Topic C Assignment 4

Andrew Martin

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1. Use multiple scales to solve
$$y'' + y + \epsilon(y')^3 = 0$$
$\epsilon \ll 1$, $y(0) = 1$ and $y'(0) = 0$.

Let $y(\tau) \sim y_0(t, T)$ where $T = \epsilon t$ is a slow timescale.

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T}$$
$$\frac{\partial^2}{\partial \tau^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2}$$

Subbing this into the ODE gives:

$$\frac{d^2 y}{dt^2} + y + \epsilon \left(\frac{dy}{dt}\right)^3 = 0$$
$$\frac{\partial^2 y}{\partial t^2} + 2\epsilon \frac{\partial^2 y}{\partial t \partial T} + \epsilon^2 \frac{\partial^2 y}{\partial T^2} + y + \epsilon \left(\frac{\partial y}{\partial t} + \epsilon \frac{\partial y}{\partial T}\right)^3 = 0$$

With initial conditions

$$y_0(0, 0) = 1$$
$$\frac{\partial y_0(0, 0)}{\partial t} = 0$$

And

$$y_1(0, 0) = 0$$
$$\frac{\partial y_1(0, 0)}{\partial t} + y_0(0, 0) = 0$$

To leading order

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$$
$$y_0 = R(T)\cos(t + \theta(T))$$

Boundary conditions:

$$y_0(0, 0) = 1 \implies R(0) = 1$$
$$\frac{\partial y_0(0, 0)}{\partial t} = 0 \implies R(0)(-\sin(\theta(0))) = 0 \implies \theta(0) = 0$$

To obtain the full forms of R and θ, find the second order:

$$\begin{aligned}
&\frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial^2 y_0}{\partial t \partial T} + y_1 + \left(\frac{\partial y_0}{\partial t}\right)^3 = 0 \\
&\frac{\partial^2 y_1}{\partial t^2} + 2\left(-R'(T)\sin(t+\theta(T)) - R(T)\cos(t+\theta(T))\theta'(T)\right) \\
&+ y_1 + \left(-R(T)\sin(t+\theta(T))\right)^3 = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= 2R'\sin(t+\theta) + 2R\theta'\cos(t+\theta) + R^3\sin^3(t+\theta) \\
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= 2R'\sin(t+\theta) + 2R\theta'\cos(t+\theta) + \frac{R^3}{4}\left(3\sin(t+\theta) - \sin(3(t+\theta))\right) \\
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= (2R' + \frac{3}{4}R^3)\sin(t+\theta) + 2R\theta'\cos(t+\theta) - R^3\left(\sin(3(t+\theta))\right)
\end{aligned}$$

Hence we require

$$\begin{aligned}
(2R' + \frac{3}{4}R^3) &= 0 \\
2R\theta' &= 0
\end{aligned}$$

For non-trivial solutions this means

$$\begin{aligned}
\theta' = 0 &\implies \theta = c \\
2R' + \frac{3}{4}R^3 &= 0 \\
\frac{R'}{R^3} &= -\frac{3}{8} \\
-\frac{1}{2R^2} &= -\frac{3}{8}T + d_* \\
2R^2 &= \frac{1}{\frac{3}{8}T - d_*} \\
R &= \pm\frac{1}{\sqrt{\frac{3}{4}T + d}}
\end{aligned}$$

And using the condition from before, $R(0) = 1$

$$\begin{aligned}
R &= \frac{1}{\sqrt{d}} \\
\implies d &= 1
\end{aligned}$$

Hence

$$\boxed{y_0 = \frac{1}{\sqrt{3T+1}}\cos(t)}$$

Figure 1 shows the two solutions obtained. Clearly the two overlap very nicely even for $\epsilon = 0.1$.

