Numerical Methods Assignment 1

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Question 1

Given 3 potentially unevenly spaced points

$$(x_j, f_j), (x_{j+1}, f_{j+1}), (x_{j+2}, f_{j+2})$$

The general Lagrange polynomial has form:

$$p_n(x) = \sum_{j=0}^{n} f_j L_j(x), \qquad L_j(x) = \prod_{\substack{i=0 \\ i\neq j}}^{n} \frac{x - x_i}{x_j - x_i}$$

So for quadratic polynomial interpolation we get:

$$p_2(x) = f_j(\frac{x - x_{j+1}}{x_j - x_{j+1}})(\frac{x - x_{j+2}}{x_j - x_{j+2}}) + f_{j+1}(\frac{x - x_j}{x_{j+1} - x_j})(\frac{x - x_{j+2}}{x_{j+1} - x_{j+2}}) + f_{j+2}(\frac{x - x_j}{x_{j+2} - x_j})(\frac{x - x_{j+1}}{x_{j+2} - x_{j+1}})$$

So the first derivative of this will be given by:

$$p_2'(x) = \frac{d}{dx}\left[f_j(\frac{x - x_{j+1}}{x_j - x_{j+1}})(\frac{x - x_{j+2}}{x_j - x_{j+2}}) + f_{j+1}(\frac{x - x_j}{x_{j+1} - x_j})(\frac{x - x_{j+2}}{x_{j+1} - x_{j+2}}) + f_{j+2}(\frac{x - x_j}{x_{j+2} - x_j})(\frac{x - x_{j+1}}{x_{j+2} - x_{j+1}})\right]$$

Noting that f_n, x_k are constants

$$\implies p_2'(x) = \frac{d}{dx}\left[f_j(\frac{x - x_{j+1}}{x_j - x_{j+1}})(\frac{x - x_{j+2}}{x_j - x_{j+2}}) + f_{j+1}(\frac{x - x_j}{x_{j+1} - x_j})(\frac{x - x_{j+2}}{x_{j+1} - x_{j+2}}) + f_{j+2}(\frac{x - x_j}{x_{j+2} - x_j})(\frac{x - x_{j+}}{x_{j+2} - x_j}\right.$$

$$p_2'(x) = \frac{f_j}{(x_j - x_{j+1})(x_j - x_{j+2})}\frac{d}{dx}[(x - x_{j+1})(x - x_{j+2})] + \frac{f_{j+1}}{(x_{j+1} - x_j)(x_{j+1} - x_{j+2})}\frac{d}{dx}[(x - x_j)(x - x_{j+2})]$$

$$\implies p_2'(x) = \frac{f_j\,(2x - x_{j+1} - x_{j+2})}{(x_j - x_{j+1})(x_j - x_{j+2})} + \frac{f_{j+1}\,(2x - x_j - x_{j+2})}{(x_{j+1} - x_j)(x_{j+1} - x_{j+2})} + \frac{f_{j+2}\,(2x - x_j - x_{j+1})}{(x_{j+2} - x_j)(x_{j+2} - x_{j+1})}$$

$$p_2'(x) = \sum_{j=0}^{2}\left(f_j \sum_{\substack{i=0 \\ i\neq j}}^{2} \frac{x - x_i}{x_j - x_i}\right)$$

