# Fluid Thesis

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**Abstract**

Just so I don't forget that there is an abstract environment...

# Contents



## 0.1 Introduction

## 0.2 Derivation of the Squire-Long equation

Squire-long / Bragg-Hawthorne equation for the stream function of axisymmetric inviscid fluid, using cylindrical coordinates

$$\frac{\partial^2\Psi}{\partial z^2}+\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r}=r^2\frac{dH}{d\Psi}-C\frac{dC}{d\Psi}$$

radial component u, azimuthal (swirl) is v, axial component w

stream function satisfies

$\nabla\cdot u=0\longrightarrow$ streamfunction exists

Remember for cylindrical coordinates:

$$u=\frac{1}{r}\frac{\partial\Psi}{\partial z},\quad w=-\frac{1}{r}\frac{\partial\Psi}{\partial r}$$

$\Psi$ is the stream function
$r$ is the radius
$C=rv$
$H=\frac{p}{\rho}+\frac{1}{2}(u^2+v^2+w^2)$
$H$ is conserved on stream surfaces
$C$ is conserved on stream surfaces

vorticity

$$w=w_re_r+w_\theta e_\theta+w_ze_z$$

where $w_r,w_\theta,w_z$ can be written in terms of the velocity

Considering cylindrical coordinates ($z$,$r$,$\theta$) with corresponding velocity ($u$,$v$,$w$), vorticity components ($w_z, w_r, w_\theta$). Axisymmetric flow as:

$$\omega_z = \frac{1}{r}\frac{\partial rv}{\partial r}, \quad \omega_r = -\frac{\partial rv}{\partial z}, \quad \omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

The continuity equation (conservation of mass) is satisfied by setting

$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad u = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$$

Where $\Psi$ is the stream function This gives the azimuthal component for $w_\theta$:

$$\begin{aligned}
\omega_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\
&= -\frac{1}{r}\frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \Psi}{\partial r} \\
&= -\frac{1}{r}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r}\right)
\end{aligned}$$

Use the vorticity equation

$$w \times v - \frac{\partial w}{\partial t} = \nabla H$$

Where

$$H = \frac{1}{2}(w^2 + u^2 + v^2) + \frac{p}{\rho}$$

This gives:

$$\begin{aligned}
u\omega_\theta - v\omega_r - \frac{\partial w}{\partial t} &= \frac{\partial H}{\partial x} \\
v\omega_z - w\omega_\theta - \frac{\partial u}{\partial t} &= \frac{\partial H}{\partial r} \\
w\omega_r - u\omega_z - \frac{\partial v}{\partial t} &= 0
\end{aligned}$$

The last one is equivalent to the material derivative of $rw$ set to 0:

$$\frac{D(rv)}{Dt} = 0$$

From the Bernoulli equation:

$$rv = C(\Psi)$$

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}|\mathbf{w}|^2 + \frac{p}{\rho} = H(\Psi)$$

Where $H(\Psi)$ and $C(\Psi)$ are arbitrary functions.

Rewriting $\omega$:

$$\omega_z = w\frac{dC}{d\Psi}, \quad \omega_r = u\frac{dC}{d\Psi}$$

Giving

$$\frac{\omega_\theta}{r} = \frac{v\omega_r}{ru} + \frac{1}{ru}\frac{dH}{d\Psi}\frac{\partial \Psi}{\partial z} = \frac{C}{r^2}\frac{dC}{d\Psi} - \frac{dH}{d\Psi}$$

Which is the form taken by the second of the dynamic equations. Now, combining this last statement with the equation for $\omega_\theta$:

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$

Taken from Batchelor's An Introduction to Fluid Dynamics

## 0.3 Rotating Flow

Considering the flow far upstream where there is constant uniform axial velocity and rotates with angular velocity $\Omega$

$$\Psi_{\text{upstream}} = \frac{1}{2}Wr^2$$
$$v = \Omega r, w = W$$

And

$$C = rv = \frac{v^2}{\Omega} = \Omega r^2 = 2\Omega\Psi/W$$
$$\frac{dC}{d\Psi} = 2\Omega/W$$

Since the flow is steady, the radial equation of motion yields:

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{w^2}{r} = \frac{C^2}{r^3}$$

$$\begin{aligned}
H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\
&= \frac{1}{2}(\Omega^2 r^2 + W^2) + \frac{p}{\rho} \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \frac{p}{\rho} \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \frac{1}{\rho}\frac{dp}{dr}dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \frac{C^2}{r^3}dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \frac{\Omega^2 r^4}{r^3}dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \int \Omega^2 r dr \\
&= \frac{\Omega^2\Psi}{W} + \frac{1}{2}W^2 + \frac{1}{2}\Omega^2 r^2 \\
&= \frac{2\Omega^2\Psi}{W} + \frac{1}{2}W^2
\end{aligned}$$

$$\begin{aligned}
\frac{dH}{d\Psi} &= \frac{\partial \frac{2\Omega^2\Psi}{W}}{\partial\Psi} \\
&= \frac{2\Omega^2}{W}
\end{aligned}$$

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$
$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = \frac{2r^2\Omega^2}{W} - \frac{4\Omega^2}{W^2}\Psi$$

Or in a more 'standard' form

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{4\Omega^2}{W^2}\Psi = \frac{2r^2\Omega^2}{W}$$

### 0.3.1 Homogeneous ODE

Considering the case where $\Psi$ is just a function of the radius, $r$. So $\Psi$ does not depend on $z$, and $\frac{\partial^2\Psi}{\partial z^2} = 0$

To simplify it into a homogeneous ODE, a change of variables is used:

$$\Psi = \frac{1}{2}Wr^2 + \psi = \frac{1}{2}Wr^2 + rF$$

$$\frac{\partial\Psi}{\partial r} = Wr + F + r\frac{\partial F}{\partial r}$$
$$\frac{\partial^2\Psi}{\partial r^2} = W + 2\frac{\partial F}{\partial r} + r\frac{\partial^2 F}{\partial r^2}$$

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} + \Psi(\frac{4\Omega^2}{W^2} - \frac{1}{r^2}) = 0$$

$$r^2\frac{d^2F}{dr^2} - r\frac{dF}{dr} + F(r^2k^2 - 1) = 0$$

Letting $k = \frac{2\Omega}{W}$ If we take $x = kr$, $\frac{dF}{dr} = \frac{dF}{dx}\frac{dx}{dr} = k$ and $\frac{d^2F}{dr^2} = k^2\frac{d^2F}{dx^2}$

$$\frac{x^2}{k^2}k^2\frac{d^2F}{dx^2} - \frac{x}{k}k\frac{dF}{dx} + F(\frac{x^2}{k^2}k^2 - 1) = 0$$
$$x^2\frac{d^2F}{dx^2} - x\frac{dF}{dx} + F(x^2 - 1) = 0$$

Which is the form of a bessel differential equation of order $\nu = 1$, giving solutions

$$F = AJ_1(kr) + BY_1(kr)$$

Returning to the streamfunction:

$$\Psi = \frac{1}{2}Wr^2 + r\left(AJ_1(kr) + BY_1(kr)\right)$$

And hence

$$w = \frac{1}{r}\frac{\partial\Psi}{\partial r} = W + AkJ_0(kr) + BkY_0(kr)$$

$A$, and $B$ rely on boundary conditions. In this case, it is necessary forthe streamlines to be the same as at the inlet along the boundary. Also introduce a vortex breakdown condition in the core of the stream, i.e. a region $0 < r < r_*$ where the streamfunction becomes zero:

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

Consider it as a matrix system

$$\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ RJ_1(kR) & RY_1(kR) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ 0 \end{pmatrix}$$

Giving

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{r_* R\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \begin{pmatrix} RY_1(kR) & -r_* Y_1(kr_*) \\ -RJ_1(kR) & r_* J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ 0 \end{pmatrix}$$

$$\begin{aligned} A &= \frac{-\frac{1}{2}RWr_*^2 Y_1(kR)}{r_* R\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \\ B &= \frac{\frac{1}{2}RWr_*^2 J_1(kR)}{r_* R\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \end{aligned}$$

And hence

$$\begin{aligned} A &= \frac{-\frac{1}{2}Wr_* Y_1(kR)}{\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \\ B &= \frac{\frac{1}{2}Wr_* J_1(kR)}{\left(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR)\right)} \end{aligned}$$

With the requirement that $r_* \neq R$ so as to not divide by zero.

