LECTURE 4

In this lecture we will introduce a new axiom, Axiom III, which we will later show is equivalent to the previous Axiom III$'$. Describing this new axiom will take some preparation though.

Upper and lower bounds

We make the following definition.

Definition 1.1: If $S \subset \mathbb{R}$ is non-empty, a number $x \in \mathbb{R}$ is said to be an *upper bound* for S if $s \leq x$ for all $s \in S$. A number $y \in \mathbb{R}$ is said to be a *lower bound* for S if $y \leq s$ for all $s \in S$. We say that S is *bounded above* if there is an upper bound for S; we say that S is *bounded below* if there is a lower bound for S; we say that S is *bounded* if it is bounded above and bounded below.

Example: Let $S = (0,1) = \{\, s \in \mathbb{R} \mid 0 < s < 1 \,\}$. Any number $x \geq 1$ is an upper bound for S; clearly if $s \in S$ then $s < 1 \leq x$, hence $s \leq x$ for all $s \in S$. Therefore S is bounded above, 1 is the *smallest* upper bound for S and the set of *all* upper bounds for S is $U = \{\, x \in \mathbb{R} \mid \ 1 \leq x \,\} = [1, \infty)$. Any number $y \leq 0$ is a lower bound for S; clearly if $s \in S$ then $y \leq 0 < s$, hence $y \leq s$ for all $s \in S$. Therefore S is bounded below, 0 is the *largest* lower bound, and the set of *all* lower bounds is $L = \{\, y \in \mathbb{R} \mid \ 0 \leq y \,\} = (-\infty, 0]$. Since S is bounded above and below it is bounded.

You might wonder why we required that S is non-empty in Definition 1.1 above; strictly speaking there was no reason to do this other than the fact that the empty set is so simple it becomes hard to think about. If we allow S to be the empty set $\emptyset$ in Definition 1.1 then *every* real number x is both an upper and lower bound for $S = \emptyset$, the reason being that the inequalities $s \leq x$ and $x \leq s$ are true *vacuously* for all $s \in \emptyset$ (because there are no such s).

Note: in the example where $S = (0,1)$ above, there are lots of upper and lower bounds for S, but none of them belong to S. The case where an upper bound or a lower bound belongs to the set in question is special, and gets a definition of its own.

Definition 1.2: Let $S \subset \mathbb{R}$. A *maximum element* of S is a number s_0 in S such that $s \leq s_0$ for all $s \in S$. A *minimum element* of S is a number s_1 in S such that $s_1 \leq s$ for all $s \in S$.

Exercise: if S has a maximum (respectively minimum) element, then this maximum (respectively minimum) element is unique. Thus we write $\max(S)$ for the maximum element of S (if it exists) and we write $\min(S)$ for the minimum element of S (if it exists).

Example: the set $S = (0,1)$ does not have a maximum or a minimum element; the set $T = [0,1]$ has $\max(T) = 1$, $\min(T) = 0$.

Example: Let $S = \{\, 1/n \mid \ n = 1, 2, 3, \ldots \,\}$. Then S has a maximum element, $\max(S) = 1$. But S does not have a minimum element since for any $n \in \mathbb{N}$, $1/(n+1) < 1/n$. S is bounded below though — any number $y \leq 0$ is a lower bound for S. There are no lower bounds for S strictly larger than 0, but this is not easy to see at the moment.

Before we continue, let's try and understand what it means for a real number N to *not* be an upper bound of a set $S \subset \mathbb{R}$. If N is not an upper bound, then that means that it is not true that the inequality $s \leq N$ is true for all $s \in S$. In other words, there must be some $s \in S$ for which this inequality is not true, i.e. there must exist $s \in S$ such that $N < s$.

There is an important principle at work here that we will constantly use in the course. We'll see lots of statements of the form

$$\text{`}\forall x\, P(x)\text{'}$$

which in non-math speak reads as 'for all x, the statement $P(x)$ is true'. The symbol $\forall$ is there to *quantify* for which values of the variable x the statement is true for. In a case like this the statement is true for all x, and so we say that $\forall$ is a *universal quantifier*. We will be interested in when a statement of the form '$\forall x\, P(x)$' is *not* true. Such a statement is not true if there is some x for which the statement $P(x)$ is not true, in other words, 'there exists x such that not $P(x)$ is true'. In symbolic logic this last statement would be written as

$$\text{`}\exists x\, \neg P(x)\text{'}$$

The symbol $\exists$ is also a quantifier — it is called an *existential quantifier*. Here $\neg$ means 'not', so that $\neg P(x)$ means 'not $P(x)$', or the *negation* of $P(x)$. Thus in symbols

$$\neg(\forall x\, P(x)) = \exists x\, \neg P(x).$$

We can also contemplate the negation of the statement $\exists x\ P(x)$; if this statement is not true, then there does not exist x such that the statement is true — in other words the statement is false for every x. Thus

$$\neg(\exists x\ P(x)) = \forall x\ \neg P(x).$$

In our example above, the statement that x is an upper bound is the statement $(\forall s \in S)(s \leq x)$. Therefore

$$\neg((\forall s \in S)(s \leq x)) = (\exists s \in S)\neg(s \leq x) = (\exists s \in S)(x < s).$$

The next definition is a critically important one for the course — make sure you know what it means, and what it does not mean.

