Stochastic Assignment 3

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Question 1. Marks for mathematical language

Question 2. Consider the problem

$$\begin{aligned} \min\ z = 3x_1 + 2x_2 + \mathbb{E}_\zeta[15y_1 + 18y_2] \\ \text{s.t.} \quad 3y_1 + 2y_2 \leq x_1 \\ 2y_1 + 5y_2 \leq x_2 \\ 0.6d_1 \leq y_1 \leq d_1 \\ 0.8d_2 \leq y_2 \leq d_2 \\ x_i \geq 0 \\ y_i \geq 0 \end{aligned}$$

Where $\zeta = (d_1, d_2)^T$ has two realisations:

$$\begin{aligned} \epsilon_1 = (d_1, d_2) = (4, 4)^T \text{ with probability } 0.5 \\ \epsilon_2 = (d_1, d_2) = (6, 8)^T \text{ with probability } 0.5 \end{aligned}$$

(a) Perform Steps 0, 1, and 2 of the L-shaped algorithm on the above problem. For the purpose of this assignment, do each step only once; that is, stop when the algorithm asks you to either go back to Step 1 or proceed to Step 3. Explain all your working and provide all MATLAB code and relevant output.

Solution Rewriting it in a more standard form:

$$\begin{aligned} \min\ z = 3x_1 + 2x_2 + \mathbb{E}_\zeta[15y_1 + 18y_2] \\ \text{s.t.} \quad 3y_1 + 2y_2 \leq x_1 \\ 2y_1 + 5y_2 \leq x_2 \\ y_1 \leq d_1 \\ -y_1 \leq -0.6d_1 \\ y_2 \leq d_2 \\ -y_2 \leq -0.8d_2 \\ x_i \geq 0 \\ y_i \geq 0 \end{aligned}$$

i. Step 0: Initialise counters

$$t = s = v = 0, \quad \mathbf{D}_0 = \mathbf{0}, \quad d_0 = 0, \quad \mathbf{E}_0 = \mathbf{0}, \quad e_0 = 0$$

ii. Step 1: Increment v ($v = 1$) and solve the simplified first stage LP. I.e. solve (given $\theta = 0$):

$$\begin{aligned} \min\ z = 3x_1 + 2x_2 \\ \text{s.t.} \quad x_i \geq 0 \end{aligned}$$

Which is trivially minimised for $\mathbf{x} = (0, 0)^T$ giving $z^{(1)} = 0$. Now set $\theta^{(1)} = -\infty$ as its constraint is missing.

iii. Step 2: Feasibility cuts Apparently we must note that $Wy \leq h - Tx$ Solve the LP:

$$\begin{aligned} \min\ \tilde{w} &= e^T v^+ + e^T v^- \\ \text{such that } W y_k^{(v)} - v^- &= h_k - T_k x^{(v)} \\ y_k^{(v)}, v^+, v^- &\geq 0 \end{aligned}$$

Considering both h_k's we get the constraints:

$$\begin{aligned} 3y_1 + 2y_2 - v_1^- &= 0 \\ 2y_1 + 5y_2 - v_2^- &= 0 \\ y_1 - v_3^- &= 4 \\ -y_1 - v_4^- &= -3.6 \\ y_2 - v_5^- &= 4 \\ -y_2 - v_6^- &= -6.4 \\ y_i &\geq 0 \end{aligned}$$

(IN CODE)

```
W = [3, 2;
     2, 5;
     1, 0;
    -1, 0;
     0, 1;
     0,-1];
A =[W,-eye(6)];

f = [0,0,1,1,1,1,1,1];
h= [0;0;4;-3.6;4;-6.4];

%Tx = 0 since x = 0

[u,w]=linprog(f,A,h,[],[],zeros(1,8));
```

Gives:

```
u =

         0
         0
         0
         0
         0
    3.6000
         0
    6.4000


w =

    10
```

