Fluid Thesis

Andrew Martin

March 17, 2020

Abstract

Just so I don't forget that there is an abstract environment...

Contents

0.1 Introduction

0.2 Background

Fluid dynamics This paper will be concerned with strictly dynamic flows.

The flow considered will be an axisymmetric, incompressible, inviscid, cylindrical flow. The stream function, ψ corresponds to the cylindrical coordinate system (r, θ, z), where θ denotes angle around the z axis. The corresponding velocity components are (u, v, w), the radial, azimuthal and axial components of the flow respectively.

It will always be assumed that at the inlet

$$\psi(z=0) = \psi_u = \frac{1}{2}Wr^2$$

Incompressibility is defined as:

$$\nabla \cdot \mathbf{u} = 0$$

Leibovich (1978) defines vortex breakdown as "a disturbance characterized by the formation of an internal stagnation point on the vortex axis, followed by reversed flow in a region of limited axial extent"

0.3 Derivation of the Squire-Long equation

The Squire-Long / Bragg-Hawthorne equation for the stream function of axisymmetric inviscid fluid, using cylindrical coordinates is:

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = r^2\frac{dH}{d\psi} - C\frac{dC}{d\psi}$$

Where C is the prescribed circulation, $C(\psi_u) = rv$ and $H(\psi_u)$ is the prescribed flow head.

Vorticity

$$w = w_r e_r + w_\theta e_\theta + w_z e_z$$

where w_r, w_θ, w_z can be written in terms of the velocity

The following derivation largely follows that of An Introduction to Fluid Dynamics by Batchelor:

Considering cylindrical coordinates (r,θ,z) with corresponding velocity (u,v,w), vorticity components (w_r, w_θ, w_z). Axisymmetric flow as:

$$\omega_z = \frac{1}{r}\frac{\partial rv}{\partial r}, \quad \omega_r = -\frac{\partial rv}{\partial z}, \quad \omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

The stream function will exist if the continuity equation (corresponding to conservation of mass) is satisfied, I.e.

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad w = \frac{1}{r}\frac{\partial\psi}{\partial r}$$

Where ψ is the stream function Hence the azimuthal component for w_θ is

$$\begin{aligned}\omega_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ &= -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial\psi}{\partial r} \\ &= -\frac{1}{r}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\end{aligned}$$

Use the vorticity equation

$$w \times v - \frac{\partial w}{\partial t} = \nabla H$$

Where

$$H = \frac{1}{2}(w^2 + u^2 + v^2) + \frac{p}{\rho}$$

This gives:

$$u\omega_\theta - v\omega_r - \frac{\partial w}{\partial t} = \frac{\partial H}{\partial x}$$
$$v\omega_z - w\omega_\theta - \frac{\partial u}{\partial t} = \frac{\partial H}{\partial r}$$
$$w\omega_r - u\omega_z - \frac{\partial v}{\partial t} = 0$$

The last one is equivalent to the material derivative of rw set to 0:

$$\frac{D(rv)}{Dt} = 0$$

From the Bernoulli equation:

$$rv = C(\psi)$$
$$\frac{\partial \psi}{\partial t} + \frac{1}{2}|\mathbf{w}|^2 + \frac{p}{\rho} = H(\psi)$$

Where $H(\psi)$ and $C(\psi)$ are arbitrary functions.

Rewriting ω:

$$\omega_z = w\frac{dC}{d\psi}, \quad \omega_r = u\frac{dC}{d\psi}$$

Giving

$$\frac{\omega_\theta}{r} = \frac{v\omega_r}{ru} + \frac{1}{ru}\frac{dH}{d\psi}\frac{\partial \psi}{\partial z} = \frac{C}{r^2}\frac{dC}{d\psi} - \frac{dH}{d\psi}$$

Which is the form taken by the second of the dynamic equations. Now, combining this last statement with the equation for ω_θ:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = r^2\frac{dH}{d\psi} - C\frac{dC}{d\psi}$$

0.4 Rotating Flow

Considering the flow far upstream where there is constant uniform axial velocity and rotates with angular velocity Ω

$$v = \Omega r, w = W$$

And

$$C = rv = \frac{v^2}{\Omega} = \Omega r^2 = 2\Omega\psi/W$$
$$\frac{dC}{d\psi} = 2\Omega/W$$

Since the flow is steady, the radial equation of motion yields:

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{w^2}{r} = \frac{C^2}{r^3}$$

$$
\begin{aligned}
H &= \frac{1}{2}(u^2+v^2+w^2)+\frac{p}{\rho} \\
&= \frac{1}{2}(\Omega^2 r^2+W^2)+\frac{p}{\rho} \\
&= \frac{\Omega^2\psi}{W}+\frac{1}{2}W^2+\frac{p}{\rho} \\
&= \frac{\Omega^2\psi}{W}+\frac{1}{2}W^2+\int\frac{1}{\rho}\frac{dp}{dr}dr \\
&= \frac{\Omega^2\psi}{W}+\frac{1}{2}W^2+\int\frac{C^2}{r^3}dr \\
&= \frac{\Omega^2\psi}{W}+\frac{1}{2}W^2+\int\frac{\Omega^2 r^4}{r^3}dr \\
&= \frac{\Omega^2\psi}{W}+\frac{1}{2}W^2+\int\Omega^2 r\,dr \\
&= \frac{\Omega^2\psi}{W}+\frac{1}{2}W^2+\frac{1}{2}\Omega^2 r^2 \\
&= \frac{2\Omega^2\psi}{W}+\frac{1}{2}W^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{dH}{d\psi} &= \frac{\partial\frac{2\Omega^2\psi}{W}}{\partial\psi} \\
&= \frac{2\Omega^2}{W}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= r^2\frac{dH}{d\psi}-C\frac{dC}{d\psi} \\
\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= \frac{2r^2\Omega^2}{W}-\frac{4\Omega^2}{W^2}\psi
\end{aligned}
$$

Or in a more standard form

$$
\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{4\Omega^2}{W^2}\psi=\frac{2r^2\Omega^2}{W}
$$

0.4.1 Analysis

Considering some point far downstream, such that $\psi(z,r) = \psi(r)$, i.e. that locally, ψ is just a function of the radius, r. The ODE is:

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} = \frac{2r^2\Omega^2}{W} - \frac{4\Omega^2}{W^2}\psi$$

A change of variables can be used to obtain a homogeneous ODE:

$$\psi(r) = \frac{1}{2}Wr^2 = \frac{1}{2}Wr^2 + rF(r)$$

Where $F(r)$ is a function of r

$$\frac{d\psi}{dr} = Wr + F + r\frac{dF}{dr}$$
$$\frac{d^2\psi}{dr^2} = W + 2\frac{dF}{dr} + r\frac{d^2F}{dr^2}$$

Reducing the ODE to:

$$r^2\frac{d^2F}{dr^2} - r\frac{dF}{dr} + F(r^2k^2 - 1) = 0$$

By making the substitutions: $k = \frac{2\Omega}{W}$ and $x = kr$, Such that

$$\frac{dF}{dr} = \frac{dF}{dx}\frac{dx}{dr} = k, \quad \frac{d^2F}{dr^2} = k^2\frac{d^2F}{dx^2}$$

The ODE simplifies to

$$\frac{x^2}{k^2}k^2\frac{d^2F}{dx^2} - \frac{x}{k}k\frac{dF}{dx} + F(\frac{x^2}{k^2}k^2 - 1) = 0$$
$$x^2\frac{d^2F}{dx^2} - x\frac{dF}{dx} + F(x^2 - 1) = 0$$

Which is the form of a Bessel differential equation of order $\nu = 1$, giving solutions

$$F = AJ_1(kr) + BY_1(kr)$$

Hence the stream function to obtain this is
Returning to the stream function:

$$\psi(r) = \frac{1}{2}Wr^2 + r\left(AJ_1(kr) + BY_1(kr)\right)$$

And hence

$$w = \frac{1}{r}\frac{\partial\psi}{\partial r} = W + AkJ_0(kr) + BkY_0(kr)$$

A, and B rely on boundary conditions. In this case, it is necessary for the streamlines to be the same as at the inlet along the boundary. Also introduce what will hereby by the vortex breakdown condition on the core of the stream, i.e. a region $0 < r < r_*$ where the stream function becomes zero:

$$\psi(R) = \frac{1}{2}WR^2$$
$$\psi(r_*) = 0$$

Consider it as a matrix system

$$\begin{pmatrix} r_* J_1(kr_*) & r_* Y_1(kr_*) \\ R J_1(kR) & R Y_1(kR) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} W r_*^2 \\ 0 \end{pmatrix}$$

