Lecture 9: Equilibrium distributions and stationarity

Concepts checklist

At the end of this lecture, you should be able to:

- *Define* a *stationary distribution* and *equilibrium distribution* of a CTMC;
- Appreciate that for our purposes *these are equivalent*; and,
- *Solve the global balance equations*, both algebraically for simple CTMCs and with the assistance of a computer otherwise.

Project Description

Discussion of the Project.

Moments

Whilst the moments approach seems promising, it also quickly becomes complicated.

Example 5: Quadratic birth process

Consider now a *quadratic (pure) birth process*. This has a state transition diagram

$$\begin{tikzcd} 1 \arrow[r, bend left, "\lambda"] & 2 \arrow[r, bend left, "4\lambda"] & 3 \arrow[r, bend left, "9\lambda"] & 4 \arrow[r, bend left, "16\lambda"] & \cdots \end{tikzcd}$$

Here, we have (try to show this)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X(t)] = \lambda\mathbb{E}[X(t)^2].$$

Hence, we have that the mean (first moment) is dependent upon the second moment. This higher-order moment dependence persists, meaning we can not get a closed system of differential equations to solve using this approach.

As stated, explicit solution of the KFDEs is rarely achievable, and moment equations also quickly become infeasible. We now consider another measure of a CTMC which is of interest in its own right, and is typically more tractable.

Stationary and equilibrium distributions

Let's consider a distribution (a probability mass function), $\pi = (\pi_i)_{i\in\mathcal{S}}$ such that if we started the CTMC according to π we have that the distribution of the process is π for all t.

Definition 7. *Assume we have a CTMC $(X(t), t \geq 0)$ with transition function $P(t)$. An $|S|$-dimensional vector $\pi = (\pi_i)_{i\in\mathcal{S}}$ with $\pi_i \geq 0$ for all i and $\sum_{i\in S}\pi_i = 1$ is called a stationary distribution if*

$$\pi = \pi P(t) \quad \textit{for all} \ \ t \geq 0.$$

This appears to be a difficult quantity to evaluate, as it requires knowledge of the transition function. However, let's consider this system of equations, in particular for the cases we are interested in where

$$P(t) = e^{Qt}.$$

We then have

$$\begin{aligned}\pi &= \pi P(t) \\ &= \pi e^{Qt} \\ &= \pi \sum_{n=0}^{\infty} Q^n \frac{t^n}{n!}.\end{aligned}$$

Now, subtract π from each side,

$$0 = \pi \sum_{n=1}^{\infty} Q^n \frac{t^n}{n!},$$

and since this must hold for all $t \geq 0$, it implies that

$$0 = \pi Q^n \quad \text{for all } \; n \geq 1.$$

Hence, $\pi Q = 0$.

So, we have shown that for CTMCs in which the matrix exponential solution to the KFDEs exists, we can determine the stationary distribution by solving $\pi Q = 0$, with the constraint $\sum_{i \in \mathcal{S}} \pi_i = 1$; much simpler than solving $\pi = \pi P(t)$! In fact,

$$\pi Q = 0, \quad \sum_{i \in \mathcal{S}} \pi_i = 1,$$

is simply a system of linear equations. We can attempt to solve these algebraically, but a variety of algorithms (i.e., numerical methods) also exist; for example, you can use the / (slash operator) in MATLAB.

In general, a solution to $\pi Q = 0, \sum_{i \in \mathcal{S}} \pi_i = 1$ is called an *equilibrium distribution*. As stated, for the CTMCs of interest to us in this course, this is equal to the stationary distribution.

Let us know take a closer look at what is happening to the dynamics under π. Let's differentiate both sides of

$$\pi = \pi P(t)$$

with respect to t, and then substitute the KFDEs. Considering the $(i,j)^{\text{th}}$ entry we have

$$\begin{aligned}0 = \sum_{i \in \mathcal{S}} \pi_i \sum_{k \in \mathcal{S}} P_{ik}(t) q_{kj} = \sum_{k \in \mathcal{S}} \sum_{i \in \mathcal{S}} \pi_i P_{ik}(t) q_{kj} = \sum_{k \in \mathcal{S}} \pi_k q_{kj} = \pi_j q_{jj} + \sum_{\substack{k \neq j \\ k \in \mathcal{S}}} \pi_k q_{kj} \\ \Rightarrow \pi_j \sum_{\substack{k \neq j \\ k \in \mathcal{S}}} q_{jk} = \sum_{\substack{k \neq j \\ k \in \mathcal{S}}} \pi_k q_{kj} \qquad \text{(global balance equation)}.\end{aligned}$$

We can interpret $\pi_j q_{jk}$ as the (probability) flux from state j to state k. Thus,

$$\sum_{\substack{k \neq j \\ k \in \mathcal{S}}} \pi_j q_{jk} = \sum_{\substack{k \neq j \\ k \in \mathcal{S}}} \pi_k q_{kj}$$

$$\text{Flux out of state } j = \text{Flux into state } j,$$

which we can expect to hold in equilibrium. Hence, the above equations are also referred to as the flux balance equations or the equilibrium equations for a CTMC.

Example 6. Linear pure-death process

The linear pure-death process has state space $\mathcal{S} = \{0, 1, 2, \dots, N\}$ and for $j = 0, 1, \dots, N$

$$\begin{aligned} q_{j,j-1} &= j\mu, \\ q_{jj} &= -j\mu. \end{aligned}$$

The equilibrium equations are

$$\pi_N N\mu = 0, \tag{6}$$
$$\pi_j j\mu = \pi_{j+1}(j+1)\mu, \qquad \text{for } j = 1, 2, \dots, N-1 \tag{7}$$
$$0 = \pi_1\mu. \tag{8}$$

Equation (7) gives us

$$\begin{aligned} \pi_j &= \frac{j+1}{j}\pi_{j+1} \qquad \text{for } j = 1, 2, \dots, N-1, \\ &= \left(\frac{j+1}{j}\right)\left(\frac{j+2}{j+1}\right)\pi_{j+2} \\ &\vdots \\ &= \frac{N}{j}\pi_N. \end{aligned}$$

However, $\pi_N N\mu = 0$ implies that $\pi_N = 0$, and thus $\pi_j = 0$ for all $j = 1, 2, \dots, N$. We have no information about π_0. However, we require $\sum_{i\in\mathcal{S}} \pi_i = 1$ and hence,

$$\sum_{k=0}^{N} \pi_k = 1 \quad \Rightarrow \quad \pi_0 = 1.$$

This corresponds to the intuitive result that if we have no individuals / working machines then we will always remain there. This also corresponds to what we'd intuitively expect to see in the long-term, starting from any number of individuals / working machines: all the components will die / break down as there are no births / repairs. Thus, the CTMC will eventually enter state 0 and remain in state 0 forever. We'll revisit this long term, or limiting behaviour.