# STATS 3006 Mathematical Statistics III
# Assignment 5
# 2018

**Assignment 5 is due by 23:59 Tuesday $5^{th}$ June 2018.**

**Assignments are to be submitted as a single pdf file online on MyUni.**

1. Suppose $X_1,\ X_2,\ \ldots,\ X_n$ are IID $U(0,\theta)$ random variables.
   (a) Find the method of moments estimator.
   (b) Prove that the method of moments estimator is unbiased and find its variance.
   (c) Explain briefly why it is also the best linear unbiased estimator.

2. Suppose $X_1, X_2, \ldots, X_n$ are IID $U(0,\theta)$ random variables and let $T = \max(X_1, X_2, \ldots, X_n)$.
   (a) Show that the PDF of $T$ is

   $$f(t) = \begin{cases} \frac{nt^{n-1}}{\theta^n} & \text{if } 0 < t < \theta \\ 0 & \text{otherwise.} \end{cases}$$

   **Hint:** Find the CDF and then differentiate.

   (b) Show that

   $$E(T) = \frac{n}{n+1}\theta \text{ and } \operatorname{var}(T) = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

   (c) Find $k$ such that $kT$ is unbiased and compare the resulting variance to that of the method of moments estimator.
   (d) Calculate the Cramér-Rao lower bound for an estimate of $\theta$ or explain why it is not possible.

3. Suppose $X \sim B(n,\theta)$ and consider the hypotheses

   $$H_0 : \theta = \theta_0 \text{ vs } H_A : \theta = \theta_a$$

   for constants $0 < \theta_0 < \theta_a < 1$.

   (a) Show that the most powerful test is to reject $H_0$ for $x \geq c$.
   **Hint.** Use the fact that $\log \frac{\theta}{1-\theta}$ is an increasing function of $\theta$.
   (b) Explain how $c$ can be determined to achieve significance level $\alpha$.
   **Hint:** What is the distribution of $X$ under $H_0$.
   (c) Evaluate $c$ for $n = 100$, $\theta_0 = 0.25$, and $\alpha = 0.05$.
   (d) Find the power of the test if $\theta_a = 0.4$.

4. Suppose $X_1, X_2, \ldots, X_n$ are IID $N(0, \theta)$ random variables, where $\theta$ denotes the variance of the normal distribution.
   (a) Show that $E(X_i^2) = \theta$ and $\text{var}(X_i^2) = 2\theta^2$.
   (b) Find the log-likelihood function, score and Fisher information.
   (c) Find the maximum likelihood estimator $\hat{\theta}$.
   (d) Prove that $\hat{\theta}$ is the minimum variance unbiased estimator for $\theta$
   (e) Find the score statistic for the hypothesis
   $$H_0 : \theta = \theta_0.$$
   (f) Explain how the score test statistic could be modified to produce the Wald test statistic in this case.
5. Consider two coins, $C_1$ and $C_2$ such that
   $$P(\text{Head}|C_1) = 0.5 \text{ and } P(\text{Head}|C_2) = 0.4.$$
   Suppose one of the two coins is selected at random and tossed repeatedly.
   (a) If the first two tosses are both tails, find the conditional probability that $C_2$ was chosen.
   (b) If the first two tosses are both tails, what is the expected number of heads to occur in the following 10 tosses.
6. *Placenta Previa* is an unusual condition in pregnancy in which the placenta is implanted very low in the uterus, obstructing normal delivery of the baby. In an early study of 980 *placenta previa* births, $X = 437$ were female. The purpose of this question is to assess the evidence that the proportion of females amongst *placenta previa* births, $\theta$, is less than the value 0.485 derived from the general population.
   (a) The prior distribution for $\theta$ will be a Beta distribution. If $\alpha + \beta = 50$. Find $\alpha$ and $\beta$ for which the prior expectation satisfies $E(\theta) = 0.485$ and obtain a plot of the prior density.
   (b) State the posterior distribution for $\theta$ and obtain a plot of its density.
   (c) Calculate the posterior probability $P(\theta < 0.485|X = 437)$ and state your conclusion.

## Honours Questions

The following questions are compulsory for Honours and Masters students and may be attempted for bonus marks by all students.

7. For the estimator of the form $kT$ in question 2c), find the value of $k$ that gives the minimum mean squared error. Compare the performance of this estimator to the unbiased estimator.
8. Consider data $X_1, X_2, \ldots, X_n$ IID $\text{Exp}(\lambda)$ and the prior distribution,
   $$\lambda \sim \text{Gamma}(\alpha, \beta).$$

(a) Find the posterior distribution, $p(\lambda|\boldsymbol{x})$.

(b) Find the posterior mean for $\lambda$.

(c) Describe the behaviour of the posterior mean when $n$ is large relative to $\alpha$ and $n\bar{x}$ is large relative to $\beta$.

(d) Describe the behaviour of the posterior mean when $n$ is small relative to $\alpha$ and $n\bar{x}$ is small relative to $\beta$.

(e) Interpret the two cases for the posterior mean in 8c and 8d.

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