Assignment 5, Mathematical Statistics III

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1. Suppose $X_1, X_2, \ldots, X_n$ are IID $U(0, \theta)$ RVs

   (a) Find the method of moments estimator
   Solution The method of moments estimator is defined as the solution to

   $$\bar{x} = \mu(\tilde{\theta})$$

   Where $\mu(\tilde{\theta}) = E(X)$.
   $E(X_i) = \frac{\theta}{2}$ So the method of moments estimator is:

   $$\bar{x} = \frac{\tilde{\theta}}{2}$$

   Which gives:

   $$\tilde{\theta} = 2\bar{x}$$

   As Required

   (b) Prove that the method of moments estimator is unbiased and find its variance
   Solution
   Unbiased if:

   $$b_T(\theta) = E(T) - \theta = 0$$

   I.e.

   $$\begin{aligned} b_{\tilde{\theta}}(\theta) &= E(\tilde{\theta}) - \theta \\ &= E(2\bar{x}) - \theta \\ &= 2E(\bar{x}) - \theta \\ &= 2E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \theta \\ &= 2\left(\frac{1}{n}\sum_{i=1}^{n} E(x_i)\right) - \theta \\ &= 2\left(\frac{1}{n}\sum_{i=1}^{n} \frac{\theta}{2}\right) - \theta \\ &= 2\frac{\theta}{2} - \theta = 0 \end{aligned}$$

   Variance:

   $$\begin{aligned} Var(\tilde{\theta}) &= var(2\bar{x}) \\ &= 4var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &= \frac{4}{n^2}var\left(\sum_{i=1}^{n} x_i\right) \\ &= \frac{4}{n^2}\left(\sum_{i=1}^{n} var(x_i)\right) \quad \text{IID} \\ &= \frac{4}{n^2}\left(\sum_{i=1}^{n} \frac{1}{12}\theta^2\right) \\ &= \frac{4}{n^2}\frac{n}{12}\theta^2 \\ &= \frac{\theta^2}{3n} \end{aligned}$$

   As Required

(c) Explain briefly why it is also the BLUE

Solution Since $E(\bar{x}) = \frac{\theta}{2}$, $\bar{x}$ is the BLUE for $\theta/2$, and hence $2\bar{x}$ is the BLUE for θ **As Required**

2. Suppose $X_1, \ldots, X_n$ are IID $U(0, \theta)$ RVs and let $T = \max(X_1, \ldots, X_n)$

(a) Show that the PDF of T is:

$$f(t) = \begin{cases} \frac{nt^{n-1}}{\theta^n} & \text{if } 0 < t < \theta \\ 0 & \text{otherwise} \end{cases}$$

Hint: Find the CDF and then differentiate

Solution Find the CDF: Recall the CDF of the uniform distribution $U(0, \theta)$ is:

$$F(x) = \frac{x}{\theta}$$

The maximum is attained by at least one of the variables, i.e.

$$\begin{aligned} F(t) = P(T \leq t) &= P(max(X_1, X_2, \ldots, X_n) \leq t) \\ &= P(X_1 \leq t, X_2 \leq t, \ldots, X_n \leq t) \\ &= \prod_{i=1}^{n} P(X_i \leq t) \text{ by independence} \\ &= \prod_{i=1}^{n} F_i(t) \\ &= \prod_{i=1}^{n} F(t) \text{ identically distributed} \\ &= \prod_{i=1}^{n} \frac{t}{\theta} \\ \implies F(t) &= \left(\frac{t^n}{\theta^n}\right)^n \end{aligned}$$

For $0 < t < \theta$. And 0 otherwise.
Differentiating the CDF:

$$\begin{aligned} f(t) &= \frac{d}{dt} F(t) \\ &= \frac{d}{dt}\left(\frac{t^n}{\theta^n}\right) \\ &= \frac{nt^{n-1}}{\theta^n} \end{aligned}$$

For $0 < t < \theta$. And 0 otherwise.

