Modelling with ODEs Assignment 2

Andrew Martin

April 11, 2019

1. (a) $\dot{x} = r + x - \log(1+x)$ Bifurcation when $f(x:r) = r + x - \log(1+x) = 0$ and $f'(x:r) = 0$ the solutions (x, r) are the bifurcation point value pair.

$$\begin{aligned} f(x:r) = r + x - \log(1+x) &= 0 \\ r &= \log(1+x) - x \\ f'(x:r) = 1 - \frac{1}{1+x} &= 0 \\ \implies x &= 0 \\ f(0:r) &= 0 \end{aligned}$$

So the bifurcation value is $\bar{r} = 0$ at $(\bar{x}, \bar{r}) = (0, 0)$
For $r < 0$ there are 2 fixed points for x. The fixed points with $x > 0$ will produce

$$\dot{x} > 0$$

Which implies they will be unstable. The values where $x < 0$ will give

$$\dot{x} < 0$$

And thus they are stable. For $r = 0$ there is only 1 fixed point (the bifurcation), and for $r > 0$ there are no fixed points. This is all shown in figure 1a. Hence this is a saddle node bifurcation.

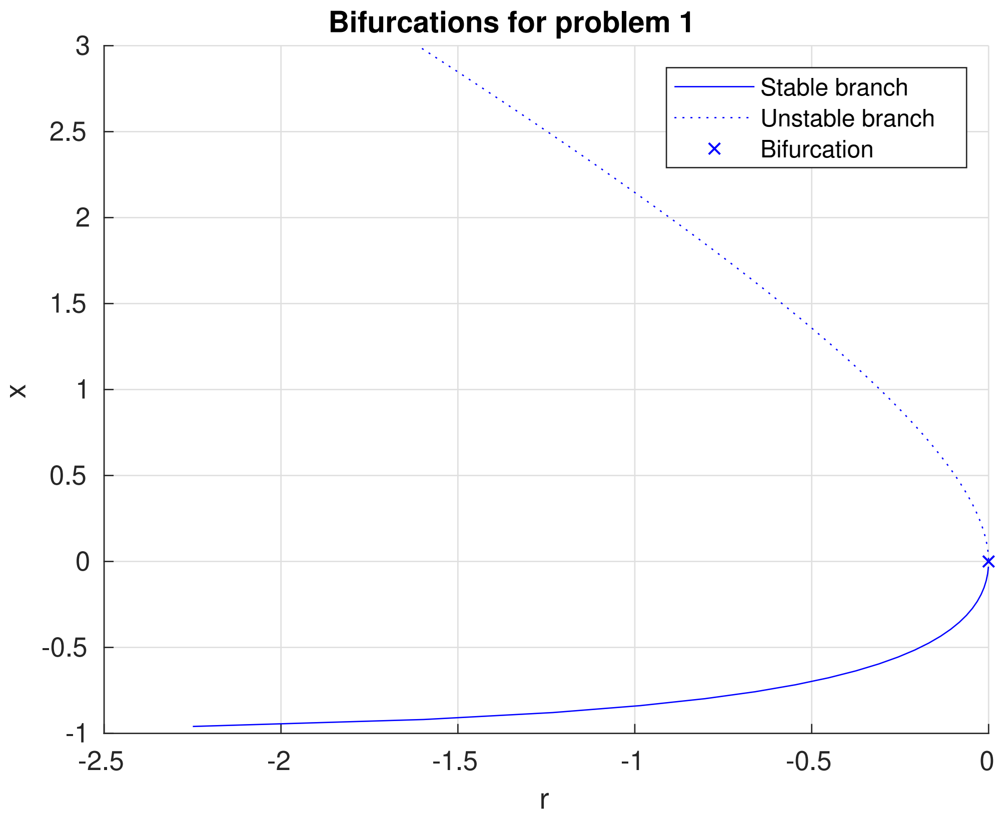

Figure 1: Bifurcation diagram for $\dot{x} = r + x - \log(1 + x)$

As shown in figure 1a, the stable solution disappears after the bifurcation, and the branch of fixed points is unstable.

