Some Exact Solutions of the Flow Through Annular Cascade Actuator Discs*

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Summary

Exact solutions are presented for two types of flow of an incompressible inviscid fluid through an annular passage with a discontinuity in tangential velocity at one section, as would occur at a cascade of airfoil blades of negligible chord. When stagnation conditions are constant throughout the flow region, it is shown that the product of the tangential vorticity component (η) and the radius (r) is constant along a stream line. The particular case when this product is a linear multiple of the stream function (ψ) is solved by dividing ψ into persistent and perturbation components and separating the variables. A second exact solution is obtained when the product (ηr) is a function of radius alone and when the stagnation conditions are allowed to vary. Two solutions of either of the above kinds can be exactly matched at the discontinuity section, thus determining the complete flow pattern for arbitrarily selected conditions at infinity. The axial velocity at this section is almost exactly the mean of upstream and downstream values at the same radius. The family of exact solutions may be of use in estimating the accuracy of approximate solutions and, as shown in a numerical example, may match some desired flow conditions sufficiently closely to be of interest in design.

Introduction

The essential feature of axial compressors and turbines is the arrangement of alternate stationary and rotating cascades of airfoils; these alter the angular momentum of the working fluid as it passes so that work is done on or by it, thus raising or lowering its pressure and energy levels. In designing such turbomachinery, the required work distribution is known, and it is necessary to find the blade profiles that will produce it.

An analysis of the flow through a single annular cascade is a first step in the study of this problem. This cannot be done by the classical two-dimensional methods because of the radial pressure gradients and velocities induced by the fluid rotation, although attempts may be made to correlate data with results from two-dimensional cascades. The radial pressure gradient is associated with centrifugal effects due to the tangential velocities produced by the blades. Conservation of angular momentum demands that, in the absence of fluid shear, these persist as the fluid flows downstream to infinity after a cascade. The radial velocities, however, are purely local phenomena caused by the adjustment in mass flow distribution associated with the different velocity profiles on either side of the cascade. We may then regard the flow as divided into two parts—upstream and downstream. In each, the flow conditions are basically those that would obtain at infinity, but, in the region of the cascade, perturbation components are introduced to smooth out the transition from one region to another.

Various authors have suggested analytical approaches to the determination of the flow through cascades in order to obtain the blade profiles for a given duty. Meyer[1] has studied the flow through free vortex blades. Marble[2] has presented a general solution, but this involves some mathematical approximation in order to linearize the equation of motion.

This paper describes two physically possible types of incompressible nonviscous flow for which exact solutions of the equations of motion are found and applied directly to the limiting case of a cascade of negligible chord width. This is a special case of the actuator disc of propeller theory. The flow is assumed to be steady and uninfluenced by external forces, except the normal forces at the parallel walls of the annulus. At one particular section, the actuator disc, there is a discontinuity in the tangential—though not in the other velocity components. Rotational symmetry is assumed—that is, there is no variation in flow conditions at a given radius with angular position. The disc is, in effect, the limiting case of a cascade containing an infinite number of blades of infinitesimal chord which exert an impulse on the fluid in the tangential direction. The basic equations employed are Euler's Equation of Motion, the equation of continuity, and the thermodynamic entropy and energy equation. Heat transfer to the fluid is neglected, and it is assumed that the fluid flows isentropically from a region where it is all at the same entropy.

Basic Equations of Annular Flow

In the absence of body forces, Euler's equations of motion for rotationally symmetric steady flow of a nonviscous fluid referred to cylindrical polar coordinates r, θ, and z (Fig. 1) may be written:

$$c_r \frac{\partial c_r}{\partial r} + c_z \frac{\partial c_r}{\partial z} - \frac{c_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{(radially)} \tag{1}$$

$$c_r \frac{\partial c_\theta}{\partial r} + c_z \frac{\partial c_\theta}{\partial z} + \frac{c_r c_\theta}{r} = 0 \quad \text{(tangentially)} \tag{2}$$

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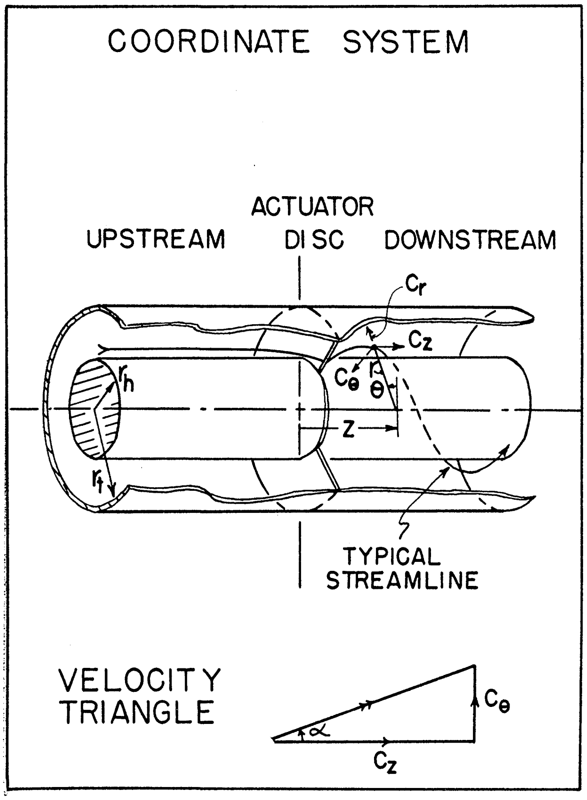

