LECTURE 28

We begin with a generalisation of the Mean Value Theorem, which we shall shortly need when discussing the Lagrange Remainder Theorem.

Theorem 6.22 (Cauchy Mean Value Theorem): Suppose that $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0$ for all $x \in (a,b)$ then there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{(f(b) - f(a)}{g(b) - g(a)}.$$

Notes: we make the following observations:

(a) The hypothesis that g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$ implies (by Rolle's Theorem) that $g(b) \neq g(a)$.

(b) The Cauchy Mean Value Theorem reduces to the ordinary Mean Value Theorem in the special case that $g(x) = x$ for all x.

Proof: Exercise. ∎

The Lagrange Remainder Theorem

Suppose that f is $(n+1)$-times differentiable on an open interval I containing x_0. Let $F(x) = f(x) - p_n(x)$, where $p_n(x)$ denotes the n-th Taylor polynomial for f at x_0. Then F is $(n+1)$-times differentiable on I and

$$0 = F(x_0) = F'(x_0) = F''(x_0) = \cdots = F^{(n)}(x_0).$$

The polynomial function $G(x) = (x - x_0)^{n+1}$ has these same properties:

$$0 = G(x_0) = G'(x_0) = G''(x_0) = \cdots = G^{(n)}(x_0).$$

Let $x \in I$, $x \neq x_0$. Observe that the derivative G is non-zero on the open interval with endpoints x and x_0. Therefore we may apply the Cauchy Mean Value Theorem to F and G on this interval to conclude that there exists a point x_1 between x_0 and x such that

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)} = \frac{F'(x_1)}{G'(x_1)}.$$

The function $G''(x)$ is non-zero for $x \neq x_0$ and hence we may apply the Cauchy Mean Value Theorem to F' and G' on the interval with endpoints x_0 and x_1 to conclude that there exists a point x_2 between x_0 and x_1 such that

$$\frac{F'(x_1)}{G'(x_1)} = \frac{F'(x_1) - F'(x_0)}{G'(x_1) - G'(x_0)} = \frac{F''(x_2)}{G''(x_2)}.$$

We continue in this manner to find a sequence of points $x > x_1 > x_2 > \cdots x_n > x_0$ with

$$\frac{F(x)}{G(x)} = \frac{F'(x_1)}{G'(x_1)} = \cdots = \frac{F^{(n)}(x_n)}{G^{(n)}(x_n)}.$$

Since $G^{(n)}(x)$ is non-zero on the open interval with end-points x_0 and x_n, we may apply the Cauchy Mean Value Theorem one more time to conclude that there exists x_{n+1} between x_0 and x_n such that

$$\frac{F^{(n)}(x_n)}{G^{(n)}(x_n)} = \frac{F^{(n)}(x_n) - F^{(n)}(x_0)}{G^{(n)}(x_n) - G^{(n)}(x_0)} = \frac{F^{(n+1)}(x_{n+1})}{G^{(n+1)}(x_{n+1})}$$

We have $F^{(n+1)}(x_{n+1}) = f^{(n+1)}(x_{n+1})$ and $G^{(n+1)}(x_{n+1}) = (n+1)!$. Therefore we have

$$\frac{F(x)}{G(x)} = \frac{F'(x_1)}{G'(x_1)} = \cdots = \frac{F^{(n)}(x_n)}{G^{(n)}(x_n)} = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}$$

and so

$$f(x) - p_n(x) = F(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}(x - x_0)^{n+1}$$

Putting these statements together we conclude

Theorem 6.23 (Lagrange Remainder Theorem): If f is $(n+1)$-times differentiable on an open interval I containing x_0 then for any $x \in I$ there exists a point z between x_0 and x such that

$$f(x) = p_n(x) + \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1}.$$

Returning to the example from last lecture we have

Example: Let $f(x) = 1/x$ and let $x_0 = 1$. Then if x is close to x_0 then

$$f(x) = p_n(x) + \frac{(-1)^{n+1}}{z}{}^{n+1}(x-1)^{n+1}$$

Suppose that $1 < x < 2$. Then $1 < z < x$ and so $1/z < 1$. Therefore

$$\left| \frac{(-1)^{n+1}}{z^{n+1}}(x-1)^{n+1} \right| < (x-1)^{n+1} \to 0 \text{ as } n \to \infty$$

since $0 < x - 1 < 1$.