(b) $\dot{x} = x - rx(1 - x)$

$$
\begin{aligned}
f(x : r) &= x - rx(1 - x) = 0 \\
x(1 - r(1 - x)) &= 0 \\
& x = 0 \ or \ r - rx - 1 = 0 \\
& x = 0 \ or \ x = 1 - \frac{1}{r} \\
f'(x : r) &= 1 - r - 2rx = 0 \\
& 1 - r = 0 \ or \ 1 - r - 2r(1 - \frac{1}{r}) = 0 \\
& r = 1 \ or \ 1 - 3r + 2 = 0 \\
& \implies r = 1
\end{aligned}
$$

Noting $f'(x : r) = 1 - r - 2rx$, we get $x = 0$, $r > 1$ is stable, $x = 0$, $r < 1$ is unstable. For $x = 1 - 1/r$, $r > 0$ it is unstable, and the same x for $r < 0$ is stable. So this suggests that this is a transcritical bifurcation.

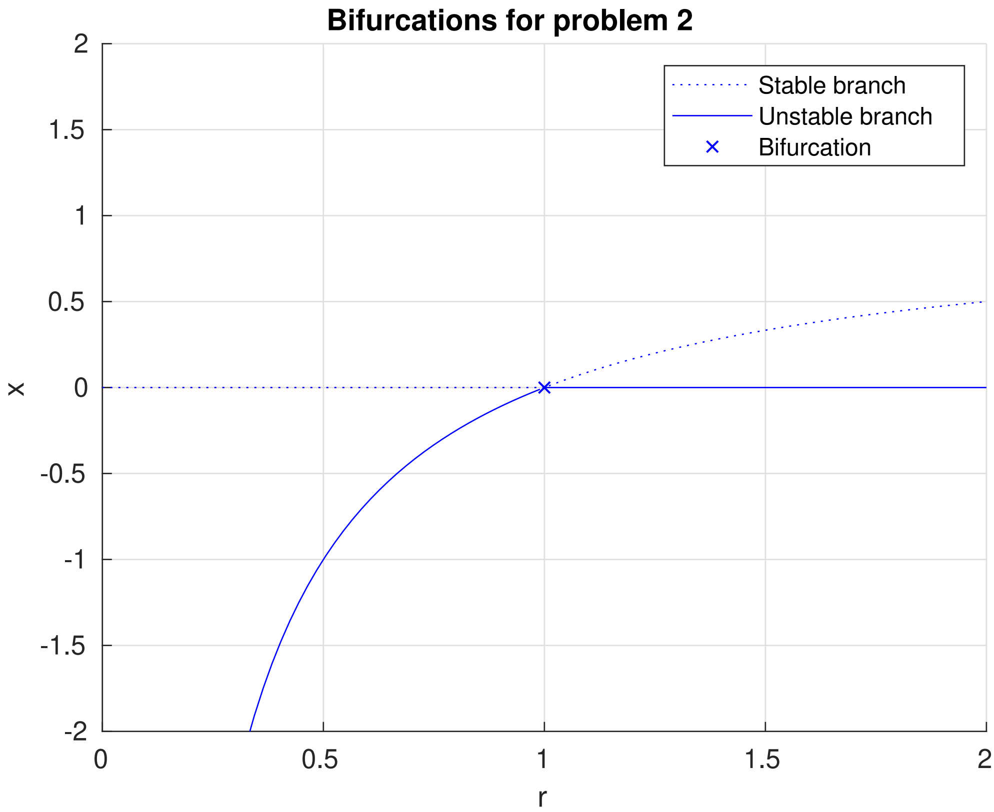

Figure 2: Bifurcation diagram for $\dot{x} = x - rx(1-x)$

Figure 1b is the bifurcation diagram for this problem. This is a form of a transcritical bifurcation, since on both sides of the bifurcation there is one stable and an unstable fixed point.

(c) $\dot{x} = rx - 4x^3$

$$\begin{aligned} \dot{x} &= 0 \\ x(r - 4x^2) &= 0 \\ x = 0 \quad x &= \pm\frac{1}{2}\sqrt{r} \end{aligned}$$

