THE UNIVERSITY of ADELAIDE

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Examination in the School of Mathematical Sciences

Semester 1, 2018

108732	**APP MTH 4121**	**Modelling with Ordinary Differential Equations Hon**

Time for completing booklet: 180 mins (plus 10 mins reading time).

Question	Marks	
1	/15	
2	/20	
3	/16	
4	/20	
5	/14	
6	/15	
Total	/100	

Instructions to candidates

- Attempt all questions and write your answers in the space provided below that question.
- If there is insufficient space below a question, then use the space to the *right* of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

- Calculators are not permitted.

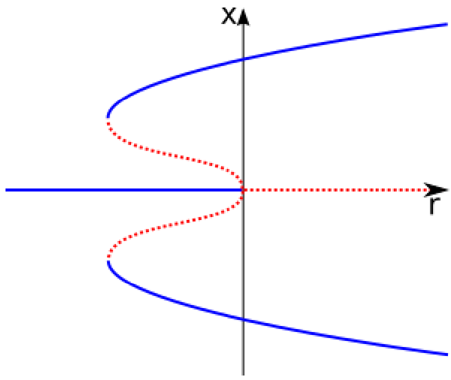
Do not commence writing until instructed to do so.

15 Total

Question 1.

Decide whether each of the following statements is true or false. Write 1–2 sentences of explanation to support each answer. You may also show an example or draw a figure if this assists your explanation.

1(a) The ODE $\dot{x} = f(x; r)$ with the bifurcation diagram below can demonstrate hysteresis. (Solid blue lines indicate stable branches and broken red lines denote unstable branches.)

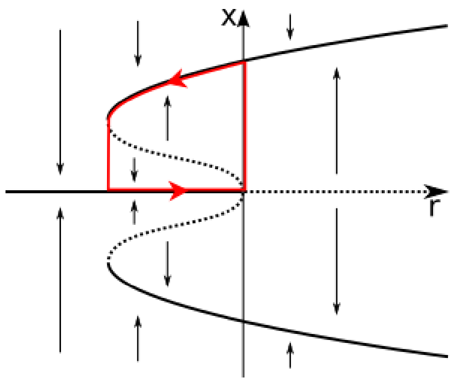


/5 marks

[2 marks]
[3 marks]

Solution **True.** ※

A possible hysteresis loop is shown below.※



1(b) The solution $(x(t), y(t))$ of a two-dimensional autonomous system of the form

$$\dot{x}(t) = f(x, y) \quad \text{and} \quad \dot{y}(t) = g(x, y),$$

for smooth f and g, can only tend towards fixed points or become unbounded as $t \to \infty$.

/5 marks

Solution False. ※ [2 marks]

It could tend to a limit cycle or lie on a closed orbit. ※ [3 marks]

Please turn over for page 4.

1(c) The backwards Euler method is unconditionally stable for the IVP

$$\frac{dx}{dt} = -(1+t)\,x \quad \text{with} \quad x(0) = 1. \tag{1}$$

/5 marks

Solution **True.** ※

[2 marks]

The stability condition for $\dot{x} = f(t,x)$ is

$$|1 - h\,f_x| > 1 \quad \Rightarrow \quad h\,f_x < 0 \quad \text{or} \quad h\,f_x > 2.$$

Here, $f = -(1+t)\,x$ and $f_x = -(1+t)$, so that

$$h\,f_x < 0 \quad \text{for} \quad t, h > 0,$$

which proves unconditional stability. ※

[3 marks]

Please turn over for page 5.

20 Total

Question 2.

Consider the ordinary differential equation

$$\frac{dX}{dT} = R\,X\left(1 - \frac{X}{K}\right) - C \quad \text{for} \quad X(T) \in \mathbb{R}, \quad T > 0, \qquad (2)$$

where R, K and C are positive parameters.

2(a) Show that ODE (2) can be transformed into the non-dimensional ODE

$$\frac{dx}{dt} = x\,(1 - x) - c \quad \text{for} \quad x(t) \in \mathbb{R}, \quad t > 0, \qquad (3)$$

using suitable scalings $X = X_c\,x$ and $T = T_c\,t$. Express the scales X_c and T_c, and non-dimensional parameter $c > 0$ in terms of R, K and C.