As Required

(b) Show that

$$E(T) = \frac{n}{n+1}\theta \text{ and } var(T) = \frac{n\theta^2}{(n+2)(n+1)^2}$$

Solution

$$\begin{aligned} E(T) &= \int_0^\theta t \frac{nt^{n-1}}{\theta^n} dt \\ &= \frac{n}{\theta^n} \int_0^\theta t^n dt \\ &= \frac{n}{\theta^n} \frac{t^{n+1}}{n+1} \Big|_0^\theta \\ &= \frac{n}{n+1}\theta - 0 = \frac{n}{n+1}\theta \end{aligned}$$

$$
\begin{aligned}
Var(T) &= E[T^2] - E[T]^2 \\
&= \int_0^\theta t^2 \frac{nt^{n-1}}{\theta^n} dt - \left(\frac{n\theta}{n+1}\right)^2 \\
&= \frac{n}{\theta^n} \int_0^\theta t^{n+1} dt - \left(\frac{n\theta}{n+1}\right)^2 \\
&= \frac{n}{\theta^n} \frac{t^{n+2}}{n+2} \Big|_0^\theta - \left(\frac{n\theta}{n+1}\right)^2 \\
&= \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} \\
&= \frac{(n+1)^2 n\theta^2 - (n+2)n^2\theta^2}{(n+2)(n+1)^2} \\
&= \frac{(n^2+2n+1)n\theta^2 - n^3\theta^2 - 2n^2\theta^2}{(n+2)(n+1)^2} \\
&= \frac{n^3\theta^2 + 2n^2\theta^2 + n\theta^2 - n^3\theta^2 - 2n^2\theta^2}{(n+2)(n+1)^2} \\
&= \frac{n\theta^2}{(n+2)(n+1)^2}
\end{aligned}
$$

As Required

(c) Find k such that kT is unbiased and compare the resulting variance to that of the method of moments estimator

Solution For kT to be unbiased:

$$
\begin{aligned}
b_{kT}(\theta) &= E(kT) - \theta = 0 \\
&= kE(T) - \theta \\
&= k\frac{n}{n+1}\theta - \theta = 0 \\
\implies k &= \frac{n+1}{n}
\end{aligned}
$$

Resulting variance:

$$
\begin{aligned}
Var(kT) &= k^2 Var(T) \\
&= \left(\frac{n+1}{n}\right)^2 \frac{n\theta^2}{(n+2)(n+1)^2} \\
&= \frac{\theta^2}{n(n+2)}
\end{aligned}
$$

Method of moments estimator: Using (1b) $\tilde{\theta} = 2\bar{x}$ and $var(\tilde{\theta}) = \frac{\theta^2}{3n}$ For $n > 1$, $Var(kT) < var(\tilde{\theta})$. So kT is a better estimator (not linear though). **As Required**

(d) Calculate the Cramér-Rao lower bound for an estimate of θ **or** explain why it is not possible

Solution Lower bound relies on regularity conditions which do not hold here, as T is nonlinear. You can calculate it but it the bound will not have any meaning, as shown:
Fisher information for this case:

$$
\begin{aligned}
\mathcal{I}(\theta) &= E\left(\sum_{i=1}^n \frac{\partial}{\partial\theta} \log f(x,\theta))^2\right) \\
&= E\left[n\left(\frac{\partial}{\partial\theta}\log\frac{1}{\theta}\right)^2\right] \\
&= E\left[n\left(\frac{\partial}{\partial\theta}\log\frac{1}{\theta}\right)^2\right] \\
&= E\left[n\left(-\frac{1}{\theta}\right)^2\right] \\
&= \frac{n}{\theta^2}
\end{aligned}
$$

So the Cramér-Rao lower bound would be

$$\frac{\theta^2}{n}$$

But the estimator found in c clearly had smaller variance $\forall n$. So one of the regularity conditions mustn't hold.
As Required

