3 Boundary layer theory and asymptotic matching

We've already seen that singular perturbation problems must be treated with care. Recall the dominant balance analysis of equation (1.7),

$$\epsilon x^5 + x = 1,$$

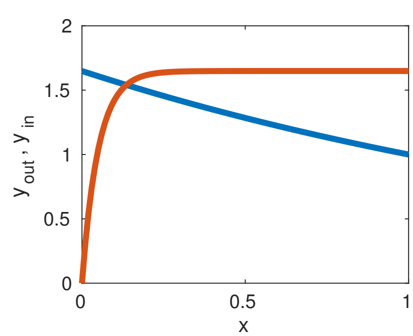
which revealed that as $\epsilon \to 0$ one of the roots was $x = 1$, while the four complex roots shot off to infinity; that is the solution exhibits behaviour on different scales (the real root is 'small'; the complex roots are 'big').

A similar kind of **multi-scale** structure is often seen in the solutions to singularly perturbed differential equations.

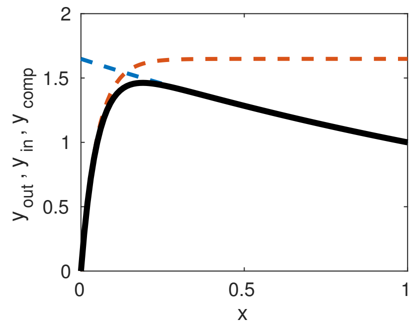
Such equations are used, for instance, to model phenomena which change rapidly in one region but more slowly in another region. A classic example (and origin of the term 'boundary layer') is from viscous fluid mechanics where fluid flows past objects/walls. In such cases there is usually a no-slip condition at the object/wall and therefore the flow behaves differently in a layer near this boundary than in the rest of the flow. Similar structure arise in a surprising array of applications.

Mathematically speaking in such cases is that a different distinguished limit of a governing equation is valid in different regions of the solution domain. To solve such problems we need to:

1. Construct solutions which are (asymptotically) valid in the various parts of the domain. Typically an **inner solution** near the edge of a domain, and an **outer solution** in the rest of the domain.



2. Join these solutions together by taking advantage of the fact that their regions of validity overlap: this is called **asymptotic matching**.



Shortly we'll go about constructing the above picture (which arise from a singularly perturbed second-order ODE). Before proceeding, note that

- Not all singularly perturbed DEs exhibit boundary layers (although it is very common).
- Asymptotic matching can be used in other contexts, not just for boundary layers (as we'll see in a later example).

3.1 Ordinary differential equations

3.1.1 An example: boundary layer in a second-order ODE

Consider the following second-order ODE:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \tag{3.1}$$

as $\epsilon \to 0$ with boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

Setting $\epsilon = 0$, the equation and boundary conditions become

$$2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

That's a first-order ODE with **two** boundary conditions. Fortunately, this equation has an exact, closed form solution. This is

$$y(x) = \frac{\exp\left(\frac{-1+\sqrt{1-\epsilon}}{\epsilon}x\right) - \exp\left(\frac{-1-\sqrt{1-\epsilon}}{\epsilon}x\right)}{\exp\left(\frac{-1+\sqrt{1-\epsilon}}{\epsilon}\right) - \exp\left(\frac{-1-\sqrt{1-\epsilon}}{\epsilon}\right)},$$

which features a $\mathrm{e}^{-x/\epsilon}$-type dependence that is characteristic of boundary layers problems.

Construct inner and outer solutions to (3.1)

3.1.2 Brief guide to solving boundary layer problems

The solution to the above problem (and many problems like it involved the following steps:

- **Find outer solution:** Try for a regular perturbation series solution (as in the previous chapter), of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \cdots .$$

  If (as in the above example) the boundary conditions cannot be satisfied by a solution of this form then the solution likely features boundary layers and further analysis is required.

- **Determine the distinguished limits:** Use a dominant balance analysis to determine appropriate scalings; each balance corresponds to either the outer solution, or an inner solution. In the previous example we rescaled only the independent variable (in that case $x = \epsilon X$), but it is sometimes also necessary to scale the dependent variable too.

- **Find the inner solution(s):** Having determined the presence (and location) of a boundary layer, seek a regular perturbation series solution to the rescaled problem. In the above example, the inner solution was of the form

$$Y(x) = Y_0(X) + \epsilon Y_1(X) + \cdots .$$

- **Matching:** Apply an asymptotic matching procedure to connect the inner and outer solutions. The matching condition in the above problem was that: the outer limit of the inner solution equals the inner limit of the outer solution, which we wrote as

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x).$$

  This is a condition on the leading order solution; more sophisticated conditions are necessary at higher order.