/5 marks

Solution Using the scalings, the ODE becomes

$$\frac{X_c}{T_c}\frac{dx}{dt} = R\,X_c\,x\,(1 - X_c\,x\,/\,K) - C$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{T_c\,R\,X_c}{X_c}\,x\,(1 - X_c\,x\,/\,K) - \frac{X_c\,C}{T_c}$$

$$= T_c\,R\,x\,(1 - X_c\,x\,/\,K) - \frac{X_c\,C}{T_c}.$$

[2 marks] ※ We choose

$$X_c\,/\,K = 1 \quad \text{and} \quad T_c\,R = 1$$

$$\Rightarrow \quad X_c = K \quad \text{and} \quad T_c = 1\,/\,R.$$

[2 marks] ※ Thus, the parameter

$$c \equiv \frac{X_c\,C}{T_c} = \frac{C}{R\,K} > 0.$$

[1 mark] ※

Please turn over for page 6.

/3 marks

2(b) Determine the steady states $x = x_*$ of ODE (3).

Solution The steady states satisfy

$$x_* (1 - x_*) - c = 0$$

$$\Rightarrow \quad x_*^2 - x_* + c = 0$$

$$\Rightarrow \quad x_* \equiv x_*^{(\pm)} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\,c}\right)$$

[2 marks] ※ Therefore, there are two steady states for $c < 0.25$, a single steady state for $c = 0.25$ and no steady states for $c > 0.25$.

[1 mark] ※

2(c) Use your answer to 2(b) to determine the bifurcation value $c = \overline{c}$. Hence calculate the bifurcation point $x = \overline{x}$.

/2 marks

Solution From 2(b) we see that the number of steady states goes from two for $0 < c < 0.25$ to zero for $c > 0.25$.

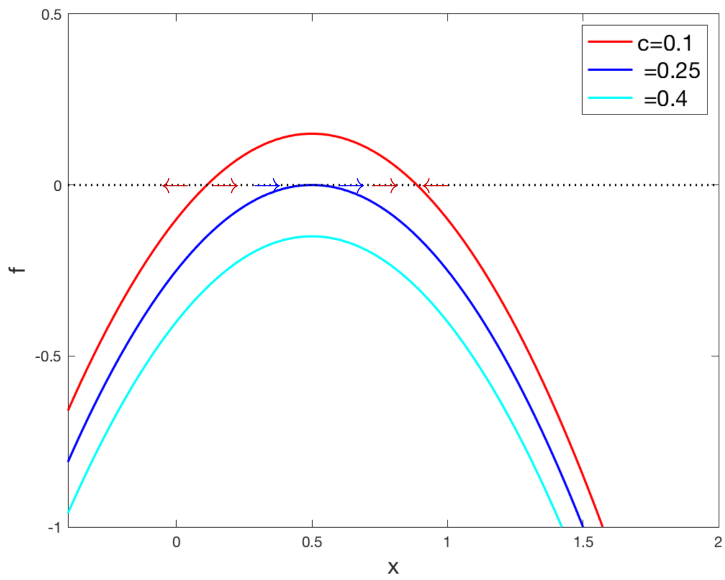
[1 mark] Clearly there is a bifurcation at the ※ bifurcation value $\overline{c} =$

[1 mark] 0.25 and the ※ bifurcation point is $\overline{x} = -\frac{1}{2}$ (the steady state for $c = 0.25$).

/4 marks

2(d) Perform a phase-line analysis for $c < \bar{c}$, $c = \bar{c}$ and $c > \bar{c}$, including arrows to indicate the stability of the fixed points. Also state the stability of the fixed points, explaining your reasoning.

Solution



[2 marks] ※ Figure (must include arrows).

[2 marks] For $c = \bar{c} = 0.25$ the single fixed point is semi-stable, as it is stable below and unstable above. ※

For $c < \bar{c}$ the smaller fixed point $(x_*^{(-)})$ is unstable as the gradient through it is positive, and the larger fixed point is stable $(x_*^{(+)})$ as the gradient is negative.