Using

$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r}$$

Gives

$$w = W + k(AJ_0(kr) + BY_0(kr))$$

Solving this for a given $k$ (or alternatively a desired $r_*$) is done numerically using `MATLAB`.

The set of valid solutions to this problem are those which satisfy the constraint

$$w(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

The plot figure 0.3.1 shows the $k, r_*$ combinations which satisfy the constraint.

Clearly this can only occur for values of $kR > 3.8$.

The first branch of this (extending from $kR \approx 3.8$) corresponds to natural solutions, whereas further branches give unwanted behaviour, which introduce reversed flow.

Code - homogeneousODE.m

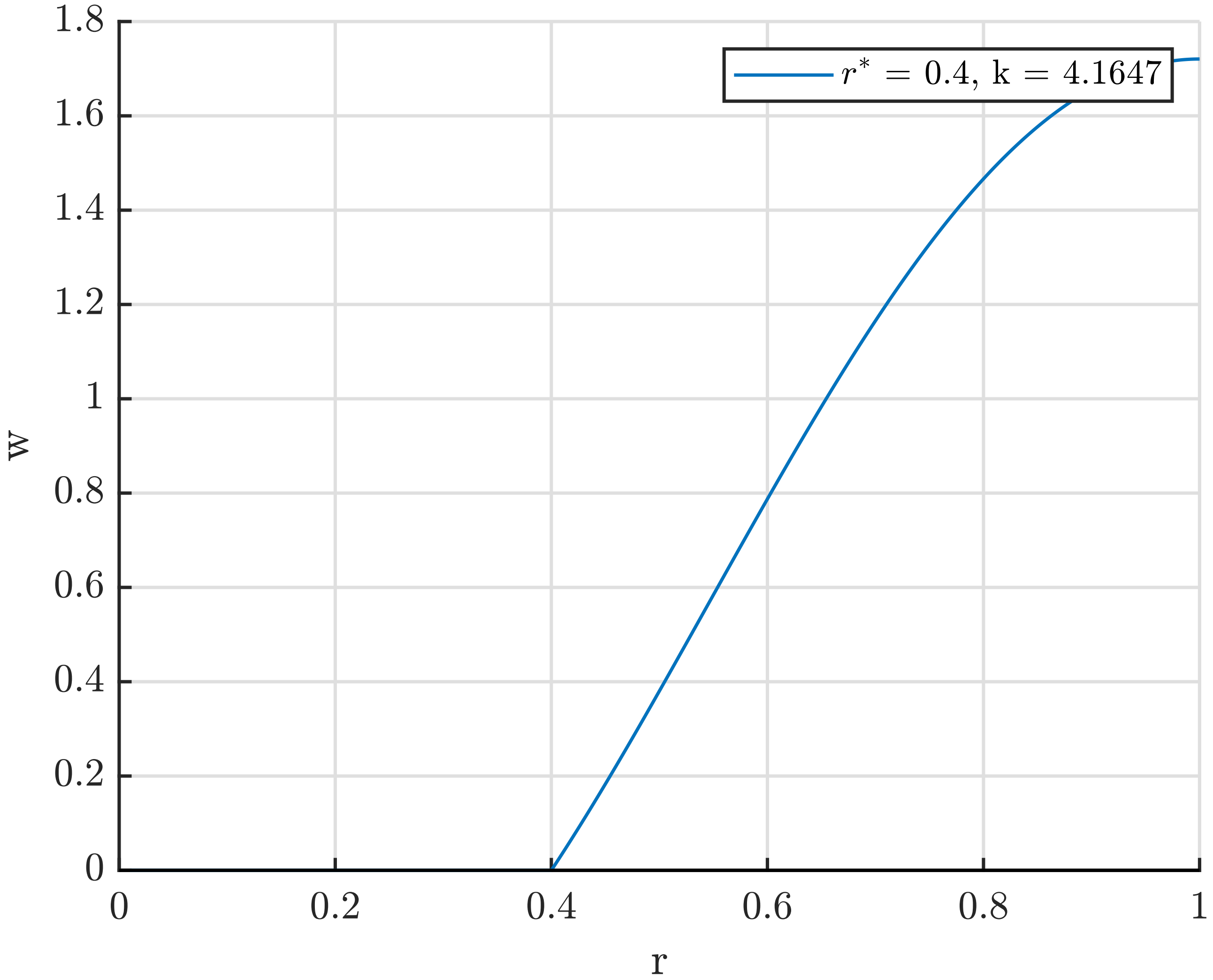


Figure 1: An example solution plot

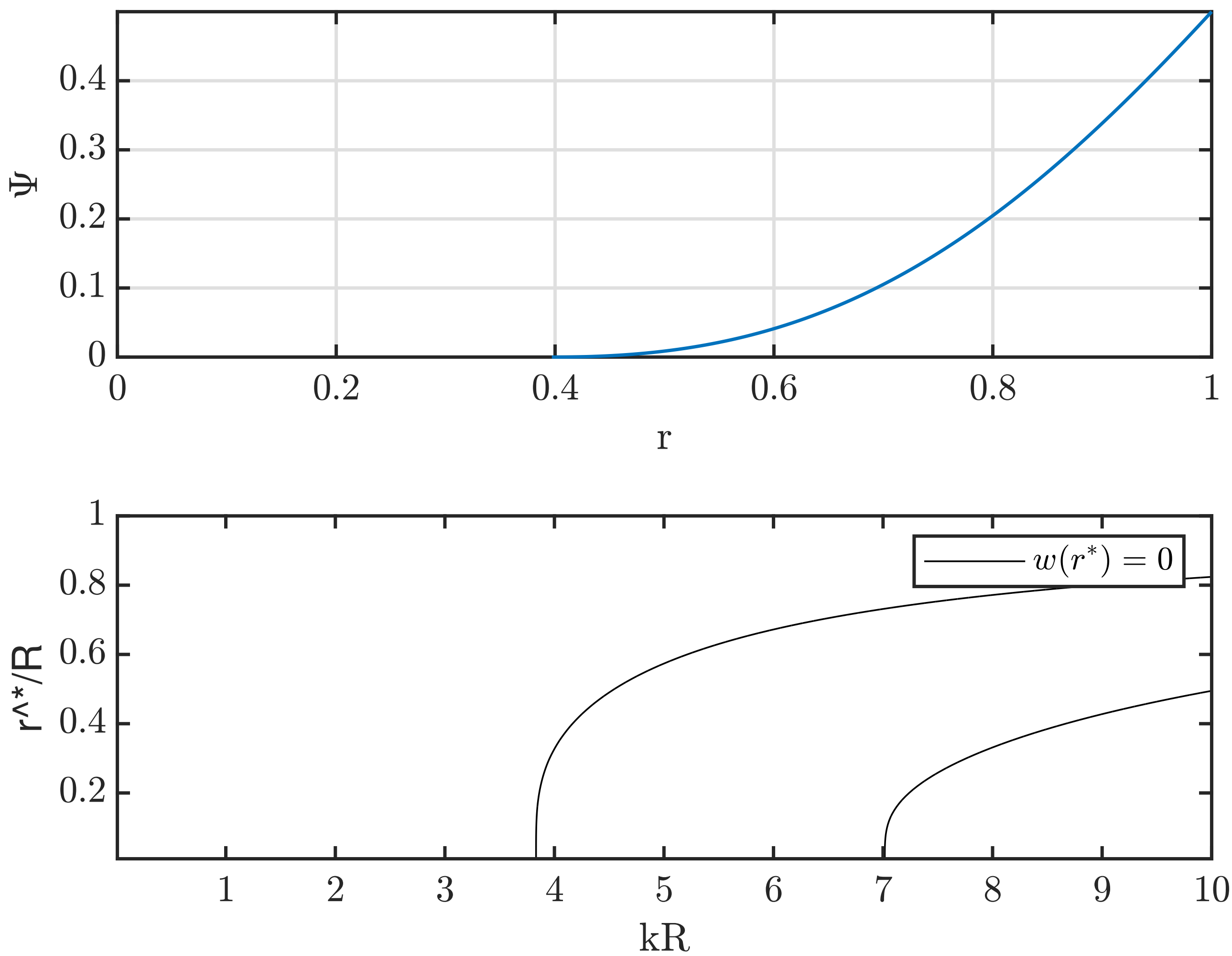


Figure 2: Solution set for the simplified problem

have to assume things for outside of the region for $\Psi$. I.e. if we go above the maximum input value then some assumption, and if we go below the minimum then it is a stagnation point

see if we can do it for the wall stagnation zones (i.e. psi goes to 0 near R) so when $\Psi > \frac{1}{2}WR^2$ Plug it into H and C

$$H = (\Omega R)^2 + \frac{1}{2}W^2$$

$$\frac{\partial H}{\partial \psi} = 0$$
$$C = \Omega R^2$$
$$\frac{\partial C}{\partial \Psi} = 0$$

Which then yields the separable first order ODE

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = 0$$

And hence

$$\frac{\partial \Psi}{\partial r} = Ar$$
$$\Psi = \frac{1}{2}Ar^2 + B$$

our left hand side could be written as

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)$$

using staggered grid

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{r}\frac{\partial \Psi}{\partial r}$$

at the boundary r=0

### 0.3.2 Numerics

Solving the ODE numerically:

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = r^2\frac{\partial H}{\partial \Psi} + C\frac{\partial C}{\partial \Psi}$$

finite difference - divide r as a grid of $N$ intervals. So our grid spaces over $R$,

$$r_i = \Delta r_i, \quad \Delta = \frac{R}{N}$$

So (check this)

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}}{\Delta^2}$$
$$\frac{\partial \Psi}{\partial r} = \frac{\Psi_{i+1} - \Psi_{i-1}}{2\Delta}$$
$$\Psi_0 = 0, \quad \Psi_N = \frac{1}{2}WR^2$$

Which should work for the index $i$ until we reach the bifurcations/stagnations

Should end up with a matrix equation

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ & & \mathbf{A} & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \mathbf{\Psi} \\ \Psi_N \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ \frac{1}{2}WR^2 \end{pmatrix}$$

