Differential Equations II

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Contents

Bibliography

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1 First order ordinary differential equations

(Kreyszig 2011, Chapt. 1)

Revision and extension of some first order ordinary differential equations (ODEs) from Maths 1B: both solving ODEs and their applications.

Contents

⋘begin lecture 1⋙

1.1 Modelling with ODEs

(Kreyszig 2011, §1.1)
(Stewart 2012, §9.1)

Modelling (Kreyszig 2011, Fig. 1): mathematical methods, algebra and proofs ('pure maths') is just in the middle step; applied maths covers all three steps.

Address situations where the mathematical model is a differential equation: typically we seek a function $y(x)$ (or $y(t)$) and then modelling gives us an equation involving its derivative dy/dx, often denoted y' or $\dot{y}$.

Many applications (Kreyszig 2011, Fig. 2) and examples (Kreyszig 2011, p.3).

Definition 1.1. *(Kreyszig 2011, p.4) A **first order ODE** is an equation involving only one variable x, one function $y(x)$ and its derivative dy/dx in either an* implicit form

$$F(x, y, y') = 0 \quad \textit{or the } \text{explicit form} \quad y' = f(x, y). \tag{1.1}$$

Chauvet Cave contains possibly the earliest known examples of prehistoric writing, and has been dated to around ≈ 30000 years BC using radiocarbon dating. Scientifically, this finding depends on the following observations:

(i) C-14 is present in all living things (in the same proportion),

(ii) in Chauvet Cave, 4% of C-14 remains,

(iii) the amount which decays in a small time Δt is proportional to Δt and the amount of C-14 present,

(iv) the half life of C-14 is 5730 years.

This gives us enough information with which to build a mathematical model that can be used to date the paintings in the cave.

Example 1.1 (Radioactivity. Exponential decay (Kreyszig 2011, Example 5, p.7)). Observed that radioactive material has a half-life.

(a) ODE modelling: *things change only by ways that we can quantify. Define* what is being addressed: let $y(t)$ be the amount of radioactive material at time t. *Observation* implies that the amount of material decaying in a (small) time interval Δt is proportional to both Δt and y: the change
$$y(t + \Delta t) - y(t) = -k\Delta t\, y(t)$$
for some constant k (such as for radium). Divide by Δt, and take $\lim_{\Delta t \to 0}$ to derive the model ODE $dy/dt = -ky$.

(b) Solution: you must instantly know to write down that $y(t) = ce^{-kt}$ satisfies this basic ODE for all constants c. But it is separable, for example.

(c) Interpretation: amount of radioactive material decays to zero on a time scale of $1/k$.

♣

Definition 1.2. *(Kreyszig 2011, pp.5–6) A function $y = h(x)$ is called a* ***solution****, or a* ***particular solution****, of an* ODE (1.1) *if it satisfies the* ODE *on some open interval. A solution with an* arbitrary *constant c, $y = h(x; c)$, is called a* ***general solution****.*

Initial conditions make solutions unique (Kreyszig 2011, p.6).

Definition 1.3. *(Kreyszig 2011, p.6) An* ***initial value problem*** *(*IVP*) is formed by an* ODE (1.1) *together with an* ***initial condition*** *$y(x_0) = y_0$ for the solution to satisfy.*

1.2 Solving and modelling with separable ODEs

(Kreyszig 2011, §1.3)
(Stewart 2012, §9.3)

Definition 1.4. *(Kreyszig 2011, p.12) An* ODE *is called* ***separable*** *if it can be put in the form*
$$\frac{dy}{dx} = \frac{f(x)}{g(y)} .$$

Example 1.2 (Kreyszig 2011, Example 3, p.13) Solve $y' = -2xy$, $y(0) = 1.8$. ♣

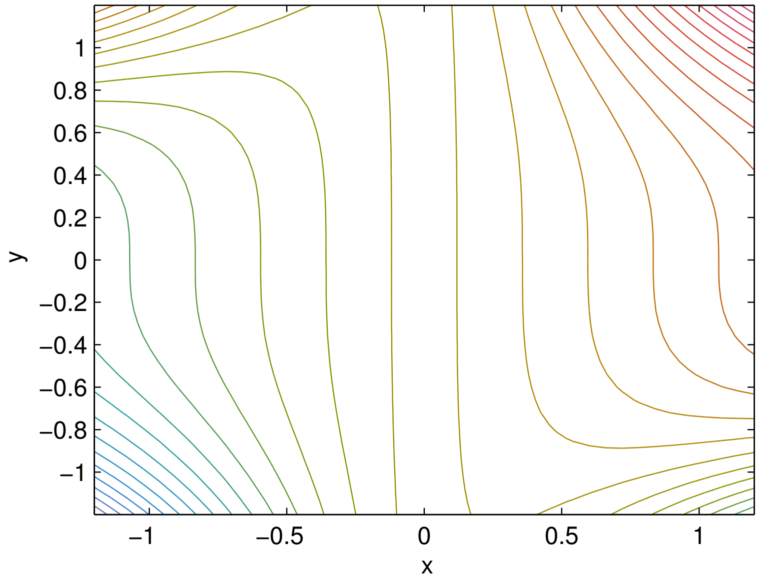

Figure 1.1: solution curves of the exact ODE *in Example 1.4.*

⋘*begin lecture 2*⋙

Modelling: in exercises and examinations you will be given sufficient observational/experimental information to do the modelling using common steps. For example, you do not have to remember *Torricelli's law* and similar, just how to use similar information.

Example 1.3 (Kreyszig 2011, Example 7, p.16)

(a) ODE modelling: *things change only by ways that we can quantify. Define* what is being addressed: here you are asked for the height of water $h(t)$. *Observation* gives us Torricelli's law to use (note the physical units make sense).

(b) Solutions via separation of variables.

(c) Interpretation tells when the tank is empty, for example.

♣

1.3 Solving exact ODEs

(Kreyszig 2011, §1.4)

Example 1.4 (backwards example). Given a family of curves $x + x^2y^3 = c$, see Figure 1.1, what ODE might they satisfy? Answer: consider any small change (dx, dy) in (x, y) that stays on the curve, $dc = 0$. (Kreyszig 2011, p.20) ♣

Definition 1.5. *(Kreyszig 2011, p.21) A first order* ODE $F(x, y, y')$

when written as

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \textit{such that } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1.2}$$

*is called an **exact differential equation**.*

For the exact differential equation (1.2), there exists some function $u(x,y)$ such that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = M\,dx + N\,dy,$$

and so an **implicit solution** to the exact ODE (1.2) is $u(x,y) = c$ for any constant c (Kreyszig 2011, p.21). Find the implicit solution through the union of the terms in the two expressions $u = \int M\,dx + k(y)$ and $u = \int N\,dy + l(x)$ (Kreyszig 2011, p.22).

Example 1.5 (Kreyszig 2011, Example 1, p.22) Solve $\cos(x+y)dx + [3y^2 + 2y + \cos(x+y)]dy = 0$. ♣

⋘*begin lecture 3*⋙

Example 1.6 (Kreyszig 2011, Example 3, p.23) Solve the initial value problem $-y\,dx + x\,dy = 0$ with $y(1) = 2$. It is not exact. But multiply by $1/x^2$. ♣

Integrating factors may reduce to exact form Given an ODE in the form $P(x,y)dx + Q(x,y)dy = 0$, a function $F(x,y)$ is called an *integrating factor* if $FP\,dx + FQ\,dy = 0$ is an exact differential equation (Kreyszig 2011, p.24).

Theorem 1.6. *(Kreyszig 2011, pp.24–5) Given a differential equation in the form $P(x,y)dx + Q(x,y)dy = 0$:*

- *if $(\partial P/\partial y - \partial Q/\partial x)/Q$ depends only upon x, then an integrating factor is*

$$F(x) := \exp\left[\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\right];$$

- *if $(\partial Q/\partial x - \partial P/\partial y)/P$ depends only upon y, then an integrating factor is*

$$F(y) := \exp\left[\int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy\right].$$

Ignore integration constants in these integrating factors.

Example 1.7 (Kreyszig 2011, Example 5, p.25) Find an integrating factor and solve the initial value problem $(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$ such that $y(0) = -1$. ♣

1.4 Solving and modelling with linear ODEs

(Kreyszig 2011, §1.5)
(Stewart 2012, §9.5)
⋘*begin lecture 4*⋙

Definition 1.7. *(Kreyszig 2011, p.27) A first order* ODE *is said to be* ***linear*** *if it can be written*

$$y' + p(x)y = r(x), \tag{1.3}$$

otherwise it is ***nonlinear****. If the right-hand side function* $r(x) = 0$*, then it is called* ***homogeneous****, otherwise it is* ***nonhomogeneous****.*

Solve such linear ODEs through multiplying by the integrating factor (Kreyszig 2011, pp.28–9)

$$F(x) := \exp\left[\int p(x)\,dx\right].$$

It leads to the general solution

$$\begin{aligned} y &= \frac{1}{F}\int Fr\,dx + \frac{c}{F} \\ &= \text{response to forcing } r + \text{response to initial data.} \end{aligned}$$

Example 1.8 (Kreyszig 2011, Example 1, p.29) Solve the initial value problem $y' + y\tan x = \sin 2x$ such that $y(0) = 1$. ♣

Example 1.9 (hormone level). (Kreyszig 2011, Example 3, pp.30–1)

Observations: Hormone levels in the blood vary in time; rate of change is the difference between a sinusoidal, 24-hour-period, input from the thyroid and a continuous removal rate proportional to the current level.

(a) ODE modelling: define variables, including units such as time in hours; identify given/assumed laws; mathematically express changes.

(b) Solution: here via integrating factor and the initial condition.

(c) Interpretation: **transients** decay exponentially leaving a long term oscillation about a mean hormone level.[1]

♣

[1] Most would not call such oscillation a *steady-state*.

1.5 Solving ODEs using computer algebra

Computer algebra systems solve many routine ODEs. For example, *Maple*, *Mathematica*, *WolframAlpha*, *Maxima*, *Reduce*, and the one available in *Matlab* that we introduce.

Example 1.10 First declare to Matlab that we are interested in the function $y(x)$ as an algebraic expression:

```
>> syms y(x)
```

Then solve ODEs analogous to the following examples.

(a) Solve $y' = -xy$ with (note the double `==`),

```
>> dsolve(diff(y,x)==-x*y)
ans = C38*exp(-x^2/2)
```

(b) Adjoin initial conditions such as $y(0) = 1.8$

```
>> dsolve(diff(y,x)==-x*y,y(0)==1.8)
ans = (9*exp(-x^2/2))/5
```

(c) (cf. Example 1.6) Solve $-y\,dx + x\,dy = 0$ with

```
>> dsolve(-y+x*diff(y,x)==0)
ans = C44*x
```

(d) (cf. Example 1.5) Although computer algebra has deficiencies as seen in trying to solve $\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0$

```
>> dsolve(cos(x+y) ...
   +(3*y^2+2*y+cos(x+y))*diff(y,x)==0)
Warning: Explicit solution could not be found.
> In dsolve at 197
ans = [ empty sym ]
```

(e) (cf. Example 1.8) Nonetheless, computer algebra will solve most things that we can, such as $y' + y\tan x = \sin 2x$:

```
>> dsolve(diff(y,x)+y*tan(x)==sin(2*x),y(0)==1)
ans = 3*cos(x) - cos(2*x) - 1
```

(f) Of course, it also solves second and higher order ODEs such as $\ddot{x}+5\dot{x}+6x = 3e^{-2t}+e^{3t}$ which Chapter 2 explores. Start by now seeking $x(t)$.

```
>> syms x(t)
>> dsolve(diff(x,t,2)+5*diff(x,t)+6*x ...
   ==3*exp(-2*t)+ exp(3*t))
ans = exp(-2*t)*(3*t + exp(5*t)/5)
    - (exp(-2*t)*(exp(5*t) + 18))/6
    + C82*exp(-2*t) + C83*exp(-3*t)
```

Matlab can also find a preferable expression of the answer above by simplifying.

```
>> expand(ans)
ans = exp(3*t)/30 - 3*exp(-2*t) + 3*t*exp(-2*t)
    + C82*exp(-2*t) + C83*exp(-3*t)
>> simplify(ans)
ans = (exp(-3*t)*(30*C83 + exp(6*t) - 90*exp(t)
    + 30*C82*exp(t) + 90*t*exp(t)))/30
```

(g) It also solves some systems of ODEs, although we will not cover this in-depth in this course: for example, $\dot{x} = y$ and $\dot{y} = -x$

```
>> syms y(t)
>> [x,y]=dsolve(diff(x,t)==y,diff(y,t)==-x)
x = C23*cos(t) + C22*sin(t)
y = C22*cos(t) - C23*sin(t)
```

♣

When the computer can solve so many ODEs, why do we need to learn how to solve ODEs? The reason is that the specific solutions are not really the important aspect, but it is the pattern and structure of both the solutions and the methodology that is important. Finding solutions by hand is the means to the end of learning about these patterns and structures.

1.6 Existence and uniqueness of solutions for IVPs

(Kreyszig 2011, §1.7)

⋘*begin lecture 5*⋙

Example 1.11 (may or may not exist). Explore the ODE $-xy' + y = 0$ with various initial conditions, $y(x_0) = y_0$. ♣

We must obtain some idea about when a solution exists, and how many solutions to look for (Kreyszig 2011, p.39).

