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Scope of the book

This monograph elaborates a fundamental topic of the theory of fluid dynamics which is introduced in most textbooks on the theory of flow of a viscous fluid. A knowledge of this introductory background, for which reference may be made to Batchelor (1967), will be assumed here. However, it will be helpful to summarise a little of the background wherever we need it. In particular, we begin by introducing the scope of the book by loosely defining the terms of the title.

The *Navier–Stokes equations* are the system of non-linear partial differential equations governing the motion of a *Newtonian fluid*, which may be liquid or gas. In essence, they represent the balance between the rate of change of momentum of an element of fluid and the forces on it, as does Newton's second law of motion for a particle, where the stress is linearly related to the rate of strain of the fluid. Newton himself did not understand well the nature of the forces between elemental particles in a continuum, but he did (Newton 1687, Vol. II, Section IX, *Hypothesis, Proposition LI*) initiate the theory of the dynamics of a uniform viscous fluid in an intuitive and imaginative way. It was many years later that the Navier–Stokes equations, as we now know them, were deduced from various physical hypotheses, and in various forms, by Navier (1827), Poisson (1831), Saint-Venant (1843) and Stokes (1845). Stokes (1846) himself reviewed the methods and hypotheses of these authors, and presented a short rational derivation of the equations. We present a summary here; for a formal derivation the reader is referred to modern treatments by Batchelor (1967), Long (1961a), and Whitham (1963).

The equations may be expressed in the form

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{F} + \nabla . \boldsymbol{\sigma} \tag{1.1}$$

where $\rho(\mathbf{x}, t)$ is the density, $\mathbf{v}(\mathbf{x}, t)$ the velocity vector, $\mathbf{F}(\mathbf{x}, t)$ the body-force vector per unit mass, and $\boldsymbol{\sigma}$ the stress tensor of the fluid at a point with position vector $\mathbf{x}$ at time t; the *material* or *convective* derivative of a field varying with $\mathbf{x}$ and t is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla. \tag{1.2}$$

For a Newtonian fluid the stress tensor is a linear function of the rate of strain, from which it may be deduced that its components are

$$\sigma_{ij} = -p\delta_{ij} + \lambda \frac{\partial v_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.3}$$

where λ is the second coefficient of viscosity and μ the coefficient of viscosity of the fluid. In general λ and μ depend upon the thermodynamic properties of the fluid, and may vary in space and time. However, we shall assume they are uniform in space and constant in time.

These equations have been widely accepted as an excellent model of the macroscopic motions of most real fluids, including air and water, and are used by countless engineers, physicists, chemists, mathematicians, meteorologists, oceanographers, geologists and biologists. There are, however, notable and useful models of fluids whose motions are not governed by the Navier–Stokes equations. For example there are *non-Newtonian fluids* which are governed by a non-linear stress tensor, and *visco-elastic* fluids in which the stress depends on the strain as well as on the rate of strain of the fluid and retains a 'memory' of previous deformations; see Spencer (1980).

In addition to conservation of momentum (1.1) we have conservation of mass which leads to the well known continuity equation

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \nabla.\mathbf{v} = 0. \tag{1.4}$$

In addition to the above equations the body force $\mathbf{F}$ must be specified, together with an equation of state, if we are to model the motion of the fluid. For the example of a uniform incompressible fluid, which we adopt throughout, the equation of state is simply that the density is constant, so the equation of continuity becomes

$$\nabla.\mathbf{v} = 0. \tag{1.5}$$

The implication of (1.5) is that the stress (1.3) is independent of λ.

For any given problem in fluid dynamics, boundary conditions and initial conditions must be specified. Whilst such conditions are specific to the problem under consideration, it should be noted that an exact solution typically arises with certain boundary conditions sharing the same symmetries as assumed in deriving the solution. In other words conditions specific to a problem do not in general coincide with those necessary to find an exact solution. At a solid boundary it is now generally accepted that there is no relative motion between the boundary and the fluid. This not only implies non-permeability, but also that there is no slip between the fluid and solid. The no-slip condition proved controversial throughout the nineteenth century; Goldstein (1938, pp. 676–680) has given an account of the long history of the controversy. At an interface between two immiscible fluids the velocity and stress components must be continuous; at an air–liquid interface, as in water-wave theory, it is common practice to approximate these conditions by vanishing shear stress, and constant pressure.

