LECTURE 11

Recall from last time the following definition.

Definition 2.8: We say (a_n) *diverges* to ∞, and we write $a_n \to \infty$, if for all $K > 0$ there exists $N \in \mathbb{N}$ such that $a_n > K$ for all $n \geq N$. Similarly we say that (a_n) *diverges* to $-\infty$, and we write $a_n \to -\infty$, if for all $K > 0$ there exists $N \in \mathbb{N}$ such that $-K < a_n$ for all $n \geq N$.

Example: Let (a_n) be the sequence defined by $a_n = \sqrt{n}$. Then $a_n \to \infty$. We prove this as follows. Let $K > 0$. We want to show that there is an $N \in \mathbb{N}$ such that $n \geq N \implies \sqrt{n} > K$. Observe that $\sqrt{n} > K \iff n > K^2$: if $\sqrt{n} > K$ then clearly $n = (\sqrt{n})^2 > K^2$. On the other hand, suppose $n > K^2$. If $\sqrt{n} \leq K$ then $n = (\sqrt{n})^2 \leq K^2$, contradiction. Hence $\sqrt{n} > K$. This observation motivates our choice of N: if $N > K^2$ then $n \geq N \implies n \geq K^2$ and hence $\sqrt{n} > K$ by what we have just observed. So let N be a natural number such that $N > K^2$ (such an N exists since $\mathbb{N}$ is not bounded above). Then $n \geq N \implies \sqrt{n} > K$. Since $K > 0$ was arbitrary it follows that $\sqrt{n} \to \infty$.

Here is an example using the Squeeze Theorem (Theorem 2.6)

Example: Let $a > 0$. Then $a^{\frac{1}{n}} \to 1$. This is obvious if $a = 1$, since then $a^{\frac{1}{n}} = 1$ for all n and hence $a^{\frac{1}{n}} \to 1$. Let's suppose that $a > 1$ — we'll see how to deal with the case where $0 < a < 1$ later.

There's an inequality that will be useful here — it's called *Bernoulli's inequality*. Suppose $x \geq 0$, then $(1 + x)^n \geq 1 + nx$ for any $n \in \mathbb{N}$. We can prove this by induction as follows. If $n = 1$ then $(1 + x)^n = 1 + x = 1 + nx$ and so the statement is true in this case. Suppose the statement is true for a natural number $n \geq 1$. Then $(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x)$ by the inductive hypothesis. We have $(1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$. Hence the statement is true for the natural number $n + 1$. Therefore the statement is true for all $n \in \mathbb{N}$ by the Principle of Mathematical Induction.

Returning to the problem at hand, observe next that since $a > 1$, $a^{\frac{1}{n}} > 1$. If not then $a^{\frac{1}{n}} \leq 1$ and hence $a = (a^{\frac{1}{n}})^n \leq 1$, a contradiction. Since $a^{\frac{1}{n}} > 1$ we have $a^{\frac{1}{n}} = 1 + x_n$, where $x_n > 0$. Therefore

$$a = (1 + x_n)^n \geq 1 + nx_n$$

by Bernoulli's inequality. Therefore $x_n \leq (a - 1)/n$. Therefore, for each $n \in \mathbb{N}$ we have

$$0 < x_n \leq (a - 1)/n.$$

Since $1/n \to 0$, we have $(a - 1)/n \to 0$. Therefore, by the Squeeze Theorem, $x_n \to 0$. Therefore, $a^{\frac{1}{n}} = 1 + x_n \to 1$.

We still have to deal with the case where $0 < a < 1$. In this case we have $1/a > 1$. Furthermore, $(1/a)^{\frac{1}{n}} = 1/a^{\frac{1}{n}}$. Hence $1/(1/a)^{\frac{1}{n}} = a^{\frac{1}{n}}$. By the case we have dealt with above, $(1/a)^{\frac{1}{n}} \to 1$. Therefore, by the Algebraic Limit Theorem,

$$a^{\frac{1}{n}} = 1/(1/a)^{\frac{1}{n}} \to 1.$$

Here is another example using the Squeeze Theorem.

Example: Prove that $(2^n + 3^n)^{\frac{1}{n}} \to 3$. The only difficulty with this example is finding the right inequality to which we can apply the Squeeze Theorem. Observe that $3^n < 2^n + 3^n \leq 2(3^n)$

for all n. Hence, $3 = (3^n)^{\frac{1}{n}} < (2^n + 3^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}}3$. By the previous example $2^{\frac{1}{n}} \to 1$ and hence $3(2^{\frac{1}{n}} \to 3$. Therefore, by the Squeeze Theorem, $2^n + 3^n)^{\frac{1}{n}} \to 3$.

You should now be able to prove without any difficulty that for any $a, b > 0$, we have $(a^n+b^n)^{\frac{1}{n}} \to \max\{ a, b \}$.

Monotonic Sequences

Definition 2.9: Let (a_n) be a sequence. We say (a_n) is

increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,

decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$,

strictly increasing if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,

strictly decreasing if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

We say (a_n) is *monotonic* if it is either increasing or decreasing.

Note: Clearly (a_n) strictly increasing $\implies$ (a_n) increasing, and (a_n) strictly decreasing $\implies$ (a_n) decreasing. Beware that the meaning of (a_n) is monotonic is *not* that sometimes it is increasing and sometimes it is decreasing!! (a_n) monotonic means that either $a_n \leq a_{n+1}$ *for all* $n \in \mathbb{N}$, or $a_n \geq a_{n+1}$ *for all* $n \in \mathbb{N}$.