Giving

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{r_* R \left(J_1(kr_*) Y_1(kR) - Y_1(kr_*) J_1(kR)\right)} \begin{pmatrix} R Y_1(kR) & -r_* Y_1(kr_*) \\ -R J_1(kR) & r_* J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2} W r_*^2 \\ 0 \end{pmatrix}$$

$$A = \frac{-\frac{1}{2} R W r_*^2 Y_1(kR)}{r_* R \left(J_1(kr_*) Y_1(kR) - Y_1(kr_*) J_1(kR)\right)}$$
$$B = \frac{\frac{1}{2} R W r_*^2 J_1(kR)}{r_* R \left(J_1(kr_*) Y_1(kR) - Y_1(kr_*) J_1(kR)\right)}$$

And hence

$$A = \frac{-\frac{1}{2} W r_* Y_1(kR)}{\left(J_1(kr_*) Y_1(kR) - Y_1(kr_*) J_1(kR)\right)}$$
$$B = \frac{\frac{1}{2} W r_* J_1(kR)}{\left(J_1(kr_*) Y_1(kR) - Y_1(kr_*) J_1(kR)\right)}$$

With the requirement that $r_* \neq R$ so as to not divide by zero.

Using

$$w = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

Gives

$$w = W + k(A J_0(kr) + B Y_0(kr))$$

Solving this for a given k (or alternatively a desired r_*) is done numerically using `MATLAB`.

The set of valid solutions to this problem are those which satisfy the constraint

$$w(r_*) = W + k(A J_0(kr_*) + B Y_0(kr_*)) = 0$$

The plot figure 0.4.1 shows the k, r_* combinations which satisfy the constraint.

Clearly this can only occur for values of $kR > 3.8$.

The first branch of this (extending from $kR \approx 3.8$) corresponds to natural solutions, whereas further branches give unnatural behaviour, introducing factors such as reversed flow.