FIG. 1. Coordinate system.

$$c_r \frac{\partial c_z}{\partial r} + c_z \frac{\partial c_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{(axially)} \tag{3}$$

c_r, c_θ, and c_z represent the velocity components in the three coordinate directions.

The equation of continuity under similar conditions becomes:

$$(\partial/\partial z)\ (\rho r c_z) + (\partial/\partial r)\ (\rho r c_r) = 0 \tag{4}$$

The components of vorticity are defined by the relations:

$$\xi = -\partial c_\theta/\partial z \quad \text{(radial component)} \tag{5}$$

$$\eta = (\partial c_r/\partial z) - (\partial c_z/\partial r) \quad \text{(tangential component)} \tag{6}$$

$$\zeta = (1/r)\ (\partial/\partial r)\ (r c_\theta) \quad \text{(axial component)} \tag{7}$$

The enthalpy of the fluid at a point, h, and the stagnation enthalpy, H—i.e., the value of the enthalpy that would be reached if the fluid were brought to rest adiabatically—are connected by the steady flow energy equation:

$$H = h + (1/2)\ (c_r^2 + c_\theta^2 + c_z^2)$$

If we assume that all the fluid was initially at the same entropy and that all subsequent changes occur isentropically, the thermodynamic relation $\delta h = \delta p/\rho$ applies, and so, for all variation through the flow,

$$\delta H = (\delta p/\rho) + (1/2)\delta(c^2) \tag{8}$$

Combining Eqs. (1), (2), and (3) with Eq. (8) to eliminate pressure differentials and introducing the vorticity components from Eqs. (5), (6), and (7),

$$\xi c_\theta - \eta c_z = \partial H/\partial r \tag{9}$$

$$\zeta c_z - \xi c_r = 0 \tag{10}$$

$$\eta c_r - \zeta c_\theta = \partial H/\partial z \tag{11}$$

Eq. (4) shows that it is possible to define a "Stream Function," ψ, of r and z, such that

$$\rho r c_r = \rho_0(\partial\psi/\partial z) \tag{12}$$

$$\rho r c_z = -\rho_0(\partial\psi/\partial r) \tag{13}$$

where ρ_0 is a constant reference density.

From the tangential equation of motion, Eq. (2),

$$[c_r(\partial/\partial r) + c_z(\partial/\partial z)](r c_\theta) = 0$$

This agrees with the physical concept that in the absence of external forces angular momentum is preserved, since it shows that $r c_\theta$ is constant along a stream line. Thus, $r c_\theta = \theta$ is a function of ψ. Also, H is a function of ψ (reference 3, article 165). The relation between θ and the vorticity components is analogous to that between ψ and the corresponding velocities, for

$$r\xi = -\partial\theta/\partial z \tag{14}$$

$$r\zeta = \partial\theta/\partial r \tag{15}$$

From Eqs. (6), (12), and (13),

$$dH/d\psi = (1/r^2)[\theta(d\theta/d\psi) + (\rho_0/\rho)r\eta] \tag{16}$$

$$\eta r = \frac{\rho_0}{\rho}\left(\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r\right) - \frac{\rho_0}{\rho^2}\left(\psi_z\frac{\partial\rho}{\partial z} + \psi_r\frac{\partial\rho}{\partial r}\right) \tag{17}$$

where ψ_r, ψ_z, etc., denote partial differentiation of ψ with respect to r and z, etc.

It is of interest to note the similarity between Eq. (16) and the relation derived by Crocco[5] for flow behind an axially symmetric shock, which is

$$\frac{dH}{d\psi} - T\frac{ds}{d\psi} = \frac{1}{r^2}\frac{\rho_0}{\rho} r\eta \tag{16a}$$

where s is the entropy and T is the absolute temperature.

Hence, Eq. (16) may be extended to a more general one allowing for entropy variation:

$$\frac{dH}{d\psi} - \frac{Tds}{d\psi} = \frac{1}{r^2}\left(\theta\frac{d\theta}{d\psi} + \frac{\rho_0}{\rho} r\eta\right) \tag{16b}$$

In incompressible flow, the relations (16) and (17) simplify to

$$dH/d\psi = (1/r^2)[\theta(d\theta/d\psi) + \eta r] \tag{18}$$

where

$$\eta r = \psi_{rr} + \psi_{zz} - (1/r)\psi_r \tag{19}$$

Features of various types of flow may be distinguished by examining Eq. (18).

