– 2015 exam –

① (a) x_0 is a limit point of S if $\forall\ \varepsilon>0$
$I_\varepsilon(x_0)\cap S\setminus\{x_0\}\neq\emptyset$

or/ $\exists$ seq. (x_n) in $S\setminus\{x_0\}$ s.th. $x_n\to x_0$.

(b) $\sqrt{5}$ is a limit point of $\mathbb{Q}$: every open interval $(\sqrt{5}-\varepsilon,\sqrt{5}+\varepsilon)$ contains rational numbers since $\mathbb{Q}$ is dense in $\mathbb{R}$.

(c) $\forall\ \varepsilon>0\ \ \exists\ \delta>0$ s.th. $\forall\ x\in S$ if $0<|x-x_0|<\delta$ then $|F(x)-L|<\varepsilon$.

(Aside: we should be able to negate this definition).

(d) $F:S\to\mathbb{R}$, x_0 limit point of S

Know: if (x_n) is a seq. in $S\setminus\{x_0\}$ s.th. $x_n\to x_0$ then $F(x_n)\to L$.

Show: $\lim\limits_{x\to x_0}F(x)=L$ (Use proof by contradiction).

Suppose $F(x)$ does not approach L as $x\to x_0$.
$\therefore\ \exists\ \varepsilon>0$ s.th. $\forall\ \delta>0\ \exists\ x\in S$ s.th.
$0<|x-x_0|<\delta$ and $|F(x)-L|\geq\varepsilon$.

Idea: Let $\delta=\frac{1}{n}$, $n=1,2,3,\ldots$

For each $n\in\mathbb{N}$ $\exists\ x_n\in S$ s.th. $0<|x_n-x_0|<\frac{1}{n}$ and $|F(x_n)-L|\geq\varepsilon$.

Since $|x_n-x_0|>0$, $x_n\neq x_0$ $\therefore$ (x_n) is a seq. in $S\setminus\{x_0\}$. Squeeze $Th^{\underline{m}}$ $\Rightarrow |x_n-x_0|\to 0$ $\therefore x_n\to x_0$.

Contradiction. ($F(x_n)\not\to L$).

② (a) $L(f,\mathcal{P}) = \sum_{i=1}^{N} m_i(f)(x_i - x_{i-1})$

where $m_i(f) = \inf_{x\in[x_{i-1},x_i]} f(x)$

$U(f,\mathcal{P}) = \sum_{i=1}^{N} M_i(f)(x_i - x_{i-1})$,

$M_i(f) = \sup_{x\in[x_{i-1},x_i]} f(x)$.

(b) $L(f) = \sup\{L(f,\mathcal{P}) \mid \mathcal{P}$ partition of $[a,b]\}$

$U(f) = \inf\{U(f,\mathcal{P}) \mid$ —— " —— $\}$.

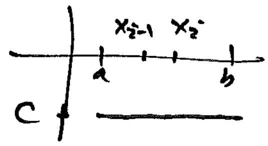
($L(f,\mathcal{P}) \leq U(f,\mathcal{P}') \quad \forall\, \mathcal{P}, \mathcal{P}'$

$\Rightarrow$ inf & sup exist).

f is integrable on $[a,b]$ if $L(f) = U(f)$.

(could you use $\forall \varepsilon > 0\ \exists$ partition $\mathcal{P}_\varepsilon$ of $[a,b]$ s.t. $U(f,\mathcal{P}_\varepsilon) - L(f,\mathcal{P}_\varepsilon) < \varepsilon \iff f$ integrable).

(c) <u>Show</u> : $L(f) = U(f)\ \ (= C(b-a))$.



Let $\mathcal{P}$ be a partition of $[a,b]$.

$m_i(f) = C = M_i(f) \qquad f(x) = C\ \forall x$.

$\therefore\ L(f,\mathcal{P}) = \sum_{i=1}^{N} C(x_i - x_{i-1}) = C(x_1 - x_0 + x_2 - x_1 + \dots + x_N - x_{N-1})$

$= C(b-a)$.

$U(f,\mathcal{P}) = \sum_{i=1}^{N} C(x_i - x_{i-1}) = C(b-a)$.

$\therefore\ L(f) = \sup\{C(b-a)\} = C(b-a) = \inf\{C(b-a)\} = U(f)$

$\therefore\ f$ is int. & $\int_a^b f(x)\,dx = U(f) = L(f) = C(b-a)$.

③ (a) Show $\left|\int_c^d g(t)\,dt\right| \leq M|c-d|$, Know: $|g(t)| \leq M$ $\forall t$.

