Random Processes III 2018: Assignment 4,

to be submitted by 1pm on Friday 5^{th} October.

[39 marks in total]

Question 0. [4 marks]

Make sure that in all your answers you:
(a) use full and complete sentences,
(b) include units where necessary,
(c) give ranges/constraints for parameters in formulae, and
(d) make sure your layout and mathematical arguments are clear and concise.

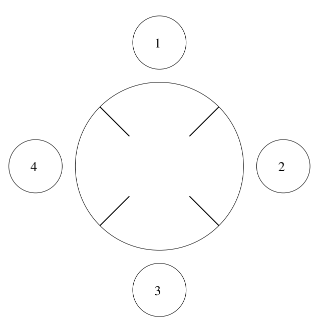
Question 1. [6 marks]

Consider a game of tennis when deuce is reached. If a player wins the next point, they are said to have advantage. On the following point, the player with advantage wins if they win the point, and the game returns to deuce if they lose the point. Each point tends to last for $1/\lambda_D$ minutes, on average, in deuce, and for $1/\lambda_A$ minutes, on average, if a player has advantage. Assume that the probability Player A wins any point is $0 < p_A < 1$. Let the state space be $S = \{1, 2, 3, 4, 5\}$, where 1 : deuce; 2 : advantage A; 3 : advantage B; 4 : A wins; 5 : B wins.

(a) Assume a game has reached deuce. Perform a first step analysis to get a system of equations which govern the expected time until the game has finished and either player has won.
(b) Calculate the expected duration of a game starting from deuce.

Question 2. [11 marks]

The *Dining Philosophers* problem is often used in computer science as a model for systems which share resources. It works as follows.



N philosophers sit around a table. In between each pair of philosophers is a single chopstick. The philosophers alternately think and eat. When they wish to eat they pick up the chopsticks either side of them if both chopsticks are free, but if one of them is being used, they cannot eat and return immediately to thinking mode.

Assume $N = 4$ and that each philosopher eats for a period which is exponentially distributed with parameter 1 and thinks for a period which is exponentially distributed with parameter γ.

(a) What is an appropriate state space for a continuous-time Markov chain model of this system, if I wish to know which philosophers are dining or thinking at any one time?

(b) What are the transition rates of this model?

(c) Write down the equilibrium equations for the model.

(d) Calculate the equilibrium distribution for the model.
Hint: Exploit Theorem 14.

(e) What is the equilibrium probability that the number of eating philosophers is zero, one, two, three or four?

Question 3. [18 marks]

Consider customers arrive to the office of the SA Department of Planning, Transport and Infrastructure (DPTI) according to a Poisson process with rate λ. The office can only contain N customers (which includes the customers currently being served). Any customer who arrives to a full office immediately leaves. Each customer has one enquiry that will be classified into only one of five categories. The probability that a customer has an enquiry related to category i is p_i, such that $\sum_{i=1}^{5} p_i = 1$. There is a distinct queue associated with each enquiry and customers immediately join the appropriate queue on arrival.

(a) Assuming that N is infinite, we begin by proving that the arrivals to each queue are independent Poisson processes via the following steps.

  (i) What is the probability that in the time interval $[0, t]$, there were n arrivals to the office and x of those arrivals joined queue i?

  (ii) Using the law of total probability, calculate the probability that in the time interval $[0, t]$, x arrivals joined queue i, $i \in \{1, 2, 3, 4, 5\}$. In other words, if $N_i(t)$ is the number of customers who arrive for queue i by time t, calculate the PMF, $p_i = \Pr(N_i(t) = x)$ for all $x \in \{0,\ 1,\ \dots\}$ and hence show that the arrivals to the ith queue follow a Poisson process with rate $p_i\lambda$.

(b) Continue to assume that N is infinite, and also assume that queue i has i servers available, each with an exponential service rate of μ.

  (i) Specify an appropriate CTMC for modelling the number of customers in all of the queues.

  (ii) Determine the stationary distribution for this CTMC.

(c) Now assumming that N is finite, determine the equilibrium distribution of the queues.