

# Fluid Thesis

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## **Abstract**

Just so I don't forget that there is an abstract environment...

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## 0.1 Introduction

## 0.2 Derivation of the Squire-Long equation

Squire-long / Bragg-Hawthorne equation for the stream function of axisymmetric inviscid fluid, using cylindrical coordinates

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

radial component u, azimuthal (swirl) is v, axial component w  
stream function satisfies

$$\nabla \cdot u = 0 \longrightarrow \text{streamfunction exists}$$

Remember for cylindrical coordinates:

$$u = \frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad w = -\frac{1}{r} \frac{\partial \Psi}{\partial r}$$

$\Psi$  is the stream function

$r$  is the radius

$$C = rv$$

$$H = \frac{p}{\rho} + \frac{1}{2}(u^2 + v^2 + w^2)$$

$H$  is conserved on stream surfaces

$C$  is conserved on stream surfaces

vorticity

$$w = w_r e_r + w_\theta e_\theta + w_z e_z$$

where  $w_r, w_\theta, w_z$  can be written in terms of the velocity

Considering cylindrical coordinates  $(z, r, \theta)$  with corresponding velocity  $(u, v, w)$ , vorticity components  $(\omega_z, \omega_r, \omega_\theta)$ . Axisymmetric flow as:

$$\omega_z = \frac{1}{r} \frac{\partial r v}{\partial r}, \quad \omega_r = -\frac{\partial r v}{\partial z}, \quad \omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

The continuity equation (conservation of mass) is satisfied by setting

$$w = \frac{1}{r} \frac{\partial \Psi}{\partial r}, \quad u = -\frac{1}{r} \frac{\partial \Psi}{\partial z}$$

Where  $\Psi$  is the stream function This gives the azimuthal component for  $w_\theta$ :

$$\begin{aligned} \omega_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ &= -\frac{1}{r} \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r} \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \\ &= -\frac{1}{r} \left( \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \end{aligned}$$

Use the vorticity equation

$$w \times v - \frac{\partial w}{\partial t} = \nabla H$$

Where

$$H = \frac{1}{2}(w^2 + u^2 + v^2) + \frac{p}{\rho}$$

This gives:

$$\begin{aligned} u\omega_\theta - v\omega_r - \frac{\partial w}{\partial t} &= \frac{\partial H}{\partial x} \\ v\omega_z - w\omega_\theta - \frac{\partial u}{\partial t} &= \frac{\partial H}{\partial r} \\ w\omega_r - u\omega_z - \frac{\partial v}{\partial t} &= 0 \end{aligned}$$

The last one is equivalent to the material derivative of  $rw$  set to 0:

$$\frac{D(rv)}{Dt} = 0$$

From the Bernoulli equation:

$$\begin{aligned} rv &= C(\Psi) \\ \frac{\partial \Psi}{\partial t} + \frac{1}{2}|\mathbf{w}|^2 + \frac{p}{\rho} &= H(\Psi) \end{aligned}$$

Where  $H(\Psi)$  and  $C(\Psi)$  are arbitrary functions.

Rewriting  $\omega$ :

$$\omega_z = w \frac{dC}{d\Psi}, \quad \omega_r = u \frac{dC}{d\Psi}$$

Giving

$$\frac{\omega_\theta}{r} = \frac{v\omega_r}{ru} + \frac{1}{ru} \frac{dH}{d\Psi} \frac{\partial \Psi}{\partial z} = \frac{C}{r^2} \frac{dC}{d\Psi} - \frac{dH}{d\Psi}$$

Which is the form taken by the second of the dynamic equations. Now, combining this last statement with the equation for  $\omega_\theta$ :

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

Taken from Batchelor's An Introduction to Fluid Dynamics

Considering the flow far upstream where there is constant uniform axial velocity and rotates with angular velocity  $\Omega$

$$\Psi_{\text{upstream}} = \frac{1}{2}Wr^2$$

$$v = \Omega r, w = W$$

And

$$C = rv = \frac{v^2}{\Omega} = \Omega r^2 = 2\Omega\Psi/W$$

$$\frac{dC}{d\Psi} = 2\Omega/W$$

Since the flow is steady, the radial equation of motion yields:

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{w^2}{r} = \frac{C^2}{r^3}$$

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2}(\Omega^2 r^2 + W^2) + \frac{p}{\rho} \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \frac{p}{\rho} \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \frac{1}{\rho} \frac{dp}{dr} dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \frac{C^2}{r^3} dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \frac{\Omega^2 r^4}{r^3} dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \int \Omega^2 r dr \\ &= \frac{\Omega^2 \Psi}{W} + \frac{1}{2}W^2 + \frac{1}{2}\Omega^2 r^2 \\ &= \frac{2\Omega^2 \Psi}{W} + \frac{1}{2}W^2 \end{aligned}$$

$$\begin{aligned} \frac{dH}{d\Psi} &= \frac{\partial \frac{2\Omega^2 \Psi}{W}}{\partial \Psi} \\ &= \frac{2\Omega^2}{W} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\ \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= \frac{2r^2 \Omega^2}{W} - \frac{4\Omega^2}{W^2} \Psi \end{aligned}$$

Or in a more 'standard' form

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{4\Omega^2}{W^2} \Psi = \frac{2r^2 \Omega^2}{W}$$

### 0.2.1 Homogeneous ODE

Considering the case where  $\Psi$  is just a function of the radius,  $r$ . So  $\Psi$  does not depend on  $z$ , and  $\frac{\partial^2 \Psi}{\partial z^2} = 0$

To simplify it into a homogeneous ODE, a change of variables is used:

$$\Psi = \frac{1}{2}Wr^2 + \psi = \frac{1}{2}Wr^2 + rF$$

$$\begin{aligned}\frac{\partial \Psi}{\partial r} &= Wr + F + r \frac{\partial F}{\partial r} \\ \frac{\partial^2 \Psi}{\partial r^2} &= W + 2 \frac{\partial F}{\partial r} + r \frac{\partial^2 F}{\partial r^2}\end{aligned}$$

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \Psi \left( \frac{4\Omega^2}{W^2} - \frac{1}{r^2} \right) = 0$$

$$r^2 \frac{d^2 F}{dr^2} - r \frac{dF}{dr} + F(r^2 k^2 - 1) = 0$$

Letting  $k = \frac{2\Omega}{W}$  If we take  $x = kr$ ,  $\frac{dF}{dr} = \frac{dF}{dx} \frac{dx}{dr} = k$  and  $\frac{d^2 F}{dr^2} = k^2 \frac{d^2 F}{dx^2}$

$$\begin{aligned}\frac{x^2}{k^2} k^2 \frac{d^2 F}{dx^2} - \frac{x}{k} k \frac{dF}{dx} + F \left( \frac{x^2}{k^2} k^2 - 1 \right) &= 0 \\ x^2 \frac{d^2 F}{dx^2} - x \frac{dF}{dx} + F(x^2 - 1) &= 0\end{aligned}$$

