Lecture 10: Equilibrium distributions continued

Example 4. Reliability (Birth and Death) (continued)

Here, the components can be repaired; we have the state space $\mathcal{S} = \{0, 1, 2, \ldots, N\}$ and

$$\begin{aligned} \text{for } \ j = 0, 1, \ldots, N-1: \quad q_{j,j+1} &= \lambda, \\ q_{jj} &= -(j\mu + \lambda), \\ \text{for } \ j = 1, \ldots, N: \quad q_{j,j-1} &= j\mu, \\ q_{NN} &= -N\mu. \end{aligned}$$

The equilibrium equations are

$$\pi_N N\mu = \pi_{N-1}\lambda, \tag{9}$$

$$\pi_j j\mu + \pi_j \lambda = \pi_{j-1}\lambda + \pi_{j+1}(j+1)\mu \quad \text{for } j = 1, \ldots, N-1, \tag{10}$$

$$\pi_0 \lambda = \pi_1 \mu. \tag{11}$$

By (10), we get

$$\pi_j j\mu - \pi_{j-1}\lambda = \pi_{j+1}(j+1)\mu - \pi_j \lambda,$$

which is of the form $A_j = A_{j+1}$, where $A_j = \pi_j j\mu - \pi_{j-1}\lambda$.

$$\Rightarrow A_j = A_1 \quad \text{for all } j = 1, \ldots, N.$$

By (9) and (11), $A_N = A_1 = 0$. Therefore,

$$\pi_j j\mu = \pi_{j-1}\lambda \quad \text{for all } j = 1, 2, \ldots, N,$$

which are known as detailed balance equations (these will be discussed more generally later). Hence,

$$\pi_j = \frac{\pi_{j-1}\lambda}{j\mu} = \frac{\pi_{j-2}\lambda^2}{j(j-1)\mu^2} = \pi_0 \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} \quad \text{for all } j = 1, 2, \ldots, N.$$

We need $\displaystyle\sum_{j=0}^{N} \pi_j = 1$ so that $\displaystyle\sum_{j=0}^{N} \pi_0 \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} = 1$ yields

$$\pi_0 = \left[\sum_{j=0}^{N} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!}\right]^{-1},$$

$$\pi_j = \frac{\left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!}}{\sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}} \quad \text{for all } j = 1, 2, \ldots, N.$$

Example 3. M/M/1 Queue (Single Server Queue) (cont.)

Recall that $\mathcal{S} = \{0, 1, 2, \dots\}$ and the transition rates are

$$\begin{aligned} q_{j,j+1} &= \lambda, \quad \text{for } j = 0, 1, 2, \dots \\ q_{j,j-1} &= \mu, \quad \text{for } j = 1, 2, \dots \\ q_{jj} &= -(\lambda + \mu), \quad \text{for } j = 1, 2, \dots \\ q_{00} &= -\lambda. \end{aligned}$$

The equilibrium equations are

$$\pi_0 \lambda = \pi_1 \mu \tag{12}$$

$$\pi_j(\lambda + \mu) = \pi_{j-1}\lambda + \pi_{j+1}\mu \quad \text{for } j = 1, 2, \dots. \tag{13}$$

We could solve these equations using the same method as for Example 4, but we can also use generating functions.

Let $P(z) := \sum_{j=0}^{\infty} \pi_j z^j$. If the equilibrium probabilities π_j exist, then $P(z)$ is analytic for $|z| \leq 1$, because

$$\begin{aligned} \left| \sum_{j=0}^{\infty} \pi_j z^j \right| &\leq \sum_{j=0}^{\infty} \pi_j \left| z^j \right| \quad \text{by the triangle inequality} \\ &\leq \sum_{j=0}^{\infty} \pi_j \quad \text{as } |z| \leq 1 \\ &= 1. \end{aligned}$$

We multiply (13) by z^j and sum from $j = 1$ to ∞ and then add (12) to get

$$\begin{aligned} \sum_{j=0}^{\infty} \pi_j \lambda z^j + \sum_{i=1}^{\infty} \pi_i \mu z^i &= \sum_{j=1}^{\infty} \pi_{j-1} \lambda z^j + \sum_{i=0}^{\infty} \pi_{i+1} \mu z^i \\ \Rightarrow \lambda P(z) + \mu(P(z) - \pi_0) &= \lambda z P(z) + \frac{\mu}{z}(P(z) - \pi_0) \\ \Rightarrow P(z)\left(\lambda + \mu - \lambda z - \frac{\mu}{z}\right) &= \pi_0 \left(\mu - \frac{\mu}{z}\right). \end{aligned}$$

Thus,

$$\Rightarrow \quad P(z) = \frac{\pi_0 \left(\mu - \frac{\mu}{z}\right)}{\lambda + \mu - \lambda z - \frac{\mu}{z}} = \frac{\pi_0(\mu z - \mu)}{(\lambda + \mu)z - \lambda z^2 - \mu} = \frac{\pi_0(\mu z - \mu)}{\left(1 - \frac{\lambda z}{\mu}\right)(\mu z - \mu)} = \frac{\pi_0}{1 - \frac{\lambda z}{\mu}}.$$

As $P(z)$ converges if and only if the π_j exist, we need $\left.\dfrac{\pi_0}{1 - \frac{\lambda z}{\mu}}\right|_{z=1}$ to converge.

Note that $\sum_{j=0}^{\infty} x^j = \dfrac{1}{1-x}$ if and only if $|x| < 1$

$\Rightarrow$ the equilibrium probabilities π_j exist if and only if $\left|\dfrac{\lambda z}{\mu}\right| < 1$ when $z = 1$

$\Rightarrow \pi_j$ exist if and only if $\lambda/\mu < 1$, which is a natural stability condition, where the service rate is higher than the arrival rate.

In this case,

$$1 = P(z)\bigg|_{z=1} = \frac{\pi_0}{1-\dfrac{\lambda}{\mu}} \quad \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu},$$

and consequently

$$P(z) = \frac{1-\dfrac{\lambda}{\mu}}{1-\dfrac{\lambda z}{\mu}} = \left(1-\frac{\lambda}{\mu}\right)\sum_{j=0}^{\infty}\left(\frac{\lambda z}{\mu}\right)^j = \sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j z^j \overset{\text{by def}}{=} \sum_{j=0}^{\infty}\pi_j z^j.$$

So, we have for all $j \in \mathcal{S}$

$$\pi_j = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j \text{ if } \lambda < \mu,$$

and the π_j do not exist, otherwise.

Physically, we have a solution to the equilibrium equations if and only if $\lambda < \mu$.