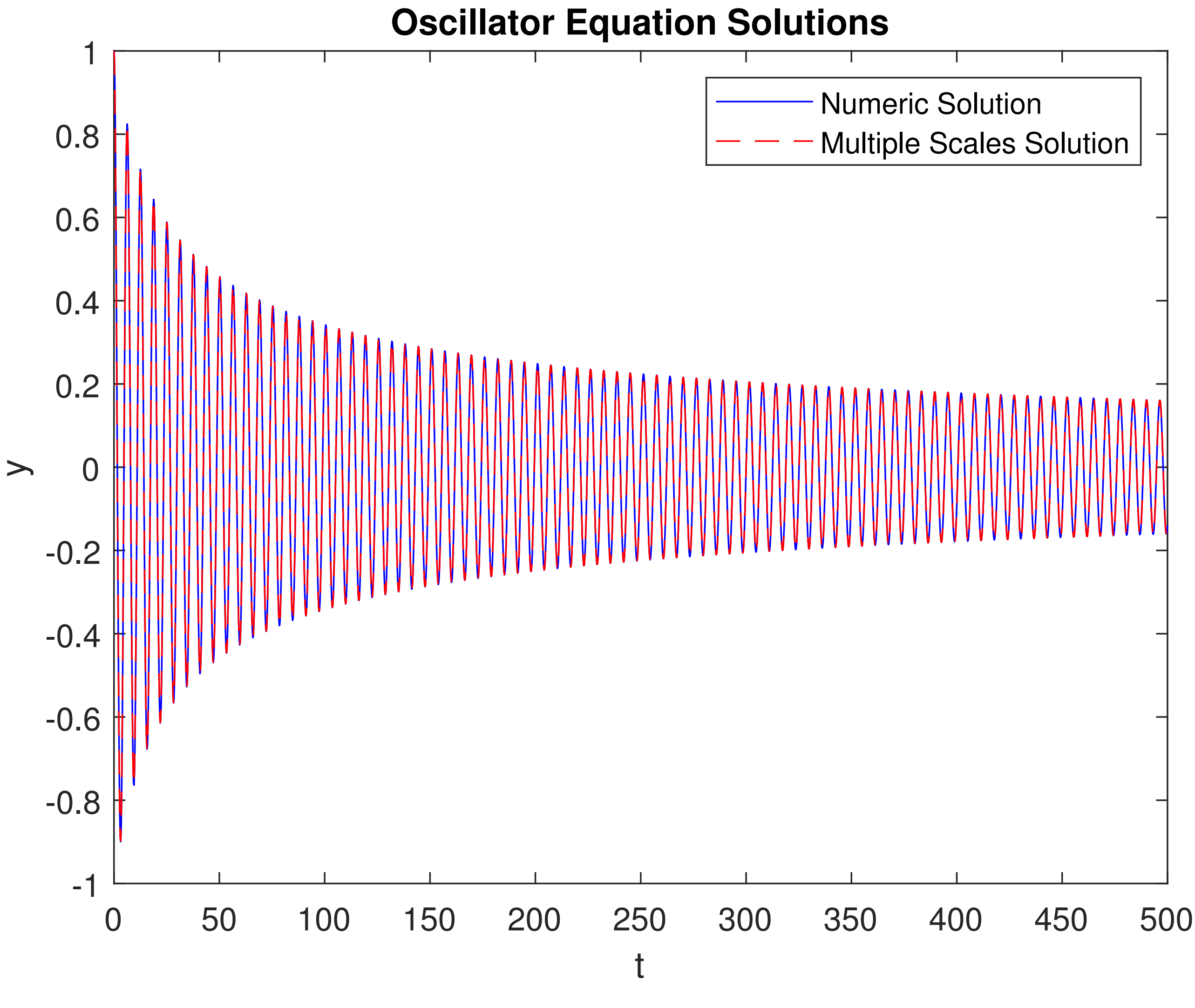

Figure 1: Comparison of numerical and multi-scale solutions for $\epsilon = 0.1$

2.
$$\frac{d^2y}{dt^2} + \epsilon(y^2-1)\frac{dy}{dt} + y = 0, \quad y(0)=1,\ y'(0)=0, \quad \epsilon \ll 1$$

(a)
$$y(t) = y(t, T, \tau)$$

$T = \epsilon t$ and $\tau = \epsilon^2 t$. This gives the partial derivative expansions:

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T} + \epsilon^2\frac{\partial}{\partial \tau}\\
\frac{d^2}{dt^2} &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T} + \epsilon^2\frac{\partial}{\partial \tau}\right) + \epsilon\frac{\partial}{\partial T}\left(\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T} + \epsilon^2\frac{\partial}{\partial \tau}\right) + \epsilon^2\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T} + \epsilon^2\frac{\partial}{\partial \tau}\right)\\
&= \frac{\partial^2}{\partial t^2} + \epsilon^2\frac{\partial^2}{\partial T^2} + \epsilon^4\frac{\partial^2}{\partial \tau^2} + 2\epsilon\frac{\partial}{\partial t\partial T} + 2\epsilon^2\frac{\partial}{\partial t\partial \tau} + 2\epsilon^3\frac{\partial}{\partial T\partial \tau}
\end{aligned}$$