Theorem 1.8 (existence and uniqueness). *(Kreyszig 2011, pp.39–40) Consider the* IVP

$$y' = f(x,y), \quad y(x_0) = y_0\,. \tag{1.4}$$

If, in some rectangle $R := \{(x,y) : |x - x_0| < a,\ |y - y_0| < b\}$, *the function* $f(x,y)$ *is continuous and bounded,* $|f| \leq K$, *then the* IVP (1.4) *has at least one solution* $y(x)$ *for* $|x - x_0| < \min(a, b/K)$.

Further, if also $f_y = \partial f/\partial y$ *is continuous and bounded in the rectangle* R, $|f_y| \leq M$, *then the* IVP (1.4) *has a unique solution* $y(x)$ *for* $|x - x_0| < \min(a, b/K)$.

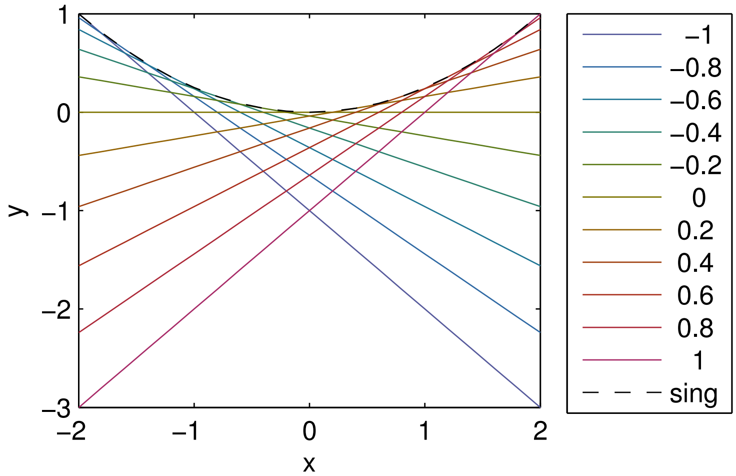

Figure 1.2: solutions of the ODE $(y')^2 - xy' + y = 0$*: general solutions* $y = cx - c^2$*; and singular solution* $y = x^2/4$.

For a proof see the level III course *Topology and Analysis*.

Example 1.12 (Kreyszig 2011, Example 1, p.41) Consider the IVP $y' = 1 + y^2$ with $y(0) = 0$. ♣

Example 1.13 (Kreyszig 2011, Example 2, p.42) Consider the IVP $y' = \sqrt{|y|}$ with $y(0) = 0$. ♣

Example 1.14 Reconsider Example 1.11. ♣

⋘*begin lecture 6*⋙

Definition 1.9 (singular solutions)**.** *(Kreyszig 2011, p.8) Sometimes an* ODE *with a general solution,* $y = h(x, c)$*, also has special particular solution(s),* $y = H(x)$*, which cannot be obtained by any value of the parameter* c*: these are called* ***singular solutions****.*

Beware: 'singular' also has other meanings such as dividing by zero.

Example 1.15 (Kreyszig 2011, Problem 16, p.8) Consider the ODE $(y')^2 - xy' + y = 0$ (nonlinear).

- A general solution is $y = cx - c^2$ for any constant c.
- A singular solution is $y = x^2/4$: it is a particular solution; it is not in the general solution family. Figure 1.2 shows the singular is the envelope of the general, which often happens.

What is the view of uniqueness theory? ♣

Picard iteration approximates (Kreyszig 2011, pp.42–3) When we cannot solve IVPs algebraically (the usual), we can approximate. Suppose the IVP (1.4) has a solution $y(x)$ for $x_0 \leq x \leq X$ then integrate both sides of the ODE (1.4): $\int_{x_0}^{X} dy/dx\, dx = \int_{x_0}^{X} f(x, y(x))\, dx \implies y(X) - y(x_0) = \int_{x_0}^{X} f(x, y(x))\, dx \implies y(X) = y_0 + \int_{x_0}^{X} f(x, y(x))\, dx \implies y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$ upon changing two symbols. Picard made this the basis of an iteration: starting with $y_0(x) := y_0$, recursively compute $y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$.

Example 1.16 Consider the IVP $y' = 1+y^2$ with $y(0) = 0$: Picard iteration is easily done in computer algebra: for example, in Matlab

```
syms x
y=0
y=0+int(1+y^2,x,0,x)
```

and repeat the last statement as often as desired. ♣

2 Second and higher order ODEs

Generalise second order ODEs (Kreyszig 2011, Chapts. 2–3).

Contents

2.1 Classification of ODEs

⋘*begin lecture 7*⋙

(Kreyszig 2011, §3.1)

For the notation that $y^{(k)} = d^k y/dx^k$ for any k, consider the implicit equation for $y(x)$

$$F(x, y, y', \ldots, y^{(n)}) = 0\,. \tag{2.1}$$

Type : If there is only one independent variable (here x) then (2.1) is an **ordinary differential equation** (ODE). If there is more than one independent variable, then the equation is a **partial differential equation** (PDE) (Chapter 3); for example, solve $\partial^2 y/\partial x^2 - \partial y/\partial t = 0$ for $y(x, t)$.

Order : The **order** of an ODE (2.1) is the degree of the highest derivative (Kreyszig 2011, p.105):

- $y' = x^3 y^2$ is first order, whereas
- $y'' + y' = e^x$ is second order.

Potentially the order can be fractional, but this is outside the scope of our courses.

Linearity : An ODE (2.1) that can be rearranged to the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x), \quad (2.2)$$

is called **linear**, otherwise it is **nonlinear** (Kreyszig 2011, p.105):

- $y' = x^3y^2$ is nonlinear due to the y^2,
- $y'' + x^2y' = e^x$ is linear, but
- $y'' + x^2y' = e^y$ is nonlinear due to the e^y, and
- $y'' + (y^2 - 1)y' + y = 0$ is nonlinear due to y^2y' (called the van der Pol ODE).

Homogeneity : If a linear ODE (2.2) has right-hand side function $r(x) = 0$ for all x, then it is called **homogeneous**, otherwise (when $r(x) \neq 0$ for at least some x) a linear ODE is called **nonhomogeneous** (Kreyszig 2011, p.106):

- $y'' + x^2y' = e^x$ is nonhomogeneous, but
- $y'' + x^2y' = e^xy$ is homogeneous.

Be wary: even within differential equations the term "homogeneous" is used to mean several different things!

Extra conditions : An **initial value problem** (IVP) is formed from an nth order ODE (2.1) together with n **initial conditions** all applying at the same x (Kreyszig 2011, p.107),

$$y(x_0) = K_0, \quad y'(x_0) = K_1, \quad \ldots, \quad y^{(n-1)}(x_0) = K_{n-1}\,. \quad (2.3)$$

If n conditions are specified at *two or more* x values, then the ODE and so-called **boundary conditions** form a **boundary value problem** (Kreyszig 2011, p.499).

Autonomy : If the independent variable x does not appear in an ODE (2.1), then the ODE is called **autonomous**, otherwise it is called **nonautonomous** (Kreyszig 2011, pp.11,33).

The equations of the universe are presumed to be autonomous: presumed the same here, there, everywhere, and everywhen.

Determinism : By default, an ODE (2.1) has no random component. If (2.1) does have some randomness within it, then it would be called a *stochastic differential equation* and is outside the scope of this course.

2.2 Linear homogeneous ODEs

(Kreyszig 2011, §3.1)

Explore the homogeneous linear nth-order ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0\,, \quad (2.4)$$

with x on some open interval I.

2.2.1 General solution

Theorem 2.1 (superposition). *(Kreyszig 2011, p.106) If $y_1(x), y_2(x), \dots, y_m(x)$ are m solutions of the ODE (2.4) on an open interval I, then so is $y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_my_m(x)$ for any constants $c_1, c_2, \dots, c_m$.*

Example 2.1 Linear and homogeneous is essential: consider cases $y'' + y = 0$, $y'' + y = 1$, and $y''y - xy' = 0$ (Kreyszig 2011, Examples 2–3, p.48). ♣

The x-dependence of the coefficients does not change the principle. For example, consider 2nd order **Euler–Cauchy odes**, $x^2y'' - \frac{3}{4}y = 0$ which fits (2.4) upon division by x^2 (Kreyszig 2011, p.71). Verify two solutions and linearly combine them to $y = c_1x^{3/2} + c_2x^{-1/2}$. But how do we know this is a general solution? Maybe there are others.

Definition 2.2. *(Kreyszig 2011, p.106) A **general solution** of the nth-order ODE (2.4) on I is of the form*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

*where $y_1(x), y_2(x), \dots, y_n(x)$ is a **basis** of n 'linearly independent' functions on I. Obtain a **particular solution** by assigning specific values to the n constants $c_1, c_2, \dots, c_n$.*

The same terms are used as in Linear Algebra because it is precisely the same concept.

Definition 2.3. *(Kreyszig 2011, p.106) Consider n functions $y_1(x), y_2(x), \dots, y_n(x)$ defined on some open interval I. This set of functions is called **linearly independent** on I if the only solution of*

$$k_1y_1(x) + k_2y_2(x) + \cdots + k_ny_n(x) = 0 \quad \text{for all } x \in I$$

*is $k_1 = k_2 = \cdots = k_n = 0$. Otherwise the set of functions is called **linearly dependent** (there exists coefficients, not all zero, such that the linear combination is zero on the interval).*

The one set of coefficients $k_1, k_2, \dots, k_n$ has to be used for all $x \in I$.

Linear dependence is equivalent to being able to write (at least) one of the functions as a linear combination of the others on the interval (Kreyszig 2011, p.107). Two functions are linearly dependent iff one is a multiple of the other.

Example 2.2 (Kreyszig 2011, p.107)

- $y_1 = x^2$, $y_2 = 5x$, $y_3 = 2x$ are linearly dependent on any I.
- $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ are linearly independent on any I.

- $y_1 = x$, $y_2 = |x|$ are linearly dependent on $I_1 = (0, 5)$, but are linearly independent on $I_2 = (-2, 3)$: the reason is that on I_1 they are a multiple of each other, but on I_2 they are not.
- What is the most general solution we can find for $y'''' - 5y'' + 4y = 0$?

♣

Theorem 2.4. *(Kreyszig 2011, Theorem 2, p.108) If the coefficients of an homogeneous linear nth-order* ODE *(2.4) are continuous on some open interval I, and the (n) initial conditions (2.3) are at some $x_0 \in I$, then the initial value problem (2.4)+(2.3) has a unique solution on I.*

No proof given.

This theorem implies that a general solution of an homogeneous linear nth-order ODE (2.4) will have precisely n arbitrary constants in a linear combination of linearly independent functions. If they were linearly dependent, then we could eliminate one (or more). With n arbitrary constants, we uniquely satisfy any given initial condition (of n values).

Example 2.3 (Kreyszig 2011, Example 4, p.108) Solve the Euler–Cauchy ODE $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$ with initial condition $y(1) = 2$, $y'(1) = 1$, $y''(1) = -4$. The interval I must exclude $x = 0$. ♣

⋘*begin lecture 8*⋙

2.2.2 Wronskian determines linear (in)dependence

Aim: given existence and uniqueness for IVPs, now establish that the so-called 'general solution' does indeed capture all solutions of an nth order ODE. There are no singular solutions.

Definition 2.5. *(Kreyszig 2011, p.108) Given n functions $y_1(x), y_2(x), \ldots, y_n(x)$ defined on some open interval I and possessing (at least) $n-1$ derivatives, define the* ***Wronskian*** *as the determinant*

$$W(x) = W(y_1, y_2, \ldots, y_n) := \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} . \tag{2.5}$$

Example 2.4 • Linearly dependent $y_1 = x^2$, $y_2 = 5x$, $y_3 = 2x$

have Wronskian

$$
\begin{aligned}
W &= \begin{vmatrix} x^2 & 5x & 2x \\ 2x & 5 & 2 \\ 2 & 0 & 0 \end{vmatrix} \\
&\quad \boxed{\begin{array}{ccc|cc} x^2 & 5x & 2x & x^2 & 5x \\ 2x & 5 & 2 & 2x & 5 \\ 2 & 0 & 0 & 2 & 0 \end{array}} \\
&= 0 + 20x + 0 - 20x - 0 - 0 \\
&= 0\,.
\end{aligned}
$$

- Linearly independent $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ have Wronskian

$$
\begin{aligned}
W &= \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \\
&\quad \boxed{\begin{array}{ccc|cc} x & x^2 & x^3 & x & x^2 \\ 1 & 2x & 3x^2 & 1 & 2x \\ 0 & 2 & 6x & 0 & 2 \end{array}} \\
&= 12x^3 + 0 + 2x^3 - 0 - 6x^3 - 6x^3 \\
&= 2x^3\,.
\end{aligned}
$$

- $y_1 = x$, $y_2 = |x|$ do not have a Wronskian on any interval including $x = 0$ as $y_2 = |x|$ is not differentiable there.

♣

Theorem 2.6. *(Kreyszig 2011, Theorem 3, p.109) Let the homogeneous linear nth-order* ODE *(2.4) have continuous coefficients $p_0(x), \ldots, p_{n-1}(x)$ on an open interval I.*

- *Then n solutions $y_1(x), y_2(x), \ldots, y_n(x)$ of (2.4) on I are linearly dependent on I iff their Wronskian $W(x_0) = 0$ for some $x_0 \in I$.*
- *Further, if $W(x_0) = 0$ for some $x_0 \in I$, then $W(x) = 0$ for all $x \in I$.*
- *Hence, if $W(x_1) \neq 0$ for some $x_1 \in I$, then the solutions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent on I and hence form a basis for a general solution of (2.4) on I.*

Example 2.5 x, x^2, x^3 are linearly independent solutions of a 3rd order ODE. Their Wronskian $W(x) = 2x^3 \neq 0$ so they are linearly independent as found before. But what about $x = 0$? Must be excluded here by the lack of continuity in the coefficients of the ODE. ♣

If not addressing solutions of an ODE, then potentially the Wronskian may not be definitive test of linear independence.