Thus an increasing sequence is a sequence (a_n) whose terms satisfy

$$a_1 \leq a_2 \leq a_3 \leq \cdots$$

and a decreasing sequence is a sequence (a_n) whose terms satisfy

$$a_1 \geq a_2 \geq a_3 \geq \cdots$$

Suppose that (a_n) is increasing, and (a_n) is bounded above, i.e. there exists $K > 0$ such that $a_n \leq K$ for all $n \in \mathbb{N}$. Note that because (a_n) is increasing it is automatically bounded below by a_1, similarly any decreasing sequence is automatically bounded above by a_1.

Let $L = \sup\{ a_n \mid n \in \mathbb{N} \}$. Then $a_n \leq L$ for all $n \in \mathbb{N}$ since L is an upper bound for the set of terms of the sequence.

Intuitively we might think that because the terms of the sequence keep getting bigger, but they can never get bigger than L, they might start to 'bunch up' around L. The picture below supports this intuition.

draw picture

We would like to make this intuition mathematically precise. In other words, under the given hypotheses, we would like to prove that $a_n \to L$. Let $\epsilon > 0$. Then $L - \epsilon$ is not an upper bound for $\{ a_n \mid n \in \mathbb{N} \}$. Hence there exists $N \in \mathbb{N}$ such that $a_N > L - \epsilon$. Here comes the crucial observation: because (a_n) is increasing, if $n \geq N$ then $a_n \geq a_N$ and hence $a_n > L - \epsilon$. Therefore if $n \geq N$ then

$$L - \epsilon < a_N \leq a_n \leq L < L + \epsilon.$$

Hence $n \geq N \implies L - \epsilon < a_n < L + \epsilon$, i.e. $|a_n - L| < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows that $a_n \to L$.

Therefore we have proven the following important theorem:

Theorem 2.10: If (a_n) is bounded above and increasing, then (a_n) converges, and $\lim_{n\to\infty} a_n = \sup\{\, a_n \mid n \in \mathbb{N} \,\}$.

Corollary: If (a_n) is decreasing and bounded below, then (a_n) converges, and $\lim_{n\to\infty} a_n = \inf\{\, a_n \mid n \in \mathbb{N} \,\}$.

Proof: Note that $(-a_n)$ is increasing and bounded above, therefore converges to $\sup\{\, -a_n \mid n \in \mathbb{N} \,\}$. But $\sup\{\, -a_n \mid n \in \mathbb{N} \,\} = \sup(-\{\, a_n \mid n \in \mathbb{N} \,\} = -\inf\{\, a_n \mid n \in \mathbb{N} \,\}$. Therefore, by the Algebraic Limit Theorem, $a_n = -(-a_n) \to -(-\inf\{\, a_n \mid n \in \mathbb{N} \,\} = \inf\{\, a_n \mid n \in \mathbb{N} \,\}$. ∎

Corollary: Suppose that (a_n) is monotonic. Then (a_n) converges if and only if (a_n) is bounded.

Proof: If (a_n) converges then (a_n) is bounded. Conversely, suppose that (a_n) is bounded. Since (a_n) is monotonic, (a_n) is either increasing or decreasing. Suppose (a_n) is increasing. Then (a_n) is increasing and bounded, hence increasing and bounded above. Therefore (a_n) converges. The proof when (a_n) is decreasing is analogous. ∎

Example: Suppose that $0 < x < 1$. Let $a_n = x^n$. Since $0 < x < 1$ we see that $x > x^2 > x^3 > \cdots > 0$. Therefore (a_n) is decreasing and bounded below. Therefore (a_n) converges (we will see later that $a_n \to 0$).

Subsequences

If (a_n) is a sequence, then (roughly speaking) a subsequence of (a_n) is sequence obtained by deleting some of the terms of (a_n). For instance, consider the sequence $a_n = 1/n$:

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

Then

$$\begin{gathered} 1, 1/3, 1/5, 1/6, 1/9, \ldots \\ 1/2, 1/4, 1/6, 1/8, 1/10, \ldots \end{gathered}$$

are subsequences, but

$$\begin{gathered} 1, 1, 1/2, 1/5, 1/6, 1/6, \ldots \\ 1/2, 1/4, 1/5, 1, 1/8, \ldots \end{gathered}$$

are *not* subsequences (you are not allowed to repeat terms, and you are not allowed to swap the order of terms).

Notice that the first subsequence above $(1, 1/3, 1/5, 1/6, 1/9, \ldots)$ is obtained by picking out the 1st, 3rd, 5th, 6th,9th, …terms of the sequence. So to write down this subsequence what we need to know is the strictly increasing sequence $1, 3, 5, 6, 9, \ldots$ which tells us which terms to pick out. Likewise, the second subsequence above $(1/2, 1/4, 1/6, 1/8, 1/10, \ldots)$ is obtained by picking out the 2nd, 4th, 5th, 6th, 8th, …terms of the original sequence. Therefore, to write it down we need to know the strictly increasing sequence of natural numbers $2, 4, 6, 8, \ldots$ which again tells us which terms to pick out. So, we get a subsequence of a sequence every time we have a strictly increasing sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers.

Here is the formal mathematical definition of a subsequence.

Definition 2.11: Let $(a_n)_{n=1}^{\infty}$ be a sequence. A *subsequence* of $(a_n)_{n=1}^{\infty}$ is a sequence $(a_{\phi(n)})_{n=1}^{\infty}$ where $\phi\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

Suppose that the sequence (a_n) corresponds to a function $f\colon \mathbb{N} \to \mathbb{R}$, i.e. $a_n = f(n)$ for all $n \in \mathbb{N}$. Then the subsequence $(a_{\phi(n)})$ corresponds to the composite function $f \circ \phi\colon \mathbb{N} \to \mathbb{R}$.

For example suppose that $\phi(n) = 2n$. Then the function ϕ determines a subsequence (a_{2n}) whose terms are $a_2, a_4, a_6, \ldots$