Lecture 13: Hitting Probabilities continued

Concepts checklist

At the end of this lecture, you should be able to:

- *Understand that we are seeking the minimal non-negative solution* to the hitting probability equations;
- *Find the minimal non-negative solution of the hitting probability equations* for simple CTMCS; and,
- *State a Theorem regarding the solution of the hitting probability of a particular state for a general CTMC.*

Example 3. M/M/1 Queue

We extend our analysis to the case where the queue has infinite capacity, by removing the upper absorbing state N and setting $\mathcal{S} = \{0, 1, \ldots\}$. Again, let f_i be the probability that the Markov chain ever visits state 0, given that it starts in state i:

$$f_i = \Pr\big(\text{ever visits state } 0 \,\big|\, \text{starts in state } i\big).$$

Then, as for the finite capacity case we have

$$f_i = \frac{\lambda}{\lambda+\mu} f_{i+1} + \frac{\mu}{\lambda+\mu} f_{i-1}, \qquad \text{with } f_0 = 1. \tag{15}$$

Unlike the finite case, we only have one boundary condition, so using just the previous technique we cannot secure a unique solution. Solving equation (15) as in the previous example, we have

$$f_i = \begin{cases} A\left(\dfrac{\mu}{\lambda}\right)^i + B & \text{for } \mu \neq \lambda, \\ Ai + B & \text{for } \mu = \lambda. \end{cases}$$

As $f_0 = 1$, we have

$$B = \begin{cases} 1 - A & \text{for } \mu \neq \lambda, \\ 1 & \text{for } \mu = \lambda. \end{cases}$$

Thus,

$$f_i = \begin{cases} 1 + A\left[\left(\dfrac{\mu}{\lambda}\right)^i - 1\right] & \text{for } \quad \mu \neq \lambda, \\ Ai + 1 & \text{for } \quad \mu = \lambda. \end{cases}$$

We shall see later that f_i is the minimal non-negative solution to

$$x_i = \frac{\lambda}{\lambda+\mu} x_{i+1} + \frac{\mu}{\lambda+\mu} x_{i-1}, \qquad \text{subject to } x_0 = 1.$$

- If $\lambda < \mu$: then $(\mu/\lambda)^i - 1 > 0$ for all $i > 0$.

  $\Rightarrow$ the minimal non-negative solution occurs when $A = 0$ and therefore $f_i = 1$ for all i.
- If $\lambda > \mu$: then $(\mu/\lambda)^i - 1 < 0$ for all $i > 0$; as $i \to \infty$, this term approaches -1.

  $\Rightarrow$ the largest value of A for which $1 + A\left[\left(\frac{\mu}{\lambda}\right)^i - 1\right] \geq 0$ for all i is $A = 1$.

  $\Rightarrow f_i = \left(\frac{\mu}{\lambda}\right)^i$ for all i.
- If $\mu = \lambda$: the minimal non-negative solution occurs when $A = 0$ and thus $f_i = 1$ for all i.

$$\text{Hence,} \qquad f_i = \begin{cases} 1, & \text{if} \quad \mu > \lambda, \\ \left(\dfrac{\mu}{\lambda}\right)^i, & \text{if} \quad \mu < \lambda, \\ 1, & \text{if} \quad \mu = \lambda. \end{cases}$$

We can now state that the *(unmodified) single server queue* is recurrent if $\mu \geq \lambda$ and transient otherwise. Let us assume that the Markov chain starts in state 0, then the only state it can go to is state 1.

- If $\lambda \leq \mu$, we return to state 0 with probability 1. Thus by definition, 0 is a recurrent state, and therefore since it is irreducible, the whole Markov chain is recurrent.
- If $\lambda > \mu$, we return to state 0 from state 1 with probability $\frac{\mu}{\lambda} < 1$. Therefore, by definition, state 0 is a transient state and the whole of the Markov chain is transient.

For $\lambda > \mu$ or $\lambda < \mu$, we could argue that this reflects the intuitive fact that the number in the queue will drift "towards ∞" or drift "towards zero". However this doesn't tell us much about the case when $\lambda = \mu$, but the analysis has shown us that the queue is still recurrent in this case.

Minimal non-negative solution:

We now show why the hitting probability f_i is the minimal non-negative solution to the equation

$$x_i = \frac{\lambda}{\lambda + \mu} x_{i+1} + \frac{\mu}{\lambda + \mu} x_{i-1}, \qquad \text{subject to } x_0 = 1.$$

Let $f_{in} = \Pr(\text{reaches state 0 in at most } n \text{ steps} \mid \text{starts in state } i)$. Then

$$f_{i,n+1} = \left(\frac{\lambda}{\lambda + \mu}\right) f_{i+1,n} + \left(\frac{\mu}{\lambda + \mu}\right) f_{i-1,n}, \tag{16}$$

with $f_{0n} = 1$ for all $n \geq 0$ and $f_{i0} = 0$ for all $i \in \mathcal{S} \setminus \{0\}$.

Lemma 1. *Let x_i be* any *non-negative solution to the equation*

$$x_i = \frac{\lambda}{\lambda + \mu} x_{i+1} + \frac{\mu}{\lambda + \mu} x_{i-1}, \qquad \textit{subject to } x_0 = 1. \tag{17}$$

Then, the probability f_{in} of hitting state 0 in n steps or fewer (given the initial state i) satisfies the inequality

$$f_{in} \leq x_i$$

for all $n \geq 0$ and $i \in \mathcal{S}$.

Proof: Clearly, this is true for $n = 0$, so let's assume that it is true for $n = k$. By (16), we have

$$\begin{aligned} f_{i,k+1} &= \left(\frac{\lambda}{\lambda+\mu}\right) f_{i+1,k} + \left(\frac{\mu}{\lambda+\mu}\right) f_{i-1,k} \\ &\leq \left(\frac{\lambda}{\lambda+\mu}\right) x_{i+1} + \left(\frac{\mu}{\lambda+\mu}\right) x_{i-1} = x_i, \end{aligned}$$

so that $f_{i,k+1} \leq x_i$ and the result is proven by induction. □

Lemma 2. *(For Example 3) The probability f_i of hitting state 0, given that the system starts in state i, is the minimal non-negative solution to the equation*

$$x_i = \frac{\lambda}{\lambda+\mu} x_{i+1} + \frac{\mu}{\lambda+\mu} x_{i-1}, \qquad \textit{subject to } x_0 = 1.$$

Proof. Clearly, f_{in} is increasing in n, since we are allowing more and more steps to reach state 0. Therefore, f_{in} is an increasing sequence, which is bounded above and so

$$f_i = \lim_{n\to\infty} f_{in} \qquad \text{exists.}$$

Also since $f_{in} \leq x_i$ for all n we have that

$$\lim_{n\to\infty} f_{i,n} \leq x_i,$$

which implies that $f_i \leq x_i$. We have already seen that f_i is a solution to equation (17) and therefore, it must be the the minimal non-negative solution to equation (17). □

We can state this result for a general CTMC.

Theorem 11. *For a particular state j in a CTMC with generator Q, the probability f_i that the CTMC ever reaches j, given that it starts in state i is given by the minimal non-negative solution to the equations*

$$(-q_{ii})x_i = \sum_{k\in\mathcal{S}, k\neq i} q_{ik} x_k, \;\; i \in S \setminus \{j\},$$

subject to the boundary condition $x_j = 1$.