Question 2

```
function [p, px] = quadinterp(x,xj,fj)
%[p, px] = quadinterp(x,xj,fj)
%Function quadinterp uses piecewise Lagrange quadratic interpolation t
%find values for [xj,fj] over x.
%INPUTS ----
%x  - row vector of points to calculate the interpolant on
%xj - row vector of x values corresponding to the known function value
%ascending order
%fj - row vector containing the data values for xj values
%OUTPUTS ----
%p  - row vector of the value of the piecewise quadratic interpolant a
%px - row vector of the first derivative or p at each value x
%The outputs p and px will both have length x
%
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%matrix to compare X and XJ
[X,XJ] = meshgrid(x,xj);
%largest corresponding j for each point
j=sum(XJ<=X);
%j must be 1,3,5,7,...,N-2
%so reduces j by 1 if it is the endpoint, and by 1 if it is even
j=j-(j==length(xj));
j=j-(mod(j,2)==0);


%application of the equations found in 1.1
p=fj(j).*(x-xj(j+1))./(xj(j)-xj(j+1))  .*(x-xj(j+2))./(xj(j)-xj(j+2));
p=p+fj(j+1).*(x-xj(j))./(xj(j+1)-xj(j))  .*(x-xj(j+2))./(xj(j+1)-xj(j+2
p=p+fj(j+2).*(x-xj(j))./(xj(j+2)-xj(j))  .*(x-xj(j+1))./(xj(j+2)-xj(j+1
%Derivative
px=fj(j).*(2*x-xj(j+1)-xj(j+2))./((xj(j)-xj(j+1)).*(xj(j)-xj(j+2)));
px=px+fj(j+1).*(2*x-xj(j)-xj(j+2))./((xj(j+1)-xj(j)).*(xj(j+1)-xj(j+2))
px=px+fj(j+2).*(2*x-xj(j)-xj(j+1))./((xj(j+2)-xj(j)).*(xj(j+2)-xj(j+1))



end
```

Question 3

Given equally-spaced data points find N_{min} to approximate $f(x) = \cos^2(x)$ over $0 \leq x \leq 2\pi$ to an accuracy of 10^{-1}.

Polynomial interpolation error has form:

$$\epsilon_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

with n points.. We want

$$\epsilon \leq 10^{-1}$$

$$\implies \max \left(\frac{f^{(n+1)}(t_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \right) \leq 10^{-1}$$

Since we are using quadratic interpolation, this gives:

$$= \max \left(\frac{4\sin(2t_x)}{3!} (x - x_0)(x - x_1)(x - x_2) \right) \leq 10^{-1}$$

$$= \max \left(\frac{4\sin(2t_x)}{6} (x - x_0)(x - x_1)(x - x_2) \right) \leq 10^{-1}$$

This can be transformed by letting $x - x_1 = y$ i.e. $x = y + x_1$ and since $\pi/2$ is contained in the interval, sin can be maximised to 1.

$$\implies = \max \left(\frac{4}{6} (y + x_1 - x_0)(y)(y + x_1 - x_2) \right) \leq 10^{-1}$$

Since the points are equally spaced, let $h = x_{j+1} - x_j$

$$\implies = \max \left(\frac{2}{3} (y + h)(y)(y - h) \right) \leq 10^{-1}$$

$$= \max \left(\frac{2}{3} (y^3 - yh^2) \right) \leq 10^{-1}$$

$$\max y^3 - yh^2 \implies 3y^2 - h^2 = 0 \implies y = \frac{h}{\sqrt{3}}$$

$$= \max \left(\frac{2}{3} ((\frac{h}{3\sqrt{3}})^3 - \frac{h}{\sqrt{3}} h^2) \right) \leq 10^{-1}$$

$$= \max \left(\frac{2h^3}{9\sqrt{3}} - \frac{2h^3}{3\sqrt{3}} \right) \leq 10^{-1}$$

We want the maximum value so we can take the absolute value of $-4h^3$

$$= \max \left(\frac{|-4h^3|}{9\sqrt{3}} \right) \leq 10^{-1}$$

$$= \max \left(\frac{4h^3}{9\sqrt{3}} \right) \leq 10^{-1}$$

$N = 1 + \frac{2\pi}{h} \implies h = \frac{2\pi}{N-1}$

$$= \left(\frac{2\pi}{N-1}\right)^3 \left(\frac{4}{9\sqrt{3}}\right) \leq 10^{-1}$$