(*a*) *Isentropic Flows with H = Constant.*—This implies constant stagnation pressure and includes all

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isentropic flows through stationary or rotating cascades with constant circulation along the blades.

Since $dH/d\psi = 0$, ηr must be a function of ψ and constant along a stream surface. Also, from Eqs. (9), (10), and (11),

$$\xi/c_r = \eta/c_\theta = \zeta/c_z = \Omega/c \tag{20}$$

where Ω is the resultant vorticity. The vortex lines thus follow the stream lines.

(*b*) *Isentropic Flows with rc_θ Zero or Constant.*—From Eq. (18),

$$dH/d\psi = \eta/r \tag{21}$$

and, hence, η/r is constant along the stream surface.

The condition that η/r should be constant along a stream surface, mentioned by Marble[2] and in Lamb (reference 3, article 165), was applied to axially symmetric flow—that is, flow with no tangential velocity. As shown here, it also results when rc_θ is constant and, hence, when ξ and ζ are zero. It ranges from the trivial, vortex-free case when $\eta = 0$ and H = constant (the so-called free vortex flow) to flow with varying H and η.

The First Exact Solution

In order to obtain the first solution, incompressible flow with constant stagnation enthalpy has been assumed. Then, from Eq. (18),

$$\eta r = f(\psi) \tag{22}$$

The perturbation concept of the flow suggests that a suitable form for the stream function is given by the relation:

$$\psi = \psi_\infty + \sum_n e^{\pm k_n z}\psi_n' \tag{23}$$

where ψ_∞ and ψ_n' are functions of r alone. k_n is assumed positive, and the sign of the index $\pm k_n z$ is such that the terms in the summation all vanish at great distances from the origin.

Substituting in Eq. (19),

$$f(\psi) = \eta r = \psi_{\infty_{rr}} - 1/r\,\psi_{\infty_r} + \sum_n e^{\pm k_n z} \times \left(k_n^2\psi_n' + \psi_{n_{rr}}' - \frac{1}{r}\psi_{n_r}'\right) \tag{24}$$

As $|z| \to \infty$, the summation terms vanish, leaving

$$f(\psi_\infty) = \psi_{\infty rr} - (1/r)\psi_{\infty r} \tag{25}$$

In the particular case where $f(\psi)$ is a linear function of ψ, it is possible to separate the variables by subtracting Eq. (25) from Eq. (24) and so solving for ψ_n' and ψ_∞. The case where $f(\psi)$ is a constant will be considered later as part of the second exact solution.

In order to nondimensionalize the equations, the reference length r_t, equal to the annulus outer radius, and the reference velocity c_m are introduced. c_m is defined as the mean axial velocity of the flow, which is constant along the annulus in incompressible flow, and is given by

$$c_m \int_{\text{hub}}^{\text{tip}} 2\pi r\, dr = \int_{\text{hub}}^{\text{tip}} 2\pi r c_z\, dr$$

whence,

$$c_m = -2[\psi]_{\text{hub}}^{\text{tip}}/(r_t^2 - r_h^2) \tag{26}$$

Then, putting $f(\psi) = -\mu^2(\psi/r_t^2)$, where μ is a constant, Eq. (25) becomes:

$$\psi_{\infty rr} - (1/r)\psi_{\infty r} + (\mu^2/r_t^2)\psi_\infty = 0 \tag{27}$$

This has as solution a first-order Bessel function:

$$\psi_\infty/r_t^2c_m = -R(a/\mu)A_1(\mu R) \tag{28}$$

where $R = r/r_t$ and

$$A_p(\mu R) = J_p(\mu R) + (a'/a)Y_p(\mu R)$$

a and a' are constants, but only one is independently variable, for, eliminating c_m from Eqs. (26) and (28),

$$a = \mu(1 - R_h^2)/[A_1(\mu) - R_hA_1(\mu R_h)] \tag{29}$$

where $r_h/r_t = R_h$.