A should be the finite difference version of

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = 0$$

I.e. for the $i^{th}$ row of $\mathbf{A}$

$$A(i) = \frac{A(i+1) - 2 * A(i) + A(i-1)}{\Delta^2} - \frac{A(i+1) - A(i-1)}{2r(i)\Delta}$$

$$A_{ij} = \begin{cases} 1 & j = i = 1 \\ 1/\Delta^2 + 1/(2r_i\Delta) & j = i - 1 \\ 2/\Delta^2 & j = i \\ 1/\Delta^2 - 1/(2r_i\Delta) & j = i + 1 \\ 1 & j = i = N \\ 0 & otherwise \end{cases}$$

For the full equation

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \Psi(\frac{4\Omega^2}{W^2} - \frac{1}{r^2}) = 0$$

$$\Psi = \frac{1}{2}Wr^2 + rF$$

$$F = \frac{\Psi}{r} - \frac{1}{2}Wr$$

Boundary conditions for $F$ relate to those for $\Psi$.

$$\Psi(R) = \frac{1}{2}WR^2 \implies F(R) = 0$$

$$\Psi(r_*) = 0 \implies F(r_*) = \frac{1}{2}Wr_*^2$$

when we look at the vortex breakdown problem, introduce a coordinate transformation

$$\eta = \frac{r - r_*}{R - r_*}$$

$$\eta = 0, r = r_*, \eta = 1 r = R$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{1}{R - r_*}\frac{\partial \Psi}{\partial \eta}$$

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{(R - r_*)^2}\frac{\partial^2 \Psi}{\partial \eta^2}$$

use the same conditions we have used anyway where $\Psi(r_*) = w(r_*) = 0$ Rankine body problem: At some point on the radius $r_0$, we get $v = K/r_0$ for some constant $K$ find $K = \Omega r_0^2$?

## 0.4 Rankine Body

$w = W$,

$$v = \begin{cases} \frac{\Gamma}{2\pi r}, & r > r_0 \\ \Omega r, & r \leq r_0 \end{cases}$$

Where the second condition was the previous solution. Since the velocity profile is now piece-wise defined, the streamfunction must also be, i.e. it is necessary to split the streamfunction into 2 regions to solve this problem. The upstream regions:

$$\begin{cases} \Psi_{inner}, & 0 \leq r \leq r_0 \\ \Psi_{outer}, & r_0 \leq r \leq R \end{cases}$$

Note that $r_0$ is defined upstream, so the position of the region may have moved downstream to a new radius, $\hat{r}$, and hence, downstream, these regions will become around $\hat{r}$ instead of $r_0$. We enforce some similar conditions as to the normal problem:

$$\begin{aligned} \Psi(r_*) &= 0, \\ \Psi(R) &= \frac{1}{2}WR^2, \\ w(r_*) &= 0 \end{aligned}$$

With the added condition that $\Psi$ must remain continuous around $\hat{r}$ I.e.

$$\lim_{r^- \to \hat{r}} \Psi(r^-) = \lim_{r^+ \to \hat{r}} \Psi(r^+)$$

And

$$\lim_{r^- \to \hat{r}} v(r^-) = \lim_{r^+ \to \hat{r}} v(r^+)$$

Where $\Psi(r^-)$ is $\Psi$ defined for $r \leq \hat{r}$ and $\Psi(r^+)$ is defined in the region $r \geq \hat{r}$.

The region for $\Psi(r)$ with $r \in [0, r_0]$ will be the same as before, i.e.

