# School of Mathematical Sciences

# Assignment Cover Sheet

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|---|---|
| Student Name | |
| Student ID | |
| Assessment Title | **Assignment 2** |
| Due Date | Thursday, 29 August, 2019 @ 12:00 noon |
| Course / Program | APP MTH 3022–Optimal Functions & Nanomechanics |
| Date Submitted | |
| **OFFICE USE ONLY**<br>Date Received | |

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# OFN Assignment 2

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September 6, 2019

1. Extremals for the functionals with $y(0) = 0, y(1) = 1$

   (a)
$$F\{y\} = \int_0^1 \left(y^2 + y'^2 + 2ye^x\right) dx$$

Euler-Lagrange:
$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= 2y + 2e^x \\ \frac{\partial f}{\partial y'} &= 2y' \\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 2y''\end{aligned}$$

Hence

$$\begin{aligned}2y'' - 2y - 2e^x &= 0 \\ y'' - y - e^x &= 0 \\ y_h'' &= y_h \\ \implies y_h &= c_1e^x + c_2e^{-x}\end{aligned}$$

Where $y_h$ is the homogeneous solution. Since the solution already contains $e^x$ try $xe^x$ for a particular solution

$$y = c_1e^x + c_2e^{-x} + c_3xe^x$$

$$y'' = c_1e^x + c_2e^{-x} + c_3(2e^x + xe^x)$$

$$\begin{aligned}y'' - y - e^x &= c_1e^x + c_2e^{-x} + c_3(2e^x + xe^x) - \left(c_1e^x + c_2e^{-x} + c_3xe^x\right) - e^x \\ 0 &= c_3\left(2ex + xe^x - xe^x\right) - e^x \\ \implies c_3 &= \frac{1}{2}\end{aligned}$$

And hence
$$y = c_1e^x + c_2e^{-x} + \frac{1}{2}xe^x$$

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ \implies c_2 &= -c_1 \\ y(1) &= c_1 e - c_1 e^{-1} + \frac{1}{2}e = 1 \\ c_1(e - e^{-1}) &= 1 - \frac{1}{2}e \\ c_1 &= \frac{1 - \frac{1}{2}e}{e - e^{-1}} \end{aligned}$$

So

$$\boxed{y = \frac{1 - \frac{1}{2}e}{e - e^{-1}}\left(e^x - e^{-x}\right) + \frac{1}{2}xe^x}$$

(b)

$$F\{y\} = \int_0^1 (y^2 - y'^2 - 2y\sin x)dx$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= 2y - 2\sin x \\ \frac{\partial f}{\partial y'} &= -2y' \\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= -2y'' \end{aligned}$$

Euler-Lagrange gives

$$\begin{aligned} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} &= 0 \\ -2y'' - 2y + 2\sin x &= 0 \\ y'' + y - \sin x &= 0 \end{aligned}$$

The homogeneous solution:

$$y_h = c_1 \cos x + c_2 \sin x$$

And particular solution can have $x\cos x$ and $x\sin x$ terms since $\cos x, \sin x$ are already in the homogeneous solution

$$y = c_1\cos x + c_2 \sin x + c_3 x\cos x + c_4 x \sin x$$

$$y'' = -(c_1\cos x + c_2\sin x) - 2c_3\sin x - c_3 x\cos x + 2c_4\cos x - c_4 x\sin x$$

$$\begin{aligned} &y'' + y - \sin x = 0 \\ &-2c_3\sin x - c_3x\cos x + 2c_4\cos x - c_4x\sin x + c_3x\cos x + c_4x\sin x - \sin x = 0 \\ &-2c_3\sin x + 2c_4\cos x - \sin x = 0 \\ &\implies c_4 = 0, \quad c_3 = -\frac{1}{2} \end{aligned}$$

Hence

$$y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$$

Using the BCs:

$$y(0) = 0, y(1) = 1$$

$$y(0) = 0 \implies c_1 = 0$$
$$y(1) = 1 \implies c_2 \sin 1 - \frac{1}{2} \cos 1 = 1$$
$$c_2 = \frac{1 + \frac{1}{2} \cos 1}{\sin 1}$$

$$\boxed{y = \frac{1 + \frac{1}{2} \cos 1}{\sin 1} \sin x - \frac{1}{2} x \cos x}$$

2. Consider

$$F\{y\} = \int_0^1 \left( \frac{1}{2} y'^2 + yy' + y' + y \right) dx, \quad y(0) = 0,\ y(1) = \frac{3}{2}$$

(a) Determine the expression for $H$ Since the functional doesn't depend on $x$

$$H(y, y') = y' \frac{\partial f}{\partial y'} - f(y, y') = const$$

$$\frac{\partial f}{\partial y'} = y' + y + 1$$

$$H(y, y') = y'^2 + yy' + y' - \left( \frac{1}{2} y'^2 + yy' + y' + y \right) = const$$