Hence the ODE becomes

$$\frac{\partial^2 y}{\partial t^2} + \epsilon^2\frac{\partial^2 y}{\partial T^2} + \epsilon^4\frac{\partial^2 y}{\partial \tau^2} + 2\epsilon\frac{\partial y}{\partial t\partial T} + 2\epsilon^2\frac{\partial y}{\partial t\partial \tau} + 2\epsilon^3\frac{\partial y}{\partial T\partial \tau} + \epsilon(y^2-1)\left(\frac{\partial y}{\partial t} + \epsilon\frac{\partial y}{\partial T} + \epsilon^2\frac{\partial y}{\partial \tau}\right) + y = 0$$

With boundary conditions

$$y(0,0,0) = 1$$
$$\frac{\partial y(0,0,0)}{\partial t} + \epsilon\frac{\partial y(0,0,0)}{\partial T} + \epsilon^2\frac{\partial y(0,0,0)}{\partial \tau} = 0$$

(b) First expand the PDE

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} + \epsilon^2\frac{\partial^2 y}{\partial T^2} + \epsilon^4\frac{\partial^2 y}{\partial \tau^2} + 2\epsilon\frac{\partial y}{\partial t\partial T} + 2\epsilon^2\frac{\partial y}{\partial t\partial \tau} + 2\epsilon^3\frac{\partial y}{\partial T\partial \tau}\\
&+ y^2\epsilon\frac{\partial y}{\partial t} + y^2\epsilon^2\frac{\partial y}{\partial T} + y^2\epsilon^3\frac{\partial y}{\partial \tau} - \left(\epsilon\frac{\partial y}{\partial t} + \epsilon^2\frac{\partial y}{\partial T} + \epsilon^3\frac{\partial y}{\partial \tau}\right) + y = 0
\end{aligned}$$

We are only considering up to $\mathcal{O}(\epsilon^2)$, so dropping ϵ^3 and higher terms

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} + \epsilon^2\frac{\partial^2 y}{\partial T^2} + 2\epsilon\frac{\partial y}{\partial t\partial T} + 2\epsilon^2\frac{\partial y}{\partial t\partial \tau}\\
&+ y^2\epsilon\frac{\partial y}{\partial t} + y^2\epsilon^2\frac{\partial y}{\partial T} - \left(\epsilon\frac{\partial y}{\partial t} + \epsilon^2\frac{\partial y}{\partial T}\right) + y = 0
\end{aligned}$$

Let
$$y(t,T,\tau) = y_0(t,T,\tau) + \epsilon y_1(t,T,\tau) + \epsilon^2 y_2(t,T,\tau) + \dots$$

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} + \epsilon^2\frac{\partial^2 y}{\partial T^2} + 2\epsilon\frac{\partial y}{\partial t\partial T} + 2\epsilon^2\frac{\partial y}{\partial t\partial \tau}\\
&+ y^2\epsilon\frac{\partial y}{\partial t} + y^2\epsilon^2\frac{\partial y}{\partial T} - \left(\epsilon\frac{\partial y}{\partial t} + \epsilon^2\frac{\partial y}{\partial T}\right) + y = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 y_0}{\partial t^2} + \epsilon\frac{\partial^2 y_1}{\partial t^2} + \epsilon^2\frac{\partial^2 y_2}{\partial t^2} + \epsilon^2\frac{\partial^2 y_0}{\partial T^2} + 2\epsilon\frac{\partial y_0}{\partial t\partial T} + 2\epsilon^2\frac{\partial y_1}{\partial t\partial T} + 2\epsilon^2\frac{\partial y_0}{\partial t\partial \tau}\\
&+ y_0^2\epsilon\frac{\partial y_0}{\partial t} + y_0^2\epsilon^2\frac{\partial y_1}{\partial t} + y_0^2\epsilon^2\frac{\partial y_0}{\partial T} - \left(\epsilon\frac{\partial y_0}{\partial t} + \epsilon^2\frac{\partial y_1}{\partial t} + \epsilon^2\frac{\partial y_0}{\partial T}\right) + y_0 + \epsilon y_1 + \epsilon^2 y_2 = 0
\end{aligned}$$