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

For the region $r_0 < r < R$ the problem must be resolved from the SL equation

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C\frac{dC}{d\Psi} \\ C = rv &= \frac{\Gamma}{2\pi} \\ \frac{dC}{d\Psi} &= 0 \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2}(0 + \frac{\Gamma^2}{4\pi^2 r^2} + W^2) + \int \frac{C^2}{r^3} dr \\ &= \frac{1}{2}(\frac{\Gamma^2}{4\pi^2 r^2} + W^2) + \int \frac{\Gamma^2}{4\pi^2 r^3} dr \\ &= \frac{1}{2}(\frac{\Gamma^2}{4\pi^2 r^2} + W^2) - \frac{\Gamma^2}{8\pi^2 r^2} \\ &= \frac{W^2}{2} \\ \frac{dH}{d\Psi} &= 0 \end{aligned}$$

And hence the SL equation gives

$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$
$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$
$$\frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = 0$$

Which results in:

$$\Psi = Cr^2 + D, \quad r \geq \hat{r}$$
$$w = \frac{1}{r}\frac{\partial\Psi}{\partial r} = 2C$$

With the requirement that there is no discontinuity at $\hat{r}$, i.e.

$$\Psi = \frac{1}{2}W\hat{r}^2 + \hat{r}\left(AJ_1(k\hat{r}) + BY_1(k\hat{r})\right) = C\hat{r}^2 + D$$

And using the same for $w$

$$w(\hat{r}) = W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) = 2C$$

And lastly the wall condition

$$\Psi(R) = \frac{1}{2}WR^2 = C\hat{r}^2 + D$$

With

$$w(r_*) = 0$$
$$\frac{\Gamma}{2\pi r_0} = \Omega r_0 \implies \Omega = \frac{\Gamma}{2\pi r_0^2}$$
$$k_{outer} = \frac{2\Gamma}{2\pi W r_0^2} = \frac{\Gamma}{\pi W r_0^2}$$

Noting that the values for $A$ and $B$ are obtained from the $r_*$ condition.

The coefficients for $\Psi$ have to be resolved, since the condition $\Psi_{inner}(R) = \frac{1}{2}WR^2$ cannot be imposed.

Parameters

$$r_0, \hat{r}, r_*, R, k, \Gamma, W, A, B, C, D$$

We can fix $r_0$, $R$, $k$, $W$ and $\Gamma$. This is 11 parameters, where 5 are fixed. Require 6 conditions. Impose:

1). $w(r_*) = 0$ (as before)

2). $\Psi_{inner}(r_*) = 0$ (as before)

3). Since at the wall $\Psi$ must remain the same, this applies to where $v$ is changed, i.e. $\Psi_{inner}(\hat{r}) = \frac{1}{2}Wr_0^2$

4). For continuity, $\Psi_{outer}(\hat{r}) = \frac{1}{2}Wr_0^2$

5). $w_{outer}(\hat{r}) = w_{inner}(\hat{r})$

6). $\Psi_{outer}(R) = \frac{1}{2}WR^2$

Redo the problem instead getting $A, B$ from 2) and 3)

$$\Psi_{inner}(r_*) = 0$$
$$\Psi_{inner}(\hat{r}) = \frac{1}{2}Wr_0^2$$

Use this for $A, B$

$$\begin{aligned}
\Psi_{inner}(r_*) &= \frac{1}{2}Wr_*^2 + r_*(AJ_1(kr_*) + BY_1(kr_*)) = 0 \\
&= r_*(AJ_1(kr_*) + BY_1(kr_*)) = -\frac{1}{2}Wr_*^2 \\
\Psi_{inner}(\hat{r}) &= \frac{1}{2}W\hat{r}^2 + \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}Wr_0^2 \\
&= \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}W(r_0^2 - \hat{r}^2)
\end{aligned}$$

This gives the matrix system for $A, B$ below. Note that the system relies on the unknowns $r_*$ and $\hat{r}$.

$$\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ \hat{r}J_1(k\hat{r}) & \hat{r}Y_1(k\hat{r}) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{det} \begin{pmatrix} \hat{r}Y_1(k\hat{r}) & -r_*Y_1(kr_*) \\ -\hat{r}J_1(k\hat{r}) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix}$$

$$\begin{aligned}
A &= \frac{1}{\det}\left(\hat{r}Y_1(k\hat{r})\left(-\frac{1}{2}Wr_*^2\right) - r_*Y_1(kr_*)\left(\frac{1}{2}W(r_0^2 - \hat{r}^2)\right)\right) \\
B &= \frac{1}{\det}\left(-\hat{r}J_1\left(-\frac{1}{2}Wr_*^2\right) + r_*J_1(kr_*)\left(\frac{1}{2}W(r_0^2 - \hat{r}^2)\right)\right)
\end{aligned}$$

Where

$$\begin{aligned}
\det &= \hat{r}r_*Y_1(k\hat{r})J_1(kr_*) - \hat{r}r_*J_1(k\hat{r})Y_1(kr_*) \\
&= \hat{r}r_*\left(Y_1(k\hat{r})J_1(kr_*) - J_1(k\hat{r})Y_1(kr_*)\right)
\end{aligned}$$

This for $r_*$

$$w_{inner}(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

Get $C$ from:

$$\begin{aligned}
w_{outer}(\hat{r}) &= w_{inner}(\hat{r}) \\
2C &= W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) \\
C &= \frac{1}{2}\left(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r}))\right)
\end{aligned}$$

Get $D$ here:

$$\Psi_{outer}(R) = CR^2 + D = \frac{1}{2}WR^2$$
$$D = \frac{1}{2}WR^2 - C$$

Hence get $\hat{r}$ from

$$\Psi_{outer}(\hat{r}) = C\hat{r}^2 + D = \frac{1}{2}Wr_0^2$$
$$C\hat{r}^2 + \frac{1}{2}WR^2 - C = \frac{1}{2}Wr_0^2$$
$$\left(\frac{1}{2}\left(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r}))\right)\right)(\hat{r}^2 - 1) = \frac{1}{2}W(r_0^2 - R^2)$$
$$(AJ_0(k\hat{r}) + BY_0(k\hat{r}))(\hat{r}^2 - 1) = \frac{1}{k}W(r_0^2 - R^2 - 1)$$

For physically valid solutions, we must impose the condition of no net change on the momentum from upstream to downstream on the momentum (Escudier, Keller). The momentum is defined as

$$s = 2\pi \int_0^{r_t} \left(\rho w^2 + p\right) r dr$$

Which comes to:

$$\Delta s = \frac{\pi}{4}\rho U^2 k^2 r_c^2 \left[-r_b^2 + \frac{1}{4}\left(\frac{r_b^4 - r_a^4}{r_c^2}\right) + \frac{3}{4}r_c^2 + \frac{1}{2}r_c^2 \log\left(\frac{r_b^2}{r_c^2}\right)\right] = 0$$

Figure 4 shows the solution set for the problem. It displays the same results as those found in (Escudier, Keller), with the same asymptote $kr_0 \to \sqrt{2}$ as $r_0 \to 0$. The results in the two figures come from `rhatrstarmomentum.m` and `numericalSolutionSetRankine.m` respectively