3. Suppose $X \sim B(n, \theta)$ and consider the hypotheses

$$H_0 : \theta = \theta_0 \text{ vs } H_A : \theta = \theta_A$$

For constants $\theta < \theta_0 < \theta_A < 1$

(a) Show that the most powerful test is to reject H_0 for $x \geq c$
Hint use the fact that $\log \frac{\theta}{1-\theta}$ is an increasing function of θ
Solution Since $B \sim B(n, \theta)$

$$p(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

Neymann-Pearson Lemma: For the test as given,

$$\text{reject } H_0 \text{ for } \frac{f(\mathbf{x}; \theta_0)}{f(\mathbf{x}; \theta_A)} \leq k$$

$$\begin{aligned}
\frac{p(\mathbf{x}; \theta_0)}{p(\mathbf{x}; \theta_a)} &\leq k \\
\frac{\binom{n}{x}\theta_0^x (1-\theta_0)^{n-x}}{\binom{n}{x}\theta_A^x (1-\theta_A)^{n-x}} &\leq k \\
\frac{\theta_0^x (1-\theta_0)^{n-x}}{\theta_A^x (1-\theta_A)^{n-x}} &\leq k
\end{aligned}$$

Take log of both sides

$$\begin{aligned}
\log\left(\frac{\theta_0^x (1-\theta_0)^{n-x}}{\theta_A^x (1-\theta_A)^{n-x}}\right) &\leq \log k \\
\log \theta_0^x (1-\theta_0)^{n-x} - \log \theta_A^x (1-\theta_A)^{n-x} &\leq \log k \\
x \log \theta_0 + (n-x)\log(1-\theta_0) - (x \log \theta_A + (n-x)\log(1-\theta_A)) &\leq \log k \\
x \log \theta_0 + n \log(1-\theta_0) - x\log(1-\theta_0) - (x \log \theta_A + n\log(1-\theta_A) - x \log(1-\theta_A)) &\leq \log k \\
x \log \frac{\theta_0}{1-\theta_0} + n \log \frac{1-\theta_0}{1-\theta_A} - x \log \frac{\theta_A}{1-\theta_A} &\leq \log k \\
x \log \frac{\theta_0}{1-\theta_0} - x \log \frac{\theta_A}{1-\theta_A} &\leq \log k - n \log \frac{1-\theta_0}{1-\theta_A} \\
x \left(\log \frac{\theta_0}{1-\theta_0} - \log \frac{\theta_A}{1-\theta_A}\right) &\leq \log k - n \log \frac{1-\theta_0}{1-\theta_A} \\
x &\geq \frac{\log k - n \log \frac{1-\theta_0}{1-\theta_A}}{\left(\log \frac{\theta_0}{1-\theta_0} - \log \frac{\theta_A}{1-\theta_A}\right)}
\end{aligned}$$

Let $c = \frac{\log k - n \log \frac{1-\theta_0}{1-\theta_A}}{\left(\log \frac{\theta_0}{1-\theta} - \log \frac{\theta_A}{1-\theta_A}\right)}$
So reject H_0 for

$$x \geq c$$

As Required

(b) Explain how c can be determined to achieve significance level α
Hint: what is the distribution of X under H_0?
Solution To determine $\alpha = P(\text{reject } H_0 | H_0 \text{ true})$ We calculate $P(x \geq c | \theta = \theta_0) \leq \alpha$

$$\begin{aligned}
P(x \geq c | \theta = \theta_0) &\leq \alpha \\
&\leq \alpha
\end{aligned}$$

As Required

(c) Evaluate c for $n = 100$, $\theta_0 = 0.25$ and $\alpha = 0.05$
Solution
As Required

(d) Find the power of the test if $\theta_a = 0.4$
Solution
As Required

4. Suppose $X_1, \ldots, X_n$ are IID $N(0, \theta)$ RVs where θ denotes the variance of the normal distribution

   (a) Show that $E(X_i^2) = \theta$ and $var(X_i^2) = 2\theta^2$

   Solution Can convert X_i to standard normal: $X_i^* = \frac{X_i}{\sqrt{\theta}}$.
   This gives $X_i^{*2} = \frac{X_i}{\theta}$. Which will have chi-squared distribution.
   Since $\frac{X_i^2}{\theta} \sim \chi_1^2$, We have that:

$$E(X_i^2) = E(\theta\chi_1^2) = \theta E(\chi_1^2) = \theta$$

   For the same reason, we get

$$var(X_i^2) = var(\theta\chi_1^2) = \theta^2 var(\chi_1^2) = 2\theta^2$$

   As Required

   (b) Find the log-likelihood function, score and fisher information

   Solution Since the $X_i's$ are independent:

$$\begin{aligned}
\ell(\theta; \mathbf{x}) &= \sum_{i=1}^{n} \log f(x_i; \theta) \\
&= \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x_i^2}{2\theta}} \\
&= \sum_{i=1}^{n} \left(\log e^{-\frac{x_i^2}{2\theta}} - \log \sqrt{2\pi\theta} \right) \\
&= \sum_{i=1}^{n} \left(-\frac{x_i^2}{2\theta} - \frac{\log 2\pi\theta}{2} \right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}(\theta; \mathbf{x}) &= \frac{\partial \ell}{\partial \theta} \\
&= \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \left(-\frac{x_i^2}{2\theta} - \frac{\log 2\pi\theta}{2} \right) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \left(-\frac{x_i^2}{2\theta} - \frac{\log 2\pi + \log\theta}{2} \right) \\
&= \sum_{i=1}^{n} \left(\frac{x_i^2}{2\theta^2} - \frac{1}{2\theta} \right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(\theta) &= E\left(-\frac{\partial^2 \ell}{\partial \theta^2} \right) \\
&= E\left(-\frac{\partial}{\partial \theta} \sum_{i=1}^{n} \left(\frac{x_i^2}{2\theta^2} - \frac{1}{2\theta} \right) \right) \\
&= E\left(-\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \left(\frac{x_i^2}{2\theta^2} - \frac{1}{2\theta} \right) \right) \\
&= E\left(-\sum_{i=1}^{n} \left(-\frac{x_i^2}{\theta^3} + \frac{1}{2\theta^2} \right) \right) \\
&= \sum_{i=1}^{n} \left(E\left(\frac{x_i^2}{\theta^3} - \frac{1}{2\theta^2} \right) \right) \\
&= \sum_{i=1}^{n} \left(\frac{E(x_i^2)}{\theta^3} - \frac{1}{2\theta^2} \right) \\
&= \sum_{i=1}^{n} \left(\frac{\theta}{\theta^3} - \frac{1}{2\theta^2} \right) \\
&= n \left(\frac{1}{2\theta^2} \right)
\end{aligned}$$

   As Required

(c) Find the MLE $\hat{\theta}$

Solution MLE is the solution

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \ell(\theta; x)$$

I.e. the theta for which $\mathcal{U}(\hat{\theta}; x) = 0$

$$\begin{aligned}
\mathcal{U}(\theta; \mathbf{x}) = 0 &= \sum_{i=1}^{n} \left(\frac{x_i^2}{2\theta^2} - \frac{1}{2\theta} \right) \\
&= \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} - \frac{n}{2\theta} \\
\frac{n}{\theta} &= \frac{\sum_{i=1}^{n} x_i^2}{\theta^2} \\
\implies \hat{\theta} &= \frac{\sum_{i=1}^{n} x_i^2}{n} \\
&= [\bar{x^2}]
\end{aligned}$$

As Required

(d) Prove that $\hat{\theta}$ is the minimum variance unbiased estimator for θ

Solution Is the MVUE if it attains the Cramér-Rao lower bound I.e.

$$\begin{aligned}
var(\hat{\theta}) &= \frac{1}{\mathcal{I}(\theta)} \\
var(\frac{\sum_{i=1}^{n} x_i^2}{n}) &= \frac{1}{n^2} \sum_{i=1}^{n} var(x_i^2) \text{ indep} \\
&= \frac{1}{n^2} \sum_{i=1}^{n} 2\theta^2 \\
&= \frac{2\theta^2}{n}
\end{aligned}$$

Which is $\frac{1}{\mathcal{I}(\theta)}$ from (b). I.e. $\hat{\theta}$ is the MVUE **As Required**

