APP MTH 3001 Applied Probability III

Class Exercise 4 Solutions

1. (a) The probability of ultimate extinction is given by

$$\begin{aligned} U_1^{\{0\}} &= \sum_{j=0}^{\infty} p_{1,j} \left(U_1^{\{0\}}\right)^j \\ &= \frac{1}{4}\left(1 + U_1^{\{0\}} + \left(U_1^{\{0\}}\right)^2 + \left(U_1^{\{0\}}\right)^3\right) \\ \Rightarrow \qquad 0 &= \left(U_1^{\{0\}}\right)^3 + \left(U_1^{\{0\}}\right)^2 - 3U_1^{\{0\}} + 1 \\ &= \left(U_1^{\{0\}} - 1\right)\left(\left(U_1^{\{0\}}\right)^2 + 2U_1^{\{0\}} - 1\right) \end{aligned}$$

Therefore we have that $U_1^{\{0\}} = 1$ or $\dfrac{-2 \pm \sqrt{2^2+4}}{2} = -1 \pm \sqrt{2}$, but as we require the minimal non-negative solution,

$$U_1^{\{0\}} = \sqrt{2} - 1 \approx 0.4142.$$

(b) Hence, we conclude that the mean number of offspring from an individual $\mu > 1$, since there is a positive probability that the population can grow without bound. In fact $\mu = \dfrac{0+1+2+3}{4} = \dfrac{6}{4} = \dfrac{3}{2}$.

2. (a) This Markov chain is irreducible. It is also recurrent, as any finite-state Markov chain must have a recurrent state (Theorem 3.21), and by Theorem 3.20, all states in this irreducible class must also be recurrent.

(b) The period of State 1 is 1, since the set of possible return times include $\{1, 2, 3, \ldots\}$. As period is also a solidarity property, all the other states must also be aperiodic. Therefore, a limiting distribution must exist.

(c) $\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$ for example.

3. Sum the first $N-1$ Global Balance Equations to get

$$\begin{aligned} \sum_{i=1}^{N-1} \pi_i &= \sum_{i=1}^{N-1}\sum_{j=1}^{N} \pi_j p_{j,i} \\ &= \sum_{j=1}^{N} \pi_j \sum_{i=1}^{N-1} p_{j,i} \quad \text{finite sums} \\ \text{or} \quad 1 - \pi_N &= \sum_{j=1}^{N} \pi_j \sum_{i=1}^{N-1} p_{j,i}, \end{aligned}$$

since $\sum_{i=1}^{N} \pi_i = 1$.

Now, by making use of the fact that $\sum_{j=1}^{N} p_{i,j} = 1$ for all $i = 1, 2, \ldots, N$ we can re-write this as

$$\begin{aligned}
1 - \pi_N &= \sum_{j=1}^{N} \pi_j \left(1 - p_{j,N}\right) \\
&= \sum_{j=1}^{N} \pi_j - \sum_{j=1}^{N} \pi_j p_{j,N} \\
&= 1 - \sum_{j=1}^{N} \pi_j p_{j,N} \\
\Rightarrow \pi_N &= \sum_{j=1}^{N} \pi_j p_{j,N},
\end{aligned}$$

which is the N^{th} Global Balance Equation. Therefore, the N^{th} Global Balance Equation is a linear combination of the first $N-1$ Global Balance Equations and hence is redundant.

4. (a) This chain has period 2, since the set of possible return times to state 1, for example, is $\{2, 4, 6, 8, \ldots\}$.

   (b) The Global Balance Equations are:

$$\begin{aligned}
\pi_i &= p\pi_{i-1} + q\pi_{i+1}, \quad i > 1, && (1) \\
\pi_1 &= \pi_0 + q\pi_2, && (2) \\
\pi_0 &= q\pi_1. && (3)
\end{aligned}$$

   Equation (1) is an infinite set of second-order linear, homogeneous, difference equations with constant coefficients. Therefore, try a solution of the form $\pi_i = m^i$ in equation (1). This gives

$$m^i = pm^{i-1} + qm^{i+1}, \quad \forall i > 1,$$

   or, on dividing by m^{i-1},

$$\begin{aligned}
m &= p + m^2 \\
\text{or} \quad (m-1)(qm - p) &= 0.
\end{aligned}$$

   Therefore, let

$$\pi_i = A + B\left(\frac{p}{q}\right)^i, \quad \forall i \geq 1.$$

   Recall that $p < q$ and hence $\frac{p}{q} < 1$. Therefore, the fact that $\sum_{i\geq 1} \pi_i < \infty$ implies that $A = 0$. Therefore, by (3), $\pi_0 = q\pi_1 = Bp$.