$$
\begin{aligned}
\mathcal{O}(1):& \frac{\partial^2 y_0}{\partial t^2} + y_0 = 0\\
\mathcal{O}(\epsilon):& \frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial y_0}{\partial t \partial T} + y_0^2\frac{\partial y_0}{\partial t} - \frac{\partial y_0}{\partial t} + y_1 = 0\\
\mathcal{O}(\epsilon^2):& \frac{\partial^2 y_2}{\partial t^2} + \frac{\partial^2 y_0}{\partial T^2} + 2\frac{\partial y_1}{\partial t \partial T} + 2\frac{\partial y_0}{\partial t \partial \tau} + y_0^2\frac{\partial y_1}{\partial t} + y_0\frac{\partial y_0}{\partial T} - \frac{\partial y_1}{\partial t} - \frac{\partial y_0}{\partial T} + y_2 = 0
\end{aligned}
$$

With boundary conditions

$$
\begin{aligned}
\mathcal{O}(1):& y_0(0,0,0) = 1, \quad \frac{\partial y_0(0,0,0)}{\partial t} = 0\\
\mathcal{O}(\epsilon):& y_1(0,0,0) = 0, \quad \frac{\partial y_1(0,0,0)}{\partial t} + \frac{\partial y_0(0,0,0)}{\partial T} = 0\\
\mathcal{O}(\epsilon^2):& y_2(0,0,0) = 0, \quad \frac{\partial y_2(0,0,0)}{\partial t} + \frac{\partial y_1(0,0,0)}{\partial T} + \frac{\partial y_0(0,0,0)}{\partial \tau} = 0
\end{aligned}
$$

(c) Leading order equation:

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$$

Gives

$$y_0 = R(T,\tau)\cos(t + \theta(T,\tau)) = A(T,\tau)e^{it} + \overline{A(T,\tau)}e^{it}$$

To eliminate the resonant terms, obtain y_1 as per lectures (but there are now arbitrary functions of τ too)
And let $A' := \frac{\partial A}{\partial T}$

$$
\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= -(y_0^2 - 1)\frac{\partial y_0}{\partial t} - 2\frac{\partial y_0}{\partial t \partial T}\\
&= -(A^2e^{i2t} + 2A\overline{A} + \overline{A}^2e^{-i2t} - 1)i(Ae^{it} - \overline{A}e^{-it}) - 2i(A'e^{it} + \overline{A}'e^{it})\\
&= -i\left(e^{it}(A^2\overline{A} - A + 2A') + e^{it}(-A\overline{A}^2 + 2A\overline{A}') + A^3e^{i3t} - \overline{A}^3e^{i3t}\right)
\end{aligned}
$$

$$A^3\overline{A} - A + 2A' = 0$$

Let $A = \rho(T,\tau)e^{i\theta(T,\tau)}$

$$
\begin{aligned}
\rho^3 - \rho + 2\rho' + i\theta'\rho = 0\\
\rho^3 - \rho + 2\rho' = 0, \quad i\theta'\rho = 0
\end{aligned}
$$

For non-trivial solutions

$$
\begin{aligned}
\frac{\partial \theta}{\partial T} = 0\\
\implies \theta = \phi(\tau)
\end{aligned}
$$

Treating the ρ equation as an ODE gives (similar to lectures since no derivatives of τ are involved)

$$\rho = \frac{e^{T/2}}{\sqrt{c + e^T}}$$

I.e.

$$\rho = \frac{e^{T/2}}{\sqrt{F(\tau) + e^T}}$$

And noting that $\rho = \frac{1}{2}R$, gives

$$y_0 = \frac{2e^{T/2}}{\sqrt{F(\tau) + e^T}} \cos(t + \phi)$$

Applying initial conditions:

$$\begin{aligned}
\frac{\partial y_0}{\partial t}(0,0,0) &= 0 \\
\implies -\frac{2}{\sqrt{F(0)+1}} \sin(\phi(0)) &= 0 \\
\sin(\phi(0)) = 0 &\implies \phi(0) = 0
\end{aligned}$$

