Lecture 4: Sample paths and simulation – A simple, useful check / tool using estimation

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Concepts checklist

At the end of this lecture, you should be able to:

- Prove the *sample path behaviour* of a CTMC:
  - the state next visited by a CTMC at the time of a state transition is proportional to the rate of visiting that state, and that it is independent of the time of the jump;
  - the time spent in a state before transitioning to another state is exponentially-distributed with rate equal to the sum of the rates of all possible transitions out of that state;
- Produce *realisations* of a CTMC using a computer.

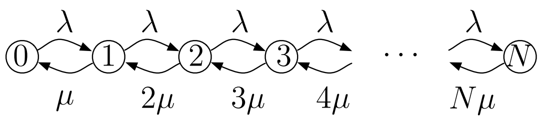
Example 4. N-machine reliability (Repairman problem), or a birth-death process, or an immigration-death process

Let us consider a process $\mathcal{X} = (X(t), t \geq 0)$ which, for example, models the number of N machines currently working, or the number of individuals in a population with a strict population ceiling of size N.

- State space $\mathcal{S} = \{0, 1, 2, \ldots, N\}$, where i is the number of machines currently working.

In terms of machines, as for the repairman problem, we have each machine fails at a constant rate, here μ, and the one repairman repairs each machine one at a time at a constant rate, here λ. In a population context, what does λ represent, and what does μ represent?

- State transition diagram



- The transition rates are

$$
\begin{aligned}
\text{(repairing)} \quad & q_{n,n+1} = \lambda \quad \text{for } n = 0, 1, 2, \ldots, N-1, \\
\text{(breaking down)} \quad & q_{n,n-1} = n\mu \quad \text{for } n = 1, \ldots, N, \\
& q_{n,n} = -(n\mu + \lambda) \quad \text{for } n = 0, \ldots, N-1, \\
& q_{N,N} = -N\mu.
\end{aligned}
$$

Thus, the generator Q is

$$Q = \begin{bmatrix} -\lambda & \lambda & \cdots & \cdots & \cdots & \cdots & 0 \\ \mu & -(\mu+\lambda) & \lambda & & & & \vdots \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & & & \vdots \\ \vdots & & 3\mu & -(3\mu+\lambda) & \lambda & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ \vdots & & & \ddots & & \ddots & \lambda \\ 0 & \cdots & \cdots & \cdots & N\mu & & -N\mu \end{bmatrix}.$$

Where does the rate $n\mu$ come from? In the 2-printer example in Lecture 2, we had that the rate of a single failure when both machines are working was twice the rate of a single failure. Extend the calculations there to answer this question! After that, we are back to a process with at most only two events possible in each state; what about a continuous-time Markov chain with an arbitrary number of possible events. What is its behaviour?

Theorem 1. *Consider a continuous-time Markov chain $\mathcal{X}$ on state space $\mathcal{S}$ with generator Q. Then, for all $i \in \mathcal{S}$,*

$$\Pr\left(\textit{moves to state } j \neq i \mid \textit{leaves state } i \textit{ at time } t\right) = \frac{q_{ij}}{-q_{ii}}. \tag{1}$$

Proof. Let T_{ij}, for $i, j \in \mathcal{S}$, denote the time to leave i for j, then $T_{ij} \sim \mathrm{Exp}(q_{ij})$. Define $M = \min_{k\neq i, k\in\mathcal{S}} T_{ik}$. Then,

$$\Pr\left(\text{moves to state } j \neq i \mid \text{leaves state } i \text{ at time } t\right)$$

$$\begin{aligned}
&= \frac{\Pr(T_{ij} = t \cap \min_{k\neq i, k\neq j, k\in\mathcal{S}} T_{ik} > t)}{\Pr(M \in [t, t+dt))} \\
&= \frac{\Pr(T_{ij} \in [t, t+dt)) \prod_{k\neq i, k\neq j} \Pr(T_{ik} > t)}{\Pr(M \in [t, t+dt))} \qquad \text{(independence)} \\
&= \frac{\Pr(T_{ij} \in [t, t+dt)) \Pr(Z > t)}{\Pr(M \in [t, t+dt))} \qquad \text{where } Z \sim \exp\left(\sum_{k\neq i, k\neq j, k\in\mathcal{S}} q_{ik}\right) \\
&= \frac{q_{ij}}{q_{ij} + \sum_{k\neq i, k\neq j, k\in\mathcal{S}} q_{ik}} \\
&= \frac{q_{ij}}{\sum_{k\neq i, k\in\mathcal{S}} q_{ik}} \\
&= \frac{q_{ij}}{-q_{ii}}.
\end{aligned}$$