§7 Series and Power Series

Definition 7.1: A *series* is an expression of the form $\sum_{n=1}^{\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers. If $N \in \mathbb{N}$ then the N-th *partial sum* of the sequence is defined to be

$$s_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n.$$

The series is said to *converge* if the sequence $(s_N)_{N=1}^{\infty}$ converges, i.e. if there is a real number S such that

$$\lim_{N \to \infty} s_N = S, \quad \text{or, in other words if } s_N \to S.$$

We say that S is the *sum* of the series and we write $S = \sum_{n=1}^{\infty} a_n$. If the series $\sum_{n=1}^{\infty} a_n$ does not converge we say that the series *diverges*.

Note: Sometimes we encounter series of the form $\sum_{n=k}^{\infty} a_n$ where $k = 0, 1, 2, \ldots$. The previous definitions apply with the obvious modifications in this case. For instance if $k = 0$ the sequence of partial sums is the sequence $(s_N)_{N=0}^{\infty}$ with $s_N = a_0 + a_1 + \cdots + a_N$.

Example: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The N-th partial sum of this series is

$$s_N = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{N \cdot (N+1)}.$$

Notice that for any natural number k we have

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Therefore we have

$$s_N = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{N} - \frac{1}{N+1})$$

whereupon it is clear that almost all of the terms in this sum cancel (sometimes we say that the sum 'telescopes') and we are left with

$$s_N = 1 - \frac{1}{N+1}.$$

It is now clear that $s_N \to 1$. Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Example: The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. To see this observe that we can group terms in the partial sums together as follows:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} + \cdots$$

Therefore, if $N > 2^n$ then we see that

$$s_N > 1 + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ times}} = 1 + \frac{n}{2}.$$

It is then clear that the sequence of partial sums $(s_N)_{N=1}^{\infty}$ is unbounded. Therefore the harmonic series diverges.

Example: The *geometric series* $\sum_{n=0}^{\infty} x^n$ is perhaps the most important of all series. If $x = 1$ then the N-th partial sum of the series is $S_N = 1 + x + \cdots + x^N = N + 1$. The sequence of partial sums clearly diverges to ∞ in this case. If $x = -1$ then the sequence of partial sums is $1, 0, 1, 0, 1, 0, \ldots$ which again does not converge. If $x \neq -1, 1$ then

$$s_N = 1 + x + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$

Since the sequence (x^{N+1}) converges for $|x| < 1$ and diverges for $|x| > 1$ we finally see that the geometric series $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$; it diverges for all other values of x.

Lemma 7.2 (Cauchy criterion): A series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\lim_{n \to \infty} (a_{n+1} + \cdots + a_{n+k}) = 0$$

for all $k \in \mathbb{N}$.

Proof: The series converges if and only if the sequence (s_N) converges, if and only if the sequence (s_N) is a Cauchy sequence. The sequence (s_N) is a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies |s_{N+k} - s_N| < \epsilon$ for all $k \in \mathbb{N}$. Therefore, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_{n+1} + \cdots + a_{n+k}| < \epsilon$ for all $k \in \mathbb{N}$ (since $s_{N+k} - s_N = a_{n+1} + \cdots + a_{n+k}$. In particular, this implies that for any $k \in \mathbb{N}$, $\lim_{n\to\infty}(a_{n+1} + \cdots + a_{n+k}) = 0$. ■

Corollary (Vanishing Criterion): If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof: This follows immediately from the Cauchy criterion for convergence above: take $k = 1$ to conclude that $\lim_{n\to\infty} a_{n+1} = 0$, i.e. $\lim_{n\to\infty} a_n = 0$. ■

Series with non-negative terms

Suppose that $\sum_{n=1}^{\infty} a_n$ is a series with non-negative terms, i.e. $a_n \geq 0$ for all n. Observe that the sequence of partial sums (s_N) is an increasing sequence:

$$s_{N+1} - s_N = a_{N+1} \geq 0.$$

Therefore the sequence (s_N) converges if and only if it is bounded. Therefore, a series with non-negative terms converges if and only if its sequence of partial sums is bounded.