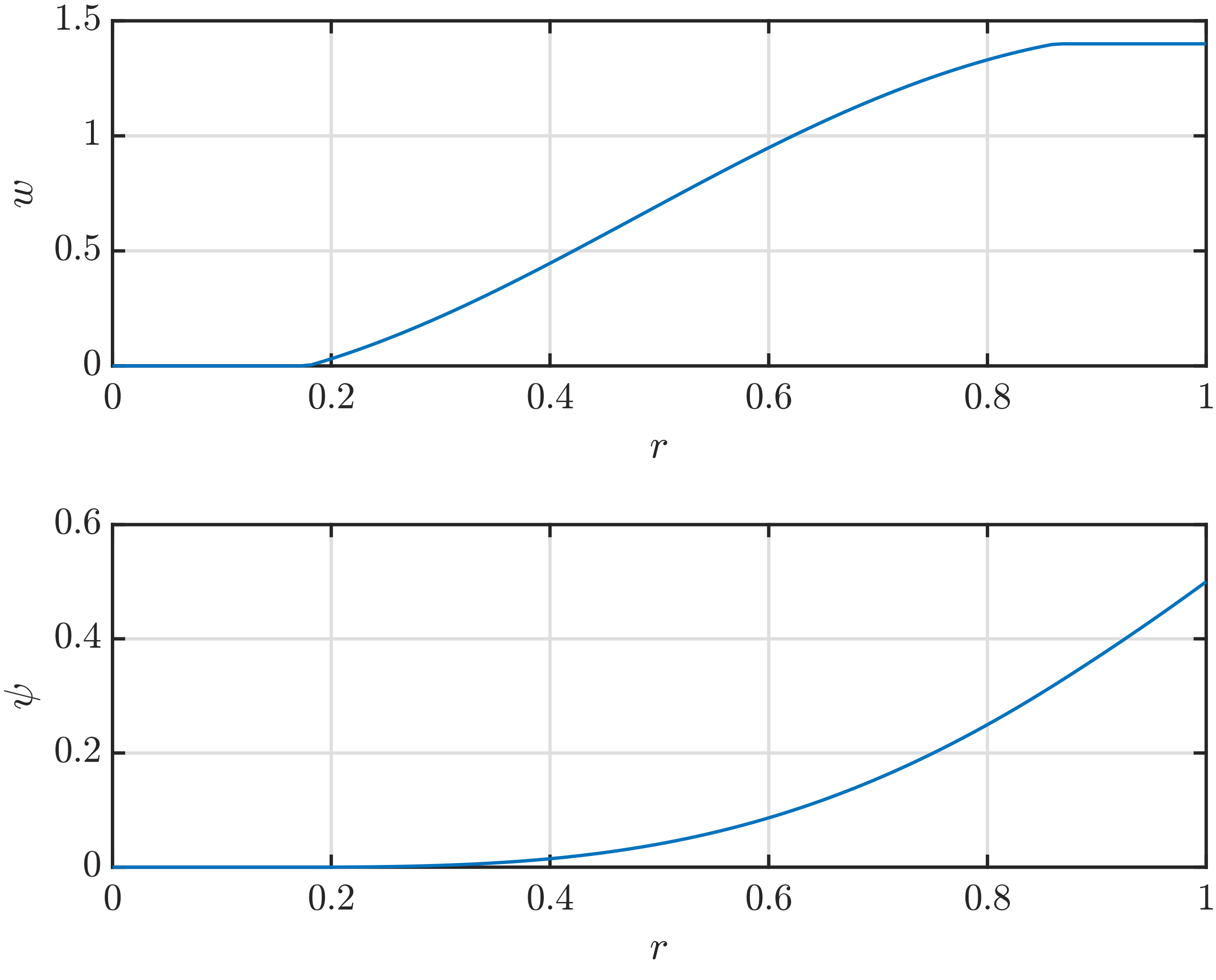


Figure 3: A solution of $w$ and $\Psi$ for the Rankine problem with 0 net momentum, $k = 3.8961, r_0 = 0.8619, r_* = 0.1784$. Obtained using

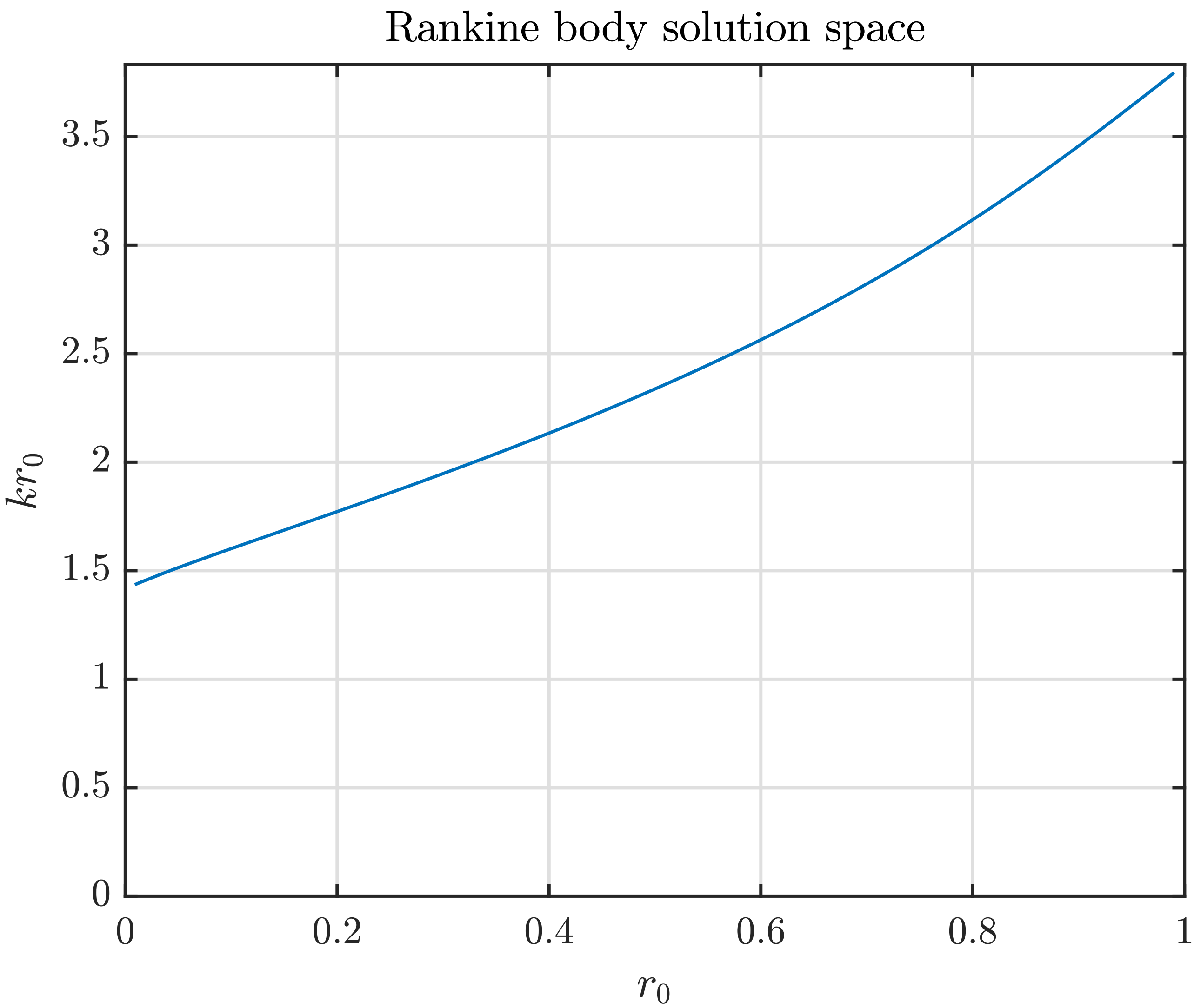


Figure 4: Solution space for the Rankine body problem

## 0.5 Lamb-Oseen Vortex

Q-vortex without a Jet. Start with

$$w = W$$
$$v = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/\delta^2}\right)$$
$$\frac{d^2\Psi}{dr^2} - \frac{1}{r}\frac{d\Psi}{dr} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$

Solve from $r_*$ to $R$ numerically.

Generate grid from $r_*$ to $R$.

