School of Mathematical Sciences

APP MTH 3022/7106 - Optimal Functions and Nanomechanics

Assignment 3 question sheet

Due: Thursday, 12 September, at 12 noon (in the hand-in box on level 6)

When presenting your solutions to the assignment, please include some explanation in words to accompany your calculations. It is not necessary to write a lengthy description, just a few sentences to link the steps in your calculation. Messy, illegible or inadequately explained solutions may be penalised. The marks awarded for each part are indicated in brackets.

All students are to attempt Questions 1–3.

1. Find the form of extremals of the following functionals

   (a) $F\{y(x), z(x)\} = \displaystyle\int_{x_0}^{x_1} (8yz - 5y^2 + y'^2 - 4z'^2)\, dx.$

   (b) $F\{\boldsymbol{q}\} = \displaystyle\int_{t_0}^{t_1} (\dot{q}_1 q_2 + \dot{q}_2 q_3 + q_1 \dot{q}_3 - \dot{q}_1^2)\, dt.$

   [8 marks]

2. Find the extremal of the following functional

$$F\{y\} = \int_0^1 (y''^2 - 360x^2 y)\, dx,$$

   subject to $y(0) = 0$, $y'(0) = 1$, $y(1) = 1$ and $y'(1) = 5/2$.

   [4 marks]

3. (a) Consider the integral definition of the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \qquad (\star)$$

   i. Write an integral expression for $B(z - \frac{1}{2}, z + \frac{1}{2})$.
   ii. Make the substitution $2t = 1 + s$ into this integral taking care to modify the integration limits appropriately.
   iii. Decompose the integral into two with integrands that are odd and even functions of s, respectively.
   iv. Use the parity of the integrand to halve the range of integration in both integrals. *Hint:* Bisect the integration interval and substitute $\sigma = -s$ in the negative interval.
   v. Use the substitution $\tau = s^2$ to put the remaining integral back into the form of $(\star)$ above.
   vi. Making use of your answers for sub-parts i–v above, and the relationship of beta function to the gamma function, derive the duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + 1/2).$$

   (b) From the integral definitions given in class show that

$$K(k) = \frac{\pi}{2} F(1/2, 1/2; 1; k^2).$$

   [12 marks]
