Assignment 1, Mathematical Statistics III

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1. Suppose $X \sim geom(p)$ with probability function:

$$p(x) = p(1-p)^x \text{ for } x = 0, 1, 2, \ldots$$

Prove that $var(X) = (1-p)/p^2$

Solution

$$\begin{aligned}
var(X) &= E(X^2) - E(X)^2 \text{ from lectures} \\
M(t) &= \frac{p}{1-(1-p)e^t} \text{ from lectures} \\
M'(t) &= \frac{(1-p)pe^t}{((p-1)e^t+1)^2} \\
M''(t) &= p\left(\frac{2(1-p)^2e^{2t}}{(1-(1-p)e^t)^3} + \frac{(1-p)e^t}{(1-(1-p)e^t)^2}\right) \\
M''(0) = E(X^2) &= p\left(\frac{2(1-p)^2}{(1-(1-p))^3} + \frac{(1-p)}{(1-(1-p))^2}\right) \\
&= p\left(\frac{2(1-p)^2}{p^3} + \frac{(1-p)}{p^2}\right) \\
&= \frac{2(1-p)^2}{p^2} + \frac{p(1-p)}{p^2} = \frac{2(1-p)^2 + p(1-p)}{p^2}
\end{aligned}$$

And given that $E(X) = \frac{1-p}{p}$, this gives:

$$\begin{aligned}
var(X) &= E(X^2) - E(X)^2 \\
&= \frac{2(1-p)^2 + p(1-p)}{p^2} - \left(\frac{1-p}{p}\right)^2 \\
&= \frac{2(1-p)^2 + p(1-p) - (1-p)^2}{p^2} \\
&= \frac{(1-p)^2 + p(1-p)}{p^2} \\
&= \frac{1-2p+p^2+p-p^2}{p^2} = \frac{1-p}{p^2}
\end{aligned}$$

As Required

2. Consider the binomial distribution with probability function

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

(a) Show directly that $E(X) = np$ (without using the MGF)

Solution Binomial distribution is discrete, so:

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} x p(x) \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x q^{(n-x)} \text{ where } q = 1-p \\
&= \sum_{x=1}^{n} np \frac{(n-1)!}{(x-1)!(n-x)!} p^{(x-1)} q^{(n-x)} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{(x-1)} q^{(n-x)}
\end{aligned}$$

The binomial theorem states that for $m > 0$

$$(a+b)^m = \sum_{r=0}^{m} \binom{m}{r} a^r b^{m-r}$$

In this case:

$a = p$, $b = q$, $m = n-1$, $r = x-1$, i.e.:

$$(p+q)^{n-1} = \sum_{x=1}^{n-1} \binom{n-1}{x-1} p^{(x-1)} q^{(n-x)}$$

$$\begin{aligned}
E(X) &= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{(x-1)} q^{(n-x)} \\
&= np(p+q)^{n-1} \\
&= np(p+(1-p))^{n-1} \\
&= np\ 1^{n-1} \\
&= np
\end{aligned}$$

As Required

(b) Show directly that $var(X) = np(1-p)$ (without using the MGF)

Solution

$$\begin{aligned} var(X) &= E\left((X-E(X))^2\right) \\ &= E(X(X-1)) + E(X) - E(X)^2 \text{ from lectures} \\ &= E(X(X-1)) + np - (np)^2 \end{aligned}$$

Solve $E[X(X-1)]$:

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)p(x) \\ &= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!} p^x(1-p)^{n-x} \\ &= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x(1-p)^{n-x} \\ &= \sum_{x=2}^{n} n(n-1)p^2\frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2}(1-p)^{n-x} \\ &= n(n-1)p^2\sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2}(1-p)^{n-x} \end{aligned}$$

Once again this can be reduced using the binomial formula, with $a = p$, $b = 1-p$, $m = n-2$ and $r = x-2$

$$\begin{aligned} E[X(X-1)] &= n(n-1)p^2\sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2}(1-p)^{n-x} \\ &= n(n-1)p^2(p+(1-p))^{n-2} \\ &= n(n-1)p^2\ 1^{n-2} \\ &= np(np-p) \end{aligned}$$

Now use this in the variance formula:

$$\begin{aligned} var(X) &= E(X(X-1) + E(X) - E(X)^2 \\ &= np(np-p) + np - (np)^2 \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= np - np^2 \\ &= np(1-p) \end{aligned}$$

As Required

(c) Consider the moment generating function, $M_n(t)$ for the binomial distribution with parameters n and p_n and suppose

$$n \to \infty;\ p_n \to 0 \text{ such that } np_n = \mu > 0$$

Find $\lim_{n\to\infty} M_n(t)$ and interpret the result.

