

Fluid Thesis

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Abstract

Just so I don't forget that there is an abstract environment...

Contents

0.1	Introduction	1
0.2	Derivation of the Squire-Long equation	1
0.2.1	Homogeneous ODE	4
0.2.2	Numerics	8
0.2.3	Rankine Body	10
0.3	Burger's Vortex	13
0.3.1	Outer vortex breakdown	19
0.4	Appendix	20
0.4.1	Supplementary Materials	20
0.4.2	Resources	20

0.1 Introduction

0.2 Derivation of the Squire-Long equation

Squire-long / Bragg-Hawthorne equation for the stream function of axisymmetric inviscid fluid, using cylindrical coordinates

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

radial component u, azimuthal (swirl) is v, axial component w
stream function satisfies

$$\nabla \cdot u = 0 \longrightarrow \text{streamfunction exists}$$

Remember for cylindrical coordinates:

$$u = \frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad w = -\frac{1}{r} \frac{\partial \Psi}{\partial r}$$

Ψ is the stream function

r is the radius

$$C = rv$$

$$H = \frac{p}{\rho} + \frac{1}{2}(u^2 + v^2 + w^2)$$

H is conserved on stream surfaces

C is conserved on stream surfaces

vorticity

$$w = w_r e_r + w_\theta e_\theta + w_z e_z$$

where w_r, w_θ, w_z can be written in terms of the velocity

Considering cylindrical coordinates (z, r, θ) with corresponding velocity (u, v, w) , vorticity components $(\omega_z, \omega_r, \omega_\theta)$. Axisymmetric flow as:

$$\omega_z = \frac{1}{r} \frac{\partial rv}{\partial r}, \quad \omega_r = -\frac{\partial rv}{\partial z}, \quad \omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

The continuity equation (conservation of mass) is satisfied by setting

$$w = \frac{1}{r} \frac{\partial \Psi}{\partial r}, \quad u = -\frac{1}{r} \frac{\partial \Psi}{\partial z}$$

Where Ψ is the stream function This gives the azimuthal component for w_θ :

$$\begin{aligned} \omega_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ &= -\frac{1}{r} \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r} \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \\ &= -\frac{1}{r} \left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \end{aligned}$$

Use the vorticity equation

$$w \times v - \frac{\partial w}{\partial t} = \nabla H$$

Where

$$H = \frac{1}{2}(w^2 + u^2 + v^2) + \frac{p}{\rho}$$

This gives:

$$\begin{aligned} u\omega_\theta - v\omega_r - \frac{\partial w}{\partial t} &= \frac{\partial H}{\partial x} \\ v\omega_z - w\omega_\theta - \frac{\partial u}{\partial t} &= \frac{\partial H}{\partial r} \\ w\omega_r - u\omega_z - \frac{\partial v}{\partial t} &= 0 \end{aligned}$$

The last one is equivalent to the material derivative of rw set to 0:

$$\frac{D(rv)}{Dt} = 0$$

From the Bernoulli equation:

$$\begin{aligned} rv &= C(\Psi) \\ \frac{\partial \Psi}{\partial t} + \frac{1}{2}|\mathbf{w}|^2 + \frac{p}{\rho} &= H(\Psi) \end{aligned}$$

Where $H(\Psi)$ and $C(\Psi)$ are arbitrary functions.

Rewriting ω :

$$\omega_z = w \frac{dC}{d\Psi}, \quad \omega_r = u \frac{dC}{d\Psi}$$

Giving

$$\frac{\omega_\theta}{r} = \frac{v\omega_r}{ru} + \frac{1}{ru} \frac{dH}{d\Psi} \frac{\partial \Psi}{\partial z} = \frac{C}{r^2} \frac{dC}{d\Psi} - \frac{dH}{d\Psi}$$

Which is the form taken by the second of the dynamic equations. Now, combining this last statement with the equation for ω_θ :

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

Taken from Batchelor's An Introduction to Fluid Dynamics

Considering the flow far upstream where there is constant uniform axial velocity and rotates with angular velocity Ω

$$\Psi_{\text{upstream}} = \frac{1}{2}Wr^2$$

$$v = \Omega r, w = W$$

And

$$C = rv = \frac{v^2}{\Omega} = \Omega r^2 = 2\Omega\Psi/W$$

$$\frac{dC}{d\Psi} = 2\Omega/W$$

Since the flow is steady, the radial equation of motion yields:

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{w^2}{r} = \frac{C^2}{r^3}$$

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2}(\Omega^2 r^2 + W^2) + \frac{p}{\rho} \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \frac{p}{\rho} \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \frac{1}{\rho} \frac{dp}{dr} dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \frac{C^2}{r^3} dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \frac{\Omega^2 r^4}{r^3} dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \Omega^2 r dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \frac{1}{2}\Omega^2 r^2 \\ &= \frac{2\Omega^2 \Psi}{W} + \frac{1}{2}W^2 \end{aligned}$$

$$\begin{aligned} \frac{dH}{d\Psi} &= \frac{\partial \frac{2\Omega^2 \Psi}{W}}{\partial \Psi} \\ &= \frac{2\Omega^2}{W} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\ \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= \frac{2r^2 \Omega^2}{W} - \frac{4\Omega^2}{W^2} \Psi \end{aligned}$$

Or in a more 'standard' form

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{4\Omega^2}{W^2} \Psi = \frac{2r^2 \Omega^2}{W}$$

0.2.1 Homogeneous ODE

Considering the case where Ψ is just a function of the radius, r . So Ψ does not depend on z , and $\frac{\partial^2 \Psi}{\partial z^2} = 0$

