Topic C Assignment 3

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1. (a)

$$\epsilon\frac{d^2y}{dx^2}+(\cosh x)\frac{dy}{dx}-y=0$$

With $y(0)=y(1)=1$ To leading order:

$$y=y_0+\mathcal{O}(\epsilon)$$

$$\begin{aligned}(\cosh x)\frac{dy_{0,out}}{dx}-y_0&=0\\ \frac{1}{y_{0,out}}\frac{dy_{0,out}}{dx}&=\text{sech}x\\ \log y_{0,out}&=2\arctan\left(\tanh x/2\right)\\ \boxed{y_{0,out}=a\exp\{2\arctan\left(\tanh x/2\right)\}}\end{aligned}$$

For the boundary conditions:
Let $x=x_*+\delta_1X$, and $y=\delta_2Y$

$$\epsilon\frac{\delta_2^2}{\delta_1^2}\frac{d^2Y}{dX^2}+(\cosh(x_*+\delta_1X))\frac{\delta_2}{\delta_1}\frac{dY}{dX}-\delta_2Y=0$$

With BCs $\delta_2Y(0)=\delta_2Y(1)=1$ Hence $\delta_2=1$.

$$\epsilon\frac{1}{\delta_1^2}\frac{d^2Y}{dX^2}+(\cosh(x_*+\delta_1X))\frac{1}{\delta_1}\frac{dY}{dX}-Y=0$$

Expand the cosh term:

$$\begin{aligned}\cosh(x_*+\delta_1X)&=\sinh(x_*)\sinh(\delta_1X)+\cosh(x_*)\cosh(\delta_1X)\\ &=\sinh(x_*)\sum_{i=0}^{\infty}\frac{(\delta_1X)^{2i+1}}{(2i+1)!}+\cosh(x_*)\sum_{i=0}^{\infty}\frac{(\delta_1X)^{2i}}{(2i)!}\\ &=\sinh(x_*)\left(\delta_1X\right)+\cosh(x_*)+\mathcal{O}(\epsilon^2)\end{aligned}$$

Noting that $x^*=0$ or $x^*=1$.

$$\epsilon\frac{1}{\delta_1^2}\frac{d^2Y}{dX^2}+(\sinh(x_*)\left(\delta_1X\right)+\cosh(x_*))\frac{1}{\delta_1}\frac{dY}{dX}-Y=0$$

$$\epsilon\frac{d^2Y}{dX^2}+\delta_1^2X\sinh(x_*)\frac{dY}{dX}+\delta_1\cosh(x_*)\frac{dY}{dX}-\delta_1^2Y=0$$

Either $\delta_1\sim\epsilon$ or $\delta_1\sim\sqrt{\epsilon}$
To leading order: Options are to neglect the δ term or the δ^2 terms.

- $\delta_1^2\sim\epsilon$ Neglecting δ_1 terms. Since $\epsilon\ll1$, $\sqrt{\epsilon}\gg\epsilon$ So this is not valid.
- $\delta_1\sim\epsilon$, neglect the δ^2 terms. This is reasonable.

$$\frac{d^2Y_0}{dX^2} + \cosh(x_*)\frac{dY_0}{dX} = 0$$
$$Y_0 = Ae^{-\cosh(x_*)X} + B$$

Since this is a negative exponential, this will not be valid as the limit

$$\lim_{X\to-\infty} Y_0(X) = \infty$$

And hence this holds provided $x_* = 0$, giving:

$$Y_0 = Ae^{-X} + B$$

$Y_0(0) = 1 \implies B = 1 - A$

$$\boxed{Y_0 = Ae^{-X} + 1 - A}$$

And applying $y(1) = 1$ gives:

$$y(1) = 1 = a\exp\{2\arctan(\tanh 1/2)\}$$
$$a = \exp\{-2\arctan(\tanh 1/2)\}$$

$$\boxed{Y_0 = 1 + (1 - \exp\{-2\arctan(\tanh(1/2))\})\left(e^{-X} - 1\right)}$$

$$\boxed{y_0 = \exp\{2\arctan(\tanh x/2) - 2\arctan(\tanh 1/2)\}}$$

Matching condition:

$$\lim_{x\to 0} y_0(x) = \lim_{X\to\infty} Y_0(X)$$
$$a\exp\{\arctan(\tanh(0))\} = Ae^{-\infty} + 1 - A$$
$$a = 1 - A$$
$$A = 1 - a = 1 - \exp\{-2\arctan(\tanh(1/2))\}$$