(e) Find the score statistic for the hypothesis

$$H_0 : \theta = \theta_0$$

Solution

$$\begin{aligned}
U &= \frac{\mathcal{U}(\theta_0, \mathbf{x})}{\sqrt{\mathcal{I}(\theta_0)}} \\
&= \frac{\sum_{i=1}^{n} \left(\frac{x_i^2}{2\theta_0^2} - \frac{1}{2\theta_0} \right)}{\sqrt{\frac{n}{2\theta_0^2}}} \\
&= \frac{\frac{\sum_{i=1}^{n} x_i^2}{2\theta_0^2} - \frac{n}{2\theta_0}}{\frac{\sqrt{n}}{\theta_0\sqrt{2}}} \\
&= \frac{\theta_0\sqrt{2}\frac{-n\theta_0 + \sum_{i=1}^{n} x_i^2}{2\theta_0^2}}{\sqrt{n}} \\
&= \frac{-n\theta_0 + \sum_{i=1}^{n} x_i^2}{\theta_0\sqrt{2}\sqrt{n}} \\
&= \frac{-n\theta_0 + n\hat{\theta}}{\theta_0\sqrt{2}\sqrt{n}} \\
&= \frac{\sqrt{n}(\hat{\theta} - \theta_0)}{\theta_0\sqrt{2}}
\end{aligned}$$

And accept for

$$\frac{\sqrt{n}(\hat{\theta} - \theta_0)}{\theta_0\sqrt{2}} > Z_{\alpha/2}$$

As Required

(f) Explain how the score test statistic could be modified to produce the Wald test statistic in this case

Solution The Wald test has form

$$W = \sqrt{\mathcal{I}(\hat{\theta})}(\hat{\theta} - \theta_0)$$

$$
\begin{aligned}
W &= \sqrt{\frac{n}{2\hat{\theta}^2}}\left(\hat{\theta}-\theta_0\right) \\
&= \frac{\sqrt{n}\left(\hat{\theta}-\theta_0\right)}{\hat{\theta}\sqrt{2}}
\end{aligned}
$$

The two would be equivalent if for the score test we took $\mathcal{I}(\hat{\theta})$ instead of $\mathcal{I}(\theta_0)$.
As Required

5. Consider two coins C_1 and C_2 such that

$$P(\text{Head}|C_1) = 0.5 \text{ and } P(\text{Head}|C_2) = 0.4$$

Suppose one of the two coins is selected at random and tossed repeatedly

(a) If the first two tosses are both tails, find the conditional probability that C_2 was chosen
Solution

$$
\begin{aligned}
P(C_2|T_1 = T_2 = Tail) &= \frac{P(T_1 = T_2 = Tail|C_2)P(C_2)}{P(T_1 = T_2 = Tail)} \\
&= \frac{(0.6)^2 0.5}{P(T_1 = T_2 = Tail|C_1)P(C_1) + P(T_1 = T_2 = Tail|C_2)P(C_2)} \\
&= \frac{(0.6)^2 0.5}{((0.5)^3) + (0.6)^2(0.5)} \\
&= \frac{(0.6)^2}{(0.5)^2 + (0.6)^2} \\
&\approx 0.590
\end{aligned}
$$

As Required

(b) If the first two tosses are both tails, what is the expected number of heads to occur in the following 10 tosses
Solution Note that $P(C_1|T_1 = T_2 = Tail) = 1 - P(C_2|T_1 = T_2 = Tail)$

$$
\begin{aligned}
E(\text{Heads}) &= (P(C_2|T_1 = T_2 = tail) * P(Head|C_2)) + (P(C_1|T_1 = T_2 = tail) * P(Head|C_1)) \\
&= (P(C_2|T_1 = T_2 = tail) * P(Head|C_2)) + ((1 - P(C_2|T_1 = T_2 = tail)) * P(Head|C_1)) \\
&= (P(C_2|T_1 = T_2 = tail) * 0.4) + ((1 - P(C_2|T_1 = T_2 = tail)) * 0.5) \\
&\approx (0.590 * 0.4) + (1 - 0.590) * 0.5 \\
&= 0.441
\end{aligned}
$$

Using this:

$$
\begin{aligned}
E(\text{Heads in 10 tosses}) &= 10 * E(\text{Heads}) \\
&\approx 10 * 0.441 \\
&= 4.41
\end{aligned}
$$

As Required

6. *Placenta Previa* is an unusual condition in pregnancy in which the placenta is implanted very low in the uterus, obstructing normal delivery of the baby. In an early study of 980 *placenta previa* births, $X = 437$ were female. The purpose of this question is to assess the evidence that the proportion of females amongst *placenta previa* births, θ is less than the value 0.485 derived from the general population.

