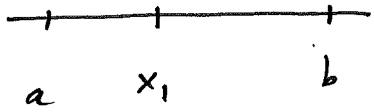
<u>Last time</u>: partitions $\mathcal{P}$ of $[a,b]$

$L(f,\mathcal{P})$, $U(f,\mathcal{P})$ $(f:[a,b]\to\mathbb{R}$ bounded$)$

$\mathcal{P}_1=\{a,b\}$



$\mathcal{P}_2=\{a,x_1,b\}$

$$m_1(f)=\inf_{x\in[a,x_1]} f(x)$$

$$m_2(f)=\inf_{x\in[x_1,b]} f(x)$$

$$L(f,\mathcal{P}_2)=m_1\Delta x_1+m_2\Delta x_2$$

$$\geq m\Delta x_1+m\Delta x_2$$

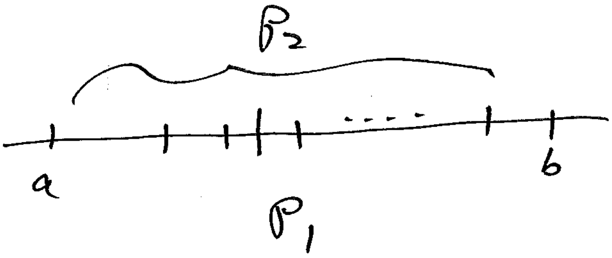
$$=L(f,\mathcal{P}_1)$$

$M_1(f)=\cdots$

$M_2(f)=\cdots$

$\therefore$ $L(f,\mathcal{P}_1)\leq L(f,\mathcal{P}_2)$

$U(f,\mathcal{P}_1)\geq U(f,\mathcal{P}_2)$.



$L(f,\mathcal{P}_1)\leq L(f,\mathcal{P}_2)$

$U(f,\mathcal{P}_1)\geq U(f,\mathcal{P}_2)$

$$U(f)=\inf\{U(f,\mathcal{P})\mid \mathcal{P} \text{ partition of } [a,b]\}$$

$$L(f)=\sup\{L(f,\mathcal{P})\mid \text{———}''\text{———}\}.$$

<u>Def^n 5.1</u>: Let $f:[a,b]\to\mathbb{R}$ be bounded. We say f is <u>integrable</u> on $[a,b]$ if $L(f)=U(f)$. If f is int. we write $\int_a^b f(x)\,dx=L(f)=U(f)$.

Suppose $\mathcal{P}_1$ & $\mathcal{P}_2$ are partitions of $[a,b]$. Then $\mathcal{P}_3=\mathcal{P}_1\cup\mathcal{P}_2$ is a common refinement of $\mathcal{P}_1$ & $\mathcal{P}_2$

$L(f,\mathcal{P}_1)\leq L(f,\mathcal{P}_3)$ ($\mathcal{P}_3$ refines $\mathcal{P}_1$).

$U(f,\mathcal{P}_2)\geq U(f,\mathcal{P}_3)$ ($\mathcal{P}_3$ refines $\mathcal{P}_2$).

$L(f,\mathcal{P}_3)\leq U(f,\mathcal{P}_3)$ (true for any partition $\mathcal{P}_3$).

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_3) \leq U(f, \mathcal{P}_3) \leq U(f, \mathcal{P}_2)$$

$$\Rightarrow \boxed{L(f, \mathcal{P}_1) \leq U(f, \mathcal{P}_2)}$$ true for all partitions $\mathcal{P}_1, \mathcal{P}_2$.

Lemma 5.2 : $L(f) \leq U(f)$ for any bded f^n $f: [a,b] \to \mathbb{R}$.

Pf: Let $\mathcal{P}_1$ be a partition of $[a,b]$. Then

$L(f, \mathcal{P}_1) \leq U(f, \mathcal{P})$ for any partition $\mathcal{P}$.

$\therefore$ $L(f, \mathcal{P}_1)$ is a lower bound for

$\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition} \ldots\}$.

$\therefore$ $L(f, \mathcal{P}_1) \leq U(f)$ (true for any $\mathcal{P}_1$).

$\therefore$ $L(f) \leq U(f)$

Ex 1. $f: [a,b] \to \mathbb{R}$ $\quad$ f is int. on $[a,b]$ &

$f(x) = c$. $\quad$ $\int_a^b f(x)\,dx = c(b-a)$.