Definition 1.3: Let $S \subset \mathbb{R}$ be non-empty. A number $x \in \mathbb{R}$ is said to be a *least upper bound* or *supremum* of S if

(1) x is an upper bound for S, and
(2) if x' is another upper bound for S then $x \leq x'$.

We make some observations about this definition.

1. A set S might not have a supremum, but if it does, the supremum is unique. To see this, observe that if b and b' are both supremums for S, then $b \leq b'$ (because b' is an upper bound and b is the least upper bound) and also $b' \leq b$ (because b is an upper bound and b' is the least upper bound). Therefore $b = b'$. Hence it makes sense to write $\sup S$ for the supremum of S (if it exists). Note that 'sup' is pronounced 'soup'.

2. If $\sup S$ exists, then

$$\sup S = \min \{ u \in \mathbb{R} \mid\ u \text{ is an upper bound for } S \}.$$

Example: as an example, consider the set $S = [0, 1)$. This set has a supremum, namely $\sup S = 1$. To prove this there are two things that we need to do: we need to prove first that 1 is an upper bound for S, and if x is another upper bound then $1 \leq x$. By definition if $s \in S$ then $s < 1$. Hence 1 is an upper bound. We prove that no number $x < 1$ can be an upper bound for S. Clearly we can't have $x < 0$ so suppose that $0 \leq x < 1$. Then $x < (x + 1)/2 < 1$ and $(x + 1)/2 \in S$. Hence x cannot be an upper bound for S. Therefore 1 is the smallest upper bound and so $\sup S = 1$.

In the example above we used the following inequality: if $x < y$ then $x < (x+y)/2 < y$. This follows immediately from the fact that if $a < b$ then $a+c < b+c$: we have $x/2 < y/2$. Therefore $x = x/2 + x/2 < x/2 + y/2 < y/2 + y/2 = y$.

We now state the new version of Axiom III$'$:

Axiom III$'$: if $S \subset \mathbb{R}$ is non-empty and bounded above then S has a least upper bound.

We will take Axiom III as the third of our Axioms for Real Analysis. In fact, if we assume Axiom III then we get Axiom III$'$ for free, conversely if we assume Axiom III$'$ then we get Axiom III for free.

Theorem 1.4: Axiom III$'$ $\implies$ Axiom III.

Proof: Assume that Axiom III$'$ holds and let $S \subset \mathbb{R}$ be non-empty and bounded above. We need to prove that S has a least upper bound. We let

$$B = \{\, x \in \mathbb{R} \mid \ x \text{ is an upper bound for } S \,\}\,.$$

Then $B \neq \emptyset$ by assumption (i.e. the assumption that S is bounded above). Let $A = \mathbb{R} \setminus B$, in other words let

$$A = \{\, x \in \mathbb{R} \mid \ x \text{ is not an upper bound for } S \,\}\,.$$

Then S is also non-empty: to see this let $s \in S$, then $s - 1 < s$ and so $s - 1$ cannot be an upper bound for S, i.e. $s - 1 \in A$. So we have that A and B are non-empty and that $A \cup B = \mathbb{R}$. To apply Axiom III$'$ we need to prove that for all $a \in A$ and for all $b \in B$, the inequality $a < b$ holds. Let $a \in A$ and let $b \in B$. Then a is not an upper bound for S and so there exists $s \in S$ such that $a < s$. On the other hand b *is* an upper bound for S and $s \leq b$. Therefore $a < s \leq b$ and hence $a < b$.

Therefore, the hypotheses of Axiom III$'$ hold and so there exists $c \in \mathbb{R}$ such that either

$$A = (-\infty, c) \quad \text{and} \quad B = [c, \infty),$$

or

$$A = (-\infty, c] \quad \text{and} \quad B = (c, \infty).$$

Suppose that $c \in A$, i.e. that $A = (-\infty, c]$ and $B = (c, \infty)$. Since $c \in A$, c is not an upper bound for S, therefore there exists $s \in S$ such that $c < s$. But then we have $c < (c+s)/2 < s$ and so $(c+s)/2 \in B$, but $(c+s)/2$ is clearly not an upper bound for S — contradiction.

Therefore it must be the case that $A = (-\infty, c)$ and that $B = [c, \infty)$. But then clearly c is a minimum element of B. Therefore $c = \sup S$ and S has a supremum. ■