Example 2.6 (Kreyszig 2011, Example 5, p.109) $e^{-2x}, e^{-x}, e^{x}, e^{2x}$ are solutions of $y'''' - 5y'' + 4y = 0$. Whereas I expect you to compute 2×2 and 3×3 determinants by hand, generally do larger ones with a computer: in Matlab/Octave

```
det([1 1 1 1;-2 -1 1 2;4 1 1 4;-8 -1 1 8])
```

♣

Theorem 2.7 (existence and completeness of general solutions). *(Kreyszig 2011, p.110) Let the coefficients $p_0(x), \dots, p_{n-1}(x)$ be continuous on an open interval I: then*

- *the ODE (2.4) has a general solution on I; and*
- *every solution of the ODE (2.4) is of the form*

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x)$$

where $y_1, y_2, \dots, y_n$ is a basis of solutions of (2.4) and $C_1, C_2, \dots, C_n$ are suitable constants.

⋘*begin lecture 9*⋙

2.2.3 Constant coefficient ODEs

(Kreyszig 2011, §3.2) Restrict attention to constant coefficient homogeneous linear nth-order ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0\,, \tag{2.6}$$

on any open interval I. Seeking solutions of the form $y = e^{\lambda x}$ derive the characteristic equation (Kreyszig 2011, p.112)

$$\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0\,. \tag{2.7}$$

If all n roots are distinct, then we write down a general solution.

As for 2nd order ODEs, three things can happen when solving the characteristic equation (2.7):

1. Distinct real roots → exponential solutions
2. Complex (conjugate) roots → trigonometric solutions
3. Repeated roots → exponentials modified by polynomials

Let's see why...

Example 2.7 (Kreyszig 2011, Example 1, p.112) Solve $y''' - 2y'' - y' + 2y = 0$. ♣

Theorem 2.8. *(Kreyszig 2011, Theorems 2 and 1, p.113)*

- *The functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_m x}$ are linearly independent on an open interval I iff all λ_j are different from each other.*
- *Solutions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$ of* ODE (2.6) *form a basis of solutions on any open interval I iff all n roots of the characteristic equation* (2.7) *are different.*

Complex roots If the coefficients of the ODE (2.6) are real, then any complex roots must occur as complex conjugate pairs, say $\lambda = \gamma \pm i\omega$. In that case, in place of functions $e^{(\gamma+i\omega)t}$ and $e^{(\gamma-i\omega)t}$, we usually prefer to use $e^{\gamma t}\cos\omega t$ and $e^{\gamma t}\sin\omega t$ in the basis (Kreyszig 2011, p.113).

Example 2.8 (Kreyszig 2011, Example 2, p.113) Solve the IVP $y''' - y'' + 100y' - 100y = 0$ such that $y(0) = 4$, $y'(0) = 11$ and $y''(0) = -299$. ♣

Multiple roots If λ is a root of multiplicity m of the characteristic equation (2.7), then m linearly independent solutions of the constant coefficient ODE (2.6) are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad \ldots, \quad x^{m-1}e^{\lambda x}.$$

Example 2.9 (Kreyszig 2011, Example 3, p.114) Solve the ODE $y''''' - 3y'''' + 3y''' - y'' = 0$. ♣

To derive the rule, substitute $e^{\lambda x}$ into the left-hand side of the ODE (2.6) and differentiate with respect to λ.

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2.3 Nonhomogeneous linear ODEs

(Kreyszig 2011, §3.3)

Explore the nonhomogeneous linear nth-order ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)\,, \qquad (2.8)$$

with x on some open interval I, and $r(x) \neq 0$. Its general solution is

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is a general solution of the corresponding homogeneous ODE (2.4), and $y_p(x)$ is any particular solution of the nonhomogeneous ODE (2.8) (Kreyszig 2011, p.116).

Example 2.10 Solve $x^2y'' - 4xy' + 6y = x$. ♣

The ODE (2.8) together with initial conditions (2.3) form an *initial value problem*: it has a unique solution (Kreyszig 2011, p.117).[1]

[1] If there were two solutions, then consider their difference.

2.3.1 Method of undetermined coefficients

Use 'systematic' guesswork to find a particular solution to the non-homogenous linear ODE (2.8), with *constant coefficients* (Kreyszig 2011, p.117, pp.81–2).

Basic rule : With arbitrary coefficients to be determined: if $r(x)$ is exponential, try the same exponential; if $r(x)$ is polynomial, try a polynomial; if $r(x)$ is sine or cosine, try a sum of corresponding sine and cosine; if $r(x)$ is sine or cosine times exponential, try a sum of corresponding sine and cosine times the same exponential.

Modification rule : If in the above, a term in y_p is a homogeneous solution, then multiply it by up to x^m where m is the multiplicity of the term.

Sum rule : When $r(x)$ is a sum of above terms, then try a corresponding sum for y_p.

Example 2.11 (Kreyszig 2011, Example 1, p.82)

- Solve the IVP $y'' + y = 0.001x^2$ with $y(0) = 0$ and $y'(0) = \frac{3}{2}$.

♣

Example 2.12
- Solve the IVP $y'' + 5y' + 6y = 3e^{-2x}$ with $y(0) = y'(0) = 0$.
- Find a general solution to $y''+5y'+6y = 3e^{-2x}+\sin 2x$.

♣

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Example 2.13 Find a particular solution of the ODE $y''' - 4y' = x + 3\cos x + e^{-2x}$. ♣

2.4 Modelling free and forced oscillations

(Kreyszig 2011, §2.4,8)

We model the spring-mass system (Kreyszig 2011, pp.62) as in car suspension or compass needle. Exactly the same mathematics applies to all sorts of oscillators such as neurones in neuroscience, interacting animals in ecology, heart beats, orbits in astronomy, electronic oscillators.

Conservation of momentum The modelling principle for objects is that momentum only changes due to applied forces; that is, the rate of change

$$\frac{d}{dt}(\text{mass} \times \text{velocity}) = \text{forces}.$$

Since the velocity is the time derivative of position, Newton's second law underpins all physical systems (Kreyszig 2011, p.63):

$$\text{mass} \times \frac{d^2}{dt^2}\text{position} = \text{forces}. \tag{2.9}$$

Define symbols: let m denote mass; let $y(t)$ denote position at time t; and according to Hooke's law, the force from a spring is $-ky$ for some constant k (Kreyszig 2011, p.62).

Undamped system This modelling gives the homogeneous ODE for the undamped system to be $my'' + ky = 0$. *Interpret* the solutions as predicting perpetual oscillation (Kreyszig 2011, p.63).

Example 2.14 (Kreyszig 2011, Example 1, p.64) Harmonic oscillations of a mass of 10 kg. ♣

odefor a damped system Another force is friction that damps motion, and modelling often leads to (Kreyszig 2011, p.64) $my'' + cy' + ky = 0$ where c is the constant of damping.[2] Solving the characteristic equation leads to three cases: overdamping, critical damping, and underdamping.

Overdamping : (Kreyszig 2011, pp.65–6) Sketching the characteristic polynomial shows that both roots are negative: *interpret* the solutions as decay to static equilibrium.

Critical damping : (Kreyszig 2011, p.66) Almost the same as overdamping: car suspension and compass needles are designed to be near critical damping.

Underdamping : (Kreyszig 2011, p.67) Complex conjugate roots with negative real-parts interpreted as damped oscillations.

Example 2.15 (Kreyszig 2011, pp.67–8) To the previous example, include effects of damping with constant $c = 100$ kg/sec, $c = 60$ kg/sec, and $c = 10$ kg/sec. ♣

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[2] At higher velocities (such as sports balls) the friction is quadratic in the velocity, whereas for other materials (such as violin strings and bows) the friction is 'stick-slip'.

2.4.1 Resonance in forced oscillations

(Kreyszig 2011, §2.8) For example: car suspension is forced by bumps; a neurone is forced by spikes from neighbouring neurones; a pendulum forced by hand.

When the additional forcing of a mass-spring is simple periodic (Kreyszig 2011, p.86), $F_0 \cos \omega t$ for some strength F_0 and frequency ω (define all terms), then Newton's second law (2.9) (conservation of momentum) becomes $y'' + cy' + ky = F_0 \cos \omega t$.

Seeking a particular solution, $y_p = a \cos \omega t + b \sin \omega t$, of this nonhomogeneous ODE leads to (Kreyszig 2011, pp.86–7)

$$\{a, b\} = F_0 \frac{\{\, m(\omega_0^2 - \omega^2)\,,\ \omega c\,\}}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}\,,$$

for $\omega_0 := \sqrt{k/m}$. Add this to the homogeneous solution.

Interpret Without damping, $c = 0$, there is a very large response near 'resonance' (Kreyszig 2011, pp.87–8). With damping the response is lessened, and phase lagged (Kreyszig 2011, pp.89–90): for example, the seasons lag behind the sun's forcing.

We can now solve a variety of linear ODEs, both homogeneous and inhomogeneous, using a number of different techniques. That's great, but most of our techniques have been restricted to constant-coefficient ODEs. Are there any techniques which can be applied for general linear ODEs, say with non-constant coefficients? Reduction of order is one such method.

2.5 Reduction of order

If one solution to a higher-order ODEis known (either by a guess or by some application of knowledge), then another solution may be found by reduction of order (Kreyszig 2011, pp.51–2).

Example 2.16 (Kreyszig 2011, Example 7, p.51) Find a basis for the ODE $(x^2 - x)y'' - xy' + y = 0$. ♣

For a general 2nd order ODE, $y'' + p(x)y' + q(x)y = 0$, one can find a general formula for a second solution $y_2(x) = y_1(x)u(x)$ given a known solution $y_1(x)$ (Kreyszig 2011, p.52). However, it is the idea that is the key.

The same trick holds for higher order ODEs.

Example 2.17 Use reduction of order to derive the multiplication by x rule in finding the general solution to $y''' - 3y' - 2y = 0$. ♣

Euler–Cauchy differential equations (Kreyszig 2011, §2.5) An nth order, homogeneous, Euler–Cauchy ODE is one that can be arranged to

$$y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = 0\,.$$

Seek solutions $y = x^m$ for some exponent m leads to the characteristic equation

$$m^n + ?m^{n-1} + \cdots + ?m + a_0 = 0\,.$$

- If there are n distinct roots $m_1, \ldots, m_n$, then $y_1 = x^{m_1}, \ldots, y_n = x^{m_n}$ are linearly independent and superposition gives a general solution.
- If any two roots are complex conjugate pair, say $m_{1,2} = \gamma \pm i\omega$, then would could use the general combination $c_1 x^{\gamma+i\omega} + c_2 x^{\gamma-i\omega}$ but typically we instead use the real combination $c_1 x^\gamma \cos(\omega \ln x) + c_2 x^\gamma \sin(\omega \ln x)$.
- If multiple roots occur, then multiply by $\ln x$ (up to a power one less than the multiplicity)!

Example 2.18 (Kreyszig 2011, Case II, p.72) Use Reduction of Order to derive that $x^2 y'' - 5xy' + 9y = 0$ has general solution $y = (c_1 + c_2 \ln x)x^3$ (Kreyszig 2011, Example 2, p.72). ♣

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2.6 Modelling in mathematical ecology

During WW1 Lotka and Volterra observed oscillations in the fish-shark population in the Adriatic Sea (east of Italy). Such oscillations are seen in many predator-prey systems in ecology.

Modelling

Define variables : let $n_1(T)$ be the number of prey at time T and $n_2(T)$ be the number of predators at time T.

Conservation : animals (dis)appear only by ways we quantify.

- The prey breed at rate a and die at a rate depending upon the number of predators, say bn_2: $dn_1/dT = an_1 - bn_2 n_1$.
- Predators would die at a rate d unless they feed in which case the predators would breed at a rate depending upon the number of prey eaten, say cn_1: $dn_2/dT = cn_1 n_2 - dn_2$.

Scale variables : experiments (expensive) could estimate values for the constants a, b, c, d, but *scaling* empowers us to answer a wide range of cases all at once. However, suppose we do know one thing from experiments: the starving death rate of predators is three times the natural birth rate of prey, that is, $d = 3a$.

Task: find some reference time T_0 and reference population numbers N_1 and N_2 that simplify the ODEs. Define $t := T/T_0$, $x_1 := n_1/N_1$ and $x_2 := n_2/N_2$. Then

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{1}{N_1}\frac{dn_1}{dt} = \frac{1}{N_1}\frac{dn_1}{dT}\frac{dT}{dt} = \frac{T_0}{N_1}\frac{dn_1}{dT} \\
&= \frac{T_0}{N_1}[an_1 - bn_2n_1] = \frac{T_0}{N_1}[aN_1x_1 - bN_2N_1x_1x_2] \\
&= (T_0a)x_1 - (T_0bN_2)x_1x_2\,, \\
\frac{dx_2}{dt} &= \cdots = (T_0cN_1)x_1x_2 - (T_0d)x_2\,.
\end{aligned}$$

Choose $T_0 := 1/a$, $N_2 := a/b$ and $N_1 := a/c$, then $T_0d = d/a = 3$ (we supposed) and the ODEs become

$$x_1' = x_1 - x_1x_2\,, \quad x_2' = x_1x_2 - 3x_2\,.$$

Understanding this one pair of ODEs empowers a much wider understanding.

Mathematical solution Unfortunately we cannot solve these nonlinear ODEs. But we do solve approximations via *linearisation*.

