Fluid Mechanics Assignment 4

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1. Consider the 2π periodic function

$$u(x) = \sin(2x)\sin(3x)$$

on the interval $0 \leq x < 2\pi$

(a) Use the discrete Fourier transform to estimate $u'(x_j)$, where $x_j = jh$, $h = 2\pi/N$ and $N = 8$. Can check answer with `Matlab`.

Solution $k = -3, \ldots, 4$

$$\begin{aligned}
u_j &= \sin(2x)\sin(3x) \\
u_j &= \frac{1}{2i}(e^{i2x_j} - e^{-i2x_j})\frac{1}{2i}(e^{i3x_j} - e^{-i3x_j}) \\
&= \frac{-1}{4}(e^{i5x_j} - e^{-ix_j} - e^{ix_j} + e^{-i5x_j}) \\
&= \frac{-1}{4}(e^{i3x_j} + e^{-i3x_j} - e^{-ix_j} - e^{ix_j}) \textit{ (5 aliases to 3)} \\
&= \frac{-1}{4}(e^{i3x_j} + e^{-i3x_j}) + \frac{1}{4}(e^{-ix_j} + e^{ix_j}) \\
&= \frac{\cos(x) - \cos(3x)}{2} \leftarrow \text{ this is just to identify the aliased result}
\end{aligned}$$

$$\implies \hat{u}_k = \begin{cases} \frac{1}{4} & k = -1, 1 \\ -\frac{1}{4} & k = -3, 3 \\ 0 & otherwise \end{cases}$$

Now get $\hat{w}$:

$$\hat{w}_k = ik\hat{u}_k$$

$$\hat{w}_k = \begin{cases} i\frac{3}{4} & k = -3 \\ -i\frac{1}{4} & k = -1 \\ i\frac{1}{4} & k = 1 \\ -i\frac{3}{4} & k = 3 \\ 0 & otherwise \end{cases}$$

Now the derivative:

$$
\begin{aligned}
u'_j &= \sum_{k=-N/2+1}^{N/2-1} \hat{w}_k e^{ikx_j} \\
&= -i\frac{3}{4}e^{i3x_j} + i\frac{3}{4}e^{-i3x_j} + -i\frac{1}{4}e^{-ix_j} + i\frac{1}{4}e^{ix_j} \\
&= \frac{1}{2}\left(-i\frac{3}{2}e^{i3x_j} + i\frac{3}{2}e^{-i3x_j} + -i\frac{1}{2}e^{-ix_j} + i\frac{1}{2}e^{ix_j}\right) \\
&\text{use the fact that } i = \frac{-1}{i} \\
&= \frac{1}{2}\left(3\frac{1}{2i}(e^{i3x_j} - e^{-i3x_j}) - \frac{1}{2i}(e^{ix_j} - e^{-ix_j})\right) \\
&= \frac{3\sin(3x_j) - \sin(x_j)}{2}
\end{aligned}
$$

As required.

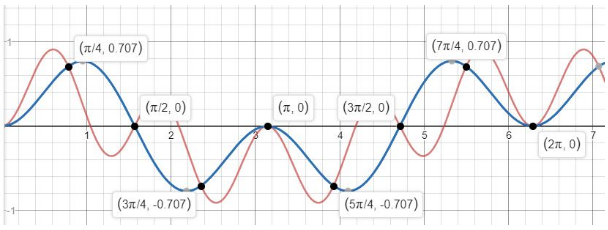
(b) Is the spectral estimate of the derivative from (a) accurate? Why or why not? If not, what is the minimum value of N that would be needed to obtain an accurate estimate of the derivative?

Solution

It will not be accurate due to aliasing. To prevent this issue, N has to be increased so that k contains -5 and 5. I.e. $N = 12$ would be the minimum N required. The aliasing is demonstrated in Figure 1, the plots coincide at all the x values considered, but are clearly different everywhere else, similarly it can be observed from the figure that the derivatives will differ considerably at most of the points. Note the exact derivative is

$$2\cos(2x)\sin(3x) + 3\cos(3x)\sin(2x)$$

Figure 1: Plot of Aliasing, red line $\sin(2x)\sin(3x)$ and blue line $\frac{\cos(x)-\cos(3x)}{2}$



As required.