And

$$\begin{aligned}
y_0(0,0,0) &= 1 \\
\implies \frac{2}{\sqrt{F(0)+1}} \cos(\phi(0)) &= 1 \\
2 &= \sqrt{F(0)+1} \\
F(0) &= 4 - 1 = 3
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{\partial y_0}{\partial t} &= -\frac{2e^{T/2}}{\sqrt{F(\tau) + e^T}} \sin(t + \phi) \\
\frac{\partial y_0}{\partial T} &= \frac{F(\tau)e^{T/2}}{(F(\tau) + e^T)^{3/2}} \cos(t + \phi) \\
\frac{\partial y_0}{\partial t \partial T} &= -\frac{F(\tau)e^{T/2}}{(F(\tau) + e^T)^{3/2}} \sin(t + \phi)
\end{aligned}$$

By putting the equation into `Matlab` and using `dsolve()` it gives

$$y_1 = -\frac{e^{\frac{3T}{2}} \sin(3\phi + 3t) - 4\alpha \cos(t) \left(F + e^T\right)^{3/2} + 4\beta \sin(t) \left(F + e^T\right)^{3/2}}{4\left(F + e^T\right)^{3/2}}$$

Where α and β are arbitrary functions of τ.
And using the boundary conditions (using `Matlab` to obtain the derivatives):

$$\begin{aligned}
\frac{\partial y_1}{\partial t}(0,0,0) + \frac{\partial y_0}{\partial T}(0,0,0) &= 0 \\
\frac{9}{32} - \beta(0) &= 0 \\
\beta(0) &= \frac{9}{32}
\end{aligned}$$

$$\begin{aligned}
y_1(0,0,0) &= 0 \\
\alpha(0) &= 0
\end{aligned}$$

(e)

$$\frac{\partial^2 y_2}{\partial t^2} + \frac{\partial^2 y_0}{\partial T^2} + 2\frac{\partial y_1}{\partial t \partial T} + 2\frac{\partial y_0}{\partial t \partial \tau} + y_0^2 \frac{\partial y_1}{\partial t} + y_0 \frac{\partial y_0}{\partial T} - \frac{\partial y_1}{\partial t} - \frac{\partial y_0}{\partial T} + y_2 = 0$$

$$\frac{\partial^2 y_2}{\partial t^2} + y_2 = -\frac{\partial^2 y_0}{\partial T^2} - 2\frac{\partial y_0}{\partial t \partial \tau} - y_0 \frac{\partial y_0}{\partial T} + \frac{\partial y_0}{\partial T} - y_0^2 \frac{\partial y_1}{\partial t} + \frac{\partial y_1}{\partial t} - 2\frac{\partial y_1}{\partial t \partial T}$$

Matlab is used for all of this.

The forcing terms are those that accompany $\cos(t+\phi)$ and $\sin(t+\phi)$ I think?

Matlab gives (collecting coefficients of sin and cos): Need to set the terms with derivatives of F and ϕ to 0.

$$\begin{aligned}
&\left(\frac{2e^T\beta(\tau)}{e^T+F} - \beta(\tau)\right)\cos(t) \\
&+ \left(\frac{3e^{\frac{3T}{2}}}{2(F+e^T)^{3/2}} - \frac{9e^{\frac{5T}{2}}}{4(F+e^T)^{5/2}} + \frac{3e^{\frac{5T}{2}}}{2(F+e^T)^{3/2}(e^T+F)}\right)\cos(3\phi+3t) \\
&+ \frac{e^T\beta(\tau)}{e^T+F}\cos(t+2\phi) \\
&+ \left(\frac{e^{2T}}{F^2+2e^TF+e^{2T}} - \frac{e^T}{e^T+F}\right)\cos(2t+2\phi) \\
&+ \frac{e^T\beta(\tau)}{e^T+F}\cos(3t+2\phi) \\
&+ \left(\frac{e^{T/2}}{2\sqrt{e^T+F}} + \frac{e^{\frac{3T}{2}}}{(e^T+F)^{3/2}} - \frac{3e^{\frac{5T}{2}}}{2(e^T+F)^{5/2}} + \frac{8e^{T/2}\frac{\partial\phi}{\partial\tau}}{\sqrt{e^T+F}}\right)\cos(t+\phi) \\
&+ \frac{3e^{\frac{5T}{2}}}{4(F+e^T)^{3/2}(e^T+F)}\cos(3\phi+5t+2\phi) \\
&+ \frac{3e^{\frac{5T}{2}}}{4(F+e^T)^{3/2}(e^T+F)}\cos(3\phi+t-2\phi) \\
&+ \left(\frac{2e^T\alpha(\tau)}{e^T+F} - \alpha\right)\sin(t) \\
&+ \left(-\frac{e^T\alpha(\tau)}{e^T+F}\right)\sin(t+2\phi) \\
&+ \frac{e^T\alpha(\tau)}{e^T+F}\sin(3t+2\phi) \\
&+ \left(-\frac{4e^{T/2}\frac{\partial F}{\partial\tau}}{(e^T+F)^{3/2}}\right)\sin(t+\phi) \\
&+ \frac{e^{2T}}{F^2+2e^TF+e^{2T}} - \frac{e^T}{e^T+F}
\end{aligned}$$