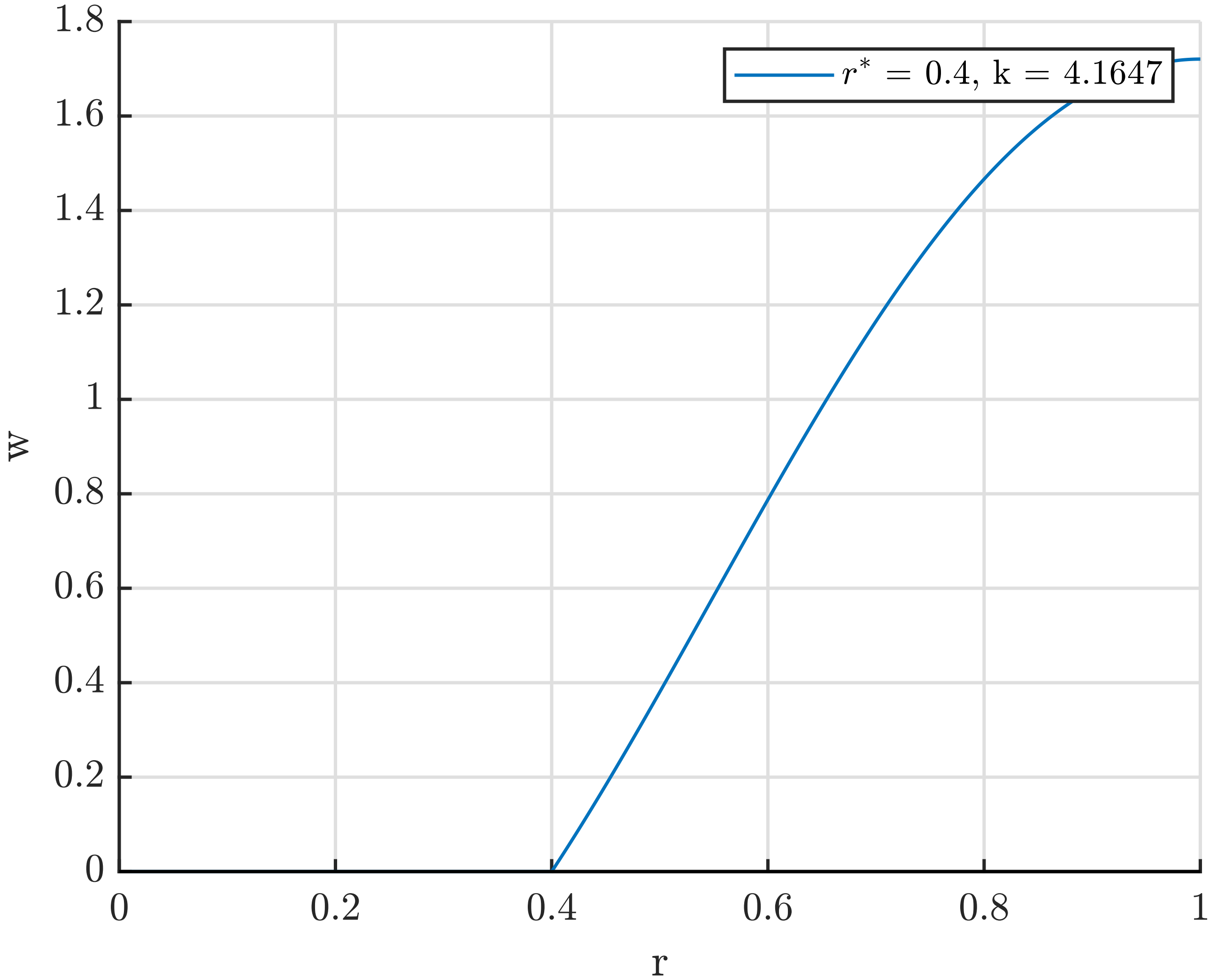

Figure 1: An example solution plot - code: RotatingFlow.m

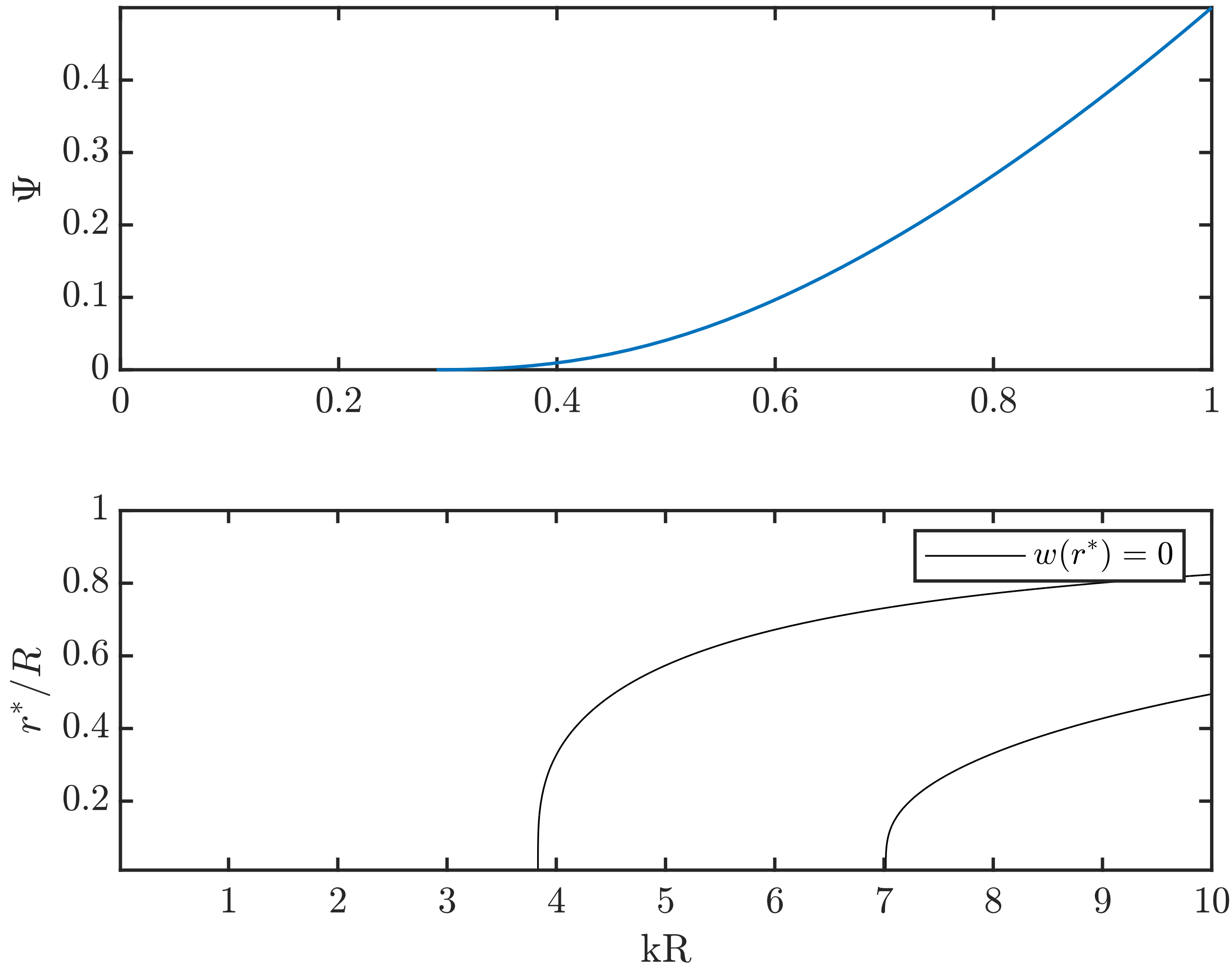

Figure 2: Solution set for the simplified problem - code: RotatingFlow.m

have to assume things for outside of the region for ψ. I.e. if we go above the maximum input value then some assumption, and if we go below the minimum then it is a stagnation point

0.5 Rankine Body

$w = W$,

$$v = \begin{cases} \frac{\Gamma}{2\pi r}, & r > r_0 \\ \Omega r, & r \leq r_0 \end{cases}$$

Where the second condition was the previous solution. Since the velocity profile is now piece-wise defined, the stream function must also be, i.e. it is necessary to split the stream function into 2 regions to solve this problem. The upstream regions:

$$\begin{cases} \psi_{inner}, & 0 \leq r \leq r_0 \\ \psi_{outer}, & r_0 \leq r \leq R \end{cases}$$

Note that r_0 is defined upstream, so the position of the region may have moved downstream to a new radius, $\hat{r}$, and hence, downstream, these regions will become around $\hat{r}$ instead of r_0. We enforce some similar conditions as to the normal problem:

$$\begin{aligned} \psi(r_*) &= 0, \\ \psi(R) &= \frac{1}{2} W R^2, \\ w(r_*) &= 0 \end{aligned}$$

With the added condition that ψ must remain continuous around $\hat{r}$ I.e.

$$\lim_{r^- \to \hat{r}} \psi(r^-) = \lim_{r^+ \to \hat{r}} \psi(r^+)$$

And

$$\lim_{r^- \to \hat{r}} v(r^-) = \lim_{r^+ \to \hat{r}} v(r^+)$$

Where $\psi(r^-)$ is ψ defined for $r \leq \hat{r}$ and $\psi(r^+)$ is defined in the region $r \geq \hat{r}$.

The region for $\psi(r)$ with $r \in [0, r_0]$ will be the same as before, i.e.

$$\psi(r) = \frac{1}{2} W r^2 + r(AJ_1(kr) + BY_1(kr))$$

For the region $r_0 < r < R$ the problem must be resolved from the SL equation

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = r^2 \frac{dH}{d\psi} - C \frac{dC}{d\psi}$$

$$C = rv = \frac{\Gamma}{2\pi}$$

$$\frac{dC}{d\psi} = 0$$

$$
\begin{aligned}
H &= \frac{1}{2}(u^2+v^2+w^2)+\frac{p}{\rho} \\
&= \frac{1}{2}(0+\frac{\Gamma^2}{4\pi^2 r^2}+W^2)+\int\frac{C^2}{r^3}dr \\
&= \frac{1}{2}(\frac{\Gamma^2}{4\pi^2 r^2}+W^2)+\int\frac{\Gamma^2}{4\pi^2 r^3}dr \\
&= \frac{1}{2}(\frac{\Gamma^2}{4\pi^2 r^2}+W^2)-\frac{\Gamma^2}{8\pi^2 r^2} \\
&= \frac{W^2}{2} \\
\frac{dH}{d\psi} &= 0
\end{aligned}
$$

And hence the SL equation gives

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= r^2\frac{dH}{d\psi}-C\frac{dC}{d\psi} \\
\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= r^2\frac{dH}{d\psi}-C\frac{dC}{d\psi} \\
\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= 0
\end{aligned}
$$

Which results in:

$$
\psi = Cr^2 + D, \quad r \geq \hat{r}
$$

$$
w = \frac{1}{r}\frac{\partial\psi}{\partial r} = 2C
$$

With the requirement that there is no discontinuity at $\hat{r}$, i.e.

$$
\psi = \frac{1}{2}W\hat{r}^2 + \hat{r}\left(AJ_1(k\hat{r})+BY_1(k\hat{r})\right) = C\hat{r}^2 + D
$$

And using the same for w

$$
w(\hat{r}) = W + k(AJ_0(k\hat{r})+BY_0(k\hat{r})) = 2C
$$

And lastly the wall condition

$$
\psi(R) = \frac{1}{2}WR^2 = C\hat{r}^2 + D
$$

With

$$
w(r_*) = 0
$$

$$
\frac{\Gamma}{2\pi r_0} = \Omega r_0 \implies \Omega = \frac{\Gamma}{2\pi r_0^2}
$$

$$
k_{outer} = \frac{2\Gamma}{2\pi W r_0^2} = \frac{\Gamma}{\pi W r_0^2}
$$

Noting that the values for A and B are obtained from the r_* condition.

The coefficients for ψ have to be resolved, since the condition $\psi_{inner}(R) = \frac{1}{2}WR^2$ cannot be imposed.

Parameters

$$r_0, \hat{r}, r_*, R, k, \Gamma, W, A, B, C, D$$

We can fix r_0, R, k, W and Γ. This is 11 parameters, where 5 are fixed. Require 6 conditions. Impose:

1). $w(r_*) = 0$ (as before)

2). $\psi_{inner}(r_*) = 0$ (as before)

3). Since at the wall ψ must remain the same, this applies to where v is changed, i.e. $\psi_{inner}(\hat{r}) = \frac{1}{2}Wr_0^2$

4). For continuity, $\psi_{outer}(\hat{r}) = \frac{1}{2}Wr_0^2$

5). $w_{outer}(\hat{r}) = w_{inner}(\hat{r})$

6). $\psi_{outer}(R) = \frac{1}{2}WR^2$

Redo the problem instead getting A, B from 2) and 3)

$$\psi_{inner}(r_*) = 0$$
$$\psi_{inner}(\hat{r}) = \frac{1}{2}Wr_0^2$$

Use this for A, B

$$\begin{aligned}
\psi_{inner}(r_*) &= \frac{1}{2}Wr_*^2 + r_*(AJ_1(kr_*) + BY_1(kr_*)) = 0 \\
&= r_*(AJ_1(kr_*) + BY_1(kr_*)) = -\frac{1}{2}Wr_*^2 \\
\psi_{inner}(\hat{r}) &= \frac{1}{2}W\hat{r}^2 + \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}Wr_0^2 \\
&= \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}W(r_0^2 - \hat{r}^2)
\end{aligned}$$

This gives the matrix system for A, B below. Note that the system relies on the unknowns r_* and $\hat{r}$.

$$\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ \hat{r}J_1(k\hat{r}) & \hat{r}Y_1(k\hat{r}) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{det} \begin{pmatrix} \hat{r}Y_1(k\hat{r}) & -r_*Y_1(kr_*) \\ -\hat{r}J_1(k\hat{r}) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix}$$

$$\begin{aligned}
A &= \frac{1}{\text{det}} \left(\hat{r}Y_1(k\hat{r}) \left(-\frac{1}{2}Wr_*^2 \right) - r_*Y_1(kr_*) \left(\frac{1}{2}W(r_0^2 - \hat{r}^2) \right) \right) \\
B &= \frac{1}{\text{det}} \left(-\hat{r}J_1 \left(-\frac{1}{2}Wr_*^2 \right) + r_*J_1(kr_*) \left(\frac{1}{2}W(r_0^2 - \hat{r}^2) \right) \right)
\end{aligned}$$

Where

$$\begin{aligned}
\text{det} &= \hat{r}r_*Y_1(k\hat{r})J_1(kr_*) - \hat{r}r_*J_1(k\hat{r})Y_1(kr_*) \\
&= \hat{r}r_* \left(Y_1(k\hat{r})J_1(kr_*) - J_1(k\hat{r})Y_1(kr_*) \right)
\end{aligned}$$