$$\frac{2\pi}{N-1} \leq \sqrt[3]{\frac{9\sqrt{3}}{4}10^{-1}}$$

$$2\pi \leq \sqrt[3]{\frac{9\sqrt{3}}{4}10^{-1}}(N-1)$$

$$\frac{2\pi}{\sqrt[3]{\frac{9\sqrt{3}}{4}10^{-1}}} \leq (N-1)$$

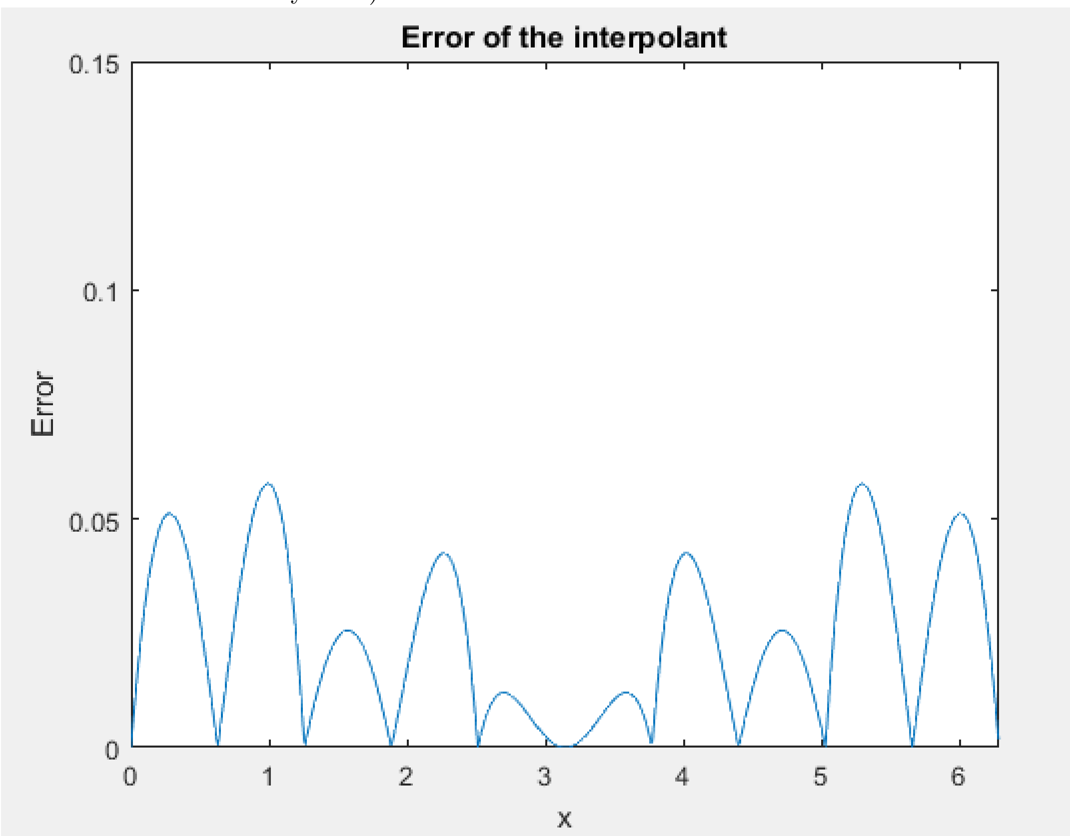
$$N \geq \frac{2\pi}{\sqrt[3]{\frac{9\sqrt{3}}{4}10^{-1}}} + 1$$