Which is the form of a bessel differential equation of order  $\nu = 1$ , giving solutions

$$F = AJ_1(kr) + BY_1(kr)$$

Returning to the streamfunction:

$$\Psi = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

And hence

$$w = \frac{1}{r} \frac{\partial \Psi}{\partial r} = W + AkJ_0(kr) + BkY_0(kr)$$

$A$ , and  $B$  rely on boundary conditions. In this case, it is necessary for the streamlines to be the same as at the inlet along the boundary. Also introduce a vortex breakdown condition in the core of the stream, i.e. a region  $0 < r < r_*$  where the streamfunction becomes zero:

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

Consider it as a matrix system

$$\begin{pmatrix} r_* J_1(kr_*) & r_* Y_1(kr_*) \\ R J_1(kR) & R Y_1(kR) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}WR_*^2 \\ 0 \end{pmatrix}$$

Giving

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{r_* R (J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))} \begin{pmatrix} RY_1(kR) & -r_*Y_1(kr_*) \\ -RJ_1(kR) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ 0 \end{pmatrix}$$

$$A = \frac{-\frac{1}{2}RW r_*^2 Y_1(kR)}{r_* R (J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

$$B = \frac{\frac{1}{2}RW r_*^2 J_1(kR)}{r_* R (J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

And hence

$$A = \frac{-\frac{1}{2}W r_* Y_1(kR)}{(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

$$B = \frac{\frac{1}{2}W r_* J_1(kR)}{(J_1(kr_*)Y_1(kR) - Y_1(kr_*)J_1(kR))}$$

With the requirement that  $r_* \neq R$  so as to not divide by zero.

Using

$$w = \frac{1}{r} \frac{\partial \Psi}{\partial r}$$

Gives

$$w = W + k(AJ_0(kr) + BY_0(kr))$$

Solving this for a given  $k$  (or alternatively a desired  $r_*$ ) is done numerically using **MATLAB**. The set of valid solutions to this problem are those which satisfy the constraint

$$w(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

The plot figure 0.2.1 shows the  $k, r_*$  combinations which satisfy the constraint.

Clearly this can only occur for values of  $kR > 3.8$ .

The first branch of this (extending from  $kR \approx 3.8$ ) corresponds to natural solutions, whereas further branches give unwanted behaviour, which introduce reversed flow.

