Lecture 16: Reversible Processes

Concepts checklist

At the end of this lecture, you should be able to:

- *Understand (and hence exploit)* relationships regarding reversible processes, reversed-time processes, and detailed-balance equations.

Recall,

Definition 16. *A continuous-time Markov chain is reversible if the reversed-time process has the same transition rates as the forward-time process, that is,*

$$q_{jk}^R = q_{jk} \quad \textit{for all } j \textit{ and } k \in \mathcal{S}.$$

A *reversible* process has special properties that we will introduce first by way of example and then by formal statement and proof.

Example 9. A general birth-and-death process.

We have the state space $\mathcal{S} = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and non-zero rates (note, can be state dependent)

$$\begin{aligned} q_{j,j+1} &= \lambda_j \quad \text{for } j \geq 0, \\ q_{j,j-1} &= \mu_j \quad \text{for } j \geq 1, \end{aligned}$$

and the equilibrium equations are

$$\begin{aligned} &\pi_j\left(\lambda_j + \mu_j\right) = \pi_{j+1}\mu_{j+1} + \pi_{j-1}\lambda_{j-1} \quad \text{for } j \geq 1 \\ &\text{with} \quad \pi_0\lambda_0 = \pi_1\mu_1. \end{aligned} \tag{18}$$

Equation (18) can be re-written as

$$\pi_{j+1}\mu_{j+1} - \pi_j\lambda_j = \pi_j\mu_j - \pi_{j-1}\lambda_{j-1}$$

which is of the form $A_{j+1} = A_j$, where $A_j = \pi_j\mu_j - \pi_{j-1}\lambda_{j-1}$ for $j \geq 1$.

From the equilibrium equations we get the boundary equation

$$\pi_1\mu_1 - \pi_0\lambda_0 = 0,$$

which implies that $A_1 = 0$, and hence that $A_j = 0$ for all $j \geq 1$, so that

$$\pi_j\mu_j = \pi_{j-1}\lambda_{j-1} \quad \text{— known as detailed balance equations.}$$

Rearranging (by repeated substitution) these equations reveals

$$\pi_j = \frac{\pi_{j-1}\lambda_{j-1}}{\mu_j} = \pi_0\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\cdots\frac{\lambda_{j-1}}{\mu_j} = \pi_0\prod_{\ell=0}^{j-1}\frac{\lambda_\ell}{\mu_{\ell+1}}.$$

The equilibrium distribution $\boldsymbol{\pi} = \{\pi_0, \pi_1, \pi_2, \dots\}$ then exists if

$$\sum_{j=0}^{\infty} \prod_{\ell=0}^{j-1} \frac{\lambda_\ell}{\mu_{\ell+1}} \; < \; \infty.$$

If the equilibrium distribution exists, the reversed-time transition rates are

$$q^R_{j,j+1} = \frac{\pi_{j+1} q_{j+1,j}}{\pi_j} = \frac{\dfrac{\pi_j \lambda_j}{\mu_{j+1}} \mu_{j+1}}{\pi_j} = \lambda_j$$

$$\text{and} \qquad q^R_{j+1,j} = \frac{\pi_j q_{j,j+1}}{\pi_{j+1}} = \frac{\dfrac{\pi_{j+1} \mu_{j+1}}{\lambda_j} \lambda_j}{\pi_{j+1}} = \mu_{j+1}.$$

Note that the reversed-time transition rates are identical to the forward-time transition rates,

$$\begin{aligned} q^R_{j,j+1} &= q_{j,j+1} = \lambda_j \\ q^R_{j+1,j} &= q_{j+1,j} = \mu_{j+1}, \end{aligned}$$

and therefore this process is reversible.

Theorem 14. *A stationary continuous-time Markov chain is reversible if and only if there exists a collection of numbers $\pi_j > 0$, summing to unity, that satisfies the detailed balance equations given by*

$$\pi_j q_{jk} = \pi_k q_{kj} \quad \text{for all } j, k \in \mathcal{S}.$$

If such a collection of π_j exists, it is the equilibrium distribution of the Markov chain.

Essentially this means that we could *assume* reversibility and then attempt to find a collection of numbers $\pi_j > 0$ summing to unity that satisfy the detailed balance equations. If we can do this we have both the equilibrium probability distribution and the knowledge that the Markov chain is reversible; otherwise we only know that the Markov chain is not reversible.

Proof:

($\Leftarrow$) Assume that the detailed balance equations have a solution. Then by summing over all states $k \neq j$, we get the global balance equations for the CTMC and therefore $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots)$ must be the equilibrium probability distribution.

Then, by using the detailed balance equations, we have

$$q^R_{jk} = \frac{q_{kj} \pi_k}{\pi_j} = q_{jk},$$

showing that the reversed-time transition rates are the same as the forward-time transition rates, and hence that the CTMC is reversible. □

($\Rightarrow$) Assume now that the CTMC is reversible. Then

$$q_{jk} = q^R_{jk} = \frac{q_{kj} \pi_k}{\pi_j} \quad \text{for all } j, k \in \mathcal{S},$$

where $\boldsymbol{\pi} = \{\pi_0, \pi_1, \dots\}$ is the equilibrium distribution of the CTMC and so

$$\pi_j q_{jk} = \pi_k q_{kj}, \quad \text{for all } j, k \in \mathcal{S},$$

which are the detailed balance equations. □

Theorem 15. *Let $X(t)$ be a stationary, not necessarily reversible, continuous-time Markov chain with transition rates q_{jk} and state space $\mathcal{S}$.*

If we can find numbers q^R_{jk} and π_j for $j, k \in \mathcal{S}$ such that

$$-q^R_{jj} = \sum_{\substack{k\in\mathcal{S}\\k\neq j}} q^R_{jk} = -q_{jj} = \sum_{\substack{k\in\mathcal{S}\\k\neq j}} q_{jk} \quad \textit{for all } j \in \mathcal{S}, \quad \textit{(equal holding times)}$$

$$\pi_j q_{jk} = \pi_k q^R_{kj} \quad \textit{for all } j,k \in \mathcal{S}, \quad (\text{"detailed balance"} \textit{ equations satisfied})$$

$$\textit{with } \sum_{j\in\mathcal{S}} \pi_j = 1 \quad \textit{and} \quad \pi_j > 0 \textit{ for all } j \in \mathcal{S}, \quad \textit{(p.m.f.)}$$

then

(i) *the q^R_{jk} are the transition rates of the reversed-time process, and*

(ii) $\boldsymbol{\pi} = \{\pi_j\}_{j\in\mathcal{S}}$ *is the equilibrium distribution of both the forward and reversed-time processes.*

Proof. For all $j \in \mathcal{S}$, since $-q_{jj} = -q^R_{jj}$ we have

$$\begin{aligned} \sum_{\substack{k\in\mathcal{S}\\k\neq j}} q_{jk} &= \sum_{\substack{k\in\mathcal{S}\\k\neq j}} q^R_{jk}, \\ &= \sum_{\substack{k\in\mathcal{S}\\k\neq j}} \frac{\pi_k q_{kj}}{\pi_j}, \\ \Rightarrow \quad \sum_{\substack{k\in\mathcal{S}\\k\neq j}} \pi_j q_{jk} &= \sum_{\substack{k\in\mathcal{S}\\k\neq j}} \pi_k q_{kj}, \end{aligned}$$

which are the global balance equations. Hence, $\boldsymbol{\pi} = \{\pi_1, \pi_2, \dots\}$ is the equilibrium distribution of both the forward and reversed-time processes. □

This is useful, in particular in the analysis of queues, because we can guess the transition rates of the reversed-time process and then verify the process by using Theorem 15.