School of Mathematical Sciences

Assessment Cover Sheet

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Assessment Title	**Assignment 5**
Due Date	**Monday 22nd October, 2018 by 5:00pm**
Course	**STATS 3005 Time Series III**
Tutorial Group Number	**1**
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Lecturer	**Professor Patty Solomon**

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Time Series A5

Andrew Martin

October 20, 2018

1. AR(1) process:

$$Y_t = 0.8Y_{t-1} + Z_t$$

Where Z_t is a white noise process with $\text{var}(Z_t) = \sigma^2$

(a) Find the spectrum $f_y(\omega)$

Solution Y_t is an AR(1) process, so the spectrum has form:

$$f_y(\omega) = \frac{\sigma^2}{|\phi(e^{i\omega})|^2}$$

Where in this case $\phi(u) = (1 - 0.8u)$

$$\begin{aligned}
|\phi(e^{i\omega})|^2 &= |1 - 0.8e^{i\omega}|^2 \\
&= |1 - 0.8\cos(\omega) - 0.8i\sin(\omega)|^2 \\
&= (1 - 0.8\cos(\omega))^2 + (0.8\sin(\omega))^2 \\
&= 1 - 1.6\cos(\omega) + 0.64\cos^2(\omega) + 0.64\sin^2\omega \\
&= 1.64 - 1.6\cos(\omega)
\end{aligned}$$

Which gives

$$f_y(\omega) = \frac{\sigma^2}{1.64 - 1.6\cos(\omega)}$$

As required.

(b) Obtain a plot of the spectrum $f_y(\omega)$ for $\sigma^2 = 1$

Solution This is found in figure 1. The code is in the appendix

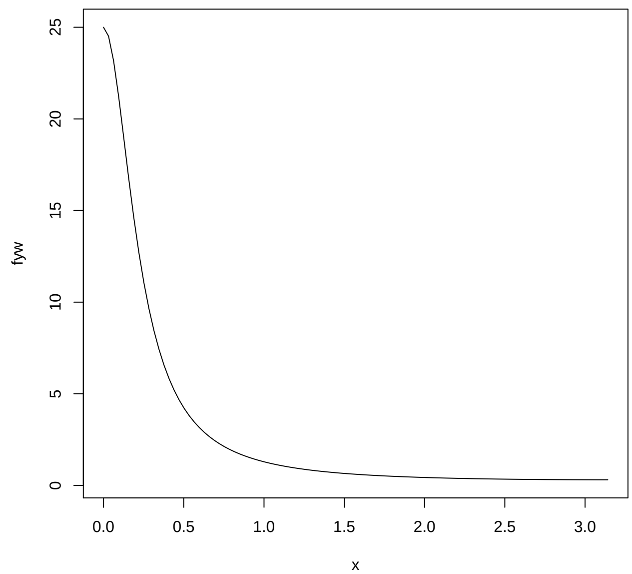

Figure 1: Plot of the spectrum $f_y(\omega) = \frac{1}{1.64 - 1.6\cos(\omega)}$

As required.

(c) Let $U_t = DY_t = (1 - B)Y_t$ be the process defined by differencing Y_t. Find the spectrum $f_u(\omega)$.

Solution If U_t is a linear filter of Y_t, with form:

$$U_t = \sum_{j=-\infty}^{\infty} a_j Y_{t-j}$$

then:

$$f_U(\omega) = |a(\omega)|^2 f_y(w)$$

Where

$$a(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{ij\omega}$$

In this case, we have

$$U_t = Y_t - Y_{t-1} = \sum_{j=-1}^{0} (-1)^{j-1} Y_{t-j}$$

This gives:

$$a(\omega) = \sum_{j=-1}^{0} (-1)^{j-1} e^{ij\omega} = e^{-i\omega} - 1$$

Giving:

$$\begin{aligned}
f_U(\omega) &= |a(\omega)|^2 f_y(\omega) \\
&= |e^{-i\omega} - 1|^2 \frac{\sigma^2}{1.64 - 1.6\cos(\omega)} \\
&= |\cos(\omega) - i\sin(\omega) - 1|^2 \frac{\sigma^2}{1.64 - 1.6\cos(\omega)} \\
&= ((\cos(\omega) - 1)^2 + \sin^2(\omega)) \frac{\sigma^2}{1.64 - 1.6\cos(\omega)} \\
&= (\cos^2(\omega) + \sin^2(\omega) - 2\cos(\omega) - 1) \frac{\sigma^2}{1.64 - 1.6\cos(\omega)} \\
&= \frac{-2\sigma^2\cos(\omega)}{1.64 - 1.6\cos(\omega)}
\end{aligned}$$

As required.

