APP MTH 3020 Stochastic Decision Theory

Tutorial 2

Week 5, Friday, August 24

1. Suppose the second-stage constraints of a two-stage problem are given by

$$W\boldsymbol{y} = \begin{pmatrix} 1 & 3 & -1 & 0 \\ 2 & -1 & 2 & 1 \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} -6 \\ -4 \end{pmatrix} \boldsymbol{\zeta} + \begin{pmatrix} 5 & -1 & 0 \\ 0 & 2 & 4 \end{pmatrix} \boldsymbol{x}$$

where $\boldsymbol{\zeta}$ is a random variable with support $\mathcal{S}_{\zeta} = \{0, 1\}$.

Assume that in the second-stage program we minimise $\boldsymbol{q}^{\top}\boldsymbol{y}$. Write down the LP(s) (both primal and dual formulations) needed to check if a given $\boldsymbol{x}$ produces a feasible second-stage program.

Note that in this case $\boldsymbol{\zeta}$ has two realisations, 0 and 1. Using the definition of the second-stage program for a given solution $\boldsymbol{x}$ to the first stage problem, we require a feasible solution to the following two primal LPs, one for each realisation of $\boldsymbol{\zeta}$:

$$\begin{aligned} \text{(P1)} \quad \min \quad & z = \boldsymbol{q}^{\top}\boldsymbol{y} \\ \text{such that} \quad & \begin{pmatrix} 1 & 3 & -1 & 0 \\ 2 & -1 & 2 & 1 \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} 5 & -1 & 0 \\ 0 & 2 & 4 \end{pmatrix} \boldsymbol{x}, \\ & \boldsymbol{y} \geq \boldsymbol{0}, \end{aligned}$$

and

$$\begin{aligned} \text{(P2)} \quad \min \quad & z = \boldsymbol{q}^{\top}\boldsymbol{y} \\ \text{such that} \quad & \begin{pmatrix} 1 & 3 & -1 & 0 \\ 2 & -1 & 2 & 1 \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} -6 \\ -4 \end{pmatrix} + \begin{pmatrix} 5 & -1 & 0 \\ 0 & 2 & 4 \end{pmatrix} \boldsymbol{x}, \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Equivalently, in the dual formulation, we require bounded solutions to the following linear programs:

$$\begin{aligned} \text{(D1)} \quad \max \quad & z = (5x_1 - x_2)u_1 + (2x_2 + 4x_3)u_2 \\ \text{s.t.} \quad & u_1 + 2u_2 \leq q_1 \\ & 3u_1 - u_2 \leq q_2 \\ & -u_1 + 2u_2 \leq q_3 \\ & u_2 \leq q_4, \quad u_1 \text{ free}, \end{aligned}$$

and

$$\begin{aligned} \text{(D2)} \quad \max \quad & z = (-6 + 5x_1 - x_2)u_1 + (-4 + 2x_2 + 4x_3)u_2 \\ \text{s.t.} \quad & u_1 + 2u_2 \leq q_1 \\ & 3u_1 - u_2 \leq q_2 \\ & -u_1 + 2u_2 \leq q_3 \\ & u_2 \leq q_4, \quad u_1 \text{ free}. \end{aligned}$$

2. (a) Is W in **Question 1** a complete recourse matrix? Justify.

As (for example) the first two columns are linearly independent, $\text{rank}(W) = 2$. In order to determine solutions to $W\boldsymbol{y} = \mathbf{0}$, there are two options.

Option 1. We can solve the system of equations:

$$\begin{aligned} y_1 + 3y_2 - y_3 &= 0 \\ 2y_1 - y_2 + 2y_3 + y_4 &= 0. \end{aligned}$$

As there are two equations and four unknowns, we can let y_3 and y_4 to be free variables.

Then, $y_1 = y_3 - 3y_2$, and substituting this into the second equation gives

$$2(y_3 - 3y_2) - y_2 = -2y_3 - y_4,$$

which implies $7y_2 = 4y_3 + y_4$ and thus $y_2 = 4/7y_3 + 1/7y_4$. So the solution to $W\boldsymbol{y} = \mathbf{0}$ is

$$\begin{pmatrix} -5/7y_3 - 3/7y_4 \\ 4/7y_3 + 1/7y_4 \\ y_3 \\ y_4 \end{pmatrix},$$

where $y_3, y_4 \in \mathbb{R}$. Clearly, there exist no values of y_3 and y_4 such that $y_3, y_4 \geq 0$ and $-5/7y_3 - 3/7y_4 \geq 1$. So W is not a complete recourse matrix.