To simplify it into a homogeneous ODE, a change of variables is used:

$$\Psi = \frac{1}{2}Wr^2 + \psi = \frac{1}{2}Wr^2 + rF$$

$$\begin{aligned}\frac{\partial \Psi}{\partial r} &= Wr + F + r \frac{\partial F}{\partial r} \\ \frac{\partial^2 \Psi}{\partial r^2} &= W + 2 \frac{\partial F}{\partial r} + r \frac{\partial^2 F}{\partial r^2}\end{aligned}$$

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \Psi \left(\frac{4\Omega^2}{W^2} - \frac{1}{r^2} \right) = 0$$

$$r^2 \frac{d^2 F}{dr^2} - r \frac{dF}{dr} + F(r^2 k^2 - 1) = 0$$

Letting $k = \frac{2\Omega}{W}$ If we take $x = kr$, $\frac{dF}{dr} = \frac{dF}{dx} \frac{dx}{dr} = k$ and $\frac{d^2 F}{dr^2} = k^2 \frac{d^2 F}{dx^2}$

$$\begin{aligned}\frac{x^2}{k^2} k^2 \frac{d^2 F}{dx^2} - \frac{x}{k} k \frac{dF}{dx} + F \left(\frac{x^2}{k^2} k^2 - 1 \right) &= 0 \\ x^2 \frac{d^2 F}{dx^2} - x \frac{dF}{dx} + F(x^2 - 1) &= 0\end{aligned}$$

Which is the form of a bessel differential equation of order $\nu = 1$, giving solutions

$$F = AJ_1(kr) + BY_1(kr)$$

Returning to the streamfunction:

$$\Psi = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

And hence

$$w = \frac{1}{r} \frac{\partial \Psi}{\partial r} = W + AkJ_0(kr) + BkY_0(kr)$$

A , and B rely on boundary conditions. In this case, it is necessary for the streamlines to be the same as at the inlet along the boundary. Also introduce a vortex breakdown condition in the core of the stream, i.e. a region $0 < r < r_*$ where the streamfunction becomes zero:

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

Consider it as a matrix system

$$\begin{pmatrix} r_* J_1(kr_*) & r_* Y_1(kr_*) \\ R J_1(kR) & R Y_1(kR) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}W r_*^2 \\ 0 \end{pmatrix}$$

Giving

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{r_* R (J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))} \begin{pmatrix} RY_1(kR) & -r_*Y_1(kr_*) \\ -RJ_1(kR) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ 0 \end{pmatrix}$$

$$A = \frac{-\frac{1}{2}RW r_*^2 Y_1(kR)}{r_* R (J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

$$B = \frac{\frac{1}{2}RW r_*^2 J_1(kR)}{r_* R (J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

And hence

$$A = \frac{-\frac{1}{2}W r_* Y_1(kR)}{(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

$$B = \frac{\frac{1}{2}W r_* J_1(kR)}{(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

With the requirement that $r_* \neq R$ so as to not divide by zero.

Using

$$w = \frac{1}{r} \frac{\partial \Psi}{\partial r}$$

Gives

$$w = W + k(AJ_0(kr) + BY_0(kr))$$

Solving this for a given k (or alternatively a desired r_*) is done numerically using **MATLAB**. The set of valid solutions to this problem are those which satisfy the constraint

$$w(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

The plot figure 0.2.1 shows the k, r_* combinations which satisfy the constraint.

Clearly this can only occur for values of $kR > 3.8$.

The first branch of this (extending from $kR \approx 3.8$) corresponds to natural solutions, whereas further branches give unwanted behaviour, which introduce reversed flow.