And $y_{overlap} = a$

Hence the composite solution is

$$y_{comp,0}(x, X) = y_0(x) + Y_0(X) - y_{overlap}$$
$$= a\exp\{2\arctan(\tanh x/2)\} + 1 + (1-a)\left(e^{-X} - 1\right) - a$$
$$y_{comp,0}(x) = a\exp\{2\arctan(\tanh x/2)\} + 1 + (1-a)\left(e^{-x/\epsilon} - 1\right) - a$$

(b) WKB ansatz solution for

$$\epsilon\frac{d^2y}{dx^2} + (\cosh x)\frac{dy}{dx} - y = 0$$

$$y(x) \sim \sum_{n=0}^{\infty} u_n(x)\epsilon^n + e^{-F(x)/\epsilon}\sum_{n=0}^{\infty} v_n(x)\epsilon^n$$

Leading order:

$$
\begin{aligned}
y &\sim u_0 + e^{-F/\epsilon} v_0 \\
y' &\sim u_0' + e^{-F/\epsilon} v_0' - \frac{F'}{\epsilon} e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots) \\
y'' &\sim u_0'' + e^{-F/\epsilon} v_0'' - 2\frac{F'}{\epsilon} e^{-F/\epsilon} v_0' + \left(\frac{F'}{\epsilon}\right)^2 e^{-F/\epsilon}\left(v_0 + \epsilon v_1 + \ldots\right) - \frac{F''}{\epsilon} e^{-F/\epsilon} v_0
\end{aligned}
$$

So the equation becomes:

$$
\begin{aligned}
\epsilon \frac{d^2 y}{dx^2} + (\cosh x)\frac{dy}{dx} - y = 0 \\
\epsilon \left(u_0'' + e^{-F/\epsilon} v_0'' - 2\frac{F'}{\epsilon} e^{-F/\epsilon} v_0' + \left(\frac{F'}{\epsilon}\right)^2 e^{-F/\epsilon}\left(v_0 + \epsilon v_1 + \ldots\right) - \frac{F''}{\epsilon} e^{-F/\epsilon} v_0 \right) \\
+ \cosh x \left(u_0' + e^{-F/\epsilon} v_0' - \frac{F'}{\epsilon} e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots) \right) - u_0 - e^{-F/\epsilon} v_0 = 0
\end{aligned}
$$

Giving the system:

$$
\begin{aligned}
\mathcal{O}(1) &: 0 = (\cosh x) u_0' - u_0 \\
\mathcal{O}(e^{-F/\epsilon}/\epsilon) &: 0 = \left(F'^2 v_0 - \cosh x F' v_0\right) \\
\mathcal{O}(e^{-F/\epsilon}) &: 0 = -2F' v_0' + F'^2 v_1 - F'' v_0 + \cosh x v_0' - \cosh x F' v_1 - v_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{u_0'}{u_0} &= \mathrm{sech} x \\
\log u_0 &= c + 2\arctan\left(\tanh x/2\right) \\
&\boxed{u_0 = a \exp\left(2\arctan\left(\tanh x/2\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(F'^2 v_0 - F' v_0 \cosh x\right) \\
0 &= v_0 \left(F'^2 - F' \cosh x\right)
\end{aligned}
$$

For non-trivial solutions, $v_0 \neq 0$ and hence

$$
\begin{aligned}
F'^2 - F' \cosh x &= 0 \\
F' &= \cosh x \\
F &= \sinh x + C
\end{aligned}
$$

Since the boundary layer is at $x_* = 0$, we want $e^{-F/\epsilon} = \mathcal{O}(1)$ and hence $F(0) = 0$

$$
\begin{aligned}
F(0) &= \sinh 0 + C \\
&= C = 0
\end{aligned}
$$

Hence

$$
F = \sinh x
$$

And the third equation becomes

$$
\begin{aligned}
0 &= -2F'v_0' + F'^2 v_1 - F'' v_0 + \cosh x v_0' - \cosh x F' v_1 - v_0 \\
0 &= v_1(F'^2 - F'\cosh x) - 2F'v_0' - F''v_0 + \cosh x v_0' - v_0 \\
0 &= -2F'v_0' - F''v_0 + \cosh x v_0' - v_0 \\
0 &= -2\cosh x v_0' - \sinh x v_0 + \cosh x v_0' - v_0 \\
0 &= -\cosh x v_0' - \sinh x v_0 - v_0 \\
\frac{v_0'}{v_0} &= \tanh x - \operatorname{sech} x
\end{aligned}
$$

