THE UNIVERSITY of ADELAIDE

Mathematics IB

Calculus Notes

School of Mathematical Sciences
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Contents

References are given by section number to the 7th Edition of *Calculus* by Stewart.

Chapter 1

DIFFERENTIAL EQUATIONS

1.1 Introduction [St:§9.1]

Definition. A *differential equation* (DE) is an equation involving derivatives of an unknown function.

◊ 1.0: Example: population growth ...

To discuss which DEs can be solved we classify them in various ways. The first method of classification is by *order*.

Definition. The *order* of a DE is the order of the highest derivative that occurs in it.

◊ 1.1: Examples ...

In this course we will only consider DEs of first and second order.

1.2 First Order Separable DEs [St:§9.3]

A general way of writing a first order DE is in the form

$$\frac{dy}{dx} = f(x, y) \tag{1.2.1}$$

where $f(x, y)$ is a given function of the two variables x and y. In the special case that $f(x, y)$ can be written as a product of a function of x and a function of y we call this a *first order separable* DE.

A first order separable DE has the form

$$\frac{dy}{dx} = F(x) \cdot G(y) \tag{1.2.2}$$

We can solve this DE by writing

$$\frac{1}{G(y)}\frac{dy}{dx} = F(x) \qquad \left(\text{or} \quad \frac{1}{G(y)}dy = F(x)\,dx\right)$$

and integrating both sides

$$\int \frac{1}{G(y)}dy = \int F(x)\,dx \tag{1.2.3}$$

Note. The standard procedure here is to use the "differentials" dy and dx, treating $\dfrac{dy}{dx}$ as the ratio of these differentials.

Note. We divided by $G(y)$ so for completeness we should also consider the case that $G(y) = 0$.

◊ 1.2: Example ...

In general our solution will involve a constant, for example it may take the form $y = f(x) + C$, or even more generally we may not be able to solve it for y, so it may be something like $g(y) = f(x) + C$ or $g(y) = Cf(x)$ or something more complicated. Each choice of C gives a different solution, and there may be multiple solutions arising from other factors as well, for example if it is of the form $y^2 = Cf(x)$ then there are two solutions for each non-zero choice of C.

This is called the *general solution* of the DE, i.e. a comprehensive formula including all (or nearly all) solutions of the DE. Here it includes one arbitrary constant C. This is typical for the general solution of a first order DE. For a second order DE the general solution will have two arbitrary constants — essentially because the solution involves two lots of integration.

◊ 1.3: Example ...

Note. In general if we are solving a DE for a function $y(x)$ we can find a particular solution if we know the value that y takes for some particular value of x, for example $y(a) = b$. By substituting these into the general solution we can solve for the constant and obtain a particular solution. In many cases the independent variable x is the time t and often the condition is at time $t = 0$. But we also say "initial condition" when we really mean some general "given condition".

◊ **1.4: Example: an initial value problem** ...

Note. One of the nice things about DEs is that we can, in principal, check that we have the correct answer.

◊ **1.5: Example: checking solution** ...

1.3 Linear First Order DEs [St:§9.4]

Consider again the general first order DE

$$\frac{dy}{dx} = f(x, y),$$

to be solved for $y(x)$. We have seen how to solve this if $f(x, y)$ is a product of functions of x and y separately. Another special case we can solve is when $f(x, y)$ is a linear function of the dependent variable y, that is,

$$\frac{dy}{dx} = f(x, y) = Q(x) - P(x)y$$

where $Q(x)$ and $P(x)$ are given functions of x. To solve this we rewrite it in a standard form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1.3.1}$$

A DE that can be written in the form 1.3.1 for some functions P and Q is called a *1st order linear DE.*

The standard method of tackling a first order linear DE is to multiply through by a function $I(x)$, called the *integrating factor* (IF), such that the LHS becomes the derivative of $I(x).y$; that is, we take

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x) \tag{1.3.2}$$

and seek to find $I(x)$ such that the LHS is

$$\frac{d}{dx}(I(x)y) = I(x)\frac{dy}{dx} + \frac{dI}{dx}y. \tag{1.3.3}$$

◊ **1.6: Derivation of $I(x)$** ...

We find that a solution is

$$I(x) = \exp\left(\int P(x)\, dx\right)$$

This gives the following method of solving first order linear DEs:

1. Check that the DE is linear, that is, it can be written in the form
$$\frac{dy}{dx} + P(x)y = Q(x).$$
2. Calculate the IF, that is, $I(x) = \exp\left(\int P(x)\,dx\right)$
3. Hence obtain $I(x)y = \int I(x)Q(x)\,dx$ and do the RHS integration to get the solution for $y(x)$.

$\Diamond$ 1.7: Examples ...

Further examples of First Order Linear DEs

1.4 Linear Second Order DE with Constant Coefficients [St:§17.1]

We consider

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x) \tag{1.4.1}$$

where $a \neq 0$; a, b and c are real numbers; $r(x)$ is a function of x only. It is convenient at this point to change notation and let

$$y' = \frac{dy}{dx}$$
$$y'' = \frac{d^2y}{dx^2}$$

This DE is clearly *2nd order* (highest derivative y''); it is called *linear* as y, y', y'' occur only linearly and *constant coefficient* as a, b, c are real constants.

If the RHS $r(x) \equiv 0$ the DE is called *homogeneous*, that is,

$$ay'' + by' + cy = 0. \tag{1.4.2}$$

If $r(x) \not\equiv 0$ then the DE (1.4.1) is called nonhomogeneous.

Recall from Algebra that every solution S of a nonhomogeneous equation $AX = B$ is of the form $S = S_0 + H$ where S_0 is a particular solution of $AX = B$ and H is a solution of the homogeneous equation $AX = 0$. The same is true for linear DE's.

Proposition 1.1. *If y_p is a particular solution of (1.4.1) then every solution of (1.4.1) is of the form $y = y_h + y_p$ where y_h is a solution of the homogeneous equation (1.4.2).*

◊ 1.8: Proof of Proposition ...

It follows that the general solution of (1.4.1) is obtained by finding the general solution to (1.4.2) and adding it to a particular solution to (1.4.1).

To find the GS to (1.4.2) we consider the algebra example again. Recall that the general solution of the equation $AX = 0$ is a linear combination of a basis for the solution space. A similar thing is true for DEs. As in Algebra, two solutions to (1.4.2) are *linearly independent* if they are not multiples of each other. Then we have

Proposition 1.2. *If $y_1(x), y_2(x)$ are linearly independent solutions of the homogenous equation*

$$ay'' + by' + cy = 0 \tag{1.4.2}$$

then the G.S. is

$$y(x) = C_1y_1(x) + C_2y_2(x),$$

for constants C_1 and C_2.
Further, if y_p is a particular solution of

$$ay'' + by' + cy = r(x)\,, \quad r(x) \not\equiv 0$$

then the G.S. of this D.E. is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p \ .$$

◊ 1.9: Partial proof of proposition ...

Now we need to take a look at methods for finding

(a) the GS of the homogeneous equation (1.4.2)

(b) a particular solution (P.S.) for the nonhomogeneous equation (1.4.1).

First we seek a GS of the homogeneous DE

$$ay'' + by' + cy = 0 \tag{1.4.2}$$

◊ 1.10: Finding a solution ...

Thus we find that $y = e^{\lambda x}$ is a solution of (1.4.2) if

$$a\lambda^2 + b\lambda + c = 0.$$

We call this the *Characteristic Equation* for the DE.

In general, since the Characteristic Equation is a quadratic, we expect two solutions,

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and the type of solution depends on the discriminant $b^2 - 4ac$. Is it > 0, < 0, $= 0$? That is are the roots, real and distinct, equal or complex?

Discriminant positive, two distinct roots.

There are two distinct real roots λ_1 and λ_2 so the GS of (1.4.2) is $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ and since $\lambda_1 \neq \lambda_2$ clearly $e^{\lambda_1 x}, e^{\lambda_2 x}$ are linearly independent.

