Random Processes Tute

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1 Tute 1

1.

$$Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

(a) State space is $\{0, 1\}$

(b) Eigen values and Eigenvectors:
Eigenvalues: Solve determinant of

$$|Q - \lambda I| = 0$$

$$\begin{aligned} (-a-\lambda)(-b-\lambda) - ab &= 0 \\ ab + (a+b)\lambda + \lambda^2 - ab &= 0 \\ \lambda(a+b+\lambda) &= 0 \\ &\implies \lambda = -a-b \text{ or } 0 \end{aligned}$$

Right eigenvector: find v such that

$$Qv = \lambda v \implies (Q - \lambda I)v = 0$$
$$\begin{pmatrix} -a-\lambda & a \\ b & -b-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Plug in the eigenvalues $\lambda = -a - b$:

$$\begin{pmatrix} -a-(-a-b) & a \\ b & -b-(-a-b) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} b & a \\ b & a \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{aligned} bv_1 + av_2 &= 0 \\ \implies v_1 &= \frac{av_2}{b} \end{aligned}$$

$$v = \begin{pmatrix} a/b \\ 1 \end{pmatrix}$$

Repeat this for $\lambda = 0$

$$\begin{pmatrix} -a & a \\ b & -b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{aligned} -av_1 + av_2 &= 0 \\ bv_1 - bv_2 &= 0 \\ \implies v_1 &= v_2 \end{aligned}$$

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Left eigenvectors are a similar problem, and give for $\lambda = 0$:

$$v = \begin{pmatrix} 1 & a/b \end{pmatrix}$$

for $\lambda = -a - b$:

$$v = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

(c) Find trans function

$$\begin{aligned} P(t) &= e^{Qt} \\ &= \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!} \\ &= \sum_{i=1}^{m} e^{\lambda_i t} M_i \end{aligned}$$

I.e. exp of the eigenvalue, and the matrix based on the eigenvectors. Where $M_i = r_i' l_i$ where r_i is the i^{th} right eigenvector and i^{th} left eigenvector. Where $r_i' l_i = 1$ Let M_1 correspond to $\lambda = 0$

$$M_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = 1 \implies M_1 = \begin{pmatrix} \frac{b}{b+a} & \frac{a}{b+a} \\ \frac{b}{b+a} & \frac{a}{b+a} \end{pmatrix}$$

Similarly solve M_2

2.

$$Q = \begin{pmatrix} -3 & 3 & 0 \\ 2 & -4 & 2 \\ 0 & 6 & -6 \end{pmatrix}$$

(a) State space $\{0, 1, 2\}$

(b) Equilibrium equations

$$\pi Q = 0$$

$$\begin{aligned} -3\pi_1 + 2\pi_2 &= 0 \\ 3\pi_1 - 4\pi_2 + 6\pi_3 &= 0 \\ 2\pi_2 - 6\pi_3 &= 0 \\ \text{such that } \pi_1 + \pi_2 + \pi_3 &= 1 \\ \pi &= \begin{pmatrix} 1/3 & 1/2 & 1/6 \end{pmatrix} \end{aligned}$$

(c) Find $\mathbb{P}$ using Q solve the equilibrium equations. I.e. solve

$$\pi = \pi \mathbb{P}$$

P =

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\pi = (1/4, 1/2, 1/4)$$

Solve equilibrium equations for this P basically.

(d) What is the relationship between these two distributions? If we multiply this π based on P by the inverse of the diagonal of Q, and then normalise, we will get the π from the previous question.

2 Tute 3

1. $P_{Nj}(t)$

(a) Write down Q. In general:

$$q_{i,j} = \begin{cases} \mu & j = i - 1 \\ -\mu & j = i \\ 0 & \text{otherwise} \end{cases}$$

In this case we only have 1 individual

$$Q = \begin{pmatrix} 0 & 0 \\ \mu & -\mu \end{pmatrix}$$

(b) Write and solve the KFDEs for $P_{10}(t)$ and $P_{11}(t)$

$$\frac{\partial P_{ij}(t)}{\partial t} = \sum_{k \in S} P_{ik}(t) q_{kj}$$

$$P' = PQ = \begin{pmatrix} P_{00} & P_{10} \\ P_{01} & P_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mu & -\mu \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial P_{00}}{\partial t} & \frac{\partial P_{01}}{\partial t} \\ \frac{\partial P_{10}}{\partial t} & \frac{\partial P_{11}}{\partial t} \end{pmatrix} = \begin{pmatrix} P_{10}\mu & -\mu P_{10} \\ P_{11}\mu & -\mu P_{11} \end{pmatrix}$$