Small populations : if the populations of prey and predators are small enough, then the product x_1x_2 is negligible compared to x_1 and x_2 so the ODEs are $x_1' \approx x_1$ and $x_2' \approx -3x_2$ with general solution of $x_1 = c_1e^t$ and $x_2 = c_2e^{-3t}$. *Interpret:* the prey increases rapidly, while the predators die out.

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Near equilibrium : Observe that if ever $x_1 = 3$ and $x_2 = 1$ then the ODEs become $x_1' = 0$ and $x_2' = 0$, then there is no change to the population. That is, $x_1 = 3$ and $x_2 = 1$ is an ecological equilibrium, a balance for all time.

Near equilibrium? Substitute $x_1 = 3+y_1(t)$ and $x_2 = 1+y_2(t)$ for ‘small’ y_1 and y_2. Then

$$\begin{aligned}
\frac{dy_1}{dt} &= \frac{dx_1}{dt} = x_1 - x_1x_2 = (3+y_1) - (3+y_1)(1+y_2) \\
&= 3 + y_1 - 3 - 3y_2 - y_1 - y_1y_2 \\
&= -3y_2 - y_1y_2 \approx -3y_2\,, \\
\frac{dy_2}{dt} &= \cdots = y_1 + y_1y_2 \approx +y_1\,.
\end{aligned}$$

Find a familiar 2nd order ODE by

$$\frac{d^2y_1}{dt^2} = \frac{d}{dt}\left[\frac{dy_1}{dt}\right] \approx \frac{d}{dt}[-3y_2] = -3\frac{dy_2}{dt} \approx -3y_1 .$$

That is, $y_1'' + 3y_1 = 0$ with general solution $y_1 = c_1\cos\sqrt{3}t + c_2\sin\sqrt{3}t$. *Interpret:* since $n_1 = N_1x_1 = N_1(3+y_1) = N_1[3 + c_1\cos(\sqrt{3}aT) + c_2\sin(\sqrt{3}aT)]$ the number of prey (fish) oscillate perpetually about the equilibrium with frequency $\sqrt{3}a$.

Such techniques and applications are taken further in the course *Modelling with* ODE*s* and research in ecology.

2.7 Systems of ODEs

(Kreyszig 2011, §4.1)

Example 2.19 Age structured populations are one case: akin to disease/epidemic modelling. ♣

Any nth order ODE may be written as a system of n coupled first order ODEs (Kreyszig 2011, p.135).

Further, when the ODE $F(y^{(n)}, \ldots, y', y, t) = 0$ is linear, then the corresponding system of ODEs takes the form $\dot{\mathbf{y}} = A(t)\mathbf{y} + \mathbf{r}(t)$.

Example 2.20 Convert $\ddot{y} - 4\dot{y} + 3y = 0$ to a system. ♣

2.8 Constant coefficient homogeneous systems

Recall that for nth order ODEs this case was the base case from which all was derived. Let's look just at this base case for systems of ODEs (Kreyszig 2011, §4.3).

Consider the system of n ODEs $\dot{\mathbf{y}} = A\mathbf{y}$. Seek solutions of the form $\mathbf{y} = \mathbf{x}e^{\lambda t}$ and find the eigenproblem from linear algebra of

$$A\mathbf{x} = \lambda\mathbf{x} . \tag{2.10}$$

Theorem 2.9. *(Kreyszig 2011, Theorem 1, p.141) If constant matrix A in system (2.10) has n linearly independent eigenvectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ then a general solution of (2.10) is*

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + \cdots + c_n\mathbf{x}^{(n)}e^{\lambda_n t}.$$

Example 2.21 Solve the system

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}\mathbf{y} .$$

♣

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Example 2.22 A more complex example is to solve the system

$$\frac{dy}{dt} = \begin{bmatrix} -2 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} y\,.$$

♣

Recall that a symmetric matrix always has n linearly independent eigenvectors. But if not symmetric, then what happens if the matrix does not have n linearly independent eigenvectors?

Example 2.23 Solve the system

$$\frac{dy}{dt} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} y\,.$$

♣

Theorem 2.10 (generalised eigenvectors). *Consider the system of* n ODE*s* $\dot{y} = Ay$*. Suppose* λ *is an eigenvalue of* A *of multiplicity two with only one linearly independent eigenvector* $\mathbf{x}$*. Then the terms in the general solution corresponding to eigenvalue* λ *are* $c_1\mathbf{x}e^{\lambda t} + c_2(\mathbf{x}t + \mathbf{v})e^{\lambda t}$ *for* ***generalised eigenvector*** $\mathbf{v}$ *satisfying*

$$(A - \lambda I)\mathbf{v} = \mathbf{x}\,.$$

Example 2.24 (orangutans on a knife edge). From Wikipedia entry on orangutans [20 Mar 2014]

> Gestation lasts for 9 months, with females giving birth to their first offspring between the ages of 14 and 15 years. Female orangutans have eight-year intervals between births, the longest inter-birth intervals among the great apes. ... Infant orangutans are completely dependent on their mothers for the first two years of their lives. The mother will carry the infant during travelling, as well as feed it and sleep with it in the same night nest. For the first four months, the infant is carried on its belly and never relieves physical contact. In the following months, the time an infant spends with its mother decreases. When an orangutan reaches the age of two, its climbing skills improve and it will travel through the canopy holding hands with other orangutans, a behaviour known as "buddy travel". Orangutans are juveniles from about two to five years of age and will start to temporarily move away from their mothers. Juveniles are usually weaned at about four years of age. Adolescent orangutans will

socialize with their peers while still having contact with their mothers. Typically, orangutans live over 30 years in both the wild and captivity.

♣

3 Partial differential equations

(Kreyszig 2011, Chapt. 12)

Contents

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3.1 Fundamental concepts of PDEs

(Kreyszig 2011, §12.1)

Definitions (Kreyszig 2011, p.541)

- A **partial differential equation** (PDE) is an equation involving derivatives in two or more independent variables.
- The **order** of a PDE is the order of the highest derivative.
- A PDE is **linear** if the dependent function (and its derivatives) appear only linearly.
- A PDE is **homogeneous** if each term contains the dependent function (or a derivative), otherwise called **nonhomogeneous**.
- **Superposition**: if u_1 and u_2 are solutions of a homogeneous linear PDE in some region R, then $u = c_1u_1 + c_2u_2$ is also a solution of that PDE in the region R for all constants c_1 and c_2.

(Note: each of these definitions are analogous to ODEs.)

Some important PDEs (Kreyszig 2011, p.541) for fields $u(x,t), u(x,y), u(x,y,t)$ or $u(x,y,z)$, and using subscripts for partial derivatives for brevity:

- $u_t + cu_x = 0$, 1D elementary wave equation;
- $u_{tt} = c^2u_{xx}$, 1D wave equation, 2nd order;
- $u_t = c^2u_{xx}$, 1D heat equation, 2nd order;
- $u_{xx} + u_{yy} = 0$, 2D Laplace equation, 2nd order;

- $u_{xx} + u_{yy} = f(x, y)$, 2D Poisson equation, 2nd order, non-homogeneous;
- $iu_t = -u_{xx} + V(x)u$, Schrödinger equation, 2nd order, complex, spatially varying; and its nonlinear version $iu_t = -u_{xx} + V(x)u + |u|^2u$—occurs in quantum physics and also the amplitude of water waves;
- $u_t - ru + (1 + \partial_{xx} + \partial_{yy})^2u + u^3 = 0$, Swift–Hohenberg equation, 2D, 4th order, nonlinear—used to model patterns such as stripes on a tiger, spots on a leopard or spatial patterns of chemical reactions in a dish. There exist some nice visualisations of pattern formation.

Example 3.1 A few PDEs can be solved using ODE methods. Find a general solution of $u_{xy} + u_x = 0$ (Kreyszig 2011, Example 3, p.542). ♣

Make a well-posed problem (Kreyszig 2011, p.541) Need boundary conditions and/or initial conditions: roughly, if nth order in a space/time independent variable, then need n boundary/initial conditions in that variable.

A general applied mathematical modelling framework derives PDEs, ODEs, ...

1. Choose a level of description (particle, lumped, continuum).
2. Define all variables and constants (with units).
3. Variables may only change in quantifiable ways—conservation. However, observations are often needed to close the problem.
4. Scale variables to usefully reduce parameter groups.
5. Invoke mathematics: theorems, methods, approximations, computations, and so on.
6. Interpret mathematical predictions.

3.2 Waves in car traffic

Figure 3.1 shows car traffic on a highway: PDEs model the movement of cars. You can find some neat interactive visualisations of this phenomena online.

1. *Choose:* model the number of cars per km, not each individual car.
2. *Define:* x measures position along the highway (km); t is time (min); $\rho(x, t)$ is the density of cars on a highway (cars/km/lane); and $u(x, t)$ is the velocity of the cars 'at' position x.

Figure 3.1: highway traffic [photo by Daniel Reiter].

3. *Conservation:* cars do not appear or disappear.
   Theorem 3.1. *If a function* $f(x)$ *is continuous on some interval* $[c,d]$ *and satisfies* $\int_a^b f\,dx = 0$ *for all* $c < a < b < d$, *then* $f(x) = 0$ *for all* $c < x < d$.

   Consequently, conservation derives the 1st order nonlinear conservation PDE

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad \text{for all } x.} \tag{3.1}$$

   Observation closes the problem: in low density cars travel at about 60 km/hr, that is, 1 km/min; at high densities cars jam and stop, typically at densities $\rho_j \approx 140$ cars/km/lane. To a fair approximation, the velocity $u \approx 1 - \rho/140$ km/min. Thus

$$\frac{\partial \rho}{\partial t} + (1-\rho/70)\frac{\partial \rho}{\partial x} = 0 \quad \text{for all } x, \quad \text{provided } \rho \le 140\,. \tag{3.2}$$

4. *Scaling:* omit here.
5. *Linearise* Constant density is an exact solution where all cars travel at the same speed.

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Example 3.2 Suppose car density is near 35 cars/km: seek solution $\rho = 35 + r(x,t)$ for small r. ♣

Generally, suppose density is $\rho(x,t) = \rho_* + r(x,t)$ for small r, then we derive the 1D, elementary, wave PDE

$$\frac{\partial \rho}{\partial t} + c_* \frac{\partial \rho}{\partial x} \approx 0 \quad \text{for } c_* = 1 - \rho_*/70 \text{ km/min}. \tag{3.3}$$

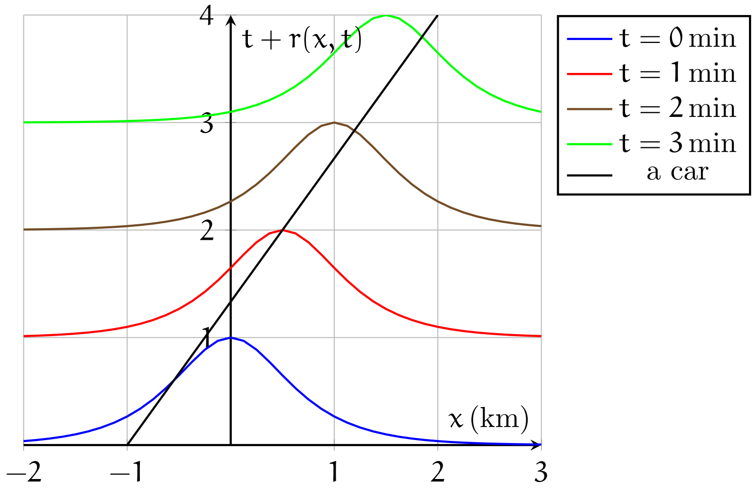

Figure 3.2: density wave of cars, and a car path. Density plots displaced vertically for clarity.

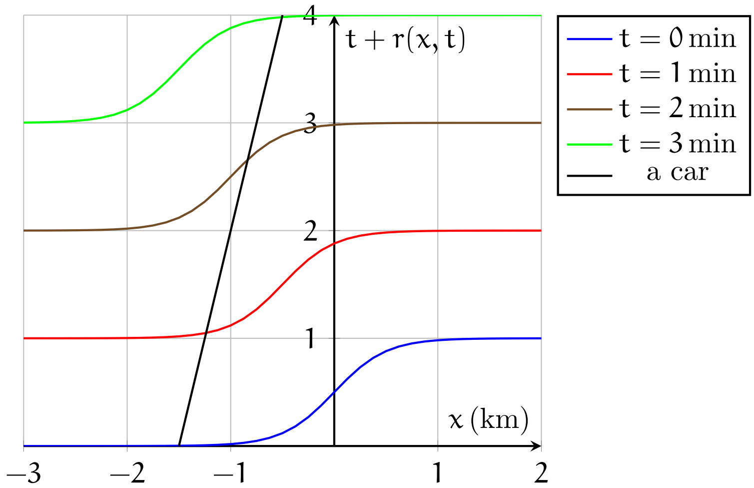

Figure 3.3: density wave of cars, and a car path. Density plots displaced vertically for clarity.

D'Alembert's solution of the wave PDE (3.3) is $\rho = f(x - c_* t)$ for any function f. Substitute and check. Initial conditions determine $f(x) = \rho_0(x)$.

6. *Interpret: car waves propagate.* The predicted density is then $\rho = \rho_0(x - c_* t)$: the $c_* t$ shifts the shape so that the shape travels at velocity c_*.

