Random Processes Assignment

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Question 0. Make sure that in all your answers you:

(a) use full and complete sentences,

(b) include units where necessary,

(c) give ranges/constraints for parameters in formulae, and

(d) make sure your layout and mathematical arguments are clear and concise.

Question 1. Let $(Y(t), t \geq 0)$ be a birth-death process on the state space $\mathcal{S} = \{0, 1, 2, \ldots\}$ with the following nonzero transition rates:

$$\begin{aligned} q_{n,n+1} &= \lambda_n, \\ q_{n,n-1} &= \mu_n, \quad n \geq 1. \end{aligned}$$

(a) Write down the KFDEs for this CTMC for initial state $i \in \mathcal{S}$.

Solution For a CTMC

$$\begin{aligned} \frac{dP_{ij}(t)}{dt} &= \sum_{k \in S} P_{ik}(t) q_{kj} \\ \frac{dP_{ij}(t)}{dt} &= P_{i,j-1}(t) q_{j-1,j} + P_{i,j}(t) q_{j,j} + P_{i,j+1}(t) q_{j+1,j} \\ &= P_{i,j-1}(t) \lambda_{j-1} + P_{i,j}(t)(-\lambda_j - \mu_j) + P_{i,j+1}(t) \mu_{j+1} \end{aligned}$$

For $i = j = 0$

$$\frac{dP_{00}(t)}{dt} = -\lambda_0 P_{0,0}(t) + \mu_1 P_{0,1}$$

As required.

(b) With reasoning, what are the initial conditions for the differential equations in part (a)?

Solution At $t = 0$, the probability of going from state i to j is 0 for $i \neq j$, and 1 for $i = j$ i.e.

$$P_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

This is because the instantaneous probability to leave the state is 0. **As required.**

Question 2. Let $(X(t), t \geq 0)$ be a CTMC on $\{1, 2\}$ with generator

$$Q = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix},$$

where $\lambda, \mu > 0$.

(a) Write down the KFDEs.
Solution

$$\frac{dP_{ij}(t)}{dt} = \sum_{k \in S} P_{ik}(t) q_{kj}$$

In matrix notation

$$\frac{dP(t)}{dt} = P(t)Q = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \times \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$$

$$\begin{pmatrix} \frac{dP_{11}(t)}{dt} & \frac{dP_{12}(t)}{dt} \\ \frac{dP_{21}(t)}{dt} & \frac{dP_{22}(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\mu P_{11} + \lambda P_{12}, & \mu P_{11} - \lambda P_{12} \\ -\mu P_{21} + \lambda P_{22}, & \mu P_{21} - \lambda P_{22} \end{pmatrix}$$

As required.

(b) By noting that

$$P_{ij}(t) = 1 - P_{ii}(t) \text{ for } i \neq j,$$

(check this!) solve these forward equations for the transition probabilities $P_{ij}(t)$, for $i, j = 1, 2$, determining and using the initial conditions. Make sure you show all working.

Solution The probabilities must satisfy

$$\sum_{k \in S} P_{ik}(t) = 1; \; \forall t, \; i \in \mathcal{S}$$

For this problem

$$\begin{aligned} \sum_{k \in S} P_{ik}(t) &= 1 \\ P_{ii}(t) + P_{ij}(t) &= 1 \quad i \neq j \\ P_{ij}(t) &= 1 - P_{ii}(t) \end{aligned}$$

So this condition holds. We get

$$\begin{aligned} \frac{dP_{11}(t)}{dt} &= -\mu P_{11} + \lambda P_{12} \\ &= -\mu P_{11} + \lambda(1 - P_{11}) \\ &= -\mu P_{11} + \lambda - \lambda P_{11} \end{aligned}$$

Has form $\frac{dy}{dx} = ay + b \implies y = c\exp\{ax\} + \frac{b}{a}$
Gives solutions:

$$P_{11}(t) = ce^{(-\lambda-\mu)t} + \frac{\lambda}{\mu + \lambda}$$

We need $P_{11}(0) = 1$

$$\begin{aligned} P_{11}(0) = 1 &= ce^0 + \frac{\lambda}{\mu + \lambda} \\ 1 &= c + \frac{\lambda}{\mu + \lambda} \\ \implies c &= 1 - \frac{\lambda}{\mu + \lambda} \end{aligned}$$

$$P_{11}(t) = \left(1 - \frac{\lambda}{\mu + \lambda}\right) e^{(-\lambda-\mu)t} + \frac{\lambda}{\mu + \lambda}$$

Using the given trait:

$$P_{12}(t) = 1 - \left(1 - \frac{\lambda}{\mu + \lambda}\right) e^{(-\lambda-\mu)t} + \frac{\lambda}{\mu + \lambda}$$

Similarly for P_{22} we have essentially swapped λ and μ So we get

$$P_{22}(t) = \left(1 - \frac{\mu}{\mu+\lambda}\right) e^{(-\lambda-\mu)t} + \frac{\mu}{\mu+\lambda}$$

And $P_{22}(t)$:

$$P_{21}(t) = 1 - \left(1 - \frac{\mu}{\mu+\lambda}\right) e^{(-\lambda-\mu)t} + \frac{\mu}{\mu+\lambda}$$

As required.

Question 3. Consider a single server queue where arrival instants occur according to a Poisson process with rate λ and the service time of each individual customer is exponentially distributed with rate μ. The interesting feature of this queue is that exactly two customers arrive at each instant.

(a) Define a suitable state space S for this continuous time Markov chain.

Solution A suitable state space would be the non-negative integers. I.e. if we denote the queue $\{X(t), t \geq 0\}$ as the number of customers in the queue, over the state space $\mathcal{S}$ we have

$$\mathcal{S} = \{0, 1, 2, \ldots\} = 0 \cup \mathbb{Z}^+$$

As required.