Solution Rearrange $np_n = \mu$ to $p_n = \frac{\mu}{n}$ The moment generating function, as given in the lecture notes is:

$$M_n(t) = \big(1 + p_n(e^t - 1)\big)^n$$

Taking the limit:

$$\lim_{n\to\infty} M_n(t) = \lim_{n\to\infty} \big(1 + p_n(e^t - 1)\big)^n$$

Using the binomial theorem as above, letting $a = p_n(e^t - 1)$, $b = 1$ and $m = n$

$$\begin{aligned}
\lim_{n\to\infty} M_n(t) &= \lim_{n\to\infty} \big(1 + p_n(e^t - 1)\big)^n \\
&= \lim_{n\to\infty} \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} (p_n(e^t - 1))^r \\
&= \lim_{n\to\infty} \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} \sum_{s=0}^{r} \binom{r}{s} (p_n e^t)^{r-s} (-p_n)^r \\
&= \lim_{n\to\infty} \sum_{r=0}^{n} \binom{n}{r} \sum_{s=0}^{r} \binom{r}{s} (p_n e^t)^{r-s} (-p_n)^r \\
&= \lim_{n\to\infty} \sum_{r=0}^{n} \binom{n}{r} \sum_{s=0}^{r} \binom{r}{s} (\frac{\mu}{n} e^t)^{r-s} (-\frac{\mu}{n})^r \\
&= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \sum_{s=0}^{r} \frac{r!}{s!(r-s)!} (\frac{\mu}{n} e^t)^{r-s} (-\frac{\mu}{n})^r
\end{aligned}$$

As Required

3. Suppose $X \sim Gamma(\alpha, \lambda)$

(a) Show directly that $E(X) = \alpha/\lambda$ (without using the MGF)

Solution Recall the PDF of the gamma distribution is:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

And the identity $\Gamma(a+1) = a\Gamma(a)$

$$\begin{aligned} E(X) &= \int_0^\infty xf(x)dx \\ &= \int_0^\infty x\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}dx \\ &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)}x^\alpha e^{-\lambda x}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\lambda x}dx \end{aligned}$$

Here we can rearrange the PDF to make x^α the subject.

$$x^\alpha = f(x)\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}\frac{1}{e^{-\lambda x}}$$

$$\begin{aligned} E(X) &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\lambda x}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty f(x)\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}\frac{1}{e^{-\lambda x}}e^{-\lambda x}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}\int_0^\infty f(x)dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}(1) \text{ as this is the CDF taken over the whole domain} \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\alpha\Gamma(\alpha)}{\lambda^{\alpha+1}} \text{ due to the gamma identity above} \\ &= \frac{\alpha}{\lambda} \end{aligned}$$

As Required

(b) Show directly that $var(X) = \alpha/\lambda^2$ (without using the MGF)

Solution

$$var(X) = E(X^2) - E(X)^2$$

$$
\begin{aligned}
E(X^2) &= \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\lambda x} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty f(x) \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}} \frac{1}{e^{-\lambda x}} e^{-\lambda x} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty f(x) \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\lambda^{\alpha+2}} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\lambda^{\alpha+2}} \int_0^\infty f(x) dx \\
&= \frac{(\alpha+1)(\alpha)}{\lambda^2}
\end{aligned}
$$

Using this:

$$
var(X) = E(X^2) - E(X)^2 = \frac{(\alpha+1)(\alpha)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}
$$

As Required

(c) Show that the MGF is given by

$$
M(t) = \left(\frac{\lambda}{\lambda - t} \right)^\alpha
$$

for $t < \lambda$

Solution

$$
\begin{aligned}
M(t) &= E\left[e^{tX}\right] \\
&= \int_{-\infty}^\infty e^{tx} f(x) dx \\
&= \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \\
&= \lambda^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} e^{tx} x^{\alpha-1} e^{-\lambda x} dx \\
&= \lambda^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} e^{x(t-\lambda)} x^{\alpha-1} dx \\
&= \lambda^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} e^{-x(\lambda-t)} x^{\alpha-1} dx \\
&= \lambda^\alpha \int_0^\infty \left(\frac{\lambda-t}{\lambda-t} \right)^\alpha \frac{1}{\Gamma(\alpha)} e^{-x(\lambda-t)} x^{\alpha-1} dx \\
&= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} e^{-x(\lambda-t)} x^{\alpha-1} dx
\end{aligned}
$$

