THE UNIVERSITY of ADELAIDE

Student ID: __________

Family name: __________

Other names: __________

Desk number: __________ Date: __________

Signature: __________

Examination in the School of Mathematical Sciences

Semester 1, 2017

107351	**APP MTH 3021**	**Modelling with Ordinary Differential Equations III**

Time for completing booklet: 120 mins (plus 10 mins reading time).

Question	Marks	
1	/13	
2	/18	
3	/20	
4	/19	
Total	/70	

Instructions to candidates

- Attempt all questions and write your answers in the space provided below that question.
- If there is insufficient space below a question, then use the space to the *right* of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

- Calculators are not permitted.

Do not commence writing until instructed to do so.

13 Total

Question 1.

Consider the ordinary differential equation

$$\frac{dx}{dt} = rx + \frac{1}{1+x} \quad \text{for} \quad x > -1, \tag{1}$$

where r is a real positive parameter.

/1 mark

1(a) Determine the steady states of this equation.

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/2 marks

1(b) Use your answer to 1(a) to determine the bifurcation value $r = \overline{r}$. Hence calculate the bifurcation point $x = \overline{x}$.

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/4 marks

1(c) Perform a phase-line analysis for $r < \overline{r}$, $r = \overline{r}$ and $r > \overline{r}$. Include arrows to indicate the stability on the fixed points, and state the stability of the fixed points, explaining your reasoning.

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/2 marks

1(d) State the type of bifurcation that occurs at $r = \overline{r}$, and give a reason.

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/4 marks

1(e) Sketch the bifurcation diagram, marking the stable and unstable branches and the bifurcation point and value.

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18 Total

Question 2.

Consider the modified predator–prey system

$$\frac{dx}{dt} = x(1-x) - xy \tag{2a}$$

$$\frac{dy}{dt} = y\left(1 - \frac{y}{x}\right) \tag{2b}$$

for $x > 0$ and $y \geq 0$.

2(a) Interpret Eq. (2a) when $y = 0$.

/2 marks

2(b) In system (2a–b), which of x and y is the predator and which is the prey? Justify your answer.

/2 marks

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2(c) Calculate the nullclines of the system, and hence calculate the two relevant fixed points.

/4 marks

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2(d) Calculate the Jacobian of the system, and hence classify the two fixed points.

/5 marks

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2(e) Explain why one of the fixed points is asymptotically stable. Sketch the trajectory in the phase plane for an initial condition $(x(0), y(0))$ close to the stable fixed point.

/5 marks

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20 Total

Question 3.

Consider the IVP

$$\frac{du}{dt} = \frac{-u}{\sqrt{t}} \quad \text{with} \quad u(1) = 0.5. \tag{3}$$

3(a) Express the IVP as an integral equation, and hence write down the associated Picard iteration scheme, including the value of the initial iterate $u^{(0)}$. Calculate the first iterate $u^{(1)}$.

/5 marks

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3(b) Show that the function

$$f(u,t) = \frac{-u}{\sqrt{t}}$$

is Lipschitz continuous for $u \in \mathbb{R}$ and $t \in [0.25, 1.75]$, and determine the smallest Lipschitz constant on this domain.

/3 marks

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3(c) Does a unique solution to IVP (3) exist for some interval of time following $t = 1$? Justify your answer.

You do **not** need to calculate the time interval, and do **not** solve the IVP.

/3 marks

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3(d) Can a unique solution be guaranteed if the initial condition is replaced by $u(0) = 1$? Explain your answer.

/2 marks

3(e) Write down the forwards/explicit Euler method for the ODE in (3), and state the local discretisation error as an order of the step size h.

/2 marks

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3(f) Suppose Euler's method is used to solve IVP (3) for $t \in [1, 4]$. Calculate the interval(s) of step sizes h for which the method is stable.

You can quote results from lectures without proof.

/3 marks

3(g) Will the numerical solution convergence to the true solution as $h \to 0$? Justify your answer.

You can ignore round-off error.

/2 marks

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19 Total

Question 4.

As in lectures, consider the second-order IVP

$$ml\frac{d^2\Theta}{d\tau^2} = -mg\sin\Theta \quad \text{with} \quad \Theta(0) = \alpha \quad \text{and} \quad \dot{\Theta}(0) = 0, \tag{4}$$

as a model of a simple pendulum.

4(a) Describe the ODE in terms of Newton's second law, and give a physical interpretation of the initial conditions.

You can add a schematic if this helps your explanation.

/2 marks

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4(b) Show that by writing $t = \tau/T$ for a suitable scale T, IVP (4) becomes

$$\frac{d^2\theta}{dt^2} + \sin\theta \quad \text{with} \quad \theta(0) = \alpha \quad \text{and} \quad \dot{\theta}(0) = 0, \qquad (5)$$

where $\theta(t) = \Theta(\tau)$.

/3 marks

4(c) Explain the assumptions that allow IVP (5) to be linearised to

$$\frac{d^2\theta}{dt^2} + \theta = 0 \quad \text{with} \quad \theta(0) = \alpha \quad \text{and} \quad \dot{\theta}(0) = 0. \qquad (6)$$

/1 mark

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4(d) Derive the centred difference formula

$$y_n'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \tag{7}$$

and find the order of the truncation error.

/5 marks

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4(e) Determine the finite difference formula given by applying the centred difference formula (7) to IVP (6). Find the local and global discretisation errors.

/5 marks

4(f) Explain how the initial conditions are incorporated into the finite difference formula.

/3 marks

End of examination questions.

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