Code - homogeneousODE.m

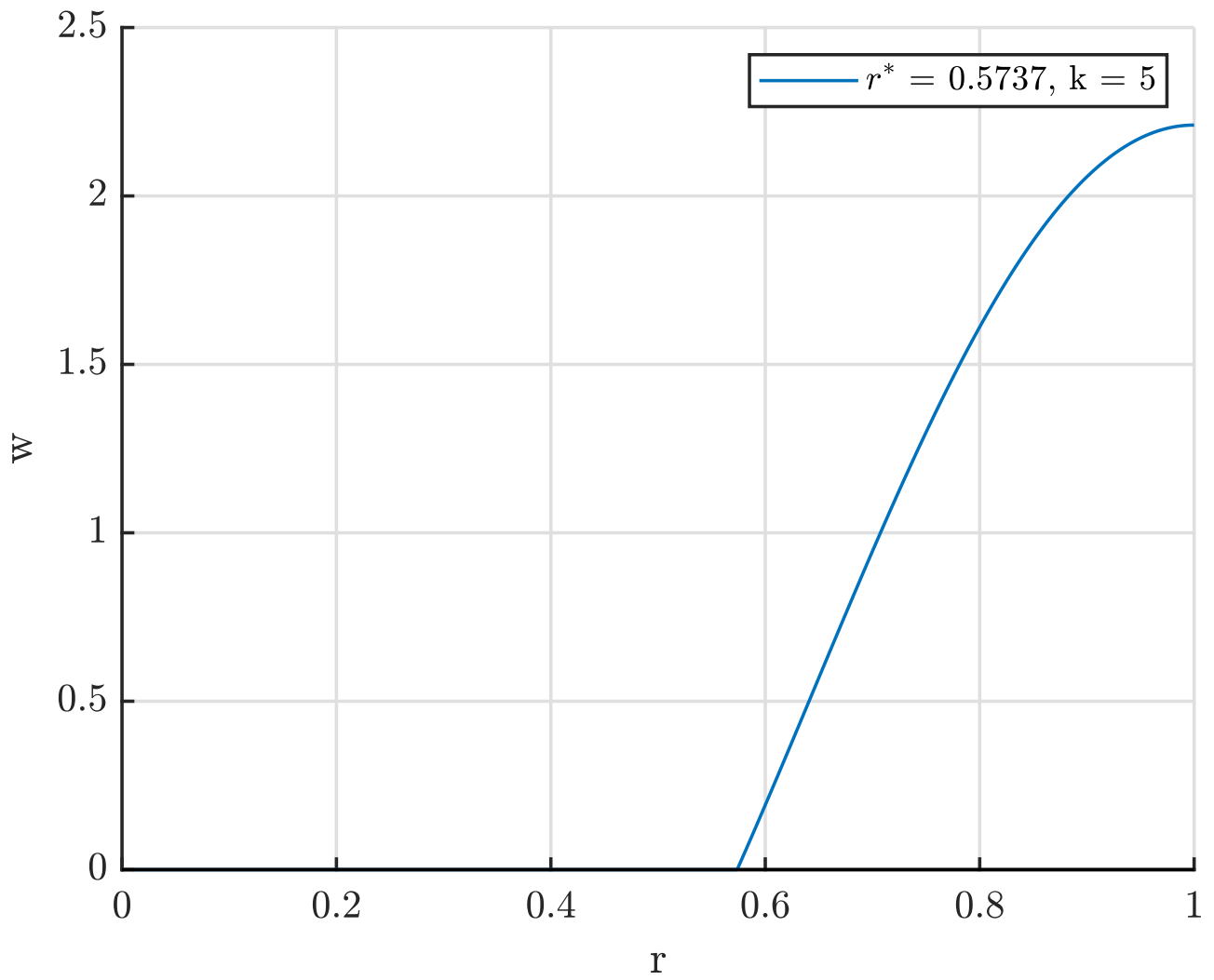


Figure 1: An example solution plot



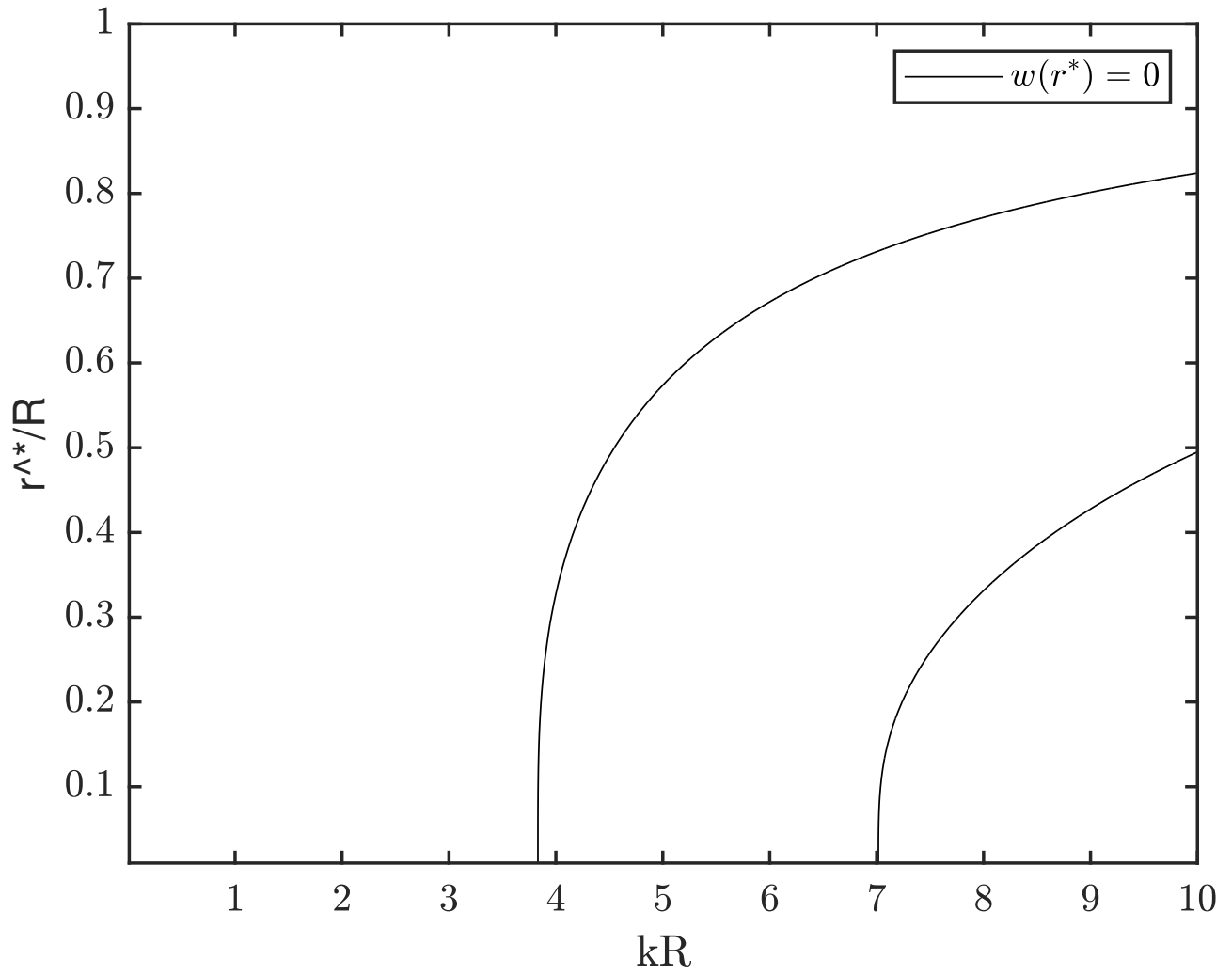


Figure 2: Solution set for the simplified problem

have to assume things for outside of the region for  $\Psi$ . I.e. if we go above the maximum input value then some assumption, and if we go below the minimum then it is a stagnation point

see if we can do it for the wall stagnation zones (i.e.  $\psi$  goes to 0 near  $R$ ) so when  $\Psi > \frac{1}{2}WR^2$   
Plug it into  $H$  and  $C$

$$H = (\Omega R)^2 + \frac{1}{2}W^2$$

$$\begin{aligned}\frac{\partial H}{\partial \psi} &= 0 \\ C &= \Omega R^2 \\ \frac{\partial C}{\partial \Psi} &= 0\end{aligned}$$

Which then yields the separable first order ODE

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = 0$$

And hence

$$\begin{aligned}\frac{\partial \Psi}{\partial r} &= Ar \\ \Psi &= \frac{1}{2}Ar^2 + B\end{aligned}$$

our left hand side could be written as

$$r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right)$$

using staggered grid

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{r} \frac{\partial \Psi}{\partial r}$$

at the boundary  $r=0$

## 0.2.2 Numerics

Solving the ODE numerically:

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{\partial H}{\partial \Psi} + C \frac{\partial C}{\partial \Psi}$$

finite difference - divide  $r$  as a grid of  $N$  intervals. So our grid spaces over  $R$ ,

$$r_i = \Delta r_i, \quad \Delta = \frac{R}{N}$$

So (check this)

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial r^2} &= \frac{\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}}{\Delta^2} \\ \frac{\partial \Psi}{\partial r} &= \frac{\Psi_{i+1} - \Psi_{i-1}}{2\Delta} \\ \Psi_0 &= 0, \quad \Psi_N = \frac{1}{2}WR^2\end{aligned}$$

Which should work for the index  $i$  until we reach the bifurcations/stagnations  
Should end up with a matrix equation

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ & \mathbf{A} & & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \mathbf{\Psi} \\ \Psi_N \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ \frac{1}{2}WR^2 \end{pmatrix}$$

A should be the finite difference version of

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = 0$$

I.e. for the  $i^{th}$  row of  $\mathbf{A}$

$$A(i) = \frac{A(i+1) - 2 * A(i) + A(i-1)}{\Delta^2} - \frac{A(i+1) - A(i-1)}{2r(i)\Delta}$$

$$A_{ij} = \begin{cases} 1 & j = i = 1 \\ 1/\Delta^2 + 1/(2r_i\Delta) & j = i - 1 \\ 2/\Delta^2 & j = i \\ 1/\Delta^2 - 1/(2r_i\Delta) & j = i + 1 \\ 1 & j = i = N \\ 0 & otherwise \end{cases}$$