Rather than directly solving that, ill just use the boundary conditions:

$$
\begin{aligned}
y(1) = 1 &\implies u(1) = 1 \\
\implies a &= \exp\{-2\arctan(\tanh 1/2)\} \\
y(0) = 1 &\implies u_0(0) + v_0 = 1 \\
v_0 &= 1 - a
\end{aligned}
$$

Hence

$$
y_{WKB,0} = u_0 + e^{-F/\epsilon} v_0
$$

$$
\boxed{y_{WKB,0} = a\exp(2\arctan(\tanh x/2)) + \exp(-\sinh x/\epsilon)(1-a)}
$$

Where

$$
a = \exp\{-2\arctan(\tanh 1/2)\}
$$

(c) First rewrite the BVP in a nicer format

$$
\frac{d^2y}{dx^2} + \frac{1}{\epsilon}\left(\cosh x \frac{dy}{dx} - y\right) = 0
$$

Figure 1 shows the comparison of the five solutions - the numerically obtained, inner, outer, composite and WKB ansatz solutions. Clearly the inner and outer solutions approach the 3 full solutions (numeric, composite, WKB) near the boundary values, and the full solutions match quite closely (with a gap forming around $x = 0.2$). The value $\epsilon = 0.2$ has been chosen to show that there is a small discrepancy in the answers, particularly around this gap.

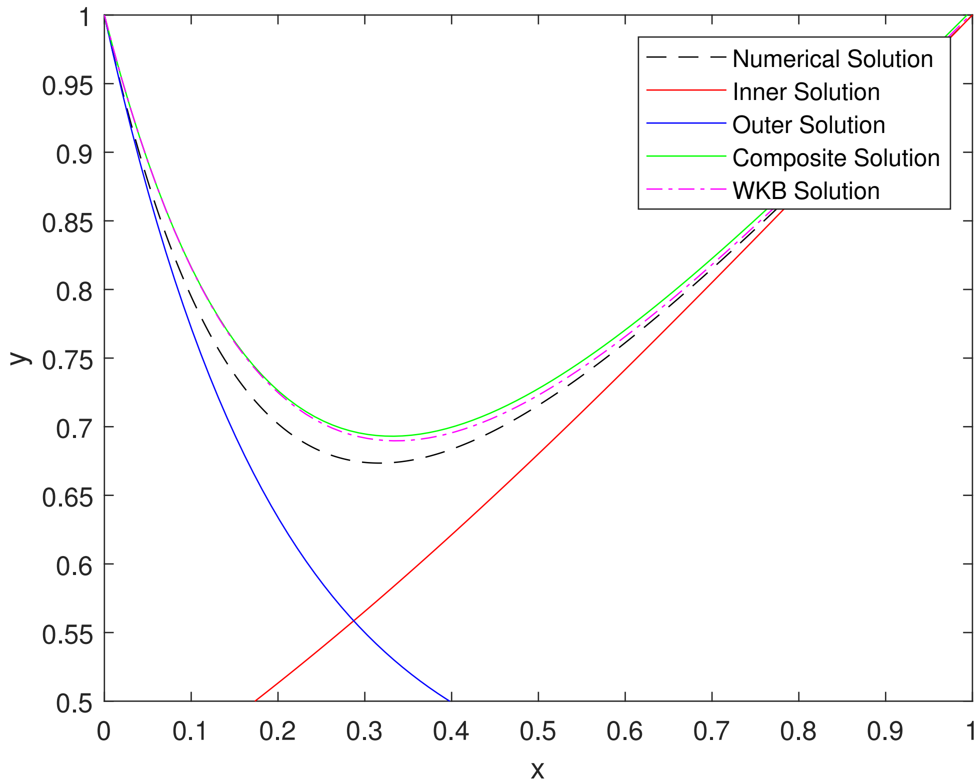

Figure 1: Comparison of Numerical, WKB and Composite solutions for $\epsilon = 0.2$

2.

$$\epsilon\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$$

With $y(-2) = -4$ and $y(2) = 2$, $\epsilon \to 0$ over $-2 \le x \le 2$. There is an internal boundary layer shown via the numerical solution, in figure 2

