Modelling with ODEs Assignment 4

Andrew Martin

June 2, 2019

1. (a) The IVP

$$\frac{dy}{dx} = x - y + 1, \quad y(1) = 2$$

Expressed as an integral equation:

$$\begin{aligned} \frac{dy}{dx} &= x - y + 1, \quad y(1) = 2 \\ \int_1^x \frac{dy}{ds} ds &= \int_1^x s - y(s) + 1 ds \\ y(x) - y(1) &= \int_1^x s - y + 1 ds \\ y(x) &= 2 + \int_1^x s - y + 1 ds \end{aligned}$$

The iterates returned by `MATLAB` (including the initial condition):

```
yp =

     2


yp =

(x - 1)^2/2 + 2


yp =

2 - ((x - 1)^2*(x - 2))/2


yp =

((x - 1)^2*(x^2 - 3*x + 3))/2 + 2
```

(b) This is a linear-inhomogeneous ODE. Solution to the homogeneous analogue:

$$\begin{aligned} \frac{dy_h}{dx} &= -y_h \\ \implies y_h &= ae^{-x} \end{aligned}$$

Using the method of undetermined coefficients, guess

$$y = y_h + bx + c$$

$$\begin{aligned} \frac{dy}{dx} &= x - y + 1 \\ -ae^{-x} + b &= x - ae^{-x} - bx - c + 1 \\ b &= x - bx - c + 1 \end{aligned}$$

$$\begin{aligned} b &= 1 \\ b + c &= 1 \\ \implies c &= 0 \end{aligned}$$

Hence

$$y = ae^{-x} + x$$

Applying the initial condition $y(1) = 2$ gives

$$\begin{aligned} y(1) = 1 &= ae^{-1} + 1 \\ a &= e \end{aligned}$$

Hence

$$y = e^{1-x} + x$$

`Matlab` gives the series:

```
- (exp(1)*x^5)/120 + (exp(1)*x^4)/24 - (exp(1)*x^3)/6 + ...
(exp(1)*x^2)/2 + (1 - exp(1))*x + exp(1)
```

Equivalently:

$$\begin{aligned} &e\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}\right) + x \\ &= x + e\sum_{n=0}^{5} \frac{(-x)^n}{n!} \\ &\approx x + e^1 e^{-x} \end{aligned}$$

Which matches the analytic solution. Comparing to the Picard iteration solution:

$$\frac{(x-1)^2(x^2 - 3x + 3)}{2} + 2$$

This quite clearly does not match the analytic or Taylor solutions.
Figure 1 plots the two solutions against each other for $0 \leq x \leq 5$. When x is small they look similar, but as x increases past 2 the values start to diverge. This is related to the $(x-1)^2$ term in the Picard solution, which grows when $|x - 1| > 1$. I.e. when $x > 2$ or $x < 0$.