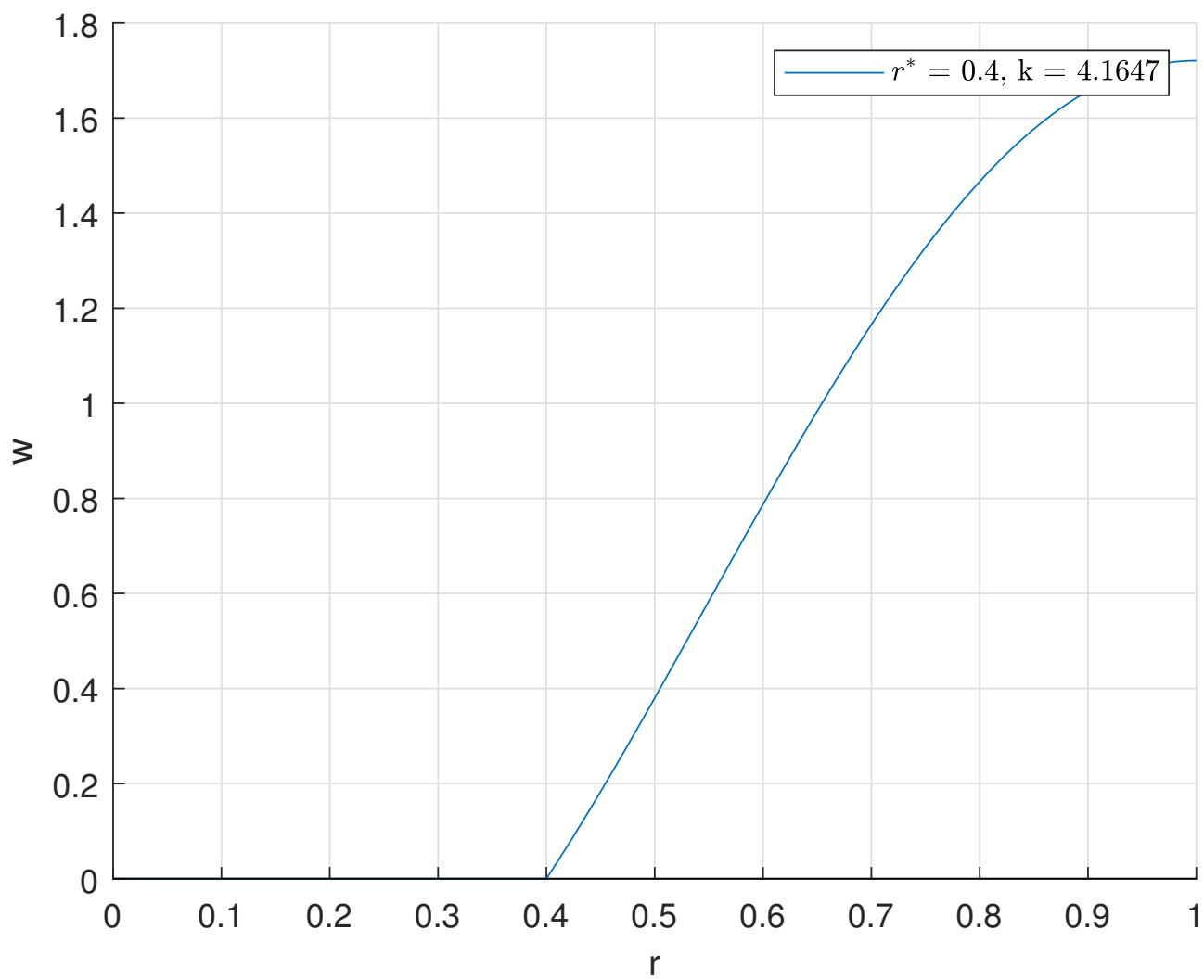


Figure 1: An example solution plot

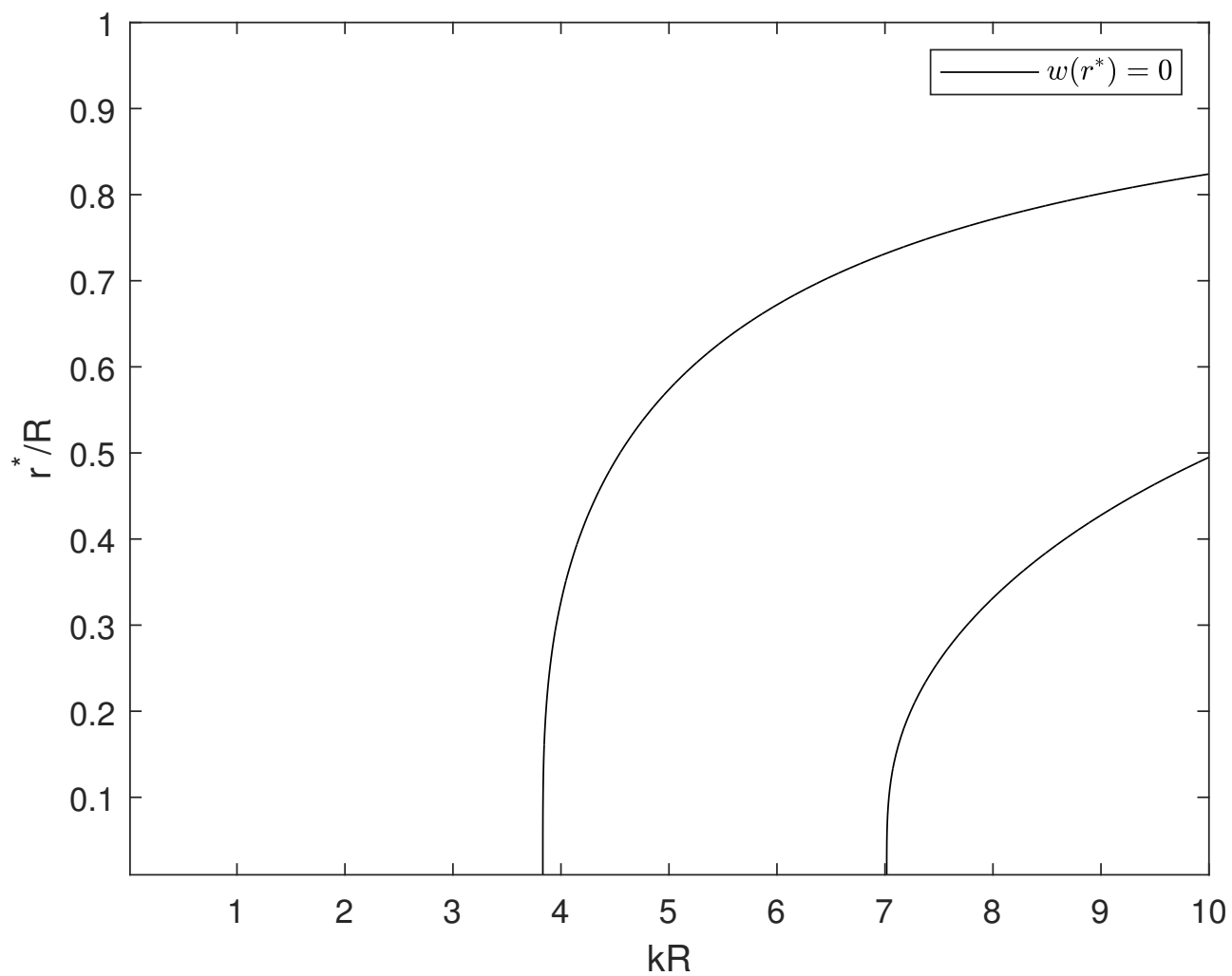


Figure 2: Solution set for the simplified problem

have to assume things for outside of the region for Ψ . I.e. if we go above the maximum input value then some assumption, and if we go below the minimum then it is a stagnation point