$c \leq d$ (g int.)

$$\left|\int_c^d g(t)\,dt\right| \leq \int_c^d |g(t)|\,dt$$

$|g(t)| \leq M$
$$\leq \int_c^d M\,dt$$
$$= M(d-c)$$
$$= M|c-d|$$

$d \leq c$

$$\left|\int_c^d g(t)\,dt\right| = \left|-\int_d^c g(t)\,dt\right|$$
$$= \left|\int_d^c (-g(t))\,dt\right|$$

$|-g(t)| = |g(t)| \leq M$

$$\therefore \leq M|c-d|$$

by this case

(b) $F(x) = \int_a^x f(t)\,dt$

L.H.S. $F(x) - F(x_0) - f(x_0)(x-x_0) = \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt - f(x_0)(x-x_0)$

$\left(\int_a^c f = \int_a^b f + \int_b^c f \quad a \leq b \leq c.\right)$ $\quad = \int_{x_0}^x f(t)\,dt - f(x_0)(x-x_0)$

$a \leq x_0 \leq x$

$\int_a^{x_0} f + \int_{x_0}^x f = \int_a^x f \Rightarrow \int_a^x f - \int_a^{x_0} f = \int_{x_0}^x f$

$a \leq x \leq x_0$

$\int_a^x f + \int_x^{x_0} f = \int_a^{x_0} \Rightarrow \int_a^x f - \int_a^{x_0} f = -\int_x^{x_0} f = \int_{x_0}^x f$

equal?

R.H.S. $\int_{x_0}^x (f(t) - f(x_0))\,dt = \int_{x_0}^x f(t)\,dt - \int_{x_0}^x f(x_0)\,dt$

$$\begin{cases} \int_{x_0}^x f(x_0)\,dt = f(x_0)(x-x_0) & \text{if } x \geq x_0 \\ -\int_x^{x_0} f(x_0)\,dt = -f(x_0)(x_0-x) & \text{if } x_0 \geq x \end{cases}$$

$$= f(x_0)(x-x_0).$$

(c) $|F(x) - F(x_0) - f(x_0)(x-x_0)| = \left|\int_{x_0}^x (f(t) - f(x_0))\,dt\right|$

$$\leq M_x |x - x_0|$$

$\sup_{t \in I_x} |f(t) - f(x_0)| = M_x$

$\therefore |f(t) - f(x_0)| \leq M_x$ $\forall t \in I_x$

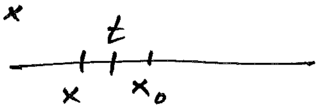
(d) Show F is diff'ble at x_0 with $F'(x_0) = f(x_0)$

$$|F(x) - F(x_0) - f(x_0)(x - x_0)| \leq M_x |x - x_0|.$$

$x \neq x_0$ $\quad \left| \dfrac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq M_x$ $\qquad \Rightarrow \qquad \lim\limits_{x \to x_0} \dfrac{F(x) - F(x_0)}{x - x_0} = f(x_0).$

Let $\varepsilon > 0$. Since f is cts at x_0 $\exists\, \delta > 0$ s.th. $\forall x \in [a,b]$ if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon/2$

$M_x = \sup\limits_{t \in I_x} |f(t) - f(x_0)|$. If $|x - x_0| < \delta$ and $t \in I_x$ then $|t - x_0| < \delta$



$\therefore |f(t) - f(x_0)| < \varepsilon/2 \quad \forall t$

$\therefore M_x \leq \varepsilon/2$

$\therefore$ if $|x - x_0| < \delta$ then $\left| \dfrac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq M_x \leq \varepsilon/2 < \varepsilon$

$\therefore$ F is diff'ble at x_0, $F'(x_0) = f(x_0)$.

4 (a) $p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \dfrac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \dfrac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

(b) if $x \in I$ then

$$f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$ where c is between x & x_0.

(c) $f(x) = \dfrac{1}{x} = x^{-1}$, $x_0 = 1$

$f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -3!\,x^{-4}$, $f^{(4)}(x) = 4!\,x^{-5}$

$f'(1) = -1$, $f''(1) = 2$, $f'''(1) = -3!$

$\therefore p_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3.$

$f(x) - p_3(x) = \dfrac{4!\,c^{-5}}{4!}(x-1)^4 = c^{-5}(x-1)^4$, c between x & 1.

