# SOLUTION KEY

1. (a) $f$ is continuous at $x_0$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \in S$ then $|f(x) - f(x_0)| < \epsilon$.

1. (b) $f$ is continuous at $x_0 \in S \iff$ for all sequences $(x_n)$ in $S$ with $x_n \to x_0$, $f(x_n) \to f(x_0)$.

1. (c) Let $(x_n)$ be a sequence in $S$ such that $x_n \to x_0$. Since $f$ and $g$ are continuous at $x_0$ we have $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. By the Limit Laws for sequences, $f(x_n) + g(x_n) \to f(x_0) + g(x_0)$. Therefore the function $f(x) + g(x)$ is continuous at $x_0$ by the proposition stated in part (b).

1. (d) Since $f$ is continuous on $[a, b]$ it attains its maximum and minimum on $[a, b]$. Hence there exists $a_1, b_1 \in [a, b]$ such that $f(a_1) \leq f(x) \leq f(b_1)$ for all $x \in [a, b]$. By the Intermediate Value Theorem, if $y \in [f(a_1), f(b_1)]$, then there exists $x \in I \subset [a, b]$ such that $f(x) = y$, where $I$ is the interval with endpoints $a_1$ and $b_1$. Therefore, the range of $f$ is equal to $[f(a_1), f(b_1)]$.

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2. (a) For all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \in S \setminus \{x_0\}$, then $|f(x) - L| < \epsilon$.

2. (b) Let $\epsilon > 0$. Let $\delta = \epsilon/2$. Then if $|x| < \delta$ and $x \in (-1, 1) \setminus \{0\}$, then $|g(x) - 1| = |2x + 1 - 1| = 2|x| < \epsilon$.

2. (c) Let $\epsilon > 0$. Since $\lim_{x \to 0} f(x) = 0$ there exists $\delta > 0$ such that if $|x| < \delta$, $x \neq 0$, then $|f(x)| < \epsilon/M$. Therefore, if $|x| < \delta$, $x \neq 0$, then $|f(x)g(x)| \leq M|f(x)| < \epsilon$. Hence $\lim_{x \to 0} f(x)g(x) = 0$.

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3. (a) $L(f, \mathscr{P}) = \sum_{i=1}^{N} m_i \Delta x_i$ and $U(f, \mathscr{P}) = \sum_{i=1}^{N} M_i \Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$.

3. (b) $L(f) = \sup \{ L(f, \mathscr{P}) \mid \text{ where } \mathscr{P} \text{ is a partition of } [a, b] \}$ and
$U(f) = \inf \{ U(f, \mathscr{P}) \mid \text{ where } \mathscr{P} \text{ is a partition of } [a, b] \}$.

3. (c) $f$ is integrable on $[a, b] \iff$ for all $\epsilon > 0$ there exists a partition $\mathscr{P}_\epsilon$ of $[a, b]$ such that $U(f, \mathscr{P}) - L(f, \mathscr{P}) < \epsilon$.

3. (d) Let $\epsilon > 0$. Choose a partition $\mathscr{P}'_\epsilon$ such that $L(f, \mathscr{P}'_\epsilon) > L(f) - \epsilon/2$. By hypothesis we may choose a partition $\mathscr{P}''_\epsilon$ such that $U(f, \mathscr{P}''_\epsilon) < L(f) + \epsilon/2$. Define a new partition $\mathscr{P}_\epsilon = \mathscr{P}'_\epsilon \cup \mathscr{P}''_\epsilon$. Then $U(f, \mathscr{P}_\epsilon) \leq U(f, \mathscr{P}''_\epsilon)$ and $L(f, \mathscr{P}'_\epsilon) \leq L(f, \mathscr{P}_\epsilon)$. Therefore

$$U(f, \mathscr{P}_\epsilon) - L(f, \mathscr{P}_\epsilon) \leq U(f, \mathscr{P}''_\epsilon) - L(f, \mathscr{P}'_\epsilon) < L(f) + \epsilon/2 - (L(f) - \epsilon/2) = \epsilon.$$

Therefore $f$ is integrable on $[a, b]$ by the result stated in part (b).

3. (e) An example of a bounded function $g\colon [0, 1] \to \mathbb{R}$ which is not integrable is the function $g$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

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4. (a) Let $f\colon [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

4. (b) Let $x_1, x_2 \in (a,b)$ and suppose without loss of generality that $x_1 < x_2$. Then $f$ is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$$

by the hypothesis on $f$. Therefore $f(x_2) - f(x_1) = 0$. Hence $f(x_1) = f(x_2)$ for all $x_1, x_2 \in (a,b)$. Therefore $f$ is a constant function.

4. (c) By part (b) above it suffices to prove that $f'(x) = 0$ for all $x$. Let $x, y \in \mathbb{R}$ and suppose that $y \neq x$. Then $\left|\frac{f(y)-f(x)}{y-x}\right| \leq \frac{|x-y|^2}{|x-y|} \leq |x-y|$ Therefore, as $y \to x$, $\left|\frac{f(y)-f(x)}{y-x}\right| \to 0$ and hence $\frac{f(y)-f(x)}{y-x} \to 0$ Therefore $f$ is differentiable at $x$ and $f'(x) = 0$.

4. (d) By FTOC Part I, $F$ is differentiable on $[a,b]$ with $F'(x) = f(x)$. By the Inverse Function Theorem, $F^{-1}$ is differentiable at $F(x)$ with derivative

$$(F^{-1})'(F(x)) = \frac{1}{F'(x)} = \frac{1}{f(x)}.$$

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5. (a) A series $\sum_{n=1}^{\infty} a_n$ is said to converge if the sequence $(s_n)$ of partial sums is convergent.

5. (b) Suppose that $a_n \geq 0$ for all $n$. Then $(s_n)$ is a monotonic sequence. Hence it converges if and only if it is bounded above.

5. (c) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $0 \leq a_n \leq b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

5. (d) We have $\ln(n+1) = \int_1^{n+1} \frac{1}{t} dt \leq (n+1-1) = n$ for all $n \geq 1$. Therefore $1/n \leq \ln(n+1)$. Since $\sum_{n=1}^{\infty} 1/n$ diverges, the Comparison Test implies that $\sum_{n=1}^{\infty} 1/\ln(n+1)$ diverges also.

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6. (a) The sequence $(f_n)_{n=1}^{\infty}$ is said to converge uniformly to $f$ on $S$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sup_{x \in S} |f_n(x) - f(x)| < \epsilon$ if $n > N$.

6. (b) If $f_n \to f$ uniformly on $S$ and $f_n$ is continuous on $S$ for all $n$, then $f$ is continuous on $S$.

6. (c) The pointwise limit of the sequence $(g_n)_{n=1}^{\infty}$ is the function $g\colon [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases}$$

However, the convergence cannot be uniform since if it were then $g$ would be continuous since each $g_n$ is continuous. But $g$ is not continuous.

6. (d) Let $\epsilon > 0$. Since each $f_n$ is continuous on $S$ and the convergence is uniform, $f$ is continuous on $S$. Therefore, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/2$. Choose $N_1 \in \mathbb{N}$ such that $n > N_1 \implies |x_n - x| < \delta$. Since $f_n \to f$ uniformly, we may choose $N_2 \in \mathbb{N}$ such that $n > N_2 \implies \sup_{x \in S} |f_n(x) - f(x)| < \epsilon/2$. Therefore, if $n > \max\{N_1, N_2\}$ then

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

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