(a) The prior distribution for θ will be a Beta distribution. If $\alpha + \beta = 50$, find α and β for which the prior expectation satisfies $E(\theta) = 0.485$ and obtain a plot of the prior density
Solution Want to find beta distribution such that

$$E(\theta|x) = 0.485$$

Mean of the beta distribution:

$$
\begin{aligned}
E(\theta|x) = \frac{\alpha}{\alpha+\beta} &= 0.485 \\
\frac{\alpha}{50} &= 0.485 \\
\alpha &= 24.25
\end{aligned}
$$

And

$$\beta = 50 - 24.25 = 25.75$$

Plot of the prior density is figure 1 **As Required**

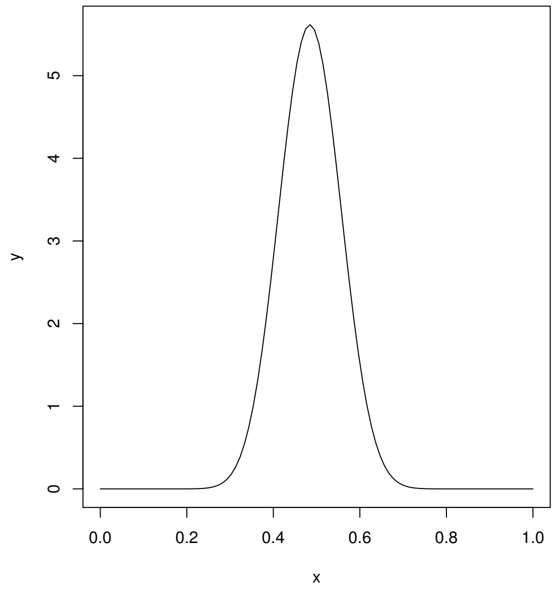

Figure 1: Plot of Beta prior density

(b) State the posterior distribution for θ and obtain a plot of its density

Solution From the lectures, $X|\theta \sim B(n, \theta)$ with the prior $\theta \sim \text{Beta}(\alpha, \beta)$ Gives the posterior:

$$\theta|x \sim \text{Beta}(\alpha + x, \beta + n - x)$$

So in this case:

$$\theta|x \sim \text{Beta}(24.25 + 437, 25.75 + 980 - 437)$$

Gives:

$$\theta|x \sim \text{Beta}(461.25, 568.75)$$

As Required

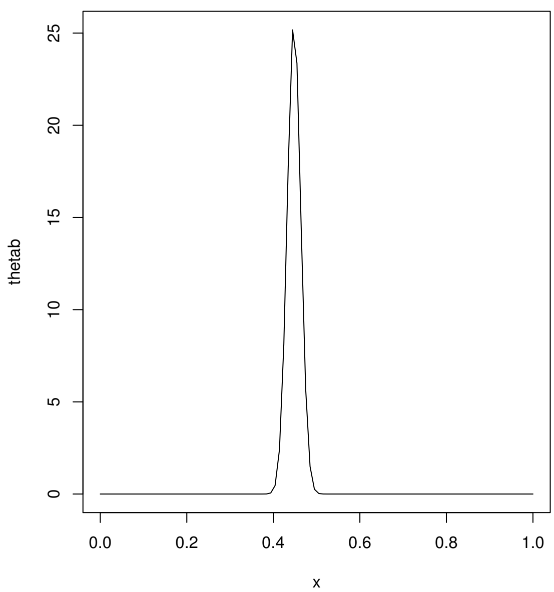

Figure 2: Plot of Beta posterior density

(c) Calculate the posterior probability $P(\theta < 0.485|X = 437)$ and state your conclusion

Solution Using R gives:

$$P(\theta < 0.485|X = 437) = 0.4471989$$

This gives us no indication of whether or not $\theta < 0.485$. I.e. we assume θ is still the same as that derived from the general population. **As Required**

The code used for question 6 is:

```
#6a
x=seq(0,1,length=100)
thetaa=dbeta(x,24.25,25.75)
plot(x,thetaa, type="l")
#6b
thetab=dbeta(x,461.25, 568.75)
plot(x,thetab, type="l")
#6c
ptheta = qbeta(0.485,461.25, 568.75)
```

Honours

7. For the estimator of the form kT in 2c, find the value of k that gives the minimum mean squared error. Compare the performance of this estimator to the unbiased estimator.

