Class Exercise 3: Applied Probability

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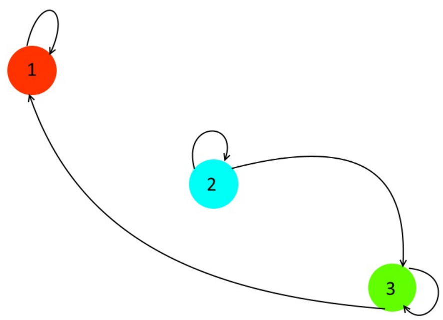
April 20, 2018

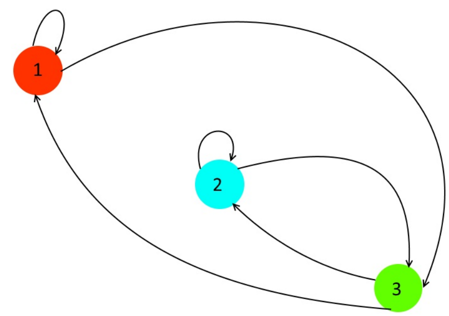
1. Given some Stochastic matrices

   (i) Draw state transition diagrams

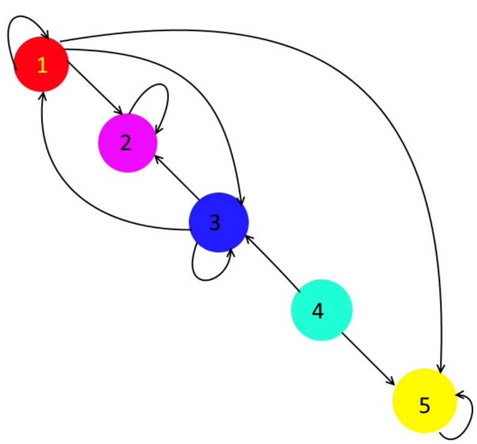
   Solution



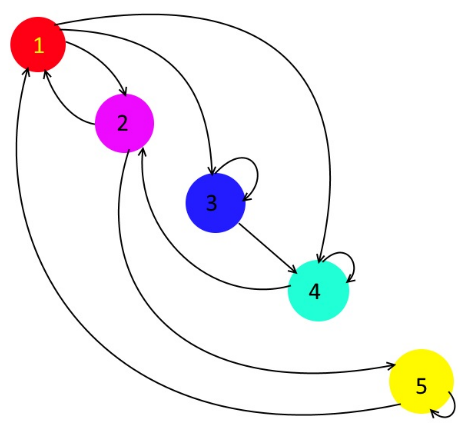
(a) State transition diagram for P_a



(b) State transition diagram for P_b



(c) State transition diagram for P_c



(d) State transition diagram for P_d

   As required.

   (ii) Determine communicating classes

   Solution For P_a - states 1, 2 and 3 are each in their own communicating class
   For P_b - states 1, 2 and 3 are in the same communicating class
   For P_c - state 5 is in its own communicating class, 2 is in its own communicating class, and states 1 and 3 are in the same class.

For P_d - states 1,2,3,4 and 5 are in the same communicating class
As required.

(iii) Classify the states as either recurrent or transient
Solution
P_a - 1 is recurrent while 2 and 3 are transient
P_b - 1, 2 and 3 are all recurrent
P_c - state 4 is ephemeral, state 2 is recurrent, state 5 is recurrent, states 1 and 3 are transient
P_d - all states are recurrent
As required.

2. Consider example 3.13 from lectures. I.e. the contest between players A and B where B is infinitely rich. Let X_n represent the fortune of player A at time n For $i \geq 1$:

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & otherwise \end{cases}$$

Find the mean hitting times $k_i = k_i^{(0)}$, $i = 0, 1, 2, \ldots$, for the cases:

(a) $p < q$
Solution Mean hitting times are minimal, non-negative solutions to

$$k_i^{\mathcal{A}} = E(H^{\mathcal{A}}|X_0 = i) = \begin{cases} 0 & i \in \mathcal{A} \\ 1 + \sum_{j \notin \mathcal{A}} p_{ij} k_j^{\mathcal{A}} & i \notin \mathcal{A} \end{cases}$$