(b) Derive $y(x)$ which is an extremal

$$y'^2 + yy' + y' - \left( \frac{1}{2} y'^2 + yy' + y' + y \right) = const$$
$$y'^2 + 2y = k$$
$$y'^2 = k - 2y$$
$$y' = \sqrt{k - 2y}$$
$$\int \frac{dy}{\sqrt{k - 2y}} = \int dx$$

Sub $u = k - 2y$, $dy = -\frac{1}{2} du$

$$\implies \int \frac{-1}{2\sqrt{u}} du = x - c$$
$$-\sqrt{u} = x - c$$
$$-\sqrt{k - 2y} = x - c$$
$$k - 2y = (c - x)^2$$
$$y = \frac{k - (c - x)^2}{2}$$

And apply BCs:

$$y(0) = 0 \implies k - c^2 = 0$$

$$y(1) = \frac{3}{2} \implies k - (c-1)^2 = 3$$

Subtract the two:

$$\begin{aligned} c^2 - (c-1)^2 &= 3 \\ 2c - 1 &= 3 \\ c &= 2 \\ \implies k &= 4 \end{aligned}$$

$$\boxed{y = \frac{4 - (2-x)^2}{2}}$$

3.

$$T\{y\} = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \sqrt{\frac{1 + y'^2}{y_0 - y}} dx \tag{1}$$

With $y(x_0) = y_0$ and $y(x_1) = y_1$. We derived

$$x = x_0 + \kappa(\theta - \sin\theta), \quad y = y_0 - \kappa(1 - \cos\theta), \quad 0 \le \theta \le \theta_1 \tag{2}$$

We must determine $\theta_1$ corresponding to $x = x_1$ and determine $\kappa$.

(a) Substitute the solution 2 into the functional 1 and evaluate for an explicit form of $T$ (in terms of $\theta_1, \kappa, g$).

Start with equation 2, and use the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{-\kappa \sin\theta}{\kappa - \kappa\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta - 1} \\ &= -\cot\frac{\theta}{2} \end{aligned}$$

$$\frac{dx}{d\theta} = \kappa(1 - \cos\theta) \implies dx = \kappa(1 - \cos\theta) d\theta$$

$$\begin{aligned} \theta_0 &\implies x_0 = x_0 + \kappa(\theta_0 - \sin\theta_0) \\ \theta_0 = \sin\theta_0 &\implies \theta_0 = 0 \end{aligned}$$

We will ignore the $x_1$ case and just label it $\theta_1$ for now.

$$
\begin{aligned}
T\{y\} &= \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \sqrt{\frac{1+y'^2}{y_0-y}} dx \\
T\{\theta\} &= \frac{1}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\frac{1+\cot^2\frac{\theta}{2}}{\kappa(1-\cos\theta)} \kappa(1-\cos\theta)} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\left(1+\cot^2\frac{\theta}{2}\right)(1-\cos\theta)} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{1+\cot^2\frac{\theta}{2} - \cos\theta - \cos\theta\cot^2\frac{\theta}{2}} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{1+\left(\frac{\sin\theta}{\cos\theta-1}\right)^2 - \cos\theta - \cos\theta\left(\frac{\sin\theta}{\cos\theta-1}\right)^2} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\frac{(\cos\theta-1)^2+\sin^2\theta-\cos\theta(\cos\theta-1)^2-\cos\theta\sin^2\theta}{(\cos\theta-1)^2}} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\frac{\cos^2\theta-2\cos\theta+1+\sin^2\theta-\cos\theta(\cos^2\theta-2\cos\theta+1)-\cos\theta\sin^2\theta}{(\cos\theta-1)^2}} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\frac{-2\cos\theta+2-\cos\theta(-2\cos\theta+1)-\cos\theta}{(\cos\theta-1)^2}} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\frac{2(1-2\cos\theta+\cos^2\theta)}{(\cos\theta-1)^2}} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{\frac{2(1-\cos\theta)^2}{(\cos\theta-1)^2}} d\theta \\
&= \frac{\sqrt{\kappa}}{\sqrt{2g}} \int_{\theta_0}^{\theta_1} \sqrt{2} d\theta \\
T\{\theta\} &= \frac{\sqrt{\kappa}}{\sqrt{g}} \theta_1
\end{aligned}
$$

(b) Assume $(x_0, y_0) = (0, 2)$ and $(x_1, y_1) = (5, 1)$ determine $\theta_1, \kappa$ for 3 different solution curves.