For the perturbation component, subtracting Eq. (25) from Eq. (24) and substituting for $f(\psi)$,

$$\sum e^{\pm k_n z}\left[\psi_{n_{rr}}' - \frac{1}{r}\psi_{n_r}' + \left(k_n^2 + \frac{\mu^2}{r_t^2}\right)\psi_n'\right] = 0$$

Omitting subscripts for clarity, it is seen that each term of the summation must satisfy the relation:

$$\psi_{rr}' - (1/r)\psi_r' + [k^2 + (\mu^2/r_t^2)]\psi' = 0$$

Solving and putting

$$\lambda^2 = k^2r_t^2 + \mu^2 \tag{30}$$

another first-order Bessel function expression is obtained:

$$\psi'/r_t^2c_m = -R(b/\lambda)B_1(\lambda R) \tag{31}$$

where

$$B_p(\lambda R) = J_p(\lambda R) + (b'/b)Y_p(\lambda R) \tag{32}$$

and b and b' are constants of integration. The full solution for the stream function is, therefore, from Eq. (23),

$$\frac{\psi}{r_t^2c_m} = -\frac{a}{\mu}RA_1(\mu R) - \sum_n e^{\pm k_n z}\frac{b_n}{\lambda_n}RB_1(\lambda_n R) \tag{33}$$

Differentiating and applying Eqs. (12) and (13),

$$c_z/c_m = aA_0(\mu R) + \sum_n e^{\pm k_n z}\, b_nB_0(\lambda_n R) \tag{34}$$

$$c_r/c_m = \mp \sum_n e^{\pm k_n z}(k_nr_t/\lambda_n)b_nB_0(\lambda_n R) \tag{35}$$

From Eqs. (18) and (27),

$$\theta(d\theta/d\psi) = -\eta r = (\mu^2/r_t^2)\psi$$

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Integrating,

$$\theta^2 = (\mu^2/r_t^2)\psi^2 + g r_t^2 c_m^2$$

where g is another nondimensional constant of integration. Therefore,

$$\left(\frac{c_\theta}{c_m}\right)^2 = \left(\frac{\mu}{R}\right)^2\left(\frac{\psi}{r_t^2 c_m}\right)^2 + \frac{g}{R^2} =$$

$$\left[aA_1(\mu R) + \sum_n \frac{\mu b_n}{\lambda_n} e^{\pm k_n z} B_1(\lambda_n R)\right]^2 + \frac{g}{R^2} \qquad (36)$$

Matching Conditions at Infinity

Putting the perturbation components equal to zero in the Eqs. (33) to (36) for the complete solution derived in the last section, the following conditions at infinity are derived:

$$c_{z\infty}/c_m = aA_0(\mu R) \qquad (37)$$

$$(c_{\theta\infty}/c_m)^2 = [(\mu/R)/(\psi_\infty/r_t^2 c_m)]^2 + (g/R^2)$$

$$\psi_\infty/r_t^2 c_m = -(a/\mu)RA_1(\mu R) \qquad (38)$$

Three of the constants involved are arbitrary, the fourth being fixed by Eq. (29). It should, therefore, be possible to make a good approximation to any desired velocity distribution at infinity.

A particular case is that of free vortex flow for which c_z and rc_θ are constant—i.e., independent of radius. This condition is satisfied when $a = 1$ and $a' = \mu = 0$, and so $\lambda_n = k_n r_t$. The full solution then becomes:

$$c_z/c_m = 1 + \sum_n e^{\pm \lambda_n(z/r_t)} b_n B_0(\lambda_n R) \qquad (39)$$

$$c_r/c_m = \sum_n e^{\pm \lambda_n(z/r_t)} b_n B_1(\lambda_n R) \qquad (40)$$

$$(c_\theta/c_m)^2 = g/R^2 \quad \text{or} \quad (rc_\theta)^2 = g(r_t c_m)^2 \qquad (41)$$

To illustrate the possibility of matching Eqs. (37) and (38) to a prescribed distribution, a flow in which c_θ varies as the square of the radius has been used, with $c_\theta = c_z$ at the tip and a hub to tip radius ratio of 0.45. The exact equations for this flow at infinity are, denoting tip axial velocity by c_{zt}:

$$c_\theta/c_{zt} = R^2 \qquad (42)$$

$$c_z/c_{zt} = \sqrt{(3/2)[(5/3) - R^4]} \qquad (43)$$

$$\frac{\psi}{r_t^2 c_{zt}} = -\frac{5}{12}\sqrt{\frac{3}{2}}\sin^{-1}\frac{R^2}{\sqrt{5/3}} - \frac{1}{4}R^2\sqrt{\frac{3}{2}\left(\frac{5}{3} - R^4\right)} \qquad (44)$$

where, from Eq. (26),

$$\frac{c_m}{c_{zt}} = \frac{-2}{1 - (0.45)^2}\left(\frac{\psi_{\text{tip}}}{r_t^2 c_{zt}} - \frac{\psi_{\text{hub}}}{r_t^2 c_{zt}}\right) \qquad (45)$$

These are plotted against radius ratio as the broken lines in Fig. 2.