(this part will apply for b and c so I will skip it for each of them)
So in this case,

$$k_0^{(0)} = 0,$$

$$k_i = 1 + pk_{i+1} + qk_{i-1}$$
$$pk_{i-1} - k_i + qk_{i+1} = -1$$

Solving the homogeneous ODE yields:

$$k_i = A_1 + A_2 \left(\frac{q}{p}\right)^i$$

Then solving the inhomogeneous part:

$$k_i = A_1 + A_2 \left(\frac{q}{p}\right)^i + \frac{i}{q-p}$$

Using the boundary condition $k_0^{(0)} = 0$:

$$k_0 = A_1 + A_2 = 0$$
$$A_1 = -A_2$$

So

This is the equation to be used in c as well

$$k_i = A_1 - A_1\left(\frac{q}{p}\right)^i + \frac{i}{q-p}$$

For $p < q$:
Want the minimal non-negative solution. So want A_2 to minimise:

$$k_i = A_1(1 - \left(\frac{q}{p}\right)^i) + \frac{i}{q-p}$$

$$\text{so set } A_2 = 0$$

$$\implies k_i = \frac{i}{q-p}$$

As required.

(b) $p = q = 1/2$

Solution Back to the ODE:

$$\frac{1}{2}k_{i+1} - k_i + \frac{1}{2}k_{i-1} = -1$$

$$k_{i+1} - 2k_i + k_{i-1} = -2$$

As required.

(c) $p > q$

Solution For $p > q$, want minimal non-negative solution to:

$$k_i = A_1(1 - \left(\frac{q}{p}\right)^i) + \frac{i}{q-p}$$

let $q/p = b < 1$ and $q - p = -\frac{1}{c} < 0$

$$\implies k_i = A_1(1 - b^i) - ic$$

For the non-negative solution:

$$k_i \geq 0 \implies A_1(1 - b^i) - ic \geq 0$$

$$\implies A_1 \geq \frac{ic}{1 - b^i}$$

$$\implies A_1 \geq \frac{\frac{-i}{q-p}}{1 - \left(\frac{q}{p}\right)^i}$$

Minimal when this is an equality. I.e.

$$k_i = \frac{\frac{-i}{q-p}}{1 - \left(\frac{q}{p}\right)^i}(1 - \left(\frac{q}{p}\right)^i) + \frac{i}{q-p}$$

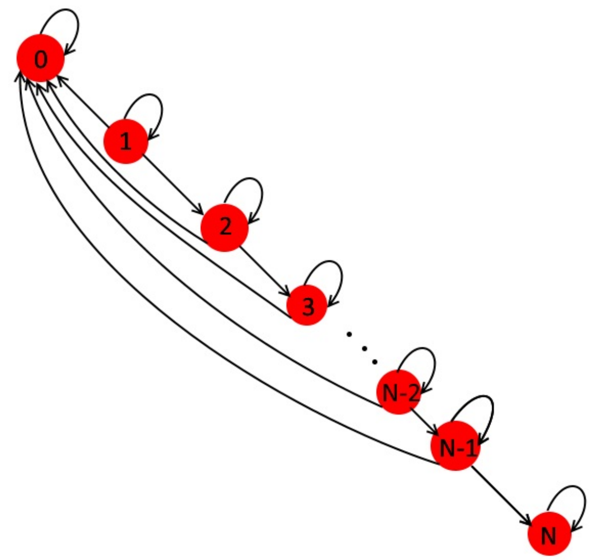
As required.

3. Consider the DTMC as given on $N+1$ states $(0, 1, 2, \ldots, N)$ where $p_i + r_i + q_i = 1$ for all $i = 1, \ldots, N-1$ and $p_i, q_i, r_i > 0 \forall i$.

(a) Identify the communicating classes, and state whether they are recurrent or transient

i. Draw a state transition diagram for this Markov chain

Solution



As required.

ii. Give a brief qualitative description (in words) of the dynamics associated with this Markov chain

Solution

In this Markov chain, from any state (apart from 0 and N) there is a chance to remain in that state, move to the next state up, or be sent to the 0 state. Once in state 0 or N The process is absorbed in that state and cannot move elsewhere.