For the full equation

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \Psi \left( \frac{4\Omega^2}{W^2} - \frac{1}{r^2} \right) = 0$$

$$\Psi = \frac{1}{2}Wr^2 + rF$$

$$F = \frac{\Psi}{r} - \frac{1}{2}Wr$$

Boundary conditions for  $F$  relate to those for  $\Psi$ .

$$\Psi(R) = \frac{1}{2}WR^2 \implies F(R) = 0$$

$$\Psi(r_*) = 0 \implies F(r_*) = \frac{1}{2}Wr_*^2$$

when we look at the vortex breakdown problem, introduce a coordinate transformation

$$\eta = \frac{r - r_*}{R - r_*}$$

$$\eta = 0, r = r_*, \eta = 1, r = R$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial r} = \frac{1}{R - r_*} \frac{\partial \Psi}{\partial \eta}$$

$$\frac{\partial^2 \Psi}{\partial r^2} = \frac{1}{(R - r_*)^2} \frac{\partial^2 \Psi}{\partial \eta^2}$$

use the same conditions we have used anyway where  $\Psi(r_*) = w(r_*) = 0$  Rankine body problem: At some point on the radius  $r_0$ , we get  $v = K/r_0$  for some constant  $K$  find  $K = \Omega r_0^2$ ?

### 0.2.3 Rankine Body

$w = W$ ,

$$v = \begin{cases} \frac{\Gamma}{2\pi r}, & r > r_0 \\ \Omega r, & r \leq r_0 \end{cases}$$

Where the second condition was the previous solution. Since the velocity profile is now piecewise defined, the streamfunction must also be, i.e. it is necessary to split the streamfunction into 2 regions to solve this problem. The upstream regions:

$$\begin{cases} \Psi_{inner}, & 0 \leq r \leq r_0 \\ \Psi_{outer}, & r_0 \leq r \leq R \end{cases}$$

Note that  $r_0$  is defined upstream, so the position of the region may have moved downstream to a new radius,  $\hat{r}$ , and hence, downstream, these regions will become around  $\hat{r}$  instead of  $r_0$ . We enforce some similar conditions as to the normal problem:

$$\begin{aligned} \Psi(r_*) &= 0, \\ \Psi(R) &= \frac{1}{2}WR^2, \\ w(r_*) &= 0 \end{aligned}$$

With the added condition that  $\Psi$  must remain continuous around  $\hat{r}$  I.e.

$$\lim_{r^- \rightarrow \hat{r}} \Psi(r^-) = \lim_{r^+ \rightarrow \hat{r}} \Psi(r^+)$$

And

$$\lim_{r^- \rightarrow \hat{r}} v(r^-) = \lim_{r^+ \rightarrow \hat{r}} v(r^+)$$

Where  $\Psi(r^-)$  is  $\Psi$  defined for  $r \leq \hat{r}$  and  $\Psi(r^+)$  is defined in the region  $r \geq \hat{r}$ .

The region for  $\Psi(r)$  with  $r \in [0, r_0]$  will be the same as before, i.e.

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

For the region  $r_0 < r < R$  the problem must be resolved from the SL equation

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}$$

$$C = rv = \frac{\Gamma}{2\pi}$$

$$\frac{dC}{d\Psi} = 0$$

$$\begin{aligned} H &= \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} \\ &= \frac{1}{2}\left(0 + \frac{\Gamma^2}{4\pi^2 r^2} + W^2\right) + \int \frac{C^2}{r^3} dr \\ &= \frac{1}{2}\left(\frac{\Gamma^2}{4\pi^2 r^2} + W^2\right) + \int \frac{\Gamma^2}{4\pi^2 r^3} dr \\ &= \frac{1}{2}\left(\frac{\Gamma^2}{4\pi^2 r^2} + W^2\right) - \frac{\Gamma^2}{8\pi^2 r^2} \\ &= \frac{W^2}{2} \end{aligned}$$

$$\frac{dH}{d\Psi} = 0$$

And hence the SL equation gives

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\ \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\ \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= 0\end{aligned}$$

Which results in:

$$\begin{aligned}\Psi &= Cr^2 + D, \quad r \geq \hat{r} \\ w &= \frac{1}{r} \frac{\partial \Psi}{\partial r} = 2C\end{aligned}$$

With the requirement that there is no discontinuity at  $\hat{r}$ , i.e.

$$\Psi = \frac{1}{2}W\hat{r}^2 + \hat{r} (AJ_1(k\hat{r}) + BY_1(k\hat{r})) = C\hat{r}^2 + D$$

And using the same for  $w$

$$w(\hat{r}) = W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) = 2C$$

And lastly the wall condition

$$\Psi(R) = \frac{1}{2}WR^2 = C\hat{r}^2 + D$$

With

$$\begin{aligned}w(r_*) &= 0 \\ \frac{\Gamma}{2\pi r_0} = \Omega r_0 &\implies \Omega = \frac{\Gamma}{2\pi r_0^2} \\ k_{outer} &= \frac{2\Gamma}{2\pi W r_0^2} = \frac{\Gamma}{\pi W r_0^2}\end{aligned}$$

Noting that the values for  $A$  and  $B$  are obtained from the  $r_*$  condition.

The coefficients for  $\Psi$  have to be resolved, since the condition  $\Psi_{inner}(R) = \frac{1}{2}WR^2$  cannot be imposed.

Parameters

$$r_0, \hat{r}, r_*, R, k, \Gamma, W, A, B, C, D$$

We can fix  $r_0$ ,  $R$ ,  $k$ ,  $W$  and  $\Gamma$ . This is 11 parameters, where 5 are fixed. Require 6 conditions. Impose:

- 1).  $w(r_*) = 0$  (as before)
- 2).  $\Psi_{inner}(r_*) = 0$  (as before)
- 3). Since at the wall  $\Psi$  must remain the same, this applies to where  $v$  is changed, i.e.  
 $\Psi_{inner}(\hat{r}) = \frac{1}{2}W r_0^2$
- 4). For continuity,  $\Psi_{outer}(\hat{r}) = \frac{1}{2}W r_0^2$

5).  $w_{outer}(\hat{r}) = w_{inner}(\hat{r})$

6).  $\Psi_{outer}(R) = \frac{1}{2}WR^2$

Redo the problem instead getting  $A, B$  from 2) and 3)

$$\begin{aligned}\Psi_{inner}(r_*) &= 0 \\ \Psi_{inner}(\hat{r}) &= \frac{1}{2}Wr_0^2\end{aligned}$$