◊ 1.11: Example ...

Discriminant zero: a double root.

Here the characteristic equation has a double root $\lambda = -\dfrac{b}{2a} = \alpha$, say, so that $a\alpha^2 + b\alpha + c = 0$. As this gives only *one* solution $e^{\alpha x}$, we must find a second independent solution. Try $y = xe^{\alpha x}$:

◊ **1.12: Calculation** . . .

So in this case $y = e^{\alpha x}$ and $y = xe^{\alpha x}$ are two linearly independent solutions.

Therefore the GS is

$$\begin{aligned} y &= Ae^{\alpha x} + Bxe^{\alpha x} \\ &= (A + Bx)e^{\alpha x} \end{aligned}$$

◊ **1.13: Example** . . .

Discriminant negative: complex roots.

Here the characteristic equation has

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \alpha \pm i\beta \quad \text{say}$$

(where $\beta = \frac{1}{2a}\sqrt{4ac - b^2}$, and $\alpha = -\frac{b}{2a}$) so that (formally) the GS is

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \end{aligned} \tag{$*$}$$

but what is $e^{i\beta x}$?

◊ **1.14: Calculation** . . .

So we take

$$y = e^{\alpha x}(C \cos \beta x + D \sin \beta x)$$

as the GS of the DE in the case that $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.

◊ **1.15: Examples** . . .

We can now can get the GS as sum of two linearly-independent solutions for the homogeneous DE $ay'' + by' + cy = 0$ (1.4.2). We next need to find a particular solution for the nonhomogeneous DE $ay'' + by' + cy = r(x)$ (1.4.1).

A particular solution for nonhomogeneous DE's [St:§17.2]

Consider the nonhomogeneous DE

$$ay'' + by' + cy = r(x). \tag{1.4.1}$$

We initially just consider the case when $r(x)$ is a polynomial. It is natural to *try* for a particular solution which is itself a polynomial of the same order. That is, if

$r(x)$	Try particular solution
5	A_0
$2x+3$	$A_1x + A_0$
$6x$	$B_1x + B_0$

This is called the "method of undetermined coefficients".

Example 1.4.1. Solve

$$y'' - y' - 2y = 2x \tag{$*$}$$

subject to $y(0) = 1/2$ and $y'(0) = 2$.

Use 4 steps:

(i) find GS to homogeneous equation

(ii) find PS of nonhomogeneous equation

(iii) then "GS to nonhomogeneous = GS to homogeneous + PS"

(iv) only apply initial conditions (if you have then) once the GS to nonhomogeneous has been found

♢ 1.16: Solution ...

1.5 The Logistic Equation [St:§9.4]

We look at some situations in population and epidemic modelling which can be modelled using 1st order DEs and illustrate how a mathematician, physicist or engineer may use DEs to model a real world situation.

♢ 1.17: Motivating example - simple exponential growth model. ...

The simple exponential growth model eventually becomes unrealistic as t increases – a population cannot grow exponentially forever. How do we model this? Our experience tells us that eventually the environment will impose some limits on growth, e.g. lack of space, food, water, disease due to overcrowding, pollution etc. A plausible model which includes this limitation is called the *Logistic Equation*:

$$
\begin{aligned}
\frac{dP}{dt} &= kP\left(1-\frac{P}{C}\right) \\
&= \frac{k}{C}P(C-P), \quad C \text{ a given positive constant}
\end{aligned}
\tag{1.5.1}
$$

which says that for small P, $\dfrac{dP}{dt} \approx kP$ as before but that as P grows, $\left(1-\frac{P}{C}\right)$ gets smaller as $P \to C$. We will see that for large t, $P(t)$ approaches C, which is called the *carrying capacity*.

The logistic equation (1.5.1) is 1st order separable.

◊ 1.18: Solving the logisitic DE ...

The solution to the logistic equation is

$$P = P(t) = \frac{CP_0}{P_0 + (C - P_0)e^{-kt}}$$

where $P_0 = P(0)$, the population at time $t = 0$.

Clearly as t becomes very large $P(t)$ becomes closer and closer to C. This justifies the term "carrying capacity" for C. Under the logistic equation model the population P approaches the value C as $t \to \infty$, whether $P_0 > C$ or $P_0 < C$.

Note. In the logistic equation $\dfrac{dP}{dt} = kP\left(1-\frac{P}{C}\right)$, if we allowed the carrying capacity $C \to \infty$, then this DE reduces to the exponential growth situation $\dfrac{dP}{dt} = kP$.

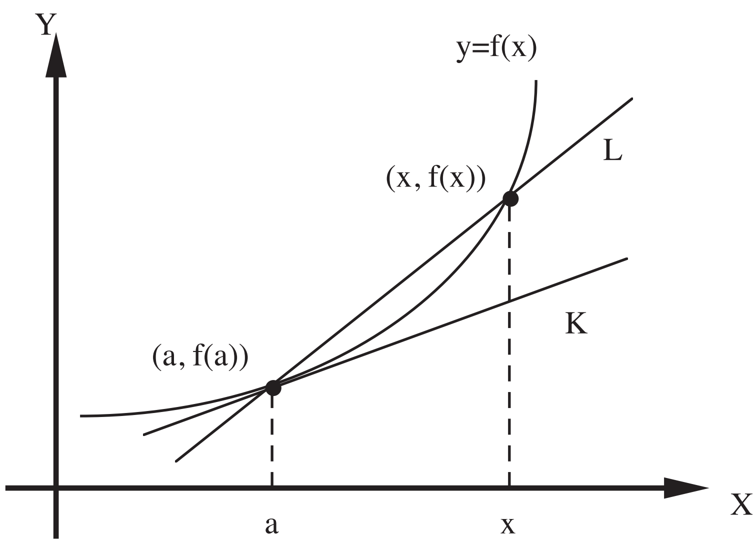
◊ 1.19: Examples of the logistic equation ...

Chapter 2

LIMITS

2.1 Introduction [St:§1.5]

Recall the definition of the derivative $f'(a)$ of a function $f(x)$ at a point a. It is the slope of the tangent line K to the graph of f through $(a, f(a))$.



◊ 2.0: Calculation of slope of tangent line . . .

We define the limit of $f(x)$ as x approaches a intuitively as:

Definition. Let f be a function whose domain includes a set of the form $(b, a) \cup (a, c) = (b, c) \setminus \{a\}$. We say that "the limit of $f(x)$ as x approaches a is (the real number) L if the values of $f(x)$ get closer and closer to L as x gets closer and closer to a".

◊ 2.1: Examples . . .

So far we have seen examples where the limit exists. However limits often do not exist. For example consider

$$f(x) = \sin \frac{1}{x}$$

As x approaches 0 this oscillates wildly between 0 and 1 and is definitely not getting closer and closer to any fixed value L.

This definition of limit is 'intuitive' in that it suffers from a number of ambiguities due to the lack of precision of the everyday language employed. For example what does 'closer and closer to' mean ? It could be read as meaning that the values of $f(x)$ are monotonically approaching L as x approaches a. This is definitely not the case as we can see by considering the example

$$f(x) = x \sin \frac{1}{x}$$

Like the preceeding example this also oscillates up and down but the amplitude of the oscillation decreases as we approach 0 because of the x factor. It has limit $L = 0$ as x approaches 0. We shall also prove this in the following lectures.

Because of the imprecision of everyday language in defining limits, mathematicians use a more formal definition that is **NOT** an examinable part of this course. However we reproduce it here as it will be met again by students taking mathematics courses in later years.

$\Diamond$ **2.2: Formal definition of the limit** ...

Our interest in this course is in calculating limits. To do this we need to collect together some basic facts about limits and some results on how to manipulate them, the so-called *limit laws*. In later years you may learn how to *prove* these facts and the limit laws from the formal definition just given.