Set the terms containing derivatives of τ and F to zero.

I.e.

$$\frac{e^{T/2}}{2\sqrt{e^T+F}} + \frac{e^{\frac{3T}{2}}}{(e^T+F)^{3/2}} - \frac{3e^{\frac{5T}{2}}}{2(e^T+F)^{5/2}} + \frac{8e^{T/2}\frac{\partial\phi}{\partial\tau}}{\sqrt{e^T+F}} = 0$$

$$-\frac{4e^{T/2}\frac{\partial F}{\partial\tau}}{(e^T+F)^{3/2}} = 0$$

The latter yields $\frac{\partial F}{\partial \tau} = 0$ and hence F is constant. From before, $F(0) = 3$ and hence $F = 3$. For the former, noting that there is no τ in the other terms,

$$\phi = \tau c$$

Where

$$\begin{aligned} c &= \sqrt{e^T + 3}\left(\frac{-\frac{e^{T/2}}{2\sqrt{e^T+3}} - \frac{e^{\frac{3T}{2}}}{(e^T+3)^{3/2}} + \frac{3e^{\frac{5T}{2}}}{2(e^T+3)^{5/2}}}{8e^{T/2}}\right) \\ &= -\frac{1}{16} - \frac{e^T}{8(e^T+3)^{1/2}} + \frac{3e^{2T}}{16(e^T+3)^{3/2}} \end{aligned}$$

Hence we finally arrive at

$$y_0 = \frac{2e^{T/2}}{\sqrt{3+e^T}} \cos\left(t + \tau\left(-\frac{1}{16} - \frac{e^T}{8(e^T+3)^{1/2}} + \frac{3e^{2T}}{16(e^T+3)^{3/2}}\right)\right)$$

Or

$$y_0 = \frac{2e^{\epsilon t/2}}{\sqrt{3+e^{\epsilon t}}} \cos\left(t\left(1 + \epsilon^2\left(-\frac{1}{16} - \frac{e^{\epsilon t}}{8(e^{\epsilon t}+3)^{1/2}} + \frac{3e^{2\epsilon t}}{16(e^{\epsilon t}+3)^{3/2}}\right)\right)\right)$$

(f) Figure 2f plots the numerical solution against the multiple scales solution. They match quite well until near the end of the plot ($t = 100$) where there appears to be a slight departure in phase. This either means ϕ is wrong, or there is some other discrepancy.