This for r_*

$$w_{inner}(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

Get C from:

$$\begin{aligned} w_{outer}(\hat{r}) &= w_{inner}(\hat{r}) \\ 2C &= W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) \\ C &= \frac{1}{2}\left(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r}))\right) \end{aligned}$$

Get D here:

$$\begin{aligned} \psi_{outer}(R) = CR^2 + D &= \frac{1}{2}WR^2 \\ D &= \frac{1}{2}WR^2 - C \end{aligned}$$

Hence get $\hat{r}$ from

$$\begin{aligned} \psi_{outer}(\hat{r}) = C\hat{r}^2 + D &= \frac{1}{2}Wr_0^2 \\ C\hat{r}^2 + \frac{1}{2}WR^2 - C &= \frac{1}{2}Wr_0^2 \\ \left(\frac{1}{2}\left(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r}))\right)\right)(\hat{r}^2 - 1) &= \frac{1}{2}W(r_0^2 - R^2) \\ \left(AJ_0(k\hat{r}) + BY_0(k\hat{r})\right)(\hat{r}^2 - 1) &= \frac{1}{k}W(r_0^2 - R^2 - 1) \end{aligned}$$

For physically valid solutions, we must impose the condition of no net change on the momentum from upstream to downstream on the momentum (Escudier, Keller). The momentum is defined as

$$s = 2\pi \int_0^{r_t} \left(\rho w^2 + p\right) r dr$$

Which comes to:

$$\Delta s = \frac{\pi}{4}\rho U^2 k^2 r_c^2 \left[-r_b^2 + \frac{1}{4}\left(\frac{r_b^4 - r_a^4}{r_c^2}\right) + \frac{3}{4}r_c^2 + \frac{1}{2}r_c^2 \log\left(\frac{r_b^2}{r_c^2}\right)\right] = 0$$

Figure 4 shows the solution set for the problem. It displays the same results as those found in (Escudier, Keller), with the same asymptote $kr_0 \to \sqrt{2}$ as $r_0 \to 0$.

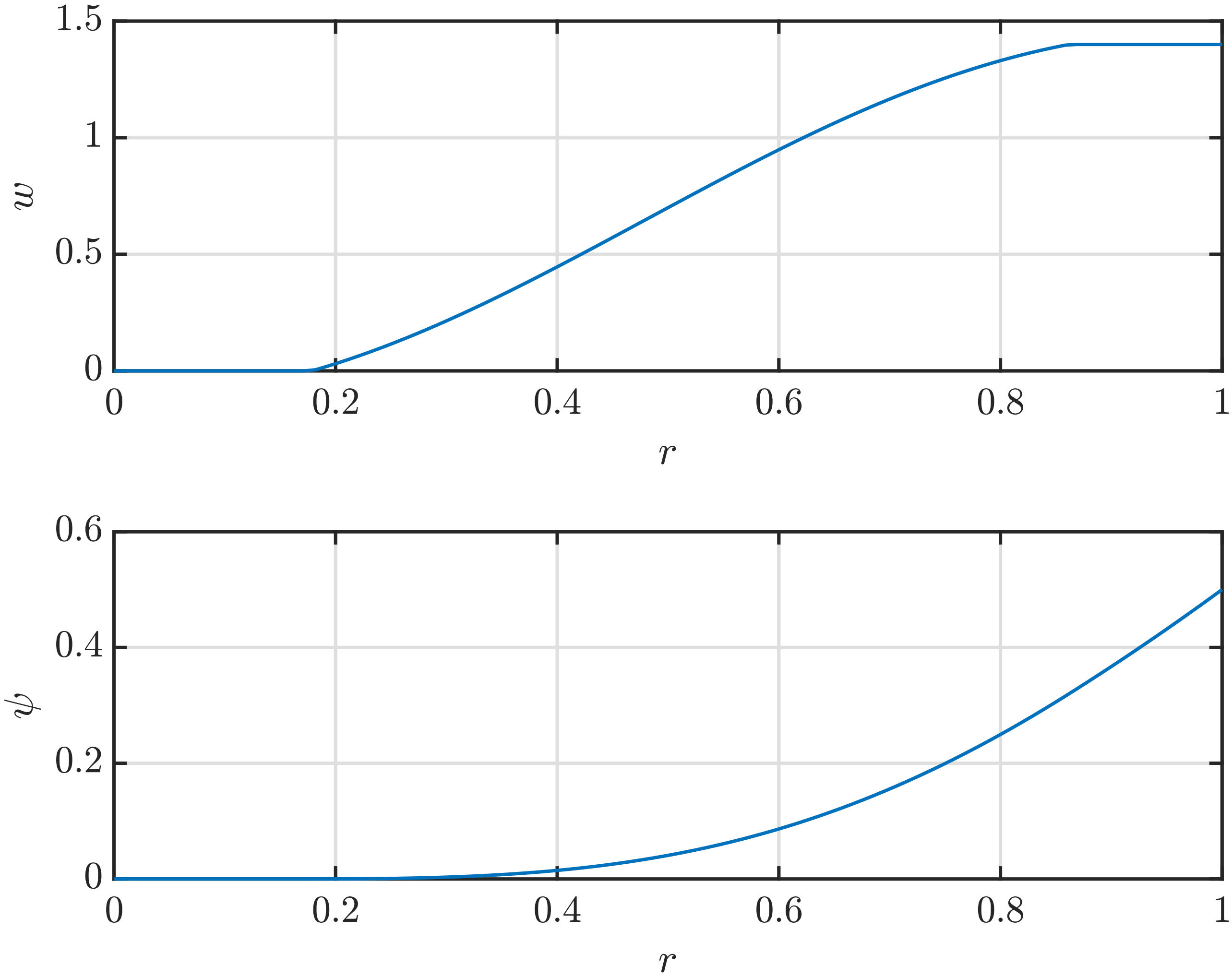

Figure 3: A solution of w and ψ for the Rankine problem with 0 net momentum, $k = 3.8961, r_0 = 0.8619, r_* = 0.1784$ - code: rhatrstarmomentum.m

