Random Processes III 2018: Assignment 1,

to be handed in at the beginning of the lecture (1pm) on Friday 10^{th} August.

[39 marks in total]

Question 0. [4 marks]

Make sure that in all your answers you:
(a) use full and complete sentences,
(b) include units where necessary,
(c) give ranges/constraints for parameters in formulae, and
(d) make sure your layout and mathematical arguments are clear and concise.

Question 1. [8 marks]

Recall the M/M/1 queue from Lecture 3. The state of the process $\mathcal{X} = (X(t), t \geq 0)$ is the queue length at time t, including the person being served.

Assume there are arrivals to the queue according to a Poisson process with rate λ; and, services occur as a Poisson process with rate μ whenever there is at least one customer.

• The state space is $\mathcal{S} = \{0, 1, 2, \ldots\}$, where each state $i \in \mathcal{S}$ means there are i people in the system.

• The transition rates are:

$$\begin{aligned} q_{i,i+1} &= \lambda \quad \text{for } i \in \mathcal{S}, \\ q_{i,i-1} &= \mu \quad \text{for } i \in \mathcal{S}\backslash\{0\}, \\ q_{ij} &= 0 \quad \text{for } j \in \mathcal{S}\backslash\{i, i+1, i-1\}. \end{aligned}$$

• The state transition diagram is

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "\mu"] \end{tikzcd}$$

(a) Show that in state $i \in \mathcal{S} \setminus \{0\}$, the sojourn time is exponentially-distributed with rate $\lambda + \mu$.

(b) Show that in state $i \in \mathcal{S} \setminus \{0\}$, the probability of an arrival at the time of an event is independent of the timing of the event and is equal to $\lambda/(\lambda + \mu)$.

Question 2. [12 marks]

Consider two time machines, maintained by a single genius. Machine $i \in \{1, 2\}$ functions for an exponentially-distributed time with mean $1/\gamma_i$ before it breaks down. Repair time for time machine i is exponentially-distributed with mean $1/\beta_i$. The genius can repair only one machine at a time.

Assume that the machines are repaired in the order in which they fail.

Construct a continuous-time Markov chain (CTMC) to model this system, where the state $X(t)$ of the system represents the status of the machines at time $t \geq 0$, by following these steps.

(a) Write down the state space of this CTMC.

(b) Draw a state transition diagram corresponding to this CTMC.

(c) Specify the generator of this CTMC.

Question 3. [3 marks]

Assume you are modelling an arrival process using a Poisson process, where the rate of arrivals is $\lambda = 24$ per day.

(a) Calculate the probability of seeing 2 arrivals in the first hour, and 5 arrivals in total during the second and third hours.

Question 4. [12 marks]

Consider Example 1 of Lecture 2, and considered again in Lecture 4.

(a) Evaluate $P_{01}(200)$, the probability of having only 1 working printer 200 days after having both printers working, given the transition function in the lecture notes.

(b) What happens to the transition function as $t \to \infty$? Compare, appropriately, to your answer in (a).

(c) Evaluate $\exp(Q * 200)$, where exp is the matrix exponential – for example, in MATLAB you can use `expm`. Compare to your answer in (b).

(d) Modify the simulation code `Repairman2.m` provided as part of Lecture 4 (or write your own code, in whatever language you like) to estimate $P_{01}(200)$. Provide details of your inputs. Compare to your answer in (a).