The right outer solution y_R with $y_R(2) = 2$ to leading order:

$$\begin{aligned} xy'_{R0} + xy_{R0} &= 0 \\ y'_{R0} + y_{R0} &= 0 \\ y_{R0} &= Ae^{-x} \end{aligned}$$

And applying the boundary condition:

$$\begin{aligned} y_{R0}(2) = Ae^{-2} &= 2 \\ A &= 2e^2 \end{aligned}$$

$$\boxed{y_{R0} = 2e^2e^{-x}}$$

The left outer solution y_L with $y_L(-2) = -4$

$$\begin{aligned} y_{L0} &= Be^{-x} \\ y_{L0}(-2) = Be^{-2} &= -4 \\ B &= -4e^{-2} \end{aligned}$$

Hence

$$\boxed{y_{L0} = -4e^{-2}e^{-x}}$$

For the inner solution $x = x_* + \delta_1 X$, and $y = \delta_2 Y$. Since the boundary conditions don't include ϵ, $\delta_2 = 1$.

$$\begin{aligned} \epsilon\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy &= 0 \\ \epsilon\frac{1}{\delta_1^2}\frac{d^2Y}{dX^2} + (x^* + \delta_1 X)\frac{1}{\delta_1}\frac{dY}{dX} + (x^* + \delta_1 X)Y &= 0 \\ \epsilon\frac{d^2Y}{dX^2} + \delta_1(x^* + \delta_1 X)\frac{dY}{dX} + \delta_1^2(x^* + \delta_1 X)Y &= 0 \\ \epsilon\frac{d^2Y}{dX^2} + \delta_1 x^*\frac{dY}{dX} + \delta_1^2 X\frac{dY}{dX} + \delta_1^2 x^* Y + \delta_1^3 XY &= 0 \\ \epsilon\frac{d^2Y}{dX^2} + \delta_1 x^*\frac{dY}{dX} + \delta_1^2\left(X\frac{dY}{dX} + x^* Y\right) + \delta_1^3 XY &= 0 \end{aligned}$$

Balances:

- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1 x^*\frac{dY}{dX}$ Hence $\delta_1 \sim \epsilon$, this is reasonable since the rejected terms will be $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ both of which are negligible.
- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1^2\left(X\frac{dY}{dX} + x^* Y\right)$ giving $\delta \sim \sqrt{\epsilon}$ neglecting terms of order $\epsilon^{1/2}$ and $\epsilon^{3/2}$. But $\epsilon^{1/2} \gg \epsilon$ so this is a contradiction.

- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1^3 XY$ with $\delta_1 \sim \epsilon^{1/3}$, meaning we have neglected the $\epsilon^{1/2}$ and $\epsilon^{1/3}$ terms in favour of ϵ. This is a contradiction since $\epsilon^{1/3} \gg \epsilon$
- $\delta_1 x^* \frac{dY}{dX} + \delta_1^2 \left(X\frac{dY}{dX} + x^* Y\right) \sim -\delta_1^3 XY$. Implying $\delta_1 \sim 1$. For $x^* = 0$ this is precisely the outer region.

Hence take $\delta_1 = \epsilon$

To leading order:

$$\begin{aligned} \frac{d^2Y_0}{dX^2} &= -x^* \frac{dY_0}{dX} \\ V' &= -x^* V \\ \implies V &= ae^{-x^* X} \\ \implies Y_0 &= a_0 e^{-x^* X} + b \end{aligned}$$

Assuming $x^* \neq 0$.

We have to match this to the left and right solutions Conditions

$$\lim_{x\to x^*} y_{R0}(x) = \lim_{X\to\infty} Y_0(X), \quad \& \quad \lim_{x\to x^*} y_{L0}(x) = \lim_{X\to-\infty} Y_0(X)$$

For non-zero x^* this gives:

$$\begin{aligned} \lim_{x\to x^*} y_{R0}(x) &= \lim_{X\to\infty} Y_0(X) \\ \lim_{x\to x^*} 2e^2 e^{-x} &= \lim_{X\to\infty} a_0 e^{-x^* X} + b \\ 2e^2 e^{-x_*} &= \lim_{X\to\infty} a_0 e^{-x^* \infty} + b \end{aligned}$$