see if we can do it for the wall stagnation zones (i.e. ψ goes to 0 near R) so when $\Psi > \frac{1}{2}WR^2$
Plug it into H and C

$$H = (\Omega R)^2 + \frac{1}{2}W^2$$

$$\begin{aligned}\frac{\partial H}{\partial \psi} &= 0 \\ C &= \Omega R^2 \\ \frac{\partial C}{\partial \Psi} &= 0\end{aligned}$$

Which then yields the separable first order ODE

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = 0$$

And hence

$$\begin{aligned}\frac{\partial \Psi}{\partial r} &= Ar \\ \Psi &= \frac{1}{2}Ar^2 + B\end{aligned}$$

our left hand side could be written as

$$r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right)$$

using staggered grid

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{r} \frac{\partial \Psi}{\partial r}$$

at the boundary $r=0$

0.2.2 Numerics

Solving the ODE numerically:

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{\partial H}{\partial \Psi} + C \frac{\partial C}{\partial \Psi}$$

finite difference - divide r as a grid of N intervals. So our grid spaces over R ,

$$r_i = \Delta r_i, \quad \Delta = \frac{R}{N}$$

So (check this)

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial r^2} &= \frac{\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}}{\Delta^2} \\ \frac{\partial \Psi}{\partial r} &= \frac{\Psi_{i+1} - \Psi_{i-1}}{2\Delta} \\ \Psi_0 &= 0, \quad \Psi_N = \frac{1}{2}WR^2\end{aligned}$$

Which should work for the index i until we reach the bifurcations/stagnations
Should end up with a matrix equation

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ & \mathbf{A} & & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \mathbf{\Psi} \\ \Psi_N \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ \frac{1}{2}WR^2 \end{pmatrix}$$

A should be the finite difference version of

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = 0$$

I.e. for the i^{th} row of \mathbf{A}

$$A(i) = \frac{A(i+1) - 2 * A(i) + A(i-1)}{\Delta^2} - \frac{A(i+1) - A(i-1)}{2r(i)\Delta}$$

$$A_{ij} = \begin{cases} 1 & j = i = 1 \\ 1/\Delta^2 + 1/(2r_i\Delta) & j = i - 1 \\ 2/\Delta^2 & j = i \\ 1/\Delta^2 - 1/(2r_i\Delta) & j = i + 1 \\ 1 & j = i = N \\ 0 & otherwise \end{cases}$$

For the full equation

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \Psi \left(\frac{4\Omega^2}{W^2} - \frac{1}{r^2} \right) = 0$$

$$\Psi = \frac{1}{2}Wr^2 + rF$$

$$F = \frac{\Psi}{r} - \frac{1}{2}Wr$$

Boundary conditions for F relate to those for Ψ .

$$\Psi(R) = \frac{1}{2}WR^2 \implies F(R) = 0$$

$$\Psi(r_*) = 0 \implies F(r_*) = \frac{1}{2}Wr_*^2$$

when we look at the vortex breakdown problem, introduce a coordinate transformation

$$\eta = \frac{r - r_*}{R - r_*}$$

$$\eta = 0, r = r_*, \eta = 1, r = R$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial r} = \frac{1}{R - r_*} \frac{\partial \Psi}{\partial \eta}$$

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{(R - r_*)^2} \frac{\partial^2 \Psi}{\partial \eta^2}$$

use the same conditions we have used anyway where $\Psi(r_*) = w(r_*) = 0$ Rankine body problem: At some point on the radius r_0 , we get $v = K/r_0$ for some constant K find $K = \Omega r_0^2$?

0.2.3 Rankine Body

$w = W$,

$$v = \begin{cases} \frac{\Gamma}{2\pi r}, & r > r_0 \\ \Omega r, & r \leq r_0 \end{cases}$$

Where the second condition was the previous solution. Since the velocity profile is now piecewise defined, the streamfunction must also be, i.e. it is necessary to split the streamfunction into 2 regions to solve this problem. The upstream regions:

$$\begin{cases} \Psi_{inner}, & 0 \leq r \leq r_0 \\ \Psi_{outer}, & r_0 \leq r \leq R \end{cases}$$

Note that r_0 is defined upstream, so the position of the region may have moved downstream to a new radius, \hat{r} , and hence, downstream, these regions will become around \hat{r} instead of r_0 . We enforce some similar conditions as to the normal problem:

$$\begin{aligned} \Psi(r_*) &= 0, \\ \Psi(R) &= \frac{1}{2}WR^2, \\ w(r_*) &= 0 \end{aligned}$$

With the added condition that Ψ must remain continuous around \hat{r} I.e.

$$\lim_{r^- \rightarrow \hat{r}} \Psi(r^-) = \lim_{r^+ \rightarrow \hat{r}} \Psi(r^+)$$

And

$$\lim_{r^- \rightarrow \hat{r}} v(r^-) = \lim_{r^+ \rightarrow \hat{r}} v(r^+)$$

Where $\Psi(r^-)$ is Ψ defined for $r \leq \hat{r}$ and $\Psi(r^+)$ is defined in the region $r \geq \hat{r}$.

