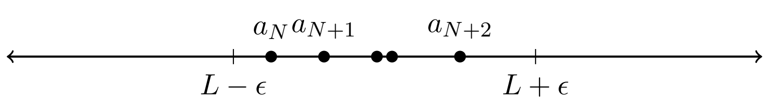
LECTURE 8

Recall from last time the definition of what it means for a sequence $(a_n)_{n=1}^{\infty}$ of real numbers to *converge* to a real number L:

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that for all } n \in \mathbb{N}, \text{ if } n \geq N, \text{ then } |a_n - L| < \epsilon.$$

We write $a_n \to L$ if $(a_n)_{n=1}^{\infty}$ converges to L. We say that a sequence $(a_n)_{n=1}^{\infty}$ converges if $a_n \to L$ for some $L \in \mathbb{R}$.

Recall that $|a_n - L| < \epsilon \iff a_n \in (L-\epsilon, L+\epsilon)$. Therefore the statement that $n \geq N \implies |a_n - L| < \epsilon$ is the same as the statement that $n \geq N \implies a_n \in (L-\epsilon, L+\epsilon)$. In turn, this is the same as the statement that all but finitely many a_n do not belong to $(L-\epsilon, L+\epsilon)$. To see this, suppose that N is the largest natural number n such that $a_n \notin (L-\epsilon, L+\epsilon)$ (we are supposing that there are only finitely many such n, therefore there is a largest one). Therefore, if $n > N$ then then $a_n \in (L-\epsilon, L+\epsilon)$. In other words $n \geq N+1 \implies a_n \in (L-\epsilon, L+\epsilon)$.



Example: Use Definition 2.1 to prove that $a_n \to 1$, where $a_n = 1 + 1/\sqrt{n}$.

The first step with any proof like this is to state 'Let $\epsilon > 0$'. So we do that:

1. Let $\epsilon > 0$.

What we need to do is to find an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - 1| < \epsilon$. This will require some investigation; what we need to do is first estimate the distance $|a_n - 1|$. Therefore we substitute for a_n and try to simplify:

2. $|a_n - 1| = |1 + 1/\sqrt{n} - 1| = |1/\sqrt{n}| = 1/\sqrt{n}$ since $1/\sqrt{n} > 0$.

The next step is to understand how big n needs to be in order for the inequality $1/\sqrt{n} < \epsilon$ to hold. So we solve this inequality for n in terms of ϵ:

3. $1/\sqrt{n} < \epsilon \iff 1/\epsilon < \sqrt{n}$ by a rearrangement of this inequality, using the fact that $\epsilon > 0$ and $\sqrt{n} > 0$. Observe that the inequality $1/\epsilon < \sqrt{n}$ holds if and only if the inequality $1/\epsilon^2 < n$. To see this, observe that since $0 < 1/\epsilon < \sqrt{n}$ we have $1/\epsilon^2 < n$. On the other hand, if $1/\epsilon^2 < n$ then we must have $1/\epsilon < \sqrt{n}$. Otherwise, if $1/\epsilon \geq \sqrt{n}$ then we have $1/\epsilon^2 \geq n$ — contradiction.

Therefore, $1/\sqrt{n} < \epsilon \iff 1/\epsilon < \sqrt{n} \iff 1/\epsilon^2 < n$.

The next step is to find a suitable N:

4. The inequality $1/\epsilon^2 < n$ is the key here: choose any natural number $N > 1/\epsilon^2$ (such an N will exist since $\mathbb{N}$ is not bounded above). Then if $n \geq N$, we have $n \geq N > 1/\epsilon^2$ so that $n > 1/\epsilon^2$ and hence $1/\sqrt{n} < \epsilon$.

We've found an N and shown that it works.

5. Therefore, since $\epsilon > 0$ was arbitrary, it follows that $a_n \to 1$.

Example: Prove that $x_n \to 3/4$ if $x_n = 3n/(4n+1)$.

