STATS 3005 Mathematical Statistics III
Tutorial 1
2018

1. Suppose X is a continuous random variable with PDF $f(x)$ and MGF $M(t)$. Assuming the order of integration and differentiation can be exchanged, prove that:
   (a) $M(0) = 1$.
   (b) $M'(0) = E(X)$.
   (c) $M''(0) = E(X^2)$.

2. Suppose $X \sim \text{geom}(p)$ with probability function

$$p(x) = p(1-p)^x \text{ for } x = 0, 1, 2, \ldots \text{ and } 0 < p < 1.$$

   (a) Show that $\sum_{x=0}^{\infty} p(x) = 1$.
   Hint: Let $q = 1 - p$ and consider the geometric series in q.
   (b) Derive the moment generating function, $M(t)$.
   (c) Show directly that $E(X) = \frac{1-p}{p}$.
   Hint: Differentiate the geometric series term by term to obtain the identity

$$\sum_{x=1}^{\infty} x q^{x-1} = (1-q)^{-2}.$$

3. For a random variable X with $E(X) = \mu$, show that:
   (a) $\text{var}(X) = E(X^2) - \mu^2$.
   (b) $\text{var}(X) = E(X(X-1)) + \mu - \mu^2$.

4. Consider the binomial distribution with parameters n and p. Show, for any *fixed* x, that

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \longrightarrow \frac{e^{-\lambda}\lambda^x}{x!}$$

   as

$$n \to \infty, \quad p \to 0, \text{ such that } np \to \lambda.$$

5. Consider the gamma function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

   (a) Prove $\Gamma(1) = 1$.
   (b) Using integration by parts, prove that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.
   (c) If $r > 0$ is an integer the show that $\Gamma(r) = (r-1)!$.

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