Stochastic Assignment 2

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Question 1. Marks for mathematical language

Question 2. Consider the two-stage SLP

$$
\begin{aligned}
\max \quad & \left\{ 2x_1 + 2x_2 + \sum_{k=1}^{3} p_k q^T y^{(k)} \right\} && (1)\\
\text{subject to} \quad & x_1 + x_2 \leq 10 && (2)\\
& x_1 + 2x_2 - y_1^{(k)} + y_2^{(k)} + 4y_3^{(k)} = h_1^{(k)} && (3)\\
& x_1 - x_2 + 2y_1^{(k)} + 3y_2^{(k)} + 2y_3^{(k)} = h_2^{(k)} && (4)\\
& x_j, y_\nu^{(k)} \geq 0 \text{ for } j = 1,2;\ k = 1,2,3;\ \nu = 1,2,3 && (5)
\end{aligned}
$$

With $q = (-2, -3, -2)^T$, $h^{(1)} = (5, 4)^T$, $h^{(2)} = (3, 5)^T$ and $h^{(3)} = (2, 2)^T$

(a) Does this SLP [have] the complete recourse property? Justify.

Solution It has complete recourse if it has $rank(W) = m_1$ and if the linear constraints

$$
\begin{aligned}
Wy &= 0\\
y_i &\geq 1 \text{ for } i = 1, \dots, m_1\\
y &\geq 0
\end{aligned}
$$

Have a feasible solution. Where W is a $m_1 \times \bar{n}$ recourse matrix. In this case,

$$W = \begin{pmatrix} -1 & 1 & 4 \\ 2 & 3 & 2 \end{pmatrix}$$

Clearly W has rank 2 (this is verified in the `Matlab` code as well) Using `Matlab` gives:

```
>> W = [-1,1,4;2,3,2]

W =

    -1     1     4
     2     3     2

     >> rank(W)

ans =

     2

>> null(W)

ans =

    0.6667
```

```
-0.6667
 0.3333
```

So the first condition is satisfied, and now we want some λ such that $\lambda * \mathrm{null}(W) \geq 0$. Clearly this is not possible. Since W does not satisfy the second condition, W is not a complete recourse matrix, and the SLP does not have the complete recourse property. **As required.**

(b) Determine the induced constraints and the corresponding induced feasibility set

Solution

$$T = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

Constraints which must be satisfied for all ϵ are

$$W\mathbf{y} = h(\epsilon) - T\mathbf{x}, y \geq 0$$

Want to find

$$K = \{x | W\mathbf{y} = h(\epsilon) - T\mathbf{x}, y \geq 0\}$$

Firstly break W into its columns

$$\begin{aligned} Wy &= W_{:,1}y_1 + W_{:,2}y_2 + W_{:,3}y_3 \\ &= W_{:,1}y_1 + (W_{:,1} + \frac{1}{2}W_{:,3})y_2 + W_{:,3}y_3 \\ &= W_{:,1}(y_1 + y_2) + W_{:,3}(\frac{1}{2}y_2 + y_3) \end{aligned}$$

Since we force $y \geq 0$:

$$Wy = W_{:,1}\lambda + W_{:,3}\mu, \quad \lambda, \mu \geq 0$$

$$Wy = h(\epsilon) - Tx \iff W_{:,1}\lambda + W_{:,3}\mu = h(\epsilon) - Tx$$

This then gives the (large) set of equations

$$(-1, 2)^T\lambda + (4, 2)^T\mu = (\epsilon_1, \epsilon_2)^T - \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} x$$

Which results in the equations

$$\begin{aligned} \epsilon_1 - x_1 - 2x_2 &= -\lambda + 4\mu \\ \epsilon_2 - x_1 + x_2 &= 2\lambda + 2\mu \end{aligned}$$

We want to remove λ from one equation, and μ from the other. This results in

$$\begin{aligned} \epsilon_1 - 2\epsilon_2 + x_1 - 4x_2 &= -5\lambda \\ 2\epsilon_1 + \epsilon_2 - 3x_1 - 3x_2 &= 10\mu \end{aligned}$$

Since $\lambda, \mu \geq 0$ we can rewrite this as

$$\begin{aligned} \epsilon_1 - 2\epsilon_2 + x_1 - 4x_2 &\leq 0 \\ 2\epsilon_1 + \epsilon_2 - 3x_1 - 3x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1 - 4x_2 &\leq -\epsilon_1 + 2\epsilon_2 \\ 3x_1 + 3x_2 &\leq 2\epsilon_1 + \epsilon_2 \end{aligned}$$

Now subbing in the h values: $h^{(1)} = (5,4)^T$, $h^{(2)} = (3,5)^T$ and $h^{(3)} = (2,2)^T$

$$\begin{aligned} x_1 - 4x_2 &\leq -5 + 8 = 3 \\ 3x_1 + 3x_2 &\leq 10 + 4 = 14 \\ x_1 - 4x_2 &\leq -3 + 10 = 7 \\ 3x_1 + 3x_2 &\leq 6 + 5 = 11 \\ x_1 - 4x_2 &\leq -2 + 4 = 2 \\ 3x_1 + 3x_2 &\leq 4 + 2 = 6 \end{aligned}$$

Which reduces to

$$x_1 - 4x_2 \leq 2 \tag{6}$$
$$x_1 + x_2 \leq 2 \tag{7}$$

This are the induced constraints.