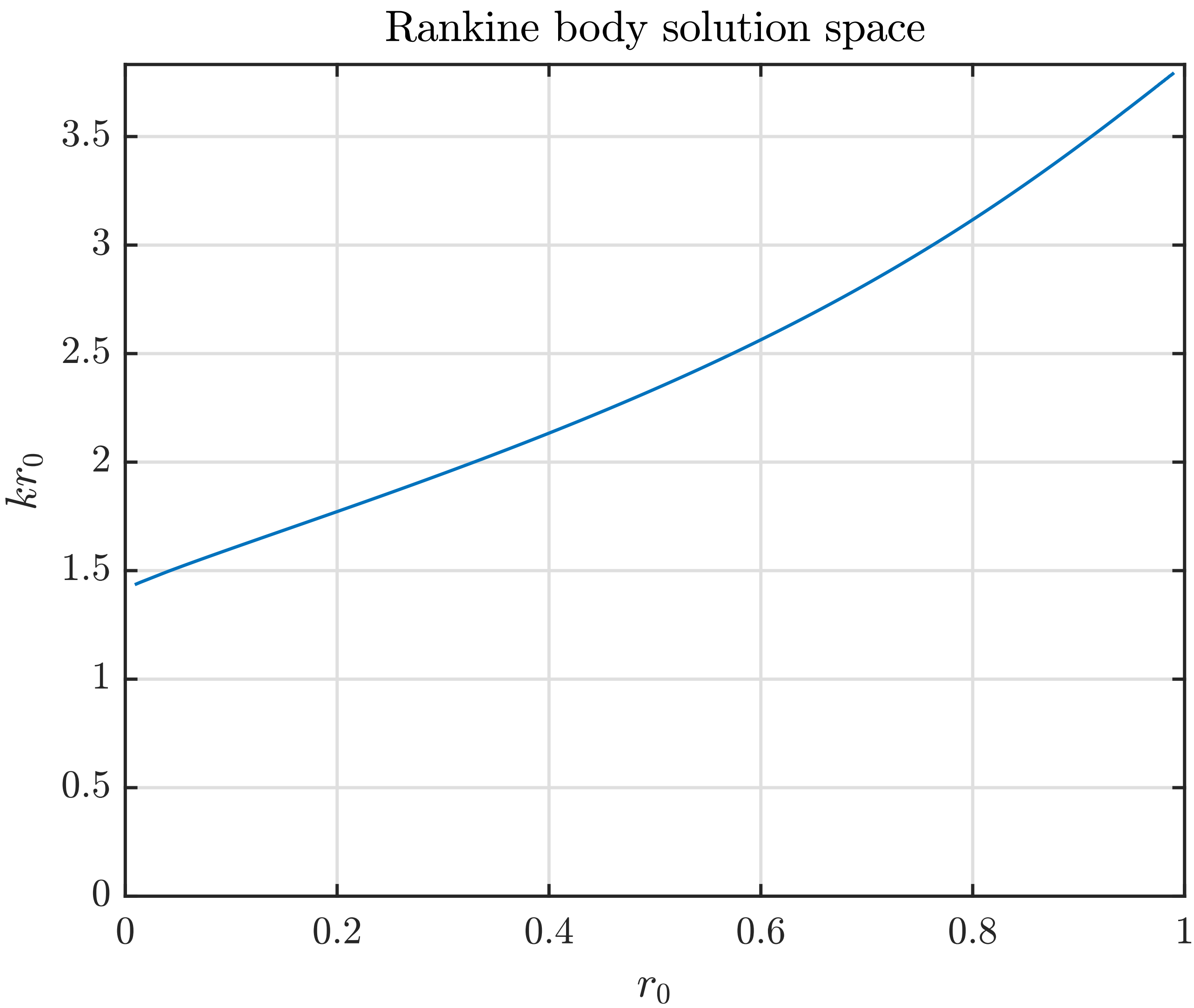

Figure 4: Solution space for the Rankine body problem - code: numericalSolutionSetRankine.m

0.6 Burgers Vortex

Q-vortex without a Jet. Start with

$$w = W$$

$$v = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/\delta^2}\right)$$

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} = r^2\frac{dH}{d\psi} - C\frac{dC}{d\psi}$$

Solve from r_* to R numerically.

Generate grid from r_* to R.

Boundary conditions as normal

$$\psi(R) = \frac{1}{2}WR^2$$

$$\psi(r_*) = 0$$

$$w(r_*) = 0$$

And the standard upstream flow

$$\psi(r) = \frac{1}{2}Wr^2$$

Non-dimensional parameter may be something like $\frac{\Gamma}{WR}$ (we can probably relate this to kr_0 for the rankine problem)

Eventually do the same thing as before with s and Δs.

$$s = \int_0^R \left(\rho w^2 + p\right) rdr = \int_0^{r_*} p(r_*)rdr + \int_{r_*}^R \left(\rho w^2 + p\right) rdr$$

Use

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v^2}{r} = \frac{\Gamma^2}{4\pi^2 r^3 +}\left(1 - e^{-r^2/\delta^2}\right)^2$$

$$\psi = \frac{1}{2}Wr^2 \implies r = \sqrt{\frac{2\psi}{W}}$$

$$\begin{aligned}
C = rv &= \frac{\Gamma}{2\pi}\left(1 - e^{-r^2/\delta^2}\right) \\
\frac{\partial C}{\partial \psi} &= \frac{\Gamma}{2\pi}\frac{\partial}{\partial \psi}\left(1 - e^{-r^2/\delta^2}\right) \\
&= \frac{-\Gamma}{2\pi}\frac{\partial}{\partial \psi}\left(e^{-r^2/\delta^2}\right) \\
&= \frac{-\Gamma}{2\pi}\frac{\partial}{\partial \psi}\left(e^{-2\psi/W\delta^2}\right) \\
&= \frac{\Gamma}{W\delta^2\pi}\left(e^{-2\psi/W\delta^2}\right) \\
&= \frac{\Gamma}{W\pi\delta^2}e^{-r^2/\delta^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{dH}{d\psi} &= \frac{dH}{dr}\frac{dr}{d\psi} \\
&= \frac{dr}{d\psi}\frac{d}{dr}\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+\frac{p}{\rho}\right) \\
&= \frac{1}{\sqrt{2W\psi}}\frac{d}{dr}\left(\frac{1}{2}v^2+\int\frac{C^2}{r^3}dr\right) \\
&= \frac{1}{Wr}\left(\frac{1}{2}\frac{dv^2}{dr}+\frac{v^2}{r}\right) \\
&= \frac{1}{Wr}\left(\frac{\Gamma^2}{4r\pi^2}\left(\frac{-1}{r^2}+2e^{-r^2/\delta^2}\left(\frac{1}{r^2}+\frac{1}{\delta^2}\right)-e^{-2r^2/\delta^2}\left(\frac{1}{r^2}+\frac{2}{\delta^2}\right)\right)+\frac{\Gamma^2}{4r\pi^2}\left(\frac{1-2e^{-r^2/\delta^2}+e^{-2r^2/\delta^2}}{r^2}\right)\right) \\
&= \frac{\Gamma^2}{4Wr^2\pi^2}\left(2e^{-r^2/\delta^2}\left(\frac{1}{r^2}+\frac{1}{\delta^2}\right)-e^{-2r^2/\delta^2}\left(\frac{1}{r^2}+\frac{2}{\delta^2}\right)+\frac{-2e^{-r^2/\delta^2}+e^{-2r^2/\delta^2}}{r^2}\right) \\
&= \frac{\Gamma^2}{2Wr^2\delta^2\pi^2}\left(e^{-r^2/\delta^2}-e^{-2r^2/\delta^2}\right) \\
&= \frac{\Gamma^2}{4\psi\delta^2\pi^2}\left(e^{-2\psi/W\delta^2}-e^{-4\psi/W\delta^2}\right)
\end{aligned}
$$

How I got the middle term:

$$
\begin{aligned}
\frac{dv^2}{dr} &= \frac{d}{dr}\left(\frac{\Gamma}{2\pi r}\left(1-e^{-r^2/\delta^2}\right)\right)^2 \\
&= \frac{\Gamma^2}{4\pi^2}\frac{d}{dr}\left(\frac{1-2e^{-r^2/\delta^2}+e^{-2r^2/\delta^2}}{r^2}\right) \\
&= \frac{\Gamma^2}{4\pi^2}\left(\frac{-2}{r^3}-2\left(-\frac{2e^{-r^2/\delta^2}}{r^3}-\frac{2e^{-r^2/\delta^2}}{r\delta^2}\right)+\left(-\frac{2e^{-2r^2/\delta^2}}{r^3}-\frac{4e^{-2r^2/\delta^2}}{r\delta^2}\right)\right) \\
&= \frac{\Gamma^2}{2r\pi^2}\left(\frac{-1}{r^2}+2e^{-r^2/\delta^2}\left(\frac{1}{r^2}+\frac{1}{\delta^2}\right)-e^{-2r^2/\delta^2}\left(\frac{1}{r^2}+\frac{2}{\delta^2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= r^2\frac{dH}{d\psi}-C\frac{dC}{d\psi} \\
&= \frac{r^2\Gamma^2}{4\psi\delta^2\pi^2}\left(e^{-2\psi/W\delta^2}-e^{-4\psi/W\delta^2}\right)-\left(\frac{\Gamma}{2\pi}\left(1-e^{-2\psi/W\delta^2}\right)\right)\left(\frac{\Gamma}{W\delta^2\pi}\left(e^{-2\psi/W\delta^2}\right)\right) \\
&= \frac{r^2\Gamma^2}{4\psi\delta^2\pi^2}\left(e^{-2\psi/W\delta^2}-e^{-4\psi/W\delta^2}\right)-\frac{\Gamma^2}{2W\delta^2\pi^2}\left(1-e^{-2\psi/W\delta^2}\right)\left(e^{-2\psi/W\delta^2}\right) \\
&= \frac{\Gamma^2}{2W\delta^2\pi^2}\left(\frac{r^2W}{2\psi}\left(e^{-2\psi/W\delta^2}-e^{-4\psi/W\delta^2}\right)-\left(e^{-2\psi/W\delta^2}-e^{-4\psi/W\delta^2}\right)\right) \\
\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r} &= \frac{\Gamma^2}{2W\delta^2\pi^2}\left(\left(\frac{r^2W}{2\psi}-1\right)\left(e^{-2\psi/W\delta^2}-e^{-4\psi/W\delta^2}\right)\right)
\end{aligned}
$$

Giving the system

$$\psi_1' = \psi_2$$
$$\psi_2' = \frac{1}{r}\psi_2 + \frac{\Gamma^2}{2W\delta^2\pi^2}\left(\left(\frac{r^2W}{2\psi_1} - 1\right)\left(e^{-2\psi_1/W\delta^2} - e^{-4\psi_1/W\delta^2}\right)\right)$$

This system is solved in `MATLAB` using `fzero()`.