Values given by Eqs. (37) and (38) have been fitted to this curve by trial and error. The solid curve shows the result of taking values for the constants of

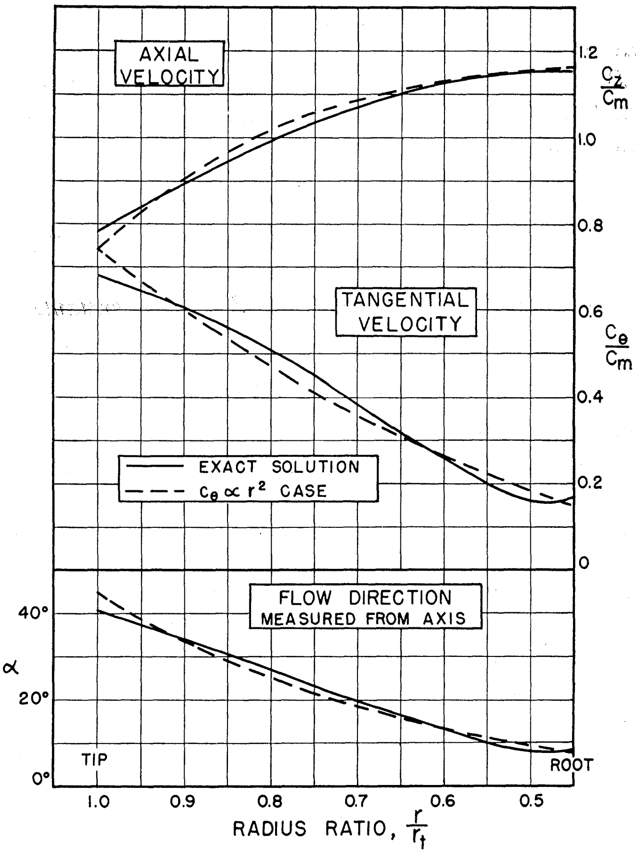

FIG. 2. Velocity distributions at infinity downstream.

$$\mu = 1.740 \qquad a = 1.415$$
$$g = 0.0055 \qquad a' = 0.551$$

There is a slight curl in the rc_θ curve at the inner radius, but agreement is otherwise close, even in such an extreme case as the one chosen.

Matching Conditions at Other Boundaries

The constants b'/b and λ are determined by the boundary conditions at the annular walls, which are assumed parallel. It is required that $c_r = 0$ for $R = R_h$ and $R = 1$ for all values of z. This condition will be satisfied if, from Eq. (35),

$$B_1(\lambda_n) = B_1(\lambda_n R_h) = 0$$

for all values of λ_n. Therefore, from Eq. (32),

$$-\frac{b_n'}{b_n} = \frac{J_1(\lambda_n)}{Y_1(\lambda_n)} = \frac{J_1(\lambda_n R_h)}{Y_1(\lambda_n R_h)} \qquad (46)$$

This provides an infinite number of possible values for b'/b and λ. The first six roots may be found in Jahnke and Emde's "Tables of Functions,"[4] and higher roots, if they were required, could be worked out from the approximate formulas:

$$J_1(x) = \sqrt{2/\pi x}\cos[x - (3\pi/4)]$$
$$Y_1(x) = \sqrt{2/\pi x}\sin[x - (3\pi/4)]$$

whence,

$$\tan[\lambda_n - (3\pi/4)] = \tan[\lambda_n R_h - (3\pi/4)]$$

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Therefore,

$$\lambda_n = N\pi/(1 - R_h) \tag{47}$$

where N is any integer. Note that, even for a hub:tip radius ratio as small as 0.5, the lowest value of λ_n is about 6. The corresponding values of k_n are obtained from Eq. (30).

Only the values of the constants b are now unknown, and these depend on matching at the blade rows. If upstream and downstream flows are both defined by solutions of our form, exact matching may be obtained in the limiting case of an actuator disc. At the disc, continuity in radial and axial velocities is required at all radii, but there may be discontinuities in tangential components. If the disc is situated for convenience at $z = 0$, exact matching of the c_r values will be obtained from Eq. (35) if

$$k_{nd}b_{nd} = -k_{nu}\, b_{nu} = k_{bn} \tag{48}$$

where suffixes u and d denote upstream and downstream conditions, respectively.

Matching of the axial velocities is obtained from Eqs. (34) and (37) if

$$\sum_n \left(\frac{k_{bn}}{k_{nd}} + \frac{k_{bn}}{k_{nu}}\right) B_0(\lambda_n R) = \left(\frac{c_{z\infty}}{c_m}\right)_u - \left(\frac{c_{z\infty}}{c_m}\right)_d \tag{49}$$