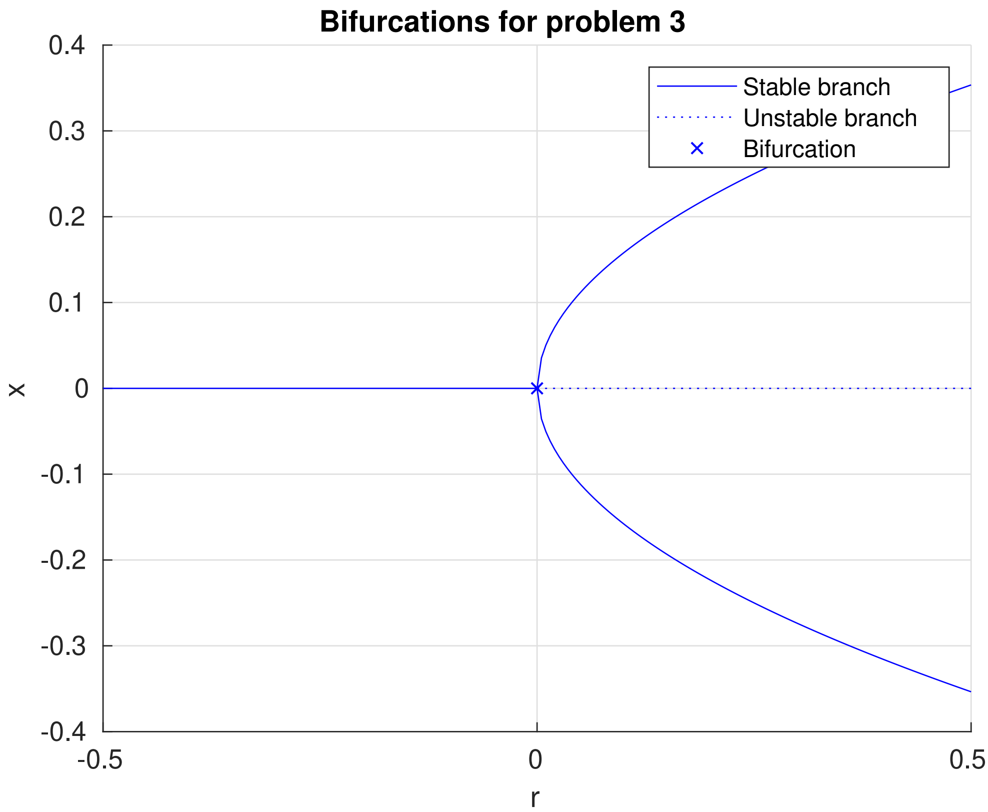

Figure 3: Bifurcation diagram for $\dot{x} = rx - 4x^3$

Figure 1c is the bifurcation diagram for this. Note that from observation the problem was very similar to the canonical form for the supercritical pitchfork bifurcation. Clearly this is a supercritical pitchfork bifurcation as after r is increased past the bifurcation point, the one stable branch splits into 2 stable branches and an unstable branch.

2. (a) Since $r = 0.4$, $x(0) = 0$ the ODE becomes

$$\frac{dx}{d\tau} = s - 0.4x + \frac{x^2}{1 + x^2}$$

i. There is a bifurcation point which is crossed when s is increased towards 0.2. Before this bifurcation point, the system will remain with no gene product, i.e. $x\tau = 0$. This is because $x = 0$ is a stable fixed point. After s is increased to the bifurcation value, the fixed point becomes semistable Since $\frac{dx}{d\tau}\Big|_{x=0} = s$. Expect a positive slope about $x = 0$. Note $x < 0$ cannot occur since when $x \to 0$, $s > x$ and hence $\frac{dx}{d\tau} > 0$. So x will always be greater than zero When $s = 0.2$
The bifurcation point, where the stability is lost:

$$\begin{aligned} 0 &= -0.4 + \frac{2x(1 + x^2) - 2x^3}{(1 + x^2)^2} \\ &= -0.4 + \frac{2x}{(1 + x^2)^2} \\ 2x &= 0.4(1 + 2x^2 + x^4) \end{aligned}$$

Which is not analytic. Solving for the first smallest positive solution numerically in `Matlab` yields $\bar{x} \approx 0.2198$ And hence $\bar{s} \approx 0.0418$. This is a semi-stable fixed point. Once $s > \bar{s}$, x is allowed to jump up to a much larger value I.e. $x \gg \bar{x}$.

Once s reaches 0.2, x will eventually reach its maximum at 2.6981 (obtained from `Matlab`). This is a stable fixed point (identifiable from figure 2(a)i).