Any reasonable explanation of stability will be accepted.

/2 marks

2(e) State the type of bifurcation that occurs at $c = \bar{c}$, and give a reason.

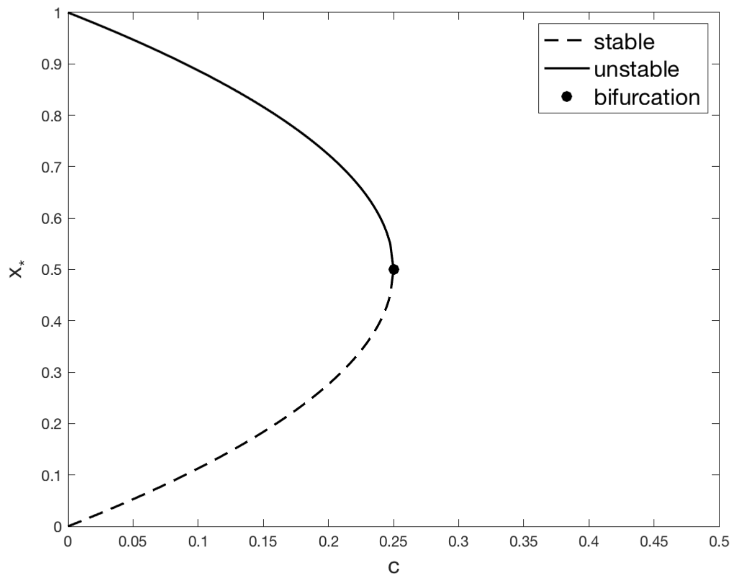
Solution A saddle-node bifurcation ※ occurs at $c = \bar{c}$, as a pair of fixed points appear/disappear as the parameter passes through that point. ※

[1 mark]
[1 mark]

2(f) Sketch the bifurcation diagram, marking the stable and unstable branches and the bifurcation point and value.

/4 marks

Solution



[2 marks] ※ for accurate bifurcation curves
[1 mark] ※ for correct stabilities
[1 mark] ※ for bifurcation point

16 Total

Question 3.

Consider the dimensionless non-linear model of competition between biological populations $x(t)$ and $y(t)$,

$$\frac{dx}{dt} = x - x\,y, \qquad x(0) = x_0, \tag{4}$$

$$\frac{dy}{dt} = \mu\,y - x\,y, \qquad y(0) = y_0, \tag{5}$$

where t is time and μ is a real positive constant.

3(a) Determine the nullclines and the two steady states.

/4 marks

Solution ※ x nullcline: $x(1-y) = 0$, i.e. $x = 0$, $y = 1$. [1 mark] [1 mark]
※ y nullcline: $y(\mu - x) = 0$, i.e. $y = 0$, $x = \mu$.
The steady states are [1 mark] [1 mark]
※ P_1: $(x, y) = (0, 0)$, ※ P_2: $(x, y) = (\mu, 1)$

Please turn over for page 10.

3(b) Let $x(t) = x_* + w(t)$ and $y(t) = y_* + z(t)$ where (x_*, y_*) denotes a steady state and $w(t)$ and $z(t)$ are small perturbations to the steady state. Linearise the system given by Eqs. (4)–(5) to obtain the approximation

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix} = J(x_*, y_*)\begin{pmatrix} w \\ z \end{pmatrix}, \tag{6}$$

where $J(x, y)$ is the Jacobian matrix, which you are to derive.

/2 marks

Solution Substituting for x and y gives

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 1 - y_* & -x_* \\ -y_* & \mu - x_* \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} + \text{higher order terms.}$$

Neglecting higher order terms gives the linear system (6) where the Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 1 - y & -x \\ -y & \mu - x \end{pmatrix}$$

[2 marks] ※

3(c) For each steady state determine, if possible, its type using the eigenvalues of the Jacobian matrix. Explain why or why not you are able to determine the type of the steady states.