The region for $\Psi(r)$ with $r \in [0, r_0]$ will be the same as before, i.e.

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

For the region $r_0 < r < R$ the problem must be resolved from the SL equation

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

$$C = rv = \frac{\Gamma}{2\pi}$$

$$\frac{dC}{d\Psi} = 0$$

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2}\left(0 + \frac{\Gamma^2}{4\pi^2 r^2} + W^2\right) + \int \frac{C^2}{r^3} dr \\ &= \frac{1}{2}\left(\frac{\Gamma^2}{4\pi^2 r^2} + W^2\right) + \int \frac{\Gamma^2}{4\pi^2 r^3} dr \\ &= \frac{1}{2}\left(\frac{\Gamma^2}{4\pi^2 r^2} + W^2\right) - \frac{\Gamma^2}{8\pi^2 r^2} \\ &= \frac{W^2}{2} \end{aligned}$$

$$\frac{dH}{d\Psi} = 0$$

And hence the SL equation gives

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\ \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\ \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= 0\end{aligned}$$

Which results in:

$$\begin{aligned}\Psi &= Cr^2 + D, \quad r \geq \hat{r} \\ w &= \frac{1}{r} \frac{\partial \Psi}{\partial r} = 2C\end{aligned}$$

With the requirement that there is no discontinuity at \hat{r} , i.e.

$$\Psi = \frac{1}{2}W\hat{r}^2 + \hat{r} (AJ_1(k\hat{r}) + BY_1(k\hat{r})) = C\hat{r}^2 + D$$

And using the same for w

$$w(\hat{r}) = W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) = 2C$$

And lastly the wall condition

$$\Psi(R) = \frac{1}{2}WR^2 = C\hat{r}^2 + D$$

With

$$\begin{aligned}w(r_*) &= 0 \\ \frac{\Gamma}{2\pi r_0} = \Omega r_0 &\implies \Omega = \frac{\Gamma}{2\pi r_0^2} \\ k_{outer} &= \frac{2\Gamma}{2\pi W r_0^2} = \frac{\Gamma}{\pi W r_0^2}\end{aligned}$$

Noting that the values for A and B are obtained from the r_* condition.

The coefficients for Ψ have to be resolved, since the condition $\Psi_{inner}(R) = \frac{1}{2}WR^2$ cannot be imposed.

Parameters

$$r_0, \hat{r}, r_*, R, k, \Gamma, W, A, B, C, D$$

We can fix r_0 , R , k , W and Γ . This is 11 parameters, where 5 are fixed. Require 6 conditions. Impose:

- 1). $w(r_*) = 0$ (as before)
- 2). $\Psi_{inner}(r_*) = 0$ (as before)
- 3). Since at the wall Ψ must remain the same, this applies to where v is changed, i.e.
 $\Psi_{inner}(\hat{r}) = \frac{1}{2}W r_0^2$
- 4). For continuity, $\Psi_{outer}(\hat{r}) = \frac{1}{2}W r_0^2$

5). $w_{outer}(\hat{r}) = w_{inner}(\hat{r})$

6). $\Psi_{outer}(R) = \frac{1}{2}WR^2$

Redo the problem instead getting A, B from 2) and 3)

$$\begin{aligned}\Psi_{inner}(r_*) &= 0 \\ \Psi_{inner}(\hat{r}) &= \frac{1}{2}Wr_0^2\end{aligned}$$

Use this for A, B

$$\begin{aligned}\Psi_{inner}(r_*) &= \frac{1}{2}Wr_*^2 + r_*(AJ_1(kr_*) + BY_1(kr_*)) = 0 \\ &= r_*(AJ_1(kr_*) + BY_1(kr_*)) = -\frac{1}{2}Wr_*^2 \\ \Psi_{inner}(\hat{r}) &= \frac{1}{2}W\hat{r}^2 + \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}Wr_0^2 \\ &= \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}W(r_0^2 - \hat{r}^2)\end{aligned}$$

This gives the matrix system for A, B below. Note that the system relies on the unknowns r_* and \hat{r} .

$$\begin{aligned}\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ \hat{r}J_1(k\hat{r}) & \hat{r}Y_1(k\hat{r}) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{\det} \begin{pmatrix} \hat{r}Y_1(k\hat{r}) & -r_*Y_1(kr_*) \\ -\hat{r}J_1(k\hat{r}) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix} \\ A &= \frac{1}{\det} \left(\hat{r}Y_1(k\hat{r}) \left(-\frac{1}{2}Wr_*^2 \right) - r_*Y_1(kr_*) \left(\frac{1}{2}W(r_0^2 - \hat{r}^2) \right) \right) \\ B &= \frac{1}{\det} \left(-\hat{r}J_1(k\hat{r}) \left(-\frac{1}{2}Wr_*^2 \right) + r_*J_1(kr_*) \left(\frac{1}{2}W(r_0^2 - \hat{r}^2) \right) \right)\end{aligned}$$

Where

$$\begin{aligned}\det &= \hat{r}r_*Y_1(k\hat{r})J_1(kr_*) - \hat{r}r_*J_1(k\hat{r})Y_1(kr_*) \\ &= \hat{r}r_*(Y_1(k\hat{r})J_1(kr_*) - J_1(k\hat{r})Y_1(kr_*))\end{aligned}$$

This for r_*

$$w_{inner}(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

Get C from:

$$\begin{aligned}w_{outer}(\hat{r}) &= w_{inner}(\hat{r}) \\ 2C &= W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) \\ C &= \frac{1}{2}(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})))\end{aligned}$$