Boundary conditions as normal

$$\Psi(R) = \frac{1}{2}WR^2$$
$$\Psi(r_*) = 0$$
$$w(r_*) = 0$$

And the standard upstream flow

$$\Psi(r) = \frac{1}{2}Wr^2$$

Non-dimensional parameter may be something like $\frac{\Gamma}{WR}$ (we can probably relate this to $kr_0$ for the rankine problem)

Eventually do the same thing as before with $s$ and $\Delta s$.

$$s = \int_0^R \left(\rho w^2 + p\right) r dr = \int_0^{r_*} p(r_*) r dr + \int_{r_*}^R \left(\rho w^2 + p\right) r dr$$

Use

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v^2}{r} = \frac{\Gamma^2}{4\pi^2 r^3 +}\left(1 - e^{-r^2/\delta^2}\right)^2$$
$$\Psi = \frac{1}{2}Wr^2 \implies r = \sqrt{\frac{2\Psi}{W}}$$

$$\begin{aligned}
C = rv &= \frac{\Gamma}{2\pi}\left(1 - e^{-r^2/\delta^2}\right) \\
\frac{\partial C}{\partial \Psi} &= \frac{\Gamma}{2\pi}\frac{\partial}{\partial \Psi}\left(1 - e^{-r^2/\delta^2}\right) \\
&= \frac{-\Gamma}{2\pi}\frac{\partial}{\partial \Psi}\left(e^{-r^2/\delta^2}\right) \\
&= \frac{-\Gamma}{2\pi}\frac{\partial}{\partial \Psi}\left(e^{-2\Psi/W\delta^2}\right) \\
&= \frac{\Gamma}{W\delta^2\pi}\left(e^{-2\Psi/W\delta^2}\right) \\
&= \frac{\Gamma}{W\pi\delta^2}e^{-r^2/\delta^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{dH}{d\Psi} &= \frac{dH}{dr}\frac{dr}{d\Psi} \\
&= \frac{dr}{d\Psi}\frac{d}{dr}\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+\frac{p}{\rho}\right) \\
&= \frac{1}{\sqrt{2W\psi}}\frac{d}{dr}\left(\frac{1}{2}v^2+\int\frac{C^2}{r^3}dr\right) \\
&= \frac{1}{Wr}\left(\frac{1}{2}\frac{dv^2}{dr}+\frac{v^2}{r}\right) \\
&= \frac{1}{Wr}\left(\frac{\Gamma^2}{4r\pi^2}\left(\frac{-1}{r^2}+2e^{-r^2/\delta^2}\left(\frac{1}{r^2}+\frac{1}{\delta^2}\right)-e^{-2r^2/\delta^2}\left(\frac{1}{r^2}+\frac{2}{\delta^2}\right)\right)+\frac{\Gamma^2}{4r\pi^2}\left(\frac{1-2e^{-r^2/\delta^2}+e^{-2r^2/\delta^2}}{r^2}\right)\right) \\
&= \frac{\Gamma^2}{4Wr^2\pi^2}\left(2e^{-r^2/\delta^2}\left(\frac{1}{r^2}+\frac{1}{\delta^2}\right)-e^{-2r^2/\delta^2}\left(\frac{1}{r^2}+\frac{2}{\delta^2}\right)+\frac{-2e^{-r^2/\delta^2}+e^{-2r^2/\delta^2}}{r^2}\right) \\
&= \frac{\Gamma^2}{2Wr^2\delta^2\pi^2}\left(e^{-r^2/\delta^2}-e^{-2r^2/\delta^2}\right) \\
&= \frac{\Gamma^2}{4\Psi\delta^2\pi^2}\left(e^{-2\Psi/W\delta^2}-e^{-4\Psi/W\delta^2}\right)
\end{aligned}
$$

How I got the middle term:

$$
\begin{aligned}
\frac{dv^2}{dr} &= \frac{d}{dr}\left(\frac{\Gamma}{2\pi r}\left(1-e^{-r^2/\delta^2}\right)\right)^2 \\
&= \frac{\Gamma^2}{4\pi^2}\frac{d}{dr}\left(\frac{1-2e^{-r^2/\delta^2}+e^{-2r^2/\delta^2}}{r^2}\right) \\
&= \frac{\Gamma^2}{4\pi^2}\left(\frac{-2}{r^3}-2\left(-\frac{2e^{-r^2/\delta^2}}{r^3}-\frac{2e^{-r^2/\delta^2}}{r\delta^2}\right)+\left(-\frac{2e^{-2r^2/\delta^2}}{r^3}-\frac{4e^{-2r^2/\delta^2}}{r\delta^2}\right)\right) \\
&= \frac{\Gamma^2}{2r\pi^2}\left(\frac{-1}{r^2}+2e^{-r^2/\delta^2}\left(\frac{1}{r^2}+\frac{1}{\delta^2}\right)-e^{-2r^2/\delta^2}\left(\frac{1}{r^2}+\frac{2}{\delta^2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r} &= r^2\frac{dH}{d\Psi}-C\frac{dC}{d\Psi} \\
&= \frac{r^2\Gamma^2}{4\Psi\delta^2\pi^2}\left(e^{-2\Psi/W\delta^2}-e^{-4\Psi/W\delta^2}\right)-\left(\frac{\Gamma}{2\pi}\left(1-e^{-2\Psi/W\delta^2}\right)\right)\left(\frac{\Gamma}{W\delta^2\pi}\left(e^{-2\Psi/W\delta^2}\right)\right) \\
&= \frac{r^2\Gamma^2}{4\Psi\delta^2\pi^2}\left(e^{-2\Psi/W\delta^2}-e^{-4\Psi/W\delta^2}\right)-\frac{\Gamma^2}{2W\delta^2\pi^2}\left(1-e^{-2\Psi/W\delta^2}\right)\left(e^{-2\Psi/W\delta^2}\right) \\
&= \frac{\Gamma^2}{2W\delta^2\pi^2}\left(\frac{r^2W}{2\Psi}\left(e^{-2\Psi/W\delta^2}-e^{-4\Psi/W\delta^2}\right)-\left(e^{-2\Psi/W\delta^2}-e^{-4\Psi/W\delta^2}\right)\right) \\
\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r} &= \frac{\Gamma^2}{2W\delta^2\pi^2}\left(\left(\frac{r^2W}{2\Psi}-1\right)\left(e^{-2\Psi/W\delta^2}-e^{-4\Psi/W\delta^2}\right)\right)
\end{aligned}
$$

Giving the system

$$
\begin{aligned}
\Psi_1' &= \Psi_2 \\
\Psi_2' &= \frac{1}{r}\Psi_2+\frac{\Gamma^2}{2W\delta^2\pi^2}\left(\left(\frac{r^2W}{2\Psi_1}-1\right)\left(e^{-2\Psi_1/W\delta^2}-e^{-4\Psi_1/W\delta^2}\right)\right)
\end{aligned}
$$

Take limit as $\Psi \to 0$ in matlab and use that in some suff area.

Could use bvp5c and break it into a system of first order odes.

So if we change the boundaries - using a 'Linear Lagrange interpolating polynomial' So that the end points are fixed.

$$\eta = \frac{r - r_*}{R - r_*}$$

$$\begin{aligned} r - r_* &= \eta(R - r_*) \\ r &= \eta(R - r_*) + r_* \end{aligned}$$

Such that $\eta \in [0, 1]$. In the function we can get $r$ from $\eta$

For finite differences we can just use a newton iteration, or use something like fsolve.

I SHOULD PLOT DOWNSTREAM $v$ for all our equations also

To use the $r_*$ and $w$ parts, we can use $w = \frac{1}{r}\frac{\partial \Psi}{\partial r}$

Try putting in the homogeneous solution to the solver to see if it works (i.e. the one with $v = \Omega r$ and $w = W$).

We should expect that the Lamb-Oseen vortex should be a more smooth version of the Rankine problem - so we should be able to compare the two.

May want to find $v_{max} = r_0$ to compare to the previous problem. Should expect the circulation goes to $\Gamma/2\pi$ as $r \to \infty$.