2. Consider the Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^4 u}{\partial \xi^4} = 0$$

We will consider L-periodic solutions, i.e. solutions for which $u(\xi, t) = u(\xi + L, t)$ subject to the initial condition

$$u(\xi, 0) = f(\xi)$$

(a) In order to apply the spectral methods that we have developed for 2π periodic functions, we will transform this into an equivalent $2\pi-$periodic problem (below) The interval $0 \le \xi < L$ is mapped to $0 < x < 2\pi$ by $x = \alpha\xi$, where $\alpha = 2\pi/L$. Show that $x = \alpha\xi$ transforms the Kuramoto-Sivashinsky equation into:

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha^4\frac{\partial^4 u}{\partial x^4} = 0$$

Where u is now 2π periodic.

Solution

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} = \frac{\partial u}{\partial x}\alpha$$
$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial}{\partial \xi}\frac{\partial u}{\partial x}\alpha = \alpha^2\frac{\partial^2 u}{\partial x^2}$$

And similarly

$$\frac{\partial^4 u}{\partial \xi^4} = \alpha^4\frac{\partial u}{\partial x}$$

Subbing this in:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^4 u}{\partial \xi^4} = 0$$
$$\implies \frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha^4\frac{\partial^4 u}{\partial x^4} = 0$$

As required.

(b) Suppose u is small. Neglecting products of u and its derivatives, the linearised equation is

$$\frac{\partial u}{\partial t} + \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha^4\frac{\partial^4 u}{\partial x^4} = 0$$

Let

$$u(x_j, t) \approx u_j(t) = \sum_{-N/2+1}^{N/2} \hat{u}_k(t)e^{ikx_j}$$

Where $x_j = jh$, $h = 2\pi/N$ and N is even. Use the pseudo-spectral method to derive a system of uncoupled ODEs for the Fourier coefficients $\hat{u}_k(t)$, $k = -N/2+1, \ldots, N/2$

Solution

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha^4\frac{\partial^4 u}{\partial x^4} = 0$$
$$\implies \frac{\partial u}{\partial t}\Big| = -\alpha u\frac{\partial u}{\partial x}\Big|_{x_j} - \alpha^2\frac{\partial^2 u}{\partial x^2}\Big|_{x_j} - \alpha^4\frac{\partial^4 u}{\partial x^4}\Big|_{x_j}$$
$$\sum_{k=-N/2+1}^{N/2} \frac{d\hat{u}_k}{dt}e^{ikx_j} = \sum_{k=-N/2+1}^{N/2} \left(-\hat{u}_k e^{ikx_j}(i^2k^2\alpha^2 + i^4k^4\alpha^4)\right)$$
$$\sum_{k=-N/2+1}^{N/2} \frac{d\hat{u}_k}{dt}e^{ikx_j} = \sum_{k=-N/2+1}^{N/2} \left(k^2\alpha^2\hat{u}_k e^{ikx_j}(1 - k^2\alpha 2)\right)$$

Equating like terms gives the system:

$$\frac{d\hat{u}_k}{dt} = k^2\alpha^2(1 - k^2\alpha^2)\hat{u}_k$$

As required.

(c) Solve the system of ODEs obtained in part (b) subject to the initial condition

$$u(\xi, 0) = f(\xi)$$

Solution The initial condition gives: $\hat{u}_k(0) = \hat{f}_k$ where $\hat{f}_k$ are the fourier coefficients of the initial condition:

$$u_j(0) = f(x_j) = \sum_{k=-N/2+1}^{N/2} \hat{f}_k e^{ikx_j}$$

$$\hat{u}_k(t) = \hat{f}_k \exp\{k^2\alpha^2(1 - k^2\alpha^2)t\}$$

As required.

(d) In the linearised model, the amplitudes of some modes may grow without limit, thereby violating the assumption that u is small.

i. Using the solution from (c) determine the range of wave numbers for which the amplitude $|\hat{u}_k(t)|$ increases with time.

ii. What is the minimum value of $L = 2\pi/\alpha$ necessary for the amplitude of any mode to increase with time?