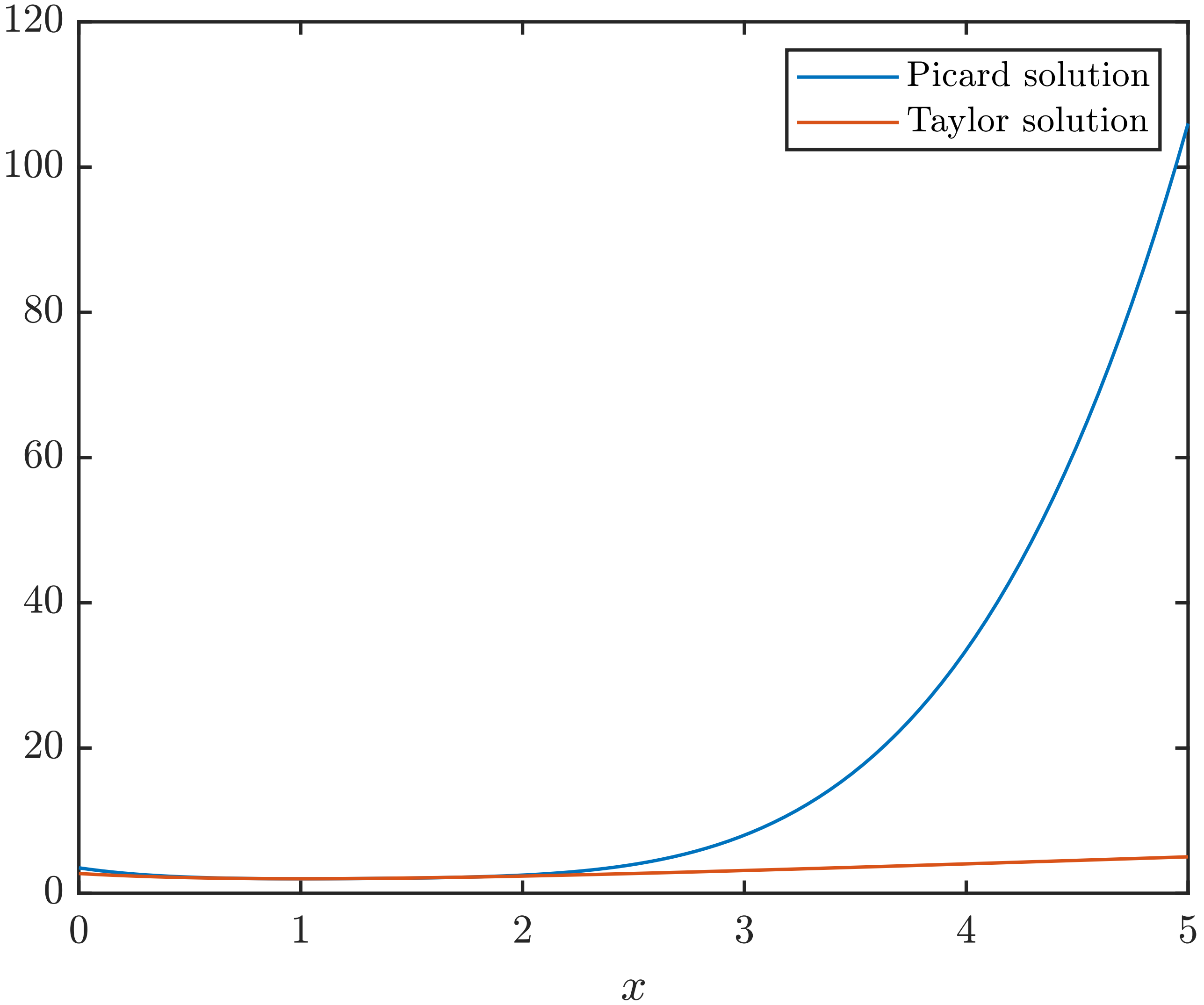

Figure 1: Comparison of Picard and Taylor solutions for the given IVP

(c) To show that the Picard-Lindelöf (PL) theorem holds, first show that the equation is Lipschitz continuous.

We know that continuous differentiability is a stronger requirement than Lipschitz continuity, Clearly the linear equation $x - y + 1$ is continuously differentiable, and hence the function is Lipschitz continuous.

Using Remark 4.4, the function is Lipschitz continuous on J if there exists L such that

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad \forall\, x, y \in J, \quad T \in I$$

If in the interval

$$I = [x_0 - \alpha, x_0 + \alpha] = [1 - \alpha, 1 + \alpha]$$
$$J = [y_0 - \delta, y_0 + \delta] = [2 - \delta, 2 + \delta]$$

Define M as

$$\begin{aligned}M = ||f|| &= \sup_{I\times J}|f| \\ &= \sup_{I\times J}|x-y+1| \\ &= \max\{|1-\alpha-2-\delta+1|, |1+\alpha-2+\delta+1|\} \\ &= \max\{|-\alpha-\delta|, |\alpha+\delta|\} \\ &= \alpha+\delta\end{aligned}$$

Since linear functions are maximised at an end point.
Hence

$$\frac{\delta}{M} = \frac{\delta}{\alpha+\delta}$$

With the limit:

$$\lim_{\delta\to\infty}\frac{\delta}{\alpha+\delta} = 1$$

$$\begin{aligned}\epsilon &= \min\{\alpha, \delta/M\} \\ &= \min\{\alpha, 1\} \\ &= 1\end{aligned}$$

By setting $\alpha \geq 1$
Hence the valid regions, I, J are

$$I = [1-\alpha, 1+\alpha] = [0, 2], \quad J = (-\infty, \infty)$$

And the maximum length of time we can guarantee a unique solution after the initial time is $\epsilon = 1$.