/5 marks

Solution For P_1: $J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$

[1 mark] ※ $\Rightarrow$ Two positive eigenvalues $\lambda_1 = 1,\ \lambda_2 = \mu$.

[1 mark] ※ $P_1 = (0,0)$ is an unstable node or source.

For P_2: $J(\mu, 1) = \begin{bmatrix} 0 & -\mu \\ -1 & 0 \end{bmatrix}$

[1 mark] ※ $\Rightarrow$ Eigenvalues given by $\lambda^2 - \mu = 0 \ \Rightarrow\ \lambda_{1,2} = \pm\sqrt{\mu}$.

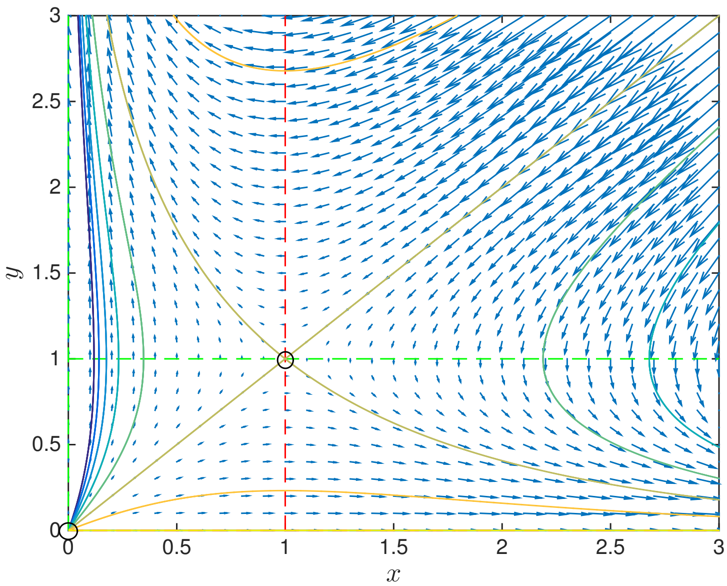
[1 mark] ※ $P_2 = (\mu, 1)$ is a saddle node.

[1 mark] ※ In each case the eigenvalues have non-zero real part so, by the Hartmann–Grobman theorem linear theory gives the type of the steady states.

/5 marks

3(d) Sketch a phase portrait in the biologically relevant part of the domain for $\mu = 1$, showing the nullclines, steady states and some trajectories. Show the direction of travel along the trajectories.

Solution



[1 mark] ※ Circles mark the two steady states.
[1 mark] ※ Green-dashed lines mark the x-nullcline.
[1 mark] ※ Red-dashed lines mark the y-nullcline.
[1 mark] ※ Other curves are trajectories.
[1 mark] ※ For direction of travel along trajectories.

20 Total

Question 4.

/2 marks

4(a) Explain what is meant by a well-posed problem.

Solution A problem with a unique solution ※ that depends continuously on the data.※

[1 mark]
[1 mark]

/4 marks

4(b) State Lax's equivalence theorem and define each of the three concepts involved.

Solution Theorem: Consistency + stability $\equiv$ convergence (for a well-posed problem). ※ [1 mark]

Consistency refers to the local discretisation error of finite difference formula tending to zero as the step size tends to zero.※ [1 mark]

Stability refers to the algorithm providing a solution to a nearby problem.※ [1 mark]

Convergence refers to the global discretisation error of finite difference formula tending to zero as the step size tends to zero. ※ [1 mark]

4(c) Consider a function $y(t)$ and discrete times $t_n = n\,h$ for some step size $h > 0$ and $n \in \mathbb{N}$, and let $y_n = y(t_n)$.

Derive a finite difference formula for the second derivative $y''_n = y''(t_n)$ by solving for the coefficients a_i in

$$y''_n = \sum_{i=-1}^{1} a_i\, y_{n+i} + O(h^m).$$

Determine the truncation error associated with the formula, i.e. determine the value of m.

Perform a simple check of the coefficients that ensures the finite difference formula is correct for constant functions.