$(x, y)$ solutions become

$$x = \kappa(\theta - \sin\theta), \quad y = 2 - \kappa(1-\cos\theta), \quad 0 \le \theta \le \theta_1$$

Solutions curves are those for which

$$
\begin{aligned}
5 &= \kappa(\theta_1 - \sin\theta_1) \\
1 &= 2 - \kappa(1-\cos\theta_1)
\end{aligned}
$$

$$\kappa = \frac{5}{\theta_1 - \sin\theta_1}$$
$$\implies 1 = 2 - \frac{5}{\theta_1 - \sin\theta_1}(1 - \cos\theta_1)$$
$$\implies \theta_1 - \sin\theta = 5(1 - \cos\theta_1)$$

Trivially $\theta = 0$ is a solution, but we will ignore this. Solutions are obtained guessed by observation (see fig 1) and then solved numerically using fzero, and are (to 4 sf)

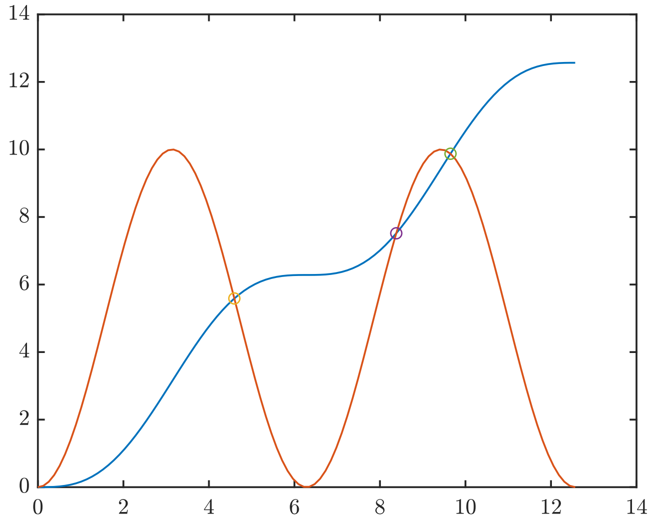


Figure 1: Intersections corresponding to solutions for $\theta_1, \kappa$

$$(\theta_1, \kappa)_1 = (4.595, 0.8948)$$
$$(\theta_1, \kappa)_2 = (8.382, 0.6651)(\theta_1, \kappa)_3 \qquad = (9.650, 0.5064)$$

(c) Take metres as the unit of length and $g = 9.807 m/s^2$, determine the value of $T$ for the three different solutions curves obtained in $(b)$. Give answers to four significant digits Corresponding to the combinations above:

$$(\theta_1, \kappa)_1 = (4.595, 0.8948) \implies T\{\theta\}_1 = 1.388$$
$$(\theta_1, \kappa)_2 = (8.382, 0.6651) \implies T\{\theta\}_2 = 2.183$$
$$(\theta_1, \kappa)_3 = (9.650, 0.5064) \implies T\{\theta\}_3 = 2.193$$

(d) Plot the curves from $(b)$ and label them with the values of $T$ calculated in $(c)$. The plots are shown here. Figures 2, 3 and 4 show the three solutions.