It is now possible to find k_b and the other coefficients by a method analogous to that employed in determining those of a Fourier series. Multiplying both sides of Eq. (49) by $RB_0(\lambda_m R)dR$ and integrating from hub to tip, we find that all terms on the left-hand side for which $m \neq n$ are zero. Performing the integration for $m = n$ and making substitutions for $c_{z\infty}$ from Eq. (37), we obtain the following expression for k_{bn}:

$$\begin{aligned}&\frac{1}{2}k_{bn}\left(\frac{1}{k_{nd}} + \frac{1}{k_{nu}}\right)r_t^2\,[B_0^2(\lambda_n) - R_h^2B_0^2(\lambda_nR_h)] = \\ &(a_d\mu_d/k_{nd}^2)[A_1(\mu_d)B_0(\lambda_n) - R_hA_1(\mu_dR_h)B_1(\lambda_nR_h)] - \\ &(a_u\mu_u/k_{nu}^2)\,[A_1(\mu_u)B_0(\lambda_n) - R_hA_1(\mu_uR_h)B_1\,(\lambda_nR_h)]\end{aligned} \tag{50}$$

This fixes all the coefficients in the series expansion, and thus the flow with given conditions at infinity up- and downstream is fully determined everywhere.

One case is of special interest. If $\mu_u = \mu_d$, then $k_{nu} = k_{nd}$, $-b_{nu} = b_{nd}$, and Eq. (34) reduces to

$$\left(\frac{c_z}{c_m}\right)_{\text{disc}} = \left(\frac{c_{z\infty}}{c_m}\right)_d + \sum_n b_{nd}B_0(\lambda_nR) = \left(\frac{c_{z\infty}}{c_m}\right)_u -$$

$$\sum_n b_{nd}B_0(\lambda_nR) = \frac{1}{2}\left[\left(\frac{c_{z\infty}}{c_m}\right)_d + \left(\frac{c_{z\infty}}{c_m}\right)_u\right] \tag{51}$$

that is, the axial velocity at the disc is exactly the arithmetic mean of those at the same radius upstream and downstream at infinity.

If μ^2 is small compared with λ_1^2, it will be extremely small compared with higher order λ_n values. Thus,

$$k_{nu}r_t \simeq \lambda_n \simeq k_{nd}r_t$$

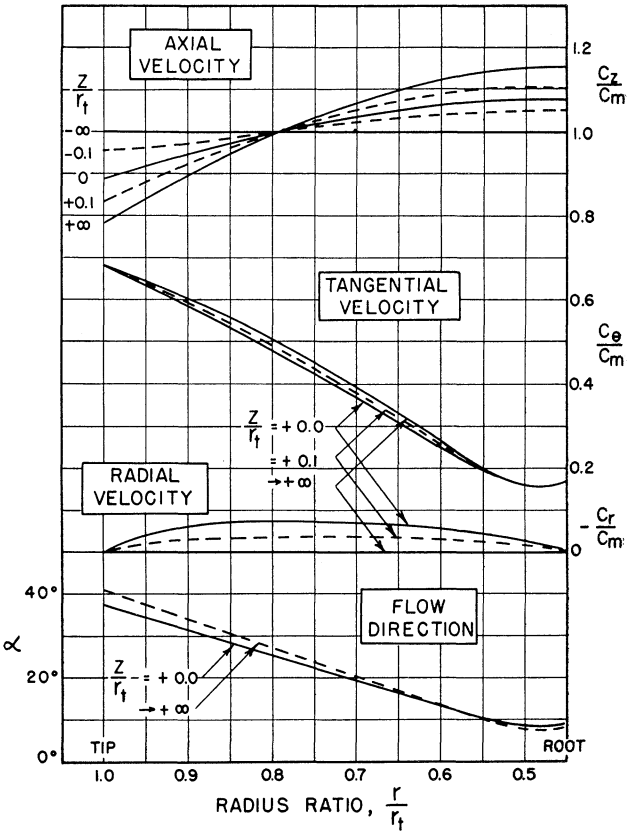

FIG. 3. Conditions near the actuator disc.

which again gives the result of Eq. (51). It seems likely that this relation will hold in many practical cases, so that, in general, the axial velocity at the disc will be almost exactly the arithmetic mean of the infinity values at the same radius.

On the other hand, $\psi = \psi_\infty$ at root and tip radii and is little different elsewhere due to the $1/\lambda_n$ factor in the coefficients of the series expansion [Eq. (33)]. Then Eq. (36) shows that there will be only a small difference at a given radius between the tangential velocity just downstream of the actuator disc and the corresponding infinity value.