Get D here:

$$\begin{aligned}\Psi_{outer}(R) &= CR^2 + D = \frac{1}{2}WR^2 \\ D &= \frac{1}{2}WR^2 - C\end{aligned}$$

Hence get \hat{r} from

$$\begin{aligned}\Psi_{outer}(\hat{r}) &= C\hat{r}^2 + D = \frac{1}{2}Wr_0^2 \\ C\hat{r}^2 + \frac{1}{2}WR^2 - C &= \frac{1}{2}Wr_0^2 \\ \left(\frac{1}{2}(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})))\right)(\hat{r}^2 - 1) &= \frac{1}{2}W(r_0^2 - R^2) \\ (AJ_0(k\hat{r}) + BY_0(k\hat{r}))(\hat{r}^2 - 1) &= \frac{1}{k}W(r_0^2 - R^2 - 1)\end{aligned}$$

For physically valid solutions, we must impose the condition of no net change on the momentum from upstream to downstream on the momentum (Escudier, Keller). The momentum is defined as

$$s = 2\pi \int_0^{r_t} (\rho w^2 + p) r dr$$

Which comes to:

$$\Delta s = \frac{\pi}{4} \rho U^2 k^2 r_c^2 \left[-r_b^2 + \frac{1}{4} \left(\frac{r_b^4 - r_a^4}{r_c^2} \right) + \frac{3}{4} r_c^2 + \frac{1}{2} r_c^2 \log \left(\frac{r_b^2}{r_c^2} \right) \right] = 0$$

0.3 Burger's Vortex

Q-vortex without a Jet. Start with

$$\begin{aligned}w &= W \\ v &= \frac{\Gamma}{2\pi r} \left(1 - e^{-r^2/\delta^2} \right) \\ \frac{d^2\Psi}{dr^2} - \frac{1}{r} \frac{d\Psi}{dr} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}\end{aligned}$$

Solve from r_* to R numerically.

Generate grid from r_* to R .

Boundary conditions as normal

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

$$w(r_*) = 0$$

Non-dimensional parameter may be something like $\frac{\Gamma}{WR}$ (we can probs relate this to kr_0 for the rankine problem)

Eventually do the same thing as before with s and Δs .

$$s = \int_0^R (\rho w^2 + p) r dr = \int_0^{r_*} p(r_*) r dr + \int_{r_*}^R (\rho w^2 + p) r dr$$

Use

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{v^2}{r} = \frac{\Gamma^2}{4\pi^2 r^3} \left(1 - e^{-r^2/\delta^2}\right)^2$$

$$\Psi = \frac{1}{2} W r^2 \implies r = \sqrt{\frac{2\Psi}{W}}$$

$$C = rv = \frac{\Gamma}{2\pi} \left(1 - e^{-r^2/\delta^2}\right)$$

$$\frac{\partial C}{\partial \Psi} = \frac{\Gamma}{2\pi} \frac{\partial}{\partial \Psi} \left(1 - e^{-r^2/\delta^2}\right)$$

$$\frac{\partial C}{\partial \Psi} = \frac{-\Gamma}{2\pi} \frac{\partial}{\partial \Psi} \left(e^{-r^2/\delta^2}\right)$$

$$\frac{\partial C}{\partial \Psi} = \frac{-\Gamma}{2\pi} \frac{\partial}{\partial \Psi} \left(e^{-2\Psi/W\delta^2}\right)$$

$$\frac{\partial C}{\partial \Psi} = \frac{\Gamma}{W\delta^2\pi} \left(e^{-2\Psi/W\delta^2}\right)$$

$$\begin{aligned} H &= \frac{1}{2} (u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2} (u^2 + v^2 + w^2) + \int \frac{1}{\rho} \frac{\partial p}{\partial r} dr \\ &= \frac{1}{2} \left(\left(\frac{\Gamma}{2\pi r} \left(1 - e^{-r^2/\delta^2}\right) \right)^2 + W^2 \right) + \int \frac{v^2}{r} dr \\ &= \frac{1}{2} \left(\left(\frac{\Gamma}{2\pi r} \left(1 - e^{-r^2/\delta^2}\right) \right)^2 + W^2 \right) + \int \frac{\frac{\Gamma^2}{4\pi^2 r^2} \left(1 - e^{-r^2/\delta^2}\right)^2}{r} dr \\ &= \frac{1}{2} W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{1}{r^2} \left(1 - e^{-r^2/\delta^2}\right)^2 + \int \frac{\left(1 - e^{-r^2/\delta^2}\right)^2}{r^3} dr \right) \\ &= \frac{1}{2} W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{1}{r^2} \left(1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}\right) + \int \frac{1}{r^3} dr + \int \frac{e^{-2r^2/\delta^2}}{r^3} dr - \int \frac{2e^{-r^2/\delta^2}}{r^3} dr \right) \\ &= \frac{1}{2} W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{1}{r^2} \left(1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}\right) - \frac{1}{2r^2} + \int \frac{e^{-2r^2/\delta^2}}{r^3} dr - 2 \int \frac{e^{-r^2/\delta^2}}{r^3} dr \right) \end{aligned}$$