Start with $\frac{\partial P_{11}}{\partial t} = -\mu P_{11}$

$$\implies P_{11}(t) = Ae^{-\mu t}$$

$$P_{11}(0) = 1 \implies A = 1 \implies P_{11}(t) = e^{-\mu t}$$

Now find $P_{10}(t)$

$$\frac{\partial P_{10}(t)}{\partial t} = \mu P_{11}(t) = \mu e^{-\mu t}$$

$$P_{10}(t) = B - e^{-\mu t}$$

But $P_{10}(t) = 1 - P_{11}(t) \implies B = 1$

$$P_{10}(t) = 1 - e^{-\mu t}$$

(c) Physical meaning:
$P_{10}(t)$ is the probability of being absorbed into 0 by time t. As $t \to \infty$ we find $P_{10}(t) = 1$ i.e. we will be absorbed into 0. Similarly, $P_{11}(t)$ is the probability of remaining in state 1 by time t. And this goes to 0 as $t \to \infty$

(d) What if we had N components? Find $P_{Nj}(t)$ We could consider the components as iid bernoulli trials

$$\begin{aligned} P_{Nj}(t) &= \binom{N}{j} P_{11}(t)^j P_{10}(t)^{N-j} \\ &= \binom{N}{j} (e^{-\mu t})^j (1 - e^{-\mu t})^{N-j} \end{aligned}$$

2. (a)
   (b)

3. Population of constant size N as a CTMC where $X(t)$ is the number of infectious people at time t. Assume infections occur at rate

$$q_{i,i+1} = \beta i(N-i)/(N-1)$$

And recoveries occur at

$$q_{i,i-1} = \gamma i$$

For $i \in S = \{0, 1, \dots, N\}$

(a) What are the communicating classes?
0 is a communicating class, and $\{1, \dots, N\}$ is a communicating class 0 is an absorbing state. Irreducible. Finite means recurrent, 0 is recurrent, and $\{1, \dots, N\}$ is transient. It has a steady state solution

$$\pi = P\pi$$

(b) System of linear equations which allow us to calculate probability of having at least j infectious individuals. at some stage over the course of the epidemic, having started with $i \leq j$ infectious individuals. Evaluate this for $i = 1$ and $j = 1, 2, 3$

$$\begin{aligned} -q_{ii} f_{ij} &= \sum_{k \neq i} q_{ik} f_{kj} \\ -q_{11} f_{1j} &= \sum_{k \neq 1} q_{1k} f_{kj} \\ (\beta 1(N-1)/(N-1) + \gamma 1) f_{1j} &= q_{10} f_{01} + q_{12} f_{2j} \\ (\beta + \gamma) f_{1j} &= q_{12} f_{2j} \\ &= \beta(N-1)/(N-1) f_{21} = \beta f_{2j} \\ (\beta + \gamma) f_{1j} &= \beta f_{2j} \end{aligned}$$

For $j = 1$ we get $f_{11} = 1$ because trivially it is 1. For $j = 2$:

$$(\beta + \gamma) f_{12} = \beta f_{22}$$
$$f_{12} = \frac{\beta}{\beta + \gamma}$$

For $j = 3$:

$$(\beta + \gamma) f_{13} = \beta f_{23}$$
$$f_{13} = \frac{\beta}{\beta + \gamma} f_{23}$$

$$\begin{aligned} f_{23} &= \frac{q_{21}}{-q_{22}} f_{13} + \frac{q_{23}}{-q_{22}} f_{33} \\ &= \frac{q_{21}}{-q_{22}} f_{13} + \frac{q_{23}}{-q_{22}} \end{aligned}$$

its fucking grotty but we end up with

$$\implies f_{13} = \frac{\beta^2 (N-2)}{\beta^2 (N-2) + \beta\gamma (N-2) + \gamma^2/(N-1)}$$

3 Tute 4

1. Infinite buffered switch. Poisson arrival rate λ, processed at rate $\mu < \lambda$. If the queue increases past K, the process rate becomes $2\mu > \lambda$. This happens until it hits 0, and then it goes back to μ.