(d) approx. $\ln 2 = \int_1^2 \frac{1}{x}\,dx$ $\quad f(x) = \frac{1}{x} \approx p_3(x)$.

$$\therefore \ln 2 = \int_1^2 \frac{1}{x}\,dx \approx \int_1^2 p_3(x)\,dx$$

$$= \int_1^2 \left(1 - (x-1) + (x-1)^2 - (x-1)^3\right) dx$$

$$= \left[x - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4\right]_1^2$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

$$= \frac{12 - 6 + 4 - 3}{12}$$

$$\int_1^2 \frac{1}{x}\,dx \qquad = \frac{7}{12}.$$

$$\left|\ln 2 - \int_1^2 p_3(x)\,dx\right| = \left|\int_1^2 \left(\frac{1}{x} - p_3(x)\right) dx\right|$$

$$\leq \int_1^2 \left|\frac{1}{x} - p_3(x)\right| dx \qquad 1 \leq c_x \leq x$$

$$= \int_1^2 \left|c_x^{-5}(x-1)^4\right| dx \qquad \therefore \frac{1}{c_x} \leq 1 \qquad \therefore c_x^{-5} \leq 1$$

$$\leq \int_1^2 (x-1)^4\,dx$$

$$= \left[\frac{1}{5}(x-1)^5\right]_1^2$$

$$= \frac{1}{5}.$$

5 (a) $\sum_{n=1}^{\infty} a_n$ converges $\iff (S_N)_{N=1}^{\infty}$ converges where $S_N = \sum_{n=1}^{N} a_n$.

(b) $\sum_{n=1}^{\infty} a_n$ converges absolutely $\iff \sum_{n=1}^{\infty} |a_n|$ converges

(c). $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but not absolutely.

(d) Suppose $\sum_{n=1}^{\infty} |a_n|$ converges.

Show $(S_N)_{n=1}^{\infty}$ converges, $S_N = \sum_{n=1}^{N} a_n$

<u>Idea</u>: show $(S_N)_{n=1}^{\infty}$ Cauchy.

$M_1 \geqslant M_2$

<u>Let $\varepsilon > 0$.</u>

$$|S_{M_1} - S_{M_2}| = \left|\sum_{n=1}^{M_1} a_n - \sum_{n=1}^{M_2} a_n\right|$$

$$= \left|\sum_{n=M_2+1}^{M_1} a_n\right|$$

$$\leqslant \sum_{n=M_2+1}^{M_1} |a_n|$$

$\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \left(\sum_{n=1}^{M} |a_n|\right)$ converges

$\Rightarrow \left(\sum_{n=1}^{M} |a_n|\right)$ Cauchy.

$\therefore \exists N \in \mathbb{N}$ s.th. $M_1 > M_2 \geqslant N$

$$\Rightarrow \left|\sum_{n=1}^{M_1} |a_n| - \sum_{n=1}^{M_2} |a_n|\right| < \varepsilon.$$

i.e. $\sum_{n=M_2+1}^{M_1} |a_n| < \varepsilon$.

$\therefore |S_{M_1} - S_{M_2}| < \varepsilon$.

6 (a) $r < s < R$

$\sum_{n=0}^{\infty} a_n x^n$ has r.o.c. $R \Rightarrow \sum_{n=0}^{\infty} a_n x^n$ converges abs st $|x| < R$.

$\therefore \sum_{n=0}^{\infty} a_n s^n$ converges $(|s| < R)$

$\therefore a_n s^n \longrightarrow 0$. $(\Rightarrow (a_n s^n)$ bounded$)$

(b) $|a_n| r^n = |a_n s^n| \cdot \frac{r^n}{s^n}$ $\quad$ ($\therefore \exists M > 0$ $|a_n s^n| \leqslant M \; \forall n$).

$\leqslant M \cdot \left(\frac{r}{s}\right)^n$ $\quad$ $0 \leqslant \frac{r}{s} < 1$

$\therefore \sum_{n=0}^{\infty} |a_n| r^n$ converges by comparison w/ geom. series $\sum M\left(\frac{r}{s}\right)^n$)

(c) $p_N \to f$ uniformly on $[-r, r]$

$\iff \sum_{n=0}^{\infty} \overbrace{a_n x^n}^{f_n(x)}$ converges uniformly on $[-r, r]$.

Weierstrass M-test: $|f_n(x)| \leq M_n$. $\sum_{n=0}^{\infty} M_n$ converges

$|x| \leq r$ $\quad |a_n x^n| \leq \underbrace{|a_n| r^n}_{= M_n} \quad (|x| \leq r) \ \forall n.$

& $\sum_{n=0}^{\infty} |a_n| r^n$ converges

$\therefore$ Weierstrass M-test $\Rightarrow \sum_{n=0}^{\infty} a_n x^n$

converges uniformly on $[-r, r]$.