$\mathcal{P}$ partition of $[a,b]$:

$$L(f, \mathcal{P}) = \sum_{i=1}^{N} m_i(f) \Delta x_i = \sum_{i=1}^{N} c \Delta x_i = c(b-a).$$

Similarly $U(f, \mathcal{P}) = c(b-a)$.

$\therefore$ $L(f) = c(b-a) = U(f)$.

Ex 2. $f: [a,b] \to \mathbb{R}$.

$$f(x) = \begin{cases} 1 & \text{if } x \in [a,b] \cap \mathbb{Q} \\ 0 & \text{if } x \in [a,b] \setminus \mathbb{Q}. \end{cases}$$

Then f is not integrable.

Let $\mathcal{P}$ be a partition. Then $m_i(f) = 0$

$\inf_{x \in [x_{i-1}, x_i]} f(x)$ $\quad$ $M_i(f) = 1$

$$L(f, \mathcal{P}) = \sum_{i=1}^{N} m_i(f) \Delta x_i = 0 \qquad L(f) = 0$$

$$U(f, \mathcal{P}) = b - a \neq 0. \qquad \text{but } U(f) = b-a > 0.$$

Ex 3. If $f : [a,b] \to \mathbb{R}$ is increasing (& bounded).
(decreasing)
then f is integrable.

- exercise.

Th^m 5.3 : Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is integrable $\iff$ $\forall\ \varepsilon > 0$ there exists a partition P_ε of $[a,b]$ s.th.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Pf: ($\Rightarrow$) Suppose f is int. Let $\varepsilon > 0$.

$\therefore\ \exists\ P_\varepsilon'$ s.th. $U(f) + \frac{\varepsilon}{2} > U(f, P_\varepsilon')$. — ①

$\exists\ P_\varepsilon''$ s.th. $L(f, P_\varepsilon'') > L(f) - \frac{\varepsilon}{2}$ — ②.

Let $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$. Then

$$U(f, P_\varepsilon) \leq U(f, P_\varepsilon') \quad \text{— ③}$$
$$L(f, P_\varepsilon) \geq L(f, P_\varepsilon''). \quad \text{— ④.}$$

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \underset{③\ \&\ ④}{\leq} U(f, P_\varepsilon') - L(f, P_\varepsilon'')$$

$$\underset{①\ \&\ ②}{<} \cancel{U(f)} + \frac{\varepsilon}{2} - \cancel{L(f)} + \frac{\varepsilon}{2}$$

$$= \varepsilon \quad (f \text{ int.} \iff L(f) = U(f)).$$

($\Leftarrow$). Let $\varepsilon > 0$. Choose a partition P_ε s.th. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

$$U(f) \leq U(f, P_\varepsilon) \quad \& \quad L(f, P_\varepsilon) \leq L(f)$$

$$\therefore\ 0 \leq U(f) - L(f) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

i.e. $0 \leq U(f) - L(f) < \varepsilon$ for all $\varepsilon > 0$.

$\therefore\ U(f) = L(f)$, i.e. f is int.

Exercise: f int. on $[a,b] \iff \exists$ partitions $P_1, P_2, P_3, \ldots$ s.th. $\lim_{n\to\infty}\big(U(f,P_n) - L(f,P_n)\big) = 0$

If this holds show $\lim_{n\to\infty} U(f,P_n) = \int_a^b f(x)\,dx$

& $\lim_{n\to\infty} L(f,P_n) = \int_a^b f(x)\,dx$.

Suppose $f: [a,b] \to \mathbb{R}$ is bounded. Suppose $\exists$ seq. of partitions (P_n) s.th. $U(f,P_n) \to I$

& $L(f,P_n) \to I$.

Is f integrable?

$\cancel{U(f) \leq U(f,P_n)}$

$$\begin{array}{ccccccc} L(f,P_n) & \leq & L(f) & \leq & U(f) & \leq & U(f,P_n) \\ \downarrow & & & & & & \downarrow \\ I & & & & & & I \end{array}$$

$\therefore$ By Preserv$^{\underline{n}}$ of Ineqs. $I \leq L(f) \leq U(f) \leq I$.

$\therefore\ U(f) = L(f) = I$.

Th$^{\underline{m}}$ 5.4: Let $f: [a,b] \to \mathbb{R}$ be bounded. Then f is int $\iff \exists$ seq. of partitions (P_n) s.th. $U(f,P_n) \to I$ & $L(f,P_n) \to I$, in which case $\int_a^b f(x)\,dx = I$.