Practical Asymptotics (APP MTH 4051/7087)
Assignment 4 (5%)

Due 27 May 2019

1. Apply the method of multiple scales to find a leading-order solution to the following oscillator equation:

$$y'' + y + \epsilon \left(y'\right)^3 = 0,$$

with $\epsilon \ll 1$, subject to $y(0) = 1$ and $y'(0) = 0$. Seek a solution of the form $y(t) \sim y_0(t, T)$, where $T = \epsilon t$ is a slow timescale. Compare this leading-order solution with a numerical solution and comment.

2. Recall from lectures that the numerical solution to the Van der Pol oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \left(y^2 - 1\right) \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0, \quad y(0) = 1, y'(0) = 0, \quad \epsilon \ll 1,$$

exhibited a phase shift, but the leading-order solution did not. To capture this phase shift we require an additional, extra slow timescale.

(a) Introduce an extra slow timescale by letting $y(t) \equiv y(t, T, \tau)$, where $T = \epsilon t$ and $\tau = \epsilon^2 t$, then use the chain rule to transform the above ODE into a PDE in terms of these three variables.

(b) Let $y(t, T, \tau) = y_0(t, T, \tau) + \epsilon y_1(t, T, \tau) + \epsilon^2 y_2(t, T, \tau) + \ldots$ and write down the leading-order, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ problems, including boundary conditions.

(c) Find y_0 by solving the leading-order problem and eliminating resonant terms from the $\mathcal{O}(\epsilon)$ equation.

[Hint: This should include arbitrary functions of τ, but otherwise be identical to that found in lectures (you may reuse working).]

(d) Having eliminated these resonant terms, find y_1 by solving the $\mathcal{O}(\epsilon)$ problem (in terms of aribtrary functions of T and τ). [Hint: strongly recommend using computer algebra for this and the next part.]

(e) Identify the resonant terms from the $\mathcal{O}(\epsilon^2)$ equation that contain derivatives of the unknown function of τ in y_0, and set these terms to zero by finding these unknown function. [Hint: One of these is easy to solve, the other needs to be considered in the 'long time' limit as $T \to \infty$.]

(f) Compare your solution for y_0 with a numerical solution and comment.