As required.

(b) This Markov chain can be viewed as a modified version of the "Two Gamblers" example seen in lectures:

i. Describe the (modified) contest that is modelled by this Markov chain

Solution

Consider a single-elimination tournament. The states of the Markov chain represent the round the player is in, with state 0 being out of the tournament, or having lost. In this tournament, a win will correspond to moving to the next round, a loss will eliminate the player from the tournament, and a tie will mean replaying that round. In each round, the player has a chance to win (proceed to next round), tie (remain in the same round and play again), or lose and be eliminated from the tournament (move to state 0). Once the player reaches state N, they have won the tournament.

As required.

(c) Show that the absorption probabilities $u_k^{(0)}$ for $k = 1, \dots, N-1$ are given by:

$$u_k^{(0)} = 1 - \left(\frac{q_k}{p_k + q_k}\right) \times \dots \times \left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right)$$

Hint: first evaluate $u_k^{(N)}$ by starting with $u_{N-1}^{(N)}$ and proceed with recursion.

Solution Note the probability to be absorbed in state N or state 0 is $u_N + u_0 = 1$

Since we are not concerned about time, the probability, $u_{N-1}^{(N)}$ will not be effected by r_{N-1} as if r_{N-1} occurs, it just repeats that state. Using this, the probability to move to N will be

$$u_{N-1}^{(N)} = \frac{\text{Prob to move to N}}{\text{Total probability to move}} = \frac{q_{N-1}}{p_{N-1} + p_{N-1}}$$

$$u_{N-2}^{(N)} = u_{N-2}^{(N-1)} \times u_{N-1}^{(N)} = \frac{q_{N-2}}{p_{N-2} + p_{N-2}} \times \frac{q_{N-1}}{p_{N-1} + p_{N-1}}$$

Using recursion this gives:

$$\begin{aligned} u_k^{(N)} &= u_k^{(2)} \times u_{k+1}^{(3)} \times \ldots \times u_{N-2}^{(N-1)} \times u_{N-1}^{(N)} \\ &= \left(\frac{q_k}{p_k + q_k}\right) \times \ldots \times \left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right) \\ &\text{Using the law of total probability} \\ \implies u_k^{(N)} &= 1 - u_k^{(N)} \\ &= 1 - \left(\frac{q_k}{p_k + q_k}\right) \times \ldots \times \left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right) \end{aligned}$$

As required.

4. Using approach 1 from the group project, determine the number of digits of accuracy you expect to lose when solving system of equations using the transition matrix P. Hence comment on the appropriateness of computing expected hitting times for this problem (i.e. part 1(h) of the project)

Solution

This involves using condition numbers (recall from Numerical methods):
A large condition number corresponds to a poorly-conditioned matrix.
Given the condition number $A = 10^k$, then you may lose up to k digits of accuracy
Using the code (below) the condition number of the transition probability matrix is 164.3494.
Using the equation $A = 10^k \implies k = \log_{10}(A)$ gives: $k = 2.2158$ This means it is expected to lose just over 2 digits of accuracy - so assume we lose 3 - due to numerical methods.

Computing expected hitting times for this problem would still be reasonable, as we do not need extremely accurate results. But there will be some inherent error.

I made this matrix manually as the matrices given in the MIDI toolbox are all weighted against duration (and the TOTAL probability is 1 i.e. $\sum_i \sum_j p_{ij} = 1$)

```
load('..\musicxml.mat');
notes= all_songs.raw_merged_nmat;
lead = getmidich(notes,2);

%Generate the transition probability matrix
numnotes = max(lead(:,4))-min(lead(:,4)) + 1
leadshifted = lead- min(lead(:,4)) + 1
transprobmatrix = zeros(numnotes);
%up to length-1, so that we don't consider the shift after the last note
%(as it is nonexistent)
for i=1:length(lead)-1
    transprobmatrix(leadshifted(i,4),leadshifted(i+1,4)) = transprobmatrix(leadshifted(i,4),leadshi

end
transprobmatrix = transprobmatrix ./ sum(transprobmatrix,2)
cond(transprobmatrix)
log10(cond(transprobmatrix))
```

As required.