Lecture 28: Renewal Theory I

Concepts checklist

At the end of this lecture, you should be able to:

- *State, prove and use* the theorem for the renewal equation satisfied by the renewal function;
- *State and apply* the one-to-one correspondence between the renewal function and the inter-arrival distribution; and,
- *State, prove and use* the theorem for the generalised renewal equation.

Theorem 26. *The renewal function $M(t)$ satisfies the renewal equation*

$$M(t) = F(t) + \int_0^t M(t-x)\mathrm{d}F(x). \tag{32}$$

Proof. Let X_1 be the random variable representing the time of the first renewal. We condition on X_1, and count the expected number of renewals thereafter. We have

$$\mathbb{E}[N(t)\big|X_1 = x] = \begin{cases} 0 & \text{if } \; x > t, \\ 1 + M(t-x) & \text{if } \; x \le t. \end{cases}$$

In the second case of the equation (above), we have the first renewal + the expected number since the first renewal, since the probabilistic structure begins anew at the instant of the first renewal.

Then,

$$\begin{aligned} M(t) &= \mathbb{E}[N(t)] \\ &= \int_0^\infty \mathbb{E}\left[N(t)\big|X_1 = x\right] \mathrm{d}F(x) \\ &= \int_0^t [1 + M(t-x)]\mathrm{d}F(x) \\ &= F(t) + \int_0^t M(t-x)\mathrm{d}F(x). \end{aligned}$$

□

Note: Much of the power of renewal theory derives from the method of reasoning used in the previous proof: conditioning on the time of the first renewal.

Theorem 27. *The only solution to the renewal equation (32), which is bounded on finite intervals, is given by*

$$M(t) = \sum_{n=1}^{\infty} F_n(t). \tag{33}$$

Proof. Taking the Laplace-Stieltjes transform of equation (32), we have

$$\begin{aligned} \widehat{M}(s) &= \widehat{F}(s) + \widehat{M}(s)\widehat{F}(s) \\ \Rightarrow \quad \widehat{M}(s) &= \frac{\widehat{F}(s)}{1-\widehat{F}(s)}. \end{aligned}$$

We have $\widehat{F}(s) < 1$, and so using the identity for the sum of a geometric series we have that

$$\begin{aligned} \widehat{M}(s) &= \sum_{n=1}^{\infty} \left[\widehat{F}(s)\right]^n \\ \Rightarrow M(t) &= \sum_{n=1}^{\infty} F_n(t), \end{aligned}$$

where $F_n(t)$ is the distribution function for the convolution of n random variables with distribution function $F(t)$.

□

Corollary 3.

$$\widehat{M}(s) = \frac{\widehat{F}(s)}{1-\widehat{F}(s)},$$

and consequently

$$\widehat{F}(s) = \frac{\widehat{M}(s)}{1+\widehat{M}(s)}.$$

Proof. This follows directly from the proof of the previous Theorem. □

Note: This corollary shows that there is a one-to-one correspondence between $M(t)$ and $F(t)$, so if we know one we can determine the other. Consequently, the Poisson process is the only renewal process having a linear mean-value function, $M(t) = \lambda t$.

Example 28.

Evaluate the renewal function corresponding to the lifetime distribution

$$F(t) = 1 - e^{-2t}(1+2t).$$

We have

$$\begin{aligned} \widehat{F}(s) &= \int_0^{\infty} e^{-st} 4te^{-2t} dt \\ &= \frac{4}{(s+2)^2}. \end{aligned}$$

Hence, we have

$$\widehat{M}(s) = \frac{4}{s(s+4)},$$

and (using the table of Laplace Transforms*)

$$M'(t) = 1 - e^{-4t},$$

and thus the renewal function is

$$M(t) = \frac{1}{4}(e^{-4t} - 1) + t.$$

**Note*: But you can see this via partial fractions:

$$\frac{4}{s(s+4)} = \frac{1}{s} - \frac{1}{(s+4)},$$

and so $M'(t) = 1 - e^{-4t}$.

Generalised Renewal Equation

When we consider renewal processes where the first lifetime is different from the rest, the generalised renewal equation for $H(t)$ (the expected number of events by time t) arises:

$$H(t) = G(t) + \int_0^t H(t-y)\mathrm{d}F(y), \tag{34}$$

where $G(t)$ is the distribution of the first lifetime, and $F(t)$ is the distribution of each of the subsequent lifetimes. In the renewal equation (32), $F(t)$ and $G(t)$ are the same function.

Theorem 28. *The solution to the generalised renewal equation is*

$$H(t) = G(t) + \int_0^t G(t-y)\mathrm{d}M(y),$$

where $M(t)$ is the solution to equation (32).

Proof.

$$\begin{aligned}
H(t) &= G(t) + \int_0^t G(t-y)\mathrm{d}M(y) \\
\Rightarrow \widehat{H}(s) &= \widehat{G}(s) + \widehat{G}(s)\widehat{M}(s) \\
&= \widehat{G}(s) + \widehat{G}(s)\,\frac{\widehat{F}(s)}{1-\widehat{F}(s)} \qquad \text{by Corollary 3} \\
&= \widehat{G}(s)\left[1 + \frac{\widehat{F}(s)}{1-\widehat{F}(s)}\right] \\
&= \frac{\widehat{G}(s)}{1-\widehat{F}(s)}.
\end{aligned}$$

Thus,

$$\widehat{H}(s) = \widehat{G}(s) + \widehat{H}(s)\widehat{F}(s) \quad \Rightarrow H(t) = G(t) + \int_0^t H(t-y)\mathrm{d}F(y).$$

□