1. Let $\epsilon > 0$.

2. Sometimes the algebra might be a little more intense. Here things are slightly more complicated than in the previous example:

$$\begin{aligned} \left|x_n - \frac{3}{4}\right| &= \left|\frac{3n}{4n+1} - \frac{3}{4}\right| \\ &= \left|\frac{12n - 3(4n+1)}{4(4n+1)}\right| \\ &= \left|\frac{-3}{4(4n+1)}\right| \\ &= \frac{3}{4(4n+1)} \end{aligned}$$

3. One possibility would be to solve the inequality $3/4(4n+1) < \epsilon$ for n in terms of ϵ: if we did that we would find that $(3/4\epsilon - 1)/4 < n$. As above, we would then take an N greater than $(3/4\epsilon - 1)/4$ and proceed as above.

Here's another strategy: $4n < 4n + 1$ and so

$$\left|x_n - \frac{3}{4}\right| = \frac{3}{4(4n+1)} < \frac{3}{16n} < \frac{1}{n}$$

since $3/16 < 1$. Estimating the difference $|x_n - 3/4|$ like this has made it a bit easier to choose an N; now we could choose N to be any natural number such that $N > 1/\epsilon$. Then, if $n \geq N$ then $1/n \leq 1/N < \epsilon$ and hence

$$\left|x_n - \frac{3}{4}\right| < \frac{1}{n} < \epsilon.$$

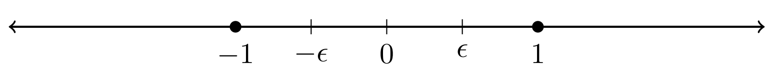
4. Therefore we have shown that there is an $N \in \mathbb{N}$ such that $n \geq N \implies |x_n - 3/4| < \epsilon$.

5. Since this is true for any $\epsilon > 0$ it follows that $x_n \to 3/4$.

Note: it is clear from these two examples that in general, when given an $\epsilon > 0$, the N that you choose will *depend* on ϵ. If ϵ is big you might be able to get away with a small N, but if ϵ is small then you might need to choose a bigger N.

Example: Let $a_n = (-1)^n$ so that (a_n) is the sequence $-1, 1, -1, 1, -1, 1, \ldots$. Prove that (a_n) does not converge to $L = 0$.

If $a_n \to 0$ then all but finitely many terms of the sequence would have to belong to $(-\epsilon, \epsilon)$ for all $\epsilon > 0$. This is clearly false as can be seen from the picture below:



For example, we may take $\epsilon = 1/2$. Then $|a_n - 0| = |(-1)^n| = 1 > 1/2$ for all n. Therefore there does not exist $N \in \mathbb{N}$ such that $|a_n - 0| < 1/2$ for all $n \geq N$.

It is important to understand what it means for a sequence (a_n) to *not converge* to a real number L.

A sequence $(a_n)_{n=1}^{\infty}$ does not converge to $L \in \mathbb{R}$ if and only if $\exists\ \epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - L| \geq \epsilon$.

Example: Prove that $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \begin{cases} \frac{1}{n}, & n \text{ even,} \\ 1 + \frac{1}{n}, & n \text{ odd} \end{cases}$$

does not converge to $L = 1$.

Let $\epsilon = 1/4$. We show that there does not exist $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - 1| < 1/4$. For, $|a_n - 1| < 1/4 \iff 3/4 < a_n < 5/4$. This inequality is not satisfied if n is even, since if n is even then $a_n \leq 1/2$.

Lemma 2.2: If $a_n \to L$ and $a_n \to M$ then $L = M$.

Proof: Suppose $L \neq M$. Then $|L - M| > 0$. Let[1] $\epsilon = |L - M|/2$.

Since $a_n \to L$ there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - L| < \epsilon$. Similarly, since $a_n \to M$ there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2 \implies |a_n - M| < \epsilon$. Therefore,

$$|L - M| \leq |L - a_n| + |a_n - M| = |a_n - L| + |a_n - M|.$$

If $n \geq \max N_1, N_2$ then $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$. Hence

$$2\epsilon = |L - M| \leq |a_n - L| + |a_n - M| < \epsilon + \epsilon = 2\epsilon.$$

But then $2\epsilon < 2\epsilon$, a contradiction. Hence $L = M$. ∎

[1] This is a standard trick that we'll use over and over in different places