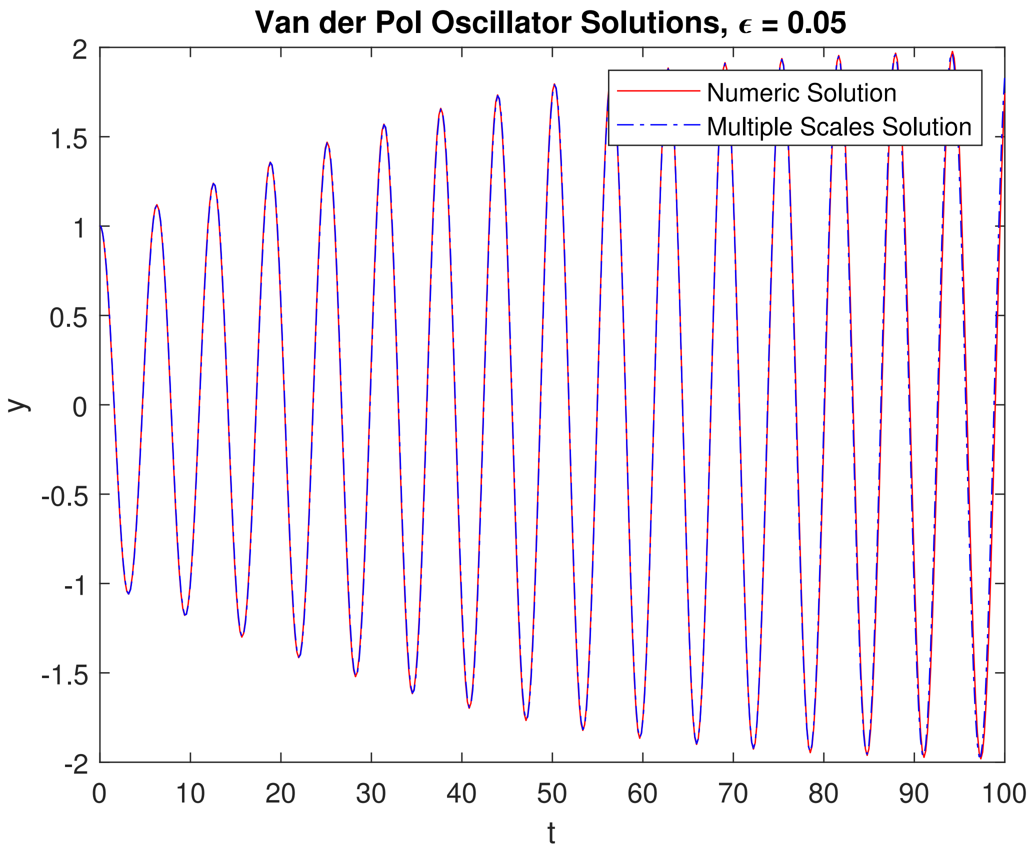

Figure 2: Comparison of numerical and multi-scale solutions for $\epsilon = 0.05$

Matlab Code

```
clear all
close all
%%Q1

epsilon = 0.1;
[t,yNumeric] = ode45(@Q1OscillatorEqn,[0,500],[1,0],[],epsilon);
T = t*epsilon;
R = 1./sqrt(0.75*T+1);
theta = 0;
yAsymp = R .* cos(t + theta);
plot(t,yNumeric(:,1),'b')
hold on
plot(t,yAsymp,'--r')
hold off
xlabel('t')
ylabel('y')
legend("Numeric Solution","Multiple Scales Solution")
title('Oscillator Equation Solutions')
saveas(gcf,'TopicCA4Q1.eps','epsc')

%%
%%Q2
%obtain symbolic solutions
syms y0(t) y1(t) R phi F T
y0 = R*cos(t+phi);
y0 = subs(y0,R,2*exp(T/2)/(sqrt(F+exp(T))));
y0t = diff(y0,t);
y0tT = simplify(diff(y0t,T));
y1eqn = simplify(diff(y1,t,2) + y1 == -2*y0tT - (y0^2-1)*y0t);

y1 = simplify(dsolve(y1eqn));
latex(y1)

%y0TT = diff(y0,T,2)
y0T = diff(y0,T);
% y1T = diff(y1,T)
y1t = diff(y1,t);
%derivative condition
alpha0 = solve(subs(subs(subs(subs(y1t + y0T,t,0),T,0),phi,0),F,3)==0)
%IC
beta0 = solve(subs(subs(subs(subs(y1,t,0),T,0),phi,0),F,3) ==0)



syms tau F(tau) phi(tau) alpha(tau) beta(tau) C4 C5

%y1 = subs(subs(subs(subs(y1,C4,alpha),C5,beta),phi,phi),F,F)
y1 = subs(subs(y1,C4,alpha),C5,beta);
```

```

%Why does this line throw an error????
% y0 = subs(subs(y0,F,F),phi,phi)
%guess ill hard code it
y0 = (2*exp(T/2)*cos(phi + t))/(exp(T) + F)^(1/2);
y0TT = diff(y0,T,T);
y0ttau = diff(y0,t,tau);
y0T = diff(y0,T);
y1t = diff(y1,t);
y1tT = diff(y1,t,T);
% y0ttau =
y2RHS = - y0TT - 2*y0ttau- 2*y0ttau...
    - y0*y0T + y0T - y0^2*y1t + y1t - 2*y1tT;

% assume(T>0)
% assume(exp(T) + F(tau) ~=0)
% assume(tau>0)
temp = combine(expand(simplify(y2RHS)),'sincos');
collection = collect(temp,{'sin' 'cos'});
latex(collection)
syms c
collection = subs(collection,F,3);
c = -1/16 - exp(T)/(8*(exp(T)+3)^(1/2)) + 3*exp(2*T)/(16*(exp(T)+3)^(5/2))


%%
%plot solutions
% close all

epsilon = 0.05;
[t,yNumeric] = ode45(@Q2VanderPol,[0,100],[1,0],[],epsilon);
figure
plot(t,yNumeric(:,1),'r')
hold on
R = @(t) 2*exp(epsilon*t/2)./(sqrt(3+exp(epsilon*t)));

c =@(T) -1/16 - exp(T)./(8*(exp(T)+3).^(1/2)) + 3*exp(2*T)./(16*(exp(T)+3).^
phi = @(t) epsilon^2*t.*(-1/16 - exp(epsilon*t)./(8*(exp(epsilon*t)+3).^(1/2
yAsymp =@(t) R(t).*cos(t + phi(t))
plot(t,yAsymp(t),'-.b')


xlabel('t')
ylabel('y')
legend("Numeric Solution","Multiple Scales Solution")
title("Van der Pol Oscillator Solutions, \epsilon = "+num2str(epsilon))
saveas(gcf,"TopicCA4Q2.eps",'epsc')


function dy = Q1OscillatorEqn(t,y,epsilon)
```

```
dy = [y(2);-y(1) - epsilon*(y(2)^3)];

end
function dy = Q2VanderPol(t,y,epsilon)

dy = [y(2);-epsilon*(y(1)^2 - 1)*y(2) - y(1)];
end
```