Solution Find the values of k which make $|\hat{u}_k(t)|$ increase in time. Look at the inside of the exponential. If the coefficient of t is > 0 then it grows in time We know $\alpha > 0$

$$\begin{aligned}
k^2\alpha^2(1 - k^2\alpha^2) &> 0 \\
k^2(1 - k^2\alpha^2) &> 0 \\
k^2 + k^4\alpha^2 &> 0 \\
k^4\alpha^2 &> -k^2 \text{ divide by } k^2 \text{ and since } -k^2 < 0 \\
k^2\alpha^2 &< 1 \\
k^2 &< \frac{1}{\alpha^2}
\end{aligned}$$

I.e. it grows in time if

$$k \in (-\frac{1}{\alpha}, \frac{1}{\alpha})$$

$\alpha = 2\pi/L$.

$$\implies k \in (-\frac{L}{2\pi}, \frac{L}{2\pi})$$

For any mode to increase with time, these fractions have to be $\in (-1, 1)$.

$$\frac{L}{2\pi} \leq 1 \implies L \leq 2\pi$$

As required.

(e) If u is not small, we must solve the non-linear equation. Describe a pseudospectral method to solve it in physical space. *Hint:* your answer must include the system of ODEs to solve, all expressions needed to compute the spatial derivatives, and any ICs

Solution This will be similar to solving burgers-equation.

$$\begin{aligned}
\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \alpha^2 \frac{\partial^2 u}{\partial x^2} + \alpha^4 \frac{\partial^4 u}{\partial x^4} &= 0 \\
\implies \frac{\partial u}{\partial t}\Big|_{x_j} &= -\alpha u \frac{\partial u}{\partial x}\Big|_{x_j} - \alpha^2 \frac{\partial^2 u}{\partial x^2}\Big|_{x_j} - \alpha^4 \frac{\partial^4 u}{\partial x^4}\Big|_{x_j} \\
\sum_{k=-N/2+1}^{N/2} \frac{d\hat{u}_k}{dt} e^{ikx_j} &= \sum_{k=-N/2+1}^{N/2-1} \left(-\hat{u}_k e^{ikx_j}(u\alpha ik + i^2k^2\alpha^2 + i^4k^4\alpha^4)\right) \\
&\quad - (-\hat{u}_{N/2} e^{i(N/2)x_j}(i^2\frac{N^2}{4}\alpha^2 + i^4\frac{N^4}{2^4}\alpha^4))
\end{aligned}$$

Equating like terms:

$$\begin{aligned}
\frac{d\hat{u}_k}{dt} &= -\hat{u}_k(u\alpha ik + i^2k^2\alpha^2 + i^4k^4\alpha^4) \\
\frac{d\hat{u}_{N/2}}{dt} &= -(u(i^2\frac{N^2}{4}\alpha^2 + i^4\frac{N^4}{2^4}\alpha^4))
\end{aligned}$$

The initial condition gives:

$$u_j(0) = f(x_j) = \sum_{k=-N/2+1}^{N/2} \hat{f}_k e^{ikx_j}$$

As required.

(f) Write a `MATLAB` script or function that uses the scheme described in (e) to solve the nonlinear equation subject to the random IC

```
f = @(x) 0.01*(2*rand(size(x))-1);
```

Upload the script/function with the assignment

Hints:

i. The system of ODEs is very stiff owing to the fourth-order *hyperdiffusion* term. You may find it better to use `ode15s` rather than `ode45`

ii. You may find that waterfall does not produce very informative plots when there are more than about $30 - 40$ times. Instead use

```
contourf(x, t, u, 'EdgeColor', 'none')
```

Read the documentation to find out more about `contourf`. Make sure to include a `colorbar` so that the reader knows what the colours mean.

iii. Make sure to document the code. The minimum documentation includes a description of the script and any functions, author name, date, and a description of any input/output arguments.