For the right, and for the left:

$$\begin{aligned} \lim_{x\to x^*} y_{L0} &= \lim_{X\to-\infty} Y_0(X) \\ \lim_{x\to x^*} -4e^{-2} e^{-x} &= \lim_{X\to-\infty} a_0 e^{-x^* X} + b \\ -4e^{-2} e^{-x_*} &= \lim_{X\to-\infty} a_0 e^{x^* \infty} + b \end{aligned}$$

For both of these to hold, we would require $x^* = 0$ (which we assumed wasn't true). So take $x^* = 0$ and resolve the DE:

$$\epsilon\frac{d^2Y}{dX^2} + \delta_1^2 X\frac{dY}{dX} + \delta_1^3 Xy = 0$$

with $\delta_1 \sim \epsilon$

$$\begin{aligned} \frac{d^2Y_0}{dX^2} + X\frac{dY_0}{dX} &= 0 \\ V' &= -XV \\ V &= ae^{-X^2} \\ Y_0 &= \int V\,dX = \int ae^{-X^2} dX \\ &= a\mathrm{erf}(\frac{X}{\sqrt{2}}) + b \end{aligned}$$

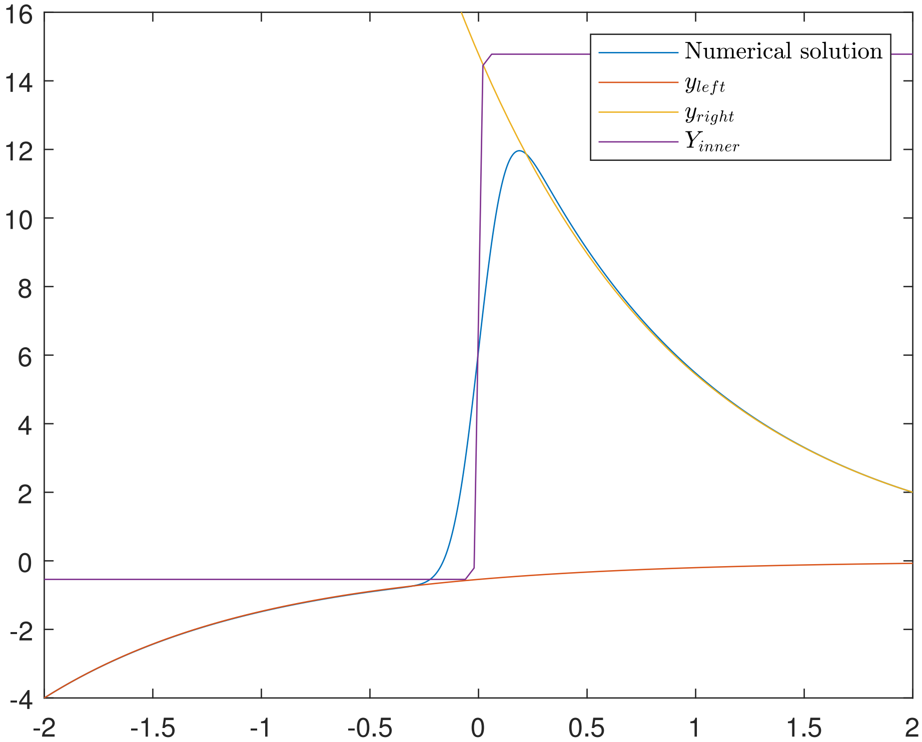

Figure 2: Plot of the internal boundary layer problem for $\epsilon = 0.1$. The left, and right outer solutions, and the inner solution are plotted.

Matching conditions:

$$\begin{aligned}
\lim_{x\to 0} y_{R0}(x) &= \lim_{X\to\infty} Y_0(X)\\
2e^2e^0 &= a\text{erf}(inf) + b\\
2e^2 &= a + b
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0} y_{L0} &= \lim_{X\to -\infty} Y_0(X)\\
-4e^{-2} &= a\text{erf}(-inf) + b\\
-4e^{-2} &= -a + b
\end{aligned}$$

$$\begin{aligned}
a + b &= 2e^2\\
-a + b &= -4e^{-2}\\
2b &= 2e^2 - 4e^{-2}\\
b &= e^2 - 2e^{-2}\\
a + e^2 - 2e^{-2} &= 2e^2\\
a &= e^2 + 2e^{-2}
\end{aligned}$$

$$\boxed{Y_0(X) = \left(e^2 + 2e^{-2}\right)\text{erf}\left(\frac{X}{\sqrt{2}}\right) + e^2 - 2e^{-2}}$$

Figure 2 shows the comparison of solutions. The outer solutions clearly match the boundary conditions, while the inner solution does not obviously match the inner region.