Example 3.3 Suppose car density is near 105 cars/km, then $c_* = 1 - 105/70 = -\frac{1}{2}$ km/min. In dense traffic the shape is a backwards wave, see Figure 3.3, even though the cars travel forwards at speed $u = 1 - 105/140 = \frac{1}{4}$ km/min. ♣

3.3 Solving the heat PDE using separation of variables

(Kreyszig 2011, §12.6)

We now consider the *parabolic* heat equation (PDE)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\,, \quad 0 \le x \le L\,, \quad t \ge 0\,, \tag{3.4}$$

with boundary and initial conditions

$$u(0,t) = u(L,t) = 0\,, \quad u(x,0) = f(x) \text{ (given)}. \tag{3.5}$$

This PDE arises from conservation of heat energy together with an experimentally observed Fourier's law (Kreyszig 2011, §12.5).

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Separation of variables (Kreyszig 2011, pp.559–561) has three main steps: seek product form solutions; satisfy boundary conditions; superpose and then initial conditions determine the solution.

1. Seek whatever solutions we can find in the form $u(x,t) = X(x)T(t)$. This form requires

   $$X'' - kX = 0\,, \quad \dot T - c^2 kT = 0\,,$$

   with boundary conditions $X(0) = X(L) = 0$.

   Linear algebra The ODE $X'' = kX$ such that $X(0) = X(L) = 0$ determines *eigenvalues* k and *eigenfunctions* X. Exactly as eigenvectors form a change of basis to diagonalise matrices (Kreyszig 2011, §8.4), so such eigenfunctions form a change to a Fourier series basis to 'diagonalise' the differential operator $\partial^2/\partial x^2$. Such 'diagonalisation' empowers solving the PDE.

2. The only non-trivial solutions for the PDE (3.4) and corresponding boundary conditions (3.5) occur for constant $k = -(n\pi/L)^2$ when $X_n = \sin\frac{n\pi x}{L}$. Solving for corresponding T_n we get an infinite family of solutions

   $$u_n(x,t) = \sin\frac{n\pi x}{L}\exp(-\lambda_n^2 t), \quad \text{for } \lambda_n = \frac{cn\pi}{L}\,.$$

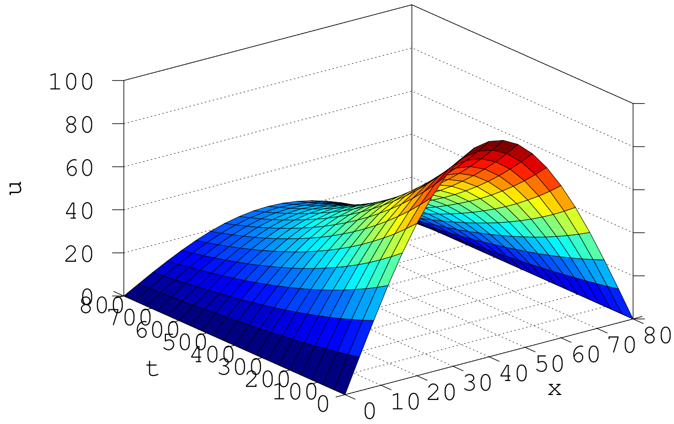
3. Superposing gives Fourier series solution

   $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\exp(-\lambda_n^2 t),$$

   where the initial condition (3.5) determines coefficients B_n.

Separation of variables is not a rigid prescription, but a flexible tool—almost the only tool for PDEs.

Example 3.4 (Kreyszig 2011, Examples 1–2, p.561) Find the temperature $u(x,t)$ in a 80 cm copper bar if the initial temperature is $100\sin(\pi x/80)^\circ$C and the ends are kept at 0°C. For copper $c^2 = 1.158\,\text{cm}^2/\text{sec}$. How long will it take the maximum temperature to drop to 50°C?




```
x=linspace(0,80,21);
t=linspace(0,800,21);
[X,T]=meshgrid(x,t);
U=100*sin(pi*X/80).*exp(-0.001785*T);
surf(x,t,U)
xlabel('x'),ylabel('t'),zlabel('u')
```

What if the initial temperature is $100\sin(3\pi x/80)^\circ$C? ♣

Specifying the field value u (the temperature) at the ends of the domain is called a *Dirichlet boundary condition*. Alternatively, specifying the gradients u_x at the ends of the domain is called a *Neumann boundary condition*.

Example 3.5 (Kreyszig 2011, Example 4, p.563) Use separation of variables to solve the heat PDE (3.4) with initial condition (3.5), but now with the 'insulating' Neumann boundary conditions of $u_x(0,t) = u_x(L,t) = 0$. Find a general solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \exp(-\lambda_n^2 t), \quad \text{for } \lambda_n = \frac{cn\pi}{L}\,.$$

The zero eigenvalue conserves heat energy. ♣

In the solution above we need to determine the coefficients A_n from the initial conditions. Substituting $t = 0$ we obtain

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x)$$

for the given function $f(x)$. So how do we find A_n? This is the subject of the next chapter.

4 Representing periodic functions by Fourier series

(Kreyszig 2011, Chapt. 11)

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Fourier series empower us to look at differential equations from a new angle and discover new information about them, as well as to satisfy given initial conditions in certain PDEs. It relates to Linear Algebra the orthogonal diagonalisation of a symmetric matrix.

4.1 Fourier series

(Kreyszig 2011, §11.1)

Definition 4.1 (periodic functions)**.** *(Kreyszig 2011, p.475) A function $f(x)$ is called a* ***periodic function*** *if it is defined for (almost) all real x and if there is some* ***period*** *$p > 0$ such that $f(x + p) = f(x)$ for all x.*

For examples: $\cos x$; $\sin(2\pi x/p)$; constant; $\tan x$; Figure 4.1; but not e^x nor x.

It follows that $f(x + np) = f(x)$ for all integer n (Kreyszig 2011, p.475).

Further, if $f(x)$ and $g(x)$ are p-periodic, then so is $c_1 f(x) + c_2 g(x)$. For example, $c_1 \cos x + c_2 \cos 2x$ is 2π-periodic.

Definition 4.2 (Fourier series)**.** *(Kreyszig 2011, p.476) For some set of coefficients $a_0, a_1, b_1, a_2, b_2, \ldots$, the infinite sum*

$$a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \tag{4.1}$$

is called a ***Fourier series****. If the sum converges, to some function $F(x)$, then the sum is called the Fourier series of $F(x)$.*

The sum $F(x)$ is 2π-periodic since all its components are. Figure 4.1 shows an example with 'randomly' chosen coefficients ($\propto \frac{1}{n^2}$). Draw such graphs with Matlab/Octave:

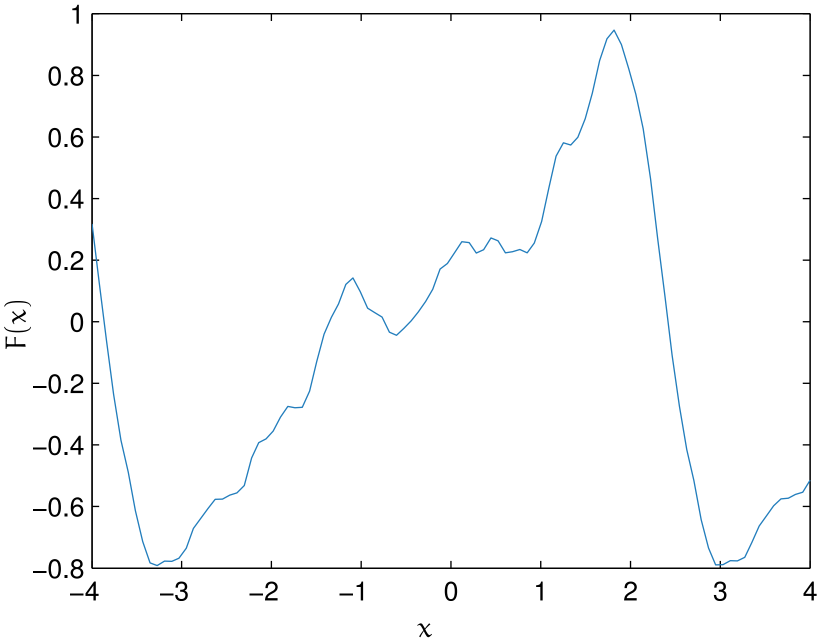

Figure 4.1: an example function with period 2π.

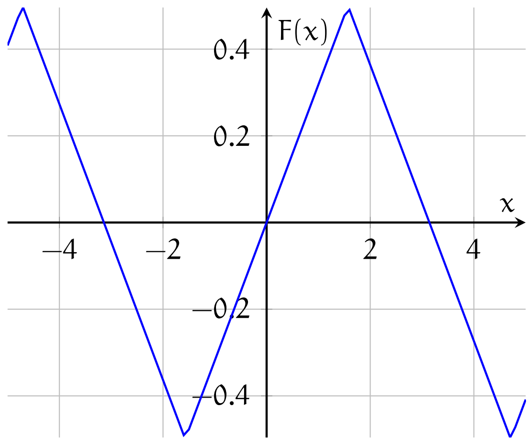

Figure 4.2: sawtooth function for Example 4.1.

```
x=linspace(-5,5);
n=1:49;
an=randn(1,49)./n.^2
F=an*cos(n'*x);
plot(x,F)
```

Example 4.1 Crudely estimate the first couple of coefficients of a Fourier series that would sum to the sawtooth function of Figure 4.2. ♣

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Theorem 4.3 (Euler formulae). *(Kreyszig 2011, p.476) If the*

Fourier series (4.1) converges to $F(x)$, then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx\,, \tag{4.2a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx\,, \tag{4.2b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx\,. \tag{4.2c}$$

Example 4.2 (a square wave). Consider the 2π-periodic function

$$F(x) = \begin{cases} 1 & |x| < \pi/2\,, \\ 0 & \pi/2 \le |x| \le \pi\,. \end{cases}$$

Assuming a Fourier series exists to sum to this function, what would be its coefficients? What identity does $F(0)$ imply? ♣

Theorem 4.4 (trigonometric orthogonality). *(Kreyszig 2011, Theorem 1, p.479) The functions $\{1, \cos nx\,, \sin nx \mid n = 1, 2, 3, \ldots\}$ are* ***orthogonal*** *on the interval $-\pi \le x \le \pi$.*

Now derive the Euler formulae (4.2) for any function $F(x)$ that is the sum of a Fourier series.

⋘*begin lecture 21*⋙

Turn the approach around and assert that most given functions $f(x)$ have a Fourier series. The functions need to be piecewise well-behaved, such as the square wave, or $|\sin x|$, or $\max(0, \cos x)$, but not $\sqrt{|\sin x|}$ nor $\sec 2x$.

Theorem 4.5 (representation by a Fourier series). *(Kreyszig 2011, Theorem 2, p.480) Let $f(x)$ be 2π-periodic, piecewise continuous, and have left-hand and right-hand derivatives at each point, then the Fourier series (4.1) with coefficients (4.2) converges. Its sum is $f(x)$, except at discontinuities where it is the average of the left and right limits at each such point.*

Omit proof (Kreyszig 2011, p.481).

Example 4.3 The square wave is discontinuous at $x = \pm\pi/2$. ♣

Example 4.4 (forced oscillation). (Kreyszig 2011, §11.3) Solve the ODE $y'' + 7y = F(x)$ for the square wave F. ♣

4.2 Generalising Fourier series

(Kreyszig 2011, §11.2)

Introduce three variations:

- generalise to arbitrary period;
- utilise symmetry;
- introduce half-range versions.

4.2.1 Arbitrary period

Suppose function $f(x)$ is $2L$-periodic. A change of variable (Kreyszig 2011, pp.483–4) tells us that it may be written as the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right], \tag{4.3}$$

with Fourier coefficients

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx\,, \tag{4.4a}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx\,, \tag{4.4b}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx\,. \tag{4.4c}$$

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Example 4.5 (square wave) (Kreyszig 2011, Example 1, p.484) Find the Fourier series of the 4-periodic function

$$f(x) = \begin{cases} 1 & -1 < x < 1\,, \\ 0 & 1 < x < 3\,. \end{cases}$$

♣

4.2.2 Even and odd functions simplify

(Kreyszig 2011, pp.486–8)

Definition 4.6. *(Kreyszig 2011, p.486)*

- *If $2L$-periodic function f is* ***even****, $f(-x) = f(x)$, then its* ***Fourier cosine series*** *is $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$ where coefficients*

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx\,, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx\,.$$

- *If $2L$-periodic function f is* ***odd****, $f(-x) = -f(x)$, then its* ***Fourier sine series*** *is $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ where coefficients*

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx\,.$$

Example 4.6 (sawtooth wave). (Kreyszig 2011, Example 5, pp.487–8) Find the Fourier series of the 2π-periodic function

$$f(x) = x + \pi, \quad -\pi < x < \pi.$$

This is π plus an odd function, so seek Fourier sine series.

```
x=linspace(-5,5);
n=1:20;
bn=-2*cos(n*pi)./n
y=pi+bn*sin(n'*x);
plot(x,y)
```

♣

4.2.3 Half-range expansion

Physical problems usually only defined on a finite interval, say $[0, L]$: violin string, metal bar, quantum potential well, ... A function defined on the interval $[0, L]$ may be periodically extended either as a $2L$-periodic even function, or as a $2L$-periodic odd function. Consequently it may be represented as either a Fourier cosine or Fourier sine series, as we choose most convenient for us.