   (a) Use a CTMC model to show that if the number of packets is less than K, it will hit K with probability 1. Starting in j then

   $$f_j = \frac{\lambda}{\lambda + \mu} f_{j+1} + \frac{\mu}{\lambda + \mu} f_{j-1}$$

   With boundaries

   $$f_k = 1, \quad f_0 = f_1$$

   $$f_j = A \left(\frac{\mu}{\lambda}\right)^j + B$$

   Using boundaries:

   $$f_k = 1 = A \left(\frac{\mu}{\lambda}\right)^k + B \quad \implies \quad B = 1 - A \left(\frac{\mu}{\lambda}\right)^k$$

   Meaning

   $$f_j = 1 + A \left(\left(\frac{\mu}{\lambda}\right)^j - \left(\frac{\mu}{\lambda}\right)^k \right)$$

   $$f_0 = f_1 \implies 1 + A \left(\left(\frac{\mu}{\lambda}\right)^1 - \left(\frac{\mu}{\lambda}\right)^k \right) = 1 + A \left(\left(\frac{\mu}{\lambda}\right)^0 - \left(\frac{\mu}{\lambda}\right)^k \right)$$
   $$\implies A = 0$$

   Which then gives

   $$f_j = 1 \text{ always}$$

   (b) Show that it will hit 0 after hitting K Similar problem, only with $f_0 = 1$ and no other boundary condition. This has to use the minimal non-negative thingo. So fun! Let f_j be hitting prob for 0.

   $$f_j = \frac{\lambda}{\lambda + 2\mu} f_{j+1} + \frac{2\mu}{\lambda + 2\mu} f_{j-1}$$

   Again we get

   $$f_j = A \left(\frac{2\mu}{\lambda}\right)^j + B$$

$$f_0 = 1 \implies B = 1 - A$$

$$f_j = 1 + \left(A\left(\frac{2\mu}{\lambda}\right)^j - 1\right)$$

We need the minimal non-negative solution. Since $2\mu > \lambda$ then $A = 0$. Giving:

$$f_j = 1$$

(c) Calculate the expected time between reaching K and zero again.
Let $t_k^{(0)}$ be the expected hitting time for state k to reach 0

$$t_k = -\frac{1}{q_{ii}} + \sum_{i \neq k} \frac{q_{ki}}{-q_{kk}} t_i$$

$t_0 = 0$.

$$\begin{aligned}
t_k &= \frac{1}{\lambda + 2\mu} + \frac{\lambda}{\lambda + 2\mu} t_{k+1} + \frac{2\mu}{\lambda + 2\mu} t_{k-1} \\
\text{guess} : \ t_k &= Ck \\
Ck &= \frac{1}{\lambda + 2\mu} + \frac{\lambda}{\lambda + 2\mu} C(k+1) + \frac{2\mu}{\lambda + 2\mu} C(k-1) \\
Ck &= Ck + \frac{\lambda}{\lambda + 2\mu} - C\frac{2\mu}{\lambda + 2\mu} + \frac{1}{\lambda + 2\mu} \\
\implies C &= \frac{1}{2\mu - \lambda}
\end{aligned}$$

Homogeneous Solution:

$$t_k = A\left(\frac{2\mu}{\lambda}\right)^k + B + \frac{1}{2\mu - \lambda} k$$

Now plug in boundary:

$$t_0 = 0 = A + B \implies B = -A$$

$$\implies t_k = A\left(\left(\frac{2\mu}{\lambda}\right)^k - 1\right) + \frac{1}{2\mu - \lambda} k$$

Minimal non-negative:

$$\implies A = 0$$

$$\therefore t_k = \frac{j}{2\mu - \lambda}$$

2. Expected hitting times and cost

$$\begin{aligned}
c_i &= \mathbb{E}\left[\text{cost until reaching } j | X(0) = i\right] \\
&= \int_0^\infty E(\$_{X(t)} | X(0) = i) dt \\
c_i &= E\left[E\left[\text{cost until reaching } j | X(0) = i, T_1 = s, X(T_1) = k\right]\right] \\
&= \int_0^\infty \sum_{\substack{k \in S \\ k \neq i}} E[\text{cost until } j | X(0) = i, T_1 = s, X(T_1) = k] \frac{q_{ik}}{-q_{ii}} (-q_{ii}) e^{q_{ii} S} ds \\
&= \int_0^\infty \sum_{\substack{k \in S \\ k \neq i}} E[\text{cost until } j | X(0) = i, T_1 = s, X(T_1) = k] q_{ik} e^{q_{ii} S} ds \text{ come back...}
\end{aligned}$$

$$\begin{aligned}
E[\text{cost until } j | X(0) = i, T_1 = s, X(T_1) = k] &= \int_0^\infty E[\$_{X(t)} | X(0) = i, T_i = S, X(T_i) = k] dt \\
&= \int_0^S \$_i dt + \int_S^\infty E[\$_{X(t)} | X(S) = k] dt \\
&= \$_i S + c_k
\end{aligned}$$