2.

$$x_j''' = \sum_{i=0}^{3} a_i x_{j+i} + \mathcal{O}(h^m)$$

(a) To calculate the coefficients, a_i first expand the sum:

$$x_j''' = a_0x_j + a_1x_{j+1} + a_2x_{j+2} + a_3x_{j+3} + \mathcal{O}(h^m)$$

Now using Taylor series on the x_{j+i} terms:

$$x_{j+n} = \sum_{k=0}^{\infty}\frac{(nh)^k}{k!}x_j^{(k)}$$

Where $x^{(k)}$ is the k^{th} derivative of x.

$$x_j''' = a_0x_j + a_1\left(\sum_{k=0}^{\infty}\frac{(h)^k}{k!}x_j^{(k)}\right) + a_2\left(\sum_{k=0}^{\infty}\frac{(2h)^k}{k!}x_j^{(k)}\right) + a_3\left(\sum_{k=0}^{\infty}\frac{(3h)^k}{k!}x_j^{(k)}\right)$$

$$\begin{aligned}&= x_j\left(a_0+a_1+a_2+a_3\right) \\ &+ x_j'h\left(a_1+2a_2+3a_3\right) \\ &+ x_j''h^2\left(\frac{1}{2}a_1+\frac{4}{2}a_2+\frac{9}{2}a_3\right) \\ &+ x_j'''h^3\left(\frac{1}{6}a_1+\frac{8}{6}a_2+\frac{27}{6}a_3\right)\end{aligned}$$

We require the coefficients of $x_j = x'_j = x''_j = 0$, and $x'''_j = 1$ to balance the LHS and RHS of the equation.

$$\begin{aligned} x_j &: \ a_0 + a_1 + a_2 + a_3 = 0 \\ x'_j &: \ a_1 + 2a_2 + 3a_3 = 0 \\ x''_j &: \ \frac{1}{2}a_2 + \frac{4}{2}a_2 + \frac{9}{2}a_3 = 0 \\ x'''_j &: \ \frac{1}{6}a_1 + \frac{8}{6}a_2 + \frac{27}{6}a_3 = 1 \end{aligned}$$

Write as a matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & \frac{1}{2} & \frac{4}{2} & \frac{9}{2} \\ 0 & \frac{1}{6} & \frac{8}{6} & \frac{27}{6} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{h^3} \end{pmatrix}$$

Solved using `MATLAB` for simplicity, giving:

```
aVals =

 -1/h^3
  3/h^3
 -3/h^3
  1/h^3
```

Corresponding to

$$\begin{aligned} a_0 &= \frac{-1}{h^3} \\ a_1 &= \frac{3}{h^3} \\ a_2 &= \frac{-3}{h^3} \\ a_3 &= \frac{1}{h^3} \end{aligned}$$

I.e. the equation is

$$x'''_j = \frac{1}{h^3}\left(-x_j + 3x_{j+1} - 3x_{j+2} + x_{j+3}\right)$$

(b) Consider the next term for all of the Taylor series (corresponding to $k = 4$):

$$\begin{aligned} &\frac{h^4}{24}a_1 + \frac{16h^4}{24}a_2 + \frac{81h^4}{24}a_3 \\ &= \frac{3h}{24} - \frac{3 * 16h}{24} + \frac{81h}{24} \\ &= \frac{h(3 - 48 + 81)}{24} = \frac{36h}{24} \end{aligned}$$

Hence $m = 1$ and the error is $\mathcal{O}(h)$.