Solution The code is:

```
% Physically solves periodic solutions for
% the Kuramoto-Sivashinsky equation
% Uses spectral methods
%
% edited from SPECTRAL_BURGERS
% original author:
% Trent Mattner
% Edited by:
% Andrew Martin
% 24/05/2018

close all
clear all

set(groot, 'DefaultLineLineWidth', 1, ...
    'DefaultAxesLineWidth', 1, ...
    'DefaultAxesFontSize', 12, ...
    'DefaultTextFontSize', 12, ...
    'DefaultTextInterpreter', 'latex', ...
    'DefaultLegendInterpreter', 'latex', ...
    'DefaultColorbarTickLabelInterpreter', 'latex', ...
    'DefaultAxesTickLabelInterpreter','latex');


for L=[3,12,14,15,20,60]
s=rng('shuffle');
N=6*L;
alpha = 2*pi / L;
f = @(x) 0.01*(2*rand(size(x))-1);
[x, t, u] = solve_kdv(N,alpha, 1000, 400, f);

%Label plots
fig = figure ;
contourf(x, t, u, 'EdgeColor','none')
c=colorbar();
c.Label.String = 'u';
xlabel('$x$')
```

```
ylabel('$t$')
zlabel('$u(x,t)$')
title(['Kuramoto-Sivashinsky Solution for $L=$' num2str(L)])
box on
%Just print them out so I have them for reference
saveas(fig,['A4KSL' num2str(L) '.jpg'])
end
function [x, t, u] = solve_kdv(N,alpha, nt, T, f)

% kurasiva solves Kuramoto-Sivashinsky equation on a
% 2*pi-periodic domain.
%
% Inputs:
%   N - number of collocation points.
%   nt - number of times for output.
%   T - final time.
%   f - function handle specifying IC.
%
% Outputs:
%   t - row vector containing output times.
%   x - row vector containing grid points.
%   u - matrix containing solution at t(j)
%       in row u(j,:).

% Set up grid.

h = 2*pi/N;
x = h*(0:N-1);
ik = 1i*[0:N/2-1 0 -N/2+1:-1]';
k2 = -[0:N/2 -N/2+1:-1]'.^2;
k4 = k2.^2;
t = linspace(0, T, nt);
% Numerical solution in physical space.

[~, u] = ode15s(@kurasiva, t, f(x));

    function dudt = kurasiva(t, u)

        uh = fft(u);
        ux = ifft(ik.*uh, 'symmetric');
        uxx = ifft(k2.*uh , 'symmetric');
        ux4 = ifft(k4.*uh, 'symmetric');
        dudt = -alpha*u.*ux -alpha^2*uxx - alpha^4 * ux4;

    end

end
```

As required.

(g) Run the code and plot the solution $u(x,t)$ for $L = 3, 12, 14, 15, 20, 60$ and $0 \leq t \leq 400$
For each plot, write a sentence or two that describes the nature of the solution (for example, reaches a steady state, travelling wave, pulsation between two states, chaos, etc.)

i. Experiment a little with N. $N \sim 6L$ seems to work reasonably well. Keep in mind the random ICs will produce different solutions each run
ii. Make sure the plots are well resolved in time, otherwise you may not be able to see details of the temporal development.

Solution The plots are appended later in the document.

- $L = 3$ (figure 2) is more-or-less constant. It varies a little based on the initial conditions, but after a very short duration (usually less than $t = 10$) it will reach a steady state. It is uniform flow
- $L = 12$ (figure 3) the solution appears to be in travelling waves, moving from $x = 0$ towards $x = 6$ with time. It is travelling in bands of u (high u followed by negative u very closely)
- $L = 14$ (figure 4) alternates in time, in a chessboard pattern.
- $L = 15$ (figure 5) alternates in space. A particular point in x after reaching a steady point (after about $t = 50$), will have constant u. Changing x changes the u value.
- $L = 20$ (figure 6) also has an alternating pattern in space. This is quite similar to figure 4, however the solutions have smaller $|u|$ values.
- $L = 60$ (figure 7) is significantly more erratic than the previous solutions. There seems to be very little pattern apart from each high u value it paired with a highly negative u value to its right.

As required.