Of course we have to have

$$\lim_{r\to 0} \frac{1}{r}\frac{\partial \Psi}{\partial r} < \infty$$

So use l'hopital's rule

$$\lim_{r\to 0} \frac{1}{r}\frac{\partial \Psi}{\partial r} = \frac{\partial^2 \Psi}{\partial r^2}$$

But we know $\psi = 0$ at $r = 0$ ( so we can ignore this )

So we will just use

$$\frac{\partial^2 \Psi}{\partial r^2}|_i - \frac{1}{r}\frac{\partial \Psi}{\partial r}|_i = f(r_i, \Psi_i)$$

With

$$\Psi_1 = 0$$

And

$$\Psi_n = \frac{1}{2} W R^2$$

We will have 1 based indexing Alternatively get the derivatives at the mid points

$$\frac{\partial \Psi}{\partial r}\bigg|_{i+1/2} = \frac{\Psi_{i+1} - \Psi_i}{h_r}$$

Where $h_r$ is the step.

Alternatively

Can rewrite the left hand side as

$$r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)\right)$$

And plug in the last thing

When we do finite differences, grab a computational variable (call it $\eta$ for now)

$$\eta = \frac{r - r_*}{R - r_*}$$

So now we are computing in $0 < \eta < 1$. So try plugging it into the ODE:

$$d\eta = \frac{dr}{R - r_*}$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{1}{R - r_*}\frac{\partial \Psi}{\partial \eta}$$

If we don't know $r_*$ the $\eta$ vector becomes

$$\begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{N-1} \\ r_* \end{pmatrix}$$

This will be a non-linear problem so we will need to solve using `fsolve`.

To get guesses could use rankine vortex stuff -

For a linear flow we could use We know that $\psi(r_*) = 0$ and $\psi(R) = \frac{1}{2}WR^2$. We could guess that $\psi$ is constant, $\psi = Ar^2 + B$.

Try plotting $w(r_*)$ for various $r_*$ to help find guesses (require it to be 0)

If using $\eta$, the ODE becomes $\psi(\eta = 0) = 0$ and $\psi(\eta = 1) = \frac{1}{2}WR^2$

$$\frac{\partial \Psi}{\partial r} = \frac{1}{R - r_*}\frac{\partial \Psi}{\partial \eta}$$

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{(R - r_*)^2}\frac{\partial^2 \Psi}{\partial \eta^2}$$

$$r = \eta(R - r_*)) + r_*$$

DE becomes

$$\frac{1}{(R - r_*)^2}\frac{\partial^2 \Psi}{\partial \eta^2} - \frac{1}{\eta(R - r_*)) + r_*}\frac{1}{R - r_*}\frac{\partial \Psi}{\partial \eta} = 0$$

$$\frac{1}{R - r_*}\frac{\partial^2 \Psi}{\partial \eta^2} - \frac{1}{\eta(R - r_*)) + r_*}\frac{\partial \Psi}{\partial \eta} = 0$$

Write a function which calculates the residual, i.e.

$$\hat{r}_i = LHS - RHS|_i$$

and then fsolve on that

Send through the vector

$$\begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_N \\ r_* \end{pmatrix}$$

Might be worth looking at setting the RHS

$$r^2 \frac{dH}{d\Psi} + C\frac{dC}{d\Psi}$$

As a function $f(r, \Psi)$ and the settings

$$f(r, \Psi) = \begin{cases} \ldots, & \text{if } 0 \leq \Psi \leq \frac{1}{2}WR^2 \\ 0 & otherwise \end{cases}$$

## 0.6 2D System

For the method - since we're stuck try using

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) = r\frac{\frac{1}{r}\frac{\partial\Psi}{\partial r}|_{i+\frac{1}{2}} - \frac{1}{r}\frac{\partial\Psi}{\partial r}|_{i-\frac{1}{2}}}{\Delta r}$$

Where $r_{i+1/2} = \frac{r_i + r_{i+2}}{\Delta r}$ and $\frac{\partial \Psi}{\partial r}|_{i+\frac{1}{2}} = \frac{\Psi_{i+1} - \Psi_i}{\Delta r}$

and using the rusak method, by letting $y = r^2/2$

We may have to make our own solver, or use an available one ().

Wang and rusak the dynamics of a swirling flow in a pipe and transitions to axisymmetric vortex breakdown

do some research on numerical solutions of axisymmetric swirling flow

Trent had a book with a means of numerically solving the system, which will work in 2D as well

$$\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = -r\eta$$

We are given an $\eta$. If we let $\Psi_{i,j} = \Psi(r = r_i, z = z_j)$

$$\frac{\partial^2\Psi}{\partial z^2}\bigg|_{i,j} = \frac{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}}{\Delta z^2}$$

$$\frac{\partial^2\Psi}{\partial r^2}\bigg|_{i,j} = \frac{\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j}}{\Delta r^2}$$

$$\frac{\partial\Psi}{\partial r}\bigg|_{i,j} = \frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2\Delta r}$$

$$\frac{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}}{\Delta z^2} + \frac{\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j}}{\Delta r^2} - \frac{1}{r_i}\left(\frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2\Delta r}\right) = -r_i\eta_{i,j}$$

Could write $r_{i,j}$ in case it changes in $j$.

Of course to write this as a linear system, we have to get form $A\mathbf{\Psi} = b$ So we would have to write the vector $\Psi$ as

$$\begin{pmatrix} \Psi_{1,1} \\ \Psi_{2,1} \\ \vdots \\ \Psi_{m,1} \\ \Psi_{1,2} \\ \vdots \\ \Psi_{1,n} \\ \vdots \\ \Psi_{m,n} \end{pmatrix}$$

So the index of $\Psi_{i,j}$ will be $i + m(j-1)$ let $v_{i+m(j-1)} = \Psi_{i,j}$, and also expanding $\eta$ in this fashion. Hence the vector $\mathbf{v}$ is

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{mn} \end{pmatrix}$$

Then the system becomes, after an index shift:

$$\frac{v_{i+m(j+1)} - 2v_{i+m(j)} + v_{i+m(j-1)}}{\Delta z^2} + \frac{v_{i+1+m(j)} - 2v_{i+m(j)} + v_{i-1+m(j)}}{\Delta r^2}$$
$$-\frac{1}{r_i}\left(\frac{v_{i+1+m(j)} - v_{i-1+m(j)}}{2\Delta r}\right) = -r_i\eta_{i+m(j)}$$

$$v_{i-1+m(j)}\left(\frac{1}{\Delta r^2} + \frac{1}{2\Delta r r_i}\right) + v_{i+m(j)}\left(-\frac{2}{\Delta z^2} - \frac{2}{\Delta r^2}\right) + v_{i+1+m(j)}\left(\frac{1}{\Delta r^2} - \frac{1}{2\Delta r r_i}\right)$$
$$+v_{i+m(j-1)}\left(\frac{1}{\Delta z^2}\right) + v_{i+m(j+1)}\left(\frac{1}{\Delta z^2}\right) = -r_i\eta_{i+m(j)}$$

Ignoring the boundary conditions on $\Psi(0,z), \Psi(r,0), \Psi(R,z), \Psi(r,Z)$

$$A_{a,b} = \begin{cases} \frac{1}{\Delta r^2} + \frac{1}{2\Delta r r_i} & b = a-1 \\ -\frac{2}{\Delta z^2} - \frac{2}{\Delta r^2} & b = a \\ \frac{1}{\Delta r^2} - \frac{1}{2\Delta r r_i} & b = a+1 \\ \frac{1}{\Delta z^2} & b = a-m \\ \frac{1}{\Delta z^2} & b = a+m \end{cases}$$

See if I can construct $A$. especially with sparse. With boundaries $\Psi(r=0)=0$, $\Psi(z=0)=f(r)$, $\Psi(r=R)=f(R)$ and $\Psi(z=Z)=$???