Use this for  $A, B$

$$\begin{aligned}\Psi_{inner}(r_*) &= \frac{1}{2}Wr_*^2 + r_*(AJ_1(kr_*) + BY_1(kr_*)) = 0 \\ &= r_*(AJ_1(kr_*) + BY_1(kr_*)) = -\frac{1}{2}Wr_*^2 \\ \Psi_{inner}(\hat{r}) &= \frac{1}{2}W\hat{r}^2 + \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}Wr_0^2 \\ &= \hat{r}(AJ_1(k\hat{r}) + BY_1(k\hat{r})) = \frac{1}{2}W(r_0^2 - \hat{r}^2)\end{aligned}$$

This gives the matrix system for  $A, B$  below. Note that the system relies on the unknowns  $r_*$  and  $\hat{r}$ .

$$\begin{aligned}\begin{pmatrix} r_*J_1(kr_*) & r_*Y_1(kr_*) \\ \hat{r}J_1(k\hat{r}) & \hat{r}Y_1(k\hat{r}) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{\det} \begin{pmatrix} \hat{r}Y_1(k\hat{r}) & -r_*Y_1(kr_*) \\ -\hat{r}J_1(k\hat{r}) & r_*J_1(kr_*) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}Wr_*^2 \\ \frac{1}{2}W(r_0^2 - \hat{r}^2) \end{pmatrix} \\ A &= \frac{1}{\det} \left( \hat{r}Y_1(k\hat{r}) \left( -\frac{1}{2}Wr_*^2 \right) - r_*Y_1(kr_*) \left( \frac{1}{2}W(r_0^2 - \hat{r}^2) \right) \right) \\ B &= \frac{1}{\det} \left( -\hat{r}J_1(k\hat{r}) \left( -\frac{1}{2}Wr_*^2 \right) + r_*J_1(kr_*) \left( \frac{1}{2}W(r_0^2 - \hat{r}^2) \right) \right)\end{aligned}$$

Where

$$\begin{aligned}\det &= \hat{r}r_*Y_1(k\hat{r})J_1(kr_*) - \hat{r}r_*J_1(k\hat{r})Y_1(kr_*) \\ &= \hat{r}r_*(Y_1(k\hat{r})J_1(kr_*) - J_1(k\hat{r})Y_1(kr_*))\end{aligned}$$

This for  $r_*$

$$w_{inner}(r_*) = W + k(AJ_0(kr_*) + BY_0(kr_*)) = 0$$

Get  $C$  from:

$$\begin{aligned}w_{outer}(\hat{r}) &= w_{inner}(\hat{r}) \\ 2C &= W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})) \\ C &= \frac{1}{2}(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})))\end{aligned}$$

Get  $D$  here:

$$\begin{aligned}\Psi_{outer}(R) &= CR^2 + D = \frac{1}{2}WR^2 \\ D &= \frac{1}{2}WR^2 - C\end{aligned}$$

Hence get  $\hat{r}$  from

$$\begin{aligned}\Psi_{outer}(\hat{r}) &= C\hat{r}^2 + D = \frac{1}{2}Wr_0^2 \\ C\hat{r}^2 + \frac{1}{2}WR^2 - C &= \frac{1}{2}Wr_0^2 \\ \left(\frac{1}{2}(W + k(AJ_0(k\hat{r}) + BY_0(k\hat{r})))\right)(\hat{r}^2 - 1) &= \frac{1}{2}W(r_0^2 - R^2) \\ (AJ_0(k\hat{r}) + BY_0(k\hat{r}))(\hat{r}^2 - 1) &= \frac{1}{k}W(r_0^2 - R^2 - 1)\end{aligned}$$

For physically valid solutions, we must impose the condition of no net change on the momentum from upstream to downstream on the momentum (Escudier, Keller). The momentum is defined as

$$s = 2\pi \int_0^{r_t} (\rho w^2 + p) r dr$$

Which comes to:

$$\Delta s = \frac{\pi}{4} \rho U^2 k^2 r_c^2 \left[ -r_b^2 + \frac{1}{4} \left( \frac{r_b^4 - r_a^4}{r_c^2} \right) + \frac{3}{4} r_c^2 + \frac{1}{2} r_c^2 \log \left( \frac{r_b^2}{r_c^2} \right) \right] = 0$$

Figure 4 shows the solution set for the problem. It displays the same results as those found in (Escudier, Keller), with the same asymptote  $kr_0 \rightarrow \sqrt{2}$  as  $r_0 \rightarrow 0$ . The results in the two figures come from `rhatrstarmomentum.m` and `numericalSolutionSetRankine.m` respectively

## 0.2.4 Lamb-Oseen Vortex

Q-vortex without a Jet. Start with

$$\begin{aligned}w &= W \\ v &= \frac{\Gamma}{2\pi r} \left( 1 - e^{-r^2/\delta^2} \right) \\ \frac{d^2\Psi}{dr^2} - \frac{1}{r} \frac{d\Psi}{dr} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi}\end{aligned}$$

Solve from  $r_*$  to  $R$  numerically.

Generate grid from  $r_*$  to  $R$ .