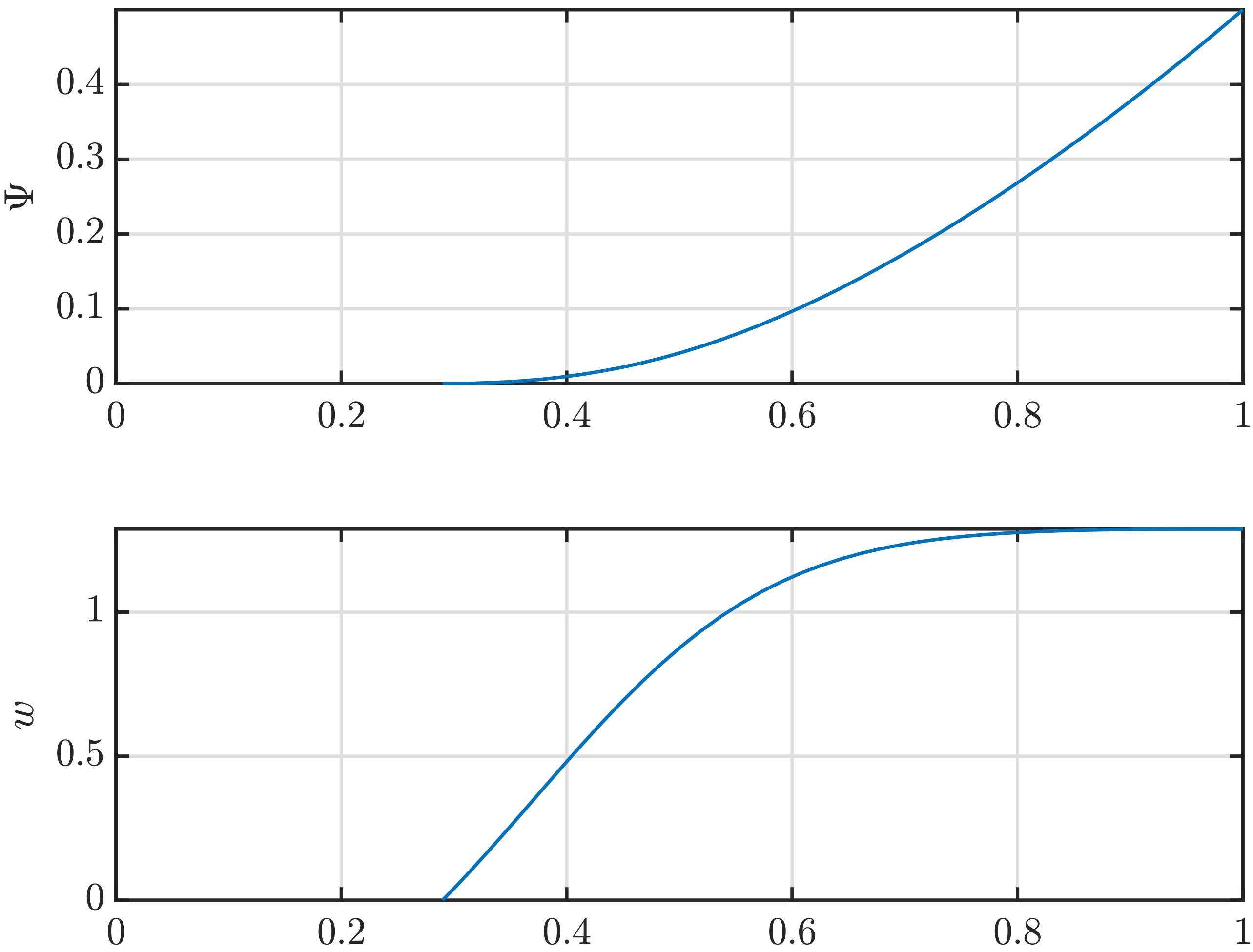

Figure 5: Example solution for the Burger vortex - code: BurgerSolver.m

0.7 Numerics

Begin by considering the ODE in $\psi(r)$:

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} = r^2\frac{dH}{d\psi} + C\frac{dC}{d\psi}$$

finite difference - divide r as a grid of N intervals, such that

$$r_i = \Delta r_i, \quad \Delta = \frac{R}{N}$$

$$\frac{\partial^2\psi}{\partial r^2} = \frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{\Delta^2}$$

$$\frac{\partial \psi}{\partial r} = \frac{\psi_{i+1} - \psi_{i-1}}{2\Delta}$$

$$\psi_0 = 0, \quad \psi_N = \frac{1}{2}WR^2$$

Which should work for the index i until we reach the bifurcations/stagnations

Should end up with a matrix equation

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ & & \mathbf{A} & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi \\ \psi_N \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ \frac{1}{2}WR^2 \end{pmatrix}$$

A should be the finite difference version of

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0$$

I.e. for the i^{th} row of $\mathbf{A}$

$$A(i) = \frac{A(i+1) - 2 * A(i) + A(i-1)}{\Delta^2} - \frac{A(i+1) - A(i-1)}{2r(i)\Delta}$$

$$A_{ij} = \begin{cases} 1 & j = i = 1 \\ 1/\Delta^2 + 1/(2r_i\Delta) & j = i-1 \\ 2/\Delta^2 & j = i \\ 1/\Delta^2 - 1/(2r_i\Delta) & j = i+1 \\ 1 & j = i = N \\ 0 & otherwise \end{cases}$$

For the full equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \psi(\frac{4\Omega^2}{W^2} - \frac{1}{r^2}) = 0$$

$$\psi = \frac{1}{2}Wr^2 + rF$$

$$F = \frac{\psi}{r} - \frac{1}{2}Wr$$

Boundary conditions for F relate to those for ψ.

$$\psi(R) = \frac{1}{2}WR^2 \implies F(R) = 0$$

$$\psi(r_*) = 0 \implies F(r_*) = \frac{1}{2}Wr_*^2$$

when we look at the vortex breakdown problem, introduce a coordinate transformation

$$\eta = \frac{r - r_*}{R - r_*}$$

$$\eta = 0, r = r_*, \eta = 1 r = R$$

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{1}{R - r_*}\frac{\partial \psi}{\partial \eta}$$

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{1}{(R - r_*)^2} \frac{\partial^2 \psi}{\partial \eta^2}$$

use the same conditions we have used anyway where $\psi(r_*) = w(r_*) = 0$ Rankine body problem: At some point on the radius r_0, we get $v = K/r_0$ for some constant K find $K = \Omega r_0^2$?

When we do finite differences, grab a computational variable (call it η for now)

$$\eta = \frac{r - r_*}{R - r_*}$$

So now we are computing in $0 < \eta < 1$. So try plugging it into the ODE:

$$d\eta = \frac{dr}{R - r_*}$$

$$\frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial r} = \frac{1}{R - r_*} \frac{\partial \psi}{\partial \eta}$$

If we don't know r_* the η vector becomes

$$\begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{N-1} \\ r_* \end{pmatrix}$$

This will be a non-linear problem so we will need to solve using `fsolve`.

To get guesses could use rankine vortex stuff -

For a linear flow we could use We know that $\psi(r_*) = 0$ and $\psi(R) = \frac{1}{2} W R^2$. We could guess that ψ is constant, $\psi = Ar^2 + B$.

Try plotting $w(r_*)$ for various r_* to help find guesses (require it to be 0)

If using η, the ODE becomes $\psi(\eta = 0) = 0$ and $\psi(\eta = 1) = \frac{1}{2} W R^2$

$$\frac{\partial \psi}{\partial r} = \frac{1}{R - r_*} \frac{\partial \psi}{\partial \eta}$$

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{1}{(R - r_*)^2} \frac{\partial^2 \psi}{\partial \eta^2}$$

$$r = \eta(R - r_*)) + r_*$$

DE becomes

$$\frac{1}{(R - r_*)^2} \frac{\partial^2 \psi}{\partial \eta^2} - \frac{1}{\eta(R - r_*)) + r_*} \frac{1}{R - r_*} \frac{\partial \psi}{\partial \eta} = 0$$

$$\frac{1}{R - r_*} \frac{\partial^2 \psi}{\partial \eta^2} - \frac{1}{\eta(R - r_*)) + r_*} \frac{\partial \psi}{\partial \eta} = 0$$

Write a function which calculates the residual, i.e.

$$\hat{r}_i = LHS - RHS|_i$$

and then fsolve on that

Send through the vector

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ r_* \end{pmatrix}$$

Might be worth looking at setting the RHS

$$r^2 \frac{dH}{d\psi} + C\frac{dC}{d\psi}$$

As a function $f(r,\psi)$ and the settings

$$f(r,\psi) = \begin{cases} \ldots, & \text{if } 0 \leq \psi \leq \frac{1}{2}WR^2 \\ 0 & otherwise \end{cases}$$

0.8 Unsteady 2D System

So far, only a reduced version of the problem has been considered - one in an infinitely long channel, where ψ has become steady in the z direction.