Practical Asymptotics (APP MTH 4051/7087)
Assignment 4 (5%)

Due 27 May 2019

1. Apply the method of multiple scales to find a leading-order solution to the following oscillator equation:

$$y'' + y + \epsilon \left(y'\right)^3 = 0,$$

with $\epsilon \ll 1$, subject to $y(0) = 1$ and $y'(0) = 0$. Seek a solution of the form $y(t) \sim y_0(t, T)$, where $T = \epsilon t$ is a slow timescale. Compare this leading-order solution with a numerical solution and comment.

2. Recall from lectures that the numerical solution to the Van der Pol oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \left(y^2 - 1\right) \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0, \quad y(0) = 1, y'(0) = 0, \quad \epsilon \ll 1,$$

exhibited a phase shift, but the leading-order solution did not. To capture this phase shift we require an additional, extra slow timescale.

(a) Introduce an extra slow timescale by letting $y(t) \equiv y(t, T, \tau)$, where $T = \epsilon t$ and $\tau = \epsilon^2 t$, then use the chain rule to transform the above ODE into a PDE in terms of these three variables.

(b) Let $y(t, T, \tau) = y_0(t, T, \tau) + \epsilon y_1(t, T, \tau) + \epsilon^2 y_2(t, T, \tau) + \ldots$ and write down the leading-order, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ problems, including boundary conditions.

(c) Find y_0 by solving the leading-order problem and eliminating resonant terms from the $\mathcal{O}(\epsilon)$ equation.

[Hint: This should include arbitrary functions of τ, but otherwise be identical to that found in lectures (you may reuse working).]

(d) Having eliminated these resonant terms, find y_1 by solving the $\mathcal{O}(\epsilon)$ problem (in terms of aribtrary functions of T and τ). [Hint: strongly recommend using computer algebra for this and the next part.]

(e) Identify the resonant terms from the $\mathcal{O}(\epsilon^2)$ equation that contain derivatives of the unknown function of τ in y_0, and set these terms to zero by finding these unknown function. [Hint: One of these is easy to solve, the other needs to be considered in the 'long time' limit as $T \to \infty$.]

(f) Compare your solution for y_0 with a numerical solution and comment.