Modelling with ODEs Assignment 4

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1. (a) The IVP

$$\frac{dy}{dx} = x - y + 1, \quad y(1) = 2$$

Expressed as an integral equation:

(b) This is a linear-inhomogeneous ODE. Solution to the homogeneous analogue:

$$\begin{aligned} \frac{dy_h}{dx} &= -y_h \\ \implies y_h &= ae^{-x} \end{aligned}$$

Using the method of undetermined coefficients, guess

$$y = y_h + bx + c$$

$$\begin{aligned} \frac{dy}{dx} &= x - y + 1 \\ -ae^{-x} + b &= x - ae^{-x} - bx - c + 1 \\ b &= x - bx - c + 1 \end{aligned}$$

$$\begin{aligned} b &= 1 \\ b + c &= 1 \\ \implies c &= 0 \end{aligned}$$

Hence

$$y = ae^{-x} + x$$

Applying the initial condition $y(1) = 2$ gives

$$\begin{aligned} y(1) = 1 &= ae^{-1} + 1 \\ a &= e \end{aligned}$$

Hence

$$y = e^{1-x} + x$$

(c)

2.

$$x_j''' = \sum_{i=0}^{3} a_i x_{j+1} + \mathcal{O}(h^m)$$

(a)

(b)

(c)

3. (a)

(b)

(c)

Matlab

School of Mathematical Sciences

Modelling with ODEs

Semester 1, 2019

Assignment 4

Due 5pm Wednesday, Week 12: Submit via MyUni

You will be marked on the presentation of your answers (including clarity of explanations)!

1. Consider the IVP
$$\frac{dy}{dx} = x - y + 1 \quad \text{with} \quad y(1) = 2.$$
   (a) Express the IVP as an integral equation, and write MATLAB code to calculate the Picard iterates. Submit your code and the first three iterates it produces.
   (b) Calculate the exact solution. Using MATLAB or otherwise, find the Taylor series of the exact solution, and hence comment on the relationship between the Picard iterates and the exact solution.
   (c) Show that the Picard-Lindelöf theorem applies to the IVP, and find the largest x-interval on which it guarantees a unique solution.

2. Consider the forward difference formula for the third derivative
$$x_j''' = \sum_{i=0}^{3} a_i \, x_{j+i} + \mathrm{O}(h^m).$$
   (a) Calculate the coefficients a_i.
   (b) Calculate the order of the truncation error m.
   (c) Perform a simple check of the coefficients that ensures the finite difference formula is correct for constant functions.

3. For the IVP
$$x' = f(t, x) \quad \text{with} \quad x(0) = a,$$
recall the leapfrog method is
$$x_{n+1} = x_{n-1} + 2\, h\, f_n.$$
   (a) Suppose you use the explicit Euler method to compute x_1. Does the use of Euler's method for the first step compromise the global error of the leapfrog method? Explain your answer.

(b) Write a Matlab code to solve the IVP

$$x' = -x \quad \text{with} \quad x(0) = 1, \tag{1}$$

using the leapfrog method with Euler's method used to find x_1. Use your code to calculate the absolute error $e(h)$ at $t = 1$ for a range of step sizes h. Plot $\log(e)$ vs. $\log(h)$, and explain how this confirms the order of accuracy of the leapfrog method.

(c) You are given that for IVPs of the form (1), solutions given by the leapfrog method can be written

$$x_n = c_+ \, \xi_+^n + c_- \, \xi_-^n \quad \text{for} \quad n = 2, \ldots$$

where

$$\xi_\pm = -h \pm \sqrt{1 + h^2},$$

and $c_\pm$ are constants. Explain why this means that the leapfrog method is not suitable to investigate the long-time behaviour of the solution of IVP (1). Use your code from part (b) to confirm the problem in calculating long-term solutions of IVP (1) using the leapfrog method.