

School of Mathematical Sciences

MATHEMATICAL BIOLOGY (HONOURS)

Assignment 1 question sheet

*Due: Friday, 30 August, by 11am
(bring to lecture, or leave in box on office door)*

1. **Dimensional analysis of rowing** A boat carries N similar rowers, each of whom we assume puts the same power, P , into propelling the boat.

- (a) Assuming that they each require the same volume of boat, V , to accommodate them, show that the wetted area of the boat is $A \propto (NV)^{\frac{2}{3}}$.
- (b) Assume that the drag force on the boat as it moves through the water depends on the wetted area of the boat, A , its speed U , and the density of the water, ρ . Hence show the drag force must be proportional to $\rho U^2 A$, and the rate of energy dissipation due to drag must be proportional to $\rho U^3 A$.
- (c) Hence show

$$U \propto N^{\frac{1}{9}} P^{\frac{1}{3}} \rho^{-\frac{1}{3}} V^{-\frac{2}{9}}.$$

- (d) If we make the further crude assumption that both P and V are proportional to body mass, would we expect size to be an advantage to a rower?

[4 marks]

2. Is it true that water flows out of a bathtub with a clockwise swirl in the southern hemisphere, and an anticlockwise swirl in the northern hemisphere?

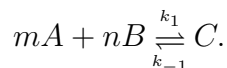
Consider a fluid flow subject to the force of gravity. In a coordinate system rotating with angular velocity, $\mathbf{\Omega}$, the dimensional Navier-Stokes equations become

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \underbrace{2\mathbf{\Omega} \times \mathbf{u}}_{\text{Coriolis acceleration}} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \underbrace{\rho(\mathbf{g} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}))}_{\text{apparent gravity}},$$

where here we will take the origin of the coordinate system to be the centre of the Earth.

The angular velocity, $\mathbf{\Omega}$, of the Earth is 2π per 24 hours, or $7.3 \times 10^{-5} \text{ s}^{-1}$ (where $\mathbf{\Omega}$ points in the direction of the Earth's axis of rotation), and the radius of the earth is around 6,400 km. Now assume that you are dealing with a bathtub with a typical lengthscale of around 1m, and the water flows at a typical speed of around 1 m s^{-1} . (You can take the density of water to be 1000 kg m^{-3} and its viscosity to be $8.9 \times 10^{-4} \text{ Pa s}$.) By nondimensionalising, determine if the Coriolis acceleration will be significant (and hence, if the swirl direction is likely to change depending on which hemisphere you are in). [4 marks]

3. Consider the following reaction where m molecules of A and n molecules of B react to produce the product, C :



- (a) In lectures, we discussed that the equation for the concentration of the product, c would take the form:

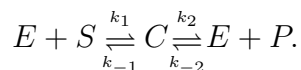
$$\frac{dc}{dt} = k_1 a^m b^n - k_{-1} c.$$

Write down the corresponding equations for a and b , the concentrations of the two reactants.

- (b) Find the two conserved quantities, and hence eliminate a and b from the equation for c above.

[4 marks]

4. **(An enzymatic reaction with reversible product formation)** Consider the Michaelis-Menten reaction from lectures, but relax the assumption that the reaction in which the product and enzyme are formed from the complex is irreversible. Hence the reaction scheme is now



- (a) Write down a modified set of equations for the concentrations s , e , c and p , based on the scheme above. The initial conditions are:

$$s(0) = s_i > 0, \quad e(0) = e_i = \epsilon s_i > 0, \quad c(0) = p(0) = 0.$$

- (b) Obtain a conservation law, and hence eliminate e from the system.
- (c) Nondimensionalise the system of three equations as in lecture, with $\epsilon = e_i/s_i \ll 1$. Show the scaled system involves 3 dimensionless parameters, which you should give in terms of the original parameters.
- (d) By neglecting terms of $O(\epsilon)$ or smaller, find a leading-order expression for the dimensionless complex concentration, and hence show that the approximate dimensionless reaction velocity takes the form

$$\frac{d\tilde{p}}{d\tilde{t}} = \frac{A_3 \tilde{s} - A_1 A_2 \tilde{p}}{\tilde{s} + A_2 \tilde{p} + A_1 + A_3},$$

where the A_i are constants.

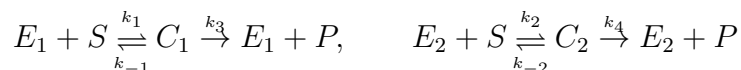
- (e) Show that, at leading order, the steady state ratio of product and substrate concentrations is

$$\frac{\bar{p}}{\bar{s}} = \frac{k_1 k_2}{k_{-1} k_{-2}}.$$

(Note: This is known as the Haldane relationship.)

[12 marks]

5. **(A substrate that can be broken down by two different enzymes)** Consider the following system of reactions:



- (a) Write down the system of equations for the concentrations of S , E_1 , E_2 , C_1 , C_2 and P . Show that there are two conserved quantities, and hence reduce your system to three equations involving only the concentrations of S , C_1 and C_2 .
- (b) Assume that the initial conditions for your system of equations are:

$$s(0) = s_i > 0, \quad e_1(0) = e_2(0) = e_i = \epsilon s_i > 0.$$

Nondimensionalise your system of three equations, to obtain a system of the form:

$$\begin{aligned} \frac{ds}{dt} &= -s(1 + \alpha) + c_1(\mu_1 + s) + \alpha c_2(\mu_2 + s) \\ \epsilon \frac{dc_1}{dt} &= s(1 - c_1) - \lambda_1 c_1 \\ \epsilon \frac{dc_2}{dt} &= \alpha[s(1 - c_2) - \lambda_2 c_2]. \end{aligned}$$

You define the dimensionless parameters in terms of the original dimensional parameters.

- (c) Find the leading-order solution for c_1 and c_2 , and hence s (Using technology to help is fine).
- (d) By making a suitable rescaling of time, find the leading-order inner (short-time) solution for s , c_1 and c_2 .

[16 marks]

Total: 40 marks