Topic C Assignment 3

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1. (a)

$$\epsilon \frac{d^2y}{dx^2} + (\cosh x)\frac{dy}{dx} - y = 0$$

To leading order:

$$y = y_0 + \mathcal{O}(\epsilon)$$

$$(\cosh x)\frac{dy_0}{dx} - y_0 = 0$$
$$\frac{1}{y_0}\frac{dy_0}{dx} = \frac{1}{\cosh x}$$

Let $x = x_* + \delta_1 X$, and $y = \delta_2 Y$

$$\epsilon \frac{\delta_2^2}{\delta_1^2}\frac{d^2Y}{dX^2} + (\cosh(x_* + \delta_1 X))\frac{\delta_2}{\delta_1}\frac{dY}{dX} - \delta_2 Y = 0$$

With BCs $\delta_2 Y(0) = \delta_2 Y(1) = 1$ Hence $\delta_2 = 1$.

$$\epsilon \frac{1}{\delta_1}\frac{d^2Y}{dX^2} + (\cosh(x_* + \delta_1 X))\frac{1}{\delta_1}\frac{dY}{dX} - Y = 0$$

Expand the cosh term:

$$\begin{aligned}\cosh(x_* + \delta_1 X) &= \cosh(x_*)\cosh(\delta_1 X) + \sinh(x_*)\sinh(\delta_1 X) \\ &= \sinh(x_*)\sum_{i=0}^{\infty}\frac{(\delta_1 X)^{2i+1}}{(2i+1)!} + \cosh(x_*)\sum_{i=0}^{\infty}\frac{(\delta_1 X)^{2i}}{(2i)!} \\ &= \sinh(x_*)(\delta_1 X) + \cosh(x_*) + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$\epsilon \frac{1}{\delta_1}\frac{d^2Y}{dX^2} + (\sinh(x_*)(\delta_1 X) + \cosh(x_*))\frac{1}{\delta_1}\frac{dY}{dX} - Y = 0$$

Possible balances:

-

$$\begin{aligned}y_{comp} &= y_{in} + y_{out} - y_{overlap} \\ &= \end{aligned}$$

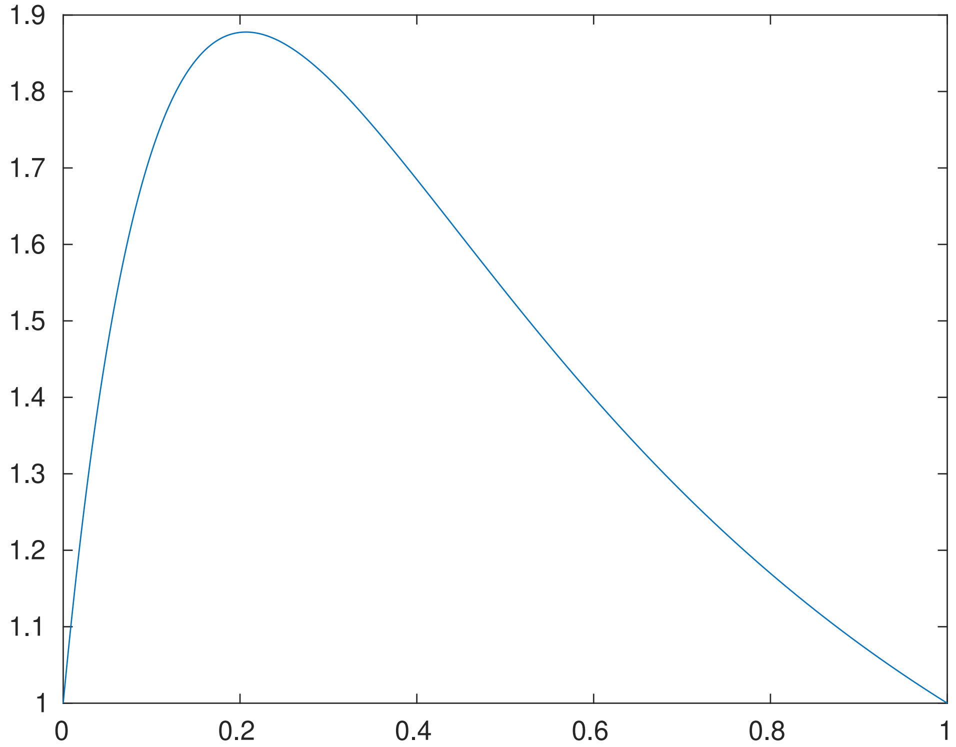

Figure 1: Comparison of Numerical, WKB and Composite solutions

(b)