(d) Obtain a plot of $|a(\omega)|^2$, where $a(\omega)$ is the transfer function of the differencing operator. Would you describe this as a high pass filter or a low pass filter?

Solution This is plotted in figure 2. As we can see it is removing the low frequencies, and amplifying the high frequencies. This makes it a high-pass filter. The code is in the appendix

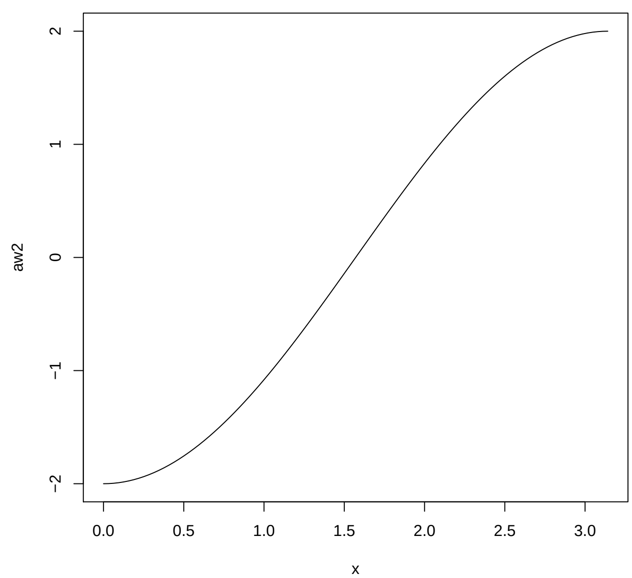

Figure 2: plot of $|a(\omega)|^2 = -2\cos(\omega)$

As required.

(e) Obtain a plot of the spectrum $f_u(\omega)$ for $\sigma^2 = 1$.

Solution This is plotted in figure 3. The code is in the appendix

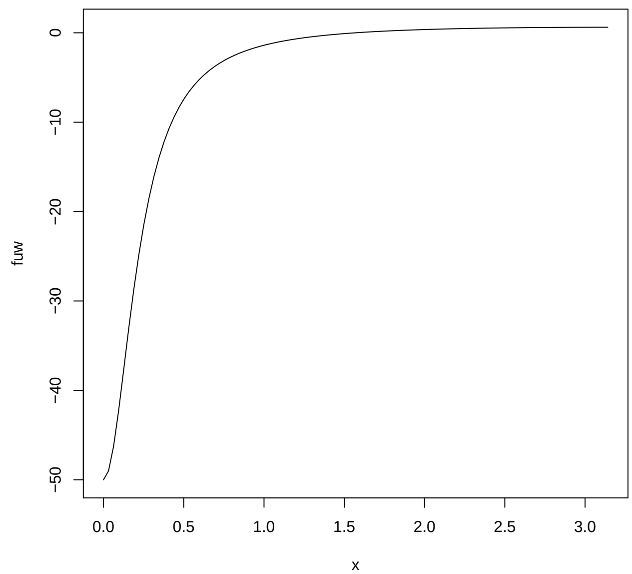

Figure 3: plot of $f_U(\omega) = \frac{-2\sigma^2\cos(\omega)}{1.64-1.6\cos(\omega)}$

As required.

2. Consider the MA(2) process:

$$Y_t = Z_t + 0.5Z_{t-1} + 0.5Z_{t-2}$$

(a) Find the spectrum by evaluating $f(\omega) = \sigma^2|\theta(e^{i\omega})|^2$

Solution

$$\begin{aligned}
\theta(u) &= 1 + 0.5u + 0.5u^2 \\
\theta(e^{i\omega}) &= 1 + 0.5e^{i\omega} + 0.5(e^{i\omega})^2 \\
|\theta(e^{i\omega})|^2 &= (1 + 0.5\cos(\omega) + 0.5\cos(2\omega))^2 + (0.5\sin(\omega) + 0.5\sin(2\omega))^2 \\
&= 1 + 0.25\cos^2(\omega) + 0.25\cos^2(2\omega) + \cos(\omega) + \cos(2\omega) + \cos(\omega)\cos(2\omega) \\
&\quad + 0.25\sin^2(\omega) + 0.25\sin^2(2\omega) + \sin(\omega)\sin(2\omega) \\
&= 1 + 0.25 + 0.25 + \cos(\omega) + \cos(2\omega) + \cos(\omega)\cos(2\omega) + \sin(\omega)\sin(2\omega) \\
&= 1.5 + \cos(\omega) + \cos(2\omega) + \cos(\omega)\cos(2\omega) + \sin(\omega)\sin(2\omega) \\
&= 1.5 + 2\cos(\omega) + \cos(2\omega) \\
&= 0.5 + 2\cos(\omega) + 2\cos^2(\omega) \\
&= 2(\cos(\omega) + 0.5)^2
\end{aligned}$$

$$\begin{aligned}
f(\omega) &= \sigma^2|\theta(e^{i\omega})|^2 \\
&= 2\sigma^2(\cos(\omega) + 0.5)^2
\end{aligned}$$

As required.

