Lecture 29: Renewal Theory II – Renewal Theorems

Concepts checklist

At the end of this lecture, you should be able to:

- *State, prove, and use* a Corollary for the expected time from the start of the process until the first event after time t; and,
- *state and use* a Theorem regarding the almost sure convergence of the long-term average number of events per unit time.

Note, in the Proof of Theorem (28), that no assumptions were made about the functions involved except the existence of their Laplace-Stieltjes Transforms.

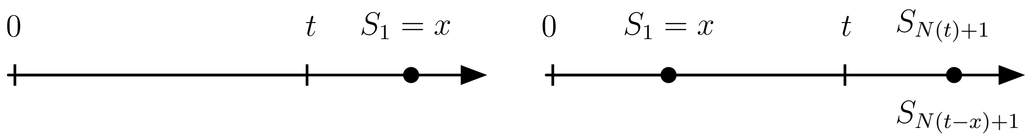
Corollary 4. *For any renewal process, the expected time from the start of the process until the first event after time t,*

$$\mathbb{E}[S_{N(t)+1}] = \mu(M(t)+1), \quad \textit{where } \mu \textit{ is the mean inter-event time.}$$

Proof. Let $H(t) = \mathbb{E}[S_{N(t)+1}]$, then

$$\mathbb{E}(S_{N(t)+1} \mid X_1 = x) = \begin{cases} x, & \text{if } \ t < x, \\ x + \mathbb{E}[S_{N(t-x)+1}], & \text{if } \ t \geq x. \end{cases}$$

The two cases correspond to the left and the right figures below



For $t \geq x$, we have by definition that $H(t-x) = \mathbb{E}[S_{N(t-x)+1}]$ and so

$$\begin{aligned} H(t) &= \int_0^\infty H(t \mid X_1 = x)\mathrm{d}F(x) \\ &= \int_t^\infty x\mathrm{d}F(x) + \int_0^t (x + H(t-x))\,\mathrm{d}F(x) \\ &= \int_0^\infty x\mathrm{d}F(x) + \int_0^t H(t-x)\mathrm{d}F(x) \\ &= \mu + \int_0^t H(t-x)\mathrm{d}F(x) \end{aligned}$$

so that $H(t)$ satisfies the generalised renewal equation (34) with $G(t) = \mu$. By Theorem 28, the solution is given by

$$H(t) = \mu + \int_0^t \mu\mathrm{d}M(y) = \mu(1 + M(t)).$$

□

Example 29. Engaged Signals and Reattempts

The behaviour of telephone users upon receiving an engaged signal can be modelled roughly as follows:

- with probability $1-p$ the caller does not attempt to retry,
- with probability p, the caller waits a time X with distribution $F(x)$ and then tries again.
- Upon retrying, the caller gets the engaged signal with probability r; in which case the process is repeated,
- or gets through with probability $1-r$, and the process ends.

A telephone company might want to know the average number of re-attempts $R(t)$ that occur during the time interval $[0, t)$ after the first engaged signal at time 0. We have,

$$R(t \,|\text{reattempts after first engaged at time } x) = \begin{cases} 0 & \text{if } \; x > t \\ 1 + rR(t-x) + (1-r) \times 0 & \text{if } \; x \leq t \end{cases}$$

whereas $R(t \,|\text{no reattempt}) = 0$. Then removing the conditioning of a reattempt and the time of the first wait yields

$$\begin{aligned} R(t) &= \int_0^t [p\big(1 + rR(t-x)\big) + (1-p) \times 0] \mathrm{d}F(x) \\ &= pF(t) + pr \int_0^t R(t-x) \mathrm{d}F(x). \end{aligned}$$

Taking Laplace-Stieltjes transforms, we get

$$\widehat{R}(s) = p\widehat{F}(s) + pr\widehat{R}(s)\widehat{F}(s) \quad \Rightarrow \quad \widehat{R}(s) = \frac{p\widehat{F}(s)}{1 - pr\widehat{F}(s)}.$$

Inverting this transform (not always easy!) gives $R(t)$. Inversion is easy analytically if $F(t) = 1 - e^{-\alpha t}$ since $\widehat{F}(s) = \frac{\alpha}{\alpha + s}$ then

$$\begin{aligned} \widehat{R}(s) &= \frac{p\alpha}{\alpha + s - pr\alpha} \\ &= \frac{p\alpha}{s + \alpha(1-pr)} \\ &= \frac{p}{(1-pr)} \frac{\alpha(1-pr)}{s + \alpha(1-pr)} \\ \text{and therefore} \quad R(t) &= \frac{p}{(1-pr)} \left(1 - e^{-\alpha(1-pr)t}\right). \end{aligned}$$

Theorem 29. *Let $\{N(t) : t \geq 0\}$ be a counting process. Then,*

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \textit{with probability } 1,$$

where μ is the mean time between events.

There are many sorts of convergence of random variables (see Additional Materials on MyUni). In Theorem 29, the phrase "with probability 1" (or, equivalentally "almost surely") means that:

of all the possible sample paths of the process $N(t)$, the quantity $\dfrac{N(t)}{t}$ converges to $\dfrac{1}{\mu}$ as $t \to \infty$ for a set *of realisations* **and** *this set has probability* 1 *of happening.*

This doesn't necessarily mean that $N(t)/t \to 1/\mu$ for all realisations, but the set of realisations on which this does not happen, has probability 0.

Example 30. A counting process

Let the time between two consecutive events, $i-1$ and i, be $X_i = \begin{cases} 0 & \text{with probability } 1/2, \\ 2 & \text{with probability } 1/2. \end{cases}$

Then $\mathbb{E}[X_i] = 1$. If we consider the counting process $N(t)$, then according to Theorem 29,

$$\lim_{t\to\infty} \frac{N(t)}{t} = 1 \qquad \text{with probability 1.}$$

A possible realisation of the process, however, is that the lifetime $X_i = 2$ *every* time, in which case

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{2}.$$

The probability of this realisation is

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0,$$

which is, of course, the probability of any particular realisation.