Since $\tilde{w} > 0$, let $\sigma_k^{(v)}$ be the associated vector of Lagrange multipliers (which we obtain by solving the dual of the last LP). CODE:

```
[sigma,~]=linprog(-h,A',f',[],[],[],zeros(1,6))
```

Output:

```
sigma =

   -1.0000
   -1.0000
         0
   -1.0000
         0
   -1.0000
```

Lastly, define:

$$\mathbf{D}_{t+1} = (\sigma_k^{(v)})^T T_k)$$

$$d_{t+1} = (\sigma_k^{(v)})^T h_k)$$

In code:

```
D = sigma'*T;
d1 = sigma' * h1;
d2 = sigma' * h2;
```

Gives:

```
D =

    1.0000    1.0000

d1 =

    5.6000

d2 =

    10
```

So we get (want the smaller d as it corresponds to a harsher cut)

$$D_0 = [1, 1]$$

$$d_0 = 5.6$$

increment t ($t = 1$ now) and go back to step 1 (so we will terminate here).

As required.

Question 3. The Adelaide University Mathematics Society is having a BBQ; there are loads of free food. You stand in line to get a hot dog, and n students are in front of you. With probability p the student at the head of the queue will finish being served in the next minute, independently of what happens in all other minutes. At the beginning of every minute, you need to decide to continue waiting or leave.

Suppose there is a cost of $\$a$ for every minute spent waiting, and the hot dog is $\$b$. Let J_n denote the expected reward obtained by employing an optimal waiting policy when there are n people ahead of you in the queue. *Hints: The hotdog is worth $\$b$ to us, as in it costs nothing, but is worth $\$b$.*

(a) Write the optimality equation for $n = 0$ and $n > 0$ with justification

Solution Every minute there is a decision to continue waiting or leave.
Waiting time is $X \sim Geom(p)$. Recall it has mean $\frac{1}{p}$.

If at the front of line you want to wait if the value of the hot dog is greater than the expected cost of waiting (the time multiplied by the cost). Note that this is a pure-death process. I.e.

$$p_{ij} = \begin{cases} p, & j = i-1 \\ 1-p, & j = i \\ 0, & otherwise \end{cases}$$

$$\begin{aligned}
J_n &= \max_{u_n \in U_n} \mathbb{E}\left[c(u_n) + \sum_{j\in S} p_j(u_n)J_{n-1}\right] \\
&= \max_{\{stay, leave\}} \mathbb{E}\left[c(\{stay, leave\}) + \sum_{j\in S} p_j(\{stay, leave\})J_{n-1}\right] \\
&= \max\left\{\mathbb{E}(-c(stay) + p_{i-1}(stay)J_{n-1} + p(stay)J_n), \mathbb{E}(c(leave) + p_{i-1}(leave)J_{n-1})\right\} \\
&= \max\{-a + pJ_{n-1} + (1-p)J_n, 0\}
\end{aligned}$$

For $n = 0$: If you leave, the cost is \$0. If you wait, the cost will be the reward (hotdog worth \$$b$) minus the expected waiting cost. I.e.:

$$J_0(i) = \max\{\mathbb{E}[b - aX], 0\} = \max\{b - a\mathbb{E}[X], 0\} = \max\{b - a/p, 0\}$$

and then for $n > 0$

As required.

(b) Show that the optimality equation can be rewritten as

$$J_n = \max\{J_{n-1} - a/p, 0\} \text{ for } n > 0$$

Solution Ignore the zero case as that won't change. Assuming that the 0 case is not taken:

$$\begin{aligned}
J_n &= \max\{-a + pJ_{n-1} + (1-p)J_n, 0\} \\
&= -a + pJ_{n-1} + J_n - pJ_n \\
pJ_n &= -a + pJ_{n-1} \\
J_n &= J_{n-1} - a/p \\
\implies J_n &= \max\{J_{n-1} - a/p, 0\}
\end{aligned}$$

As required.

