APP MTH 3020 Stochastic Decision Theory

Tutorial 3

Week 7, Friday, September 7

1. One morning, Bobby McGee wakes up and decides to toss 5 fair coins. Let N be a random variable representing the number of tails that Bobby sees.

   a. Calculate the pmf $\boldsymbol{\pi}_N(n)$ of N, for $n = 0, 1, 2, 3, 4, 5$. What is the name of this distribution?

   b. Suppose Bobby loses \$1 (to himself!) for every tail that he sees, but wins \$1 for every head. Let M be the random variable representing the total amount that Bobby wins. What values can the random variable M take?

   c. Write down the pmf $\boldsymbol{\pi}_M(m)$.

   Suppose Bobby adopts a different strategy, in which he keeps tossing a single fair coin until he sees a head and then stop.

   e. Write down the pmf $\boldsymbol{\pi}_N(n)$, $n \geq 0$, for the number of *tails* that Bobby sees. What is the name of this distribution?

   g. Using the same betting rules as above, and assuming that Bobby has enough funds to finance this strategy, write down the pmf $\boldsymbol{\pi}_M(m)$ for the amount of money that he wins.

2. Consider a butterfly who occupies one of four patches. The patches are located at co-ordinates $\{(2,2), (2,1), (1,3), (1,1)\}$. Given the butterfly is in patch i, it migrates overnight to patch $j \neq i$ with probability $\exp(-d_{ij})$ where d_{ij} is the distance from patch i to patch j, and remains in patch i otherwise.

   a. Specify a discrete-time Markov chain $\{X_n\}$ to model the position of the butterfly. That is, define a state space $\mathcal{S}$ and transition probability matrix P.

   b. Given the butterfly is in the patch located at $(1,1)$ now, what is the probability it will be in either of the patches located at positions $(1,3)$ and $(2,2)$ tomorrow, two days later, and seven days later?

   c. How might you modify the Markov chain specified in part (a) to account for possible death of the butterfly?

3. Elli the Elephant wants to decide whether or not to go running each morning, over the course of a week, starting from Sunday (Day 0). Her health can be in one of the four states, $\{1, 2, 3, 4\}$, where 1 is when she is in her healthiest, 2 is slightly less, 3 is even less, and 4 is her worst.

   Let $f(i)$ be the cost of taking care of Ellie during a single stage (one day) when she is in state i, such that
   $$f(1) = 1, f(2) = 3, f(3) = 5, f(4) = 7.$$

   Two actions are possible each day, Run, at a cost of $R = 4$, which can take place at the beginning of the day, or Not Run.

   Each day, without running, Ellie can either remain the same, or deteriorate to a worse state of health according to $P(\text{Not Run})$. If Run is chosen, Ellie's health cannot always transition to the perfect state, but behaves according to $P(\text{Run})$. We assume the transition happens at the beginning of a stage/day and she remains in the new state until the end of the day.

   Also, assume that Ellie does not have to make a decision on the final day, Saturday, and that the cost associated with finishing in any particular state is 0, which implies $J_6(i) = 0$ for all $i = 1, 2, 3, 4$.

$$P(\text{Run}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 \\ 0.5 & 0.4 & 0.1 & 0 \end{bmatrix}, \quad P(\text{Not Run}) = \begin{bmatrix} 0.6 & 0.3 & 0.1 & 0 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

a. Find the expression for, and calculate, $J_{T-1}(i)$ for each of the states $i \in \{1, 2, 3, 4\}$. Show a detailed analysis similar to that in lectures for the calculation of these cost-to-go quantities.

b. Calculate, either by hand or with the help of MATLAB, the remaining $J_t(i)$ for each state $i \in \{1, 2, 3, 4\}$ and for all stages t over the finite horizon $T = 6$. Generate a table showing the optimal policy and costs for this problem.

c. Interpret this policy in words for Ellie.

d. If Ellie is considering her exercise routine over a much larger time horizon, $T \gg 6$, what would you suggest as the optimal policy for each state as a measure of time-to-go (that is, of the number of remaining time periods)?