THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2015

105929 APP MTH 3020 Stochastic Decision Theory III

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 70

Instructions

- Attempt all questions.
- This examination comprises 70% of the total assessment for this course.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Formulae sheets are provided.
- Calculators without remote communications or CAS capability are allowed.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

Stochastic Linear Programming

1. A white goods manufacturer makes freezers and fridges, where due to the physical layout of the premises, there is a total production limit of 1500 units of white goods per production period. Both assembly and finishing are required resources for the manufacturing process. The requirements measured in hours per unit are known and shown in the table below along with the profit per unit of product.

Product parameters	freezers	fridges
Assembly hours	8	16
Finishing hours	6	4
profit per unit	\$150	\$250

Our problem is to select the product mix to maximise total profit when the availability of the labour resources are not known. Rather we have two equally likely realisations of the hours available for manufacture.

$$\varepsilon_1 = \begin{cases} 4950 & \text{assembly hours} \\ 3636 & \text{finishing hours} \end{cases}$$

$$\varepsilon_2 = \begin{cases} 5850 & \text{assembly hours} \\ 4064 & \text{finishing hours} \end{cases}$$

For the example, the expected time available for assembly is 5400 hours and the expected time available for finishing is 3850 hours.

(a) Write down a LP in terms of the two primal variables x_1, x_2, where x_1 is the number of freezer units and x_2 is the number of fridge units produced. This LP should be able to be used to find the maximum profit when the expected available times for assembly and finishing are used to define the problem constraints.

(b) Verify that 625 freezers and 25 fridges is a feasible solution of the original LP and find the profit.

(c) Write down the dual LP in terms of dual variables y_1, y_2, y_3.

(d) Verify that $y_1 = 14.0625, y_2 = 6.25, y_3 = 0$ is a feasible solution to the dual LP and find the value of the objective function.

(e) Giving an explanation, are these solutions optimal in each case?

[14 marks]

Please turn over for page 3

Stochastic Linear Programming

2. A solution to the averaged value LP in the previous question is not very acceptable because it does not allow for the stochastic variation of available assembly and finishing hours.

   Assume that additional assembly hours may be purchased at \$50 per hour and that extra finishing hours may be purchased at \$120 per hour.

   Also assume that any unused base assembly hours are wasted and must be costed at \$40 per hour and similarly any unused base hours of finishing must be costed at \$90 per hour.

   (a) Giving some explanation, write down the expanded version of a recourse model considering both realisations as outlined.
   Hint: You will have a $\boldsymbol{y}$ vector of length $2 \times 4 = 8$.

   (b) Define A, $\boldsymbol{b}$, $\boldsymbol{q}^T, W, T(\boldsymbol{\xi})$ and $\boldsymbol{h}(\boldsymbol{\xi})$, to rewrite the above problem in the form

$$\begin{aligned} \max \quad & z = \boldsymbol{c}^T\boldsymbol{x} - E[Q(\boldsymbol{x}, \boldsymbol{\xi})]_{\boldsymbol{\varepsilon}_i} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

   with $Q(\boldsymbol{x}, \boldsymbol{\varepsilon}_i)$ for each realisation $\boldsymbol{\varepsilon}_i$ of $\boldsymbol{\xi}$ given by

$$\begin{array}{rrcl} & \min & & \boldsymbol{q}^T\boldsymbol{y} \\ \text{s.t.} & W\boldsymbol{y} & = & \boldsymbol{h}(\boldsymbol{\varepsilon}_i) - T(\boldsymbol{\varepsilon}_i)\boldsymbol{x}, \\ & \boldsymbol{y} & \geq & \boldsymbol{0}, \end{array}$$

   where $\boldsymbol{y}^T = (y_1, y_2, y_3, y_4)$.

   (c) Giving an explanation, is W a complete recourse matrix?

   (d) Give a brief explanation of how the L-shaped algorithm is beneficial in finding a solution to these types of problem.

[14 marks]

Markov Decision Problems

3. Consider a finite state Markov Decision Process, where for some chosen policy, over a finite horizon T we have a value function given by

$$V_k(i) \quad = \quad \sum_{j=1}^{n} p_{i,j}\big(r_{i,j} + V_{k+1}(j)\big), \tag{1}$$

on states $i \in \{1, \ldots, n\}$, where $r_{i,j}$ is a reward associated with a transition from state i to state j with probability $p_{i,j}$ under this chosen policy. Assume also that for $T >> k$, we have that the process is such that there exists a positive constant g such that for each $i \in \{1, \ldots, n\}$

$$V_k(i) \quad = \quad g \times (T - k) + v_i, \tag{2}$$

where v_i is the value of starting in state i at time k.

(a) Using the above information show that for each $i \in \{1, \ldots, n\}$

$$v_i + g \quad = \quad \left(\sum_{j=1}^{n} p_{i,j} r_{i,j}\right) + \left(\sum_{j=1}^{n} p_{i,j} v_j\right).$$

(b) Explain how the equations in part (a) may be used to define the policy improvement routine for each state i.

