# TUTORIAL 6

1// $J\{y\} = \int_{x_0}^{x_1} K(x,y)\, e^{\tan^{-1} y'} \sqrt{1+y'^2}\, dx$

where $K(x,y) \neq 0$. $(x_0, y_0)$ prescribed.

$(x_1, y_1)$ lies on $y = \phi(x)$

Now

$$p = \frac{\partial f}{\partial y'} = K\left( \frac{e^{\tan^{-1} y'}}{1+y'^2}\sqrt{1+y'^2} + e^{\tan^{-1} y'} \frac{y'}{\sqrt{1+y'^2}} \right)$$

$$= \frac{K e^{\tan^{-1} y'}}{\sqrt{1+y'^2}} (1+y')$$

$$H = y'\frac{\partial f}{\partial y'} - f = K e^{\tan^{-1}} \left( y' \frac{1+y'}{\sqrt{1+y'^2}} - \sqrt{1+y'^2} \right)$$

$$= \frac{K e^{\tan^{-1} y'}}{\sqrt{1+y'^2}} (y'-1)$$

So the vector $(-H, p) = \frac{K e^{\tan^{-1} y'}}{\sqrt{1+y'^2}} (1-y', 1+y')$

The curve is parameterises as $(x, \phi)$ and so its tangent vector is $(1, \phi')$ and so the traverselity condition is

$$\frac{K e^{\tan^{-1} y'}}{\sqrt{1+y'^2}} (1-y', 1+y') \cdot (1, \phi') = 0$$

Now $\frac{K e^{\tan^{-1} y'}}{\sqrt{1+y'^2}} \neq 0$ so dividing by that we have.

$$(1-y', 1+y') \cdot (1, \phi') = 0.$$

Further we ~~can~~ divide both sides by $\sqrt{2}$

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}y', \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}y'\right)\cdot(1, \phi') = 0$$

and now replace $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}, \sin\frac{\pi}{4}$

$$\left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}y', \cos\frac{\pi}{4}y' + \sin\frac{\pi}{4}\right)\cdot(1, \phi') = 0$$

And we note the left vector is a rotation of the extremal tangent vector by an angle of $\pi/4$.

Therefore, in this case, extremals will intersect with $\phi(x)$ at an angle of $\pi/4$.

2/ Consider the two dimension problem of finding the shortest distance from $(1,1)$ to the circle $x^2+y^2=1$. In this case we would have a functional

$$F\{y\} = \int_{x_1}^{1} \sqrt{1+y'^2}\, dx$$

The Euler-Lagrange eqn is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Lets parametrise the circle by $(\sqrt{1-y^2}, y)$ and call $\phi(y) = \sqrt{1-y^2}$, then the traversality condition would be

$$\left[\frac{\partial f}{\partial y'} - \frac{d\phi}{dy}\left(y'\frac{\partial f}{\partial y'} - f\right)\right]_{x=x_1} = 0 .$$

So now using similar arguments to those used to derive the traversality condition in the 2D problem, the 3D problem is as follows:

Find shortest distance from $(1,1,1)$ to unit sphere $x^2+y^2+z^2=1$. The functional is

$$F\{y,z\} = \int_{x_1}^{1} \sqrt{1+y'^2+z'^2}\, dx$$

The Euler-Lagrange equations are

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = 0$$

We parameterise the sphere by $(\sqrt{1-y^2-z^2}, y, z)$ and call $\phi(y,z) = \sqrt{1-y^2-z^2}$ and we have two traversality conditions

$$\left[\frac{\partial f}{\partial y'} - \frac{\partial \phi}{\partial y}\left(y'\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z'} - f\right)\right]_{x=x_1} = 0$$

$$\left[\frac{\partial f}{\partial z'} - \frac{\partial \phi}{\partial z}\left(y'\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z'} - f\right)\right]_{x=x_1} = 0$$