N min when this is equality (or as close to as possible). $N_{min} = 9.60...$ But N_{min} must be an integer so the minimum value, and it must be odd since we are using interpolation so:

$$N_{min} = 11$$

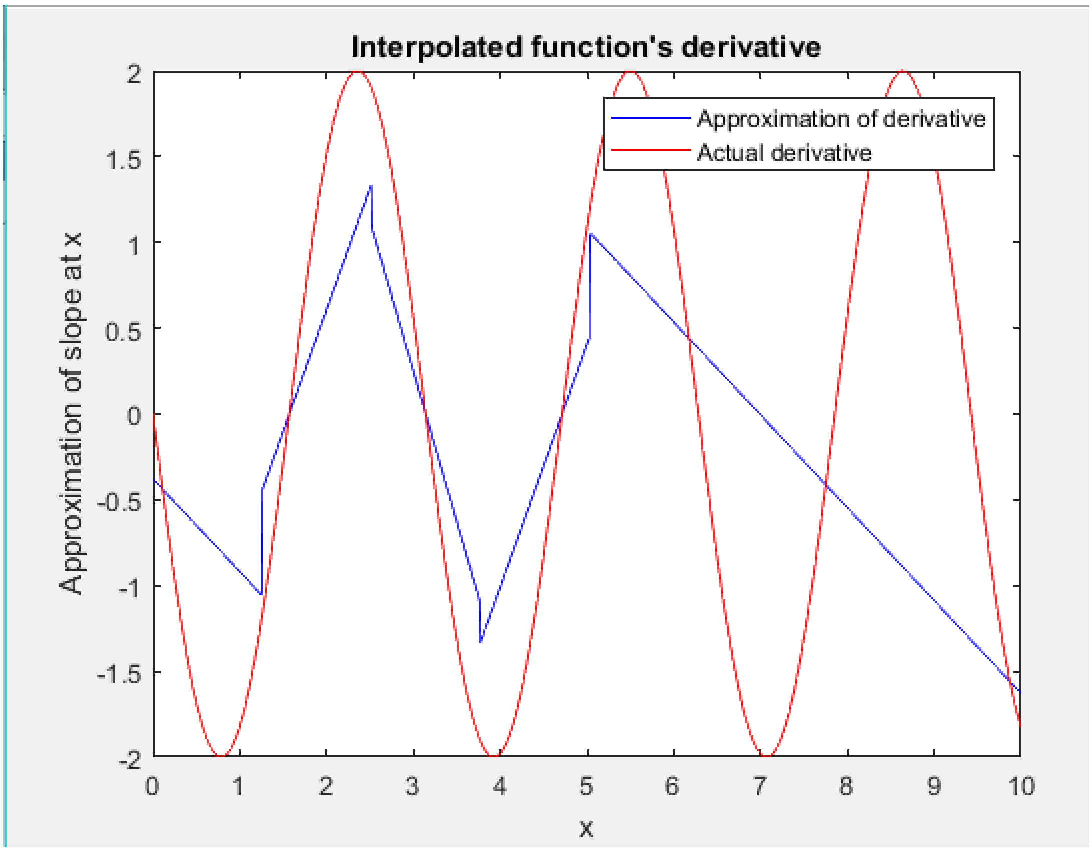
Question 4
MATLAB script quadtest to sample $f(x) = \cos^2(x)$ over $0 \leq x \leq 2\pi$ then calls quadinterp from 1.2 to evaluate the interpolant and its first derivative on 1000 equi-spaced points over the same interval.
Create two plots

(a) The absolute error i.e. $|f(x) - P(x)|$ over the domain (check the absolute error is bounded above by 10^{-1}).



Over the domain $0 \leq x \leq 2\pi$, this is always under 0.1.

(b) The first derivatives $f'(x)$, $P'(x)$ over the interval



From this it is quite visible that there is significant error in the interpolated derivative.