[8 marks]

Markov Decision Problems

4. A co-op produces toys and at the end each year may be in one of two states $X \in \{1, 2\}$, where 1 is a healthy state and 2 is an unhealthy state. At the beginning of each year, they can take a decision to research the market overseas (at a cost of 4) or not such that the state of the co-op and a reward is given by the following state transition and reward matrices

$$\mathrm{P}\,(\text{research}) = \begin{pmatrix} 0.75 & 0.25 \\ 0.8 & 0.2 \end{pmatrix} \quad \text{and} \quad \mathrm{P}\,(\text{no research}) = \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$$

$$\mathrm{R}\,(\text{research}) = \begin{pmatrix} 4 & -2 \\ -2 & -8 \end{pmatrix} \quad \text{and} \quad \mathrm{R}\,(\text{no research}) = \begin{pmatrix} 8 & 2 \\ 2 & -4 \end{pmatrix}.$$

   (a) Giving explanation, write down a Bellman equation for the expected monetary value $V_0^{\Psi}(i)$ for the co-op under a policy Ψ starting from state $i \in \{1, 2\}$ at the beginning of stage $k = 0$ over a finite horizon T.

   (b) Perform a value iteration procedure on this problem to find the optimal policy and value over a time horizon of $T = 4$ years for starting in state $i \in \{1, 2\}$, assuming that there is a zero reward for finishing in either state. That is, $V_4(1) = V_4(2) = 0$.

   (c) From the value iteration you have just performed what would you propose as an optimal stationary policy if the co-op was to run indefinitely?

[14 marks]

5. The Greek adventurer Theseus is trapped in a room from which lead n passages. Theseus knows that if he enters passage i for $i \in \{1, \ldots, n\}$, one of three fates will befall him:

   - he will escape with probability p_i,
   - he will be killed with probability q_i, and
   - with probability $r_i = 1 - p_i - q_i$ he will find the passage to be a dead end and be forced to return to the room.

   The fates associated with different passages are independent. Using an interchange argument, establish the order in which Theseus should attempt the passages if he wishes to maximise his probability of eventual escape.

[8 marks]

Hidden Markov Models

6. (a) Given a model $(\boldsymbol{\theta}_p, \boldsymbol{\phi}_p, \boldsymbol{p}_0)$, and a sequence of observations $\boldsymbol{O}$ of the random variable Y, where

- $\boldsymbol{\theta}_p$ are the parameters of a homogeneous Markov chain $\{X_t | t \geq 0\}$ that describes the state transitions of a Hidden Markov chain $\mathrm{P}\left(X_t = j \middle| X_{t-1} = i\right)$.
- $\boldsymbol{\phi}_p$ are the parameters of the observation probability mass function $\mathrm{P}\left(Y_t = y \middle| X_t = i\right)$.
- $\boldsymbol{p}_0$ is the initial state distribution $\mathrm{P}\left(X_0 = i\right)$.

Describe the three classical problems associated with the above hidden Markov model.

(b) Consider that a process can be in one of two states, high (H) or low (L), initially with equal probability. Consider that state H characterises coding DNA while L characterises a non-coding DNA. DNA code is represented by a sequence of four letters A, C, G and T that encode genetic information. Assume that we know the following information.

$$\begin{aligned}
\log_2\left(\mathrm{P}\left(X_t = H \mid X_{t-1} = H\right)\right) &= -1, \quad \log_2\left(\mathrm{P}\left(X_t = L \mid X_{t-1} = H\right)\right) = -1, \\
\log_2\left(\mathrm{P}\left(X_t = H \mid X_{t-1} = L\right)\right) &= -1.3219, \\
\log_2\left(\mathrm{P}\left(X_t = L \mid X_{t-1} = L\right)\right) &= -0.7370, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\log_2\left(\mathrm{P}\left(Y_t = A \mid X_t = H\right)\right) &= -2.3219, & \log_2\left(\mathrm{P}\left(Y_t = A \mid X_t = L\right)\right) &= -1.7370, \\
\log_2\left(\mathrm{P}\left(Y_t = C \mid X_t = H\right)\right) &= -1.7370, & \log_2\left(\mathrm{P}\left(Y_t = C \mid X_t = L\right)\right) &= -2.3219, \\
\log_2\left(\mathrm{P}\left(Y_t = G \mid X_t = H\right)\right) &= -1.7370, & \log_2\left(\mathrm{P}\left(Y_t = G \mid X_t = L\right)\right) &= -2.3219, \\
\log_2\left(\mathrm{P}\left(Y_t = T \mid X_t = H\right)\right) &= -2.3219, & \log_2\left(\mathrm{P}\left(Y_t = T \mid X_t = L\right)\right) &= -1.7370.
\end{aligned}$$

Calculate the most probable state sequence and its $\log_2$ probability that corresponds to the observation sequence $GGCACTG$ using the Viterbi algorithm.

[12 marks]

	Primal (Dual)	Dual (Primal)
	$\max z = \sum_{j=1}^{n} c_j x_j + z_0$	$\min w = \sum_{i=1}^{m} y_i b_i + z_0$
1.	$\sum_{j=1}^{n} a_{ij} x_j = b_i$	y_i free
2.	$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$	$y_i \geq 0$
3.	$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$	$y_i \leq 0$
4.	$x_j \geq 0$	$\sum_{i=1}^{m} y_i a_{ij} \geq c_j$
5.	$x_j \leq 0$	$\sum_{i=1}^{m} y_i a_{ij} \leq c_j$
6.	x_j free	$\sum_{i=1}^{m} y_i a_{ij} = c_j$