2.2 Basic properties of limits

Theorem 2.1. *If $\lim_{x \to a} f(x)$ exists then it is unique. That is, if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$ then $L = M$.*

Note that this is perfectly plausible as if L and M are not equal they must be some distance apart and if $f(x)$ is very close to one of them it cannot also be very close to other. The formal proof uses the definition and this intuitive idea.

Theorem 2.2. *The following statements of limits are equivalent*

(i) $\lim_{x \to a} f(x) = L$ *(ii)* $\lim_{x \to a} (f(x) - L) = 0$

(iii) $\lim_{x \to a} |f(x) - L| = 0$ *(iv)* $\lim_{h \to 0} f(a + h) = L$ *putting $x = a + h$ in (i)*

We will often use the particular case $\lim_{x \to a} f(x) = 0$ if and only if $\lim_{x \to a} |f(x)| = 0$.

2.3 The Limit Laws [St:§1.6]

To use the definition of limit to prove that $\lim_{x\to 0}(x^{34}+3x^7+2)=2$ would be horrendous. Hence we rely on the following rules.

Theorem 2.3 (Elementary limits)**.** *(i)* $\lim_{x\to a} x = a$ *; (ii)* $\lim_{x\to a} c = c$ *, if c is a constant.*

Again these are very plausible as if x gets closer and closer to a then x gets closer and closer to a! Also if c is a constant then as x gets close to a c does not change and so can only be close to itself.

To calculate results such as $\lim_{x\to 0}(x^{34}+3x^7+2)=2$ we use these elementary limits and the limit laws.

Theorem 2.4 (Limit Laws)**.** *Let f and g be functions both defined in $(b,c)\backslash\{a\}$, a deleted neighbourhood of a, and suppose that $\lim_{x\to a} f(x)=L$, $\lim_{x\to a} g(x)=M$.*

Then for any real numbers α and β we have

(1) $\lim_{x\to a}\{\alpha f(x)+\beta g(x)\}=\alpha L+\beta M$

(2) $\lim_{x\to a}\{f(x)g(x)\}=LM$

(3) $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\begin{cases}\dfrac{L}{M} & \text{if } M\neq 0\\ DNE & \text{if } M=0, L\neq 0\\ \text{may/may not exist} & \text{if } M=0, L=0\end{cases}$

(4) If $\lim_{x\to a} g(x)=M$ and $\lim_{x\to M} f(x)=f(M)$ then, if f is defined in an open interval containing M, $\lim_{x\to a}(f\circ g)(x)=f(M)$. (Composition of functions).

◊ 2.3: Examples ...

Sometimes a limit does not exist because as x approaches 0, $f(x)$ becomes arbitrarily large so that no matter what number L you choose $f(x)$ will not get closer and closer to it. We sometimes denote this limit by $+\infty$ but be aware that $+\infty$ is not a number and writing

$$\lim_{x\to 0} f(x)=+\infty$$

is just a shorthand way of saying that $f(x)$ gets larger and larger as x gets closer and closer to 0.

◊ 2.4: Example of unbounded limit ...

2.4 The derivative [St:§2.2]

Just as limits may not always exist, the limit we use to define the derivative may not exist. If it does we call the function differentiable. More formally we have

Definition. We say that f is *differentiable* at $a \in \mathcal{D}(f)$ if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists. In this case we call the value of this limit the *derivative* of f at a, and denote it by $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

◊ 2.5: Examples . . .

Recall the sum and product formulae for differentiation given in Maths IA. These can be proved using the limit laws.

◊ 2.6: Proof of sum formula . . .

2.5 The Squeeze Theorem [St:§1.6]

Theorem 2.5 (The Squeeze Theorem). *Suppose that for all x (except possibly at a) in some interval containing a,*

$$f(x) \leq g(x) \leq h(x).$$

Then if $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} h(x) = L$ then $\lim\limits_{x \to a} g(x) = L$ also.

Again this is clearly plausible if both $f(x)$ and $h(x)$ are getting closer to L as x gets closer to a then $g(x)$, being trapped between them, must get close to L as well.

Often the Squeeze Theorem is used in the form where $f(x) = 0$. For example if $0 \leq g(x) \leq h(x)$ and $\lim\limits_{x \to a} h(x) = 0$ then $\lim\limits_{x \to a} g(x) = 0$. Similarly if $0 \leq |g(x)| \leq |h(x)|$ and $\lim\limits_{x \to a} |h(x)| = 0$ then $\lim\limits_{x \to a} |g(x)| = 0$, that is $\lim\limits_{x \to a} g(x) = 0$.

◊ 2.7: Squeeze Theorem example . . .

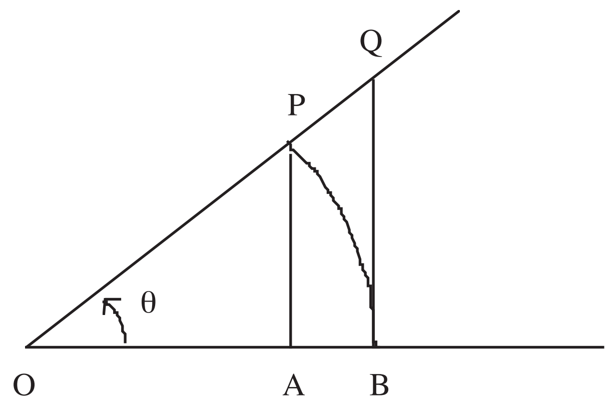
We will use the Squeeze Theorem to derive the *Fundamental Trigonometric Limit*:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{or} \qquad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

2.6 The Fundamental Trigonometric Limit [St:§2.4]

From the definitions of $\sin(\theta)$ and $\cos(\theta)$ it is clear that $\lim_{\theta \to 0} \sin\theta = 0, \lim_{\theta \to 0} \cos\theta = 1$.

We now look at $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$. (It is conventional to use θ in trig rather than x. Also note that θ is measured in radians unless told otherwise). Let θ be an angle in the first quadrant, that is $0 < \theta < \frac{\pi}{2}$.



◊ **2.8: Proof of fundamental trig limit** ...

◊ **2.9: Examples using the fundamental trig limit** ...

You have known for some time that the derivative of $\sin x$ is $\cos x$. We can use the fundamental trigonometric limit and the limits we have derived from it to prove this fact.

◊ **2.10: Derivative of** $\sin x$ ...

2.7 One-sided limits [St:§1.6]

◊ 2.11: Motivating example ...

Definition. Let f be a function defined on an interval (a, b) (i.e. the domain of f contains (a, b)). The limit of $f(x)$ as x approaches a *from above* (or *from the right*) is L, if $f(x)$ gets closer and closer to L as x gets closer and closer to a for $a < x < b$.

We write $\lim_{x\to a^+} f(x) = L \qquad$ (or $\lim_{x\uparrow a} f(x) = L$).

A similar definition holds for limits *from below* (or from the left), $\lim_{x\to b^-} f(x)$.

As with the other definitions of limit there is a formal definition that is not part of this course.

Note. All the Limit Laws (1),(2),(3),(4) are valid for one-sided limits and appropriate versions of the Squeeze Theorem and Theorem 2.2 on equivalent statements hold.

Also a fairly obvious but useful result is.

Theorem 2.6.

$$\lim_{x\to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \textit{and} \quad \lim_{x\to a^+} f(x) = L.$$

That is, a two-sided limit exits if and only if both the one-sided limits exist and are equal.

◊ 2.12: Examples of one-sided limits ...

2.8 Limits at Infinity [St:§3.4]

Often it is of interest to know how a function behaves as x gets more and more positive or more and more negative. We call this behaviour the limits "at ∞" or "at $-\infty$". For example

$$\lim_{x\to\infty} \frac{3x^2 + 2x + 1}{x^2 + 1}.$$

Just as for the limit as $x \to a$, there is an intuitive notion of the limit as $x \to \infty$.