Option 2. We can find the null space of W by using command `null(W)` in MATLAB, which gives

ans =

−0.4768	−0.3453
0.4124	0.0900
0.7606	−0.0752
−0.1552	0.9311.

Non-trivial vectors $\boldsymbol{y}$ that are solutions to $W\boldsymbol{y} = \mathbf{0}$ are therefore linear combinations of the two null-space vectors above:

$$\boldsymbol{y} = \lambda \begin{pmatrix} -0.4768 \\ 0.4124 \\ 0.7606 \\ -0.1552 \end{pmatrix} + \mu \begin{pmatrix} -0.3453 \\ 0.0900 \\ -0.0752 \\ 0.9311 \end{pmatrix} \quad \text{for } \lambda, \mu \in \mathbb{R}.$$

Noting that the first 2 columns of W are linearly independent, we need $\boldsymbol{y} \geq \mathbf{0}$, such that $y_1, y_2 \geq 1$ and $y_3, y_4 \geq 0$, for W to be a complete recourse matrix, which is not possible.

(b) Is the following matrix W a complete recourse matrix? Justify.

$$W = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

MATLAB (or we can check ourselves!) reveals that $\text{rank}(W) = 3 = m_1$ (where m_1 is the number of rows of a matrix W), and we see that the first 3 columns of W are linearly independent. Then, for

$$\boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}^\top \geq \mathbf{0},$$

we see that $y_1, y_2, y_3 \geq 1$ and $y_4, y_5, y_6 \geq 0$. This, together with the fact that $W\boldsymbol{y} = \mathbf{0}$, by Theorem 3.2 we have that W is a complete recourse matrix.

(c) What is the implication of a complete recourse matrix?

Every first stage solution will have a feasible second-stage solution.

(d) Give a numerical example of a simple recourse matrix.

$$W = (I\ , -I) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

(e) What is the implication of a simple recourse matrix?

Every first-stage solution will have a feasible second-stage solution.

3. Consider a farmer who has a total of 500 acres of land available for growing *wheat*, *corn*, and *sugar beets*, with the planting costs per acre being \$150, \$230, and \$260, respectively.

The farmer needs at least 200 tons (T) of wheat and 240 T of corn for cattle feed which can be grown on the farm or bought from a wholesaler. The purchase prices per ton are \$238 for wheat and \$210 for corn. The amount produced in excess will be sold at prices of \$170 per ton for wheat and \$150 per ton for corn. For sugar beets there is a quota on production which is 6000 T for the farmer. Any amount of sugar beets up to the quota can be sold at \$36 per ton, the amount in excess of the quota is limited to \$10 per ton.

The farmer knows the average yield on his land is 2.5 T, 3.0 T and 20.0 T per acre for wheat, corn and sugar beets. The data are shown the following table.

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3.0	20.0
Planting cost (\$/acre)	150	230	260
Purchase price (\$/T)	238	210	–
Selling price (\$/T)	170	150	36 (under 6000 T) 10 (above 6000 T)
Minimum requirement (T)			–
Total available land : 500 acres			

(a) Formulate, and solve, an LP to find a solution to the farmer's problem.

Denote by

- x_1, x_2 and x_3 the amount of acres of land for wheat, corn and sugar beets, respectively,
- y_1 and y_2 the amount in tons of wheat and corn purchased, respectively,
- y_3 and y_4 the amount in tons of wheat and corn sold, respectively,
- y_5 and y_6 the amount of sugar beets sold at the favourable price and the reduced price, respectively.

From the problem statement,

- the planting cost is $150x_1 + 230x_2 + 260x_3$,
- the cost for buying additional wheat and corn is $238y_1 + 210y_2$,
- the profit from selling excess wheat and corn is $170y_3 + 150y_4$, and
- the profit from selling sugar beats is $36y_5 + 10y_6$.