/6 marks

Solution Using Taylor series

$$y_{n-1} = y_n - h\,y'_n + \frac{h^2}{2}y''_n - \frac{h^3}{6}y'''_n + \ldots$$

and

$$y_{n+1} = y_n + h\,y'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + \ldots$$

Substituting into the finite difference formula, and equating coefficients on the left- and right-hand sides

$$a_{-1} + a_0 + a_1 = 0$$

$$h\,(-a_{-1} + a_1) = 0$$

$$\frac{h^2}{2}\,(a_{-1} + a_1) = 1.$$

[2 marks] ※

Solving this system, the coefficients are

$$a_0 = \frac{-2}{h^2}, \quad \text{and} \quad a_{\pm 1} = \frac{1}{h^2},$$

[1 mark] ※ so that the finite difference formula is

$$y''_n = \frac{y_{n-1} - 2\,y_n + y_{n+1}}{h^2}$$

[1 mark] ※

For the error, we go to the next term (order h^3), and find

$$\frac{h^3}{6}\,(-a_{-1} + a_1) = 0.$$

Therefore, it's necessary to go to the 4th order term, and we find

$$\frac{h^4}{24}\,(a_{-1} + a_1) = \frac{h^4}{24}\frac{2}{h^2} = O(h^2),$$

[2 marks] and thus $m = 2$ ※

4(d) Consider the finite difference formula

$$x_{n+1} = 4\,x_n - (3 - 4\,h)\,x_{n-1}, \tag{7}$$

where $h > 0$ is the step size. Perform a consistency analysis to show that the finite difference formula is consistent with the ODE

$$\frac{dx}{dt} = -2x \quad \text{for} \quad x(t). \tag{8}$$

/4 marks

Solution Similar to above

$$x_{n+1} = x_n + hx_n' + \frac{h^2}{2}x_n'' + \frac{h^3}{6}x_n''' + \dots$$
$$x_{n-1} = x_n - hx_n' + \frac{h^2}{2}x_n'' - \frac{h^3}{6}x_n''' + \dots$$

and substituting into the finite difference formula gives

$$\begin{aligned}
& x_n + hx_n' + \frac{h^2}{2}x_n'' + \frac{h^3}{6}x_n''' + \dots \\
=& 4x_n - (3 - 4h)\{x_n - hx_n' + \frac{h^2}{2}x_n'' - \frac{h^3}{6}x_n''' + \dots\} \\
=& x_n + 3h\{x_n' - \frac{h}{2}x_n'' + \frac{h^2}{6}x_n''' + \dots\} \\
& + 4h\{x_n - hx_n' + \frac{h^2}{2}x_n'' - \frac{h^3}{6}x_n''' + \dots\} \\
\Rightarrow \quad & h\{x_n' + \frac{h}{2}x_n'' + \frac{h^2}{6}x_n''' + \dots\} \\
=& 3h\{x_n' - \frac{h}{2}x_n'' + \frac{h^2}{6}x_n''' + \dots\} \\
& + 4h\{x_n - hx_n' + \frac{h^2}{2}x_n'' - \frac{h^3}{6}x_n''' + \dots\} \\
\Rightarrow \quad & x_n' = -2x_n + h(x_n'' + 2x_n') + h^2(-\frac{1}{6}x_n''' - x_n'') + O(h^3)
\end{aligned}$$

[3 marks] ※ In the limit $h \to 0$

$$x_n' = -2\,x_n$$

so that the finite difference formula is consistent with the ODE

$$\frac{dx}{dt} = -2x.$$

[1 mark] ※

4(e) Show that the local discretisation error in using the finite difference formula (7) to approximate $x(t_n + h)$ is $O(h^3)$.

/2 marks

Solution From above

$$x'_n = -2x_n + h(x''_n + 2x'_n) + h^2(-\frac{1}{6}x'''_n - x''_n) + O(h^3),$$

and using the ODE

$$x' = -2\,x \quad \Rightarrow \quad x''_n + 2x'_n = 0.$$

Therefore,

$$x'_n = -2x_n + h^2(-\frac{1}{6}x'''_n - x''_n) + O(h^3),$$

[1 mark] ※ so that the truncation error is $O(h^2)$, and it follows that
[1 mark] the local discretisation error is $O(h^3)$. ※

4(f) State the problem encountered when using the finite difference formula (7) to time step, starting with some known initial condition $x_0 = x(0)$. How could you overcome this problem using Euler's method? Does this compromise the global error in using the method?