(c) For a constant function, x:

$$x''' = 0$$

Hence

$$\sum_{i=0}^{3} a_i = 0$$

I.e.

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 0 \\ \frac{-1+3-3+1}{h^3} &= 0 \end{aligned}$$

Clearly this is true. This was also verified to be true in the `MATLAB` code.

3.

$$x' = f(t, x), \quad x(0) = 1$$

With leapfrog method

$$x_{n+1} = x_{n-1} + 2hf_n$$

(a) The local error of the explicit Euler method is $\ell_e(h) = \mathcal{O}(h^2)$, whereas the error for the leapfrog method $\ell_l(h) = \mathcal{O}(h^3)$. The global error of the leapfrog method is normally $g_l(h) = \mathcal{O}(h^2)$. Locally we introduce a lower order error $\mathcal{O}(h^2)$, rather than $\mathcal{O}(h^3)$.
I.e.

$$x_1 = x_0 + hf_0 + \mathcal{O}(h^2)$$

And when we consider

$$\begin{aligned} x_3 &= x_1 + 2hf_2 + \mathcal{O}(h^3) \\ &= x_0 + hf_0 + \mathcal{O}(h^2) + 2hf_2 + \mathcal{O}(h^3) \\ &= x_0 + hf_0 + 2hf_2 + \mathcal{O}(h^2) \end{aligned}$$

Hence we now have local error of order $\mathcal{O}(h^2)$ rather than $\mathcal{O}(h^3)$.
This means the global error will become $\mathcal{O}(h)$ rather than $\mathcal{O}(h^2)$.

(b) Solve:

$$x' = -x, \quad x(0) = 1$$

Using FD gives (leapfrog method)

$$x_{n+1} = x_{n-1} - 2hx_n$$

And obtain x_1 with

$$x_1 = x_0 - hx_0 = x_0(1-h)$$

To compare the error with the real IVP, note that the solution of the ODE is

$$x = e^{-t}$$

And at $t = 1$,

$$x(1) = e^{-1}$$

Figure 2 plots the logged error against logged step size. As the power of h increases towards 1, the order of the error increases linearly, respecting the $\mathcal{O}(h)$ global error claim made earlier.