(c) First rewrite the BVP in a nicer format

$$\frac{d^2y}{dx^2} + \frac{1}{\epsilon}\left(\cosh x \frac{dy}{dx} - y\right) = 0$$

Figure 1

$$\epsilon\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (2x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

2.

$$\epsilon\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$$

With $y(-2) = -4$ and $y(2) = 2$, $\epsilon \to 0$ over $-2 \leq x \leq 2$. There is an internal boundary layer shown via the numerical solution, in figure 2

The right outer solution y_R with $y_R(2) = 2$ to leading order:

$$\begin{aligned} xy'_{R0} + xy_{R0} &= 0 \\ y'_{R0} + y_{R0} &= 0 \\ y_{R0} &= Ae^{-x} \end{aligned}$$

And applying the boundary condition:

$$\begin{aligned} y_{R0}(2) = Ae^{-2} &= 2 \\ A &= 2e^2 \end{aligned}$$

The left outer solution y_L with $y_L(-2) = -4$

$$
\begin{aligned}
y_{L0} &= Be^{-x} \\
y_{L0}(-2) &= Be^{-2} = -4 \\
B &= -4e^{-2}
\end{aligned}
$$

For the inner solution $x = x_* + \delta_1 X$, and $y = \delta_2 Y$. Since the boundary conditions don't include ϵ, $\delta_2 = 1$.

$$
\begin{aligned}
\epsilon\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy &= 0 \\
\epsilon\frac{1}{\delta_1^2}\frac{d^2Y}{dX^2} + (x^* + \delta_1 X)\frac{1}{\delta_1}\frac{dY}{dX} + (x^* + \delta_1 X)Y &= 0 \\
\epsilon\frac{d^2Y}{dX^2} + \delta_1(x^* + \delta_1 X)\frac{dY}{dX} + \delta_1^2(x^* + \delta_1 X)Y &= 0
\end{aligned}
$$

Balances:

- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1^2 X\frac{dY}{dX}$, neglect $\delta_1^3 XY$, giving $\delta_1 = \sqrt{\epsilon}$, and since $\delta_1^3 = \epsilon^{3/2} \ll \epsilon = \delta_1^2$, this is reasonable.
- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1^3 XY$, neglect $\delta_1^2 X\frac{dY}{dX}$. This gives $\delta_1 = \epsilon^{1/3}$, and $\delta_1^3 = \epsilon \gg \epsilon^{2/3} = \delta_1^2$, which is a contradiction.
- $\delta_1^2 X\frac{dY}{dX} \sim -\delta_1^3 XY = 0$, neglect $\epsilon\frac{d^2Y}{dX^2}$ This gives $\delta_1 = 1$, which is the outer solution we have already solved.

Hence $\delta_1 = \sqrt{\epsilon}$

So $x = x_* + \epsilon^{1/2}X$. Take the expansion $Y(X) = Y_0 + \epsilon^{1/2}Y_1 + \dots$.

To leading order:

$$
\begin{aligned}
\frac{d^2Y_0}{dX^2} + X\frac{dY_0}{dX} &= 0 \\
\frac{V'}{V} &= -X \\
V &= e^{-X}
\end{aligned}
$$

Matlab Code

```
%%
%%1c
epsilon  = 0.1;
%obtain a numerical solution to the bvp
 solinit1 =bvpinit(linspace (0,1,11),[0   1]);
sol1=bvp4c(@(x,y)BVPODE1(x,y,epsilon),@boundaries1,solinit1);
xout1=linspace(0,1,1001);
yout1=deval(sol1,xout1);

plot(xout1,yout1(1,:))

saveas(gcf,"TopicCA3Q1.eps",'epsc')
%%
```