/2 marks

Solution To take the first time step, we use the finite difference formula with $n = 1$, i.e.

$$x_2 = 4x_1 - (3 - 4h)x_0.$$

We know x_0, but we do not know x_1. To overcome this we use Euler's method to find x_1, i.e.

$$x_1 = x_0 + hf_0 = x_0(1 - 2h) = 1 - 2h,$$

[1 mark] ※ with an error of $O(h^2)$.

From above, the local discretisation error is $O(h^3)$, which means a global discretisation error $O(h^2)$. Thus, using Euler's method for the first step does not compromise the global error.

[1 mark] ※

14 Total

Question 5.

Consider the IVP

$$\frac{dx}{dt} = x^3 \quad \text{with} \quad x(0) = 1. \tag{9}$$

5(a) Find the exact solution. Comment on existence and uniqueness of the solution.

/5 marks

Solution The ODE is separable and hence the solution can be found as

$$\int x^{-3}dx = \int dt$$
$$\Rightarrow \quad \frac{-1}{2}x^{-2} = t + c_1$$

[1 mark] ※ Applying the IC $x(0) = 1$, gives

$$c_1 = \frac{-1}{2}.$$

Therefore, the solution is

$$x(t) = \frac{1}{\sqrt{1-2t}},$$

[1 mark]
[1 mark] ※ where the positive branch of the squareroot is taken to satisfy the IC. ※

It follow that a unique solution exists, but only for $t < 0.5$,
[2 marks] as the solution blows up at $t = 0.5$.※

Please turn over for page 18.

/1 mark

5(b) Define what it means for a function $f(x) : J \to \mathbb{R}$ to be Lipschitz continuous on J.

Solution Lipschitz continuity on the interval J means

$$|f(x_2) - f(x_1)| \leq L\,|x_2 - x_1| \quad \text{for} \quad x_1, x_2 \in J.$$

[1 mark] ※

/1 mark

5(c) State the intervals on which $f(x) = x^3$ is Lipschitz continuous.

Solution For $f = x^3$ and any finite J we can find a suitable Lipschitz constant L, and hence $f = x^3$ is Lipschitz continuous. However, it is not globally Lipschitz continuous, i.e. for $J = \mathbb{R}$. ※ [1 mark]

/2 marks

5(d) Does the Picard–Lindelof theorem guarantee a unique solution to IVP (9) for some interval of t following $t = 0$?

Solution From 5(c), we have that x^3 is Lipschitz continuous for $J = [1-\delta, 1+\delta]$ for any δ. Also note that $f(x) = x^3$ is continuous on $I \times J$ for any $I = [-\epsilon, \epsilon]$. ※ [1 mark] Thus, the Picard–Lindelof theorem applies, and this guarantees that a unique solution, $x(t)$, exists for some interval of time following the initial time $t = 0$. ※ [1 mark]

/5 marks

5(e) Express the IVP (9) as an integral equation, and hence write down the associated Picard iteration scheme, including the value of the initial guess $x^{(0)}$. Calculate the first iterate $x^{(1)}$.

Solution The integral equation version of the IVP is

$$x(t) = 1 + \int_0^t x^3(s)ds,$$

[1 mark] ※ and the associated Picard iteration scheme is

$$x^{(k)}(t) = 1 + \int_0^t \left(x^{(k-1)}(s)\right)^3\, ds, \quad \text{for} \quad k = 1, 2, \ldots$$

[2 marks] with $x^{(0)} = 1$.※ Therefore

$$\begin{aligned} x^{(1)}(t) &= 1 + \int_0^t 1\, ds \\ &= 1 + [s]_0^t \\ &= 1 + t. \end{aligned}$$

[2 marks] ※

15 Total

Question 6.