Here a numerical analysis of the two dimensional system is introduced where the flow is not assumed to be steady.

0.8.1 Alterations

A new form for the system is used which is easier to handle numerically in an updating system.

$$\eta = -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)$$
$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = -r\eta$$

$$\frac{\partial v}{\partial t} = \frac{1}{r}J(v) + \frac{1}{r}\frac{v}{r}\frac{\partial \psi}{\partial z}$$
$$\frac{\partial \eta}{\partial t} = J(\frac{\eta}{r}) + 2\frac{v}{r}\frac{\partial v}{\partial z}$$

Where

$$J(x) := \frac{\partial \psi}{\partial z}\frac{\partial x}{\partial r} - \frac{\partial \psi}{\partial r}\frac{\partial x}{\partial z}$$

Which are the governing equations from Lopez (1990) for $Re \to \infty$.

Time is scaled by Ω and pipe length by $1/R$

The boundary conditions used are:

At $z = 0$, use w to get two of these.

$$\psi = f(r)$$
$$v = g(r)$$
$$\eta = -\frac{\partial w}{\partial r} = \frac{1}{r^2}\frac{\partial \psi}{\partial r}(r,0) - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2}(r,0)$$

at $r = 0$, the trivial BCs.

$$\psi = 0$$
$$v = 0$$
$$\eta = 0$$

at $r = R$, the inlet conditions

$$\psi = f(R)$$
$$v = g(R)$$
$$\eta = -\frac{\partial w}{\partial r}\Big|_{r=R} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\Big|_{r=R} = \frac{1}{R^2}\psi'(R,z) - \frac{1}{R}\psi''(R,z)$$

$v(R) = g(R)$ taken from (7.5.7) from bachelors book.

at $z = Z$ (outlet)

$$\frac{\partial \psi}{\partial z} = 0$$
$$\frac{\partial \eta}{\partial z} = 0$$
$$\frac{\partial v}{\partial z} = 0$$

For solving the problem numerically the stepping equations outlined in Lopez (1990) are altered to work inside the `Matlab` `ode45()` ODE solver. The initial conditions and boundary conditions reflect those used by Zhang et al (2019). These papers represent the pipe with a transformation of r, where $y := r^2/2$.

The time stepping scheme relies on time-stepping the azimuthal vorticity, velocity and the stream-function, $\psi(r,z,t), \eta(r,z,t)$, and $v(r,z,t)$ respectively. Their initial values are:

$$\psi(r,z,0) = \frac{1}{2}r^2 + \delta\phi_B(r)\sin\left(\frac{\pi z}{2Z}\right)$$
$$\eta(r,z,0) = -\delta r\left(\phi_{Byy}(y) - (\frac{\pi}{2Z})^2\frac{\phi_B(y)}{2y})\right)\sin(\frac{\pi z}{2 * Z})$$
$$v(r,z,0) = \frac{1}{r}\left((2\omega y) + 2\delta\omega\phi_B(y)\sin\left(\frac{\pi z}{2Z}\right)\right)$$

Where δ is the size of the initial perturbation, $\phi_B(r) = rJ_1(r\omega_B)$, ω_B is Benjamin's critical swirl ratio for solid-body rotation, $\phi_{Byy} = -4\omega_B^2(\phi_B)/r^2$ (Rusak et al 2019), Z is the length of the pipe in the z direction, and ω is the swirl ratio.

0.8.2 Finite Differences

Backward difference:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \mathcal{O}(h)$$

Forward Difference

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

Central Differences

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$$

Backwards differences for a second derivative

$$f''(x) = \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$

Forwards

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

Central

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

and using the rusak method, by letting $y = r^2/2$

do some research on numerical solutions of axisymmetric swirling flow

0.8.3 Discretisation

To discretise the system, discretise r into r_i such that $i = 1, \dots, m$ and z to z_j such that $j = 1, \dots, n$. Finite differences then give:

$$\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta z^2} + \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta r^2} - \frac{1}{r_i}\left(\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta r}\right) = -r_i \eta_{i,j}$$

Could write $r_{i,j}$ in case it changes in j.

Of course to write this as a linear system, we have to get form $A\boldsymbol{\psi} = b$, or more precisely

$$A\boldsymbol{\psi} = -\mathbf{r}\boldsymbol{\eta}$$

So we would have to write the vector $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi} = \begin{pmatrix} \psi_{1,1} \\ \psi_{2,1} \\ \vdots \\ \psi_{m,1} \\ \psi_{1,2} \\ \vdots \\ \psi_{1,n} \\ \vdots \\ \psi_{m,n} \end{pmatrix}$$

The vector $\boldsymbol{\eta}$ will follow the same pattern, and the vector $\mathbf{r}$ will just be $[\mathbf{r}]_{i,j} = r_i$

So the index of $\psi_{i,j}$ will be $i + m(j-1)$ let $x_{i+m(j-1)} = \psi_{i,j}$, and also expanding η in this fashion. Hence the vector $\mathbf{x}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{mn} \end{pmatrix}$$

Then the system becomes, after an index shift:

$$\frac{x_{i+m(j+1)} - 2x_{i+m(j)} + x_{i+m(j-1)}}{\Delta z^2} + \frac{x_{i+1+m(j)} - 2x_{i+m(j)} + x_{i-1+m(j)}}{\Delta r^2} - \frac{1}{r_i}\left(\frac{x_{i+1+m(j)} - x_{i-1+m(j)}}{2\Delta r}\right) = -r_i \eta_{i+m(j)}$$

$$x_{i-1+m(j)}\left(\frac{1}{\Delta r^2} + \frac{1}{2\Delta r r_i}\right) + x_{i+m(j)}\left(-\frac{2}{\Delta z^2} - \frac{2}{\Delta r^2}\right) + x_{i+1+m(j)}\left(\frac{1}{\Delta r^2} - \frac{1}{2\Delta r r_i}\right) + x_{i+m(j-1)}\left(\frac{1}{\Delta z^2}\right) + x_{i+m(j+1)}\left(\frac{1}{\Delta z^2}\right) = -r_i \eta_{i+m(j)}$$

Ignoring the boundary conditions on $\psi(0,z), \psi(r,0), \psi(R,z), \psi(r,Z)$

$$A_{a,b} = \begin{cases} \frac{1}{\Delta r^2} + \frac{1}{2\Delta r r_i} & b = a-1 \\ -\frac{2}{\Delta z^2} - \frac{2}{\Delta r^2} & b = a \\ \frac{1}{\Delta r^2} - \frac{1}{2\Delta r r_i} & b = a+1 \\ \frac{1}{\Delta z^2} & b = a-m \\ \frac{1}{\Delta z^2} & b = a+m \end{cases}$$

$$\eta|_{r=R} \approx \frac{1}{R^2}\left(\frac{\psi_{nr,j}-\psi_{nr-1,j}}{\Delta r}\right) - \frac{1}{R}\left(\frac{\psi_{nr,j}-2\psi_{nr-1,j}+\psi_{nr-2,j}}{\Delta r^2}\right)$$
$$\eta|_{z=0} \approx \frac{1}{r^2}\left(\frac{\psi_{i+1,1}-\psi_{i-1,1}}{2\Delta r}\right) - \frac{1}{r}\left(\frac{\psi_{i+1,1}-2\psi_{i,1}+\psi_{i-1,1}}{\Delta r^2}\right)$$

0.9 Papers

Do a write up and look at some of the research - explain them, bifurcations etc. Talk about how vortex breakdown is studied a lot and how we've reproduced results, differences, agreement.

0.9.1 An Introduction to Fluid Dynamics

Bachelor.

Best place to start - really goes to the basics and has derivations for the Squire-Long equation as well as some examples, analytic solutions. A bit limited however for the relevant area - truly just gives an introduction to vortex breakdown.

We re-derived the SL equation

0.9.2 Simulations of axisymmetric, inviscid swirling flows in circular pipes with various geometries

Zhang, Wang, Rusak paper. Looks at a computational method for solving the problem, with the extra condition of allowing the pipes to have varying geometry.

