Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 9

Last lecture

- Saw that the Euler-Lagrange equation is independent of the coordinate system.
- Considered the fourth special case of degenerate functionals $f(x, y, y') = A(x, y)y' + B(x, y)$.
- We found that the extremals of these functions did not depend on the curve y except through the end-points.

Extensions

Now we consider extensions to the simple E-L equations presented so far:

- when f includes higher-order derivatives, e.g., $f(x, y, y', y'')$, e.g., the shape of a bent bar.
- when there are several dependent variables (i.e., $\boldsymbol{y}$ is a vector), e.g., calculating a particle trajectory in nD.
- when there are several independent variables (i.e., $\boldsymbol{x}$ is a vector), e.g. calculating extremal surface.

Standard Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} \;=\; 0}$$

Higher-order derivatives

Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', y'')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, y', and y'', and $x_0 < x_1$. As before, the necessary condition for the extremum is that the first variation be zero, e.g.

$$\delta F(\eta, y) = 0$$

Taylor's theorem

As before we perturb y to get $\hat{y} = y + \epsilon\eta$
Once again we apply Taylor's theorem to derive

$$f(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'') = \\ f(x, y, y', y'') + \epsilon\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''}\right) + \mathcal{O}(\epsilon^2)$$

and hence that

$$F\{y + \epsilon\eta\} = \int_{x_0}^{x_1}\left[f(x, y, y', y'') + \epsilon\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''}\right)\right]dx \\ + \mathcal{O}(\epsilon^2).$$

First Variation

So, now the first variation will be given by

$$
\begin{aligned}
\delta F &= \lim_{\epsilon\to 0} \frac{F\{y+\epsilon\eta\} - F\{y\}}{\epsilon} \\
&= \int_{x_0}^{x_1} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \eta'' \frac{\partial f}{\partial y''} \right) dx \\
&= \left[\eta \frac{\partial f}{\partial y'} + \eta' \frac{\partial f}{\partial y''} \right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(\eta \frac{\partial f}{\partial y} - \eta \frac{d}{dx} \frac{\partial f}{\partial y'} - \eta' \frac{d}{dx} \frac{\partial f}{\partial y''} \right) dx \\
&= \left[\eta \frac{\partial f}{\partial y'} + \eta' \frac{\partial f}{\partial y''} - \eta \frac{d}{dx} \frac{\partial f}{\partial y''} \right]_{x_0}^{x_1} \\
&\quad + \int_{x_0}^{x_1} \left(\eta \frac{\partial f}{\partial y} - \eta \frac{d}{dx} \frac{\partial f}{\partial y'} + \eta \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} \right) dx.
\end{aligned}
$$

New boundary conditions

We require new fixed-end point conditions

$$\begin{aligned} y(x_0) &= y_0 & \qquad y(x_1) &= y_1 \\ y'(x_0) &= y'_0 & \qquad y'(x_1) &= y'_1 \end{aligned}$$

which implies that

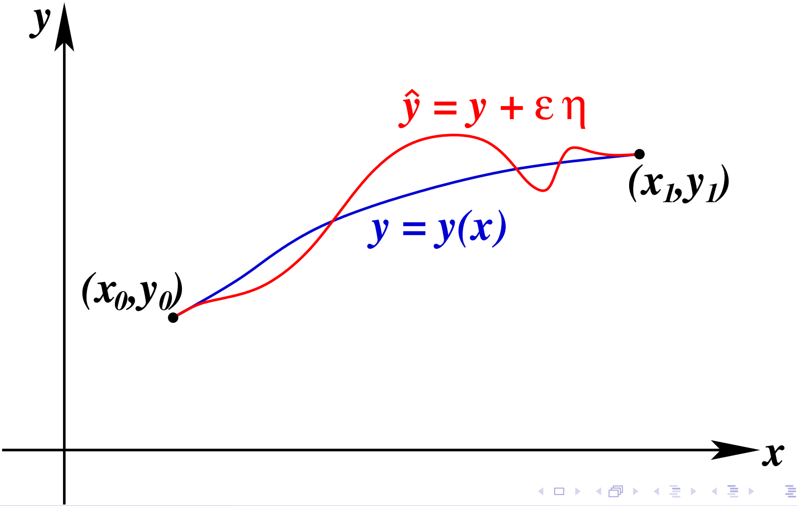
$$\begin{aligned} \eta(x_0) &= 0 & \qquad \eta(x_1) &= 0 \\ \eta'(x_0) &= 0 & \qquad \eta'(x_1) &= 0 \end{aligned}$$

Which gives

$$\delta F = \int_{x_0}^{x_1} \eta \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} \right) dx$$

Fixing the end-points

We now fix the derivative and value of y at the end points.



4th Order Euler-Lagrange equation

$\delta F(\eta, y) = 0$ for arbitrary η satisfying the boundary conditions, so the result is the 4th order Euler-Lagrange equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0}$$

This is a 4th order differential equation.