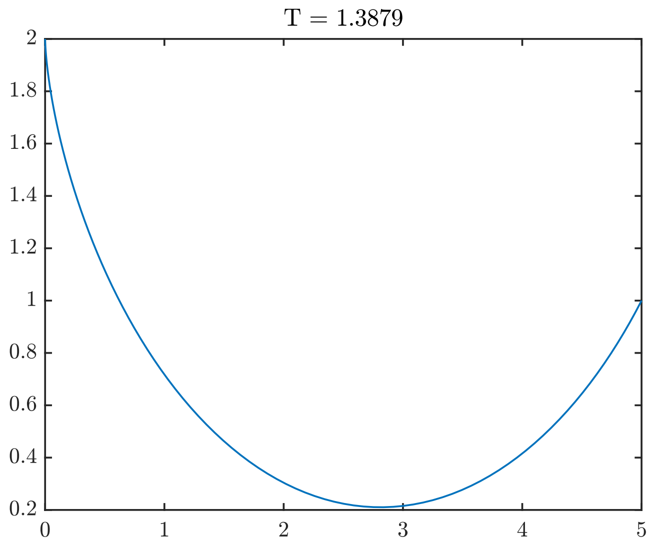


Figure 2: Solution plot for $(\theta_1, \kappa) = (4.595, 0.8948)$

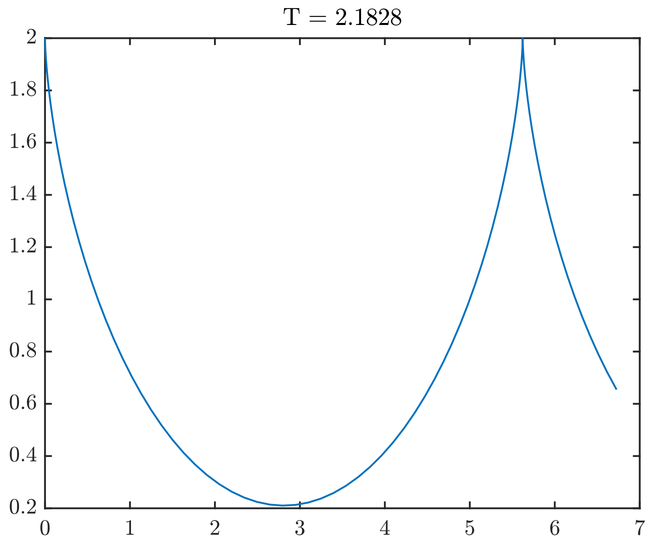


Figure 3: Solution plot for $(\theta_1, \kappa) = (8.382, 0.6651)$

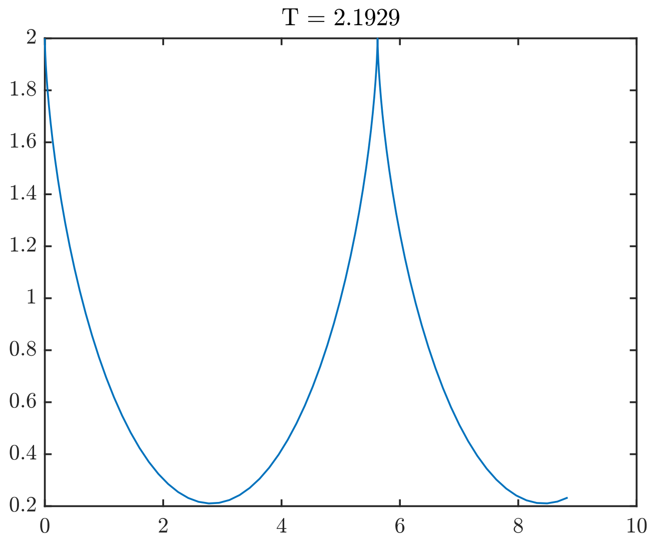


Figure 4: Solution plot for $(\theta_1, \kappa) = (9.650, 0.5064)$

The code used is below:

```
%Make plots less repulsive
set(groot, 'DefaultLineLineWidth', 1, ...
    'DefaultAxesLineWidth', 1, ...
    'DefaultAxesFontSize', 12, ...
    'DefaultTextFontSize', 12, ...
    'DefaultTextInterpreter', 'latex', ...
    'DefaultLegendInterpreter', 'latex', ...
    'DefaultColorbarTickLabelInterpreter', 'latex', ...
    'DefaultAxesTickLabelInterpreter','latex');
close all
clear all
%intersection curves (observation)
t = linspace(0,4*pi)
plot(t,t -sin(t))
hold on
plot(t,5*(1-cos(t)))

%solve the nonlinear equation for theta ...
%(guesses based on observation)
t1 = fzero(@(t) 1 - 5/(t-sin(t))*(1-cos(t)),4)
t2 = fzero(@(t) 1 - 5/(t-sin(t))*(1-cos(t)),8)
t3 = fzero(@(t) 1 - 5/(t-sin(t))*(1-cos(t)),10)
%get kappa from the theta solutions
k1 = 5/(t1 - sin(t1))
k2 = 5/(t2 - sin(t2))
k3 = 5/(t3 - sin(t3))

%add them to plot to check
scatter(t1,t1-sin(t1))
scatter(t2,t2-sin(t2))
scatter(t3,t3-sin(t3))
saveas(gcf,'IntersectionPlot.eps','epsc')
%parametric solutions to plot
x = @(k,t) k*(t-sin(t));
y = @(k,t) 2 - k*(1-cos(t))
T = @(k,t1) sqrt(k)/sqrt(9.807) * t1;
%first sol
theta1 = linspace(0,t1);
figure
plot(x(k1,theta1),y(k1,theta1))
T1 = T(k1,t1)
title("T = " + num2str(T1))
saveas(gcf,'Sol1.eps','epsc')
%second sol
theta2 = linspace(0,t2);
figure
plot(x(k1,theta2),y(k1,theta2))
T2 = T(k2,t2)
title("T = " + num2str(T2))
```

```
saveas(gcf,'Sol2.eps','epsc')

%third sol
theta3 = linspace(0,t3);
figure
plot(x(k1,theta3),y(k1,theta3))
T3 =T(k3,t3)
title("T = " + num2str(T3))
saveas(gcf,'Sol3.eps','epsc')
```