0.9.3 The Structure and Dynamics of Bubble-Type Vortex Breakdown

Spall, Ash, Gatski

The paper doesn't fully explain what they do. They do give some initial conditions however, including a boundary condition I don't think i've seen in any other papers

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = const$$

The paper doesn't really explain very much of the maths, but ends up with some interesting plots. Including plots of the radial velocity which a lot of the other papers simply disregard.

0.9.4 Computational Design for Long-Term Numerical Integration of the Equations of Fluid Motion: Two-Dimensional Incompressible Flow Part 1

Arakawa

Uses a strange syntax, particularly using ζ as the vorticity. The paper has a lot of equations in it, but none of them really seem to relate... Part 2 may be a little more insightful?

It does aim to obtain the jacobian for the numerical method. The paper kind of goes backwards - describing an average/finite different quantity and then explaining what continuous quantity it represents.

0.9.5 VORTEX BREAKDOWN: A TWO-STAGE TRANSITION

Escudier, Keller This considered the rankine vortex and applied a zero net force condition to the flow to look for 'realistic' solutions given no external force applied. The paper does not fully derive this however. It also looks at the same thing for potential flow. Not a huge amount of information in this paper, although the force condition is useful as it ensures that solutions are natural.

We used the Δs condition here and were able to reproduce their results for the rankine vortex.

0.9.6 Theory of the vortex breakdown phenomenon

Benjamin A highly verbose paper. Attempts to explain how the flow states can change, and how vortex breakdown can actually occur. Uses linearisation mostly, and calculus of variations to explain vortex breakdown. Doesn't actually have all that much information...

0.9.7 Some Exact Solutions of the Flow Through Annular Cascade Actuator Discs

Bragg, Hawthorne

Derivation of the vorticity form of the S-L equation. A bit harder to read than the other papers, quite packed full of equations

0.9.8 Swirling flow states in finite-length diverging or contracting circular pipes

Wang Rusak Another verbose paper. Particularly interested in looking at steady states with vortex breakdown. Enforces the condition of zero radial velocity at the outlet - this may be questionable. Goes a little in depth to explain how the shape of a pipe affects the occurrence of vortex breakdown.

They use the $y = \frac{r^2}{2}$ trick in this paper.

This is similar to their 'simulations of ...' paper , except this one looks more at analysis and analytics instead of simulation. Calculus of variations.

0.9.9 Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe

Wang, Rusak

Another verbose paper. Looks at bifurcations for vortex breakdown. Looks at finding critical swirl ratios where bifurcations occur. This paper limits its scope to regular straight pipes. This also uses calculus of variations.

This paper is a lot harder to understand than the others since CoV can be quite hard to interpret.

0.9.10 Axisymmetric vortex breakdown Part 1. Confined swirling flow

Lopez

Introduces a procedure to numerically solve the system. Unfortunately the paper doesn't clearly indicate some factors for solving - e.g. initial conditions. It also does not derive or explain the prediction equations that it uses.

We are trying to reproduce the code used in this paper.

0.10 Andrew's Notes

When breakdown is present, require

$$C = \frac{u_x \delta t}{\delta x} + \frac{u_y \delta t}{\delta y} \leq C_{max} = 0.05$$

Otherwise 0.2.

The system is

$$\begin{aligned}
\psi(r,z,0) &= \frac{1}{2}r^2 + \delta\phi_B(r)\sin\left(\frac{\pi z}{2Z}\right) \\
\eta(r,z,0) &= -\delta r\left(\phi_{Byy}(r) - \left(\frac{\pi}{2Z}\right)^2 \frac{\phi_B(r)}{r^2}\right)\sin\left(\frac{\pi z}{2Z}\right) \\
v(r,z,0) &= \frac{\omega}{r}\left(r^2 + 2\delta\phi_B(r)\sin\left(\frac{\pi z}{2Z}\right)\right)
\end{aligned}$$

$$\phi_B = rJ_1(2\omega_B r)$$

With BCs:

$$\begin{aligned}
\psi(x=0) &= y \\
\psi(y=1/2) &= 1/2 \\
\psi(y=0) &= 0 \\
\psi_x(x=L) &= 0 \\
K(x=0) &= 2(\omega y) \\
\chi(x=0) &= -\psi_0 yy = 0
\end{aligned}$$

$$\begin{aligned}
\psi(x,y,0) &= y + \delta\phi_B(y)\sin(\pi x/2L) \\
\chi(x,y,0) &= -\delta[\phi_{Byy}(y) - (\pi/2L)^2\phi_B(y)/2y]\sin(\pi x/2L) \\
K(x,y,0) &= 2\omega y + 2\delta\omega\phi_B(y)\sin(\pi x/2L)
\end{aligned}$$

$$\phi_B(y) = \phi_1(y) = \sqrt{2y}J_1(2\omega_B\sqrt{2y})$$

ω_B is the first zero to the bessel function J_1. And $\phi_m(y)$ is the solution to

$$\phi_{myy} + \left(4\omega^2_{(m,n)} - \frac{(2n-1)^2\pi^2}{4L^2}\right)\frac{\phi_m}{2y} = 0$$
$$\phi_m(0) = \phi_m(\frac{1}{2}) = 0$$

$$K = rv$$
$$\eta = \sqrt{2y}\chi = r\chi$$

Length of pipe $R = 1$, $L = 6$, $\omega = \{1.9, 1.95, 1.97\}$ for the three sims.
$y = r^2/2$
try playing around with outlet conditions. wang,rusak use the $\frac{\partial\phi}{\partial z} = 0$
trent suggested at $z = Z$

$$\frac{\partial\phi}{\partial t} = -U_a\frac{\partial\phi}{\partial z}$$

where ϕ would be both v and η, i.e.

$$\frac{\partial v}{\partial t} = -U_a \frac{\partial v}{\partial z}$$

$$\frac{\partial \eta}{\partial t} = -U_a \frac{\partial \eta}{\partial z}$$

and U_a is a parameter.

set U_a to some value, perhaps the mean velocity in the tube - i.e. try 1

If the flow is steady however, this is equivalent to $\frac{\partial \phi}{\partial z} = 0$, so we want to check if this makes any difference.

The current issue is I am not obtaining psi in the post processing.

I SHOULD PLOT DOWNSTREAM v for all our equations also

To use the r_* and w parts, we can use $w = \frac{1}{r}\frac{\partial \psi}{\partial r}$

Try putting in the homogeneous solution to the solver to see if it works (i.e. the one with $v = \Omega r$ and $w = W$).

We should expect that the Burgers vortex should be a more smooth version of the Rankine problem - so we should be able to compare the two.

May want to find $v_{max} = r_0$ to compare to the previous problem. Should expect the circulation goes to $\Gamma/2\pi$ as $r \to \infty$.

Of course we have to have

$$\lim_{r\to 0} \frac{1}{r}\frac{\partial \psi}{\partial r} < \infty$$

So use l'hopital's rule

$$\lim_{r\to 0} \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{\partial^2 \psi}{\partial r^2}$$

But we know $\psi = 0$ at $r = 0$ (so we can ignore this)

So we will just use

$$\frac{\partial^2 \psi}{\partial r^2}|_i - \frac{1}{r}\frac{\partial \psi}{\partial r}|_i = f(r_i, \psi_i)$$

With

$$\psi_1 = 0$$

And

$$\psi_n = \frac{1}{2} W R^2$$

We will have 1 based indexing Alternatively get the derivatives at the mid points

$$\frac{\partial \psi}{\partial r}\Big|_{i+1/2} = \frac{\psi_{i+1} - \psi_i}{h_r}$$

Where h_r is the step.

Alternatively

Can rewrite the left hand side as

$$r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right)$$

Process: start with (3),(4) then (5) including BCs for psi. now obtain BCs of eta.

Research applications.

f18 vortex breakdown - occurs in aerodynamic flows - why its a problem (interacts with tail of jets)

vortical flows used in combustion devices

0.11 Appendix

0.11.1 Supplementary Materials

This is where all the basic fluid mechanics knowledge should be (definitions, etc.)

0.11.2 Resources

Books: An Introduction to Fluid Dynamics Batchelor

Swirling flow states in finite-length diverging or contracting circular pipes Zvi Rusak

Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe Zvi Rusak, and Shixiao Wang

The Navier-Stokes equations: a classification of flows and exact solutions Drazin, Riley

The Structure of Vortex Breakdown, Leibovich

Simulations of axisymmetric, inviscid swirling flows in circular pipes with various geometries Zhang, Rusak, Wang (2019)