Back to the come back.. line

$$
\begin{aligned}
c_i &= \int_0^\infty \sum_{\substack{k\in S\\ k\neq i}} (\$_i S + c_k) q_{ik} e^{q_{ii}S} dS \\
&= \sum_{\substack{k\in S\\ k\neq i}} \left[\$_i \int_0^\infty S e^{q_{ii}S} dt + c_k \int_0^\infty e^{q_{ii}S} dS \right] \\
&= \frac{\$_i}{-q_{ii}} + \sum_{\substack{k\in S\\ k\neq i}} \frac{q_{ik}}{-q_{ii}} c_k
\end{aligned}
$$

3. 2D birth death process. Room 1 has R_1 spots for customers of type 1 for queue 1, similarly R_2 for type 2 for queue 2 and R_3 holds both 1 and 2 as overflow.These customers then move to their rooms when they can.
   Let λ_1, λ_2 be the poisson arrivals and μ_1, μ_2 be the service rates.

   (a) Restrict the state space:
   Let n_1, n_2 be the number of people in queue 1 and queue 2 respectively.

   $$A = \{(n_1, n_2) \in \mathbb{N}^2 : [n_1 - (R_1+1)]^+ + [n_2 - (R_2+1)^+] \leq R_3$$

   (b) Write down the joint equilibrium
   Since all the rates are independent, we just use the formula. For when we un-restrict it:

   $$\pi(n) = C \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$$

   $$C = (1 - \frac{\lambda_1}{\mu_1})(1 - \frac{\lambda_2}{\mu_2})$$

   Truncated version;

   $$\pi(n) = C_{trunc} \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$$

   $$C_{trunc} = \sum_{n\in A} \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$$

4 Tute 5

1. Open Jackson network with 3 queues. Each queue has a single server which serves at rate μ. Calls arrive to the first queue from outside the network according to a Poisson process at rate λ and when their service is complete, are routed according to:

   $$[\gamma_{ij}] = \begin{pmatrix} 0 & 1/8 & 7/8 \\ 3/4 & 0 & 1/4 \\ 1/2 & 1/4 & 0 \end{pmatrix}$$

   (a) Solve the traffic equations to determine the average arrival rate at each queue
   Note that the $[\gamma_{ij}]$ is a probability matrix.

   $$
   \begin{aligned}
   \mathbf{y} &= \lambda + \mathbf{y}\gamma \\
   y_1 &= \lambda + \frac{3}{4}y_2 + \frac{1}{2}y_3 \\
   y_2 &= \frac{1}{2}y_1 + \frac{1}{4}y_3 \\
   y_3 &= \frac{7}{8}y_1 + \frac{1}{4}y_2
   \end{aligned}
   $$

   Solve this system of linear equations to get

   $$
   \begin{aligned}
   y_1 &= \frac{120}{29}\lambda \\
   y_2 &= \frac{44}{29}\lambda \\
   y_3 &= 4\lambda
   \end{aligned}
   $$

(b) Write down an expression for the equilibrium distribution for the network
The invariant measures are:

$$Q_i(n_i) = \prod_{l=1}^{n_1} \frac{y_i}{\mu_i(l)}$$

But in this case all the $\mu_i(l) = \mu$

$$Q_i(n_i) = \left(\frac{y_1}{\mu}\right)^{n_i}$$

The equilibrium distribution is then (given C exists)

$$\pi(n_1, n_2, n_3) = \left(\frac{y_1}{\mu}\right)^{n_1} \left(\frac{y_2}{\mu}\right)^{n_2} \left(\frac{y_3}{\mu}\right)^{n_3} C$$

$$= \left(1 - \frac{120\lambda}{29\mu}\right)\left(\frac{120\lambda}{29\mu}\right)^{n_1}\left(1 - \frac{44\lambda}{29\mu}\right)\left(\frac{44\lambda}{29\mu}\right)^{n_2}\left(1 - \frac{4\lambda}{\mu}\right)\left(\frac{4\lambda}{\mu}\right)^{n_3}$$

Now note we need

$$\frac{120\lambda}{29\mu} < 1, \quad \lambda < \frac{\mu 29}{120}, \quad \lambda < \frac{29}{44\mu}, \quad \lambda < \frac{1}{4}\mu$$

So we need

$$\lambda < \min\{\frac{29}{120}\mu, \frac{29}{44}\mu, \frac{1}{4}\mu\} = \frac{29}{120}\mu$$

(c) If $\lambda = 9/2$ and $\mu = 10$. Does the equilibrium exist? And how stable is the network
The equilibrium won't exist as $\frac{9}{2} > \frac{29}{12}$. The network is not stable.

