

## Random Processes III 2018: Assignment 1,

to be handed in at the beginning of the lecture (1pm) on Friday 10<sup>th</sup> August.

[39 marks in total]

Question 0. [4 marks]

Make sure that in all your answers you:

- (a) use full and complete sentences,
- (b) include units where necessary,
- (c) give ranges/constraints for parameters in formulae, and
- (d) make sure your layout and mathematical arguments are clear and concise.

Question 1. [8 marks]

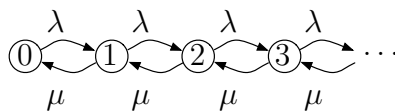
Recall the M/M/1 queue from Lecture 3. The state of the process  $\mathcal{X} = (X(t), t \geq 0)$  is the queue length at time  $t$ , including the person being served.

Assume there are arrivals to the queue according to a Poisson process with rate  $\lambda$ ; and, services occur as a Poisson process with rate  $\mu$  whenever there is at least one customer.

- The state space is  $\mathcal{S} = \{0, 1, 2, \dots\}$ , where each state  $i \in \mathcal{S}$  means there are  $i$  people in the system.
- The transition rates are:

$$\begin{aligned} q_{i,i+1} &= \lambda && \text{for } i \in \mathcal{S}, \\ q_{i,i-1} &= \mu && \text{for } i \in \mathcal{S} \setminus \{0\}, \\ q_{ij} &= 0 && \text{for } j \in \mathcal{S} \setminus \{i, i+1, i-1\}. \end{aligned}$$

- The state transition diagram is



- (a) Show that in state  $i \in \mathcal{S} \setminus \{0\}$ , the sojourn time is exponentially-distributed with rate  $\lambda + \mu$ .
- (b) Show that in state  $i \in \mathcal{S} \setminus \{0\}$ , the probability of an arrival at the time of an event is independent of the timing of the event and is equal to  $\lambda/(\lambda + \mu)$ .

Question 2. [12 marks]

Consider two time machines, maintained by a single genius. Machine  $i \in \{1, 2\}$  functions for an exponentially-distributed time with mean  $1/\gamma_i$  before it breaks down. Repair time for time machine  $i$  is exponentially-distributed with mean  $1/\beta_i$ . The genius can repair only one machine at a time.

Assume that the machines are repaired in the order in which they fail.

Construct a continuous-time Markov chain (CTMC) to model this system, where the state  $X(t)$  of the system represents the status of the machines at time  $t \geq 0$ , by following these steps.

- (a) Write down the state space of this CTMC.
- (b) Draw a state transition diagram corresponding to this CTMC.
- (c) Specify the generator of this CTMC.

Question 3. [3 marks]

Assume you are modelling an arrival process using a Poisson process, where the rate of arrivals is  $\lambda = 24$  per day.

- (a) Calculate the probability of seeing 2 arrivals in the first hour, and 5 arrivals in total during the second and third hours.

Question 4. [12 marks]

Consider Example 1 of Lecture 2, and considered again in Lecture 4.

- (a) Evaluate  $P_{01}(200)$ , the probability of having only 1 working printer 200 days after having both printers working, given the transition function in the lecture notes.
- (b) What happens to the transition function as  $t \rightarrow \infty$ ? Compare, appropriately, to your answer in (a).
- (c) Evaluate  $\exp(Q * 200)$ , where  $\exp$  is the matrix exponential – for example, in MATLAB you can use `expm`. Compare to your answer in (b).
- (d) Modify the simulation code `Repairman2.m` provided as part of Lecture 4 (or write your own code, in whatever language you like) to estimate  $P_{01}(200)$ . Provide details of your inputs. Compare to your answer in (a).