School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Tutorial 1 (Week 2)

1. Revision / Basic Skills

   - Draw a diagram showing the general shape of a cubic. Now draw one for a quartic, a quintic and a polynomial of degree 6. What sort of degenerate cases are there?
   - If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two given vectors, and $t \in [0,1]$, what does $t\boldsymbol{x} + (1-t)\boldsymbol{y}$ (or $\boldsymbol{y} + t(\boldsymbol{x} - \boldsymbol{y})$) represent geometrically?
   - Write the definition of a partial derivative, and find the partial derivatives with respect to x of
     (a) $f(x,y) = x^2 + y^4 - 3x^4 y$.
     (b) $f(x,y,z) = x\sin(2x + yz)$.
     (c) $f(x,y,z) = (x^2 - y)^2 e^z$.
   - Find the gradient of the following scalar fields
     (a) $f(x,y) = x^2 + y^4 - 3x^4 y$.
     (b) $f(x,y,z) = x\sin(2x + yz)$.
     (c) $f(x_1,x_2,x_3) = (x_1^2 - x_2)^2 e^{x_3}$.
     (d) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$, where $\boldsymbol{x} \in \mathbb{R}^n$.
   - Find the Taylor series expansion about $x = 0$ for the following functions
     (a) $\cos x$.
     (b) $\log(1 + x)$.

2. Use the chain rule to find dz/dt, where

$$z = 2x^2 + 3xy - 4y^2, \quad \text{and} \quad (x,y) = (\cos t, \sin t).$$

3. Use Taylor's Theorem to derive a polynomial approximation (of at least degree 2) for

$$f(x,y) = \sin(x + y^2).$$

4. Find the cylinder of largest volume that can be placed inside the unit sphere.

5. $\star$ In lectures the slope of steepest descent from $(0,1)$ to $(1,0)$ of the form $y = (1-x)^\epsilon$ was given as having an optimal value of $\epsilon \approx 2.5$. This is a numerical approximation. Using your favourite numerical package (Matlab, Maple, Julia, etc) find a more accurate approximation for the optimal (quickest descent) value of ϵ.

   Hint #1: The functional you should be considering for this version of the problem is

$$F\{y\} = \int_0^1 \sqrt{\frac{1 + y'^2}{1 - y}}\, dx.$$

   Hint #2: You might find the following result useful

$$F = \int_0^1 f(x,\epsilon)\, dx \qquad \Rightarrow \qquad \frac{dF}{d\epsilon} = \int_0^1 \frac{\partial f}{\partial \epsilon}\, dx.$$

6. ⋆ You have an empty soft drink can weighing $m_{\mathrm{c}} = 50$ g in the form of a perfect cylinder with radius $r = 3.5$ cm and height $h = 13$ cm. You want to make this can as difficult as possible to tip over by adding water (with density of $\rho = 1\ \mathrm{g{\cdot}cm^{-3}}$) into the can so that, when upright, the combined can and water system has the lowest possible centre of mass. What height of water z in the can achieves this minimum and is it possible to find this minimum without appealing to calculus at all? (That is, just using algebra, not performing a series of experimental trials.)

   Hint #1: The height of the centre of mass of the combined system is a weighted average of the heights of the centres of mass of the component systems. In other words, the can has mass m_{c} and centre of mass height $h_{\mathrm{c}} = h/2$. Whereas the water has mass $m_{\mathrm{w}} = \rho\pi r^2 z$ and centre of mass height $h_{\mathrm{w}} = z/2$. Thus the combined system has a centre of mass height

$$H = \frac{m_{\mathrm{c}} h_{\mathrm{c}} + m_{\mathrm{w}} h_{\mathrm{w}}}{m_{\mathrm{c}} + m_{\mathrm{w}}}.$$

   Hint #2: If you are having trouble finding a non-calculus solution, try solving the problem for different values of m_{c} and investigating the relationship between H and z. There is an elegant physical argument which explains this relationship.

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