The E-L equations can be integrated once to yield

$$\frac{y'}{\sqrt{1+y'^2+z'^2}} = \text{const}, \qquad \frac{z'}{\sqrt{1+y'^2+z'^2}} = \text{const}.$$

which can be simplified to

$$y' = c_1 \qquad z' = c_3 \qquad \text{(constants)}$$

so

$$y = c_1 x + c_2 \qquad z = c_3 x + c_4$$

As expected extremals in $\mathbb{R}^3$ are just straight lines. Since the line must pass through $(1,1,1)$ we have

$$c_1 + c_2 = 1 \qquad c_3 + c_4 = 1.$$

Now the traverselity conditions are

(1) $$\left[\frac{y'}{\sqrt{1+y'^2+z'^2}} + \frac{y}{\sqrt{1-y^2-z^2}}\left(\frac{y'^2}{\sqrt{1+y'^2+z'^2}} + \frac{z'^2}{\sqrt{1+y'^2+z'^2}} - \sqrt{1+y'^2+z'^2}\right)\right]_{x=x_1} = 0$$

$$\left[y' - \frac{y}{\sqrt{1-y^2-z^2}}\right]_{x=x_1} = 0$$

at $x = x_1$, $y = y_1$ and $\sqrt{1-y^2-z^2} = x_1$

Also $y' = C_1$ and so we have

$$C_1 - \frac{y_1}{x_1} = 0$$

Likewise the second traverselity condition gives

(2) $$\left[z' - \frac{z}{\sqrt{1-y^2-z^2}}\right]_{x=x_1} = 0$$

$$\Rightarrow \quad C_3 - \frac{z_1}{x_1} = 0$$

In other words the traverselity conditions reduce to

$$y_1 = C_1 x_1 \qquad z_1 = C_3 x_1$$

We also have from the solution

$$y_1 = C_1 x_1 + C_2 \qquad z_1 = C_3 x_1 + C_4$$

So $C_2 = C_4 = 0$

and since $C_1 + C_2 = 1$, $C_3 + C_4 = 1$ we have $C_1 = C_3 = 1$

So the solution for the extremal is

$$y = x \qquad z = x \qquad \text{so} \quad x_1 = y_1 = z_1$$

Since it lies on the sphere: $x_1^2 + x_1^2 + x_1^2 = 1$

$$x_1^2 = \frac{1}{3}$$

So $x_1 = y_1 = z_1 = \frac{1}{\sqrt{3}}$

Where we have chosen the positive root since from Pythagoras the distance from $(1,1,1)$ to

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \quad \text{is} \quad \sqrt{3}-1$$

and to

$$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \quad \text{is} \quad \sqrt{3}+1$$

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3// $$M\{y\} = \int_0^1 y'^2(1+y')^2\,dx \qquad y(0)=0 \quad y(1)=m$$

where $-1<m<0$

The integrand depends only on $y'$ so the extremals will be straight lines and by examination

$$y' = \begin{cases} 0 & \text{or} \\ -1 \end{cases}$$

will yield $M=0$, a global minimum.

A single line of the form $y_1=\alpha$ or $y_2=\beta-x$ cannot satisfy both endpoints so we seek an extremal with a corner

$$f = y'^2 + 2y'^3 + y'^4$$

$$\phi = \frac{\partial f}{\partial y'} = 2y' + 6y'^2 + 4y'^3$$

for $y_1=\alpha$ $\quad y_1'=0 \quad$ and so $\phi=0$

for $y_2=\beta-x$ $\quad y_2'=-1 \quad$ and $\phi = -2+6-4=0$

$$H = y'p - f$$
$$= 2y'^2 + 6y'^3 + 4y'^4 - y'^2 - 2y'^3 - y'^4$$
$$= y'^2 + 4y'^3 + 3y'^4$$

for $y_1 = \alpha$ $\quad H = 0$

for $y_2 = \beta - x$ $\quad H = 1 - 4 + 3 = 0$

So for these two solution $p|_{y_1} = p|_{y_2}$ $\quad H|_{y_1} = H|_{y_2}$

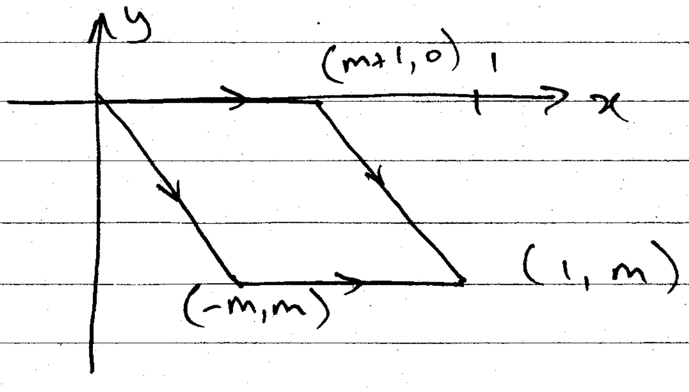
and so the Weierstrauss-Erdmann conditions are satisfied everywhere.