As required.

(c) Does this SLP have the relatively complete recourse property? Justify.

Solution

Using (b), it would have relatively complete recourse if the resulting induced feasibility set would still encapsulate the feasible region. If we consider equation 7:

$$x_1 + x_2 \leq 10$$

and compare to the initial constraint 2:

$$x_1 + x_2 \leq 2$$

Since $x_1, x_2 \geq 0$, the induced constraint region is *clearly* a subset of this region. As such, we get $K \cap X \subset X$, and the SLP does not have the relatively complete recourse property.

As required.

Question 3. Suppose that the recourse function $Q(x, \zeta)$ is given by

$$Q(x, \zeta) = \min_y \{y | y \geq \zeta, y \geq x\}$$

Where $x > 0$ and the RV ζ has density

$$f_\zeta(w) = \frac{2}{\omega^3}, \quad \omega \geq 1$$

(a) Determine the closed form expression for $\mathbb{E}_\zeta[Q(x, \zeta)]$.

Solution Consider $Q(x, \zeta)$

$$\begin{aligned} Q &= \min_y \{y | y \geq \zeta, y \geq x\} \\ Q &= \max\{\zeta, x\} \end{aligned}$$

Since the minimium will be the y equal to the greater of ζ and x.

$$\begin{aligned} \mathbb{E}_\zeta[Q(x, \zeta)] &= \int_{S_\zeta} Q(x, \zeta) f_\zeta(\zeta) d\zeta \\ &= \int_1^\infty \max\{\zeta, x\} \frac{2}{\omega^3} d\omega \end{aligned}$$

The max can be broken into two by separating the integral.

$$\begin{aligned}\mathbb{E}_\zeta[Q(x,\zeta)] &= \int_1^x x\frac{2}{\omega^3}d\omega + \int_x^\infty \omega\frac{2}{\omega^3}d\omega \\ &= \int_1^x x\frac{2}{\omega^3}d\omega + \int_x^\infty \frac{2}{\omega^2}d\omega \\ &= x\left[-\frac{1}{\omega^2}\right]_1^x + \left[-\frac{2}{\omega}\right]_x^\infty \\ &= x(-\frac{1}{x^2}+1) + \frac{2}{x} \\ &= x + \frac{1}{x}\end{aligned}$$

As required.

(b) Consequently, solve

$$\min_{x\geq 0}\{x + \mathbb{E}_\zeta[Q(x,\zeta)]\}$$

Solution

$$\begin{aligned}\min_{x\geq 0}\{x + \mathbb{E}_\zeta[Q(x,\zeta)]\} &= \min_{x\geq 0}\{x + x + \frac{1}{x}\} \\ &= \min_{x\geq 0}\{2x + \frac{1}{x}\}\end{aligned}$$

Find critical points for $x \geq 0$

$$\frac{\partial}{\partial x}(2x + \frac{1}{x}) = 2 - \frac{1}{x^2}$$
$$2 - \frac{1}{x^2} = 0$$
$$\implies x = \pm\frac{1}{\sqrt{2}}$$

We only care for positive values of x

$$x = \frac{1}{\sqrt{2}}$$

Check that this is a minimum

$$\frac{\partial^2}{\partial x^2}(2 - \frac{1}{x^2}) = \frac{2}{x^3} > 0 \text{ for } x > 0$$

Plugging this into the min function gives

$$\min_{x\geq 0}\{x + \mathbb{E}_\zeta[Q(x,\zeta)] = \frac{2}{\sqrt{2}} + \sqrt{2} = 2\sqrt{2}$$

As required.

Question 4. Qantas would like to determine how to best partition their new plane for a direct Adelaide-Paris route. The plane can accommodate 200 economy passengers. A portion of the plane can be reserved for first class seats, each of which takes up twice the space of an economy seat. A business section can also be considered, but each business seat takes up 1.5 times as much space as an economy seat. Once the plane is designed to section into these three classes, it is permanent.

A first class seat and a business seat generate three times and two times, respectively, as much profit as does an economy seat.

Clearly, for each flight the plane will not necessarily be full in every section. There are three scenarios, each happens with equal probability: (a) weekday morning and evening traffic, (b) weekend traffic, and (c) weekday midday traffic.

For Scenario (a), it is expected that up to 20 first class, 50 business, and 200 economy seats can be sold. For Scenario (b), the numbers are 10, 25 , and 175, respectively. For Scenario (c), 5, 10, and 150, respectively. We assume that they cannot sell more tickets than seats, for each section. (In reality overbooking happens all the time!)