Boundary conditions as normal

$$\Psi(R) = \frac{1}{2}WR^2$$

$$\Psi(r_*) = 0$$

$$w(r_*) = 0$$

And the standard upstream flow

$$\Psi(r) = \frac{1}{2}Wr^2$$

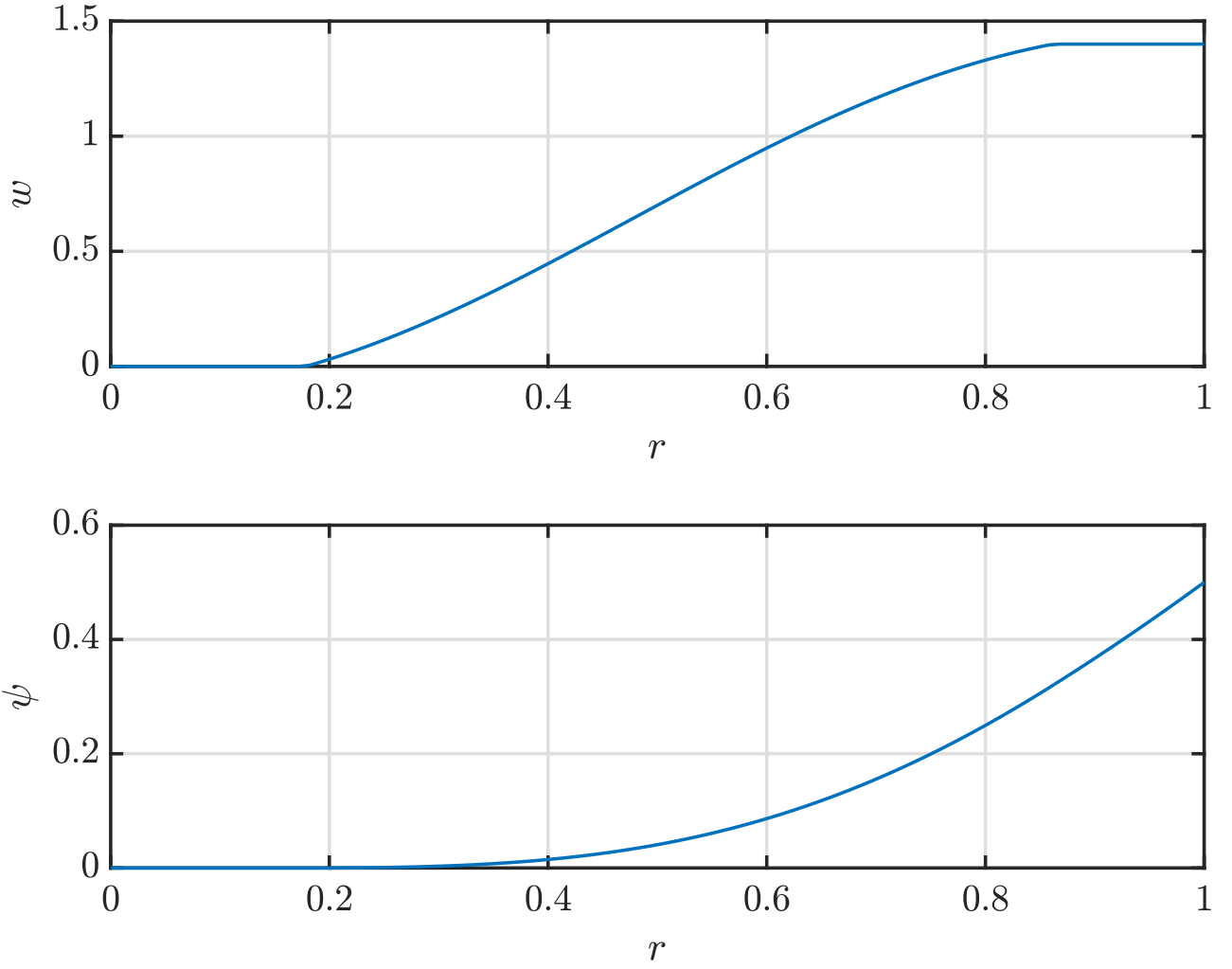


Figure 3: A solution of  $w$  and  $\Psi$  for the Rankine problem with 0 net momentum,  $k = 3.8961$ ,  $r_0 = 0.8619$ ,  $r_* = 0.1784$ . Obtained using

Non-dimensional parameter may be something like  $\frac{\Gamma}{WR}$  (we can probably relate this to  $kr_0$  for the rankine problem)

Eventually do the same thing as before with  $s$  and  $\Delta s$ .

$$s = \int_0^R (\rho w^2 + p) r dr = \int_0^{r_*} p(r_*) r dr + \int_{r_*}^R (\rho w^2 + p) r dr$$

Use

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{v^2}{r} = \frac{\Gamma^2}{4\pi^2 r^3} \left(1 - e^{-r^2/\delta^2}\right)^2$$