Focus on the two integrals:

$$\begin{aligned}
1 : & \int \frac{e^{-2r^2/\delta^2}}{r^3} dr \\
& \text{integrate by parts} \quad \left(u = e^{-2r^2/\delta^2}, \quad v' = \frac{1}{r^3} \right) \\
& = -\frac{e^{-2r^2/\delta^2}}{2r^2} - \int \frac{2e^{-2r^2/\delta^2}}{\delta^2 r} dr \quad \left(\text{let } u = \frac{2r^2}{\delta^2} \implies dr = \frac{\delta^2}{4r} du \right) \\
& = -\frac{e^{-2r^2/\delta^2}}{2r^2} - \frac{1}{\delta^2} \int \frac{e^{-u}}{u} du \\
& = -\frac{e^{-2r^2/\delta^2}}{2r^2} - \frac{-E_1(u)}{\delta^2} \\
& = \frac{E_1\left(\frac{2r^2}{\delta^2}\right)}{\delta^2} - \frac{e^{-2r^2/\delta^2}}{2r^2} \\
& = -E_i\left(-2r^2/\delta^2\right)/\delta^2 - \frac{e^{-2r^2/\delta^2}}{2r^2}
\end{aligned}$$

Same process for the second:

$$\begin{aligned}
2 : & \int \frac{2e^{-r^2/\delta^2}}{r^3} dr \\
& = -\frac{e^{-r^2/\delta^2}}{r^2} - 2 \int \frac{e^{-r^2/\delta^2}}{\delta^2 r} dr \\
& = -\frac{e^{-r^2/\delta^2}}{r^2} - \frac{1}{\delta^2} \int \frac{-e^{-u}}{u} dr, \quad u = \frac{r^2}{\delta^2}, \quad dr = \frac{\delta^2}{2r} du \\
& = -\frac{e^{-r^2/\delta^2}}{r^2} - \frac{-E_1(r^2/\delta^2)}{\delta^2} \\
& = \frac{E_1(r^2/\delta^2)}{\delta^2} - \frac{e^{-r^2/\delta^2}}{r^2} \\
& = -E_i\left(-r^2/\delta^2\right)/\delta^2 - \frac{e^{-r^2/\delta^2}}{r^2}
\end{aligned}$$

Giving

$$\begin{aligned}
H &= \frac{1}{2}W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{1}{r^2} \left(1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2} \right) - \frac{1}{2r^2} + \int \frac{e^{-2r^2/\delta^2}}{r^3} dr - \int \frac{2e^{-r^2/\delta^2}}{r^3} dr \right) \\
&= \frac{1}{2}W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{1}{r^2} \left(1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2} \right) - \frac{1}{2r^2} + \frac{E_1\left(\frac{2r^2}{\delta^2}\right)}{\delta^2} - \frac{e^{-2r^2/\delta^2}}{2r^2} - \frac{E_1(r^2/\delta^2)}{\delta^2} + \frac{e^{-r^2/\delta^2}}{r^2} \right) \\
&= \frac{1}{2}W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{1}{2r^2} + \frac{E_1\left(\frac{2r^2}{\delta^2}\right)}{\delta^2} + \frac{e^{-2r^2/\delta^2}}{2r^2} - \frac{E_1(r^2/\delta^2)}{\delta^2} - \frac{e^{-r^2/\delta^2}}{r^2} \right)
\end{aligned}$$

Or in terms of Ψ

$$= \frac{1}{2}W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{W}{4\Psi} + \frac{E_1(4\Psi/W\delta^2)}{\delta^2} + \frac{We^{-4\Psi/W\delta^2}}{4\Psi} - \frac{E_1(2\Psi/W\delta^2)}{\delta^2} - \frac{We^{-2\Psi/W\delta^2}}{2\Psi} \right)$$

Hence using $r^2 = \frac{2\Psi}{W}$

$$\begin{aligned}
\frac{\partial H}{\partial \Psi} &= \frac{\partial}{\partial \Psi} \left(\frac{1}{2}W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{W}{4\Psi} + \frac{E_1(4\Psi/W\delta^2)}{\delta^2} + \frac{We^{-4\Psi/W\delta^2}}{4\Psi} - \frac{E_1(2\Psi/W\delta^2)}{\delta^2} - \frac{We^{-2\Psi/W\delta^2}}{2\Psi} \right) \right) \\
&= \frac{\Gamma^2}{4\pi^2} \frac{\partial}{\partial \Psi} \left(\frac{W}{4\Psi} + \frac{E_1(4\Psi/W\delta^2)}{\delta^2} + \frac{We^{-4\Psi/W\delta^2}}{4\Psi} - \frac{E_1(2\Psi/W\delta^2)}{\delta^2} - \frac{We^{-2\Psi/W\delta^2}}{2\Psi} \right) \\
&= \frac{\Gamma^2}{4\pi^2} \left(-\frac{W}{4\Psi^2} + \frac{e^{4\Psi/W\delta^2}}{\Psi\delta^2} - \frac{e^{-4\Psi/W\delta^2}}{\Psi\delta^2} - \frac{e^{2\Psi/W\delta^2}}{\Psi\delta^2} - \frac{We^{-2\Psi/W\delta^2}}{2\Psi} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial \Psi} &= \frac{\partial}{\partial \Psi} \left(\frac{1}{2}W^2 + \frac{\Gamma^2}{4\pi^2} \left(\frac{W}{4\Psi} - \frac{E_i(-4\Psi/W\delta^2)}{\delta^2} + \frac{We^{-4\Psi/W\delta^2}}{4\Psi} + \frac{E_i(-2\Psi/W\delta^2)}{\delta^2} - \frac{We^{-2\Psi/W\delta^2}}{2\Psi} \right) \right) \\
&= \frac{\Gamma^2}{4\pi^2} \frac{\partial}{\partial \Psi} \left(\frac{W}{4\Psi} - \frac{E_i(-4\Psi/W\delta^2)}{\delta^2} + \frac{We^{-4\Psi/W\delta^2}}{4\Psi} + \frac{E_i(-2\Psi/W\delta^2)}{\delta^2} - \frac{We^{-2\Psi/W\delta^2}}{2\Psi} \right) \\
&= \frac{\Gamma^2}{4\pi^2} \left(-\frac{W}{4\Psi^2} - \frac{e^{-4\Psi/W\delta^2}}{\delta^2\Psi} + \frac{-4e^{-4\Psi/W\delta^2}}{4\delta^2\Psi} + \frac{e^{-2\Psi/W\delta^2}}{\delta^2\Psi} - \frac{-2We^{-2\Psi/W\delta^2}}{2\delta^2\Psi} \right)
\end{aligned}$$