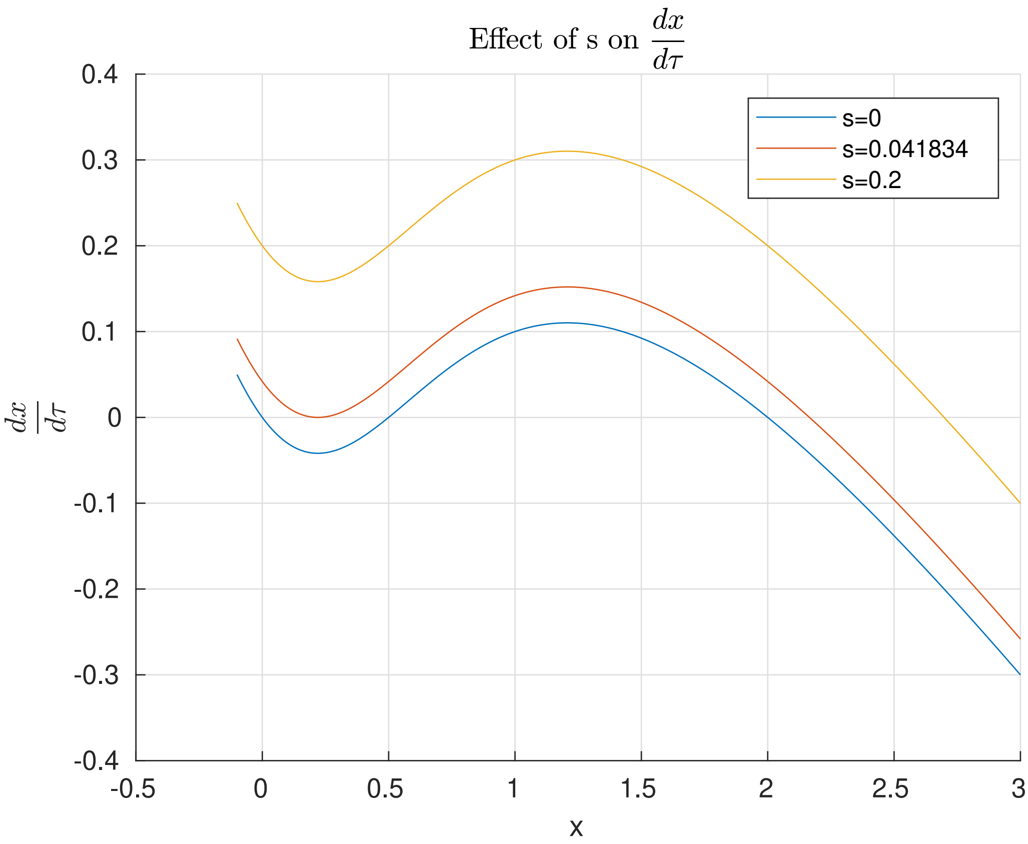

Figure 4: plot of $\frac{dx}{d\tau}$ against x, for $s = 0$ $\bar{s}$, and $s = 0.2$

ii. If s is now decreased back to 0, x will shift left due to the negative derivative. The derivative becomes zero for $s = 0$, and $\tilde{x} = 2$. This is a stable fixed point since the slope of $\frac{dx}{dt}$ is negative around this point.
The fact that increasing s and then decreasing it gives different steady states suggests there is hysteresis.

(b) Now we are varying both s and r in

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2}$$

Noting that $r > 0$ and $s \geq 0$.

i.

$$
\begin{aligned}
f(x:s,r) = s - rx + \frac{x^2}{1+x^2} &= 0 \\
f'(x:s,r) = -r + \frac{2x}{(x^2+1)^2} &= 0 \\
\implies r = \frac{2x}{(x^2+1)^2} \\
\implies s - \frac{2x^2}{(x^2+1)^2} + \frac{x^2}{1+x^2} &= 0 \\
\implies s = \frac{2x^2}{(x^2+1)^2} - \frac{x^2}{1+x^2} \\
\implies s = \frac{x^2(1-x^2)}{(x^2+1)^2}
\end{aligned}
$$