Solution

The MSE of the estimator T of θ is defined by:

$$MSE_{Tk}(\theta) = E\left((Tk-\theta)^2\right)$$

$$\begin{aligned}
MSE_{Tk}(\theta) &= E\left((Tk-\theta)^2\right) \\
&= E(T^2k^2 - 2Tk\theta + \theta^2) \\
&= E(T^2k^2) - E(2Tk\theta) + E(\theta^2) \\
&= k^2E(T^2) - 2k\theta E(T) + \theta^2 \\
&= k^2E(T^2) - 2k\theta\frac{n\theta}{n+1} + \theta^2 \\
&= k^2\frac{n\theta^2}{n+2} - 2k\theta\frac{n\theta}{n+1} + \theta^2 \text{ using variance calc in 2b} \\
&= k^2\frac{n\theta^2}{n+2} - \frac{2kn\theta^2}{n+1} + \theta^2 \text{ using variance calc in 2b}
\end{aligned}$$

Want to find k which minimises MSE:

$$\begin{aligned}
\frac{\partial}{\partial k}MSE_{Tk}(\theta) = 2k\frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1} &= 0 \\
2k\frac{n\theta^2}{n+2} &= \frac{2n\theta^2}{n+1} \\
k &= \frac{\frac{2n\theta^2}{n+1}}{\frac{2n\theta^2}{n+2}} \\
&= \frac{n+2}{n+1}
\end{aligned}$$

This k clearly $\neq$ the k from 2c. Although as $n \to \infty$ they both approach 1.

$$\begin{aligned}
b_{kT}(\theta) &= kE(T) - \theta \\
&= \frac{n+2}{n+1}\left(\frac{n\theta}{n+1}\right) - \theta \\
&= \frac{n\theta(n+2)}{(n+1)^2} - \theta \neq 0
\end{aligned}$$

So some bias is introduced.

As Required

8. Consider data $X_1, \ldots, X_n$ IID Exp(λ) and the prior dist

$$\lambda \sim Gamma(\alpha, \beta)$$

(a) Find the posterior distribution $p(\lambda|\mathbf{x})$

Solution First, the joint PDF of $\mathbf{X}|\lambda$, since the X_i are IID:

$$\begin{aligned}
p(\mathbf{x}|\lambda) &= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \\
&= \lambda^n \exp\left(-\lambda\sum_{i=1}^{n} x_i\right) \\
&= \lambda^n e^{-\lambda n\bar{x}}
\end{aligned}$$

(given in notes as genesis for gamma dist)

$$\begin{aligned}
p(\lambda|\mathbf{x}) &\propto p(\lambda)p(\mathbf{x}|\lambda) \\
&\propto \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\lambda^n e^{-\lambda n\bar{x}} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{n+\alpha-1}e^{-\lambda(\beta+n\bar{x})}
\end{aligned}$$

Need to find constant such that it integrates to 1

$$\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{n+\alpha-1}e^{-\lambda(\beta+n\bar{x})}d\lambda$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \lambda^{n+\alpha-1}e^{-\lambda(\beta+n\bar{x})}d\lambda$$

As Required

(b) Find the posterior mean for λ

Solution

$$E(\lambda|\mathbf{x}) =$$

As Required

(c) Describe the behaviour of the posterior mean when n is large relative to α and $n\bar{x}$ is large relative to β

Solution

As Required

(d) Describe the behaviour of the posterior mean when n is small relative to α and $n\bar{x}$ is small relative to β

Solution

As Required

(e) Interpret the two cases for the posterior mean in $8c$ and $8d$

Solution

As Required