Example 4.7 (triangle wave). (Kreyszig 2011, pp.489–90) Find the two half-range expansions of function

$$f(x) = \begin{cases} \frac{2}{L}x & 0 < x < \frac{L}{2}, \\ \frac{2}{L}(L-x) & \frac{L}{2} < x < L. \end{cases}$$

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```
L=4
x=linspace(-L-1,L+1);
n=1:10;
an=4/pi^2./n.^2.*(2*cos(n*pi/2)-cos(n*pi)-1)
ycos=0.5+an*cos(n'*x*pi/L);
bn=8/pi^2./n.^2.*sin(n*pi/2)
ysin=bn*sin(n'*x*pi/L);
plot(x,ycos,x,ysin)
```

♣

We can now (at last) complete our Example 3.5.

Example 4.8 (temperature in a bar revisited). (Kreyszig 2011, Example 12.5, p.563) Consider the heat equation in a bar with insulated ends, with initial temperature of the bar:

$$f(x) = \begin{cases} x & 0 < x < \frac{L}{2}, \\ L-x & \frac{L}{2} < x < L. \end{cases}$$

♣

5 PDEs and waves

(Kreyszig 2011, Chapt. 12)

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5.1 The wave equation

The first aim of this section is to show that water waves on a flat bed satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{5.1}$$

where $u(x, t)$ is the velocity of water above the bed. Many physical systems satisfy this wave PDE (Kreyszig 2011, §12.2, e.g.). This section then looks at two important classes of solution.

5.1.1 Modelling water waves, floods and tsunamis

Define as in Figure 5.1: $h(x, t)$ is water depth; and $u(x, t)$ is the lateral water velocity (*choose* to resolve only the vertical average). These change in ways we quantify.

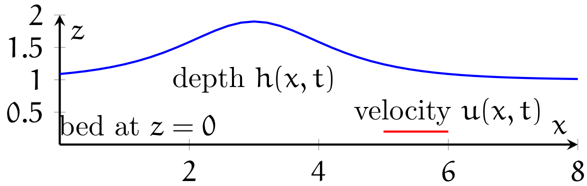

Figure 5.1: schematic diagram of water wave on a flat bed.

Water is conserved The 'density' of water at any point (in kg/m) is $\rho(x,t) := \rho_* w h(x,t)$ where $\rho_* \approx 1000\,\text{kg/m}^3$ is the 3D density of water, and w is the width of the channel/river/estuary. The conservation PDE (3.1) applies to imply

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0\,, \tag{5.2}$$

when the water moves with lateral velocity $u(x,t)$.

Momentum is forced Recall Newton's law that momentum is conserved except when acted upon by forces: $d/dt(\text{momentum}) = (\text{forces})$; a consequence is that the water acceleration is driven by depth gradients, $u_t + uu_x = -gh_x$. But how? We derive the **momentum equation** for water waves

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}\,. \tag{5.3}$$

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Linearise For small velocity u and water nearly the constant depth D we find

$$\frac{\partial h}{\partial t} + D\frac{\partial u}{\partial x} \approx 0\,, \quad \frac{\partial u}{\partial t} \approx -g\frac{\partial h}{\partial x} \implies \frac{\partial^2 u}{\partial t^2} \approx c^2\frac{\partial^2 u}{\partial x^2}$$

for 'wave speed' $c := \sqrt{gD}$.

5.1.2 Separation of variables is the algebraic method

(Kreyszig 2011, §12.3)

Let's solve the wave PDE (5.1). We need two initial and two boundary conditions as the PDE is 2nd order in both time and space: say a harbour/bathtub/well of length L so

$$\begin{aligned} &u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for all } t, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } 0 \le x \le L\,. \end{aligned}$$

Separation of variables

1. (Kreyszig 2011, pp.545–6) Seeking $u = X(x)T(t)$ leads to

$$\ddot{T} - c^2 kT = 0\,, \quad X'' - kX = 0\,, \quad X(0) = X(L) = 0\,.$$

2. (Kreyszig 2011, pp.546–7) Hence a family of solutions for the PDEand corresponding boundary conditions are

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= \left(B_n \cos\frac{cn\pi t}{L} + B_n^* \sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi}{L}x\,. \end{aligned}$$

3. (Kreyszig 2011, pp.548–9) Superpose to get 'general' solution

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\frac{cn\pi t}{L}+B_n^*\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi}{L}x\,,$$

where initial conditions determine the coefficients as a Fourier sine series.

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Example 5.1 (Kreyszig 2011, p.542) Separation of variables finds a general solution of the PDE $u_{xy}+u_x=0$. ♣

5.1.3 D'Alembert suggests classification

(Kreyszig 2011, §12.4)

For the wave, when an initial condition is $u_t(x,0)=0=g(x)$ then, using $\cos A\,\sin B=\frac{1}{2}[\sin(B-A)+\sin(B+A)]$, our solution becomes a sum of two waves (Kreyszig 2011, p.549):

$$u(x,t)=\tfrac{1}{2}[f(x-ct)+f(x+ct)].$$

D'Alembert's solution of the wave PDE generalises this to (Kreyszig 2011, pp.553–4)

$$u(x,t)=\phi(x-ct)+\psi(x+ct)$$

for any twice differentiable functions ϕ and ψ.

Classification (Kreyszig 2011, p.553–5) Consider the wide class of 2nd order PDEs

$$Au_{tt}+2Bu_{tx}+Cu_{xx}=F(t,x,u,u_t,u_x). \tag{5.4}$$

Changing variables to $v:=x-c_1t$ and $w:=x-c_2t$ the PDE becomes the normal form PDE $u_{vw}=G$ when 'speeds' c_1 and c_2 are the two roots of the **characteristic equation**

$$Ac_j^2-2Bc_j+C=0\,;\quad\text{that is, } c_j=\left(B\pm\sqrt{B^2-AC}\right)/A\,.$$

Example 5.2 When right-hand side $F=0$ then the PDE has the general D'Alembert solution $u=f(x-c_1t)+g(x-c_2t)$. ♣

A PDE (5.4) is classified as

- **hyperbolic** when $B^2-AC>0$ (e.g. wave PDE),
- **parabolic** when $B^2-AC=0$ (e.g. heat PDE),
- **elliptic** when $B^2-AC<0$ (e.g. Laplace's PDE).

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5.2 Solving Laplace's PDE using separation of variables

In 2D space, *steady in time*, the wave PDE for membrane, the heat PDE for temperature field $u(x, y)$, or a PDE for a soap film reduces to the *elliptic* **Laplace's equation** (Kreyszig 2011, pp.564)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,. \tag{5.5}$$

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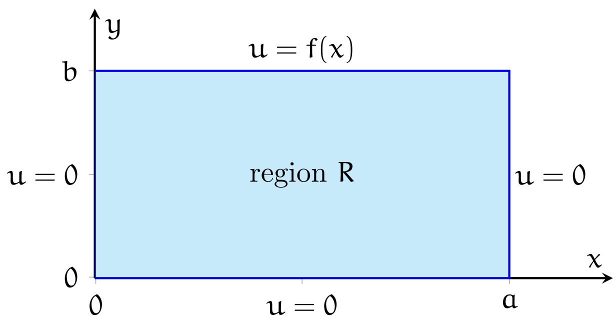
Three types of boundary conditions commonly form a well-posed **boundary value problem**:

- **Dirichlet** specifies the value u on the boundary;
- **Neumann** specifies the normal derivative, $\partial u/\partial n = \hat{n} \cdot \boldsymbol{\nabla} u$, on the boundary;
- **mixed** is with u specified on part of the boundary, and $\partial u/\partial n$ specified on the rest.

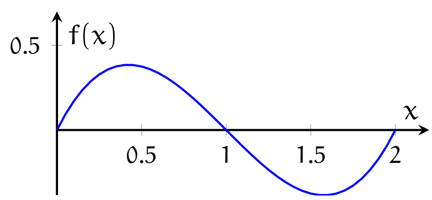
Let's explore a Dirichlet BVP in a rectangle R:

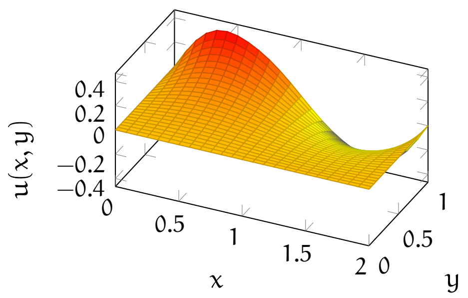


Separation of variables (Kreyszig 2011, pp.563–4) leads to the solution

$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}\,,$$
$$A_n^* = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\,dx\,.$$

Example 5.3 Let's take length $a = 2$, width $b = 1$ and $f(x) = x(x-1)(x-2)$.





To check, perhaps draw in Matlab/Octave with

