THE UNIVERSITY of ADELAIDE

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Examination in the School of Mathematical Sciences

Semester 1, 2017

107354 APP MTH 7035 Modelling with Ordinary Differential Equations — PG

Time for completing booklet: 180 mins (plus 10 mins reading time).

Question	Marks	
1	/15	
2	/13	
3	/18	
4	/20	
5	/19	
6	/15	
Total	/100	

Instructions to candidates

- Attempt all questions and write your answers in the space provided below that question.
- If there is insufficient space below a question, then use the space to the *right* of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

- Calculators are not permitted.

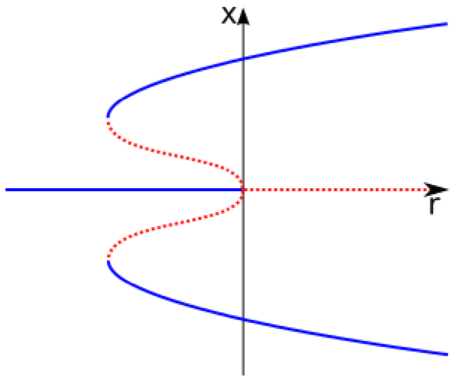
Do not commence writing until instructed to do so.

15 Total

Question 1.

Decide whether each of the following statements is true or false. Write a paragraph of explanation to support each answer. Try to be as precise as possible, referring to theorems from lectures where appropriate. You may also show an example or draw a figure if this assists your explanation.

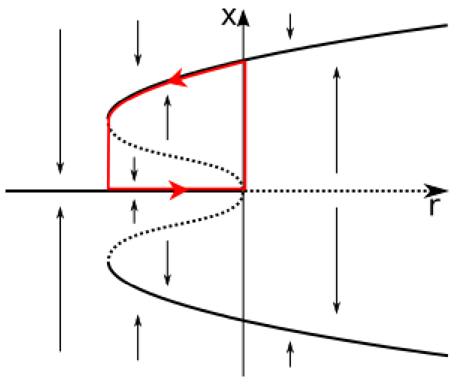
1(a) The ODE $\dot{x} = f(x; r)$ with the bifurcation diagram below can demonstrate hysteresis. (Solid blue lines indicate stable branches and broken red lines denote unstable branches.)



/5 marks

Solution **True.** ※ [2 marks]

A possible hysteresis loop is shown below.※ [3 marks]



/5 marks

1(b) Every closed orbit is a limit cycle.

Solution False. ※

[2 marks]

A limit cycle must be an *isolated* closed orbit. This is not the case, e.g., for the linear 2D system

$$\dot{x} = \beta y \quad \text{and} \quad \dot{y} = -\beta x$$

where all trajectories around the origin, which is a centre, are closed orbits. ※

[3 marks]

Please turn over for page 4.

1(c) The finite difference formula

$$x_{n+2} = (2 + 6h^2)x_{n+1} - x_n$$

is consistent with the ODE

$$\frac{d^2x}{dt^2} = 6x.$$

/5 marks

Solution True. ※

[2 marks] First shift the indices down 1 to give:

$$x_{n+1} = (2 + 6h^2)x_n - x_{n-1},$$

then expand around $t = t_n$:

$$x_n + hx'_n + \frac{h^2}{2!}x''_n + \frac{h^3}{3!}x'''_n + \frac{h^4}{4!}x^{iv}_n + \dots$$
$$= (2 + 6h^2)x_n - \left(x_n - hx'_n + \frac{h^2}{2!}x''_n - \frac{h^3}{3!}x'''_n + \frac{h^4}{4!}x^{iv}_n + \dots\right)$$

[1 mark] ※ and rearranging, we have

$$2\left(\frac{h^2}{2!}x''_n + \frac{h^4}{4!}x^{iv}_n + O(h^6)\right) = 6h^2x_n$$
$$x''_n + O(h^2) = 6x_n.$$

[1 mark]
[1 mark] ※ In the limit $h \to 0$ we have the ODE $x'' = 6x$.※

13 Total

Question 2.

Consider the ordinary differential equation

$$\frac{dx}{dt} = rx + \frac{1}{1+x} \quad \text{for} \quad x > -1, \tag{1}$$

where r is a real positive parameter.

