APP MTH 3001 Applied Probability III
Class Exercise 4, 2018

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May 20, 2018

1. In a discrete time population branching process,

   the probability that an individual has $j = 0, 1, 2, 3$ offspring is given by $p_{1,j} = \dfrac{1}{4}$.

   (a) Find the probability of ultimate extinction of the line of descent from an individual.
   Solution Extinction probability for a single individual is $U_1^{(0)}$:

   $$U_1^{(0)} = \frac{1}{4} + \frac{1}{4}U_1^{(0)} + \frac{1}{4}\left(U_1^{(0)}\right)^2 + \frac{1}{4}\left(U_1^{(0)}\right)^3$$

   Consider this as a cubic:

   $$y(x) = \frac{1}{4} + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{4}x^3$$

   Want to solve

   $$\begin{aligned} x &= \frac{1}{4} + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{4}x^3 \\ \implies 0 &= x^3 + x^2 - 3x + 1 \\ 0 &= (x-1)(x^2 + 2x - 1)) \end{aligned}$$

   I.e. the solutions are $x = 1$ and

   $$x = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

   So the minimal, non-negative solution is $x = -1 + \sqrt{2} \approx 0.414$.
   I.e. the probability of extinction from a line of descent it $U_1^{(0)} = -1 + \sqrt{2}$.
   As required.

   (b) Hence deduce whether the mean number of offspring per individual μ is greater than 1 or not, fully justifying your answer. Verify your conclusion by finding μ.
   Solution From lectures, the mean number of offspring per individual, μ is:

   1. $\mu > 1 \implies U_1^{(0)} < 1$
   2. $\mu = 1 \implies U_1^{(0)} = 1$
   3. $\mu < 1 \implies U_1^{(0)} = 1$

   I.e. in this case we would expect $\mu > 1$, as $U_1^{(0)} < 1$, which appears sensible, given the problem.
   Checking:

   $$\frac{dy}{dx} = \frac{1}{4} + \frac{1}{2}x + \frac{3}{4}x^2 = \mu$$

Set $x = 1$ which gives:

$$\mu = \frac{1}{4} + \frac{1}{2} + \frac{3}{4} = \frac{3}{2} = 1.5 > 1$$

As required.

2. A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$\mathbb{P}_a = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

(a) Is this Markov chain irreducible? Is the Markov chain recurrent or transient? Explain your answers.

Solution

This Markov chain is irreducible and recurrent.

This is because the three states all belong to the same communicating class, and the class is recurrent (it is impossible to leave the class). Since recurrence is a solidarity property, all states in the class are recurrent.

As required.

(b) What is the period of state 1? Hence deduce the period of the remaining states. Does this Markov chain have a limiting distribution?

Solution

The entire Markov chain is aperiodic, i.e. it has period $d(i) = 1 \ \forall i$. Each state can be accessed in one step from any other state, and each state is accessible from itself in one step. Since periodicity is a solidarity principle, all elements in this Markov chain are aperiodic (since they are all in the same communicating class).

It does have a limiting distribution. (Using `MATLAB`) it can be quickly verified that

$$P^{10} \approx P^{11} \approx P^{12}, \ldots$$

Which would imply that they are approaching a limiting distribution.

The limiting distribution is (to 4 decimal places):

$$\lim_{n\to\infty} \mathbb{P}_\alpha^n = \begin{pmatrix} 0.4697 & 0.2424 & 0.2879 \\ 0.4697 & 0.2424 & 0.2879 \\ 0.4697 & 0.2424 & 0.2879 \end{pmatrix}$$

As required.

(c) Consider a general three-state Markov chain with transition matrix

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Give an example of a specific set of probabilities $p_{i,j}$ for which the Markov chain is *not* irreducible (there is no single right answer to this, of course !).

Solution

$$\mathbb{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This Markov chain is **not irreducible**, as state 3 is disjoint from the other states. **As required.**

3. Consider a general irreducible Markov chain on the finite state space $\{1, 2, \ldots, N\}$. Show that one of the Global Balance equations

$$\pi_i = \sum_{j=1}^{N} \pi_j p_{j,i}, \quad i = 1, 2, \ldots, N,$$

is always redundant.

Solution We know this is true because there will be $N+1$ equations with N unknowns, and trivially they must be linearly dependent, giving a redundant equation.
Want to show that you can write one of the equations in terms of all the others.
We know that $\sum_{i=1}^{N} p_{i,j} = 1$.
Sum over all the π_i, since the sums aren't divergent, we can swap the order of summation

$$\begin{aligned}\sum_{i=1}^{N} \pi_i &= \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_j p_{j,i} \\ &= \sum_{j=1}^{N} \pi_j \sum_{i=1}^{N} p_{j,i} \\ \sum_{i=1}^{N} \pi_i &= \sum_{j=1}^{N} \pi_j \sum_{i=1}^{N} p_{j,i}\end{aligned}$$

As required.

4. Consider the random walk on the state space $\{0, 1, 2, \ldots\}$, with transition probabilities for $i = 1, 2, \ldots$, given by

$$p_{i,j} = \begin{cases} p & \text{if } \; j = i+1 \\ q & \text{if } \; j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_{0,1} = 1.$$

As usual, $p + q = 1$, and we assume that $p, q > 0$.