The LP for the above problem is:

$$
\begin{aligned}
\text{minimise} \quad & z = 150x_1 + 230x_2 + 260x_3 + 238y_1 + 210y_2 \\
& \quad - 170y_3 - 150y_4 - 36y_5 - 10y_6 \\
\text{such that} \quad & x_1 + x_2 + x_3 = 500 \quad \text{– available land} \\
& 2.5x_1 + y_1 - y_3 = 200 \quad \text{– wheat demand} \\
& 3x_2 + y_2 - y_4 = 240 \quad \text{– corn demand} \\
& y_5 + y_6 = 20x_3 \quad \text{– yield for sugar beets} \\
& y_5 \leq 6000 \quad \text{– quota for sugar beets} \\
& x_1, x_2, x_3, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0.
\end{aligned}
$$

Using `linprog`, we obtain the following solution listed in Table 1.

	Wheat	Corn	Sugar Beets
Planted acres	120	80	300
Yield (T/acre)	300	240	6,000
Purchases (T)	–	–	–
Sales (T)	100	–	6,000
Maximum profit: $118,600			

Table 1: Solution for average yields.

(b) The yield is actually sensitive to the weather and the previous problem and solution can be described as that based on average yields. The yields may vary by a margin of $\pm 20\%$. Find the optimal solution for both the below average and above average yield scenarios.

	Wheat	Corn	Sugar Beets
Planted acres	100	25	375
Yield (T)	200	60	6000
Purchases (T)	–	180	–
Sales (T)	–	–	6000
Maximum profit: $59,950			

Table 2: Solution for below average yields.

	Wheat	Corn	Sugar Beets
Planted acres	183.3333	66.6667	250
Yield (T)	550	240	6000
Purchases (T)	–	–	–
Sales (T)	350	–	6000
Maximum profit: $167,667			

Table 3: Solution for above average yields.

(c) What is the average profit of all three scenarios?

The average profit of all three scenarios is $115,406.

(d) The farmer now realises he cannot make a perfect decision that is best in all circumstances. Decisions on land assignment need to be made immediately, but sales and purchases that depend on the yields are taken later. To maximise his long run profit, he seeks a solution which maximises his expected profit.

Assuming each of the three previous scenarios occur w.p. 1/3 each, that is, the yields are under by 20%, average or over by 20%. Formulate the farmer's problem as a two-stage SLP with recourse. Write in expanded form.

$$\begin{aligned}
\text{minimise} \quad & z = 150x_1 + 230x_2 + 260x_3 \\
& \quad + \frac{1}{3}\left(238y_{1,1} + 210y_{2,1} - 170y_{3,1} - 150y_{4,1} - 36y_{5,1} - 10y_{6,1}\right) \\
& \quad\quad + \frac{1}{3}\left(238y_{1,2} + 210y_{2,2} - 170y_{3,2} - 150y_{4,2} - 36y_{5,2} - 10y_{6,2}\right) \\
& \quad\quad\quad + \frac{1}{3}\left(238y_{1,3} + 210y_{2,3} - 170y_{3,3} - 150y_{4,3} - 36y_{5,3} - 10y_{6,3}\right) \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \leq 500 \\
& 3.0x_1 + y_{1,1} - y_{3,1} \geq 200 \\
& 2.5x_1 + y_{1,2} - y_{3,2} \geq 200 \\
& 2.0x_1 + y_{1,3} - y_{3,3} \geq 200 \\
& 3.6x_2 + y_{2,1} - y_{4,1} \geq 240 \\
& 3.0x_2 + y_{2,2} - y_{4,2} \geq 240 \\
& 2.4x_2 + y_{2,3} - y_{4,3} \geq 240 \\
& y_{5,1} + y_{6,1} \leq 24x_3 \\
& y_{5,2} + y_{6,2} \leq 20x_3 \\
& y_{5,3} + y_{6,3} \leq 16x_3 \\
& y_{5,1} \leq 6000 \\
& y_{5,2} \leq 6000 \\
& y_{5,3} \leq 6000 \\
& \boldsymbol{x}, \boldsymbol{y} \geq \mathbf{0}.
\end{aligned}$$

(e) Solve this recourse DEP.

		Wheat	Corn	Sugar Beets
First Stage	Planted acres	170	80	250
Below	Yield (T)	340	192	4000
	Purchases (T)	–	48	–
	Sales (T)	140	–	4000
Average	Yield (T)	425	240	5000
	Purchases (T)	–	–	–
	Sales (T)	225	–	5000
Above	Yield (T)	510	288	6000
	Purchases (T)	–	–	–
	Sales (T)	310	48	6000
Maximum profit: $108,390				

Table 4: Solution for the stochastic linear program.