Now consider

$$
\begin{aligned}
\sum_{i=0}^{\infty} \pi_1 &= 1 \\
\pi_0 + \sum_{i\geq 1} \pi_i &= 1 \\
Bp + \sum_{i\geq 1} B\left(\frac{p}{q}\right)^i &= 1 \\
Bp + B\left(\frac{\frac{p}{q}}{1-\frac{p}{q}}\right) &= 1 \\
B\frac{p\left(1-\left(\frac{p}{q}\right)\right)+\frac{p}{q}}{1-\frac{p}{q}} &= 1 \\
B\frac{p+(1-p)\left(\frac{p}{q}\right)}{1-\frac{p}{q}} &= 1 \\
B\frac{p+q\left(\frac{p}{q}\right)}{1-\frac{p}{q}} &= 1 \\
B\frac{2p}{1-\frac{p}{q}} &= 1 \\
B &= \frac{1-\frac{p}{q}}{2p}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\pi_i &= \frac{1}{2p}\left(1-\frac{p}{q}\right)\left(\frac{p}{q}\right)^i, \quad \forall i \geq 1, \\
\pi_0 &= q\pi_1 = \frac{1}{2}\left(1-\frac{p}{q}\right).
\end{aligned}
$$

(c) The Partial Balance Equations on the sets $\{0, 1, \ldots, n\}$ can be written as

$$
\begin{aligned}
p\pi_n &= q\pi_{n+1}, \quad \forall n \geq 1, \\
\pi_0 &= q\pi_1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\pi_{n+1} &= \pi_n\left(\frac{p}{q}\right) \\
&= \pi_{n-1}\left(\frac{p}{q}\right)^2 \\
&= \pi_1\left(\frac{p}{q}\right)^n, \quad n \geq 1.
\end{aligned}
$$

We now have an expression for all π_n in terms of π_1, so let's now normalise to evaluate π_1.

$$\begin{aligned}
\sum_{n=0}^{\infty} \pi_n &= 1 \\
\pi_0 + \pi_1 + \sum_{n=2}^{\infty} \pi_n &= 1 \\
q\pi_1 + \pi_1 + \sum_{n=1}^{\infty} \pi_1 \left(\frac{p}{q}\right)^n &= 1 \\
\pi_1 \left[q + \sum_{n=0}^{\infty} \left(\frac{p}{q}\right)^n \right] &= 1 \\
\pi_1 \left[q + \frac{1}{1 - \left(\frac{p}{q}\right)} \right] &= 1, \quad \text{because } p < q, \\
\pi_1 \left[\frac{(q-p) + 1}{1 - \left(\frac{p}{q}\right)} \right] &= 1 \\
\pi_1 \left[\frac{(q-p) + (p+q)}{1 - \left(\frac{p}{q}\right)} \right] &= 1 \\
\pi_1 \left[\frac{2q}{1 - \left(\frac{p}{q}\right)} \right] &= 1 \\
\pi_1 &= \frac{1 - \left(\frac{p}{q}\right)}{2q} \\
\pi_1 &= \frac{1 - \left(\frac{p}{q}\right)}{2p} \left(\frac{p}{q}\right).
\end{aligned}$$

Hence

$$\pi_0 = q\pi_1 = q\frac{1 - \left(\frac{p}{q}\right)}{2p}\left(\frac{p}{q}\right) = \frac{1 - \left(\frac{p}{q}\right)}{2}.$$

(d) Consider the above argument, however, stop at the line

$$\pi_1 \left[q + \sum_{n=0}^{\infty} \left(\frac{p}{q}\right)^n \right] = 1.$$

If $p \geq q$, then $\frac{p}{q} \geq 1$, in which case the geometric series diverges. This implies that there is no value of π_1 that can solve the above equation and hence there is no solution to the Partial Balance Equations (and the normalising condition). Therefore, the stationary distribution can not exist.