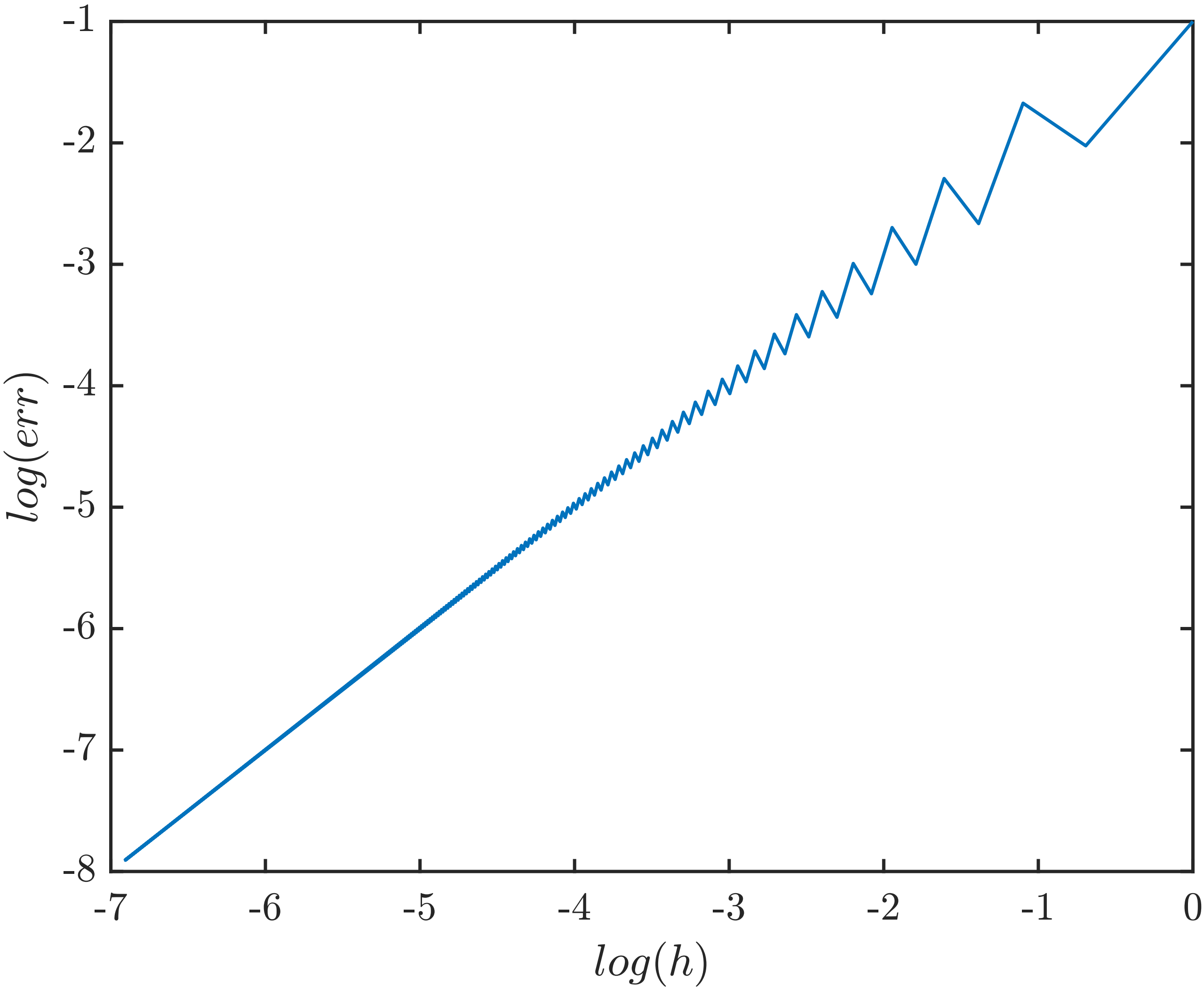

Figure 2: Comparison of the log step size and log absolute errors for the IVP at $t = 1$

(c)

$$x_n = c_+\xi_+^n + c_-\xi_-^n, \quad n = 2, \dots$$

Where

$$\xi_\pm = -h \pm \sqrt{1+h^2}$$

If $c_+ \neq 0$ looking at ξ_+^n, and assuming $h > 0$:

$$\begin{aligned} \xi_+^n &= \left(-h + \sqrt{1+h^2}\right)^n \\ \lim_{n\to\infty} \xi_+^n &= \lim_{n\to\infty} (k)^n, \quad k < 1 \\ &= 0 \end{aligned}$$

The assertion that $-h + \sqrt{1+h^2} < 1$ comes from the fact that

$$\begin{aligned} \sqrt{1+h^2} &< 1 + h \\ \implies \sqrt{1+h^2} - h &< 1 =: k \end{aligned}$$

If $c_- \neq 0$ and we focusing on ξ_-^n:

$$\begin{aligned}
\xi_-^n &= \left(-h - \sqrt{1+h^2}\right)^n \\
&= (-1)^n \left(h + \sqrt{1+h^2}\right)^n \\
&= (-1)^n (k)^n, \quad k > 1 \\
\lim_{n\to\infty} \xi_-^n &= \lim_{n\to\infty} (-1)^n (k)^n \\
&= \pm\infty
\end{aligned}$$

This divergence in value and sign does not match the long-term behaviour of the IVP:

$$\begin{aligned}
\lim_{t\to\infty} x(t) &= \lim_{t\to\infty} e^{-t} \\
&= 0
\end{aligned}$$

Where there is no variation in sign, and the solution converges.

This is shown in figure 3. The left plot shows the logged values of the analytic and numeric solutions to the IVP. It appears that for small $\log(t)$ the leapfrog solution matches the analytic solution reasonably well, but as $\log(t)$ grows, there is clear divergence.