- **Construct composite solution:** Add the inner and outer solutions together; subtract the overlaps to prevent doubling-up.

3.1.3 Another second-order ODE

The first example we considered had an exact solution (and so we knew about its boundary layer structure from the form of that). Let's know consider a problem with no exact solution:

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x, \tag{3.2}$$

with $\epsilon \to 0$ over $0 \leq x \leq \pi$, and the boundary conditions

$$y(0) = 2, \qquad y(\pi) = -1.$$

Solve this problem.

3.1.4 Yet another second-order ODE

Consider the following second-order ODE:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \left(2 - x^2\right) y = -1, \quad \text{as } \epsilon \to 0,$$

subject to the boundary conditions $y'(0) = 0$ and $y(1) = 0$.

Solve this problem.

This illustrated two interesting points, namely that

- To obtain the outer solution we didn't even have to solve a differential equation!
- The form of the inner solution was in powers of $\epsilon^{1/2}$.

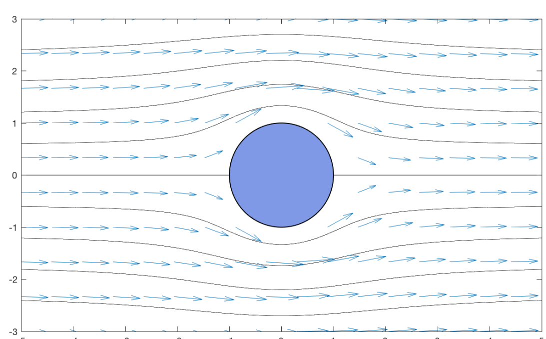
Taken as a whole, these ODE examples illustrate that there is no one-size-fits-all method for analysing boundary layers. Rather the issues around inner solutions, distinguished limits and matching must all be carefully considered to develop a consistent, coherent picture of a given problem.

All the examples so far involved constructing leading-order solutions, as will the next few applied examples. Techniques for developing higher-order solutions (particularly higher-order matching) will be discussed later.

3.2 Partial differential equations

3.2.1 Heat transfer from a cylinder to a fluid in uniform flow

Recall the flow of inviscid fluid past a cylinder looked like this:



Let's add an extra layer of detail on top of this: what if the cylinder is hot, and is heating the fluid as it goes past. Assume that the flow is quite fast, and so the heat from the cylinder doesn't penetrate very far into the fluid before the warmed fluid is advected downstream. This results in a warm **layer** around the cylinder.

This can be seen mathematically. Assume that the diffusion of temperature is small compared to advection of temperature (high Peclet number), that is

$$\mathbf{u} \cdot \nabla T = \epsilon \nabla^2 T \quad \text{in } r \geq 1.$$

We're only solving for T here since we already know that

$$\mathbf{u} = \nabla \phi, \quad \phi = \left(r + \frac{1}{r} \right) \cos \theta.$$

The boundary conditions are

$$T = 1, \quad \text{on } r = 1, \qquad T \to 0, \quad \text{as } r \to \infty.$$

That is the temperature is fixed at 1 on the surface of the cylinder, and far away from the cylinder the temperature is zero.

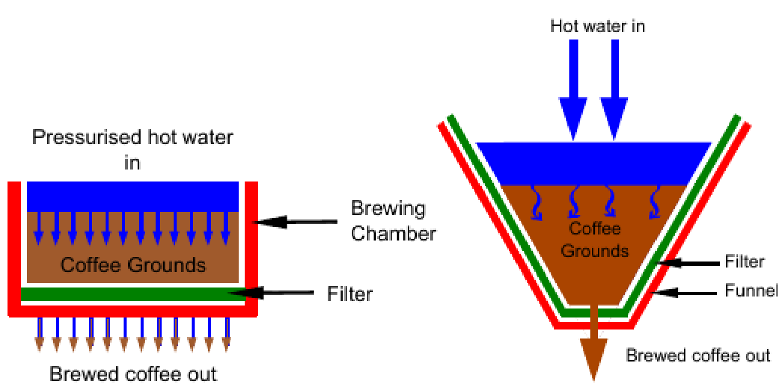
Boundary-layer solution for this problem.