Which using the PDF of the gamma distribution is:

$$= \frac{\lambda^\alpha}{(\lambda - t)^\alpha} \int_0^\infty f(x)dx$$
$$= \left(\frac{\lambda}{(\lambda - t)}\right)^\alpha$$

As Required

4. Suppose $U \sim U(0,1)$ and let $X = \sqrt{U}$

(a) Find the PDF of X

Solution Theorem 2 from lectures states if U is continuous with PDF $f_U(u)$ with $X = h(U)$, where h is differentiable and monotonic, then the PDF of X is:

$$f_X(x) = f_U\left(h^{-1}(x)\right)|h^{-1}(x)'|$$

In this case, $U(u) = 1\ \forall u, \quad h(U) = \sqrt{U}, \implies h^{-1}(x) = x, \implies h^{-1}(x)' = 1$

$$\begin{aligned} f_X(x) &= f_U\left(h^{-1}(x)\right)|h^{-1}(x)'| \\ &= f_U(x^2)|2x| \\ &= |2x|\frac{1}{b-a} = 2x, \quad a \le x^2 \le b \end{aligned}$$

As Required

(b) Calculate $E(X)$ directly from its PDF and also from the distribution of U and check that the answers agree.

Solution From the PDF:

$$\begin{aligned} E(X) &= \int_{\sqrt{a}}^{\sqrt{b}} x2xdx \\ &= \int_0^1 2x^2dx \\ &= \frac{2}{3}x^3|_0^1 \\ &= 2/3 \end{aligned}$$

The other way:

$$E(X) = E(\sqrt{U}) = \int_a^b \sqrt{u}\frac{1}{b-a}du = \int_0^1 \sqrt{u}du$$
$$= \frac{2}{3}u^{3/2}|_0^1 = 2/3 - 0 = 2/3$$

The two match up. **As Required**

5. Suppose $U \sim U(0,1)$ and let $X = 3U + 2$

   (a) Find the MGF of X

   Solution $h(u) = 3u + 2$, $h^{-1}(u) = \frac{u-2}{3}$, $h^{-1}(u)' = \frac{1}{3}$.

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= E[e^{th(U)}] \\
&= \int_{-\infty}^{\infty} e^{th(U)} f(u) du \\
&= \int_0^1 e^{t(3u+2)} 1 du \\
&= e^{2t} \int_0^1 e^{3ut} du \\
&= e^{2t} \frac{1}{3t} \left(e^{3ut}\right) |_0^1 \\
&= \frac{e^{2t}\left(1 + e^{3t}\right)}{3t} \\
&= \frac{e^{2t} + e^{5t}}{3t}
\end{aligned}
$$

   As Required

   (b) Hence, identify the distribution of X.

   Solution X shares the moment generating function of the $U(2,5)$ distribution I.e. $X \sim U(2,5)$.

   As Required

6. Suppose $Z \sim N(0,1)$

   (a) Show that $E(Z) = 0$

   Solution

$$
\begin{aligned}
E(Z) &= \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{\frac{-z^2}{2}} dz
\end{aligned}
$$

Integration by substitution: $u = -z^2/2$ and $du/dz = -z$

$$\begin{aligned}
E(Z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -e^u du \\
&= \frac{-1}{\sqrt{2\pi}} e^u|_{-\infty}^{\infty} \\
&= \frac{-1}{\sqrt{2\pi}} e^{-z^2/2}|_{-\infty}^{\infty} \\
&= \frac{-1}{\sqrt{2\pi}} \left(e^{-\infty} - e^{-\infty}\right) \\
&= 0 - 0 = 0
\end{aligned}$$

As Required

(b) Show that $var(Z) = 1$

Solution

$$\begin{aligned}
Var(Z) &= E(Z^2) - E(Z)^2 \\
&= E(Z^2) - 0 \\
&= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz
\end{aligned}$$

First use integration by parts, $v' = ze^{-z^2/2}$, $u = z$, $u' = 1$, $v = -e^{-z^2/2}$ (reverse of above)

$$\begin{aligned}
\implies Var(Z) &= \frac{1}{\sqrt{2\pi}} \left(uv'|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} vu' dz \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(z^2 e^{-z^2/2}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-z^2/2} dz \right) \\
&= \frac{-1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} -e^{-z^2/2} dz \right) \quad \textit{the uv' function is symmetric} \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \\
&= 1
\end{aligned}$$