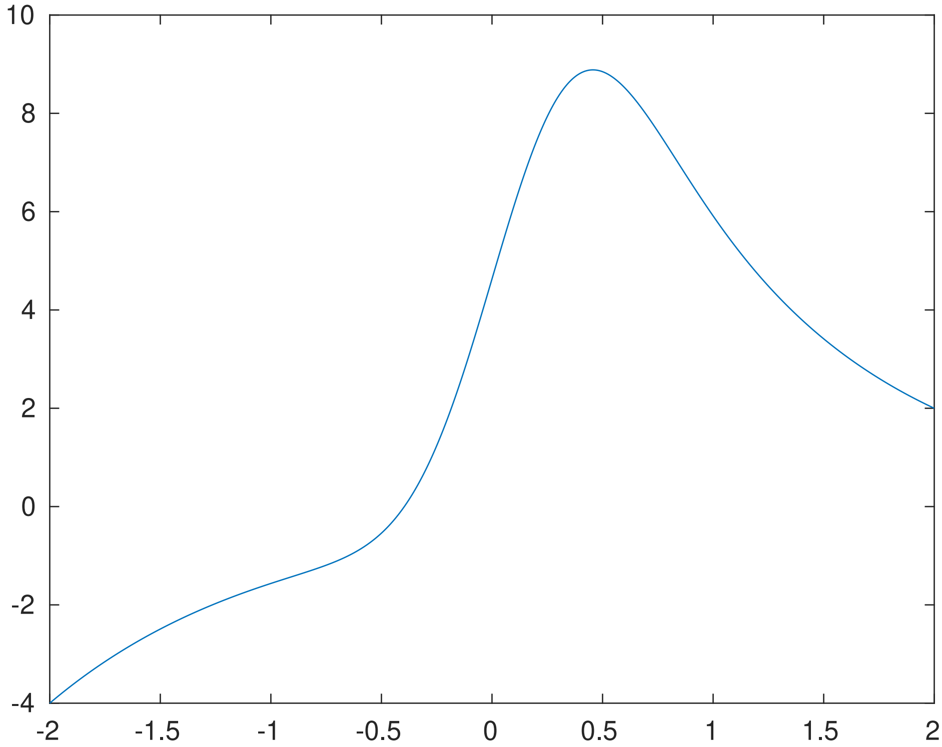

Figure 2: Caption here

```
%%2
epsilon = 0.1;
%numerical solution to the bvp
 solinit2 =bvpinit(linspace(−2,2,11),[0  1]);
sol2=bvp4c(@(x,y)BVPODE2(x,y,epsilon),@boundaries2,solinit2);
xout2=linspace(−2,2,1001);
yout2=deval(sol2,xout2);

plot(xout2,yout2(1,:))
saveas(gcf,"TopicCA3Q2.eps",'epsc')



%%%FUNCTIONS
function res=boundaries1(ya,yb)
res=[ya(1)−1;yb(1)−1];
end
function dy=BVPODE1(x,y,epsilon)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=(1/epsilon)∗(−(cosh(x)∗y(2))−y(1));
end


function res=boundaries2(ya,yb)
res=[ya(1)+4;yb(1)−2];
end
```

```
function dy=BVPODE2(x,y,epsilon)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=(1/epsilon)*(-x*y(2)-x*y(1));
end
```

Practical Asymptotics (APP MTH 4051/7087)
Assignment 3 (5%)

Due 10 May 2019

1. Consider the following boundary value problem.

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\cosh x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to $y(0) = y(1) = 1$, for $\epsilon \to 0$ over the interval $0 \leq x \leq 1$.

   (a) Find a leading-order composite solution to this problem.

   (b) Apply a leading-order WKB ansatz to find a different approximate solution.

   (c) Compare these approximations with a numerical solution and comment briefly. How well do the approximate solutions satisfy the outer boundary condition?

2. This question involves an **internal boundary layer**, a region of rapid variation located away from the edges of the domain. Find leading-order outer and inner solutions to the following problem:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0,$$

subject to $y(-2) = -4$ and $y(2) = 2$, for $\epsilon \to 0$ over the interval $-2 \leq x \leq 2$. As part of your solution you should identify where the internal layer is located (discussing in detail why there is no boundary layer at $x = \pm 2$).

Compare the inner and outer solutions with a numerical solution.

(Note: A composite solution is not required here, although coming up with one might be fun!)

[Hints: it would be a **very** good idea to look at a numerical solution before starting your analysis. Different outer solutions are required each side of the internal layer (perhaps call these y_L and y_R) which require their own matching conditions.]