Using $y_1$ on the left and $y_2$ on the right we would have, $\alpha = 0$, $\beta = m+1$, corner: $(m+1, 0)$

$$y = \begin{cases} 0 & x \le m+1 \\ m+1-x & x > m+1 \end{cases}$$

Alternatively with $y_2$ on the left and $y_1$ on the right we have $\alpha = m$, $\beta = 0$, corner: $(-m, m)$

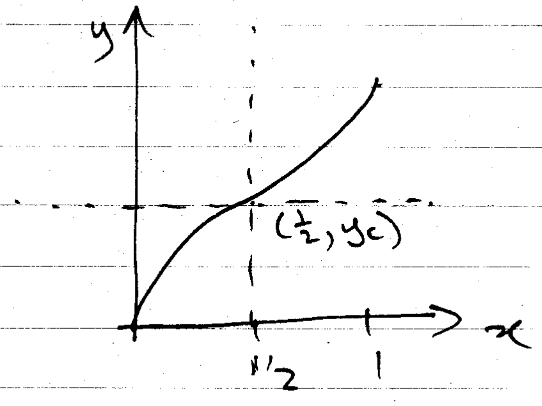
$$y = \begin{cases} -x & x \le -m \\ m & x > -m \end{cases}$$




Two solutions.

4//

The solution will look something like




It will be symmetric about the point $(\frac{1}{2}, y_c)$ so that once we have a parametric solution for $0 \le x \le \frac{1}{2}$ say $(x(\phi), y(\phi))$ for $\phi_0 \le \phi \le \phi_c$

then $x(\phi_0) = 0, \; y(\phi_0) = 0$
$x(\phi_c) = \frac{1}{2} \;\; y(\phi_c) = y_c$

Then the solution for the region $\frac{1}{2} \le x \le 1$ will be given by $(1 - x(\phi), 2y_c - y(\phi))$ for the same range of $\phi$ and $y(1) = 2y_c$

So we restrict our considerations to the half-problem

$$y(0) = 0 \quad y'(0) \to \infty \,, \quad \underbrace{\boxed{y(\tfrac{1}{2}) = y_c}}_{} \quad K\big|_{x=\frac{1}{2}} = 0$$
not prescribed.

The arclength constraint is $\int_0^{\frac{1}{2}} \sqrt{1+y'^2}\, dx = L/2$

(a) The functional with the arclength constraint is

$$H\{y\} = \int_0^{1/2} \left( \frac{y''^2}{(1+y'^2)^{5/2}} + \lambda\sqrt{1+y'^2} \right) dx$$

The working of the first part of this problem follows that of lecture 24. So I just quote the key equations and omitting much of the working we will save time.

Since $f$ is $y$-absent we have

$$\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} = \text{const}$$

but due to the free end point at $x = \frac{1}{2}$ the constant must be zero. So

$$\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{\partial f}{\partial y''}$$

Since it is $x$ & $y$ absent the chain rule yields

$$\frac{df}{dx} = y''\frac{\partial f}{\partial y'} + y'''\frac{\partial f}{\partial y''}$$

Substituting for $\frac{\partial f}{\partial y'}$ from the E-P equation we have.

$$\frac{df}{dx} = y''\frac{d}{dx}\frac{\partial f}{\partial y''} + y'''\frac{\partial f}{\partial y''}$$

Hence.

$$f - y''\frac{\partial f}{\partial y''} = \text{const.} = -\beta.$$

from this, by substituting the functional, we derive.

$$K = -\left(\lambda + \frac{\beta}{(1+y'^2)^{1/2}}\right)^{\frac{1}{2}}$$

where we choose $K$ -ve because the curve is concave down ($y'' < 0$) for $0 < x < \frac{1}{2}$

for $\theta = \tan^{-1} y' \implies K = -(\lambda + \beta\cos\theta)^{\frac{1}{2}}$

So $\quad \frac{dx}{d\theta} = -\frac{\cos\theta}{(\lambda+\beta\cos\theta)^{\frac{1}{2}}} \qquad \frac{dy}{d\theta} = -\frac{\sin\theta}{(\lambda+\beta\cos\theta)^{1/2}}$