Definition. Let f be a function whose domain contains (a, ∞) (for some real number a). Then $\lim_{x\to+\infty} f(x) = L$ if $f(x)$ gets closer and closer to L as x gets larger and larger.

There is an analogous definition for $x \to -\infty$.

Note. Just as for one-sided limits, appropriate versions of the Limit Laws, the Squeeze Theorem and Theorem 2.2 on equivalent statements hold.

As x values approach $+\infty$, $t = \dfrac{1}{x}$ approaches 0. Hence, we get the following very useful result.

Theorem 2.7.

$$\lim_{x\to\infty} f(x) = L \quad \textit{if and only if} \quad \lim_{t\to 0^+} f\left(\frac{1}{t}\right) = L$$

$$\lim_{x\to-\infty} f(x) = M \quad \textit{if and only if} \quad \lim_{t\to 0^-} f\left(\frac{1}{t}\right) = M$$

♢ **2.13: Examples of limits at infinity** ...

2.9 Improper integrals [St:§7.8]

In Maths IA we considered integrals of the form $\int_a^b f(x)dx$ where f is a bounded function on the interval $[a, b]$. In practice more complicated things than this will arise. The theory of limits gives a method for handling these, so-called, *improper integrals*. We consider two cases.

Unbounded intervals [St:566]

If we want to integrate over unbounded intervals such as (a, ∞) , $(-\infty, b)$ or even $(-\infty, \infty)$ we use the following definition.

Definition. If $\displaystyle\int_a^b f(x)\,dx$ exists for all $b > a$ and if $\displaystyle\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists then we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx,$$

Similarly we define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$$

and for the case unbounded at both ends we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\alpha f(x)\,dx + \int_\alpha^\infty f(x)\,dx$$

for any real number α.

♢ **2.14: Example of integrals on unbounded intervals** ...

Unbounded functions

The other type of improper integral we wish to consider occurs when the integrand $f(x) \to \pm\infty$ at some point in the region of integration.

Definition. If $f(x) \to \pm\infty$ as $x \to a^+$

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if the limit exists. We have a similar definition if $f(x) \to \pm\infty$ as $x \to b^-$.

◊ 2.15: Examples of integrals of unbounded functions ...

Important note: Whenever calculating improper integrals of either type it is essential to express it explicitly in terms of limits rather than just finding the answer.

2.10 Linear Approximation [St:§2.9]

If we wish to approximate the curve $y = f(x)$ by a straight line near $x = a$, where $y = f(a)$, it is clear that the natural straight line to choose is the tangent at the point $(a, f(a))$ (provided f is differentiable at a). We will now look at the approximation which the equation to the tangent provides to the function $f(x)$.

◊ 2.16: Derivation of formula ...

For x is close to a

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ near } a.$$

If we write $x = a + h$ then

$$f(a+h) \approx f(a) + f'(a)h$$

and we expect the approximation to be better for smaller h.

Alternative Notation

If we set

$$\Delta x = h$$
$$\text{and } (\Delta f)(a) = f(a + \Delta x) - f(a)$$

then the linear approximation can be written in the form

$$(\Delta f)(a) \approx f'(a)\Delta x$$

which uses differentiation to estimate the change Δf in the function f for a (small) change Δx in the value of x.

◊ 2.17: Examples of linear approximation ...

2.11 L'Hôpital's Rule [St:§6.8]

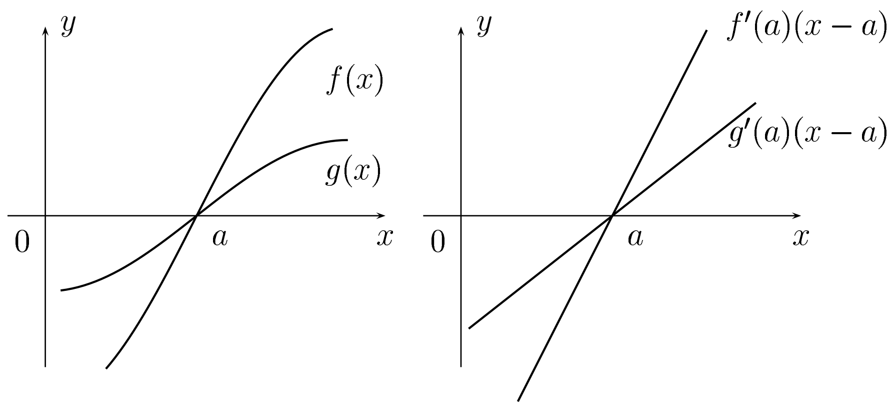
Theorem 2.8 (L'Hôpital's Rule.). *Suppose functions f and g are differentiable in an interval containing point a but not necessarily at a and that $g'(x) \neq 0$ in that interval except possibly at a. Suppose also that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the right-hand side limit exists. Moreover, if the right hand side limit approaches $\pm\infty$, so also does the left hand side limit.

In order to give a formal proof of L'Hôpital's Rule, we must first prove a theorem called Cauchy's Mean Value Theorem. For the moment we give a possible argument to show why L'Hôpital's Rule works in a special case.

$\Diamond$ 2.18: Justification for L'Hôpital's Rule ...



$\Diamond$ 2.19: Examples of L'Hôpital's Rule ...

Note. L'Hôpital can only be applied if the limit has an indeterminate form. If the limit actually exists application of l'Hôpital will give rubbish.

◊ 2.20: Example ...

Moral. L'Hôpital is only applicable to indeterminate forms.

Note. L'Hôpital may be applied in cases where the $\frac{0}{0}$ form occurs for limits as x tends to ∞ or $-\infty$ instead of to a finite number such as a.

Note. L'Hôpital's Rule also applies when $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{\infty}{\infty}$ (or $\frac{-\infty}{\infty}$ or $\frac{\infty}{-\infty}$). Again we can take a here to be finite or $\pm\infty$.

◊ 2.21: Further examples ...

Chapter 3

CONTINUITY

3.1 The Definition of Continuity [St:§1.8]

In talking of $\lim_{x\to a} f(x) = L$ we considered a function f containing a set of the form $(b,a)\cup(a,c)$ in its domain. Now we discuss the case where $\lim_{x\to a} f(x) = L = f(a)$.

Definition. Let $f(x)$ be defined on (b,c) and let a be a point in (b,c). We say that f is *continuous at* a if $\lim_{x\to a} f(x) = f(a)$.

Note. There are three essential requirements for f to be continuous at a

(i) the function f is defined at a ($a \in \mathcal{D}(f)$)

(ii) the limit $\lim_{x\to a} f(x)$ exists, and

(iii) the value of the limit is $f(a)$.

Failure of any one or more of these will lead to a "discontinuity" for $f(x)$ at $x = a$. That is, if f is not continuous at $x = a$, we say it is *discontinuous*.

Definition (Continuity on an Interval)**.** We say that f is continuous on the open interval (b,c) if f is continuous at every point of (b,c).

We say that f is continuous on the closed interval $[b,c]$ if f is continuous at every point of (b,c) and if $\lim_{x\to b^+} f(x) = f(b)$ and $\lim_{x\to c^-} f(x) = f(c)$ (that is, f continuous from the right at b and from the left at c).

(Similar definitions hold for $(b,c]$ and $[b,c)$)

$\Diamond$ 3.0: Examples ...

The Limit Laws imply the following results about continuous functions.

Theorem 3.1 (Rules for Continuity). *If f and g are continuous at a, and α and β are any real numbers, then*

(i) $\alpha f(x) + \beta g(x)$ *is continuous at a.*

(ii) $f(x).g(x)$ *is continuous at a.*

(iii) $f(x)/g(x)$ *is continuous at a, provided $g(a) \neq 0$.*

Note. These follow directly from Limit Laws (1), (2), (3).

We also have

Theorem 3.2 (Composition of Continuous Functions). *If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.*

◊ 3.1: Example ...