/1 mark

2(a) Determine the steady states of this equation.

Solution

$$x^* = \frac{1}{2}\Big\{ -1 \pm \sqrt{1 - 4/r} \Big\}$$

[1 mark] ※

/2 marks

2(b) Use your answer to 2(a) to determine the bifurcation value $r = \overline{r}$. Hence calculate the bifurcation point $x = \overline{x}$.

Solution From 2(a) we see that the number of steady states goes from zero for $0 < r < 4$ to two for $r > 4$. Clearly there is a bifurcation at the ※ bifurcation value $r = 4$ and the ※ bifurcation point $x^* = -\frac{1}{2}$ (the steady state for $r = 4$).

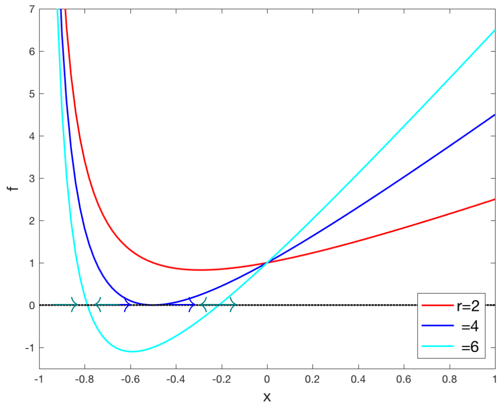
[1 mark]
[1 mark]

Please turn over for page 6.

/4 marks

2(c) Perform a phase-line analysis for $r < \overline{r}$, $r = \overline{r}$ and $r > \overline{r}$. Include arrows to indicate the stability on the fixed points, and state the stability of the fixed points, explaining your reasoning.

Solution



[2 marks] ※ Figure (must include arrows).

For $r = \overline{r} = 4$ the single fixed point is semi-stable, as it is stable below and unstable above. ※

[2 marks] For $r > \overline{r}$ the smaller fixed point is stable as the gradient through it is negative, and the larger fixed pint is unstable as the gradient is positive.

Any reasonable explanation of stability should be accepted.

/2 marks

2(d) State the type of bifurcation that occurs at $r = \overline{r}$, and give a reason.

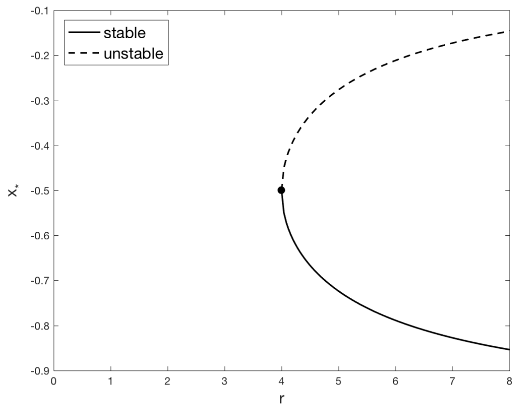
[1 mark] **Solution** A saddle-node bifurcation ※ occurs at $r = \overline{r}$, as a pair of fixed points appear/disappear as the parameter

[1 mark] passes through that point.※

/4 marks

2(e) Sketch the bifurcation diagram, marking the stable and unstable branches and the bifurcation point and value.

Solution



[2 marks] ※ for accurate bifurcation curves

[1 mark] ※ for correct stabilities

[1 mark] ※ for bifurcation point

Please turn over for page 8.

18 Total

Question 3.

Consider the modified predator–prey system

$$\frac{dx}{dt} = x(1-x) - xy \tag{2a}$$

$$\frac{dy}{dt} = y\left(1 - \frac{y}{x}\right) \tag{2b}$$

for $x > 0$ and $y \geq 0$.

/2 marks

3(a) Interpret Eq. (2a) when $y = 0$.

Solution This is the logistic model

$$\frac{dx}{dt} = x(1-x).$$

[1 mark] ※ For small x the rate of increase is x (i.e. exponential growth). As x approaches the carrying capacity $x = 1$ the growth rate

[1 mark] decreases to zero. ※

/2 marks

3(b) In system (2a–b), which of x and y is the predator and which is the prey? Justify your answer.

Solution The term $-xy$ reduces the x-population as the y-population increases.