As Required

(c) Derive the MGF, $M(t)$

Solution

$$\begin{aligned}
M(t) &= E[e^{tZ}] \\
&= \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{2tz-z^2}{2}}dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{2tz-z^2+t^2-t^2}{2}}dz \\
&= e^{t^2/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{2tz-z^2-t^2}{2}}dz \\
&= e^{t^2/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-(t-z)^2}{2}}dz \\
&= e^{t^2/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dz \text{ letting } y = t-z \\
&= e^{t^2/2} * 1 \text{ as the integral was over the pdf} \qquad = e^{t^2/2}
\end{aligned}$$

As Required

7. Suppose $X \sim N(\mu, \sigma^2)$ and let $Y = aX + b$ for constants a, b with $a \neq 0$ Prove that $Y \sim N(a\mu + b, a^2\sigma^2)$

   Solution Expectation:

$$\begin{aligned}
E(Y) &= E(aX + b) \\
&= E(aX) + E(b) \\
&= aE(X) + b \\
&= a\mu + b
\end{aligned}$$

   Variance:

$$\begin{aligned}
var(Y) &= var(aX + b) \\
&= var(aX) \\
&= a^2 var(X) \\
&= a^2\sigma^2
\end{aligned}$$

   PDF: (show that it has the same distribution) Denote $\sigma_y^2 = a^2\sigma^2$ and $\mu_y = a\mu + b$ $h(X) = aX + b \implies h^{-1}(Y) = \frac{y-b}{a}$ $h^{-1}(y)' = \frac{1}{a}$ (and h is monotonic)

$$
\begin{aligned}
f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \\
&= \frac{1}{|a|}\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(\frac{y-b}{a}-\mu)^2}{2\sigma^2}) \\
&= \frac{1}{\sqrt{2\pi}\sigma a}\exp(-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}) \text{ absolute value disappears because} \sqrt{a^2} = a \\
&= \frac{1}{\sqrt{2\pi}\sigma a}\exp(-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}) \\
&= \frac{1}{\sqrt{2\pi}\sigma_y}\exp(-\frac{(y-\mu_y)^2}{2\sigma_y^2})
\end{aligned}
$$

Which is the standard normal with variance $a^2\sigma^2$ and expectation $a\mu + b$

As Required

Honours Questions

8. Suppose X is Cauchy distributed. Find the distribution of $Y = 1/X$

   Solution Let $X \sim Cauchy(x)$
   pdf: $f(x) = \frac{1}{x} \times \frac{1}{1+x^2}$
   cdf: $F(x)\frac{1}{2} + \frac{1}{\pi}\arctan(x)$
   $Y = h(X)$, $h(X) = 1/X$, $h^{-1}(X) = ln(X)$ Moment generating function of Y:

$$
\begin{aligned}
M_Y(t) &= E(e^{th(X)}f(x)) \\
&= \int_{-\infty}^{\infty} e^{\frac{t}{X}}\frac{1}{x} \times \frac{1}{1+x^2}dx \\
&= \int_{-\infty}^{\infty} e^{\frac{t}{X}}\frac{1}{x+x^3}dx
\end{aligned}
$$

   PDF of Y:

$$
\begin{aligned}
f_Y(y) &= f_X(h^{-1}(y))|h^{-1}(y)'| \\
&= ln(x)(x+x^3)
\end{aligned}
$$

   Cant use CDF as Cauchy is not continuous. **As Required**

9. Consider the Poisson process with rate λ and suppose it is given that there is exactly 1 occurrence in the interval $[0, t)$. Show that conditionally on this information, the exact time, X, of the occurrence is $U(0, t)$.
Hint: Find the conditional CDF of X using the usual definition of conditional probability.

Solution The Poisson process with rate $\lambda > 0$ is a point process on $[0, \infty)$ which satisfies the following axioms:

(a) The numbers of occurrences in disjoint intervals are independent

(b) The probability of **1 or more** occurrences in any interval $[t, t+h)$ is $\lambda h + o(h)$ as $h \to 0$

(c) The probability of **more than one** occurrence in any interval $[t, t+h)$ is $o(h)$ as $h \to 0$.

We are given that the first interval $[0, t)$ contains an occurrence.

As Required