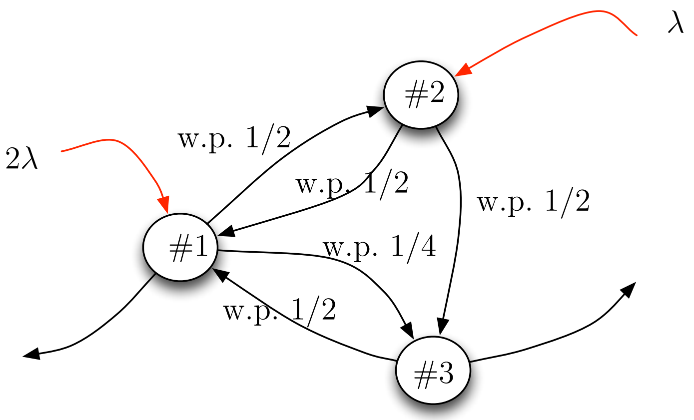
Lecture 22: Jackson Network example, and observed distributions in CTMCs

Concepts checklist

At the end of this lecture, you should be able to:

- *use* Jackson's Theorem to specify the equilibrium distribution of Open Jackson Networks;
- *state* and *prove* relationships between observed distributions by event streams; and,
- *state* PASTA theorem.

Example 17. Three single-server queues with feed-back



Consider a network of three single-server queues, with

- service rates μ_1, μ_2 and μ_3 at each respective queue when there are customers,
- exogenous arrival rates are marked (in red) as rates 2λ and λ,
- transition probabilities between nodes and those which leave the system are also marked.

Here, we have

$$\text{exogenous arrival rates} \quad \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (2\lambda, \lambda, 0),$$

$$\text{traffic routing probabilities} \quad \gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 \end{bmatrix},$$

$$\text{network departure probabilities} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \end{bmatrix}.$$

Denote by $\boldsymbol{y} = (y_1, y_2, y_3)$ total average throughput rates; then, the traffic equations are $\boldsymbol{y} = \boldsymbol{\lambda} + \boldsymbol{y}\boldsymbol{\gamma}$, or

$$y_1 = 2\lambda + \frac{1}{2}y_2 + \frac{1}{2}y_3, \tag{21}$$

$$y_2 = \lambda + \frac{1}{2}y_1, \tag{22}$$

$$y_3 = \frac{1}{4}y_1 + \frac{1}{2}y_2. \tag{23}$$

We can solve for each y_i, for example by substituting (**??**) into (**??**) and (**??**) to get

$$y_1 = 2\lambda + \frac{\lambda}{2} + \frac{1}{4}y_1 + \frac{1}{2}y_3 \quad \text{and} \quad y_3 = \frac{1}{4}y_1 + \frac{\lambda}{2} + \frac{1}{4}y_1.$$

These imply

$$y_1 = 2\lambda + \frac{\lambda}{2} + \frac{1}{4}y_1 + \frac{1}{4}y_1 + \frac{\lambda}{4} \Rightarrow \quad y_1 = \frac{11\lambda}{2}, \quad y_2 = \frac{15\lambda}{4}, \quad y_3 = \frac{13\lambda}{4}$$

and so

$$\begin{aligned}
\pi(\boldsymbol{n}) &= \pi(n_1, n_2, n_3) \\
&= \pi_1(n_1)\pi_2(n_2)\pi_3(n_3) \\
&= \left(1 - \frac{11\lambda}{2\mu_1}\right)\left(\frac{11\lambda}{2\mu_1}\right)^{n_1}\left(1 - \frac{15\lambda}{4\mu_2}\right)\left(\frac{15\lambda}{4\mu_2}\right)^{n_2}\left(1 - \frac{13\lambda}{4\mu_3}\right)\left(\frac{13\lambda}{4\mu_3}\right)^{n_3}, \\
&\text{iff } \lambda < \min\left\{\frac{2\mu_1}{11}, \frac{4\mu_2}{15}, \frac{4\mu_3}{13}\right\}.
\end{aligned}$$

Recall David Kendall's notation for queues: A/B/n/m, where A describes the arrival process, B describes the service time distribution, n is the number of servers, m is the number of waiting and service spaces (capacity) = the maximum number of customers that may occupy the system.

A and B can have many forms, some of which are M (Markov or memoryless, i.e., exponential distributions), G (General), and D (Deterministic).

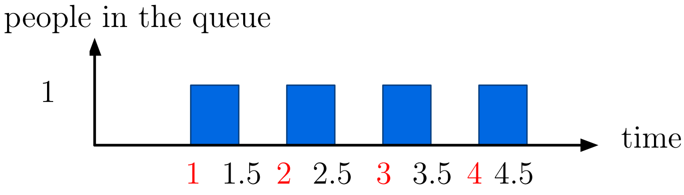
Equilibrium Distributions as seen by Arrivals

Recall that the equilibrium probabilities π_j can be interpreted as either

1. **Ergodic:** The long term proportion of time the system is in state j, or
2. **Limiting:** The probability that, at an arbitrary time point (in equilibrium $\equiv$ far away from any initial conditions), the system is in state j.