The Kermack–McKendrick model for the evolution of an epidemic is

$$\dot{x} = -kxy, \quad \dot{y} = kxy - ly, \quad \text{and} \quad \dot{z} = ly, \tag{10}$$

where t is time, $x(t)$ denotes the number of healthy people, $y(t) > 0$ denotes number of sick people, $z(t)$ denotes the number of dead people, and k and l are positive constants.

6(a) Determine the rates at which (i) healthy people get sick, and (ii) sick people die.

/1 mark

Solution Healthy people get sick at a rate $k\,y$, and sick people die at a constant rate l. ※ [1 mark]

6(b) Show that

$$x + y + z = N, \tag{11}$$

where N is constant. Interpret this equation.

/2 marks

Solution Summing the three equations gives

$$\dot{x} + \dot{y} + \dot{z} = -kxy + kxy - ly + ly = 0 \quad \Rightarrow \quad x + y + z = N.$$

[1 mark]
[1 mark]
※ This says that the total population remains constant in size, except for deaths due to the epidemic. ※

Please turn over for page 20.

6(c) Use the first and third components of the model given in Eq. (10) to show that

$$x(t) = x_0 \exp(-kz/l), \tag{12}$$

where $x(0) = x_0$ and $z(0) = 0$. Hence show that

$$\dot{z} = l\,[N - z - x_0 \exp(-kz/l)]\,. \tag{13}$$

/4 marks

Solution Combining the first and third components gives

$$\dot{x} = -\frac{k}{l} x \dot{z} \quad \Rightarrow \quad \log(x) = -\frac{kz}{l} + \text{const.}$$

and using the ICs

$$x(t) = x_0 \exp(-kz(t)/l).$$

[2 marks] ※

Now

$$x + y + z = N \quad \Rightarrow \quad y = N - z - x_0 \exp(-kz(t)/l),$$

and using the z-equation

$$\dot{z} = l[N - z - x_0 \exp(-kz(t)/l)].$$

[2 marks] ※

Please turn over for page 21.

6(d) You are given that ODE (13) can be nondimensionalised to

$$\frac{du}{d\tau} = a - bu - e^{-u}, \tag{14}$$

where $u(\tau) \geq 0$, and constants $a > 1$ and $b > 0$.

Use the definition of a bifurcation in terms of the right-hand side of the ODE and its derivative to show that for these biologically relevant ranges of a and b, ODE (14) has no bifurcations.

/4 marks

Solution Write

$$f(u) = a - bu - e^{-u} \quad \text{and} \quad f'(u) = -b + e^{-u}.$$

The conditions for a bifurcation are

$$a - bu - e^{-u} = 0 \quad \text{and} \quad -b + e^{-u} = 0$$

[2 marks] ※

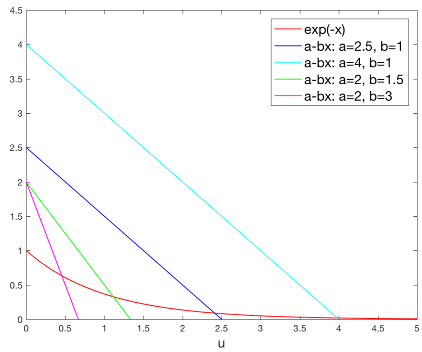
$$\Rightarrow \quad a = e^{-u}(1 + u) \leq 1, \quad \text{for all } u \geq 0,$$

which contradicts $a > 1$. Therefore, no bifurcations occur in this parameter range. ※ [2 marks]

/3 marks

6(e) By sketching the right-hand side of ODE (14) or otherwise, show that there is a single biologically relevant fixed point, u^*, and determine its stability.

Solution The single (biologically relevant) fixed point ($u \geq 0$) is the intersection of $\exp(-u)$ and $a - bu$, as in the figure below.



[2 marks] ※ As $f = a - bu - \exp(-u)$, the derivative f_u is negative at
[1 mark] the fixed point, so that the fixed point is stable. ※

6(f) What modification might be made to this model to make it more appropriate for a flu epidemic?

/1 mark

Solution Most obviously, the model should allow sick people to recover (possibly with immunity). ※ [1 mark]

Any sensible suggestion should be accepted.

End of examination questions.