Both sin and cos are continuous on $\mathbb{R}$.

◊ 3.2: Proof ...

3.2 Discontinuities [St:§1.8]

Definition. Suppose f is defined near a (in other words, f is defined on an interval (b, c) containing a, except possibly at a) then f is **discontinuous** at a (or f has a **discontinuity** at a) if f is **not** continuous at a.

Note. Note that f can have a discontinuity at a because

(i) it is not defined at a ($a \notin \mathcal{D}(f)$)

(ii) the limit $\lim_{x\to a} f(x)$ DNE, or

(iii) or $\lim_{x\to a} f(x) = L \neq f(a)$.

Definition. If $\lim_{x\to a} f(x)$ DNE, we say that f has an *essential discontinuity* at a.

If $\lim_{x\to a} f(x) = L \neq f(a)$ we say that the discontinuity is *removable* since if we define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \neq a \\ L & \text{if} \quad x = a \end{cases}$$

then the function $\tilde{f}(x)$ is continuous at a and differs from $f(x)$ only at $x = a$.

$\Diamond$ **3.3: Examples** ...

3.3 Continuity and differentiability [St:§2.2]

Consider the function $f(x) = |x|$. This is continuous at 0 but not differentiable at 0, so that continuity *does not* imply differentiability. However differentiable functions are continuous.

Theorem 3.3. *If f is differentiable at a then f is continuous at a.*

$\Diamond$ **3.4: Proof of Theorem** ...

We can use this to prove the product formula for derivatives. Recall that this says that

$$\frac{d}{dx}(fg)(x) = \frac{d}{dx}\{f(x)\,g(x)\} = f'(x)\,g(x) + f(x)\,g'(x).$$

$\Diamond$ **3.5: Proof of Product Rule** ...

3.4 The Intermediate Value Theorem [St:§1.8]

Functions continuous on closed bounded intervals, i.e. $[a, b]$, have some very important properties which depend critically on the continuity and on the fact that the interval is closed and bounded.

Theorem 3.4 (The Intermediate Value Theorem)**.** *Let f be continuous on $[a, b]$. For each real number K between $f(a)$ and $f(b)$ there is at least one $c \in (a, b)$ such that $f(c) = K$.*

◊ 3.6: Examples using the IVT ...

Note. Recall that in the first lectures of the year we discussed the fact that the real numbers had no "gaps" in them and that equations such as $x^2 - 2 = 0$ could be solved for x a real number but did not necessarily have a solution when x is a rational number. The IVT gives a precise statement of this general property of the real numbers.
In the special case where the graph of a continuous function cuts the x-axis, there is a real number c with $f(c) = 0$; that is, c is a root of the equation $f(x) = 0$. We now consider two methods for solving equations of the form $f(x) = 0$.

3.5 Bisection Method

The *method of bisection* is a simple, but effective method for finding roots of an equation $f(x) = 0$.
If we know that f is continuous on the interval $[a, b]$ and $f(a) < 0 < f(b)$ then the IVT tells us that there is a root of the equation $f(x) = 0$ in the interval (a, b). We choose a first approximation to the root as $x = (a + b)/2$ and consider $f((a + b)/2)$. If this is zero we have a root; if this value is less than zero there is a root in $((a + b)/2, b)$; while if the value is greater than zero there is a root in $(a, (a + b)/2)$. Now we can repeat the procedure and obtain an approximation to the root with greater accuracy.
This method obviously works equally well if f is continuous on $[a, b]$ and $f(a) > 0 > f(b)$.
Such approximation methods are very important because even for polynomial equations

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

there is no general formula for $n \geq 5$; and although there are formulae for $n = 3$, 4, it is quicker and easier to use a method to approximate a solution.

3.6 Newton's Method [St: §3.8]

Newton gave an alternative method of solving the equation $f(x) = 0$, provided that f is a differentiable function. If you have a reasonably close initial guess x_0, you construct the local tangent to $y = f(x)$ at the point $(x_0, f(x_0))$.

Since the tangent is a good local approximation to the curve, we find where the tangent cuts the x-axis and call that point x_1, an improved guess. Then we go back to the point $(x_1, f(x_1))$ on the curve and repeat the procedure to get an x_2. We want to find a formula for the point x_n obtained by repeating this procedure n times.

$\diamondsuit$ **3.7: Derivation of formula** ...

Newton's Method for solving $f(x) = 0$

Given a sufficiently close initial guess x_0 then we have an algorithm for generating successive iterations $x_1, x_2, \ldots$ that will often approximate the true root.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \ldots$$

$\diamondsuit$ **3.8: Examples** ...

Space for further examples (if needed)

Chapter 4

APPLICATIONS OF THE DERIVATIVE

4.1 Maximum and Minimum Values of Continuous Functions [St:§3.1]

Definition. A function $f : \mathcal{D} \to \mathcal{R}$ is said to have a *maximum* at $a \in \mathcal{D}$ if $f(x) \leq f(a)$ for every $x \in \mathcal{D}$.

Similarly, for a minimum.

◊ 4.0: Example ...

Theorem 4.1 (Extrema of a Continuous Function)**.** *Let f be a continuous function on $[a, b]$. Then f* attains *a maximum and a minimum value in $[a, b]$.*

◊ 4.1: Example ...

Now we introduce the notion of local maxima and minima.

Definition. We say that f has a *local maximum at a* if there is an open interval $I \subset \mathcal{D}(f)$ containing a such that f has a maximum on I at a. A *local minimum* is defined similarly. A *local extremum* is a local maximum or minimum.

Theorem 4.2. *Let f have a local extremum at a. If f is differentiable at a then $f'(a) = 0$.*

Note. The converse is not necessarily true, e.g. $f(x) = x^3$ has $f'(0) = 0$ but there is not a local extremum there.

◊ 4.2: Proof ...

This immediately leads us to a theorem for the systematic seeking of maxima/minima for a continuous function f on a closed interval $[a, b]$.

Theorem 4.3. *Let f be a continuous function on $[a, b]$. Then the extrema of f on this interval must occur either*

(i) at an endpoint a or b; or

(ii) at points in (a, b) where $f'(x) = 0$; or

(iii) at points in (a, b) where f' fails to exist.

It is convenient to group types (ii) and (iii) by using

Definition. A point $c \in \mathcal{D}(f)$ is called a *critical point of f* if c is an interior point (i.e. not an end point of any interval) of $\mathcal{D}(f)$ and either $f'(c)$ DNE or $f'(c) = 0$.

Hence Theorem 4.3 says: a continuous function f on $[a, b]$ attains its max/min at endpoints or critical points. So the method of finding the max/min is

(1) Locate critical points of f in (a, b), namely $c_1, c_2, \ldots, c_n$.

(2) Calculate $f(a), f(b), f(c_1), \ldots, f(c_n)$ and pick out the largest/smallest.

Note. Finding all the critical points may not always be easy.

$\Diamond$ **4.3: Examples** ...

We can use the above ideas also for f continuous on open or on unbounded intervals but without the certainty that a maximum/minimum is attained for the case of f continuous on $[a, b]$. An internal extremum must still occur at a critical point but end points (or $\pm\infty$) need separate analysis.

$\Diamond$ **4.4: Example** ...

4.2 Applied Maximum/Minimum Problems [St:§3.1]

Practical maximum/minimum problems differ widely but it is possible to indicate some fairly general steps although they may not all be relevant in every problem. Treat these just as guidelines which we illustrate by a number of examples.

The steps are:

(1) Draw a picture (if possible). Label the variables.

(2) Identify the quantity to be maximized or minimized. This is the dependent variable.

(3) Identify the quantities on which the dependent variable depends. Write down the relations between these variables.

(4) Select *one* of the quantities from step 3 and express the dependent variable as a function of this variable – the independent variable – alone. Use the physical constraints of the problem to fix the domain of the function. [Note: frequently one variable may be a better choice than others, so a little thought should be put into this step].