It is worth noting that figure 3 (left) has omitted complex parts when taking the logarithm. Figure 3 (right) shows the values of x from the leapfrog method on the regular t scale. The values grow extremely quickly, and oscillate between positive and negative values.

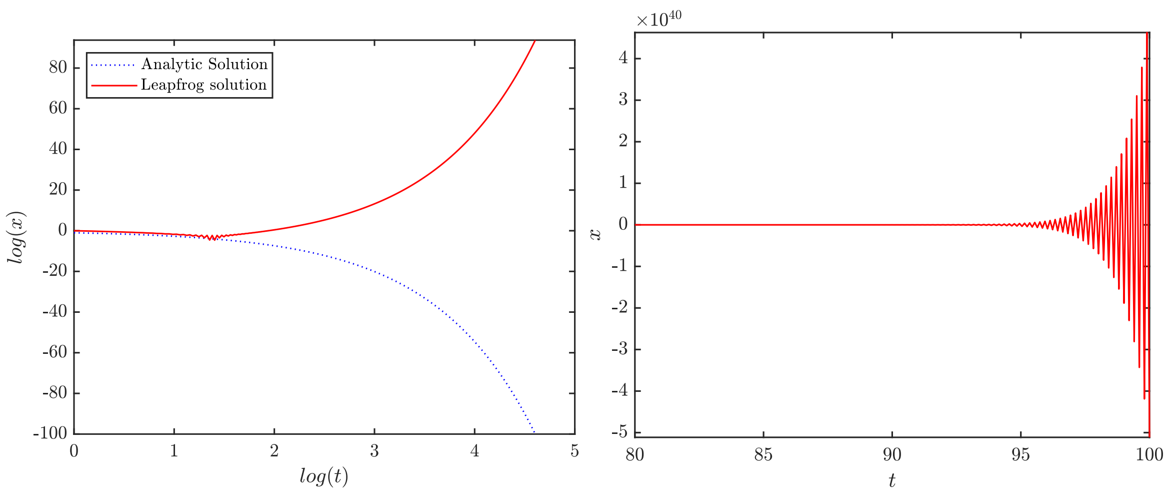

Figure 3: Long term behaviour of the Leapfrog solution compared with analytic solution. Left: logged scales comparison of solutions, right: value of the leapfrog solution

Matlab

```
close all
clear all

%Make plots less repulsive
set(groot, 'DefaultLineLineWidth', 1, ...
     'DefaultAxesLineWidth', 1, ...
     'DefaultAxesFontSize', 12, ...
     'DefaultTextFontSize', 12, ...
     'DefaultTextInterpreter', 'latex', ...
     'DefaultLegendInterpreter', 'latex', ...
     'DefaultColorbarTickLabelInterpreter', 'latex', ...
     'DefaultAxesTickLabelInterpreter','latex');


%%1a
%calculate picard iterates of the DE
%y' = x-y+1, y(1) = 2
syms x s
yp = 2
for k=1:3
    yp =2+ int(s-yp+1,s,1,x)
end


%%1b
%verify solution using sym
syms y(x)
eqn = diff(y,x,1) == x - y +1;
%solve the DE with IC y(1)=2
y = dsolve(eqn, y(1)==2)
taylor(y)

%compare taylor solution to picard

%try statement in case i've already ran this section

yp = matlabFunction(yp);

y = matlabFunction(y);
x = linspace(0,5);
plot(x,yp(x))
hold on
plot(x,y(x))
xlabel("$$x$$")
legend("Picard solution","Taylor solution")
saveas(gcf,"ODEsA4Q1b,eps",'epsc')
%%
%%2a
```

```
%solve the system Ax=b for the coefficients
%x = [a0;a1;a2;a3]
syms h
A = [1, 1,   1,   1;
     0, 1,   2,   3;
     0, 1/2, 4/2, 9/2;
     0, 1/6, 8/6, 27/6];
 b = [0;0;0;1/h^3];
aVals = A\b
sum(aVals)

%%
%%3b

vals = zeros(1,1000);
h = zeros(1,1000);
for nPts =1:1000
    x = FrogLeap(nPts,1);
    vals(nPts) = x(end);
    h(nPts) = 1/nPts;
end
err = abs(vals - exp(-1));
plot(log(h),log(err))
xlabel("$$log(h)$$")
ylabel("$$log(err)$$")
saveas(gcf,"ODEsA4Q3b.eps",'epsc')

%%
%%3c
npts = 1000;
endTime = 100;

x = FrogLeap(npts,endTime);

t = linspace(1,endTime,npts);
xAnalytic= exp(-t);
plot(log(t),log(xAnalytic),':b')
hold on
plot(log(t),log(x),'r')
axis([0,5,-inf,inf])

xlabel("$$log(t)$$")
ylabel("$$log(x)$$")
legend("Analytic Solution","Leapfrog solution",'location','Northwest')
saveas(gcf,"ODEsA4Q3c.eps",'epsc')
hold off

plot(t,x,'r')
axis([80,100,-inf,inf])