Again we define $\quad k = \left(\frac{\lambda+\beta}{2\beta}\right)^{\frac{1}{2}} \qquad \gamma = \left(\frac{2}{\beta}\right)^{1/2}$

$$k\sin\phi = \sin\frac{\theta}{2}$$

So
$$\frac{dx}{d\phi} = -\gamma \frac{1-2k^2\sin^2\phi}{(1-k^2\sin^2\phi)^{1/2}}$$
$$\frac{dy}{d\phi} = -2\gamma k \sin\phi$$

Now. integrating.

$$x(\phi) = c_1 - \gamma\left[2E(\phi,k) - F(\phi,k)\right]$$
$$y(\phi) = c_2 + 2\gamma k\cos\phi.$$

now at $x=0$ $y' \to \infty$ so we choose $\theta = \pi/2$

so
$$\phi_0 = \sin^{-1}\left(\frac{1}{k}\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}k}\right)$$

not if $\sin\phi_0 = \frac{1}{\sqrt{2}k}$ then $\cos\phi_0 = \pm\sqrt{\frac{2k^2-1}{2k^2}}$

So $$c_1 = \gamma\left[2E(\phi_0,k) - F(\phi_0,k)\right]$$
$$c_2 = \mp 2\gamma k\sqrt{\frac{2k^2-1}{2k^2}} = \mp\gamma\sqrt{4k^2-2}.$$

Note that under our change of parameter.

$$K = -(\lambda+\beta\cos\theta)^{\frac{1}{2}} = -2\frac{k}{\gamma}\cos\phi$$

and at $x=\frac{1}{2}$ $K=0$ so this corresponds to $\phi_1 = \frac{\pi}{2}$

Looking now at the arclength constraint we have.

$$\int_0^{\frac{1}{2}}(1+y'^2)\,dx = \gamma\int_{\phi_0}^{\pi/2}\frac{d\phi}{(1+k^2\sin^2\phi)^{1/2}} = L/2$$
$$= \gamma\left(K(k) - F(\phi_0,k)\right) = L/2.$$

Our other piece of information is

$$x\left(\frac{\pi}{2}\right) = \frac{1}{2}.$$

So $\gamma\left\{\left[2E(\phi_0,k) - F(\phi_0,k)\right] - \left[2E(k) - K(k)\right]\right\} = \frac{1}{2}$.

Solving these two equations numerically we can determine $k$ and $\gamma$ and therefore the solution from $0 \le x \le \frac{1}{2}$ and by the symmetry argument also $\frac{1}{2} \le x \le 1$.

(b) for $L = 2$ using MAPLE we determine

$$k = 0.9254\ldots$$
$$\gamma = 0.6990\ldots$$

and $y_c = 0.83455\ldots$

so at $y(1) = 2y_c = 1.6691\ldots$

(c) See attached

Example MAPLE code for parts (b) and (c) of question 4 is

```
> restart;
> with(plots);
> Digits := 15;
> sinphi0 := proc (k) options operator, arrow; 1/(sqrt(2)*k) end proc;
> X := proc (phi, k, gam) options operator, arrow; gam*(2*EllipticE(sinphi0(k), k)-
  EllipticF(sinphi0(k),k)-2*EllipticE(sin(phi), k)+EllipticF(sin(phi), k)) end proc;
> Y := proc (phi, k, gam) options operator, arrow; gam*(2*k*cos(phi)+sqrt(4*k^2-2))
  end proc;
> L := proc (k, gam) options operator, arrow; gam*(EllipticK(k)-EllipticF(sinphi0(k), k))
  end proc;
> mu := L(k, gam)/X((1/2)*Pi, k, gam);
> plot([mu, 2], k = .85 .. 1);
> k0 := fsolve(mu = 2, k = .9 .. .95);
> g0 := fsolve(X((1/2)*Pi, k0, gam) = 1/2);
> yc := evalf(Y((1/2)*Pi, k0, g0));
> plot([X(phi, k0, g0), Y(phi, k0, g0), phi = (1/2)*Pi .. Pi-arcsin(sinphi0(k0))],
  color = red);
> plot([1-X(phi, k0, g0), 2*yc-Y(phi, k0, g0), phi = (1/2)*Pi .. Pi-arcsin(sinphi0(k0))],
  color = blue);
> display(%, %%, scaling = constrained);
```

The graph produced by the last line of this code is