Generalisation

Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\, dx,$$

where f has continuous partial derivatives of second order with respect to $x, y, y', \ldots, y^{(n)}$, and $x_0 < x_1$, and the values of $y, y', \ldots, y^{(n-1)}$ are fixed at the end-points, then the extremals satisfy the condition

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} = 0$$

This is sometimes called the **Euler-Poisson Equation**.

Example 1

Find y that gives an extremal of

$$F\{y\} = \int_0^1 (1 + y''^2)\, dx$$

subject to $y(0) = 0$, $y(1) = 1$, $y'(0) = 1$, $y'(1) = 1$.

$$\begin{aligned}
\frac{\partial f}{\partial y} &= 0 \\
\frac{d}{dx}\frac{\partial f}{\partial y'} &= 0 \\
\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} &= \frac{d^2}{dx^2} 2y'' = 2\frac{d^4 y}{dx^4}
\end{aligned}$$

Example 1 solution

The E-P equation gives

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 2\frac{d^4 y}{dx^4} = 0$$

The solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

Given the end-points

$$\begin{array}{llll} y(0) = 0 \Rightarrow c_1 = 0. & & y(1) = 1 \Rightarrow 1 + c_3 + c_4 = 1. \\ y'(0) = 1 \Rightarrow c_2 = 1. & & y'(1) = 1 \Rightarrow 1 + 2c_3 + 3c_4 = 1. \end{array}$$

Final solution is $y(x) = x$. Is this a maximum or minimum?

Example 2

Consider

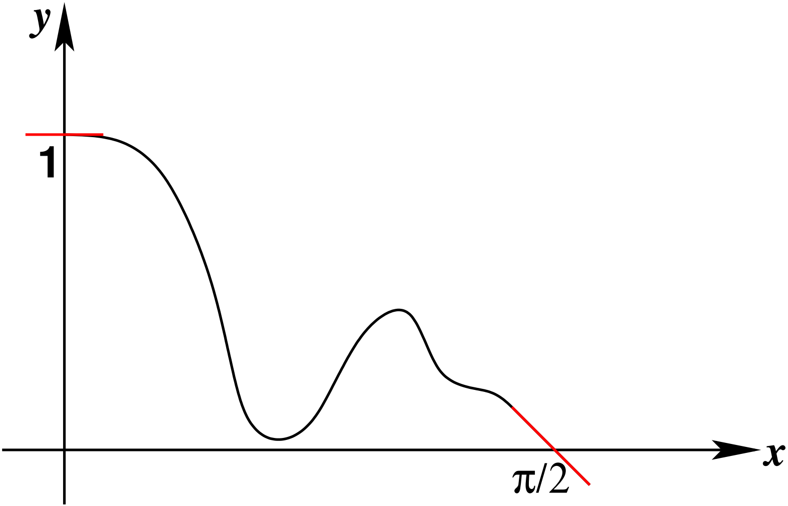
$$F\{y\} = \int_0^{\pi/2} \left(y''^2 - y^2 + x^2\right)\,dx$$

subject to $y(0) = 1$, $y(\pi/2) = 0$, $y'(0) = 0$, $y'(\pi/2) = -1$

$$\begin{aligned} \frac{\partial f}{\partial y} &= -2y, \\ \frac{d}{dx}\frac{\partial f}{\partial y'} &= 0, \\ \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} &= 2\frac{d^4 y}{dx^4}. \end{aligned}$$

Notice that the x^2 term doesn't influence the form of extremal!

Example 2



Example 2 solution

The E-P equation gives

$$\frac{\partial f}{\partial y} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = -2y + 2\frac{d^4 y}{dx^4} = 0$$

The solution is

$$y(x) = A\sinh x + B\cosh x + C\sin x + D\cos x$$

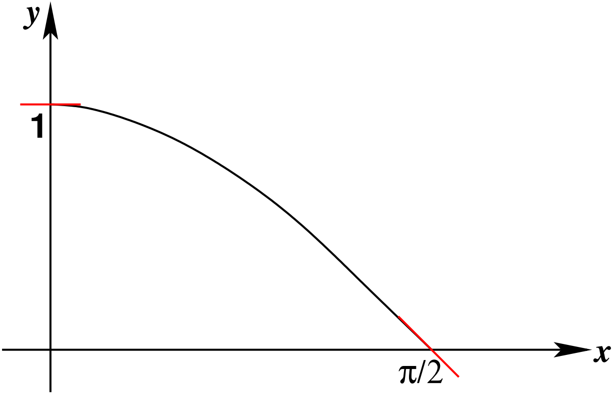
Given the end-points

$$\begin{aligned} y(0) = 1 \quad &\Rightarrow \quad B + D = 1 \\ y'(0) = 0 \quad &\Rightarrow \quad A + C = 0 \\ y\left(\frac{\pi}{2}\right) = 0 \quad &\Rightarrow \quad A\sinh\frac{\pi}{2} + B\cosh\frac{\pi}{2} + C = 0 \\ y'\left(\frac{\pi}{2}\right) = -1 \quad &\Rightarrow \quad A\cosh\frac{\pi}{2} + B\sinh\frac{\pi}{2} - D = -1 \end{aligned}$$