An alternative form of equation (1.1) is

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \wedge \boldsymbol{\omega} = -\frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho\mathbf{v}^2\right) + \mathbf{F} - \nu\nabla \wedge \boldsymbol{\omega}, \tag{1.6}$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ is the vorticity vector and $\nu = \mu/\rho$ is the kinematic viscosity. Physically, the vorticity at any point in a fluid flow is proportional to the instantaneous angular velocity of an elementary spherical particle of fluid centred on the point. Flows with $\boldsymbol{\omega} \equiv \mathbf{0}$ are known as irrotational flows for which, of course, $\mathbf{v} = \nabla\phi$. Taking the 'curl' of equation (1.6), and acknowledging the solenoidal property of $\boldsymbol{\omega}$, gives

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}.\nabla)\mathbf{v} + \nabla \wedge \mathbf{F} + \nu\nabla^2\boldsymbol{\omega}, \tag{1.7}$$

where the terms on the right-hand side of (1.7) represent contributions to the rate of change of vorticity at a fluid particle. Thus, the contribution of the first term is due to the stretching and bending of vortex lines, the second is a distributed source term that is only relevant when the body force is non-conservative, whilst the last term represents the diffusion of vorticity between fluid particles. In general we shall assume that the body force is zero. There is a sense in which the vorticity may be viewed as fundamental, since it is the only flow quantity whose values are not propagated instantaneously in an incompressible fluid.

For an inviscid fluid, Lagrange's theorem tells us that a fluid which initially has zero vorticity, will have vanishing vorticity for all time. This cannot be true in a viscous fluid, since even in the absence of non-conservative body forces diffusion of vorticity from nearby particles can occur. But diffusion alone cannot *create* vorticity, and a question we must ask is how vorticity is introduced into the fluid where none is present initially? The answer is that a solid boundary acts as a source of vorticity. To appreciate this consider a two-dimensional flow, under the action of a pressure gradient, in the x-direction over the plane boundary $y = 0$. With $\mathbf{v} = (u, v, 0)$, $\boldsymbol{\omega} = (0, 0, \zeta)$ we have the flow of vorticity out of the surface $y = 0$, using (1.6) with $\mathbf{F} = \mathbf{0}$, as

$$-\nu\frac{\partial\zeta}{\partial y} = -\nu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial p}{\partial x}.$$

We see that (negative) vorticity flows into the fluid from the boundary in a favourable pressure gradient $\partial p/\partial x < 0$; when this changes sign vorticity is removed from it at the surface. This may rapidly reduce the magnitude of the surface vorticity to zero, corresponding to separation of the flow from the boundary. We shall often find it convenient to interpret our exact solutions in terms of their vortex dynamics.

An *exact solution* may seem to be no more nor less than a solution, because either a given set of fields $\mathbf{v}$, p, for given ρ, μ and body force $\mathbf{F}$ satisfies the governing equations or it does not. However, usage has given the phrase 'exact solution' a special meaning. It often denotes a solution which has a simple explicit form, usually an expression in finite terms of elementary or other well known special functions. Sometimes an exact solution is taken to be one which can be reduced to the solution of an ordinary differential equation or a system of a few ordinary differential equations. Rarely we go even further, and take an exact solution to be the solution of a partial differential equation, provided that the equation has fewer independent variables than the Navier–Stokes equations themselves. This is in contrast to an 'approximate solution', which is taken to be a field, simple or complicated, which approximates a solution either in a numerical sense or in an asymptotic limit, for example vanishingly small viscosity. Thus the logical distinctions between solutions, exact solutions and approximate solutions are blurred, but in practice the distinctions made are usually clear and useful. The exact solutions are, essentially, a subset of the solutions of the Navier–Stokes equations which happen to have relatively simple mathematical expressions and which are, mostly, simple physically. The essence of this account, then, is the explicitness and relative simplicity of the expression of the solutions. Many exact solutions of the Navier–Stokes equations are unstable and therefore unobservable in practice. The stability of flows is a large and important topic of fluid mechanics in its own right. We choose not to introduce it in this work, but refer the reader to two modern accounts of the subject, by Drazin and Reid (1981), and Drazin (2002).