$$\Psi = \frac{1}{2} W r^2 \implies r = \sqrt{\frac{2\Psi}{W}}$$



## Rankine body solution space

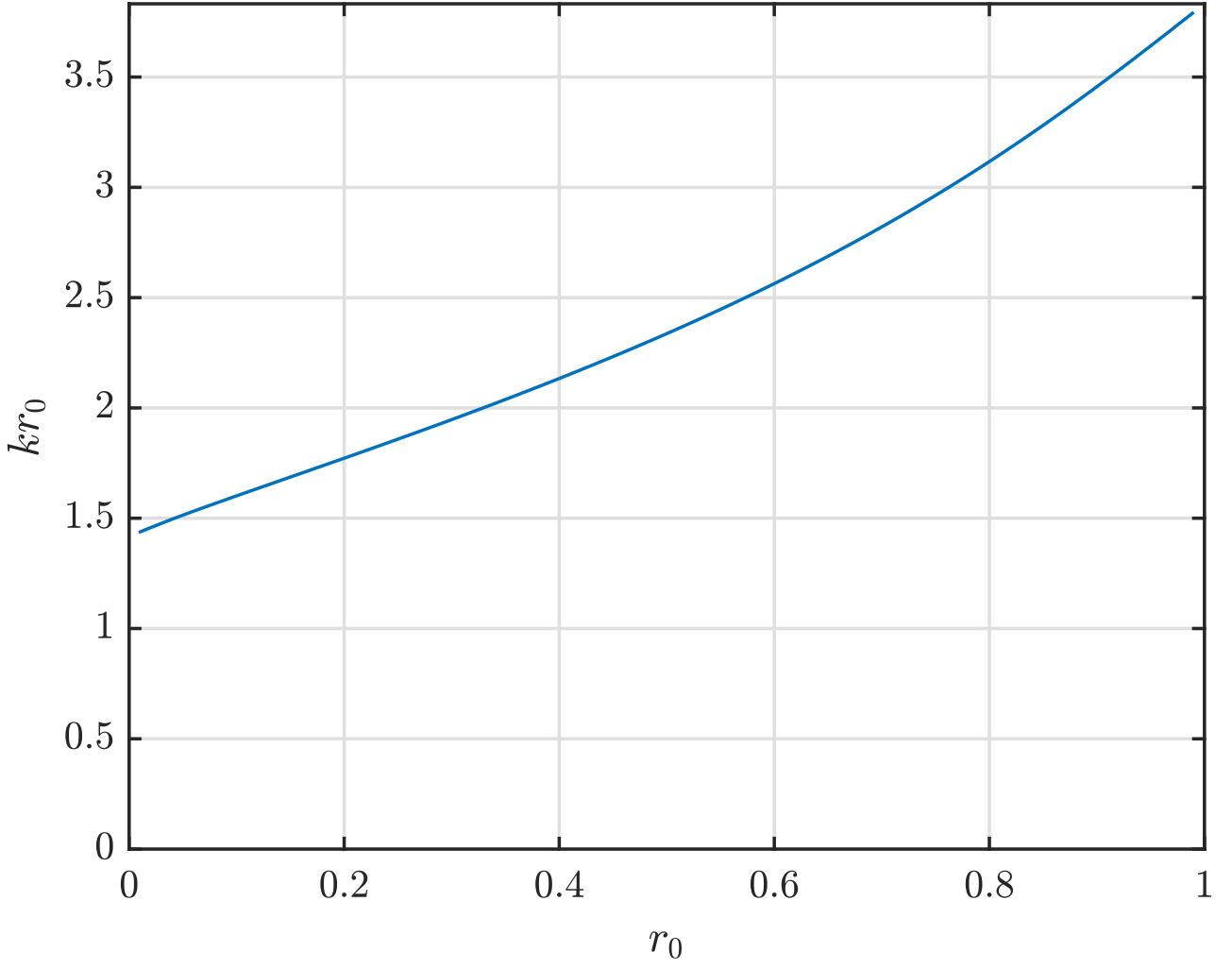


Figure 4: Solution space for the Rankine body problem

$$\begin{aligned}
 C &= rv = \frac{\Gamma}{2\pi} \left(1 - e^{-r^2/\delta^2}\right) \\
 \frac{\partial C}{\partial \Psi} &= \frac{\Gamma}{2\pi} \frac{\partial}{\partial \Psi} \left(1 - e^{-r^2/\delta^2}\right) \\
 &= \frac{-\Gamma}{2\pi} \frac{\partial}{\partial \Psi} \left(e^{-r^2/\delta^2}\right) \\
 &= \frac{-\Gamma}{2\pi} \frac{\partial}{\partial \Psi} \left(e^{-2\Psi/W\delta^2}\right) \\
 &= \frac{\Gamma}{W\delta^2\pi} \left(e^{-2\Psi/W\delta^2}\right) \\
 &= \frac{\Gamma}{W\pi\delta^2} e^{-r^2/\delta^2}
 \end{aligned}$$

$$\begin{aligned}
\frac{dH}{d\Psi} &= \frac{dH}{dr} \frac{dr}{d\Psi} \\
&= \frac{dr}{d\Psi} \frac{d}{dr} \left( \frac{1}{2} (u^2 + v^2 + w^2) + \frac{p}{\rho} \right) \\
&= \frac{1}{\sqrt{2W\psi}} \frac{d}{dr} \left( \frac{1}{2} v^2 + \int \frac{C^2}{r^3} dr \right) \\
&= \frac{1}{Wr} \left( \frac{1}{2} \frac{dv^2}{dr} + \frac{v^2}{r} \right) \\
&= \frac{1}{Wr} \left( \frac{\Gamma^2}{4r\pi^2} \left( \frac{-1}{r^2} + 2e^{-r^2/\delta^2} \left( \frac{1}{r^2} + \frac{1}{\delta^2} \right) - e^{-2r^2/\delta^2} \left( \frac{1}{r^2} + \frac{2}{\delta^2} \right) \right) + \frac{\Gamma^2}{4r\pi^2} \left( \frac{1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}}{r^2} \right) \right) \\
&= \frac{\Gamma^2}{4Wr^2\pi^2} \left( 2e^{-r^2/\delta^2} \left( \frac{1}{r^2} + \frac{1}{\delta^2} \right) - e^{-2r^2/\delta^2} \left( \frac{1}{r^2} + \frac{2}{\delta^2} \right) + \frac{-2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}}{r^2} \right) \\
&= \frac{\Gamma^2}{2Wr^2\delta^2\pi^2} \left( e^{-r^2/\delta^2} - e^{-2r^2/\delta^2} \right) \\
&= \frac{\Gamma^2}{4\Psi\delta^2\pi^2} \left( e^{-2\Psi/W\delta^2} - e^{-4\Psi/W\delta^2} \right)
\end{aligned}$$

How I got the middle term:

$$\begin{aligned}
\frac{dv^2}{dr} &= \frac{d}{dr} \left( \frac{\Gamma}{2\pi r} \left( 1 - e^{-r^2/\delta^2} \right) \right)^2 \\
&= \frac{\Gamma^2}{4\pi^2} \frac{d}{dr} \left( \frac{1 - 2e^{-r^2/\delta^2} + e^{-2r^2/\delta^2}}{r^2} \right) \\
&= \frac{\Gamma^2}{4\pi^2} \left( \frac{-2}{r^3} - 2 \left( -\frac{2e^{-r^2/\delta^2}}{r^3} - \frac{2e^{-r^2/\delta^2}}{r\delta^2} \right) + \left( -\frac{2e^{-2r^2/\delta^2}}{r^3} - \frac{4e^{-2r^2/\delta^2}}{r\delta^2} \right) \right) \\
&= \frac{\Gamma^2}{2r\pi^2} \left( \frac{-1}{r^2} + 2e^{-r^2/\delta^2} \left( \frac{1}{r^2} + \frac{1}{\delta^2} \right) - e^{-2r^2/\delta^2} \left( \frac{1}{r^2} + \frac{2}{\delta^2} \right) \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= r^2 \frac{dH}{d\Psi} - C \frac{dC}{d\Psi} \\
&= \frac{r^2 \Gamma^2}{4\Psi\delta^2\pi^2} \left( e^{-2\Psi/W\delta^2} - e^{-4\Psi/W\delta^2} \right) - \left( \frac{\Gamma}{2\pi} \left( 1 - e^{-2\Psi/W\delta^2} \right) \right) \left( \frac{\Gamma}{W\delta^2\pi} \left( e^{-2\Psi/W\delta^2} \right) \right) \\
&= \frac{r^2 \Gamma^2}{4\Psi\delta^2\pi^2} \left( e^{-2\Psi/W\delta^2} - e^{-4\Psi/W\delta^2} \right) - \frac{\Gamma^2}{2W\delta^2\pi^2} \left( 1 - e^{-2\Psi/W\delta^2} \right) \left( e^{-2\Psi/W\delta^2} \right) \\
&= \frac{\Gamma^2}{2W\delta^2\pi^2} \left( \frac{r^2 W}{2\Psi} \left( e^{-2\Psi/W\delta^2} - e^{-4\Psi/W\delta^2} \right) - \left( e^{-2\Psi/W\delta^2} - e^{-4\Psi/W\delta^2} \right) \right) \\
\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} &= \frac{\Gamma^2}{2W\delta^2\pi^2} \left( \left( \frac{r^2 W}{2\Psi} - 1 \right) \left( e^{-2\Psi/W\delta^2} - e^{-4\Psi/W\delta^2} \right) \right)
\end{aligned}$$

Take limit as  $\Psi \rightarrow 0$  in matlab and use that in some suff area.

