THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2013

104831	**MATHS 2100**	**Real Analysis II**
104830	**MATHS 7100**	**Real Analysis**

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 8 TOTAL MARKS: 90

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- No calculators, books, notes, or other aids are permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. (a) (i) Let $f : \mathbb{R} \to \mathbb{R}$ and $(A_k)_{k\in K}$ be a family of subsets of $\mathbb{R}$. Show that

$$f^{-1}\big(\bigcup_{k\in K} A_k\big) = \bigcup_{k\in K} f^{-1}(A_k).$$

(ii) Let $K = \mathbb{N}$, the natural numbers, $A_k = [-k, k]$, and $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^4$. Find $\bigcup_{k\in K} f^{-1}(A_k)$.

(b) Let $f : A \to B$ be a function. Show that if

$$f(C \cap D) = f(C) \cap f(D)$$

for all $C, D \subset A$ then f is 1-1.

[5+3 = 8 marks]

2. (a) Let $A \subset \mathbb{R}$. Define what is meant by

(i) an *upper bound* of A

(ii) $\sup A$, the *supremum* of A.

(b) Let $A \subset \mathbb{R}$ be a set with maximum value b. That is, $b \in A$ and if $x \in A$ then $x \leq b$. Prove that $\sup A = b$.

(c) (i) Define what it means for a set to be *countable.*

(ii) Show that $\mathbb{Z}$, the integers, is a countable subset of $\mathbb{R}$.

(iii) Let K be the collection of all sequences of zeros and ones. That is, K is the set of all maps from $\mathbb{N}$ to the two element set $\{0, 1\}$. Show that K is not countable.

[2+3+8 = 13 marks]

3. (a) Let (a_n) be a sequence in $\mathbb{R}$. Define what it means for (a_n) to *converge* to a real number a.

(b) Show that if $a_n = \dfrac{n}{n^2 + 1}$ then (a_n) converges to 0.

(c) (i) Show that if (a_n) is a convergent sequence then there is a real number M such that $|a_n| \leq M$ for all n in $\mathbb{N}$.

(ii) Show that if (a_n) converges to $a \neq 0$ and (b_n) converges to $b \neq 0$ then $(a_n b_n)$ converges to ab.

[2+3+8 = 13 marks]

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4. (a) (i) Let A be a subset of $\mathbb{R}$. State what it means for A to be *open*.

   (ii) Use your definition in (i) to show that $(-\infty, 10)$ is open.

   (b) Show that if A is a non-empty open set that is bounded above then $\sup A \notin A$.

   (c) Prove that if A_i is open for $i \in \{1, 2, \ldots, n\}$ then $\bigcap_{i=1}^{n} A_i$ is open.

   (d) Show that the intersection of infinitely many open sets may not be open.

   (e) (i) Let A be a subset of $\mathbb{R}$. State the definition of $\bar{A}$, the *closure* of A.

   (ii) Show that if $A = [-1, 1)$ then $\bar{A} = [-1, 1]$. You may assume that if a sequence (a_n) in a closed set C converges to a then $a \in C$, and that $[-1, 1]$ is closed.

[3+3+3+2+5 = 16 marks]

5. (a) (i) Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and let c be a limit point of A. Define what it means for the limit of f at c to exist and be equal to the real number L.

   (ii) Use your definition to show that $\lim\limits_{x \to 5} 11x + 2 = 57$.

   (b) (i) State what it means for a function $f : A \to \mathbb{R}$ to be *uniformly continuous* on A.

   (ii) Give an example of a function that is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. You do *not* need to demonstrate that your function is not uniformly continuous.

[5+3 = 8 marks]

6. (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ (for $n \in \mathbb{N}$) then f is differentiable on $\mathbb{R}$, and $f'(x) = nx^{n-1}$.

   (b) State the Mean Value Theorem.

   (c) Let I be an open interval and $f : I \to \mathbb{R}$ be a differentiable function with $f'(x) \geq 0$ for all $x \in I$. Show that f is increasing on I.

   (d) Prove that there is no continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(\mathbb{I}) \subset \mathbb{Q}$ and $g(\mathbb{Q}) \subset \mathbb{I}$, where $\mathbb{I}$ denotes the irrational numbers and $\mathbb{Q}$ the rational numbers.

[3+2+3+4 = 12 marks]

7. (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and for any partition $\mathcal{P}$ of $[a, b]$ write $L(f, \mathcal{P})$for the lower sum of f with respect to $\mathcal{P}$ and $U(f, \mathcal{P})$ for the corresponding upper sum. Define what it means for f to be *integrable* on $[a, b]$.

   (b) Show that if f is integrable on $[a, b]$ then for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that
   $$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

   (c) Show that every increasing function $f : [0, 1] \to \mathbb{R}$ is integrable on $[0, 1]$.

   (d) Is every bounded function on $[a, b]$ integrable on $[a, b]$? Explain your answer.

[2+4+4+2 = 12 marks]

8. (a) State what it means for a sequence (f_n) of functions on $A \subset \mathbb{R}$ to converge to a function $f : A \to \mathbb{R}$

   (i) pointwise

   (ii) uniformly.

   (b) Let $f_n : [0, 1] \to \mathbb{R}$ be given by
   $$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

   (i) Find f, the pointwise limit of the sequence (f_n).

   (ii) Does $f_n \to f$ uniformly on $[0, 1]$ as $n \to \infty$?

[3+5 = 8 marks]