(c) By induction the above implies that $J_n \leq J_{n-1}$. Explain why intuitively this is expected

Solution Intuitively we expect to get greater reward the closer to the front of the line we are - we will be spending more time waiting if we are further back in the line. If we are in the i^{th} position and we wait, we will be waiting for *at least* i minutes, so intuitively the $i + 1^{th}$ position will wait for *at least* $i + 1$ minutes. Alternatively if at the j^{th} position in line, the choice to leave is taken, then all positions behind j should employ the same policy, giving equality.

As required.

(d) Determine the threshold N such that the form of the optimal policy is to wait only if $n \leq N$. Find J_n in terms of a, b and p.

Solution Easiest to find J_n in terms of a, b, p first:

$$\begin{aligned}
J_n &= \max\{J_{n-1} - a/p, 0\} \\
&= \max\{\max\{J_{n-2} - a/p, 0\} - a/p, 0\}\} \\
&= \max\{J_{n-2} - 2a/p, 0\} \\
&\vdots \\
&= \max\{J_0 - \frac{(n-1)a}{p}, 0\} \\
&= \max\{b - \frac{na}{p}, 0\}
\end{aligned}$$

So to find the optimal policy: Want $J_N > 0$

$$\begin{aligned}J_N = \max\{b - \frac{Na}{p}, 0\} &> 0 \\ \implies b - \frac{Na}{p} &> 0 \\ \frac{Na}{p} &< b \\ \therefore N &< \frac{bp}{a}\end{aligned}$$

I.e. the optimal policy is to wait only if

$$n < \frac{bp}{a}$$

And leave the queue otherwise. **As required.**

Question 4. The bad news is you have just been bitten by a poisonous spider. The good news is you have m potion bottles which might just save your life. If you drink the i^{th} bottle, there is a probability of α_i that you will stay alive, a probability of β_i that you will die instantaneously, and a probability of $1 - \alpha_i - \beta_i$ that the potion will do absolutely nothing.

Hints:
in order to maximise your probability of staying alive, you first need to determine the expression for the probability of staying alive.

So, if the order in which you will be drinking the bottles is $1, 2, 3, \ldots, i, j, \ldots n$ *then what is the probability that you will stay alive? If you switch i and j, then what is the probability that you will stay alive?*

If a potion does "absolutely nothing", it doesn't mean you are cured from the poison. It just means you will live to drink the next potion (if there's any left!) in hope of saving yourself.

(a) Determine the order in which you should drink the potion bottles to maximise your probability of staying alive, in terms of the ratios α_i/β_i .

Solution As per the hint, calculate probability of survival. Consider drinking the i^{th} and then the j^{th} (and vice-versa) Let $P_{ij} := P(\text{cured after } i \textit{ or } j)$ where i is first:

$$\begin{aligned}P_{ij} &= P(i \text{ cures you}) + P(i \text{ doesn't kill you}|i \text{ didn't cure you})P(j \text{ cures you}) \\ &= \alpha_i + (1 - \alpha_i - \beta_i)\alpha_j \\ P_{ji} &= P(j \text{ cures you}) + P(j \text{ doesn't kill you}|j \text{ didn't cure you})P(i \text{ cures you}) \\ &= \alpha_j + (1 - \alpha_j - \beta_j)\alpha_i\end{aligned}$$

If we consider that $P_{ij} > P_{ji}$

$$\begin{aligned}P_{ij} &> P_{ji} \\ \alpha_i + (1 - \alpha_i - \beta_i)\alpha_j &> \alpha_j + (1 - \alpha_j - \beta_j)\alpha_i \\ \alpha_i + \alpha_j - \alpha_i\alpha_j - \beta_i\alpha_j &> \alpha_j + \alpha_i - \alpha_j\alpha_i - \alpha_i\beta_j \\ -\beta_i\alpha_j &> -\alpha_i\beta_j \\ \beta_i\alpha_j &< \alpha_i\beta_j \\ \implies \frac{\alpha_i}{\beta_i} &> \frac{\alpha_j}{\beta_j}\end{aligned}$$

Since this must hold true for all i, j. The most effective technique is to drink the potions with the highest ratio $\frac{\alpha_i}{\beta_i}$ first.
As required.