Not sure where I have gone wrong but I should get:

$$H = \frac{1}{2}W^2 + \frac{\Gamma^2}{4\delta^2\pi^2} \left(E_i \left(\frac{4\psi}{\delta^2 W} \right) - E_i \left(\frac{2\psi}{\delta^2 W} \right) \right)$$

$$\frac{dH}{d\Psi} = \frac{\Gamma^2}{4\delta^2\Psi\pi^2} \left(e^{-\frac{4\Psi}{\delta^2 W}} - e^{-\frac{2\Psi}{\delta^2 W}} \right)$$

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = \frac{\Gamma^2}{2\delta^2 W \pi^2} \left(e^{-\frac{r^2}{\delta^2}} \left(e^{-\frac{r^2}{\delta^2}} - 1 \right) + e^{-\frac{2r^2}{\delta^2}} \left(e^{\frac{r^2}{\delta^2}} - 1 \right) \right)$$

`dsolve()` gives

$$\Psi(r) = ar^2 + b$$

When we do finite differences, grab a computational variable (call it η for now)

$$\eta = \frac{r - r_*}{R - r_*}$$

So now we are computing in $0 < \eta < 1$. So try plugging it into the ODE:

$$d\eta = \frac{dr}{R - r_*}$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial r} = \frac{1}{R - r_*} \frac{\partial \Psi}{\partial \eta}$$

If we don't know r_* the eta vector becomes

$$\begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{N-1} \\ r_* \end{pmatrix}$$

This will be a non-linear problem so we will need to solve using fsolve.

To get guesses could use rankine vortex stuff -

For a linear flow we could use We know that $\psi(r_*) = 0$ and $\psi(R) = \frac{1}{2}WR^2$. We could guess that ψ is constant, $\psi = Ar^2 + B$.

Try plotting $w(r_*)$ for various r_* to help find guesses (require it to be 0)

The equation for s comes from the Euler equations (Don't need to use this / know this - we did it with tensors) gives

$$\frac{\partial u_i u_j}{\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

$$\int_v \frac{\partial u_i u_j}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} dv$$

Use divergence theorem

$$\int \nabla \cdot v dv = \int v \cdot \hat{n} ds$$

$$\int_v \frac{\partial u_i u_j}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} dv = \int_s \frac{\partial u_i u_j n_i}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} ds$$

$$v_i = p \delta_{ij}, \quad p \delta_{ij} n_i = p n_j$$

$$= \int_s u_i u_j n_i + \frac{p}{\rho} n_j ds$$

Since we are considering a cylinder, we get 0 along the boundary of the cylinder, and so we only care about the inlet and outlet. At the inlet the normal faces in the negative w

$$u_i n_i = -w$$

$$u_i n_i = w$$

$$\int_{(1)} -w^2 - \frac{p}{\rho} dS + \int_{(2)} \int w^2 + \frac{p}{\rho} dS = 0$$

Which is literally in flux = out flux.

0.3.1 Outer vortex breakdown

Considering the initial problem for vortex breakdown, except perhaps the breakdown is a pocket expanding from R rather than 0. I.e. the breakdown occurs about the wall rather than the center. So assuming r^\dagger is our outer vortex breakdown radius

This simply means obtaining a new A , B and k .

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

$$w(r) = W + k(AJ_0(kr) + BY_0(kr))$$

Such that

$$w(r^\dagger) = 0, \quad \Psi(0) = 0, \quad \text{and} \quad \Psi(r^\dagger) = 0$$

To enforce $\Psi(0) = 0$ note that $\lim_{r \rightarrow 0} \frac{Y_1(kr)}{r} = -\infty$. Hence it is necessary to set $B = 0$.

$$\Psi(r) = \frac{1}{2}Wr^2 + rAJ_1(kr), \quad w(r) = W + kAJ_0(kr)$$

And to enforce $\Psi(r^\dagger) = 0$

$$\implies Ar^\dagger J_1(kr^\dagger) = -\frac{1}{2}Wr^{\dagger 2}$$

$$A = \frac{-Wr^\dagger}{2J_1(kr^\dagger)}$$

And obtain k using

$$w(r^\dagger) = 0$$

$$kAJ_0(kr) = -W$$

$$\Psi(r^\dagger) = \Psi(R) = \frac{1}{2}WR^2$$

0.4 Appendix

0.4.1 Supplementary Materials

This is where all the basic fluid mechanics knowledge should be (definitions, etc.)

0.4.2 Resources

Books: An Introduction to Fluid Dynamics Batchelor

Swirling flow states in finite-length diverging or contracting circular pipes Zvi Rusak

Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe Zvi Rusak, and Shixiao Wang