)^2}\frac{\partial^2\Psi}{\partial\eta^2} - \frac{1}{\eta(R - r_*)) + r_*}\frac{1}{R - r_*}\frac{\partial\Psi}{\partial\eta} = \frac{\Gamma^2}{2\delta^2W\pi^2}\left(e^{-\frac{(\eta(R-r_*))+r_*)^2}{\delta^2}}\left(e^{-\frac{(\eta(R-r_*))+r_*)^2}{\delta^2}} - 1\right) + e^{-\frac{2(\eta(R-r_*))+}{\delta^2}}\right.
$$

The equation for s comes from the Euler equations (Don't need to use this / know this - we did it with tensors) gives

$$\frac{\partial u_i u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

$$\int_v \frac{\partial u_i u_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} dv$$

Use divergence theorem

$$\int \nabla \cdot v dv = \int v \cdot \hat{n} ds$$

$$\int_v \frac{\partial u_i u_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} dv = \int_s \frac{\partial u_i u_j n_i}{\partial x_i} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} ds$$

$$v_i = p\delta_{ij}, \quad p\delta_{ij} n_i = p n_j$$

$$= \int_s u_i u_j n_i + \frac{p}{\rho} n_j ds$$

Since we are considering a cylinder, we get 0 along the boundary of the cylinder, and so we only care about the inlet and outlet. At the inlet the normal faces in the negative w

$$u_i n_i = -w$$

$$u_i n_i = w$$

$$\int_{(1)} -w^2 - \frac{p}{\rho} dS + \int_{(2)} \int w^2 + \frac{p}{\rho} dS = 0$$

Which is literally in flux = out flux.

0.3.1 Outer vortex breakdown

Considering the initial problem for vortex breakdown, except perhaps the breakdown is a pocket expanding from R rather than 0. I.e. the breakdown occurs about the wall rather than the center. So assuming $r^\dagger$ is our outer vortex breakdown radius

This simply means obtaining a new A, B and k.

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

$$w(r) = W + k(AJ_0(kr) + BY_0(kr))$$

Such that

$$w(r^\dagger) = 0, \qquad \Psi(0) = 0, \quad and \quad \Psi(r^\dagger) = 0$$

To enforce $\Psi(0) = 0$ note that $\lim_{r\to 0} \frac{Y_1(kr)}{r} = -\infty$. Hence it is necessary to set $B = 0$.

$$\Psi(r) = \frac{1}{2}Wr^2 + rAJ_1(kr), \quad w(r) = W + kAJ_0(kr)$$

And to enforce $\Psi(r^\dagger) = 0$

$$\implies Ar^\dagger J_1(kr^\dagger) = -\frac{1}{2}Wr^{\dagger 2}$$

$$A = \frac{-Wr^\dagger}{2J_1(kr^\dagger)}$$

And obtain k using

$$w(r^\dagger) = 0$$

$$kAJ_0(kr) = -W$$

$$\Psi(r^\dagger) = \Psi(R) = \frac{1}{2}WR^2$$

0.4 Appendix

0.4.1 Supplementary Materials

This is where all the basic fluid mechanics knowledge should be (definitions, etc.)

0.4.2 Resources

Books: An Introduction to Fluid Dynamics Batchelor

Swirling flow states in finite-length diverging or contracting circular pipes Zvi Rusak

Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe Zvi Rusak, and Shixiao Wang