2. Consider a queue with poisson arrival with rate λ. Exponentially distributed service time with parameter μ. Let the residence time be the total of the waiting time and the service time

(a) If the queue is $M/M/\infty$ what is the distribution of the residence time?
$Exp(\mu)$

(b) If it is $M/M/N$ what is the mean waiting time? So what is the mean residence time?

$$E[W] = \frac{C(N, \frac{\lambda}{\mu}}{N\mu - \lambda}$$

$$E[\text{residence time}] = \frac{C(N, \frac{\lambda}{\mu}}{N\mu - \lambda} + \frac{1}{\mu}$$

(I.e. the expected waiting time plus the service time

(c) If the queue is a $M/M/1$ queue, what is the mean waiting time? So what is the mean residence time?
Using (ii)

$$E[W] = \frac{C(1, \frac{\lambda}{\mu}}{\mu - \lambda} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$E[\text{residence time}] = \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = \frac{1}{\mu - \lambda}$$

(d) How does the answer to (iii) relate to the conditional expected waiting time in such a queue, given you have to wait? Can you explain the relationship?
The conditional expected waiting time given you have to wait:

$$E[W_Q | W_Q > 0] = \frac{1}{\mu - \lambda}$$

(Using the thing from lectures)
These are the same!

$$\begin{aligned}
E(residence\ time) &= E\left[E[residence\ time | j \text{ people in the queue}]\right] \\
&= \sum_{j=0}^{\infty} \pi_j (E[W_Q | \text{see j in queue}] + \frac{1}{\mu} \\
&= \sum_{j=0}^{\infty} \pi_j (\frac{j}{\mu} + \frac{1}{\mu}) \\
&= \sum_{j=0}^{\infty} \pi_j \frac{j+1}{\mu}
\end{aligned}$$

$$\pi_j = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^j, \quad j \geq 0$$

$$\hat{\pi}_k = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^{k-1}, \quad k \geq 1$$

$$\pi_j = \hat{\pi}_{j+1}$$

$$\begin{aligned} E[res\ time] &= \sum_{j=0}^{\infty} \pi_j (\frac{j+1}{\mu} \\ &= \sum_{j=0}^{\infty} \hat{\pi}_{j+1} \frac{j+1}{\mu} \\ &= \sum_{k=1}^{\infty} \hat{\pi}_k \frac{k}{\mu} \\ &= \sum_{k=1}^{\infty} \hat{\pi}_k E[Q_Q|\text{j customers}]] \\ &= E[W_Q | W_Q > 0] \end{aligned}$$

Basically showed that the two things are the same holy shit.

3. A hospital operates like a complex queueing system with patients moving between areas and waiting for services

   (a) Consider the hospital operates at 90% occupancy, and admits on average 100 people per day. Assuming it has 500 beds, use Little's Law to determine the average stay of a patient.
   We need average queue length and entry rate. So $\bar{Q} = (90\%) * 500 = 450$
   $\lambda = 100$ per day

   $$\bar{Q} = \lambda \bar{W} \implies \bar{W} = \frac{\bar{Q}}{\lambda} = \frac{450}{100} = 4.5$$

   (b) Now consider the ED of the hospital. Here, the waiting time between arrival and the start of treatment is recorded. Assuming the average waiting time is 3 hours and that the ED starts to treat on average 10 new patients per hour, determine the average number of patients waiting for service. List any assumptions made.

   $$\bar{W} = 3 \text{ hours} \quad \lambda = 10 \text{ people / hour} \quad \bar{Q} = \lambda \bar{W} = 3 * 10 = 30$$

   (c) Consider a single resuscitation room within the ED. This operates like a M/G/1/1 queue. Show that the probability that the resuscitation room is occupied is given by $\frac{\lambda}{\mu+\lambda}$, where λ is the arrival rate of patients requiring the resuscitation room, and $1/\mu$ is the mean time that a patient occupied the room.

   $$\bar{W} = \frac{1}{\mu} \quad \bar{\lambda} = \lambda(1 - P(occupied))$$

   $$\bar{Q} = 0 \times (1 - P(occupied)) + 1 * P(occupied)$$

   $$\bar{Q} = P(occupied)$$

   Now use littles law

   $$\bar{Q} = \bar{\lambda} \bar{W}$$

   $$P(occupied) = \frac{1}{\mu}(\lambda(1 - P(occupied)) \implies P(occupied) = \frac{\lambda}{\mu + \lambda}$$

Littles Law:

$$\bar{Q} = \lambda \bar{W}$$

I.e. Average queue length = entry rate times average waiting time.