# APP MTH 3002 Fluid Mechanics III
# Assignment 3
# Due: 12 noon, Friday 11 May.

When presenting your solutions to the assignment, please include some explanation in words to accompany your calculations. It is not necessary to write a lengthy description, just a few sentences to link the steps in your calculation. Messy, illegible or inadequately explained solutions may be penalised. The marks awarded for each part are indicated in boxes.

This assignment has 2 questions, for a total of 30 marks.

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1. The interior of an incompressible axisymmetric vortex ring in spherical coordinates has a velocity field $\boldsymbol{u} = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta$, where

$$u_r = C\left(1 - \frac{r^2}{a^2}\right)\cos\theta, \quad u_\theta = -C\left(1 - 2\frac{r^2}{a^2}\right)\sin\theta, \quad 0 \le r \le a \tag{1}$$

and $C$ is a constant.

$\boxed{3}$ (a) Stagnation points are points where $\boldsymbol{u} = \mathbf{0}$. Find any stagnation points in the velocity field of the vortex ring (1).

$\boxed{4}$ (b) Find the streamfunction $\psi(r,\theta)$ of the vortex ring for $0 \le r \le a$.

$\boxed{2}$ (c) Uniform flow past a sphere of radius $a$ is given by the stream function

$$\psi(r,\theta) = \frac{1}{2}Ur^2\left(1 - \frac{a^3}{r^3}\right)\sin^2\theta, \quad r \ge a, \tag{2}$$

where $U$ is the free-stream velocity. Show that it is possible to match this flow to (1) in such a way that the velocity is continuous at $r = a$. Find the value of $C$ that is necessary to achieve this.

$\boxed{2}$ (d) Use the stream function found in part (a) for $0 \le r \le a$ and the stream function from part (c) for $r \ge a$ to plot streamlines in the $x$-$z$ plane for $0 \le r \le 4a$.

*Hint:* You might need to experiment with the contour levels in order to see the streamlines inside the vortex ring ($r < a$).

2. Consider two-dimensional incompressible unidirectional flow between two fixed infinite parallel plates separated by a distance $h$, as shown in figure 1. Suppose that a constant pressure gradient is applied,

$$\frac{\partial p}{\partial z} = -\frac{\Delta p}{L},$$

where $\Delta p$ is the pressure drop over a distance $L$, and that the fluid is initially stationary.

$\boxed{4}$ (a) Derive the partial differential equation for $w$ from the Navier–Stokes and continuity equations.

$\boxed{1}$ (b) Write down the boundary and initial conditions for $w$.

$\boxed{2}$ (c) Find a *steady* solution $w = W(y)$ that satisfies the partial differential equation from part (a) for $\partial W/\partial t = 0$ subject to the boundary conditions from part (b) (but *not* the initial condition).

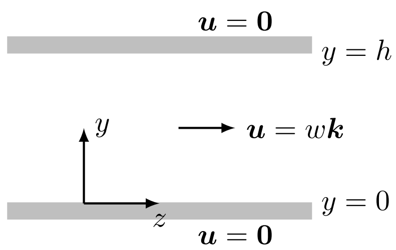


Figure 1: Two-dimensional unidirectional flow between fixed plates.

[2] (d) Let $w = W + \widetilde{w}$, where $\widetilde{w}$ is the perturbation of $w$ from the steady solution $W$. Write down the partial differential equation and boundary and initial conditions for $\widetilde{w}$.

[6] (e) Use separation of variables and Fourier series to find the solution $\widetilde{w}$ that satisfies the problem specified in part (d).

*Hints:* You are given that the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$Y'' - \lambda Y = 0, \quad Y(0) = Y(h) = 0,$$

are

$$\lambda_n = -\left(\frac{n\pi}{h}\right)^2, \quad Y_n = \sin\left(\frac{n\pi y}{h}\right), \quad n = 1, 2, \ldots$$

You are also given the integral

$$\int_0^h y(h-y)\sin\left(\frac{n\pi y}{h}\right)\,\mathrm{d}y = 2\left[1 - (-1)^n\right]\left(\frac{h}{n\pi}\right)^3, \quad n = 1, 2, \ldots$$

[4] (f) Write your solution for $w$ in terms of the nondimensional variables

$$\hat{w} = \frac{w}{\left(\dfrac{h^2 \Delta p}{2\mu L}\right)}, \quad \hat{y} = \frac{y}{h} \quad \text{and} \quad \hat{t} = \frac{\nu t}{h^2}.$$

Plot $\hat{w}$ for $0 \le \hat{y} \le 1$ and $0 \le \hat{t} \le 1$. Briefly describe the motion of the fluid as time goes by.

*Hints:* You'll need to truncate your series for plotting. Make sure that you keep enough terms for your solution to converge to graphical accuracy.

Plotting $\hat{w}$ at $\hat{t} = 0$ will give you a pretty good idea if your Fourier coefficients are correct. You'll find that the *shape* of the velocity profile at very early times is quite different to later times, so you may wish to plot a few profiles for $0 \le \hat{t} \le 0.01$ on one plot, and a few profiles for $0.01 \le \hat{t} \le 1$ on another.