The term $1-y/x$ reduces the y-population as the x-population

[1 mark] decreases. ※

[1 mark] Therefore, y is the predator and x is the prey. ※

3(c) Calculate the nullclines of the system, and hence calculate the two relevant fixed points.

/4 marks

Solution The y-nullcline is $y(1-y/x) = 0$, ※ which implies that

[1 mark]

$$y = 0 \qquad \text{or} \qquad y = x.$$

The x-nullcline is

$$x(1-x) - xy = 0 \quad \Rightarrow \quad y = \frac{x(1-x)}{x}.$$

[1 mark] ※

Substituting $y = 0$ gives $x(1-x) = 0$, which implies that

$$x = 0 \qquad \text{or} \qquad x = 1.$$

Substituting $y = x$ into the x-nullcline gives $x(1-2x) = 0$, which implies that

$$x = 0 \qquad \text{or} \qquad x = 0.5.$$

The three fixed points are

$$(x, y) = (0, 0), \quad (0.5, 0.5), \quad (1, 0).$$

However, the origin is out of the applicable range, leaving two relevant fixed points. ※

[2 marks]

Please turn over for page 10.

/5 marks

3(d) Calculate the Jacobian of the system, and hence classify the two fixed points.

Solution The Jacobian of the system is

$$J = \begin{pmatrix} 1-2x-y & -x \\ (y/x)^2 & 1-2y/x \end{pmatrix}.$$

[1 mark] ※

At $(x, y) = (1, 0)$ the Jacobian is

$$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

[2 marks] This has trace and determinant $T = 0$ and $D = -1$, respectively. This shows that it is a saddle.※

At $(x, y) = (0.5, 0.5)$ the Jacobian is

$$\begin{pmatrix} -0.5 & -0.5 \\ 1 & -1 \end{pmatrix}.$$

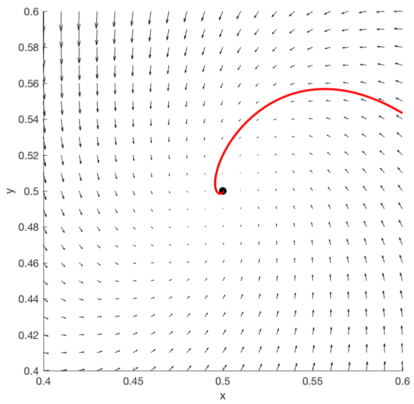
[2 marks] This has trace and determinant $T = -1.5$ and $D = 1$, respectively. Noting that $T < 0$ and $T^2 = 2.25 < 4 = 4D$, the fixed point is a stable spiral. ※

3(e) Explain why one of the fixed points is asymptotically stable. Sketch the trajectory in the phase plane for an initial condition $(x(0), y(0))$ close to the stable fixed point.

/5 marks

Solution All solutions in a neighbourhood of the fixed point at $(x, y) = (0.5, 0.5)$ move towards that fixed point, meaning it is asymptotically stable. The other fixed point does not have this property. ※ [2 marks]

A trajectory is shown in the figure below.



A simple spiral is sufficient, but it should have the correct direction with an argument, e.g. on the y-nullcline $y = x$ where trajectories are horizontal, they point rightwards (in the positive x-direction) if $x < 0.5$, and leftwards (negative x-direction) if $x > 0.5$. ※ [3 marks]

20 Total

Question 4.

Consider the IVP

$$\frac{du}{dt} = \frac{-u}{\sqrt{t}} \quad \text{with} \quad u(1) = 0.5. \tag{3}$$

4(a) Express the IVP as an integral equation, and hence write down the associated Picard iteration scheme, including the value of the initial iterate $u^{(0)}$. Calculate the first iterate $u^{(1)}$.

/5 marks

Solution The integral equation version of the IVP is

$$u(t) = 0.5 - \int_1^t \frac{u(s)}{\sqrt{s}} ds,$$

[1 mark] ※ and the associated Picard iteration scheme is

$$u^{(k)}(t) = 0.5 - \int_{0.1}^t \frac{u^{(k-1)}(s)}{\sqrt{s}} ds \quad \text{for} \quad k = 1, 2, \ldots$$

[2 marks] with $u^{(0)} = 0.5$.※ Therefore

$$u^{(1)}(t) = 0.5 - \int_1^t \frac{0.5}{\sqrt{s}} ds$$

$$= 0.5 - \left[\sqrt{s}\right]_1^t = 0.5 - \left[\sqrt{t} - 1\right] = 1.5 - \sqrt{t}.$$

[2 marks] ※

4(b) Show that the function

$$f(u,t) = \frac{-u}{\sqrt{t}}$$

is Lipschitz continuous for $u \in \mathbb{R}$ and $t \in [0.25, 1.75]$, and determine the smallest Lipschitz constant on this domain.