□

Theorem 2. *Consider a continuous-time Markov chain $\mathcal{X}$ on state space $\mathcal{S}$ with generator Q. Once moving to some state $i \in \mathcal{S}$, the time the system stays in i until it moves out of i is exponentially distributed with rate $-q_{ii}$.*

Proof. Note that:

- The time the system stays in i until it moves out of i, is the same as the time until the system moves to some state $j \neq i$.
- The latter time is $\min_{j \neq i, j \in \mathcal{S}} T_{ij} \sim \text{Exp}\left(\sum_{j \neq i, j \in \mathcal{S}} q_{ij}\right)$.
- This implies that $\min_{j \neq i, j \in \mathcal{S}} T_{ij} \sim \text{Exp}(-q_{ii})$, which completes the proof. □

We have discussed earlier that the transition function is a powerful tool that we'd like to evaluate, but our specification of models, and the *sample-path behaviour* has been in terms of rates of events/transitions. Before starting to explore the relationship between these in more detail, we should revisit the algorithm mentioned in Lecture 1 in light of the two theorems we just proved.

Simulating a continuous-time Markov chain

Pseudo-code for simulation of a CTMC:

1. INPUTS: initial state, state $\in \mathcal{S}$; total time to run simulation for, T; and set $t = 0$, $ts = 0$, and $ss =$ state.
2. Calculate the rate of each possible transition from the current state, $q_{\text{state},j}$.
3. Calculate the sum of these rates, q_{state}.
4. While $t < T$ and $q_{\text{state}} > 0$:
5. Generate an exponential random variable with rate q_{state}, te, and update time $t = t + te$ and store $ts = [ts; t]$.
6. Choose the state next visited, j, with probability $q_{\text{state},j}/q_{\text{state}}$, and update the current state, state $= j$ and store $ss = [ss; \text{state}]$.
7. End while

Let's consider Example 1, the problematic printers in maths. I am interested in the probability of both printers not working in 6 months time, given both are currently working. How can I evaluate this probability?

First, I could evaluate from the transition function presented in Lecture 1. The probability both machines are not working in t time units starting with both machines working is

$$(1 + \exp(-t/6) - 2\exp(-t/12))/25.$$

Hence, the desired probability is $(1 + \exp(-180/6) - 2\exp(-180/12))/25 \approx 0.04$.

However, I could also estimate this probability by simulation. Figure 4 below shows the estimate of this probability based upon increasing numbers of simulations.

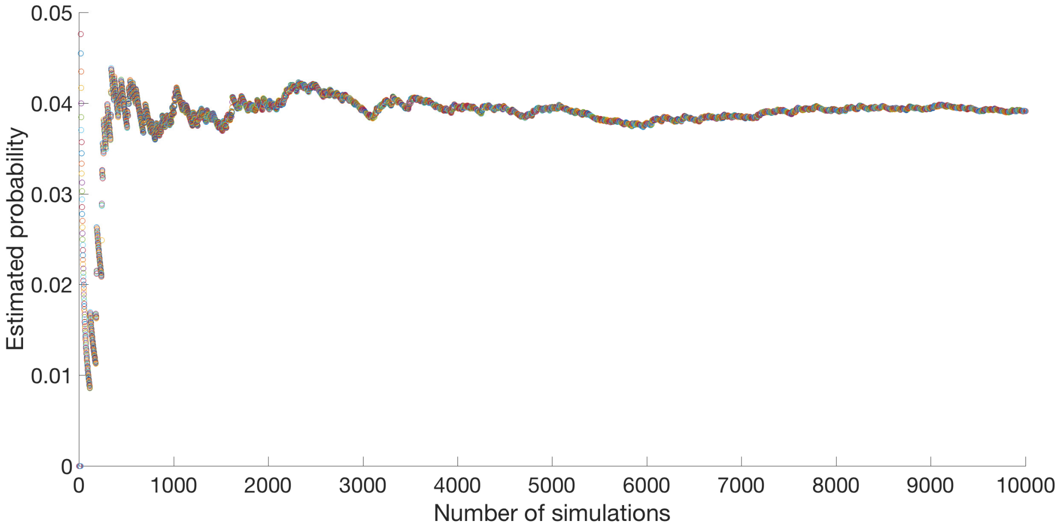


Figure 4: Estimate of the probability of both printers not working after 180 days, given both initially working, versus the number of simulations used.

Do you think $10,000$ simulations is sufficient to estimate the probability? How would you assess this?

In the exact expression for the probability, what happens as $t \to \infty$? What value does this converge to? What is the interpretation of this phenomenon?