Question 5

Using more points $N > N_{min}$, discuss the accuracy of the interpolant and derivative as N increases. Explain why the derivative $P'(x)$ generically doesn't match $f'(x)$ as well as $P(x)$ matches $f(x)$

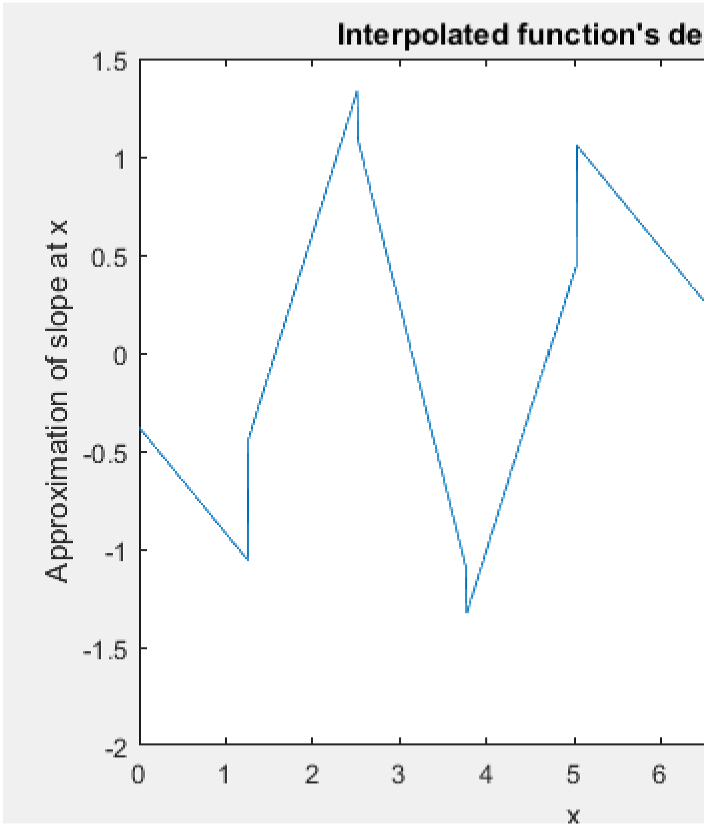
```
%N comparison
xj=linspace(0,2*pi,2*Nmin-1);
fj=cos(xj).^2;

[p2, ~] = quadinterp(x,xj,fj);
error2 = abs(fx-p2);
xj=linspace(0,2*pi,4*Nmin-3);
fj=cos(xj).^2;
[p3,~]= quadinterp(x,xj,fj);

error3=abs(fx-p3);
figure
plot(x,error,'b-',x,error2,'y-',x,error3,'r-');
title("Comparison of varying Ns");
xlabel('x');
ylabel('approximation');
legend('N=11','N=21','N=41')
axis([0,2*pi , 0 ,0.15]);
```

From the maths done in question 3, and from the plot above, it is apparent that As N increases, the error of the approximation decreases. Doubling N (and subtracting 1 to keep it odd), approximately decreases the error by a factor of