In the early decades of the development of the mathematical theory of the motion of a viscous fluid, exact solutions were the only solutions available. Researchers solved what problems they could, rather than solving the practical problems in hand. Inevitably the solvable problems were the simple ones, usually idealised with a strong symmetry. From the mid-nineteenth century, and early twentieth century, asymptotic methods were developed, and thereafter numerical methods, albeit crude ones at first, to extend somewhat the range of tractable problems. However, the development of computational fluid dynamics over the past half century has changed the emphasis of research from, as one might express it, finding exact solutions of approximate problems to finding approximate solutions of exact problems. Nevertheless, the exact solutions remain a valuable and irreplaceable resource. They immediately convey more physical insight than a numerical table or sheaf of diagrams. This is especially true when the system is governed by one or more parameters, when a complete numerical tabulation would be voluminous. It may be argued that the exact solutions convey more than does the watching of a video of a numerical experiment or attendance at an experiment in a laboratory. However, the visual and aural appreciation of laboratory experiments and natural phenomena must never be underestimated. The very simplicity of exact solutions, in many cases, causes them to be used as prototypes which form our concepts of flow; for example when looking at a flow in a channel whose walls are nearly parallel we interpret observations of the flow by reference to our knowledge of plane Poiseuille flow, with a parabolic velocity profile, in a channel with fixed parallel walls. The exact solutions are first approximations to more general flows in many asymptotic theories as well as in our concepts. The exact solutions are also valuable when, as special cases of more general results, they enable theories or computer programs giving those results to be tested for errors and accuracy; in particular, they may provide benchmark tests for computational fluid dynamics. Exact solutions often arise as similarity solutions associated with systems of ordinary non-linear differential equations; many of these have provided a rich source of challenging and fruitful analytical problems and have attracted the interest of mathematical analysts. Such similarity solutions may emerge as asymptotic limits of more general solutions after a long time, or as the far-field solution in some spatial limit. Finally we remark that exact solutions are invaluably useful for examples and exercises that help students to learn and understand the theory of fluid mechanics. In teaching they can provide a bridge to the use of pictures of laboratory and computer experiments and the use of difficult mathematical methods.

We note that the Navier–Stokes equations are invariant under the action of various discrete and continuous groups of transformations of the independent and dependent variables. The set of all solutions of the equations is invariant under the same groups, although an individual solution may change under the action of one or more of the groups, and a given set of boundary conditions and initial conditions may change similarly. The groups of transformations under which the Navier–Stokes equations are invariant are: translations of time and space, Galilean transformations, parity or reversal and the group of transformations $t, \mathbf{x}, \mathbf{v}, p \longmapsto k^2 t, k\mathbf{x}, k^{-1}\mathbf{v}, k^{-2} p$. The groups of transformations due to scalings of time and space are closely related to the ideas of dynamical similarity and Buckingham's pi theorem. If we define $t' = t/T, \mathbf{x}' = \mathbf{x}/L, \mathbf{v}' = \mathbf{v}/U, p' = p/\rho U^2$ for arbitrary scales of time T, length L and velocity U then the Navier–Stokes equations for a uniform incompressible fluid in the absence of a body force are transformed to the dimensionless form

$$St \frac{\partial \mathbf{v}'}{\partial t} + \mathbf{v}'.\nabla'\mathbf{v}' = -\nabla' p' + R^{-1}\nabla'^2\mathbf{v}', \quad \nabla'.\mathbf{v}' = 0, \tag{1.8}$$

where $St = L/UT$ is the Strouhal number and $R = UL/\nu$ is the Reynolds number. For flows in which there is a natural time scale ω^{-1} where, for example, ω is a characteristic frequency, then we see from (1.8) that our unsteady flow is characterised by two independent dimensionless parameters. However, for an unsteady flow, for example a flow started from rest, in which the only time scale is L/U then, as for steady flows, a single parameter R characterises the flow.

Invariance under groups of scaling transformations leads to many exact solutions of the Navier–Stokes equations, known as similarity solutions, which arise as solutions of ordinary differential equations. Whilst most books on viscous flow theory pay some attention to dynamical similarity and similarity solutions, there are some books, for example Sedov (1959), Birkhoff (1960) and Barenblatt (1979, 1996), devoted to the theory of dynamical similarity and scaling transformations in similarity solutions. In particular Barenblatt (1996) gives a stimulating assessment of the significance and importance of the exact solutions of partial differential equations, as well as their relationship to asymptotic properties of other solutions. There are a few books, for example Bluman and Kumei (1989), Olver (1992) and Fushschich, Shtelen and Serov (1993), on the systematic use of continuous groups to find solutions of linear and non-linear partial differential equations, where it can be seen that Lie's theory has been used to find some valuable new solutions of equations other than the Navier–Stokes equations. Nevertheless the maturity of both the theory of the Navier–Stokes equations and the Lie theory of differential equations makes it unlikely that many more exact solutions of the Navier–Stokes equations remain to be discovered. For the same reason, those that do remain are probably of little importance. However, recent developments by Ludlow, Clarkson and Bassom (1998, 1999) in which non-classical reduction methods are employed, as opposed to the classical Lie group method, may offer an alternative way ahead.