(b) Write down the autocovariance function directly

Solution

$$
\begin{aligned}
\gamma_k &= \text{cov}(Y_t, Y_{t+k}) \\
&= \text{cov}(Z_t + 0.5Z_{t-1} + 0.5Z_{t-2}, Z_{t+k} + 0.5Z_{t+k-1} + 0.5Z_{t+k-2}) \\
&= \text{cov}(Z_t, Z_{t+k}) + 0.5\text{cov}(Z_t, Z_{t+k-1}) + 0.5\text{cov}(Z_t, Z_{t+k-2}) + 0.5\text{cov}(Z_{t-1}, Z_{t+k}) + 0.25\text{cov}(Z_{t-1}, Z_{t+k-1}) \\
&\quad + 0.25\text{cov}(Z_{t-1}, Z_{t+k-2}) + 0.5\text{cov}(Z_{t-2}, Z_{t+k}) + 0.25\text{cov}(Z_{t-2}, Z_{t+k-1}) + 0.25\text{cov}(Z_{t-2}, Z_{t+k-2})
\end{aligned}
$$

If we let $g_k = \text{cov}(Z_t, Z_{t+k})$ and apply covariance laws, this gives:

$$
\begin{aligned}
\gamma_k &= g_k + 0.5g_{k-1} + 0.5g_{k-2} + 0.5g_{k+1} + 0.25g_k + 0.25g_{k-1} + 0.5g_{k+2} + 0.25g_{k+1} + 0.25g_k \\
&= 0.5g_{k+2} + 0.75g_{k+1} + 1.5g_k + 0.75g_{k-1} + 0.5g_{k-2}
\end{aligned}
$$

Note that $g_0 = \sigma^2$, and $g_k = 0$ for $k \geq 1$
This gives:

$$
\gamma_k = \begin{cases} 1.5\sigma^2, & k = 0 \\ 0.75\sigma^2, & k = \pm 1 \\ 0.5\sigma^2, & k = \pm 2 \\ 0, & |k| \geq 3 \end{cases}
$$

As required.

(c) Using the inversion formula, find γ_0.

Solution Using the nicer inversion formula:

$$
\begin{aligned}
\gamma_k &= \frac{1}{\pi}\int_0^\pi \cos(k\omega)f(\omega)d\omega \\
&= \frac{1}{\pi}\int_0^\pi \cos(k\omega)2\sigma^2(\cos(\omega)+0.5)^2 d\omega \\
&= \frac{2\sigma^2}{\pi}\int_0^\pi \cos(k\omega)(\cos(\omega)+0.5)^2 d\omega \\
\gamma_0 &= \frac{2\sigma^2}{\pi}\int_0^\pi (\cos(\omega)+0.5)^2 d\omega \\
&= \frac{2\sigma^2}{\pi}\int_0^\pi 0.25 + \cos(\omega) + \cos^2(\omega) d\omega \\
&= \frac{2\sigma^2}{\pi}\left(\int_0^\pi 0.25 d\omega + \int_0^\pi \cos(\omega)d\omega + \int_0^\pi \cos^2(\omega)d\omega\right) \\
&= \frac{2\sigma^2}{\pi}\left(0.25\pi + \int_0^\pi \frac{1}{2} + \cos(2\omega)d\omega\right) \\
&= \frac{2\sigma^2}{\pi}\left(0.25\pi + \frac{\pi}{2}\right) \\
&= \frac{2\sigma^2}{\pi}\left(0.25\pi + \frac{\pi}{2}\right) \\
\therefore \gamma_0 &= 1.5\sigma^2
\end{aligned}
$$

Which is the same as the answer to b. **As required.**

Code

The code used to produce the plots is below:

```
pdf(file="A5Plots.pdf")
fyw = function(w){1/(1.64 - 1.6*cos(w))}
plot(fyw,0,pi)

aw2 = function(w){-2*cos(w)}
plot(aw2,0,pi)

fuw = function(w){-2*cos(w)/(1.64 - 1.6*cos(w))}
plot(fuw,0,pi)

dev.off()
```