(a) Formulate the above problem as a SLP, where Qantas wants to maximise the expected profit.

Solution Breaking down the problem:

- At most 200 economy seats.
- First class seats equal 2 economy seats (in size), and are worth 3 times as much as economy.
- Business seats equal 1.5 economy seats, and are worth 2 times as much as economy.
- Scenario (a): 20 first class, 50 business, 200 economy
- Scenario (b): 10 first class, 25 business, 175 economy
- Scenario (c): 5 first class, 10 business, 150 economy
- Scenarios all have same probability
- Cannot sell more tickets than seats
- Want to adjust the number of seats to maximise profit

Let x_1, x_2, x_3 denote the number of economy, business and first class seats respectively on the plane, and y_1, y_2, y_3 be the number sold respectively. $h^{(1)} = (200, 50, 20)^T$, $h^{(2)} = (175, 25, 10)^T$, $h^{(3)} = (150, 10, 5)^T$

The expected profit SLP will have form:

$$\begin{aligned} \max \left\{ z = \frac{1}{3} \sum_{k=1}^{3} y_1^{(k)} + 2y_2^{(k)} + 3y_3^{(k)} \right\} \\ \text{subject to} \quad x_1 + 1.5x_2 + 2x_3 &\leq 200 \\ y_j^{(k)} &\leq h_j^{(k)} \\ y_j^{(k)} &\leq x_j \\ x_j, y_\nu^{(k)} &\geq 0 \text{ for } j = 1,2,3;\ k = 1,2,3;\ \nu = 1,2,3 \end{aligned}$$

Converting this into standard form

$$\begin{aligned} \min \left\{ -z = -\frac{1}{3} \left(\sum_{k=1}^{3} y_1^{(k)} + 2y_2^{(k)} + 3y_3^{(k)} \right) \right\} \\ \text{subject to} \quad x_1 + 1.5x_2 + 2x_3 &\leq 200 \\ y_j^{(k)} &\leq h_j^{(k)} \\ -x_j + y_j^{(k)} &\leq 0 \\ x_j, y_\nu^{(k)} &\geq 0 \text{ for } j = 1,2,3;\ k = 1,2,3;\ \nu = 1,2,3 \end{aligned}$$

As required.

(b) Solve the above SLP using MATLAB. In addition to the solution, include code, output, and an interpretation of the solution

Solution As shown in the code and output we see that the optimal solution is

$$[x, y] = \big(150, 20, 10, 150, 20, 10, 150, 20, 10, 150, 10, 5\big)$$

I.e. the optimal set up of the plane is to design the plane with 150 economy seats, 20 business class seats, and 10 first class seats. The last triple $(150, 10, 5)$ correspond to the scenario where only 150 economy, 10 business and 5 first class seats are **sold**. This does not change how we should organise the plane, it simply means there will be 10 empty business class seats and 5 empty first class seats.

The maximised expected profit generated per flight is 208.3333 economy seats worth of value. This would have to be multiplied by the cost of an economy seat to find out how much money would be earned, however.

The code:

```
%Generating the z more safely...
%yk is the y_j^(k) coefficients
yk = [1/3,2/3,1];
%change into a min function and repeat the yk
z = -[0,0,0,yk,yk,yk];
%big constraint matrix, doesnt acknowledge the y_j^(k) <= h_j^(k)
A = [ 1,1.5,  2, 0, 0, 0, 0, 0, 0, 0, 0, 0;
     -1,  0,  0, 1, 0, 0, 0, 0, 0, 0, 0, 0;
      0, -1,  0, 0, 1, 0, 0, 0, 0, 0, 0, 0;
      0,  0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0;
     -1,  0,  0, 0, 0, 0, 1, 0, 0, 0, 0, 0;
      0, -1,  0, 0, 0, 0, 0, 1, 0, 0, 0, 0;
      0,  0, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0;
     -1,  0,  0, 0, 0, 0, 0, 0, 0, 1, 0, 0;
      0, -1,  0, 0, 0, 0, 0, 0, 0, 0, 1, 0;
      0,  0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1];
%other side of the inequalities for Ax <=b
b = [200; zeros(9,1)];
%upper bounds corresponding to the hjk (the first 3 are max number of seats
%(not totally necessary)
upper = [200,200/1.5,100,200,50,20,175,25,10,150,10,5];
%lower bounds
lower = zeros(1,12);

%solve the problem
[x,zval] = linprog(z,A,b,[],[],lower,upper);
%convert the value back to a maximum
zval=-zval;
%print the x matrix and optimal value
x
zval
```

And the output:

```
x =

  150.0000
   20.0000
   10.0000
  150.0000
   20.0000
   10.0000
```

```
  150.0000
   20.0000
   10.0000
  150.0000
   10.0000
    5.0000


zval =

  208.3333
```

As required.