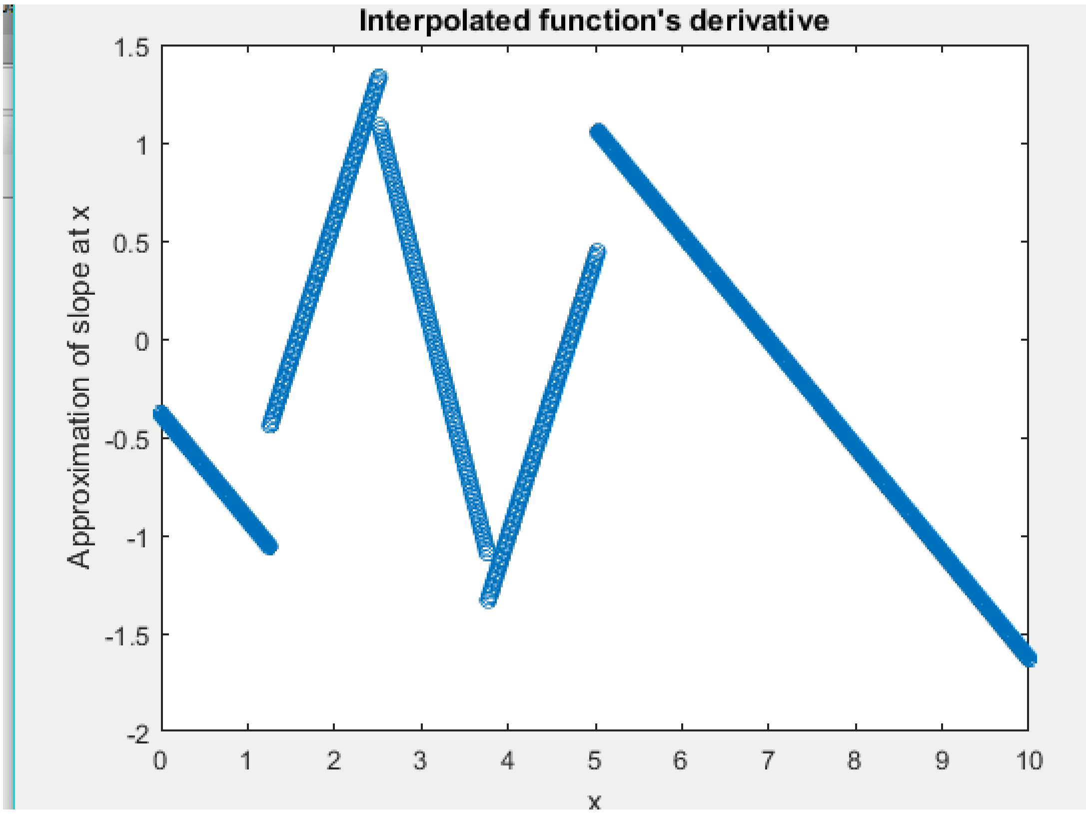
```
>> max(error)

ans =

    0.0577

>> max(error2)

ans =

    0.0076

>> max(error3)

ans =

   9.8800e-04
```

8.

The derivative $P'(x)$ won't usually match $f'(x)$. This is due to points where the derivative isn't defined - $P'(x)$ will have 2 values at the same point. I.e. for x_j, $j = 1, 3, 5...