(a) What is the period of this Markov chain?

Solution
$d(i) = 2$; it takes any multiple of two steps to get from any i to i again
As required.

(b) For the case $p < q$, use the Global Balance Equations to show that the stationary distribution for this Markov chain is given by

$$\begin{aligned}\pi_i &= \frac{1}{2p}\left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^i, \quad i \geq 1, \\ \pi_0 &= \frac{1}{2}\left(1 - \frac{p}{q}\right).\end{aligned}$$

Hint: the state space is not finite. One way of approaching this situation is to solve the system of difference equations by trying a solution of the form $\pi_i = m^i$. For the general form of the solution, you will have two constant coefficients that need to be determined. In order to determine one of the coefficients, use the fact that $\sum_i \pi_i < \infty$, then use the normalization constraint $\sum_i \pi_i = 1$ to determine the other coefficient.

Solution The global balance equations are given by:

$$\begin{aligned}
\pi_j &= \sum_{i \in \mathcal{S}} \pi_i p_{i,j} \\
\implies \pi_j &= \pi_{j-1} p_{j-1,j} + \pi_{j+1} p_{j+1,j} \\
\implies \pi_j &= \pi_{j-1} p + \pi_{j+1} q
\end{aligned}$$

And subject to the condition

$$\sum_{j=0}^{\infty} \pi_j = 1$$

Try guessing $\pi_j = A + Bm^j$:

$$\sum_i \pi_i < \infty$$

$$\sum_i A + Bm^i = \infty A + B \sum_{i=1}^{N}$$

As required.

(c) For the case $p < q$, use the Partial Balance equations on the set $\mathcal{B} = \{0, 1, 2, \ldots, n\}$, for all n, to again find the stationary distribution, $\boldsymbol{\pi}$.

Solution This is skip-free and irreducible, so the partial balance are simple!

$$\pi_{j-1} p_{j-1,j} = \pi_j p_{j,j-1}$$

$$\pi_{j-1} p = \pi_j q \quad \forall j > 0$$

And we also have $\pi_0 = q\pi_1$.
Subject to $\sum_j \pi_j = 1$.

$$\begin{aligned}
\pi_{j-1} p &= \pi_j q \\
\pi_j &= \pi_{j-1} \frac{p}{q} \\
&= \pi_{j-2} \left(\frac{p}{q}\right)^2 \\
&= \pi_1 \left(\frac{p}{q}\right)^{j-1}
\end{aligned}$$

Use the summation constraint:

$$\begin{aligned}
\sum_{j=0}^{N} \pi_j &= \pi_0 + \sum_{j=1}^{N} \pi_1 \left(\frac{p}{q}\right)^{j-1} \\
1 &= \pi_1 \left(q + 1 + \sum_{j=2}^{N} \left(\frac{p}{q}\right)^{j-1} \right) \\
\implies \pi_1 &= \frac{1}{q + 1 + \sum_{j=2}^{N} \left(\frac{p}{q}\right)^{j-1}}
\end{aligned}$$

Since $p < q$ we can use geometric series:

$$\pi_1 = \frac{1}{\frac{1}{1-\frac{p}{q}} + q}$$

As required.

(d) For the case $p \geq q$, use the Partial Balance equations on the set $\mathcal{B} = \{0, 1, 2, \ldots, n\}$, for all n, to show that the stationary distribution does not exist for this Markov chain.
Solution Since $p \geq q$, we have that the system is positive recurrent. Which then means that $\pi_i > 0$. Again, the partial balance equations are:

$$\pi_0 = q\pi_1$$

And

$$\begin{aligned}
\pi_{j-1} p &= \pi_j q \\
\pi_j &= \pi_{j-1} \frac{p}{q} \\
&= \pi_{j-2} \left(\frac{p}{q}\right)^2 \\
&= \pi_1 \left(\frac{p}{q}\right)^{j-1}
\end{aligned}$$

Since $p \geq q$ we have $\left(\frac{p}{q}\right)^j \geq 1$, and since $\pi_i > 0$, it is clear that $\left(\frac{p}{q}\right)^j \pi_0 \geq 0$.
Also, since $\left(\frac{p}{q}\right)^j \leq \left(\frac{p}{q}\right)^{j+1}$ We can then use the summation constraint:

$$\begin{aligned}
\sum_{j=0}^{N} \pi_j &= 1 \\
\sum_{j=0}^{N} \pi_j &= \pi_0 + \sum_{j=1}^{N} \pi_1 \left(\frac{p}{q}\right)^{j-1} \\
&= \pi_1 \left(q + \sum_{j=1}^{N} \left(\frac{p}{q}\right)^{j-1} \right) \\
&\geq \pi_1 \sum_{j=1}^{N} 1 \\
&= \pi_1 N
\end{aligned}$$

Since this is an infinite state space, take the limit as $N \to \infty$

$$\begin{aligned}\lim_{N\to\infty} \sum_{j=0}^{N} \pi_j &\geq \lim_{N\to\infty} \pi_1 N \\ &\geq \pi_1 \times \infty \\ \implies \pi_1 &= 0\end{aligned}$$

Which is a contradiction, as the $\pi_j > 0$ for all j. I.e. the stationary dist doesn't exist. **As required.**