(5) The problem should now be converted into a mathematical one of optimizing a certian function over a certain interval. Use calculus to solve this problem.

(6) Answer the original question.

$\Diamond$ **4.5: Examples** ...

Space for further examples (if needed)

4.3 Rolle's Theorem [St:§3.2]

For functions f, continuous on $[a, b]$ and differentiable on (a, b), there are a number of useful results.

Theorem 4.4 (Rolle's Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then if $f(a) = f(b) = 0$ there is at least one $c \in (a, b)$ such that $f'(c) = 0$.*

$\Diamond$ **4.6: Proof** . . .

Note. Rolle's Theorem says: if f is differentiable then between any two zeros of f there is at least one zero of f'. This means that if f has n zeros then f' must have at least $n - 1$ zeros. We will also use this in the form that says that if f' has precisely $n - 1$ zeros then f can have no more than n zeros.

Note. Result extends to the case in which $f(a) = f(b) \neq 0$ by applying the above theorem to the function $g(x) = f(x) - f(a)$ so $g(a) = g(b) = 0$. Then $f'(x) = g'(x)$ and $g'(c) = 0$ implies $f'(c) = 0$.

$\Diamond$ **4.7: Examples** . . .

Result. An nth degree polynomial $(n \geq 1)$ has at most n real zeros.

$\Diamond$ **4.8: Proof** . . .

$\Diamond$ **4.9: Further Examples involving Rolle's theorem** . . .

Space for further examples if needed

Many physical, chemical or biological phenomena, for example, pressure, temperature, velocity, chemical reaction rates and populations, are modelled by continuous and differentiable functions, usually of time and/or space. Although the explicit form of some functions is difficult to determine, knowing they are continuous and differentiable allows us to make significant decisions as to their form and to the values they take in real life. So far we have considered the IVT and Rolle's Theorem which allow us to deduce certain properties of continuous and differentiable functions. In the next section we consider the Mean Value Theorem which is a generalisation of Rolle's Theorem. This theorem has a number of important consequences for differentiable functions, as we shall see.

4.4 Mean Value Theorem - extending Rolle's Theorem

Theorem 4.5 (Mean Value Theorem (MVT))**.** *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there is at least one number c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e. the slope at c is the "mean value" of the slope on $[a, b]$.

◊ 4.10: Proof ...

◊ 4.11: Examples involving the MVT ...

Note. We can rewrite the MVT as follows. Let f be continuous on $[a, b]$, differentiable on (a, b) and take x and $x + h$ in (a, b). By the MVT there exists $c \in (x, x + h)$ with $f'(c) = \dfrac{f(x+h) - f(x)}{(x+h) - x}$ and rewriting we have

$$f(x + h) = f(x) + hf'(c).$$

This relationship is the basis of most numerical methods for differentiable functions.

Consequences of the MVT

Corollary 4.1 (Functions with Zero Derivative [St:237])**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all x in (a, b), then f is a constant function on $[a, b]$. That is, there exists a constant C such that $f(x) = C$ for all x in $[a, b]$.

◊ 4.12: Proof ...

Corollary 4.2 (Functions with Equal Derivatives)**.** Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f'(x) = g'(x)$ for all x in the open interval (a,b). Then f and g differ by a constant on $[a,b]$. That is, there exists a constant K such that

$$f(x) = g(x) + K$$

for all x in $[a,b]$.

◊ 4.13: Proof ...

Note. Corollary 4.2 is the reason we add a constant when we consider indefinite integrals.

Definition (Increasing and Decreasing Functions)**.** A function f is said to
(i) *increase* on the interval I if, for every two numbers x_1, x_2 in I,

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

(ii) *decrease* on the interval I if, for every two numbers x_1, x_2 in I,

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

Corollary 4.3 (Increasing/Decreasing Functions Test)**.** (a) If $f'(x) > 0$ for all x in an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ for all x in an interval, then f is decreasing on that interval.

◊ 4.14: Proof ...

◊ 4.15: Examples ...

Theorem 4.6 (The First Derivative Test). *Suppose f is continuous on an open interval, and differentiable on that interval except possibly at c, a critical point of f.*

(a) *If f' changes from positive to negative at c, then $f(c)$ is a local maximum.*

(b) *If f' changes from negative to positive at c, then $f(c)$ is a local minimum.*

(c) *If f' does not change sign at c, then f has no local maximum nor minimum at c.*

◊ 4.16: Proof ...

◊ 4.17: Examples using the First Derivative Test ...

Concavity

In discussing functions it is helpful to introduce the idea of the concavity of a curve.

Definition.

(1) If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on the interval I

(2) If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on the interval I.

Theorem 4.7 (Test for Concavity). *Let f be twice differentiable on an open interval I.*

(i) If $f''(x) > 0$ on I, f is concave up(ward) on I.

(ii) If $f''(x) < 0$ on I, f is concave down(ward) on I.

Definition. A point where the concavity of f changes is called an *inflection point*.

Note. Note that this means that the graph of f crosses the tangent line at an inflection point c and that either $f''(c) = 0$ or $f''(c)$ DNE. The converse however is not true: $f(x) = x^4$ has $f''(0) = 0$ but 0 is a minimum, but not a point of inflection.

◊ 4.18: Examples involving concavity ...

Theorem 4.8 (Second Derivative Test). *Let f be twice differentiable on the open interval I containing the critical point c at which $f'(c) = 0$.*

(i) If $f''(x) > 0$ on I then $f(c)$ is the minimum value of f on I.
(ii) If $f''(x) < 0$ on I then $f(c)$ is the maximum value of f on I.

◊ **4.19: Proof** ...

Note. The second derivative test is commonly used in the following form. If $f''(x)$ has the same sign in some interval containing c then a convenient local version of the second Derivative test exists for use at critical point c with $f'(c) = 0$.

(i) If $f''(c) > 0$ then $f(c)$ is a local minimum.
(ii) If $f''(c) < 0$ then $f(c)$ is a local maximum.

◊ **4.20: Example of the second derivative test** ...

Note. The second Derivative Test gives no information if $f''(c) = 0$ at critical point c. In fact it is possible to have maxima, minima or neither.

◊ **4.21: Example** ...

4.5 Curve Sketching [St:§3.5]

Steps for Curve Sketching

1. Determine any x- or y-intercepts and points of discontinuity.
2. Solve $f'(x) = 0$ and find where $f'(x)$ DNE to find the critical points.
3. Find intervals where $f(x)$ is increasing or decreasing.
4. Solve $f''(x) = 0$ and find where $f''(x)$ DNE to locate possible inflection points.
5. Find where f is concave up, concave down.
6. Determine any horizontal/vertical/oblique asymptotes and the behaviour of $f(x)$ and $f'(x)$ near discontinuities. Also find the behaviour of $f(x)$ as $x \to \pm\infty$.
7. Plot and label the critical points, possible inflection points and intercepts.
8. Plot the curve in a manner consistent with your findings.

Note that for some functions, it may not be possible to follow all of these steps.

$\Diamond$ **4.22: Examples** ...

Space for further examples (if needed)

Theorem 4.9 ((Cauchy's Mean Value Theorem)). *Suppose that f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is a number c in (a, b) for which*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Chapter 5

TAYLOR SERIES

5.1 Permutations and combinations

The number of ways of permuting m objects is $m! = m \times (m-1) \times (m-2) \times \ldots \times 2 \times 1$.
$m!$ is called "m factorial". Then for $m = 1, 2, \ldots$ we have
$1! = 1,\ 2! = 2 \times 1 = 2,\ 3! = 3 \times 2 \times 1 = 6,\ 4! = 24,\ 5! = 120$ etc.
For completeness we define $0! = 1$.

♢ 5.0: Example . . .