$ the j values $3, 5, 7, 9...$ will have 2 values for the derivative. which



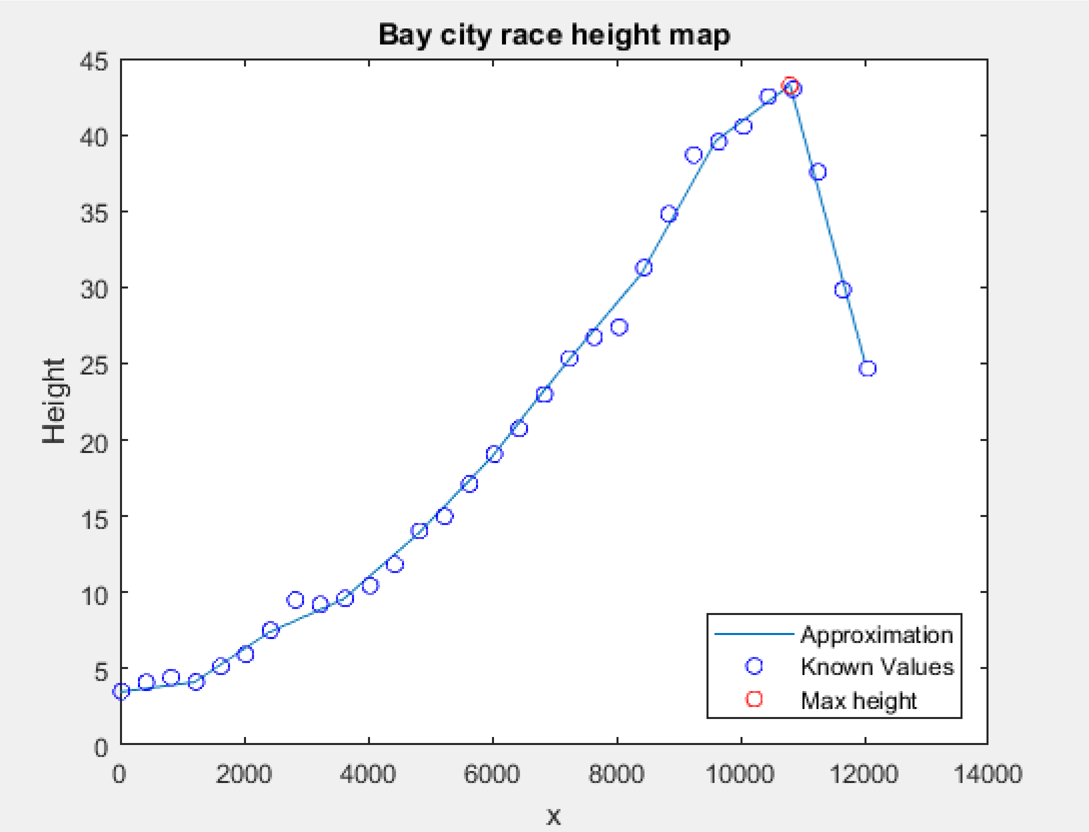
means these points will have an undefined slope.
As shown in the figure, close to the values $x = 1, 2.5, 3.5, 5$ the derivative has two values, which is apparent by the vertical lines.