```

```
xlabel("$$t$$")
ylabel("$$x$$")
saveas(gcf,"ODEsA4Q3c2.eps",'epsc')
hold off



function x = FrogLeap(npts,endTime)

h = endTime/npts;
x = zeros(1,npts);
%matlab uses 1 based indexing so x0 == x(1)
x(1) = 1;
x(2) = (1-h)*x(1);
for n = 3:npts
    x(n) = x(n-2) - 2*h*x(n-1);
end
end
```

School of Mathematical Sciences

Modelling with ODEs

Semester 1, 2019

Assignment 4

Due 5pm Wednesday, Week 12: Submit via MyUni

You will be marked on the presentation of your answers (including clarity of explanations)!

1. Consider the IVP
$$\frac{dy}{dx} = x - y + 1 \quad \text{with} \quad y(1) = 2.$$
   (a) Express the IVP as an integral equation, and write MATLAB code to calculate the Picard iterates. Submit your code and the first three iterates it produces.
   (b) Calculate the exact solution. Using MATLAB or otherwise, find the Taylor series of the exact solution, and hence comment on the relationship between the Picard iterates and the exact solution.
   (c) Show that the Picard-Lindelöf theorem applies to the IVP, and find the largest x-interval on which it guarantees a unique solution.

2. Consider the forward difference formula for the third derivative
$$x_j''' = \sum_{i=0}^{3} a_i \, x_{j+i} + \mathrm{O}(h^m).$$
   (a) Calculate the coefficients a_i.
   (b) Calculate the order of the truncation error m.
   (c) Perform a simple check of the coefficients that ensures the finite difference formula is correct for constant functions.

3. For the IVP
$$x' = f(t, x) \quad \text{with} \quad x(0) = a,$$
   recall the leapfrog method is
$$x_{n+1} = x_{n-1} + 2\,h\,f_n.$$
   (a) Suppose you use the explicit Euler method to compute x_1. Does the use of Euler's method for the first step compromise the global error of the leapfrog method? Explain your answer.

(b) Write a Matlab code to solve the IVP

$$x' = -x \quad \text{with} \quad x(0) = 1, \tag{1}$$

using the leapfrog method with Euler's method used to find x_1. Use your code to calculate the absolute error $e(h)$ at $t = 1$ for a range of step sizes h. Plot $\log(e)$ vs. $\log(h)$, and explain how this confirms the order of accuracy of the leapfrog method.

(c) You are given that for IVPs of the form (1), solutions given by the leapfrog method can be written

$$x_n = c_+\,\xi_+^n + c_-\,\xi_-^n \quad \text{for} \quad n = 2, \ldots$$

where

$$\xi_\pm = -h \pm \sqrt{1 + h^2},$$

and $c_\pm$ are constants. Explain why this means that the leapfrog method is not suitable to investigate the long-time behaviour of the solution of IVP (1). Use your code from part (b) to confirm the problem in calculating long-term solutions of IVP (1) using the leapfrog method.