If we choose or select n objects from a set or collection of m objects and the **order of selection is important** (that is, we list the chosen objects in order of selection) then there are $m \times (m-1) \times (m-2) \times \ldots \times (m-n+2) \times (m-n+1)$ ways of doing this.

♢ 5.1: Example . . .

In this topic we are much more interested in the choice of n objects from a set of m objects in which the **order of selection is not important**. That is, we only wish to know how many ways we can select a group of n objects from a given set of m objects and not the order in which the selection was made. An alternative statement is that we wish to know how many different *combinations* of n objects we can construct from a set of m objects. The number of ways of doing this is given by dividing $m \times (m-1) \times (m-2) \ldots \times (m-n+2) \times (m-n+1)$ by $n!$, as in the previous choice, the list of n objects contained all possible permutations of the n objects.

♢ 5.2: Example . . .

In general, if we choose n objects from m objects, with order not important, there are $\binom{m}{n}$ (in words "m choose n ") ways of doing this.

◊ 5.3: Calculation ...

So we get the formula

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

Note that $\binom{m}{0} = \frac{m!}{0!m!} = 1 = \binom{m}{m}$, $\quad \binom{m}{1} = \binom{m}{m-1} = m$,
$\binom{m}{2} = \binom{m}{m-2} = \frac{m(m-1)}{2}$ and so on.

One area in which these ideas are important is in the expansion of $(a+b)^m$ and the special case of $(1+x)^m$ which is known as the Binomial Theorem.

◊ 5.4: Binomial Theorem examples ...

5.2 Taylor Polynomials [St:§11.10]

We have seen how to approximate a function at a point by its tangent line. In this section we shall discuss how to achieve better approximations by using, instead of a linear function like a line, quadratic, cubic and higher order polynomials. We will find we can do even better — getting exact infinite *series* for many functions — enabling evaluation for arbitrary accuracy.

Linear Approximation

We have seen that near a point $(a, f(a))$ we can approximate a differentiable function $f(x)$ by its tangent line whose equation is

$$y = f(a) + f'(a)(x-a)$$

We will call this polynomial the first order Taylor polynomial for f at a. It is denoted by $P_1(x)$ so we have

$$P_1(x) = f(a) + f'(a) \cdot (x-a)$$

We can obtain the same result by finding a degree 1 polynomial P_1 such that the value of P_1 and its first derivative agree with those of f at the point a.

◊ 5.5: Calculation ...

♢ 5.6: Example ...

Clearly we might hope to get a better approximation if we used, say, a quadratic polynomial to approximate $f(x)$ near $x = a$. We look for a quadratic which goes through $(a, f(a))$ and has the same value and first and second derivatives at $x = a$.

♢ 5.7: Calculation ...

We get

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

as the quadratic approximation to f at a.
Notice that in order to determine the three constants in the quadratic $P_2(x)$, there must be three conditions imposed on it; namely the value, the first *and second* derivative at $x = a$.

♢ 5.8: Example ...

The nth degree polynomial approximation can be written:

$$P_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n$$

and we can use the same procedure as above.

♢ 5.9: Calculation ...

We obtain the formula

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

and this is called the *Taylor Polynomial of degree* n for $f(x)$ at $x = a$.

The special case when $a = 0$ is called the *Maclaurin Polynomial of degree* n for $f(x)$:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \ .$$

◊ 5.10: Examples of Taylor Polynomials ...

It is natural to ask how accurate Taylor approximations are. In general we expect better accuracy as n increases and if $(x-a)$ is small. This is summarised by Taylor's theorem.

Theorem 5.1 (Taylor's Theorem). *Suppose the function f has $(n+1)$ derivatives on some interval containing a and x. Then if*

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the nth Taylor polyomial of f at a, $f(x) = P_n(x) + R_n(x)$ where the remainder

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some number z between a and x.

◊ 5.11: Comments on proof of Taylor's Theorem ...

Note. Observe that the remainder (error) is essentially the next ($n+1$ degree) term in which $f^{(n+1)}$ is evaluated at a point z between a and x. We will use Taylor's Theorem to estimate the error in the approximation of $f(x)$ by $P_n(x)$. Notice that we have

$$\begin{aligned} |f(x) - P_n(x)| &= |R_n(x)| \\ &= \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \right|. \end{aligned}$$

Assume we can find a constant C such that

$$|f^{(n+1)}(z)| \leq C$$

for all z between a and x then we have

$$|f(x) - P_n(x)| \leq \frac{C}{(n+1)!}|x-a|^{n+1}$$

and this enables us to estimate the error.

◊ 5.12: Example of Taylor's Theorem ...

A natural question to ask is: How many terms are needed in a Taylor polynomial to get a specified accuracy?

◊ 5.13: Examples ...

Taylor Series

If the function f has derivatives of all orders, we can write $f(x)$ in the form given by Taylor's Theorem for all values of n. Further, if we can show the remainder $R_n(x) \to 0$ as $n \to \infty$ then

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The expression on the right hand side is called the *Taylor series* of f at $x = a$. In the special case $a = 0$ we have the *Maclaurin series* of f.

To make sense of the equation above we need to consider when it is sensible to add up an infinite string of numbers.

5.3 Infinite Series [St:§11.2]

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_i + \cdots$$

where the a_i are real numbers.

The *Nth partial sum* S_N is the sum of the first N terms

$$S_N = a_1 + a_2 + \cdots + a_N$$

We say the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ is *convergent* with *sum* S provided

$$S = \lim_{N\to\infty} S_N.$$

If $\displaystyle\lim_{N\to\infty} S_N$ does not exist we say that the series *diverges*.

Question. What does $\displaystyle\lim_{N\to\infty} S_N$ mean? It is like $\displaystyle\lim_{x\to\infty} f(x)$ but n takes only integer values. So we mean that the partial sums S_N get arbitrarily close to the value S when N gets sufficiently large (i.e. as $N \to \infty$).

Note that, just as in the case of limits as x tends to infinity there is a precise definition. This is not part of the present course but we give it here.

Definition. $S_N (N = 1, 2, 3, \cdots)$ has limit S provided that for any $\epsilon > 0$ there exists a positive number M such that $|S_N - S| < \epsilon$ whenever $n > M$.

Clearly for the partial sums to get closer and closer to some value S then the extra amount added each time must tend towards 0, so it is not surprising that using the technical definition above it is possible to prove the following

Proposition. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

The converse of this result is not true, as is shown by the following

Example 5.3.1. The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\Diamond$ **5.14: Proof** ...

Moral. The terms tending to zero is a necessary *but not sufficient* condition for the convergence of a series.

Example 5.3.2. The p**- series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

This result depends on the Integral Test for series which is not covered in this course. Here we will give a flavour of the argument by looking at a special case and showing that $\sum_{1}^{\infty} \frac{1}{n^2} \leq 2$.

$\Diamond$ **5.15: Outline of proof of special case** ...

Example 5.3.3. The series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

$\Diamond$ **5.16: Proof** ...

The Geometric Series
Consider the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots .$$

This is called the *geometric series* with common ratio x.

◊ 5.17: Calculation of partial sum . . .

$$S_N = \frac{1 - x^{N+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{N+1}}{1 - x}.$$

Consider what happens as $N \to \infty$. If $|x| < 1$ then $x^{N+1} \to 0$ so $S_N \to \frac{1}{1-x}$, i.e.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \quad \text{if} \quad |x| < 1.$$

If $|x| > 1$ then the term $x^{N+1} \to \pm\infty$ and so $\sum\limits_{n=0}^{\infty} x^n$ diverges.

Sometimes we encounter a geometric series of the form

$$\sum_{n=0}^{\infty} a x^n = a + ax + ax^2 + \dots = a(1 + x + x^2 + \dots).$$

This diverges if $|x| > 1$ and converges to $a/(1 - x)$ if $|x| < 1$.

◊ 5.18: Example . . .

There are a number of tests used to see if a given series is convergent. We look at two useful tests.

