APP MTH 3001 Applied Probability III
Class Exercise 1 Solutions, 2018

1. Use the fact that $\{A\} = \{A \cap \Omega\}$, and the fact that $B_i, i = 1, \ldots n$, is a partition of Ω. Then

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}\left(\cup_{i=1}^{n}(A \cap B_i)\right) && \text{(starting point in question)} \\ &= \sum_{i=1}^{n} \mathrm{P}(A \cap B_i) && \text{(as the } B_i \text{ are disjoint)} \\ &= \sum_{i=1}^{n} \mathrm{P}(A \mid B_i)\,\mathrm{P}(B_i), && \text{(by the definition of conditional probability)} \end{aligned}$$

where in the second to last step we have used the "addition law" for disjoint events, and in the last step we have used the formula for conditional probability.

2. (a) Expand the RHS using the formula for conditional probability, giving:

$$\text{RHS} = \mathrm{P}(A_1)\frac{\mathrm{P}(A_2 \cap A_1)}{\mathrm{P}(A_1)}\frac{\mathrm{P}(A_3 \cap A_2 \cap A_1)}{\mathrm{P}(A_2 \cap A_1)} \cdots \frac{\mathrm{P}(A_n \cap A_{n-1} \cap \cdots \cap A_1)}{\mathrm{P}(A_{n-1} \cap \cdots \cap A_1)}.$$

All but one of the terms in this expression cancel, leaving only the term equal to the LHS, as required.

(b) for the case $n = 3$, we have

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\,\mathrm{P}(A_2 \mid A_1)\,\mathrm{P}(A_3 \mid A_1 \cap A_2).$$

Divide both sides by $\mathrm{P}(A_1)$,

$$\frac{\mathrm{P}(A_1 \cap A_2 \cap A_3)}{\mathrm{P}(A_1)} = \mathrm{P}(A_2 \mid A_1)\,\mathrm{P}(A_3 \mid A_1 \cap A_2),$$

and use the conditional probability formula on the LHS to give

$$\mathrm{P}(A_2 \cap A_3 \mid A_1) = \mathrm{P}(A_3 \mid A_1 \cap A_2)\,\mathrm{P}(A_2 \mid A_1).$$

Let $A_3 = A$, $A_2 = B$ and $A_1 = C$ to get the required result.

3. $Y = \sum_{i=1}^{n} 1_{A_i}$, where 1_{A_i} is the indicator function that is 1 if the result is a success and 0 otherwise for each of n independent trials. Therefore

$$E[Y] = E\left[\sum_{i=1}^{n} 1_{A_i}\right] = \sum_{i=1}^{n} E[1_{A_i}] = \sum_{i=1}^{n} p = np,$$

where p is the probability of success at each independent trial.

4. (a) $\mathrm{P}(E_n) = \left(\frac{1}{2}\right)^n = 2^{-n}$, as this is just a sequence of independent Bernoulli trials .

(b) The quantity $\lim_{n\to\infty} \mathrm{P}(E_n)$ is the probability of the event

$$\{\text{no head ever turns up}\},$$

which is equal to zero (since $\lim_{n\to\infty} 2^{-n} = 0$).

(c)

$$\begin{aligned} \mathrm{P}\left(\text{head turns up eventually}\right) &= 1 - \mathrm{P}\left(\text{no head ever turns up}\right) \\ &= 1 - \lim_{n\to\infty} \mathrm{P}\left(E_n\right) \\ &= 1. \end{aligned}$$

(d) Using the hint, we can then conclude that

$$\begin{aligned} \mathrm{P}\left(S \text{ turns up eventually}\right) &= \lim_{N\to\infty} \mathrm{P}\left(S \text{ occurs somewhere in the } NK \text{ tosses}\right) \\ &\geq \lim_{N\to\infty} \mathrm{P}\left(\text{one of the } N \text{ groups is } S\right) \\ &= 1 - \lim_{N\to\infty} \mathrm{P}\left(\underline{\text{none}} \text{ of the } N \text{ groups is } S\right) \\ &= 1 - \lim_{N\to\infty} \left(1 - 2^{-K}\right)^N \\ &= 1, \end{aligned}$$

and therefore

$$\mathrm{P}\left(S \text{ turns up eventually}\right) = 1$$

since a probability cannot be greater than one.