To illustrate the application of the method, a flow that is axial and vortex-free upstream at infinity has been matched at the actuator disc to the approximate c_θ varying as r^2 example downstream at infinity already mentioned (shown as solid curves in Fig. 2). The first six roots of Eq. (46) for $R_h = r_h/r_t = 0.45$ are:

λ_n	5.843	11.495	17.22	22.90	28.6	34.3
b_n'/b_n	−2.31	3.86	0.89	0.126	−0.488	−1.69

and these give, with Eqs. (48) and (50),

b_{nu}	0.0818	−0.0166	−0.0230	−0.0265	−0.0155	−0.0070
b_{nd}	0.0856	−0.0167	−0.0230	−0.0265	−0.0155	−0.0070

The calculated values of c_r/c_m and c_z/c_m at the actuator disc and c_θ/c_m just downstream are shown in Fig. 3. The values of c_z/c_m at the disc are almost exactly the arithmetic mean of upstream and downstream infinity values. Because of the large difference between the axial velocity distributions at infinity on either side

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of the disc, the radial velocities are high—up to 8 per cent of the mean axial velocity. Since rc_θ is constant along a stream line and the radial velocities are inflows, the tangential velocities at the disc are less than those at the same radii at infinity downstream, except at the hub and tip where, the radial velocities being zero, no change occurs in c_θ/c_m.

The angle of flow, α, is also plotted. The effects mentioned above tend, in this case, to increase the angle at the tip and reduce it at the root as the flow moves downstream. An approximation suggested for rough design work was to calculate the flow angles near the disc using the mean axial velocity of upstream and downstream flows and the downstream infinity value of the tangential velocity. This result is also shown on Fig. 4; it appears that this is a better approximation to use than the downstream infinity values of the flow angles, especially at the root and tip.

Velocity values at $z = \pm 0.1r_t$ are also shown, and it may be seen that these are about midway between disc and infinity values. This demonstrates that the decay rate of the perturbation components is rapid as the fluid moves away from the blades.

Except in the expression for c_z/c_m, the series for the perturbation velocity components converges rapidly. Six terms were actually calculated so that c_r/c_m and c_θ/c_m might be accurately known. The values obtained by taking only two terms in the expansion are compared with the more accurate calculation on Fig. 4, and

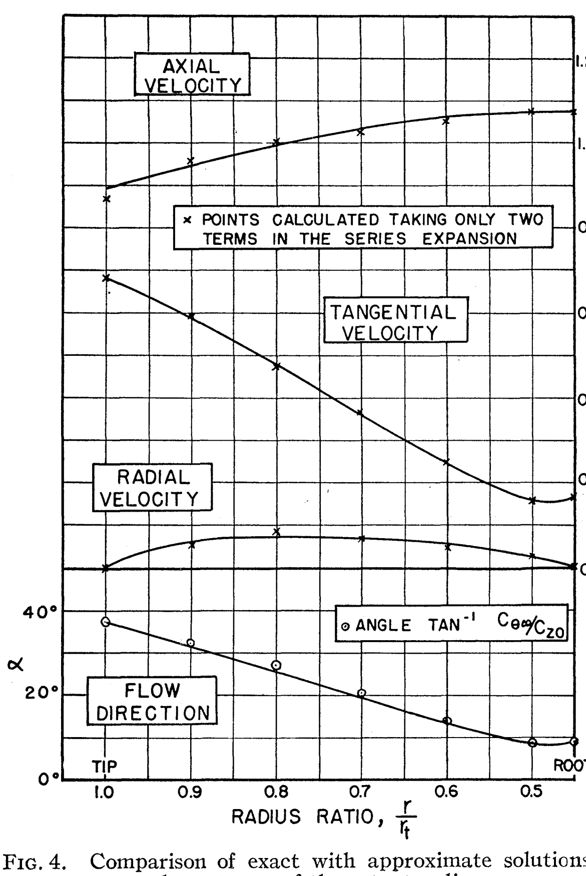

FIG. 4. Comparison of exact with approximate solutions just downstream of the actuator disc.

it may be seen that the added refinement is unnecessary, especially since the exponential $e^{\pm k_n z}$ soon kills high-order terms away from the disc. The error involved in using Eq. (51) to calculate c_z/c_m cannot be more than half the difference in the first series term—i.e., more than 0.002 in this case—since all the other series terms are virtually identical upstream and downstream.

The Second Exact Solution

Returning to the generalized equation for incompressible flow, Eq. (18),

$$dH/d\psi = (1/r^2)[\theta(d\theta/d\psi) + \eta r]$$

suppose that ηr is a function of r only. Differentiating with respect to z,

$$\frac{d^2H}{d\psi^2}\frac{\partial\psi}{\partial z} = \frac{1}{r^2}\frac{d^2[(1/2)\theta^2]}{d\psi^2}\frac{\partial\psi}{\partial z}$$

Since ψ is a function of r and z, this equation can only be satisfied everywhere if

$$d^2H/d\psi^2 = d^2(\theta^2)/d\psi^2 = 0$$

Integrating and again using nondimensional constants,

$$\frac{dH}{d\psi} = \frac{a_1 c_m}{r_t^2} \quad \therefore \quad \frac{H}{c_m^2} = a_1\frac{\psi}{c_m r_t^2} + a_3 \tag{52}$$

$$\theta\frac{d\theta}{d\psi} = a_2 c_m \quad \therefore \quad \left(\frac{rc_\theta}{r_t c_m}\right)^2 = 2a_2\frac{\psi}{c_m r_t^2} + a_4 \tag{53}$$