/3 marks

Solution Lipschitz continuity means

$$|f(u_2,t) - f(u_1,t)| \leq L|u_2 - u_1|$$

[1 mark] for all $t \in [0.25, 1.75]$.※ Here, we have

$$\left|\frac{-u_2}{\sqrt{t}} - \frac{-u_1}{\sqrt{t}}\right| = \left|\frac{1}{\sqrt{t}}\right| |u_2 - u_1|$$

$$\leq \frac{1}{\sqrt{0.25}}|u_2 - u_1| = 2|u_2 - u_1|,$$

so that the function is Lipschitz continuous, as required, with
[2 marks] smallest Lipschitz constant $L = 2$. ※

4(c) Does a unique solution to IVP (3) exist for some interval of time following $t = 1$? Justify your answer.

You do **not** need to calculate the time interval, and do **not** solve the IVP.

/3 marks

Solution The function f is continuous on $t \in [0.25, 0.75]$ and $u \in \mathbb{R}$ and Lipschitz continuous on this domain (from 4(b)). Therefore the Picard–Lindelof theorem tells us that a unique solution exists for some interval $t \in [1-\varepsilon, 1+\varepsilon]$ for some $\varepsilon > 0$, and, in particular, this means a unique solution
[3 marks] exists for $t \in [1, 1+\varepsilon]$. ※

4(d) Can a unique solution be guaranteed if the initial condition is replaced by $u(0) = 1$? Explain your answer.

/2 marks

Solution The function f is not bounded/continuous at $t = 0$, and therefore the Picard–Lindelof theorem does not
[2 marks] apply. A unique solution cannot be guaranteed. ※

4(e) Write down the forwards/explicit Euler method for the ODE in (3), and state the local discretisation error as an order of the step size h.

/2 marks

Solution Euler's method is

$$u_{i+1} = u_i - \frac{hu_i}{\sqrt{t_i}},$$

[1 mark]
[1 mark] ※ and it has local discretisation error $O(h^2)$.※

/3 marks

4(f) Suppose Euler's method is used to solve IVP (3) for $t \in [1, 4]$. Calculate the interval(s) of step sizes h for which the method is stable.

You can quote results from lectures without proof.

Solution Euler's method is stable for stepsizes h such that

$$|1 + hf_u| < 1 \quad \Rightarrow \quad |1 - h/\sqrt{t}| < 1,$$

[1 mark] ※ meaning we require

$$-2 < -h/\sqrt{t} < 0 \quad \Rightarrow \quad 0 < h/\sqrt{t} < 2.$$

The first inequality is satisfied automatically as $h > 0$. The second inequality requires

$$h < 2\sqrt{t} \leq 2\sqrt{1} = 2.$$

[2 marks] ※

/2 marks

4(g) Will the numerical solution convergence to the true solution as $h \to 0$? Justify your answer.

You can ignore round-off error.

Solution The method is consistent (local discretisation error tends to zero as $h \to 0$) and stable for $h < 2$, so by Lax's equivalence theorem it will converge. ※ [2 marks]

19 Total

Question 5.

As in lectures, consider the second-order IVP

$$ml\frac{d^2\Theta}{d\tau^2} = -mg\sin\Theta \quad \text{with} \quad \Theta(0) = \alpha \quad \text{and} \quad \dot{\Theta}(0) = 0, \tag{4}$$

as a model of a simple pendulum.

5(a) Describe the ODE in terms of Newton's second law, and give a physical interpretation of the initial conditions.

You can add a schematic if this helps your explanation.

/2 marks

Solution The ODE models the angle $\Theta(\tau)$ made by a mass m on the end of a weightless string of length l, with respect to the vertical direction, as a function of time τ.