So the bifurcation curves are

$$
r = \frac{2x}{(x^2+1)^2}, \quad s = \frac{x^2(1-x^2)}{(x^2+1)^2}
$$

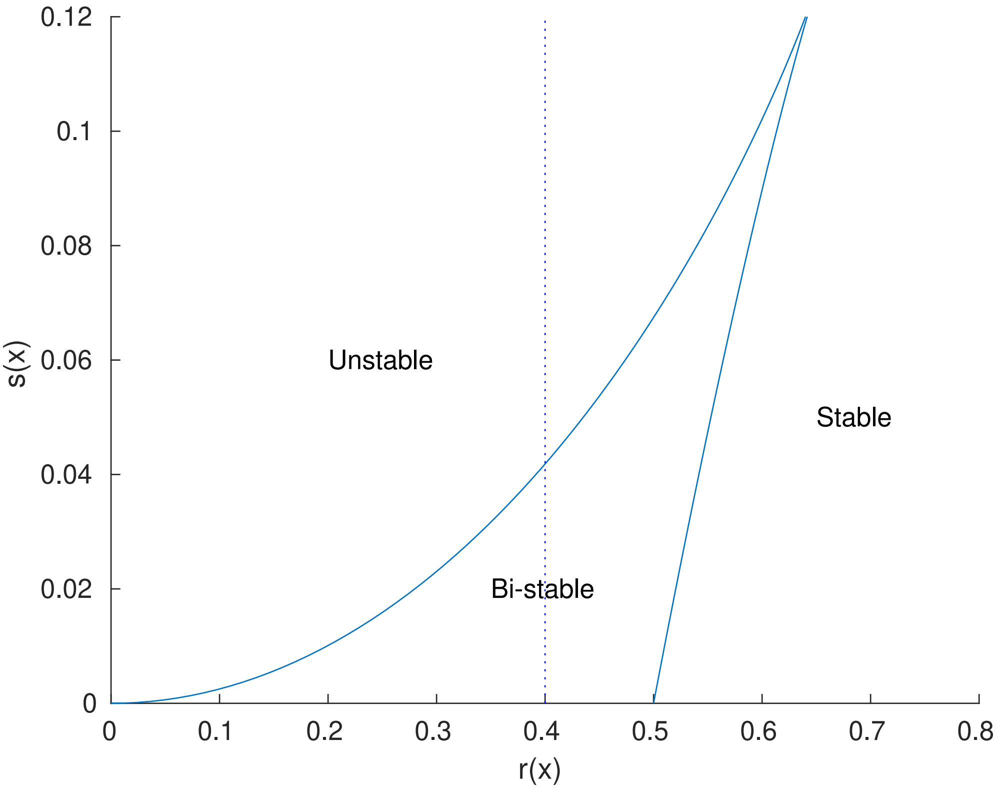

Figure 5: Bifurcation curves, in the $r(x)$, $s(x)$ plane. Dotted line at $r = 0.4$

ii. Figure 2(b)ii plots the bifurcations in the (r, s) plane. The dotted line for $r = 0.4$ shows the nature of the previous problem. Where fixing $r = 0.4$ and increasing s to 0.2 made the problem move from the bi-stable region to the unstable region, and then decreasing it led it back into the bi-stable region.

iii. The example in question 2a had $s = 0.2$, $r = 0.4$. It is clear that this lies in the unstable region. This can be seen visualy from the plot 2(b)ii.

The curve for $r(x)$, $s(x)$ is the bifurcation for the system. Along the curve there exists 2 steady states. On the region contained within the curve (the bistable region), there are 3 steady states, and outside of the curve there is only 1 steady state. These are demonstrated in figure 2(a)i, the $s = 0$ plot sits within the bistable region, the $s \approx 0.041834$ sits on the bifurcation curve, and only has one steady state, and the $s = 0.2$ is in the unstable region (outside of the steady state curve) and there is only 1 solution.