Could use bvp5c and break it into a system of first order odes.

So if we change the boundaries - using a 'Linear Lagrange interpolating polynomial' So that the end points are fixed.

$$\eta = \frac{r - r_*}{R - r_*}$$

$$r - r_* = \eta(R - r_*)$$

$$r = \eta(R - r_*) + r_*$$

Such that  $\eta \in [0, 1]$ . In the function we can get  $r$  from  $\eta$

For finite differences we can just use a newton iteration, or use something like fsolve.

I SHOULD PLOT DOWNSTREAM  $v$  for all our equations also

To use the  $r_*$  and  $w$  parts, we can use  $w = \frac{1}{r} \frac{\partial \Psi}{\partial r}$

Try putting in the homogeneous solution to the solver to see if it works (i.e. the one with  $v = \Omega r$  and  $w = W$ ).

We should expect that the Lamb-Oseen vortex should be a more smooth version of the Rankine problem - so we should be able to compare the two.

May want to find  $v_{max} = r_0$  to compare to the previous problem. Should expect the circulation goes to  $\Gamma/2\pi$  as  $r \rightarrow \infty$ .

Of course we have to have

$$\lim_{r \rightarrow 0} \frac{1}{r} \frac{\partial \Psi}{\partial r} < \infty$$

So use l'hospital's rule

$$\lim_{r \rightarrow 0} \frac{1}{r} \frac{\partial \Psi}{\partial r} = \frac{\partial^2 \Psi}{\partial r^2}$$

But we know  $\psi = 0$  at  $r = 0$  ( so we can ignore this )

So we will just use

$$\frac{\partial^2 \Psi}{\partial r^2} \Big|_i - \frac{1}{r} \frac{\partial \Psi}{\partial r} \Big|_i = f(r_i, \Psi_i)$$

With

$$\Psi_1 = 0$$

And

$$\Psi_n = \frac{1}{2} W R^2$$

We will have 1 based indexing Alternatively get the derivatives at the mid points

$$\frac{\partial \Psi}{\partial r} \Big|_{i+1/2} = \frac{\Psi_{i+1} - \Psi_i}{h_r}$$

Where  $h_r$  is the step.

Alternatively

Can rewrite the left hand side as

$$r \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \right)$$

And plug in the last thing

When we do finite differences, grab a computational variable (call it  $\eta$  for now)

$$\eta = \frac{r - r_*}{R - r_*}$$

So now we are computing in  $0 < \eta < 1$ . So try plugging it into the ODE:

$$d\eta = \frac{dr}{R - r_*}$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial r} = \frac{1}{R - r_*} \frac{\partial \Psi}{\partial \eta}$$

If we don't know  $r_*$  the  $\eta$  vector becomes

$$\begin{pmatrix} \eta_0 \\ \vdots \\ \eta_{N-1} \\ r_* \end{pmatrix}$$

This will be a non-linear problem so we will need to solve using **fsolve**.

To get guesses could use rankine vortex stuff -

For a linear flow we could use We know that  $\psi(r_*) = 0$  and  $\psi(R) = \frac{1}{2}WR^2$ . We could guess that  $\psi$  is constant,  $\psi = Ar^2 + B$ .

Try plotting  $w(r_*)$  for various  $r_*$  to help find guesses (require it to be 0)

If using  $\eta$ , the ODE becomes  $\psi(\eta = 0) = 0$  and  $\psi(\eta = 1) = \frac{1}{2}WR^2$

$$\begin{aligned} \frac{\partial \Psi}{\partial r} &= \frac{1}{R - r_*} \frac{\partial \Psi}{\partial \eta} \\ \frac{\partial^2 \Psi}{\partial r^2} &= \frac{1}{(R - r_*)^2} \frac{\partial^2 \Psi}{\partial \eta^2} \\ r &= \eta(R - r_*) + r_* \end{aligned}$$

DE becomes

$$\begin{aligned} \frac{1}{(R - r_*)^2} \frac{\partial^2 \Psi}{\partial \eta^2} - \frac{1}{\eta(R - r_*) + r_*} \frac{1}{R - r_*} \frac{\partial \Psi}{\partial \eta} &= 0 \\ \frac{1}{R - r_*} \frac{\partial^2 \Psi}{\partial \eta^2} - \frac{1}{\eta(R - r_*) + r_*} \frac{\partial \Psi}{\partial \eta} &= 0 \end{aligned}$$

### 0.2.5 Outer vortex breakdown

Considering the initial problem for vortex breakdown, except perhaps the breakdown is a pocket expanding from  $R$  rather than 0. I.e. the breakdown occurs about the wall rather than the center. So assuming  $r^\dagger$  is our outer vortex breakdown radius

This simply means obtaining a new  $A$ ,  $B$  and  $k$ .

$$\Psi(r) = \frac{1}{2}Wr^2 + r(AJ_1(kr) + BY_1(kr))$$

$$w(r) = W + k(AJ_0(kr) + BY_0(kr))$$

Such that

$$w(r^\dagger) = 0, \quad \Psi(0) = 0, \quad \text{and} \quad \Psi(r^\dagger) = 0$$

To enforce  $\Psi(0) = 0$  note that  $\lim_{r \rightarrow 0} \frac{Y_1(kr)}{r} = -\infty$ . Hence it is necessary to set  $B = 0$ .

$$\Psi(r) = \frac{1}{2}Wr^2 + rAJ_1(kr), \quad w(r) = W + kAJ_0(kr)$$

And to enforce  $\Psi(r^\dagger) = 0$

$$\implies Ar^\dagger J_1(kr^\dagger) = -\frac{1}{2}Wr^{\dagger 2}$$

$$A = \frac{-Wr^\dagger}{2J_1(kr^\dagger)}$$

And obtain  $k$  using

$$w(r^\dagger) = 0$$

$$kAJ_0(kr) = -W$$

$$\Psi(r^\dagger) = \Psi(R) = \frac{1}{2}WR^2$$

## 0.3 Appendix

### 0.3.1 Supplementary Materials

This is where all the basic fluid mechanics knowledge should be (definitions, etc.)

### 0.3.2 Resources

Books: An Introduction to Fluid Dynamics Batchelor

Swirling flow states in finite-length diverging or contracting circular pipes Zvi Rusak

Wall-separation and vortex-breakdown zones in a solid-body rotation flow in a rotating finite-length straight circular pipe Zvi Rusak, and Shixiao Wang