The Ratio Test.

Consider a series $\sum\limits_{n=0}^{\infty} a_n$, with each $a_n \neq 0$, such that

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

either exists or is infinite.
Then

$$\begin{array}{lll} \text{if} & r < 1 & \text{the series converges,} \\ \text{if} & r > 1 & \text{the series diverges,} \\ \text{if} & r = 1 & \text{the ratio test is inconclusive.} \end{array}$$

◊ **5.19: Examples using the ratio test ...**

Alternating Series

Definition 5.2. An alternating series is an infinite sum of the form ($a_n > 0$):

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

The Alternating Series Test

If $a_n \geq a_{n+1} \geq 0$ for all n and if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

◊ **5.20: Examples of alternating series ...**

Estimating sums

For an alternating series which satisfies the Alternating Series Test, we can approximate the sum s of the series with a partial sum s_n and importantly, we have a very useful estimate for the remainder or error term.

Theorem 5.3. Alternating Series Estimation Theorem

If $s = \sum_{n=0}^{\infty}(-1)^n a_n$ *is the sum of an alternating series which satisfies*

$$(i) \quad 0 \leq a_{n+1} \leq a_n \quad \textit{and} \quad (ii) \quad \lim_{n\to\infty} a_n = 0$$

then if $s_m = \sum_{n=0}^{m}(-1)^n a_n$,

$$|R_m| = |s - s_m| \leq a_{m+1}.$$

◊ **5.21: Proof ...**

◊ **5.22: Example of estimating an alternating series** ...

5.4 Power Series [St: §11.8]

For the case of $f(x)$ with all derivatives the Taylor polynomial for $f(x)$ about $x = a$ or the Maclaurin polynomial for $f(x)$ about $x = 0$ can be continued indefinitely, i.e. we can generate Taylor series or Maclaurin series – effectively "infinite polynomials".

For example, Maclaurin series for e^x, $\cos x, \sin x$ are

$$\begin{aligned} e^x: &\quad 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots &&= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x: &\quad 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdot &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \sin x: &\quad x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

These are examples of infinite series of the form $\sum\limits_{n=0}^{\infty} a_n x^n$. Such series are called **power series**.

We start our investigation of power series with (again) **the geometric series** and consider the question of **convergence**.

◊ **5.23: Example - convergence of geometric series** ...

So for the geometric series we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x| < 1, \qquad \text{but} \sum_{n=0}^{\infty} x^n \quad \text{diverges} \quad \text{if} \quad |x| \geq 1.$$

$\lozenge$ 5.24: Example - repeating decimals ...

$\lozenge$ 5.25: Further example on convergence of geometric series ...

The convergence behaviour of the geometric series is rather typical of power series, as we see in the next theorem.

Theorem 5.4 (Convergence of Power Series)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n x^n.$$

Then either

(1) the series converges for all values of x, or
(2) the series converges only for $x = 0$, or
(3) there exists a number $R > 0$ such that $\sum_{n=0}^{\infty} a_n x^n$ converges for all x with $|x| < R$ and diverges for all x with $|x| > R$.

The number R is called the radius of convergence. In case (1) we often write "$R = \infty$" and in case (2), $R = 0$.

Note. The set of all reals for which a power series converges is called *the interval of convergence*. When $0 < R < \infty$ the 4 possibilities are $(-R, R), (-R, R], [-R, R), [-R, R]$.

$\lozenge$ 5.26: Examples of intervals of convergence ...

5.5 Taylor Series [St: §11.10]

Recall that the Taylor series of a function $f(x)$ at $x = a$ is

$$f(x) = \lim_{N\to\infty} P_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and that if $a = 0$ we call this a Maclaurin series.

◊ 5.27: Example - interval of convergence for Maclaurin series of $\cos x$ ...

The ratio test tells us that this power series converges - but not what it converges to. We would hope it converges to $\cos x$. To see this we consider the remainder formula $f(x) = P_n(x) + R_n(x)$ as $n \to \infty$.

Theorem 5.5 (Taylor Series Representation)**.** *If $f(x)$ has derivatives of all orders on some interval containing a and if $\lim_{n\to\infty} R_n(x) = 0$ for each x in the interval, then the Taylor Series for $f(x)$ converges to $f(x)$ at each x in the interval. That is,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In the case that $a = 0$ (Maclaurin Series)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

◊ 5.28: Example - Maclaurin series for e^x ...

Note. Similarly $R_n(x) \to 0$ for $\sin x, \cos x$ so their Maclaurin series converge to $\sin x, \cos x$ respectively. We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

for all x and

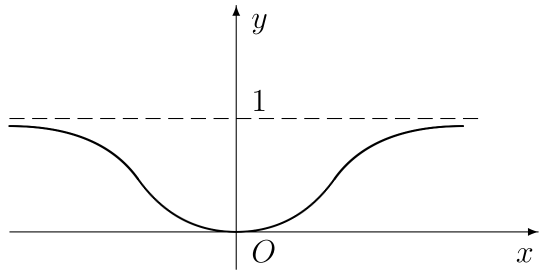
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

for all x.

Note (A Cautionary Example)**.** Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Notice that $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}} = 0 = f(x)$ so f is continuous at 0. The graph of f is



We claim that this function has a Maclaurin series which converges, but it does not converge to the values of the function.

$\Diamond$ **5.29: Proof of claim** ...

The Binomial theorem

Another very useful power series arises from the **Binomial Theorem**

Recall that for integer m;

$$(1+x)^m = 1^m + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m$$

where

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{m(m-1)\dots(m-n+1)}{n!}.$$

With power series in mind we can write this as

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + x^m.$$

We now consider $f(x) = (1+x)^k$ where k is *any* real number and find the Maclaurin series of f.

$\Diamond$ **5.30: Calculation of Maclaurin series** ...

We get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \dots$$

We test this series for convergence.

$\Diamond$ **5.31: Calculation of convergence** ...

So the series converges if $|x| < 1$. To prove the result we want we should now prove that $\lim_{n\to\infty} R_n(x) = 0$. This is quite difficult and we omit the proof. Hence we have the *binomial series*

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n$$

valid for $|x| < 1$. If we define

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

then we can write the binomial series as

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n$$

again valid for $|x| < 1$.

Special Cases of the Binomial Theorem

Case 1 If $k = m$, an integer.

Case 3 Replace x by $-x$

Case 4 Put $x = t^2$ in Case 2

◊ **5.32: Calculations** ...

The Binomial Theorem is useful for calculating expressions such as $(1+x)^{1/2}, (1+x)^{3/5}$ etc.

◊ **5.33: Binomial Theorem Example** ...

5.6 Power Series Computations [St:§11.9]

We have looked at power series of the form $\sum_{n=0}^{\infty} a_n x^n$ and the notion of radius of convergence R.

The same ideas apply to power series based about some other point a i.e. $\sum_{n=0}^{\infty} a_n (x-a)^n$ with convergence for $|x-a| < R$ and divergence for $|x-a| > R$, or convergence for all real numbers x when we write $R = \infty$). Based on a more sophisticated analysis of convergence of series the following very powerful theorem can be proved. It essentially states that the power series can be "differentiated and integrated term by term" within its radius of convergence.

Theorem 5.6 (Differentiation and Integration of Power Series)**.** *Suppose*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad \textit{for} \quad |x-a| < R$$

where R is the radius of convergence. Then

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1} \quad \textit{for} \quad |x-a| < R$$

and

$$\begin{aligned}\int_a^x f(t)dt &= \sum_{n=0}^{\infty}\left(\int_a^x a_n(t-a)^n dt\right)\\ &= \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-a)^{n+1} \quad \textit{for} \quad |x-a|<R\end{aligned}$$

Note. The theorem tells us nothing about the question of convergence at $x = a + R$ and $x = a - R$. This needs to be checked in each case.

♢ 5.34: Examples ...

Space for further examples (if needed)