Practical Asymptotics (APP MTH 4051/7087)
Assignment 5 (5%)

Due 14 June 2019

1. Consider the integral

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{x2it(1-t^2/6+3it/4)}}{t^2+9} dt$$

   (a) Identify any saddle points, then find the paths of steepest descent and ascent through each of these saddle(s).

   (b) Sketch (or plot) the saddles and paths of steepest descent/ascent.

   (c) Sketch a deformed contour that passes through a saddle point in a direction that will permit $I(x)$ to be evaluated by the method of steepest descent.

   (d) Use the deformed contour from part (c) to approximate $I(x)$ to leading-order as $x \to \infty$.

2. Use the method of steepest descents to show that

$$I(x) = \frac{1}{2} \int_{-1}^{1} e^{-4xt^2+5ixt-ixt^3} dt \sim \frac{1}{2} e^{-2x} \sqrt{\pi/x}, \quad \text{as } x \to \infty.$$

   A complete solution should go through similar steps to Question 1, but will require a few extra details. [Hints:

   - The deformed steepest descent path should have the same endpoints as the original contour (this might look a bit weird).
   - The contributions from the end points are negligible compared to that from a saddle point (you need to show this).]

The following is an extension question which you may do as an alternative the short project.

3. Continue the analysis of the Airy function $\text{Ai}(x)$. Recall that the integral representation was

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{C'} e^{i(t^3/3+xt)} dt,$$

   where the contour C' starts at infinity with $2\pi/3 < \arg(t) < \pi$ and ends at infinity with $0 < \arg(t) < \pi/3$.

   (a) Use the method of steepest descents to find $\text{Ai}(x)$ to leading-order as $x \to \infty$.

   (b) Now investigate **Stokes phenomenon**, the idea that the behaviour of these integrals depends on the direction x approaches infinity. We have already seen this is the case along the real axis, and will now extend this to the complex plane. Consider the integral:

$$\text{Ai}(z) = \frac{1}{2\pi} \int_{C'} e^{i(t^3/3+zt)} dt, \quad \text{as } |z| \to \infty.$$

   where C' is as above and $z = e^{i\theta} x$, that is $\arg(z) = \theta$.

i. Determine the location of the two saddle points, which will now vary with θ.

ii. Find expressions for the steepest descent paths through each saddle (these will now also depend on θ).

iii. Write a MATLAB code to plot the saddles and steepest descent paths for any value of θ.

iv. With reference to the original contour (thinking about how it can be deformed) and the above analysis, discuss why and for what value of θ there is a qualitative change in leading-order behaviour.

(c) The following paper (available on MyUni) extends the above analysis:

Berry, M.V., **Asymptotics, superasymptotics, hyperasymptotics**, *in Asymptotics Beyond All Orders, 1991.*

Briefly summarise the contents of this paper, and discuss how it relates to part (b). You may be particularly interested in Figure 6, which will (hopefully) look familiar.