Wang, Rusak use $\frac{\partial \Psi}{\partial z}=0$. And at the right boundary it might make sense to use a backwards difference so we don't have to deal with the $n+1$. So that $\Psi_{i,j}$ is stored in index

for the time solver, precompute the LU factorisation of the matrix.

The BCs for our system will be:

at $z=0$, use $w$ to get two of these.

$$\Psi = f(r)$$
$$v = g(r)$$
$$\eta = -\frac{\partial w}{\partial r}$$

at $r=0$, the trivial BCs.

$$\Psi = 0$$
$$v = 0$$
$$\eta = 0$$

at $r=R$

$$\Psi = f(R)$$
$$v = g(R)$$
$$\eta = ?$$

$v=g(R)$ taken from (7.5.7) from bachelors book. We don't actually need an $\eta$ condition here really

In the book he uses $\frac{\partial \Psi}{\partial r} = 0$ on rigid boundaries, which sets $w = 0$, but we can't do this for this problem.

On the wall $r = R$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = -\frac{\partial w}{\partial r} = \frac{1}{r^2}\frac{\partial \Psi}{\partial r} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2}$$

He claims

at $z = Z$ (outlet)

$$\frac{\partial \Psi}{\partial z} = 0$$
$$\frac{\partial \eta}{\partial z} = 0$$
$$\frac{\partial v}{\partial z} = 0$$

The latter two we just assumed without any checking.

$$\Psi(r = R, z) = f(R)$$

Use backwards difference on the edge of $r = R$

$$\eta = -\frac{\partial w}{\partial r} = \frac{1}{r^2}\frac{\partial \Psi}{\partial r} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2}$$

Way to solve based on this paper:

We use the streamfunction-vorticity formulation

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = -r\eta$$

Process: start with (3),(4) then (5) including BCs for psi. now obtain BCs of eta.

Try and get code working - read new paper for BCs and algorithms, etc.

Research applications.

f18 vortex breakdown - occurs in aerodynamic flows - why its a problem (interacts with tail of jets)

vortical flows used in combustion devices

## 0.7 Papers

Do a write up and look at some of the research - explain them, bifurcations etc. Talk about how vortex breakdown is studied a lot and how we're reproduced results, differences, agreement.

### 0.7.1 An Introduction to Fluid Dynamics

Bachelor.

Best place to start - really goes to the basics and has derivations for the squire-longe equation as well as some examples, analytic solutions. A bit limited however for the relevant area - truly just gives an introduction to vortex breakdown.

### 0.7.2 Simulations of axisymmetric, inviscid swirling flows in circular pipes with various geometries

Wang, Rusak paper. Looks at a computational method for solving the problem, with the extra condition of allowing the pipes to have varying geometry.

### 0.7.3 The Structure and Dynamics of Bubble-Type Vortex Breakdown

Spall, Ash, Gatski

The paper doesn't fully explain what they do. They do give some initial conditions however, including a boundary condition I don't think i've seen in any other papers

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = const$$

The paper doesn't really explain very much of the maths, but ends up with some interesting plots. Including plots of the radial velocity which a lot of the other papers simply disregard.

### 0.7.4 Computational Design for Long-Term Numerical Integration of the Equations of Fluid Motion: Two-Dimensional Incompressible Flow Part 1

Arakawa

Uses a strange syntax, particularly using $\xi$ as the vorticity. The paper has a lot of equations in it, but none of them really seem to relate... Part 2 may be a little more insightful?

It does aim to obtain the jacobian for the numerical method. The paper kind of goes backwards - describing an average/finite different quantity and then explaining what continuous quantity it represents.

### 0.7.5 VORTEX BREAKDOWN: A TWO-STAGE TRANSITION

Escudier, Keller This considered the rankine vortex and applied a zero net force condition to the flow to look for 'realistic' solutions given no external force applied. The paper does not fully derive this however. It also looks at the same thing for potential flow. Not a huge amount of information in this paper, although the force condition is useful as it ensures that solutions are natural.

### 0.7.6 Theory of the vortex breakdown phenomenon

Benjamin A highly verbose paper. Attempts to explain how the flow states can change, and how vortex breakdown can actually occur. Uses linearisation mostly, and calculus of variations to explain vortex breakdown. Doesn't actually have all that much information...

### 0.7.7 Some Exact Solutions of the Flow Through Annular Cascade Actuator Discs

Bragg, Hawthorne

Derivation of the vorticity form of the S-L equation. A bit harder to read than the other papers, quite packed full of equations

### 0.7.8 Swirling flow states in finite-length diverging or contracting circular pipes

Wang Rusak Another verbose paper. Particularly interested in looking at steady states with vortex breakdown. Enforces the condition of zero radial velocity at the outlet - this may be questionable. Goes a little in depth to explain how the shape of a pipe affects the occurrence of vortex breakdown.

They use the $y = \frac{r^2}{2}$ trick in this paper.

This is similar to their 'simulations of ...' paper , except this one looks more at analysis and analytics instead of simulation. Calculus of variations.

### 0.7.9 Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe

Wang, Rusak

ANother verbose paper. Looks at bifurcations for vortex breakdown. Looks at finding critical swirl ratios where bifurcations occur. This paper limits its scope to regular straight pipes. This also uses calculus of variations.

This paper is a lot harder to understand than the others since CoV can be quite hard to interpret.

### 0.7.10 Axisymmetric vortex breakdown Part 1. Confined swirling flow

Lopez

Introduces a procedure to numerically solve the system. Unfortunately the paper doesn't clearly indicate some factors for solving - e.g. initial conditions. It also does not derive or explain the prediction equations that it uses.

## 0.8 Outer vortex breakdown

Considering the initial problem for vortex breakdown, except perhaps the breakdown is a pocket expanding from $R$ rather than 0. I.e. the breakdown occurs about the wall rather than the center. So assuming $r^\dagger$ is our outer vortex breakdown radius

This simply means obtaining a new $A$, $B$ and $k$.

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

$$w(r) = W + k(AJ_0(kr) + BY_0(kr))$$

Such that

$$w(r^\dagger) = 0, \qquad \Psi(0) = 0, \quad and \quad \Psi(r^\dagger) = 0$$

To enforce $\Psi(0) = 0$ note that $\lim_{r\to 0} \frac{Y_1(kr)}{r} = -\infty$. Hence it is necessary to set $B = 0$.

$$\Psi(r) = \frac{1}{2}Wr^2 + rAJ_1(kr), \quad w(r) = W + kAJ_0(kr)$$

And to enforce $\Psi(r^\dagger) = 0$

$$\implies Ar^\dagger J_1(kr^\dagger) = -\frac{1}{2}Wr^{\dagger 2}$$

$$A = \frac{-Wr^\dagger}{2J_1(kr^\dagger)}$$

And obtain $k$ using

$$w(r^\dagger) = 0$$

$$kAJ_0(kr) = -W$$

$$\Psi(r^\dagger) = \Psi(R) = \frac{1}{2}WR^2$$

## 0.9 Appendix

### 0.9.1 Supplementary Materials

This is where all the basic fluid mechanics knowledge should be (definitions, etc.)

### 0.9.2 Resources

Books: An Introduction to Fluid Dynamics Batchelor

Swirling flow states in finite-length diverging or contracting circular pipes Zvi Rusak

Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe Zvi Rusak, and Shixiao Wang

The Navier-Stokes equations: a classification of flows and exact solutions Drazin, Riley