[1 mark] The left-hand side is the product of mass and the acceleration of the pendulum. The right-hand side is the gravitational force. ※

[1 mark] The initial conditions say that the pendulum is initially held at angle α (i.e. zero initial velocity). ※

5(b) Show that by writing $t = \tau/T$ for a suitable scale T, IVP (4) becomes

$$\frac{d^2\theta}{dt^2} + \sin\theta \quad \text{with} \quad \theta(0) = \alpha \quad \text{and} \quad \dot{\theta}(0) = 0, \qquad (5)$$

where $\theta(t) = \Theta(\tau)$.

/3 marks

Solution First note that

$$ml\frac{d^2\Theta}{d\tau^2} = -mg\sin\Theta \quad \Rightarrow \quad \frac{d^2\Theta}{d\tau^2} + \frac{g}{l}\sin\Theta = 0.$$

We have

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \frac{1}{T}\frac{d}{dt},$$

[1 mark] ※ so that

$$\frac{d^2}{d\tau^2} = \frac{1}{T^2}\frac{d^2}{dt^2},$$

and

$$\frac{1}{T^2}\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

Choosing

$$T = \sqrt{\frac{l}{g}}$$

[1 mark] results in the required version of the ODE,※ and the ICs are
[1 mark] unchanged.※

5(c) Explain the assumptions that allow IVP (5) to be linearised to

$$\frac{d^2\theta}{dt^2} + \theta = 0 \quad \text{with} \quad \theta(0) = \alpha \quad \text{and} \quad \dot{\theta}(0) = 0. \qquad (6)$$

/1 mark

Solution We assume that the initial angle α is small and that the angle θ remains small during motion. Then

$$\sin\theta \approx \theta$$

[1 mark] so that the ODE can be linearised into the required form.※

5(d) Derive the centred difference formula

$$y''_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \qquad (7)$$

and find the order of the truncation error.

/5 marks

Please turn over for page 17.

Solution Taylor series and manipulation:※

[4 marks]

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^2}{3!}y'''(x_n) + \frac{h^4}{4!}y''''(x_n) + \dots$$
$$y(x_n - h) = y(x_n) - hy'(x_n) + \frac{h^2}{2}y''(x_n) - \frac{h^2}{3!}y'''(x_n) + \frac{h^4}{4!}y''''(x_n) - \dots$$
$$y(x_{n+1}) + y(x_{n-1}) = 2y(x_n) + h^2y''(x_n) + O(h^4)$$
$$y''(x_n) = \frac{y(x_{n+1}) - 2y(x_n) + y(x_{n-1})}{h^2} + O(h^2)$$

[1 mark] Hence formula as given with truncation error $O(h^2)$.※

5(e) Determine the finite difference formula given by applying the centred difference formula (7) to IVP (6). Find the local and global discretisation errors.

/5 marks

Solution Applying the centred difference formula, the ODE becomes

$$\frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{h^2} + O(h^2) + \theta_n = 0$$

$$\Rightarrow \quad \theta_{n+1} - 2\theta_n + \theta_{n-1} + h^2\theta_n + O(h^4) = 0$$

so that the finite difference formula is

$$\theta_{n+1} = (2 - h^2)\theta_n - \theta_{n-1}.$$

[3 marks] ※ The local discretisation error is $O(h^4)$ and the global dis-
[2 marks] cretisation error is $O(h^3)$. ※

5(f) Explain how the initial conditions are incorporated into the finite difference formula.

/3 marks

Solution The initial conditions become

$$\theta_0 = \alpha \quad \text{and} \quad \frac{\theta_1 - \theta_{-1}}{2h} = 0 \quad \Rightarrow \quad \theta_{-1} = \theta_1.$$

[2 marks] ※ Therefore the $n = 0$ formula becomes

$$\theta_1 = (2 - h^2)\alpha - \theta_1 \quad \Rightarrow \quad \theta_1 = \frac{1}{2}(2 - h^2)\alpha,$$

and the formulae for $n = 1, \ldots$ can be used as given in 5(e).
[1 mark] ※

15 Total

Question 6.

Consider a population of a known fixed size N. The population is divided into three categories: (i) I, the number of individuals that have an infectious disease (they are contagious); (ii) S, the number of individuals that are susceptible to contracting the disease; and (iii) R, the number of individuals that have recovered from the disease.