This image makes it clearer, as gaps are evident in the plot.

Question 6

Add more code to quadtest which loads bay_city.dat and then calls quadinterp to interpolate the data and plot the elevation every $h = 1m$ along the course. Plot the original points as circles or crosses and label axes meaningfully.



Plot of an approximation of the height map of the bay-city run.

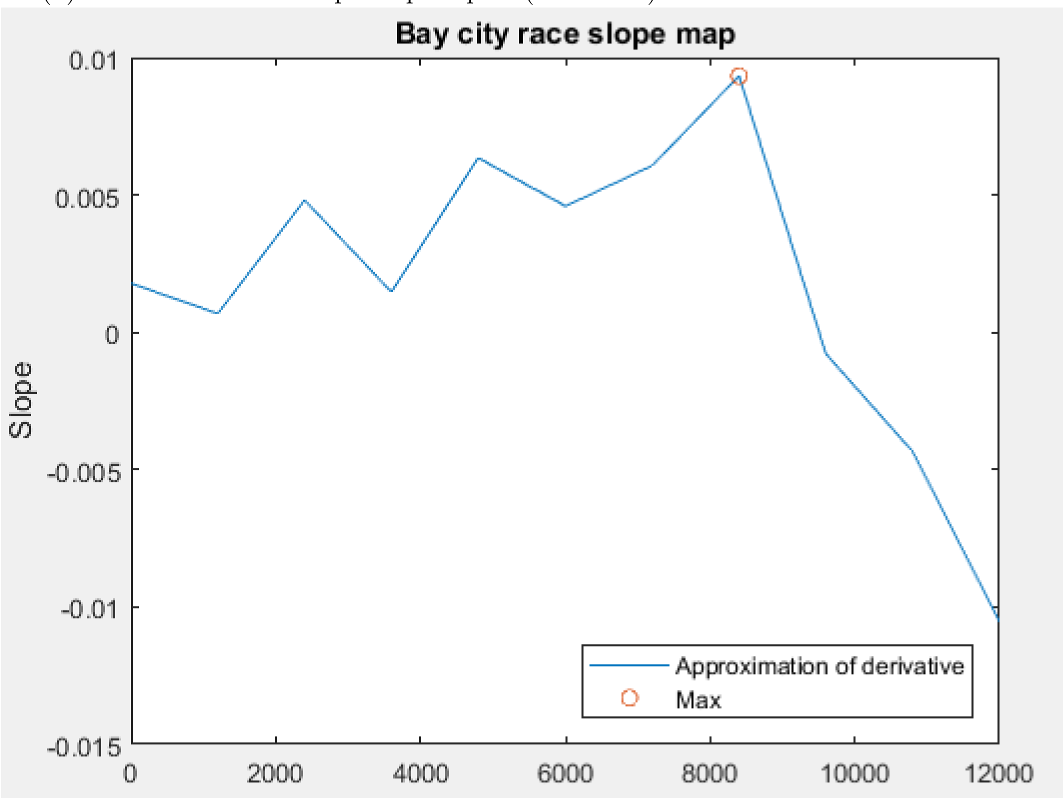
Question 7

add code which uses logical indexing to answer:

(a) How far into the course is the highest elevation (function)

As shown in the graphic in question 6, indicated with a red circle, the maximum height of the bay-city race is approximately 10800m in or 10.8Km, with an approximate height of 43m.

(b) How far in is the steepest uphill part (derivative)



It was found (and is shown in the graph above) that the point with the maximum slope is $x = 8400$m into the bay-city run. The slope at this point is 0.0093m.

Question 8
Comment on a potential numerical issue with quadratic interpolation as defined in quadinterp in finding the steepest section of the course.
As discussed in question 5 the derivative is not defined on the points $j = 3, 5, 7, 9, 11$

Another issue is there is no reference to compare the slope generated this way to the actual slope of the race. There is no given slope.

Complete quadtest code:

```
%Andrew Martin
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%14/08/2017
Nmin=11;
x=linspace(0,10,1000);
xj=linspace(0,2*pi,Nmin);
fj=cos(xj).^2;
fx=cos(x).^2;
fprimej=-2*sin(2*x);

[p, px] = quadinterp(x,xj,fj);

%removes graphs for tidyness
close all

%error plot
error = abs(fx-p);
figure
plot(x,error)
title("Error of the interpolant")
axis([0,2*pi , 0 ,0.15]);
xlabel('x');
ylabel('Error');


%N comparison
xj=linspace(0,2*pi,2*Nmin-1);
fj=cos(xj).^2;

[p2, ~] = quadinterp(x,xj,fj);
error2 = abs(fx-p2);
xj=linspace(0,2*pi,4*Nmin-3);
fj=cos(xj).^2;
[p3,~]= quadinterp(x,xj,fj);

error3=abs(fx-p3);
figure
plot(x,error,'b-',x,error2,'y-',x,error3,'r-');
title("Comparison of varying Ns");
xlabel('x');
ylabel('approximation');
legend('N=11','N=21','N=41')
axis([0,2*pi , 0 ,0.15]);
```

```
%Graph of the function
figure
plot(x,p,'-',xj,fj,'o');
title("Known data points and the Interpolated function");
xlabel('x');
ylabel('approximation');

%Graph of the derivatives
figure
plot(x,px,'b-',x,fprimej,'r-');
title("Interpolated function's derivative");
xlabel('x');
ylabel('Approximation of slope at x');
legend('Approximation of derivative','Actual derivative')



%Bay city section
%note baydata is in metres
baydata=importdata('bay_city.dat');
N=11;
xj=baydata(1,:);
fj=baydata(2,:);
x=linspace(0,12000,N); %given the race is a 12Km race
[p, px] = quadinterp(x,xj,fj);

%max elevation
[mx, indx] = max(p);
maxpoint=x(indx);
figure
plot(x,p,'-',xj,fj,'bo',maxpoint,mx,'ro');
title("Bay city race height map");
xlabel('x');
ylabel('Height');
legend('Approximation','Known Values','Max height','location','southeas
```

```
%max slope
[ma, ind] = max(px);
maxslopepoint =x(ind);

figure
plot(x,px,'-',maxslopepoint,ma,'o');
title("Bay city race slope map");
xlabel('x');
ylabel('Slope');
legend('Approximation of derivative','Max','Location','southeast')

```