Matlab Code

```
%%
%%1c
close all
clear all
epsilon = 0.2;
%obtain a numerical solution to the bvp
solinit1=bvpinit(linspace(0,1,11),[0 1]);
sol1=bvp4c(@(x,y)BVPODE1(x,y,epsilon),@boundaries1,solinit1);
xout1=linspace(0,1,1001);
yout1=deval(sol1,xout1);

plot(xout1,yout1(1,:),'--k')
hold on
%my solutions
x = linspace(0,1);
xstar = 0;
X = xstar + x/epsilon;
a = exp(-2*atan(tanh(1/2)));
youter = a*exp(2*atan(tanh(x/2)));
A = 1- exp(-2*atan(tanh(1/2)));
yinner = A*exp(-X) + 1-A;
ycomp= youter + yinner -a;

%WKB solution
u0 = a*exp(2*atan(tanh(x/2)));
F = sinh(x);
v0 = 1-a;
ywkb = u0 + exp(-F/epsilon).*v0;


plot(x,youter,'r')
plot(x,yinner,'b')
plot(x,ycomp,'g')
plot(x,ywkb,'-.m')
hold off
xlabel('x')
ylabel('y')
axis([0,1,0.5,1])
legend("Numerical Solution","Inner Solution",...
    "Outer Solution","Composite Solution","WKB Solution")
saveas(gcf,"TopicCA3Q1.eps",'epsc')
%%
%%2
epsilon = 0.01;
%numerical solution to the bvp
solinit2=bvpinit(linspace(-2,2,11),[0 1]);
sol2=bvp4c(@(x,y)BVPODE2(x,y,epsilon),@boundaries2,solinit2);
xout2=linspace(-2,2,1001);
```

```matlab
yout2=deval(sol2,xout2);
figure
plot(xout2,yout2(1,:))
hold on
%my solutions
x = linspace(-2,2);
yL = -4*exp(-2)*exp(-x);
yR = 2*exp(2)*exp(-x);
%inner sol
a = exp(2) + 2*exp(-2);
b = exp(2) -2*exp(-2);
X = x/epsilon;

Y = a*erf(X/sqrt(2)) + b;
plot(x,yL)
plot(x,yR)
plot(x,Y)
hold off
axis([-2,2,-4,16])
legend("Numerical solution","$$y_{left}$$","$$y_{right}$$",...
    "$$Y_{inner}$$",'interpreter','latex')
saveas(gcf,"TopicCA3Q2.eps",'epsc')


%%%FUNCTIONS
function res=boundaries1(ya,yb)
res=[ya(1)-1;yb(1)-1];
end
function dy=BVPODE1(x,y,epsilon)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=(1/epsilon)*(-(cosh(x)*y(2))+y(1));
end


function res=boundaries2(ya,yb)
res=[ya(1)+4;yb(1)-2];
end
function dy=BVPODE2(x,y,epsilon)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=(1/epsilon)*(-x*y(2)-x*y(1));
end
```

Practical Asymptotics (APP MTH 4051/7087)
Assignment 3 (5%)

Due 10 May 2019

1. Consider the following boundary value problem.

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\cosh x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to $y(0) = y(1) = 1$, for $\epsilon \to 0$ over the interval $0 \le x \le 1$.

   (a) Find a leading-order composite solution to this problem.

   (b) Apply a leading-order WKB ansatz to find a different approximate solution.

   (c) Compare these approximations with a numerical solution and comment briefly. How well do the approximate solutions satisfy the outer boundary condition?

2. This question involves an **internal boundary layer**, a region of rapid variation located away from the edges of the domain. Find leading-order outer and inner solutions to the following problem:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0,$$

subject to $y(-2) = -4$ and $y(2) = 2$, for $\epsilon \to 0$ over the interval $-2 \le x \le 2$. As part of your solution you should identify where the internal layer is located (discussing in detail why there is no boundary layer at $x = \pm 2$).

Compare the inner and outer solutions with a numerical solution.

(Note: A composite solution is not required here, although coming up with one might be fun!)

[Hints: it would be a **very** good idea to look at a numerical solution before starting your analysis. Different outer solutions are required each side of the internal layer (perhaps call these y_L and y_R) which require their own matching conditions.]