3.2.2 Case study 2: boundary layers for coffee brewing

A very important area of applied mathematics is 'industrial mathematics'. This (often) involves working with a partner company to address a question of interest; perhaps to better understand or optimise an industrial process. Such processes often have a lot of 'moving parts', and quantifying the relationship between these is where modelling can help.

The following paper deals with one such application (some would say the most important application): brewing coffee.

Moroney, K.M., et al, *Asymptotic analysis of the dominant mechanisms in the coffee extraction process*, **76**(6):2196–2217, SIAM. J. Appl. Math., 2016.



3.3 Higher-order matching

Our technique for matching the inner and outer solution of leading boundary layer problems was to use a matching condition of the form:

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x).$$

This is an example of a more general procedure/rule known as Van Dyke matching.

The same Van Dyke who wrote *Perturbation methods in fluid mechanics*

This more general rule is concisely (but rather confusingly) stated as:

The m-term inner expansion of the n-term outer solution

matches with

the n-term outer expansion of the m-term inner solution.

This statement doesn't really make sense on its own, but describes a very general matching procedure. The first line means:

1. Find n terms in the outer solution.
2. Rewrite this expression in terms of the inner variable.
3. Expand/simplify the expression as appropriate.
4. Retain the first m terms.

Similarly, the third line tells us to

1. Find m terms in the inner solution.
2. Rewrite this expression in terms of the outer variable.
3. Expand/simplify the expression as appropriate.
4. Retain the first n terms.

Matching involves equating the two expression to determine any unknown coefficients.

Let's see how this works in practice by considering an example,

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x, \quad \epsilon\to 0, \tag{3.3}$$

subject to $y(0) = -2$ and $y(1) = 1$, for $0 \le x \le 1$.

Construct higher-order matched solution to this boundary layer problem.

This example raised some key points which often come up in this matching procedure:

- neglect of exponentially small terms in the inner solution expressed in terms of the outer variable. In this example these were of the form $e^{(x-1)/\epsilon}$ if there was a boundary layer at $x = 0$ these might look like $e^{x/\epsilon}$;

- this is a self checking method: we found one matching coefficient from the leading order match, then found it again in the $\mathcal{O}(\epsilon)$ match;
- we determined the overlap behaviour along the way, as for leading order matching the overlap **function** is just one side of the matching criterion.

This type of matching works for a great many problems but not all problems; notably those involving $\log\epsilon$ are tricky. An approach to such problems is outlined in Hinch, Chapter 5.2.

Now let's previous example and do a match correct up to ϵ^2.

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x, \quad \epsilon\to 0, \tag{3.4}$$

subject to $y(0)=2$ and $y(\pi)=-1$, over $0\le x\le\pi$. Recall there was a boundary layer at $x=0$ and we rescaled with $x=\epsilon X$, $y=Y$.

Find composite solution correct to ϵ^2.

3.4 WKBJ approximation for boundary layers

An alternate way of solving boundary layer problems is to apply ansatz of the form:

$$y(x)\sim\sum_{n=0}^{\infty}u_n(x)\epsilon^n+\mathrm{e}^{-F(x)/\epsilon}\sum_{n=0}^{\infty}v_n(x)\epsilon^n \tag{3.5}$$

which is known as a **WKB expansion** (sometime written in a variety of different forms). Those initials stand for Wentzel, Kramers and Brillouin. The naming of this is a bit contentious, sometimes Jeffreys is also credited and the acronym becomes either **WKBJ** or **JWKB**. It's also sometimes credited to some combination of Louiville, Green, Rayleigh or Carlini; in physics slightly less general versions are referred to as either geometric or physical optics. **These are all (essentially) the same thing.**

It's easy to see why such an approximation might be useful for looking at boundary layers: there's a exponential with a $1/\epsilon$ dependency built-in, which is what we saw in the inner solution to previous examples. Consider the following ODE:

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x+1)\frac{\mathrm{d}y}{\mathrm{d}x}+2y=0, \tag{3.6}$$

with $\epsilon\to 0$ over $0\le x\le\pi$, and the boundary conditions

$$y(0)=2, \qquad y(1)=1.$$

The leading-order solution by matched asymptotics (you might like to check this) is:

$$y_{\mathrm{comp}}(x) = \frac{3}{1+2x} - \mathrm{e}^{-x/\epsilon} \tag{3.7}$$

Lets see if we can 'improve' on that using the a WKB approximation.

Find leading order WKB solution, and compare with matched.