```
nxy=21 % make x and y vectors of coordinates
x=linspace(0,2,nxy);
y=linspace(0,1,nxy);
% put all combinations in row vectors X and Y
[X,Y]=meshgrid(x,y);
X=X(:)'; Y=Y(:)';
m=1:11; % sum over these modes
am=12./(pi^3*m.^3.*sinh(m*pi))
U=reshape(am*(sin(pi*m'*X).*sinh(pi*m'*Y)),nxy,nxy);
surf(x,y,U) % surface plot
xlabel('x'), ylabel('y'), zlabel('u')
```

♣

⋘*begin lecture 27*⋙

5.3 Waves in a pool doubles the Fourier series

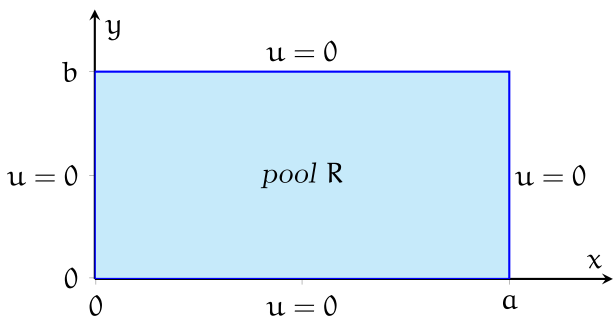
(Kreyszig 2011, §12.9)

Modelling the vibrations of a membrane (Kreyszig 2011, §12.8, not lectured) leads to the 2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{5.6}$$

Instead, here a plausible argument obtains the same PDE for water waves where in (5.6) $u(x, y, t)$ represents the water depth and $c^2 = gD$ for mean depth D.

Sloshing in a rectangular pool *Apply Dirichlet boundary conditions around the $a \times b$ rectangle that $u = 0$.*



Apply initial conditions known water depth and known initial movement:

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y) \quad \text{in } R.$$

The three steps invoke separation of variables twice (Kreyszig 2011, pp.578–584):

1. *Separate* $u(x,y,t) = F(x,y)T(t)$ *and subsequently separate* $F(x,y) = X(x)Y(y)$ *to get three familiar* ODEs*:*

$$\ddot{T} + \lambda^2 T = 0\,, \quad \frac{d^2X}{dx^2} + k^2 X = 0\,, \quad \frac{d^2Y}{dy^2} + p^2 Y = 0\,.$$

2. *The boundary conditions determine allowable* X *and* Y *to form a doubly infinite family of possible* $F_{mn}(x,y)$ *and hence possible* $u_{mn}(x,y,t)$.
3. *Superposition via a double sum empowers us to satisfy the given initial conditions.*

The algebra is complicated in any application, but this general procedure is most useful. Furthermore, even when we cannot do the algebra explicitly, the decomposition into a sum of spatially complicated modes is typical.

6 Series solutions of ODEs create special functions

⋘*begin lecture 28*⋙

Most differential equations cannot be solved algebraically. One method of approximation is to construct Taylor series of solutions, and more general series solutions. Important classes of such solutions form *special functions*. (Kreyszig 2011, Chapt. 5)

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Example 6.1 (edge waves). How can we understand the picture in Figure 6.1? Answer: solve the ODE $xy'' + (1-2x)y' + ny = 0$.

Figure 6.1: waves on the ocean edge at the shore, Port Fairy, 1991. These are not the ocean waves we usually see!

But the ODE has varying coefficients! Can use Reduction of Order to find the solution. ♣

6.1 Solving non-homogeneous ODEs by variation of parameters

(Kreyszig 2011, §2.10, §3.3)

Find a particular solution of the ODE

$$y'' + p(x)y' + q(x)y = r(x) \tag{6.1}$$

when we know a general homogeneous solution $y_h = c_1y_1(x) + c_2y_2(x)$ but the right-hand side $r(x)$ defeats 'undetermined coefficients'. Be a little devious.

Example 6.2 Find a particular solution of $y''-\frac{4}{x}y'+\frac{6}{x^2}y = x^2\ln x$ given two homogeneous solutions are $y_1 = x^2$ and $y_2 = x^3$. ♣

In the general 2nd order case (Kreyszig 2011, pp.101–2), seek a particular solution

$$y = u(x)y_1(x) + v(x)y_2(x) \quad \text{where } u'y_1 + v'y_2 = 0\,.$$

Substituting into the general ODE (6.1) eventually leads to integrals for u and v which give solution

$$y = -y_1\int\frac{y_2r}{W}\,dx + y_2\int\frac{y_1r}{W}\,dx \quad \text{where } W = y_1y_2' - y_2y_1'$$

is the Wronskian.[1] The integration constants in these integrals serve to give the general solution component.

In applications, these integrals might be done numerically. In further mathematics, the right-hand side $r(x)$ might be a nonlinear function of the unknown $y(x)$ in which case the 'particular solution' above becomes an integral equation for $y(x)$.

You may come across problems in other courses where traditionally people use the so-called Laplace Transform. The variation of parameters formula can almost always replace the Laplace Transform method, in conjunction with solution of homogeneous ODEs (perhaps via computer algebra).

What if we don't have homogeneous solutions to our ODEalready? Can't I have a general method which I can throw at any ODE? Yes, sort of, but be careful what you wish for...

6.2 Power series method

(Kreyszig 2011, §5.1)

[1] Remembering this formulae is optional, but we expect you to be able to do the derivation in general and in specific examples.

Definition 6.1. *(Kreyszig 2011, p.168) A **power series** about the **centre** x_0 is a series*

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\,, \qquad (6.2)$$

*with constant **coefficients** $a_0, a_1, a_2, \ldots$. Mostly, the centre $x_0 = 0$.*

Example 6.3 (Kreyszig 2011, Example 1, p.168) Three familiar power series are for $1/(1-x)$, e^x and $\cos x$. ♣

Example 6.4 (Kreyszig 2011, Example 2, pp.168–9) Find a power series solution of $y'-y=0$ (whose solution we know is generally $y=ce^x$). ♣

Power series method by hand (Kreyszig 2011, p.169, modified) To solve the ODE

$$h(x)y'' + p(x)y' + q(x)y = r(x) \qquad (6.3)$$

provided $h(x_0) \neq 0$:

1. represent $h(x), p(x), q(x), r(x)$ as power series in $(x-x_0)$;
2. substitute a solution y in the power series form (6.2) with as yet unknown coefficients;
3. collect like powers of $(x-x_0)$, equate coefficients;
4. solve coefficient equations starting from the lowest order and working up.

Example 6.5 Solve $y''+xy = 1/(1+x^2)$ in a power series about centre $x_0 = 0$. ♣

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Use Picard iteration with computer algebra To solve the ODE

$$y'' + p(x)y' + q(x)y = r(x)\,, \qquad (6.4)$$

rearrange to $y'' = r(x) - p(x)y' - q(x)y$ then extend Picard iteration:

1. start with initial approximation $y_0(x) := a_0 + a_1(x-x_0)$;
2. compute the right-hand side $F_{n-1}(x) := r - py'_{n-1} - qy_{n-1}$;
3. compute new approximation $y_n(x) := a_0 + a_1(x-x_0) + \int_{x_0}^{x}\int_{x_0}^{x} F_{n-1}(x)\,dx\,dx$;
4. repeat until $y_n(x)$ settles—up to some chosen power of $(x-x_0)$.

Example 6.6 Solve $y'' + xy = 1/(1+x^2)$ in a power series about centre $x_0 = 0$. ♣

The same method works for nonlinear ODEs written in the form $y'' = f(x, y, y')$ such that $y(x_0) = a_0$ and $y'(x_0) = a_1$.

Example 6.7 Solve the nonlinear ODE $y'' + xy^2 = 1/(1+x^2)$ in a power series about centre $x_0 = 0$. ♣

6.2.1 Recall power series properties

(Kreyszig 2011, pp.170–4) (Stewart 2012, Chapt.11)

- The series (6.2) has **partial sum**

$$\begin{aligned} s_n(x) &:= \sum_{m=0}^{n} a_m(x-x_0)^m \\ &= a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n. \end{aligned}$$

- The series (6.2) **converges** at x if $\lim_{n\to\infty} s_n(x)$ exists, otherwise it is **diverges** at x.

- The series (6.2) **converges absolutely** at x if the series $\sum_{m=0}^{\infty} |a_m(x-x_0)^m|$ converges: convergence absolutely implies convergence (but the converse may not).

  The previous are generic for series, the following are specific to power series.

- The ratio test shows that a power series (6.2) generally has a **convergence interval**, $|x - x_0| < R$ for some **radius of convergence** R.

  Exercise 6.1 (Kreyszig 2011, Example 4, p.172) Convergence radius of $R = \infty, 1, 0$. ♣

- Power series (6.2) with positive radius of convergence ($R > 0$), may be differentiated, added (subtracted), and multiplied: the result power series has radius of convergence at least as big as the smallest radius of convergence of the operand power series.

  Further, if a power series (6.2) has positive radius of convergence, and a sum that is identically zero in its interval of convergence, then every coefficient must be zero, $a_m = 0$.

- A function $f(x)$ is **analytic** at a point $x = x_0$ if it can be represented by a power series in $(x - x_0)$ with a radius of convergence $R > 0$.

Theorem 6.2. *(Kreyszig 2011, Theorem 1, p.172) If functions $h(x), p(x), q(x), r(x)$ in the* ODE (6.3) *are analytic at $x = x_0$ and $h(x_0) \neq 0$, then every solution of* (6.3) *is analytic at $x = x_0$ (and hence has a power series with centre x_0 with $R > 0$).*

No proof.

6.3 Legendre polynomials

(Kreyszig 2011, §5.2)

Legendre polynomials help describe structures on spheres, such as the motion of the ocean and atmosphere. Legendre's ODE is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{for } n \in \mathbb{R}. \tag{6.5}$$

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See power series solution $y = \sum_{m=0}^{\infty} a_m x^m$ to deduce that $y = a_0 y_1(x) + a_1 y_2(x)$ where (Kreyszig 2011, pp.176–7)

$$\begin{aligned} y_1(x) &= 1 - \tfrac{n(n+1)}{2!}x^2 + \tfrac{(n-2)n(n+1)(n+3)}{4!}x^4 + \cdots, \\ y_2(x) &= x - \tfrac{(n-1)(n+2)}{3!}x^3 + \tfrac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots. \end{aligned}$$

Convergence These converge for $|x| < 1$, and generally diverge for $|x| > 1$: both y_1 and y_2 are really series in x^2 so use the ratio test with (x^2) as the variable.

Polynomials For integer n one of y_1 or y_2 will terminate (and hence converge for all x) (Kreyszig 2011, pp.177–8): the terminating ones are called Legendre polynomials,

$$\begin{aligned} &P_0(x) := 1, && P_1(x) := x, \\ &P_2(x) := \tfrac{1}{2}(3x^2 - 1), && P_3(x) := \tfrac{1}{2}(5x^3 - 3x). \\ &\vdots && \vdots \end{aligned}$$

6.4 Extending to the Frobenius method

(Kreyszig 2011, §5.3)

Many ODEs arising in applications are not quite so nice: for example, Bessel's ODE

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu \text{ constant},$$

describes the radial structure of the vibrations of a drum, waves in a paddling pool, and of many other problems in circular/cylindrical geometries. Because of x^2y'' we apparently cannot find power series solutions about $x = 0$.

Definition 6.3 (regular and singular points)**.** *(Kreyszig 2011, p.181) For the* ODE (6.3)*: if functions $h(x), p(x), q(x), r(x)$ are analytic at $x = x_0$ and $h(x_0) \neq 0$, then $x = x_0$ is called a **regular point** or **ordinary point**, otherwise it is called **singular**.*

Further, for the ODE $y'' + p(x)y' + q(x)y = 0$ *with a singular point at* $x = x_0$*: if functions* $(x - x_0)p(x)$ *and* $(x - x_0)^2 q(x)$ *are both analytic at* $x = x_0$*, then* $x = x_0$ *is called an* ***regular singular point****, otherwise it is an* ***irregular singular point****.*

Example 6.8 $x = 0$ is a regular singular point of Bessel's ODE (and Frobenius applies), but $x = \infty$ is an irregular singular point. ♣

The view is always much better from a singular point (courtesy of PhD comics).

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Mostly we arrange the definition of x so that the centre $x_0 = 0$.

Theorem 6.4 (Frobenius method)**.** *(Kreyszig 2011, p.180) If the* ODE *(6.3) has a regular singular point at centre* $x = 0$*, then it has at least one solution of the form*

$$y(x) = x^r(a_0 + a_1x + a_2x_2 + \cdots) = x^r \sum_{m=0}^{\infty} a_m x^m, \tag{6.6}$$

for some exponent $r \in \mathbb{C}$ *chosen so that* $a_0 \neq 0$*.*

The ODE *also has a second linearly independent solution of similar form, but may involve logarithms.*

6.4.1 Bessel's equation and Bessel functions $J_\nu(x)$

Consider Bessel's ODE

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0\,, \quad \nu \text{ constant}, \tag{6.7}$$

and by Frobenius seek a solution in the form (6.6). Substitute and equate coefficients (Kreyszig 2011, pp.187–8) to obtain two possible exponents $r = \pm\nu$.

The case $r = \nu \in \mathbb{C}$ (Kreyszig 2011, p.188) First find $a_1 = a_3 = a_5 = \cdots = 0$. Second, the recursion for the even numbered coefficients $(m = 2k)$

$$a_{2k} = -\frac{1}{2^{2k}(\nu + k)} a_{2k-2}$$

$$\implies \quad a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(\nu+1)\cdots(\nu+k)}\,.$$

Obtained one solution (not yet the required two) of Bessel's ODE:

$$y = a_0 x^\nu \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(\nu+1)\cdots(\nu+k)}\right].$$

Definition 6.5 (Bessel functions for integer $\nu = n \in \mathbb{N}_0$)**.** *(Kreyszig 2011, pp.188–9) Set $a_0 = 1/(2^n n!)$ and define the* ***Bessel function of the first kind*** *of order n as*

$$J_n(x) := x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+n} k!(n+k)!}\,.$$

For example, $J_0(x)$ and $J_1(x)$. Plot some in Matlab/Octave with the following and see beautiful curves:

```
[nus,xs]=meshgrid(0:3,linspace(0,15))
js=besselj(nus,xs)
plot(xs,js)
```

To cater for non-integer $\nu \in \mathbb{C}$, we introduce a new function that behaves like the factorial.

Definition 6.6 (Bessel functions for most ν)**.** *(Kreyszig 2011, pp.190–1) Set $a_0 = 1/[2^\nu \Gamma(\nu+1)]$ and define the* ***Bessel function of the first kind*** *of order ν as*

$$J_\nu(x) := x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\nu} k!\Gamma(\nu+k+1)} = \sum_{m=0}^{\infty} \frac{(-1)^k (x/2)^{2k+\nu}}{k!\Gamma(\nu+k+1)}\,.$$

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Definition 6.7 (gamma function)**.** *(Kreyszig 2011, p.190) Define the* ***gamma function****, see Figure 6.2,*

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\,dt$$

so that $\Gamma(\nu+1) = \nu\Gamma(\nu)$ and $\Gamma(n+1) = n!$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

For example, $J_{3/2}(x), \ldots, J_{-3/2}(x)$. Plot some in Matlab/Octave with the following and see beautiful curves:

```
[nus,xs]=meshgrid(-1.5:1.5,linspace(0,15));
js=besselj(nus,xs);
js(abs(js)>1)=nan;
plot(xs,js)
```

As seen in the plots, (Kreyszig 2011,) derives the special cases $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

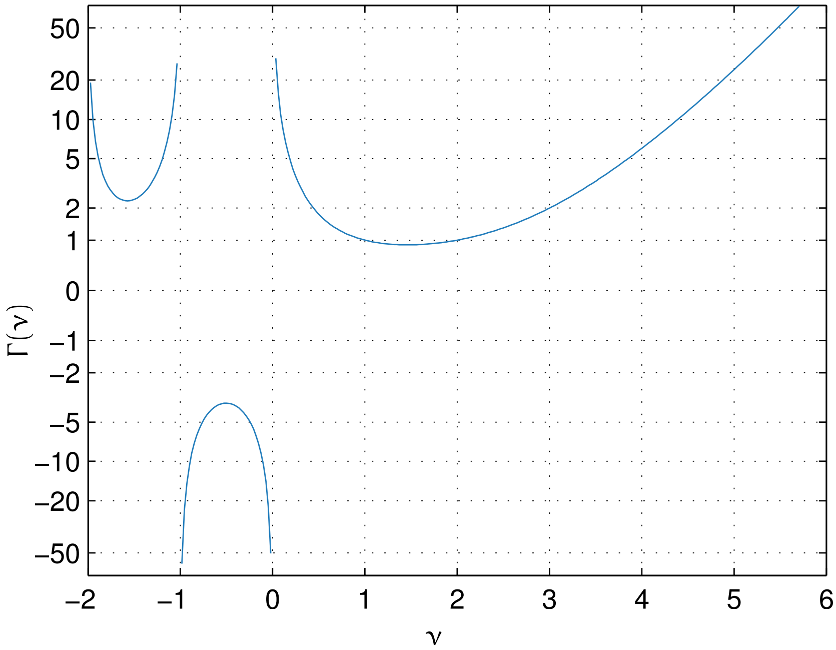

Figure 6.2: the Gamma function (note the stretched vertical axis).

General solution (Kreyszig 2011, pp.194–5) A general solution to Bessel's equation is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

for non-integer ν. Because $J_{-\nu}(x) \sim x^{-\nu} \to \infty$ as $x \to 0$, the 'physics' of the problem often requires that we use only $J_\nu(x)$.

For integer $\nu = n$, $J_{-n}(x) = (-1)^n J_n(x)$ are linearly dependent. We have to do more work, maybe reduction of order, to find a linearly independent partner for $J_n(x)$.

6.4.2 Indicial equation identifies three cases

Consider a general ODE with a regular singular point at $x = 0$ (Kreyszig 2011, pp.181–2):

$$x^2 y'' + x b(x) y' + c(x) y = 0\,. \tag{6.8}$$

Seek solution $y(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)$: substitute, rearrange, and equate powers of x. Since coefficient $a_0 \neq 0$, the exponent r must satisfy the **indicial equation**

$$r(r-1) + b_0 r + c_0 = 0\,. \tag{6.9}$$

Generally for $h(x)y'' + p(x)y' + q(x)y = 0$ for analytic h, p, q but with $h(x_0) = 0$, try $y = (x - x_0)^r(1 + \cdots)$ and see what happens.

Theorem 6.8 (Frobenius). *(Kreyszig 2011, pp.182–3) Let $b(x)$ and $c(x)$ be analytic at $x = 0$, and r_1 and r_2 (possibly complex) are the roots of the indicial equation (6.9), then three cases arise.*

1. *Two distinct roots, not differing by an integer: two linearly independent solutions of the* ODE (6.8) *are*

$$\begin{aligned} y_1(x) &= x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots), \\ y_2(x) &= x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots). \end{aligned}$$

2. *A double root, $r_1 = r_2 = r$: two linearly independent solutions of the* ODE (6.8) *are*

$$\begin{aligned} y_1(x) &= x^{r}(a_0 + a_1 x + a_2 x^2 + \cdots), \\ y_2(x) &= y_1(x)\ln x + x^{r}(\qquad A_1 x + A_2 x^2 + \cdots). \end{aligned}$$

3. *Two roots differing by an integer, say $r_1 > r_2$: two linearly independent solutions of the* ODE (6.8) *are*

$$\begin{aligned} y_1(x) &= x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots), \\ y_2(x) &= k y_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots), \end{aligned}$$

   where constant k may be zero.

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Example 6.9 Characterise, over all x, the nature of solutions to the ODE

$$2x(1+x)y'' + (3+x)y' - xy = 0\,. \qquad (6.10)$$

♣

Example 6.10 (edge waves revisited). Recall that Example 6.1 looked at a beach and sought ocean edge waves travelling with water height $H = h(x)\cos[(Z - cT)2\pi/L]$ for $x := X2\pi/L$. The offshore structure function $h(x) = e^{-x}y(x)$ where $y(x)$ satisfies the ODE $xy'' + (1-2x)y' + ny = 0$ for parameter $n = \frac{2\pi c^2}{gL\alpha} - 1$. Find power series solutions for the wave structure.

♣

7 More partial differential equations

(Kreyszig 2011, Chapt. 12)

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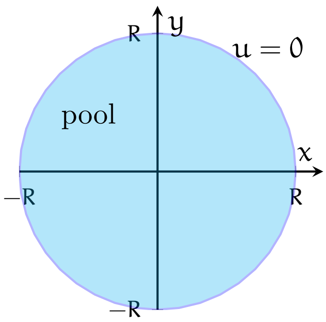
7.1 Circular pools and Bessel functions

(Kreyszig 2011, §12.10)

For separation of variables, not only does the PDE have to be amenable to separation, but so does the shape of the boundary.



Use polar coordinates in a circular pool: $x = r\cos\theta$ and $y = r\sin\theta$. The wave PDE (5.6) for $u(r,\theta,t)$ is (Kreyszig 2011, pp.585–6)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right). \tag{7.1}$$

Solve with Dirichlet boundary conditions $u(R,\theta,t) = 0$ when the pool has radius R.

First, separate the angular dependence Seek product form solutions $u(r,\theta,t) = U(r,t)Q(\theta)$ to find

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{n^2}{r^2}U\right), \quad n \text{ integer}. \tag{7.2}$$

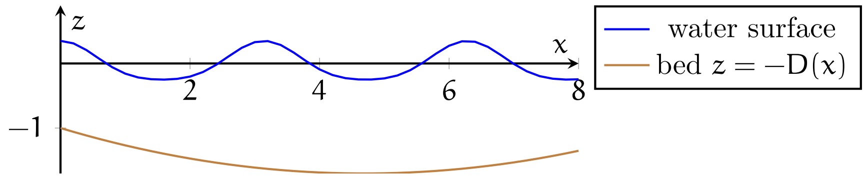


Figure 7.1: water waves above a varying bed.

Second, separate radial and time Seek product form $U = W(r)T(t)$ then obtain (Kreyszig 2011, p.587 with extra term)

$$\ddot{T} + c^2k^2T = 0\,, \quad s^2\frac{d^2W}{ds^2} + s\frac{dW}{ds} + (s^2 - n^2)W = 0\,,$$

where $s = kr$ is a scaled radial coordinate. The boundary condition $u(R,t) = 0$ becomes $W = 0$ at $r = R$.

The T-ODE has trigonometric solutions $T = A\cos kct + B\sin kct$ (for real k). The W-ODE is Bessel's ODE (6.7) with solution proportional to $W = J_n(s) = J_n(kr)$ as the other solution is unphysically like $s^{-n} \propto r^{-n}$ as $r \to 0$.

Restrict to no angular dependence, $n = 0$ Setting $k_m = \alpha_m/R$ where α_m is the mth zero of the Bessel function J_0, we find solutions (Kreyszig 2011, p.588)

$$u_m(r,t) = W_m(r)T_m(t) = (A_m\cos k_mct + B_m\sin k_mct)J_0(k_mr).$$

Hence, a general solution is

$$u(r,t) = \sum_{m=1}^{\infty}(A_m\cos k_mct + B_m\sin k_mct)J_0(k_mr).$$

where initial conditions determine coefficients A_m and B_m (Kreyszig 2011, p.589), but we do not detail.

7.1.1 Separation of variables gives edge waves

If time permits.

Model water of variable depth *Figure 7.1 shows to change the coordinate system to have $z = 0$ at the mean water level, and $z = -D(x)$ to be the varying depth of the bed.* Define *the surface to be at $z = \eta(x,t)$ so the depth of water is $h = D(x) + \eta(x,t)$.*

- *Conservation of water* (5.2) *still applies with $h = D + \eta$:*

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}(Du) \approx 0\,.$$

- *Conservation of momentum* (5.3) *still applies, with extra bed forces:*

$$\frac{\partial u}{\partial t} \approx -g\frac{\partial\eta}{\partial x}\,.$$

- *Combining the two equations gives a variable depth wave* PDE *in one space dimension:*

$$\frac{\partial^2\eta}{\partial t^2} = \cdots = \frac{\partial}{\partial x}\left(gD\frac{\partial\eta}{\partial x}\right).$$

The vector version

$$\begin{aligned} &\frac{\partial^2\eta}{\partial t^2} = \boldsymbol{\nabla}\cdot(gD\boldsymbol{\nabla}\eta) \\ \implies\quad &\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial x}\left(gD\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(gD\frac{\partial\eta}{\partial y}\right). \end{aligned}$$

in two lateral dimensions.

- *Rename variables to match Example 6.1:* $x \mapsto X$, $y \mapsto Z$, $t \mapsto T$, $\eta \mapsto H$, $D \mapsto D(X)$, *gives as before*

$$\frac{1}{g}\frac{\partial^2 H}{\partial T^2} = \frac{\partial}{\partial X}\left(D(X)\frac{\partial H}{\partial X}\right) + D(X)\frac{\partial^2 H}{\partial Z^2},$$

to govern water near beaches, in harbours, estuaries, rivers, and so on.

Separate variables *by seeking* $H(X, Z, T) = F(X, T)G(Z)$ *(and remembering* D *depends upon* X*):*

$$\frac{1}{g}F_{TT} - (DF_X)_X = -\frac{4\pi^2}{L^2}DF.$$

Separate again by seeking $F(X, T) = V(X)W(T)$*:*

$$(DV')' + \left(\frac{\omega^2}{g} - \frac{4\pi^2}{L^2}D\right)V = 0.$$

For a beach of depth $D = \alpha X$ *this becomes* $XV'' + V' + \left(\frac{\omega^2}{g\alpha} - \frac{4\pi^2}{L^2}X\right)V = 0$ *which a scaling of* X *reduces to the* ODE *discussed in Example 6.1.*

Interpret: *the most important solutions were those decaying purely exponentially away from the shoreline: the* ODE *determines these solutions occur when* $\frac{\omega^2}{g\alpha} = \frac{2\pi}{L}$ *; that is, these edge waves are of frequency* $\omega = \sqrt{2\pi g\alpha/L}$.

In general, separation of variables tells us that the general behaviour is that of a double sum of modes: a sum over all the off-shore structures $V(X)$ *multiplied by the along shore trigonometric structures* $\cos\frac{2\pi Z}{L}$ *for feasible wavelengths* L *that fit along the beach, each multiplied by the corresponding oscillation in time at the appropriate frequency.*

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7.2 Further Fourier Series

7.2.1 Orthogonality empowers other series representations

(Kreyszig 2011, §11.6)

Generalise key linear algebra from finite dimensional vectors to 'infinite dimensional' functions.

Definition 7.1 (inner product). *(Kreyszig 2011, p.500) Let $u(x)$ and $v(x)$ be functions (possibly complex valued) defined on the interval $[a, b]$. The **inner product** with (real) **weight function** $r(x)$ ($r > 0$ on the interval) is*

$$(u, v) = \int_a^b \overline{u(x)} v(x) r(x)\, dx\,, \tag{7.3}$$

*where $\overline{u}$ denotes the complex conjugate. If $(u, v) = 0$, then the two functions are called **orthogonal** (on $[a, b]$ with weight function r). The **norm** of a function $u(x)$ (on $[a, b]$ with weight function r) is*

$$\|u\| = \sqrt{(u, u)} = \sqrt{\int_a^b |u|^2 r\, dx}\,. \tag{7.4}$$

Example 7.1 With the common weight function $r = 1$ orthogonal functions are:

- $\sin x$ and $\cos x$ on $[-\pi, \pi]$;
- e^{ix} and e^{-ix} on $[0, 2\pi]$;
- but 1 and x only when $a = -b$.

Example norms are $\|\sin mx\| = \sqrt{\pi}$ on $[-\pi, \pi]$ and $\|e^{imx}\| = \sqrt{2\pi}$ on $[0, 2\pi]$. ♣

Definition 7.2 (orthogonal sets). *The set of functions $\{y_0(x), y_1(x), y_2(x), \ldots\}$ are termed an **orthogonal set** (on $[a, b]$ with weight function r) if the inner product $(y_m, y_n) = 0$ for all $m \neq n$, and $\|y_m\| > 0$ for all m.*

Example 7.2 Orthogonal sets are

- $\{\sin x, \sin 2x, \sin 3x, \ldots\}$ on $[-\pi, \pi]$ with $r = 1$;
- $\{e^{imx} : m \in \mathbb{Z}\}$ on $[0, 2\pi]$ with $r = 1$;
- Legendre polynomials $\{P_n(x) : n = 0, 1, 2, \ldots\}$ on $[-1, 1]$ with $r = 1$;
- Bessel functions $\{J_n(x) : n = 0, 1, 2, \ldots\}$ on $[0, \infty)$ with $r = 1/x$.

♣

7.2.2 Generalised Fourier Series

Definition 7.3. *For an orthogonal set* $\{y_0(x), y_1(x), y_2(x), \dots\}$ *(on* $[a, b]$ *with weight function* r*) let* $F(x)$ *be a function represented as the convergent series (Kreyszig 2011, p.504)*

$$F(x) = \sum_{m=0}^{\infty} c_m y_m(x). \tag{7.5}$$

*The sum is called an **orthogonal series**, **orthogonal expansion**, or **generalised Fourier series** of* $F(x)$.

By orthogonality, the coefficients (Kreyszig 2011, p.505)

$$c_m = \frac{(y_m, F)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b \overline{y_m(x)} F(x) r(x)\, dx\,. \tag{7.6}$$

One possible example is the 'Fourier–Legendre series' (Kreyszig 2011, p.505). But the key issue is: when can such a series reasonably represent everything we might want it to represent?

Definition 7.4 (completeness)**.** *(Kreyszig 2011, p.508) Consider some set of functions* S *on the interval* $[a, b]$*, and an orthogonal set of functions* $Y = \{y_0(x), y_1(x), y_2(x), \dots\}$*. If each* $f \in S$ *that is orthogonal to all* y_m *has norm* $\|f\| = 0$*, then the set* Y *is termed **complete** in* S*.*

Example 7.3 Let the set $Y = \{\sin x, \sin 2x, \sin 3x, \dots\}$

- If S is the set of piecewise differentiable functions on $[-\pi, \pi]$, then Y is *not* complete. For example, $\cos x \in S$ is orthogonal to all $\sin mx$, yet $\cos x$ has non-zero norm.
- Nonetheless, Y is complete in the set of piecewise continuous *odd* functions on $[-\pi, \pi]$.
- Also, Y is complete in the set of piecewise continuous functions on $[0, \pi]$.

♣

7.2.3 Complex Fourier Series

Consider the orthogonal set $\{e^{inx} : n \in \mathbb{Z}\}$ in $[-\pi, \pi]$ with weight function 1 (could rearrange into $\{y_0, y_1, \dots\}$):

- assert this set is complete;
- the norms $\|e^{imx}\| = \cdots = \sqrt{2\pi}$ (as before);
- orthogonal set as $(e^{imx}, e^{inx}) = \cdots = 0$ (as before).

Consequently, on $[-\pi, \pi]$ reasonable functions

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where } c_n = \frac{(e^{inx}, f)}{\|e^{inx}\|^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx\,.$$

This is the **complex Fourier series**. A complex Fourier series relates closely to the trigonometric Fourier series: if real function

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx],$$

then $c_0 = a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n) = \bar{c}_n$.

Example 7.4 (saw-tooth wave). Find the complex Fourier series of the 2π-periodic function

$$f(x) = \begin{cases} 0, & -\pi < x \le 0, \\ x, & 0 \le x < \pi. \end{cases}$$

Confirm the series with a graph:

```
x=linspace(-5,5);
n=-10:10;
n=n(n~=0)
cn=(i*pi*(-1).^n./n+((-1).^n-1)./n.^2)/(2*pi)
f=pi/4+cn*exp(i*n'*x);
plot(x,real(f),x,imag(f))
```

♣

The complex Fourier series generalises to arbitrary period L. Taking $L \to \infty$ in a suitable way it becomes the **Fourier transform** (Kreyszig 2011, §11.9).