6(a) Explain how the law of mass action can be used to model infection of susceptible individuals.

/1 mark

Solution Susceptible individuals are infected through contact with infected individuals so that the rate of infection is estimated to be proportional to the product of the two population sizes S and I.※ [1 mark]

Please turn over for page 20.

6(b) Assuming that individuals who have recovered from the disease have permanent immunity, write down an ODE model for each of the susceptible (S), infected (I) and recovered (R) populations that describes how they evolve over time. Explain your model and define any parameters that you introduce.

/3 marks

Solution

$$\frac{dS}{dt} = -\alpha SI,$$

$$\frac{dI}{dt} = \alpha SI - \beta I,$$

$$\frac{dR}{dt} = \beta I.$$

[1 mark] ※ The rate of decrease in S and increase in I due to infection is αSI by the law of mass action, as explained in 6(a), where
[1 mark] α is the rate constant.※ Recovery is at a rate proportional to the size of the infected population, i.e. βI, where β is the
[1 mark] rate constant.※

6(c) Explain why the ODE for R does not have to be solved.

/1 mark

Solution Since $N = S+I+R$ we have $R = N-S-I$. We can solve the coupled ODEs for S and I and then determine
[1 mark] R from known N. ※

6(d) Explain how the model can be solved using the implicit Euler method with a suitable linearisation.

/3 marks

Solution Linearise the ODE for S by writing

$$\left.\frac{dS}{dt}\right|_n = -\alpha S_n I_{n-1}.$$

[1 mark]
[1 mark] ※ Then the implicit Euler method gives※

$$(*) \quad S_n = S_{n-1} - \alpha h S_n I_{n-1} \quad \Rightarrow \quad S_n = \frac{S_{n-1}}{1+\alpha h I_{n-1}}.$$

While we could also linearise the ODE for I, if we have already
[1 mark] computed S_n then I_n is given by※

$$(**) \quad I_n = I_{n-1} + h I_n(\alpha S_n - \beta) \quad \Rightarrow \quad I_n = \frac{I_{n-1}}{1-h(\alpha S_n - \beta)}.$$

Given initial; values S_0 and I_0 we solve $(*)$ followed by $(**)$ for $n = 1, 2, \ldots$.

6(e) If having the disease provides no immunity, show that your model can be written as the scalar ODE

$$\frac{dS}{dt} = (-\alpha S + \beta)(N - S), \tag{8}$$

for parameters α and β.

/3 marks

Solution If no immunity, recovered individuals become susceptible immediately and there is no R population※ : [2 marks]

$$\frac{dS}{dt} = -\alpha SI + \beta I,$$
$$\frac{dI}{dt} = \alpha SI - \beta I.$$

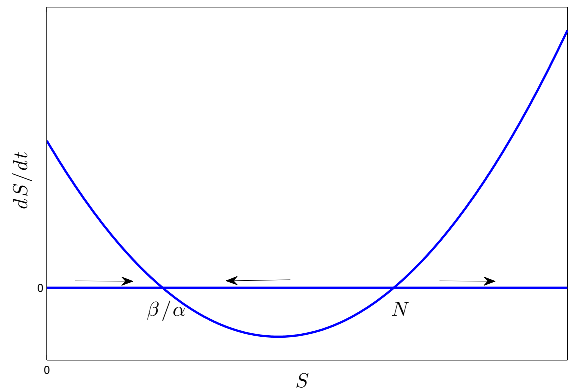
Since $S + I = N$ we can write $I = N - S$ and the model becomes※ [1 mark]

$$\frac{dS}{dt} = (-\alpha S + \beta)(N - S).$$

6(f) Assuming that $\beta/\alpha < N$, show that in the long term the solution to (8) approaches a steady state. Give the steady state solution and a physical interpretation of its meaning.

/4 marks

Solution There are two fixed point of the ODE for S: $S = N$ and $s = \beta/\alpha$.※ [1 mark] Plotting the quadratic on the RHS of the ODE and performing a phase line analysis (see below)※ [1 mark] shows that the solution goes to the stable steady state $(S, I) = (\beta/\alpha, N - \beta/\alpha)$※ [1 mark], i.e. there will always be some infected individuals and some susceptible individuals.※ [1 mark]



End of examination questions.