Appendices

Figure 2: Solution to Kurasiva Equation for $L = 3$

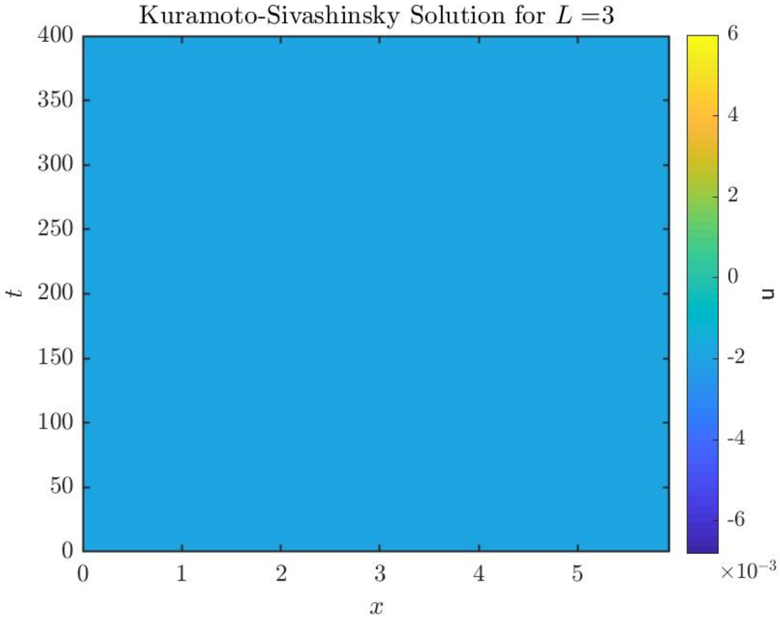


Figure 3: Solution to Kurasiva Equation for $L = 12$

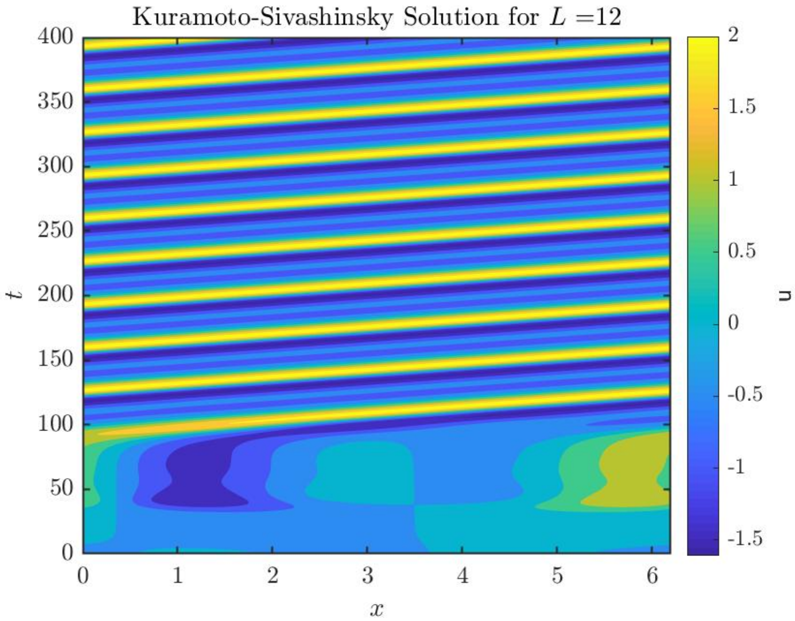


Figure 4: Solution to Kurasiva Equation for $L = 14$

