School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Tutorial 2 (Week 4)

1. Find the extremals of the functionals below subject to the fixed end point conditions prescribed.

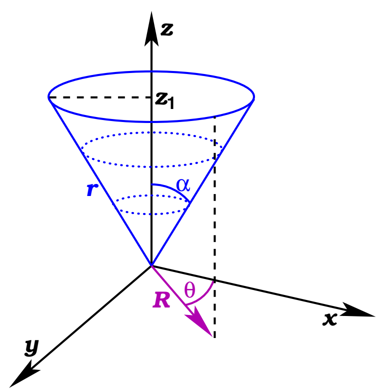
   (a) $F\{y\} = \int_0^{\pi/2} (y^2 + y'^2 - 2y\sin x)\,dx, \quad y(0) = 0, \quad y(\pi/2) = 3/2.$

   (b) $F\{y\} = \int_1^2 \frac{y'^2}{x^3}\,dx, \quad y(1) = 0, \quad y(2) = 15.$

   (c) $F\{y\} = \int_0^2 (xy' + y'^2)\,dx, \quad y(0) = 1, \quad y(2) = 0.$

2. Can light bend along a circular arc, purely through refraction? Explain your answer.

3. Find the geodesics on a right circular cone (as shown in the figure).



4. Newton's aerodynamic problem (the problem of finding the surface of revolution that minimizes drag) is often approximated by assuming the shape is long and thin, so that y' is large (and negative). In this case we can approximate

$$\frac{1}{1+y'^2} \simeq \frac{1}{y'^2}$$

and the functional of interest by

$$F\{y\} \simeq \int_0^R \frac{x}{y'^2}\,dx,$$

Derive the shape that arise from minimizing this functional.

5. $\star$ In lectures we show that extremals of functionals of the form

$$F\{y\} = \int f(y, y')\, dx,$$

satisfy the condition that

$$f - y' \frac{\partial f}{\partial y'} = \text{const.} \tag{1}$$

(a) Prove (1) by determining $\frac{df}{dx}$ using the chain rule and substituting the result for $y' \frac{\partial f}{\partial y}$ into the standard Euler-Lagrange equation (multiplied by y').

(b) If f also depends on y'' the Euler-Lagrange equation which applies is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

Again starting from the differentiating $f(y, y', y'')$ using the chain rule, determine the corresponding identity that applies in the x-absent case.