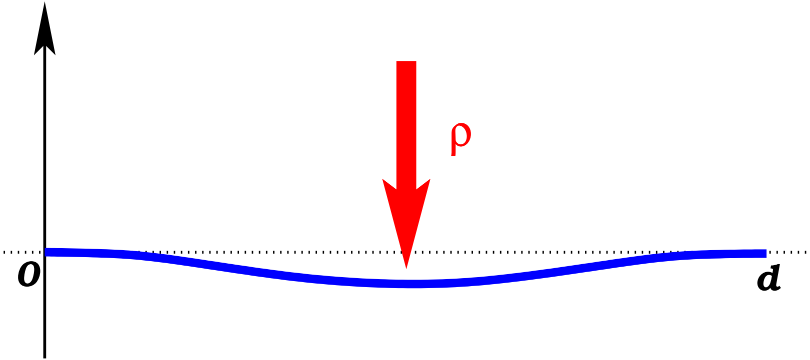
Example 2 solution

This is a linear system of four equations in four unknowns which may be solved using familiar methods. However it is also easy to verify that $A = B = C = 0$, $D = 1$ is the solution. So

$$y(x) = \cos x$$



Example 3 - Bent elastic beam



Two end-points are fixed, and clamped so that they are level, e.g. $y(0) = 0$, $y'(0) = 0$, and $y(d) = 0$ and $y'(d) = 0$.
The load (per unit length) on the beam is given by a function $\rho(x)$.

Example 3

Let $y : [0, d] \to \mathbb{R}$ describe the shape of the beam, and $\rho : [0, d] \to \mathbb{R}$ be the load per unit length on the beam.
For a bent elastic beam the potential energy from elastic forces is

$$V_1 = \frac{\kappa}{2} \int_0^d y''^2 \, dx, \qquad \kappa = \text{flexural rigidity}$$

The potential energy is

$$V_2 = -\int_0^d \rho(x) y(x) \, dx$$

Thus the total potential energy is

$$V = \int_0^d \frac{\kappa y''^2}{2} - \rho(x) y(x) \, dx$$

Example 3

The Euler-Poisson equation is

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} &= 0 \\ -\rho(x) + \kappa y^{(4)} &= 0 \\ y^{(4)} &= \frac{\rho(x)}{\kappa} \end{aligned}$$

This DE has solution

$$y(x) = P(x) + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

where the c_k's are the constants of integration, and $P(x)$ is a particular solution to $P^{(4)}(x) = \rho(x)/\kappa$.

Example 3 - uniform load

If the beam is uniformly loaded, then $\rho(x) = \rho$ and so

$$y(x) = \frac{\rho x^4}{4!\kappa} + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

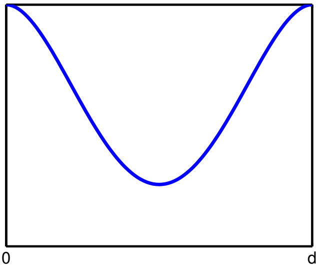
The end-conditions imply

$$\begin{aligned}
y(0) = 0 \quad &\Rightarrow \quad c_0 = 0 \\
y'(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\
y(d) = 0 \quad &\Rightarrow \quad \frac{\rho d^4}{4!\kappa} + c_2 d^2 + c_3 d^3 = 0 \\
y'(d) = 0 \quad &\Rightarrow \quad \frac{\rho d^3}{3!\kappa} + 2c_2 d + 3c_3 d^2 = 0
\end{aligned}$$

Example 3 - uniform load

Choose a solution of the form

$$y(x) = \frac{\rho(d-x)^2x^2}{24\kappa}$$



Then the derivative

$$\begin{aligned} y'(x) &= \frac{\rho(d-x)^2x}{12\kappa} - \frac{\rho(d-x)x^2}{12\kappa} \\ &= \frac{\rho x(d-x)(d-2x)}{12\kappa} \end{aligned}$$

We can see that the constraints are satisfied

$$\begin{array}{rclcrcl} y(0) &=& 0, & & y(d) &=& 0, \\ y'(0) &=& 0, & & y'(d) &=& 0. \end{array}$$

Example 3 - uniform load

$$\tilde{y}(x) = -\frac{\rho(d-x)^2 x^2}{24\kappa}$$

Maximum displacement occurs at $x = d/2$, and is given by

$$\tilde{y}(d/2) = -\frac{\rho d^4}{384\kappa}$$

Contrast this with the catenary.

$$\tilde{y}(x) = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$$

where c_1 and c_2 are determined by the end-points (there are no physical values such as m or g in the solution).