Substituting the above into Eqs. (18) and (19),

$$\psi_{zz} + \psi_{rr} - (1/r)\psi_r = \eta r = c_m(a_1R^2 - a_2) \tag{54}$$

Suppose, as before, that

$$\psi = \psi_\infty + \sum_n e^{\pm k_n z}\psi_n' \tag{23}$$

Then

$$c_m(a_1R^2 - a_2) = \psi_{\infty_{rr}} - (1/r)\psi_{\infty r} + \sum_n e^{\pm k_n z}[k_n^2\psi_n' + \psi_{nrr}' - (1/r)\psi_{nr}']$$

For this to be satisfied everywhere,

$$k_n^2\psi_n' + \psi_{nrr}' - \frac{1}{r}\psi_{nr}' = 0$$

Putting $\lambda_n = k_n r_t$, the solution for ψ' has exactly the same form as in the case previously considered—Eq. (31). Also,

$$c_m[a_1(R)^2 - a_2] = \psi_{\infty rr} - (1/r)\psi_{\infty r}$$

may be solved to give

$$\frac{\psi_\infty}{r_t^2 c_m} = \frac{a_1}{8}R^4 - \frac{a_2}{4}R^2(2\log_e R - 1) - \frac{a_5}{2}R^2 \tag{55}$$

a_5 is another constant of integration, but it is not independent of the others, being fixed by Eq. (26). Eqs.

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(23), (31), (52), (53), and (55) comprise full solutions for the flow.

By differentiation, using the relations (12) and (13),

$$\frac{c_r}{c_m} = \sum_n \mp b_n \, e^{\pm \lambda_n z/r_t} B_1(\lambda_n R) \tag{56}$$

$$\frac{c_z}{c_m} = -\frac{a_1}{2} R^2 + a_5 + a_2 \log_e R + \sum_n e^{\pm \lambda_n z/r_t} b_n B_0(\lambda_n R) \tag{57}$$

The vorticity components are then given by Eq. (54) and

$$r\xi = -a_2(c_r c_m / c_\theta) \tag{58}$$

$$r\zeta = -a_2(c_z c_m / c_\theta) \tag{59}$$

We now have a second full solution, with, as before, three constants to determine the velocity distribution at infinity plus an additional constant a_1, since variation in stagnation conditions is permitted. The perturbation effects may be treated in exactly the same way as in the first case, the values of λ depending only on the hub: tip ratio and the values of b being fixed by matching at the actuator disc.

In matching two flows of this kind, the axial velocities at the disc will be exactly halfway between the infinity values at the same radii for reasons similar to those given in discussing the case of $\mu_u = \mu_d$ in the last section. This might be explained physically by considering the flow to be caused by sheets of vortices shed by the blades that constitute the disc. Only the tangential component or ring component of these vortices will affect the axial velocity at the disc, and, since the strength of the vortex ring component at a given radius is the same everywhere, the net effect at the disc must be exactly half the effect at infinity.

For constant stagnation enthalpy throughout the flow, $a_1 = 0$, and thus Eq. (54) reduces to

$$\eta r = -a_2 c_m$$

which is constant. This is a limiting case of the $H =$ constant flows mentioned in the first section of the paper.

Types of flow with $rc_\theta =$ constant are obtained when $a_2 = 0$; then Eq. (54) becomes

$$\eta / r = a_1 c_m / r_t^2$$

which is constant, and

$$c_{z\infty}/c_m = a_5 - (1/2) a_1 R^2 \tag{60}$$

representing a limiting case of the flows with $rc_\theta =$ constant and η/r constant along a stream line discussed earlier.

Although the tangential velocity distribution is the same as free vortex in this flow, the axial velocity is not the same because of the variation in stagnation enthalpy with radius. The vorticity at a given radius is constant, and the flow consists of trailing "smoke ring" vortices traveling along the stream surfaces, but the stream lines do not follow the vortex lines.

From Eqs. (52) and (53),

$$\frac{H}{c_m^2} = \left(a_3 - \frac{a_1 a_4}{2a_2}\right) + \frac{a_1}{2a_2}\left(\frac{rc_\theta}{r_t c_m}\right)^2 \tag{61}$$

Now the change in H along a stream line passing through a rotor is proportional to the change in rc_θ. Since rc_0 does not enter linearly into Eq. (61), it is not possible to use this equation for flows both upstream and downstream of a rotor. Hence, for flow through a rotor, these solutions are restricted to cases for which $H =$ constant.

No numerical examples are shown for this second exact solution.

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