Matlab Code

```
%%Q1a
close all
x = linspace(-1,3);
r = log(1+x) -x;
figure
hold on
plot(r(x<0),x(x<0),'b')
plot(r(x>0),x(x>0),':b')
plot (0,0, 'xb')
xlabel('r')
ylabel('x')
title ("Bifurcations for problem 1")
legend(["Stable branch","Unstable branch","Bifurcation"])
grid on
saveas(gcf,"ODEsA2Q1a.eps","epsc")
%%Q1b
figure
hold on
r = linspace (0,2);
plot ([min(r ),1],[0,0], ':b')
plot ([1, max(r )],[0,0], 'b')
plot(r(r<1),1-1./r(r<1),'b','HandleVisibility','off')
plot(r(r>1),1-1./r(r>1),':b','HandleVisibility','off')
plot (1,0, 'xb')
axis([-inf, inf,-2,2])
title ("Bifurcations for problem 2")
legend(["Stable branch","Unstable branch","Bifurcation"])
xlabel('r')
ylabel('x')
grid on
saveas(gcf,"ODEsA2Q1b.eps","epsc")


%%Q1c

figure
hold on
plot ([-0.5,0],[0,0], 'b')
plot ([0,0.5],[0,0], ':b')
x = linspace (0,0.5);
%handlevisibility off makes the legend clean
plot(x,0.5*sqrt(x),'b','HandleVisibility','off')
plot(x,-0.5*sqrt(x),'b','HandleVisibility','off')
plot (0,0, 'xb')
title ("Bifurcations for problem 3")
legend(["Stable branch","Unstable branch","Bifurcation"])
xlabel('r')
ylabel('x')
grid on
```

```
saveas(gcf,"ODEsA2Q1c.eps","epsc")

%%Q2ai

% r = 0.4;
% [t,x] = meshgrid(linspace(0,5,20));
% f = -r.*x + (x.^2)./(1+x.^2);

% figure
% for s=linspace(0,0.2)
%     %f = f+s;
%     f = s-r.*x + (x.^2)./(1+x.^2);
%     df = f./(sqrt(f.^2 + ones(size(ftakes)).^2));
%     dt = ones(size(ftakes))./(sqrt(f.^2 + ones(size(ftakes)).^2));
%     quiver(t,x,dt,df)
%     drawnow
%     pause(0.05)
% end
%%Q2aii
syms x
%obtain approximate numerical solutions
%for the bifurcation
r= 0.4;
approxxbar = fsolve(@(x) -r + 2*x/((1+x^2)^2), 0.2)
approxsbar = r*approxxbar - (approxxbar^2)/(1+approxxbar^2)
%max value of x after increasing s to 0.2
maxx=fsolve(@(x) 0.2 - r*x + (x.^2)./(1+x.^2), 2.5)


x = linspace(-0.1,3,500);

figure
hold on
xsq = x.^2;
for s=[0,approxsbar,0.2]
dx = s - r*x + (xsq)./(1+xsq);
plot(x,dx)
end
legend(["s=0","s="+num2str(approxsbar),"s=0.2"])
xlabel("x")
ylabel("$$\frac{dx}{d\tau}$$",'interpreter','latex')
title('Effect of s on $$\frac{dx}{d\tau}$$','interpreter','latex')
grid on
saveas(gcf,"ODEsA2Q2a.eps","epsc")
%%Q2bii
r = 2*x./((xsq + 1).^2);
s = xsq.*(1 - xsq)./((xsq+1).^2);

figure
hold on
plot(r,s)
```

```
plot  ([0.4,0.4],[0,0.12],  ':b')
xlabel('r(x)')
ylabel('s(x)')
axis  ([0,0.8,0,0.12])
text (0.35,0.02," Bi-stable")
text (0.2,0.06," Unstable")
text (0.65,0.05," Stable")
saveas(gcf," ODEsA2Q2b.eps","epsc")
```

School of Mathematical Sciences

Modelling with ODEs

Semester 1, 2019

Assignment 2

Due 5pm Monday, Week 6: Submit via MyUni

You will be marked on the presentation of your answers (including clarity of explanations)!

1. Consider the following ODEs:

   (a) $\dot{x} = r + x - \log(1 + x)$; (b) $\dot{x} = x - r\,x\,(1 - x)$; (c) $\dot{x} = r\,x - 4\,x^3$.

   For each ODE:

   - Find the bifurcation value $\bar{r}$. You may find it helpful to use MATLAB.
   - State the type of bifurcation with reason.
   - Produce the bifurcation diagram, with the stable and unstable branches indicated.

2. In Tutorial 2 you studied the ODE

$$\frac{dx}{d\tau} = s - r\,x + \frac{x^2}{1 + x^2}, \tag{1}$$

   which models the dynamics of a (nondimensional) gene product $x(\tau)$, activated by a (nondimensional) biochemical substance $s \geq 0$, and with parameter $r > 0$.

   (a) Let $r = 0.4$, and assume that initially there is no gene product, i.e. $x(0) = 0$. Suppose that the biochemical substance is introduced by slowly increasing s from zero up to 0.2.

      i. Explain what happens to $x(\tau)$ and why.

      ii. Explain (with reasons) what happens if the biochemical substance is then slowly decreased back to zero.

   (b) Consider ODE (1) for two varying parameters, $s \geq 0$ and $r > 0$.

      i. Calculate the bifurcation curves.

      ii. Plot the bifurcation curves in the (r, s)-plane.

      iii. Determine the number of steady states and their stability in each region of your plot, and describe what happens on the bifurcation curves.