Waiting times for customers depend not on 1. or 2., but on the number of customers in the system when the customer arrives. The distribution of this number can differ from that reflected in 1. and 2.

Example 18. D/D/1 queue



Consider a $D/D/1$ queue with inter-arrival time 1 and service time 1/2.

The queue is occupied for half of the time, and so, if the queue is observed at a random moment, the probability that it will have one customer is 1/2:

$$\Rightarrow \Pr\{\text{ in equilibrium, the system has no customers in it }\} = 1/2.$$

However, arriving customers do not observe the queue at random instants, but come exactly one unit of time after the last customer. In fact, an arriving customer always sees an empty queue:

$$\Rightarrow \Pr\{\text{ an arriving customer sees no customers }\} = 1.$$

The above simple example shows that arrival time distributions are not always the same as equilibrium distributions.

Questions: Under what circumstances are they the same? If not, when can the arrival time distribution be calculated from the equilibrium distribution?

Let $\mathcal{X}$ be any CTMC with state space $\mathcal{S}$, $P_j(t)$ be the probability that the chain is in state j at time t, and γ_j be the intensity of an event stream which occurs when $\mathcal{X}$ is in state j. These *events* may, or may not, change the state of $\mathcal{X}$. For example they may be arrival or departure points, which do change the state or they may simply be an arbitrary set of points, where no state change is occurring.

We want to observe the state of the Markov Chain at these event points.

Denote by $\{\pi_j^{(E)}, j \in \mathcal{S}\}$ the distribution in equilibrium just before particular event points, and $P_j^{(E)}(t)$ the time-dependent equivalent. We can find general relationships between $\{\pi_j^{(E)}, j \in \mathcal{S}\}$ and $\{\pi_j, j \in \mathcal{S}\}$, and between $P_j^{(E)}(t)$ and $P_j(t)$.

Theorem 21.

$$P_j^{(E)}(t) = \frac{\gamma_j P_j(t)}{\sum_{k \in \mathcal{S}} \gamma_k P_k(t)} \quad \text{and} \quad \pi_j^{(E)} = \frac{\gamma_j \pi_j}{\sum_{k \in \mathcal{S}} \gamma_k \pi_k}.$$

Proof. Let $X(t)$ be the random variable representing the number in the system at time t.

$$\begin{aligned}
P_j^{(E)}(t) &= \Pr\left(\text{an event, which occurs at time } t\text{, ``sees'' state } j\right) \\
&= \lim_{h\to 0} \Pr(X(t) = j \mid \text{ event occurs in } (t, t+h)) \\
&= \lim_{h\to 0} \frac{\Pr(X(t) = j \cap \text{an event in } (t, t+h))}{\Pr(\text{an event in } (t, t+h))} \\
&= \lim_{h\to 0} \frac{\Pr(\text{an event in } (t, t+h) \mid X(t) = j) \Pr(X(t) = j)}{\sum_{k\in\mathcal{S}} \Pr(\text{event in } (t, t+h) \mid X(t) = k) \Pr(X(t) = k)} \\
&= \lim_{h\to 0} \frac{(\gamma_j h + o(h)) P_j(t)}{\sum_{k\in\mathcal{S}} (\gamma_k h + o(h)) P_k(t)} \\
&= \frac{\gamma_j P_j(t)}{\sum_{k\in\mathcal{S}} \gamma_k P_k(t)}.
\end{aligned}$$

Now, taking the limit as $t \to \infty$ gives

$$\begin{aligned}
\lim_{t\to\infty} \text{LHS} &= \lim_{t\to\infty} P_j^{(E)}(t) = \pi_j^{(E)}, \\
\lim_{t\to\infty} \text{RHS} &= \lim_{t\to\infty} \frac{\gamma_j P_j(t)}{\sum_{k\in\mathcal{S}} \gamma_k P_k(t)} = \frac{\gamma_j \pi_j}{\sum_{k\in\mathcal{S}} \gamma_k \pi_k}.
\end{aligned}$$

□

The next theorem is generally referred to as PASTA, meaning **P**oisson **A**rrivals **S**ee **T**ime **A**verages, but we present it in a more general setting. The theorem tells us that any Poisson stream, whether they are arrivals or not, sees the distribution of the Markov Chain exactly the same as if they watched it over all time.

In particular, the equilibrium distribution at the Poisson time points is exactly the same as the distribution found from the equilibrium equations (that is, the time averaged distribution).

Theorem 22 (PASTA). *If $\gamma_j = \lambda$ for all $j \in \mathcal{S}$ and $P_j(t) > 0$, then*

$$P_j^{(E)}(t) = P_j(t) \quad \textit{and} \quad \pi_j^{(E)} = \pi_j \quad \textit{for all } j \in \mathcal{S}.$$