(b) Write down the transition rates and consequently the generator Q.

Solution Arrivals occur at rate λ, and arrivals exclusively happen in pairs.
Service (leaving) occurs with rate μ. This gives

$$q_{i,i+2} = \lambda$$
$$q_{i,i-1} = \mu, \quad i \neq 0$$

As required.

(c) Under what conditions will the equilibrium distribution exist for this system? In other words, under what conditions is this system stable?

Solution $\mu > 2\lambda$. The space has to stay finite - and the system will be stable if we don't expect infinite growth. So if people are being served more frequently than pairs are arriving, we would expect stability.

As required.

(d) Under the stability conditions of part (c), write down the equilibrium equations for this system. *Do not attempt to solve this system of equations.*

Solution Equilibrium equations are the global balance

$$\pi_j \sum_{\substack{k \in S \\ k \neq j}} q_{jk} = \sum_{\substack{k \in S \\ k \neq j}} \pi_k q_{kj}$$

subject to

$$\sum_{k \in S} \pi_k = 1$$

$$\pi_j(\lambda + \mu) = \lambda \pi_{j-2} + \mu \pi_{j+1}, \; j \geq 2 \quad (1)$$
$$\pi_1(\lambda + \mu) = \mu \pi_2 \quad (2)$$
$$\pi_0 \lambda = \mu \pi_1 \quad (3)$$

This gives

$$\pi_1 = \frac{\lambda \pi_0}{\mu}$$
$$\pi_2 = \frac{\pi_1(\lambda + \mu)}{\mu} = \frac{\frac{\lambda \pi_0}{\mu}(\lambda + \mu)}{\mu}$$

As required.

(e) Use the probability generating function method to show for this system that

$$P(z) = \frac{\mu\pi_0}{\mu - \lambda z(z+1)}$$

Solution Recall

$$P(z) := \sum_{j=0}^{\infty} \pi_j z^j$$

Use eqn (1)

$$\begin{aligned}
(\lambda+\mu)\pi_j &= \lambda\pi_{j-2} + \mu\pi_{j+1} \\
\sum_{j=2}^{\infty} z^j(\lambda+\mu)\pi_j &= \sum_{j=2}^{\infty} z^j\lambda\pi_{j-2} + \sum_{j=2}^{\infty} z^j\mu\pi_{j+1} \\
(\lambda+\mu)\left(-\pi_0 - z\pi_1 + \sum_{j=0}^{\infty} z^j\pi_j\right) &= z^2\sum_{j=2}^{\infty} z^{j-2}\lambda\pi_{j-2} + \frac{\mu}{z}\sum_{j=2}^{\infty} z^{j+1}\pi_{j+1} \\
(\lambda+\mu)\left(-\pi_0 - z\pi_1 + P(z)\right) &= z^2\lambda\sum_{j=0}^{\infty} z^j\pi_j + \frac{\mu}{z}\sum_{j=3}^{\infty} z^j\pi_j \\
(\lambda+\mu)\left(-\pi_0 - z\pi_1 + P(z)\right) &= z^2\lambda P(z) + \frac{\mu}{z}\left(\sum_{j=0}^{\infty} z^j\pi_j - z^2\pi_2 - z\pi_1 - \pi_0\right) \\
(\lambda+\mu)\left(-\pi_0 - z\pi_1 + P(z)\right) &= z^2\lambda P(z) + \frac{\mu}{z}\big(P(z) - z^2\pi_2 - z\pi_1 - \pi_0\big) \\
&\text{use (2) and (3)} \\
(\lambda+\mu)\left(-\pi_0 - z\frac{\lambda\pi_0}{\mu} + P(z)\right) &= z^2\lambda P(z) + \frac{\mu}{z}\left(P(z) - z^2\frac{\frac{\lambda\pi_0}{\mu}(\lambda+\mu)}{\mu} - z\frac{\lambda\pi_0}{\mu} - \pi_0\right) \\
-\pi_0\lambda - \mu\pi_0 - z\frac{\lambda^2\pi_0}{\mu} - z\lambda\pi_0 + P(z)\lambda + P(z)\mu &= z^2\lambda P(z) + \frac{\mu}{z}P(z) - z\frac{\lambda^2\pi_0}{\mu} - z\lambda\pi_0 - \lambda\pi_0 - \frac{\mu}{z}\pi_0 \\
-\mu\pi_0 + P(z)\lambda + P(z)\mu &= z^2\lambda P(z) + \frac{\mu}{z}P(z) - \frac{\mu}{z}\pi_0 \\
(\lambda+\mu)P(z) - \mu\pi_0 &= (\lambda z^2 + \frac{\mu}{z})P(z) - \frac{\pi_0\mu}{z} \\
P(z)(\lambda z^2 + \frac{\mu}{z} - \lambda - \mu) &= \pi_0\mu(\frac{1}{z} - 1) \\
P(z) &= \frac{\pi_0\mu(\frac{1}{z}-1)}{\lambda z^2 + \frac{\mu}{z} - \lambda - \mu} \\
P(z) &= \frac{\pi_0\mu(\frac{1}{z}-1)}{(\frac{1}{z}-1)(\mu - \lambda z(z+1))} \\
P(z) &= \frac{\mu\pi_0}{\mu - \lambda z(z+1)}
\end{aligned}$$

As required.

(f) Find π_0 in terms of λ and μ. *Do not attempt to find the full equilibrium distribution.*

Solution Consider

$$\begin{aligned}
P(z=1) &= E(1) = 1 \\
\frac{\mu\pi_0}{\mu - 2\lambda} &= 1 \\
\mu\pi_0 &= \mu - 2\lambda \\
\pi_0 &= \frac{\mu - 2\lambda}{\mu}
\end{aligned}$$

As required.