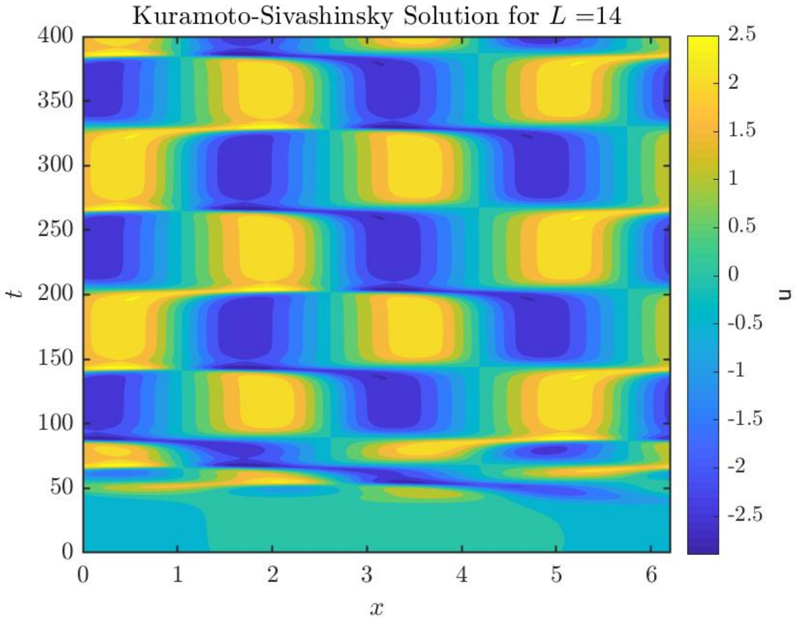


Figure 5: Solution to Kurasiva Equation for $L = 15$

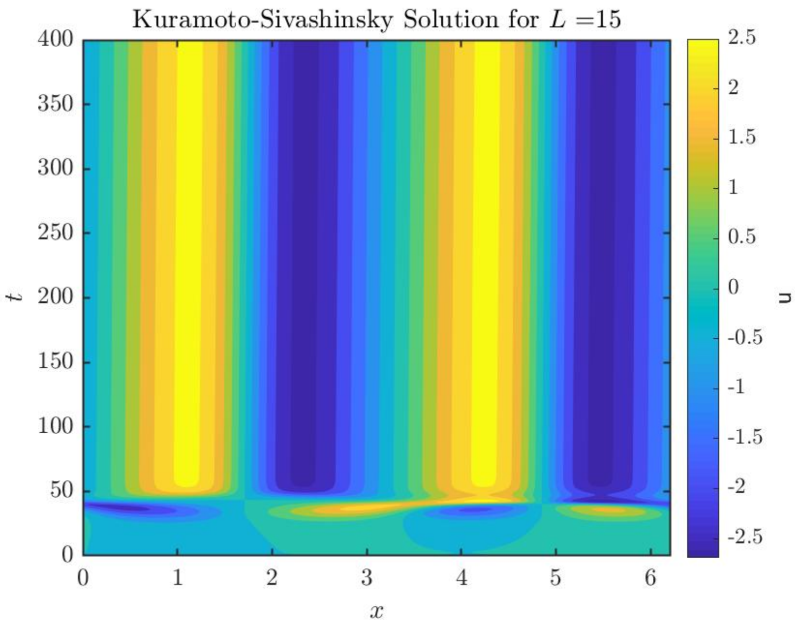


Figure 6: Solution to Kurasiva Equation for $L = 20$

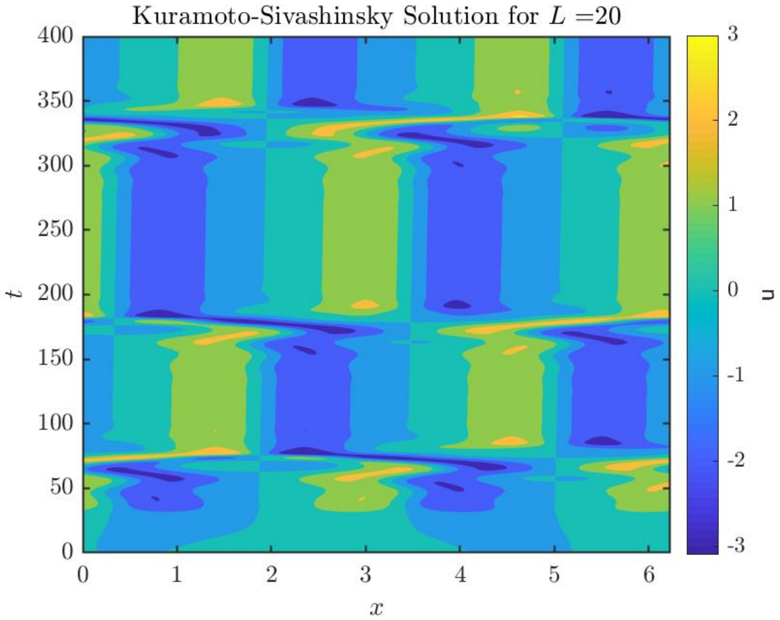


Figure 7: Solution to Kurasiva Equation for $L = 60$