Much of the early history of the exact solutions has been recorded by Truesdell (1954); also many textbooks, such as Batchelor (1967), contain brief accounts of the most important exact solutions. More extensive accounts are to be found in treatises on the flow of a viscous fluid, such as Dryden, Murnaghan and Bateman (1932), Schlichting (1979), Whitham (1963) and Lagerstrom (1996). Berker (1963) described the exact solutions in great detail; indeed at the time it was written his article was comprehensive and encyclopaedic. Wang (1989a, 1990a, 1991) has recently written review articles. Yet we believe these works leave a need for an up-to-date comprehensive integrated account of the exact solutions, and that this book fulfils that need. We do not claim that our account is exhaustive, and indeed references published before Berker's (1963) article are emphasised less than those published afterwards. Our philosophy for selection is that a solution is 'significant', either in an historical or novel sense or, perhaps most importantly, that it offers insight into the dynamical behaviour of viscous fluids. In that sense omissions are due to prejudice, but we hope not ignorance. The remaining four chapters of the book divide the exact solutions as follows. In chapter 2 we consider steady flows bounded by plane boundaries, and in chapter 3 steady flows bounded by curved boundaries, or exhibiting axisymmetry. To some extent that division between the two chapters is not clear-cut, since some exact solutions could feature in either. The remaining two chapters are devoted to unsteady flows, with a division as between the first two.

We conclude this introductory chapter by setting out, for subsequent reference, the Navier–Stokes equations for an incompressible fluid, in the three most commonly occurring co-ordinate systems. In what follows we make extensive reference to these. For general orthogonal curvilinear co-ordinates reference may be made to Whitham (1963).

(i) Cartesian co-ordinates In Cartesian co-ordinates (x, y, z) we take respective velocity components (u, v, w) so that $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. The Navier–Stokes and continuity equations (1.1), (1.5) then become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X + \nu\nabla^2 u, \tag{1.9}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y + \nu\nabla^2 v, \tag{1.10}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + Z + \nu\nabla^2 w, \tag{1.11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.12}$$

where the body force per unit mass $\mathbf{F} = (X, Y, Z)$ and here, and below, ∇^2 represents the three-dimensional Laplacian operator. The vorticity $\boldsymbol{\omega} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = (\xi, \eta, \zeta)$ where

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{1.13}$$

For two-dimensional flow, independent of the co-ordinate z, say, the continuity equation (1.12) is satisfied by introducing the stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{1.14}$$

and the only non-zero component of the vorticity is

$$\zeta = -\nabla^2 \psi, \tag{1.15}$$

which satisfies, for a conservative body force $\mathbf{F}$,

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = \nu \nabla^2 \zeta, \tag{1.16}$$

so that

$$\frac{\partial}{\partial t}(\nabla^2 \psi) - \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} = \nu \nabla^4 \psi. \tag{1.17}$$

(ii) Cylindrical polar co-ordinates We define cylindrical polar co-ordinates (r, θ, z) such that

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r \geq 0, \quad 0 \leq \theta < 2\pi, \tag{1.18}$$

with corresponding velocity components $\mathbf{v} = (v_r, v_\theta, v_z) = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{z}}$, vorticity and body-force components $\boldsymbol{\omega} = (\omega_r, \omega_\theta, \omega_z)$, $\mathbf{F} = (F_r, F_\theta, F_z)$ respectively. The components of the Navier–Stokes equations are, then,

$$\begin{aligned}\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right),\end{aligned} \tag{1.19}$$

$$\begin{aligned}\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \\ = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + F_\theta + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right),\end{aligned} \tag{1.20}$$

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z + \nu\nabla^2 v_z, \tag{1.21}$$

with the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0. \tag{1.22}$$

The components of vorticity are given by

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \quad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}. \tag{1.23}$$

For a rotationally symmetric flow, independent of θ, we introduce a different stream function ψ, first identified by Stokes (1842) such that with

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{1.24}$$

the continuity equation is satisfied identically.

(iii) Spherical polar co-ordinates We define spherical polar co-ordinates (r, θ, ϕ) such that

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad r \geq 0, \quad 0 \leq \theta \leq \pi,$$
$$0 \leq \phi < 2\pi,$$

with corresponding velocity components $\mathbf{v} = (v_r, v_\theta, v_\phi) = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\phi\hat{\boldsymbol{\phi}}$, vorticity and body-force components $\boldsymbol{\omega} = (\omega_r, \omega_\theta, \omega_\phi)$, $\mathbf{F} = (F_r, F_\theta, F_\phi)$ respectively. The components of the Navier–Stokes equations are, then,

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r + \nu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right), \tag{1.25}$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}$$
$$= -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + F_\theta + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right), \tag{1.26}$$

$$\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi\cot\theta}{r}$$
$$= -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + F_\phi + \nu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right), \tag{1.27}$$

with the continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 0. \tag{1.28}$$

The components of vorticity are given by

$$\begin{aligned}\omega_r &= \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial\phi}\right\},\\ \omega_\theta &= \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi),\\ \omega_\phi &= \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}.\end{aligned} \tag{1.29}$$

The Stokes stream function, for a rotationally symmetric flow independent of ϕ, is now defined such that

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{1.30}$$