1 Diseases (Topic B)

Exam 5th July - EMG07 from 9:20am

Exam:

- 8 questions
- 92 marks
- 3 hour exam

Questions

1.1 10 true false questions with brief justification (20 marks)

Can't put much here

1.2 ODE - SIR model (5 marks)

a

1.3 Characteristics of different model types (reasoning) (6 marks)

a

1.4 Specifying a CTMC (interpreting from words) and simulation (14 marks)

For a CTMC you must give a state space, $x(t) \in S \forall t \geq 0$, and state transitions (or a generator).

1.5 CTMC model and deterministic approximation (11 marks)

1.6 Branching processes (14 marks)

1.7 Path integrals (12 marks)

1.8 Bayesian inference (10 marks)

SI model

$$\frac{dI}{dt} = \begin{cases} \frac{\beta I(N-I)}{N}, & \text{FDT} \\ \beta I(N-I), & \text{DDT} \end{cases}$$

Non dimensionalisation by letting $i = \frac{I}{N}$, and same for s, r.

Analytic solution to the ODE

$$i(t) = \frac{i_0}{i_0 + (1-i_0)e^{-\beta t}}$$

Final size

$$r_\infty = 1 + \frac{1}{R_0}\mathcal{W}(-s_0 e^{-R_0} R_0)$$

where $\mathcal{W}$ is the lambert W function (sol to $f(w) = we^w$)

Let T_1 be the time the outbreak ends

$$T_1 = \inf\{t | i(t) > 1 - \frac{1}{N}\}$$

so sub T_1 into $i(t) = 1 - 1/N$

$$i(t) = \frac{i_0}{i_0 + (1 - i_0)e^{-\beta t}}$$

$P = (P_{ij}(t),\ i, j \in s,\ t \geq 0)$ is a transition function if

- $P_{ij}(t) \geq 0$
- $P_{ij}(0) = \delta_{ij}$
- $\sum_{j \in S} P_{ij}(t) \leq 1$
- $P_{ij}(t + s) = \sum_{k \in S} P_{ik}(s) P_{kj}(t)$

If P is a standard transition function, i.e. $\lim_{t \downarrow 0} P_{ij}(t) = \delta_{ij}$

$$q_i = \lim_{t \downarrow 0} \frac{1 - P_{ii}(t)}{t}$$
$$q_{ij} = \lim_{t \downarrow 0} \frac{P_{ij}(t)}{t}$$

With $0 \leq q_i \leq \infty$ and $0 \leq q_{ij} < \infty$

Chapman-Kolmogorov

$$P_{ij}(s + t) = \sum_{k \in S} P_{ik}(s) P_{kj}(t), \quad s, t \geq 0$$

Kolmogorov-Forward eq

$$P'(t) = P(t)Q$$
$$P(t) = P(0)e^{Qt}$$

A state is recurrent if

$$\int_0^\infty P_{jj}(t) dt = \infty$$

Transient if $< \infty$.

It is positive recurrent if the mean return time T_j is finite, null recurrent otherwise.

If S_n is the time of the n^{th} jump, with $S = \lim_{n \to \infty} S_n$

$$S_n = \sum_{i=1}^n T_i$$

If $E(S) < \infty$ then the chain performs an infinite # of jumps in a finite time with probability 1 (the chain explodes), and is not regular.

$$p(s + \tau)[I - \tau Q] \approx p(s)$$

A family of MCs is density dependent if

$$q_{k,k+l} = rf(\frac{K}{r}, l), \quad l \neq 0$$

SIS is density dependent with

$$f(i,l) = \begin{cases} \beta(1-i)i, & l = 1 \\ \gamma i, & l = -1 \end{cases}$$

Don't forget expectation

$$E(I(t)) = \sum_{I=0}^{\infty} IP_I(t)$$

Branching processes:

If the lifetime of an individual is exponentially distributed with rate μ. At the time of death an individual generates a random number of children with pmf

$$\{P_k\}_{k\geq 0}$$

and pgf

$$P(s) = \sum_{k=0}^{\infty} P_k s^k$$

So

$$P'(s) = \sum_{k=0}^{\infty} P_k s^{k-1}$$

so $P'(1) = \mathbb{E}(y) = m$, i.e. the mean number of offspring for one person

$$F(s,t) = \sum_{k\geq 0} P(X(t) = k)s^k$$

Where $F(s,t)$ is the p.g.f of $X(t)$ (the population size at t).

The first person is alive with probability $e^{-\mu t}$, since $X(t) = 1$, $F(s,t) = s$

The first person will die in $(u, u+du)$ with probability $\mu e^{-\mu u}du$, and will have N offspring with prob P_N. And so the number of people at time t is

$$X(t) = \sum_{i=1}^{N} X_i(t-u)$$

Where $X_i(t-u)$ is the size of the subprocess generated by the i^{th} child after $t-u$ units of time. So the p.g.f of $X_i(t-u)$ is $F(s,t-u)$ and each is iid so the pgf of $X(t)$ is $F(s,t-u)^N$. And since N is a random var

$$\sum_{k\geq 0} P_k F(s,t-u)^k = P(F(s,t-u))$$

Eventually get

$$\frac{\partial F(s,t)}{\partial t} = \mu(P(F(s,t)) - F(s,t))$$

since it is a PGF the expected pop at time t diff wrt s and then set $s = 1$ (use $F(1,t) = 1$) Giving the mean population size, where $m = P'(1)$ i.e. the mean num of offspring

$$M(t) = e^{\mu(m-1)t}$$

- $m > 1$ gives $\lim_{t\to\infty} M(t) = \infty$ (outbreak)
- $m = 1$, $M(t) = 1$ for all t (almost sure extinction)

- $m < 1$ with $\lim_{t\to\infty} M(t) = 0$ subcritical

If q is the probability of extinction, q is the minimal, non-negative solution to

$$q = P(q)$$

Where $P(q) = \sum_{k\geq 0} p_k q^k$ In a normal branching process in the state 1 $P(q) = P_0 + P_2 q^2 = \frac{\gamma}{\gamma+\beta} + \frac{\beta}{\gamma+\beta} q^2$

This approximation is good for a minor outbreak for large N or a major outbreak until $\sqrt{N}$ people are infected.

Household model 2 groups (internal house, external) M houses with N people in each house. Total pop MN with SIR dynamics The state space is $\mathbf{m}$ where $m_{S,I}$ corresponds to the number of hh's in each possible config. I.e. $m_{(i,j)}$ is the number of households with i susceptible and j infected

$$\mathbf{m} = \left(m_{(N,0)}, m_{(N-1,1)}, \dots, m_{(0,0)}\right)$$

3 events -

- internal hh infection, $\frac{\beta si}{N-1} m_{(s,i)}$
- external hh infection, $\frac{\alpha si}{MN} m_{s,i}$
- recovery, $\left(m_{(s,i)}, m_{(s,i-1)}\right) \to \left(m_{(s,i)} - 1, m_{(s,i-1)} + 1)\right)$

Want to find d_i, the expected amount accumulated given the process started in state i.

$$Q_B \mathbf{d} = -\mathbf{f}$$

Where Q_B is the Q matrix restricted to the non-absorbing states, and f_i is the cost per unit time in state i, and $\mathbf{f} = \mathbf{1}$ gives the expected time until absorption (extinction)

Branching process - If $X(t)$ is the population size at time t, with state space $S = \mathbb{N}$ and 0 is absorbing

$$F(s,t) = \sum_{k\geq 0} P(X(t) = k) s^k$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$(Q_B - lF)L = -a$$

Where Q_B as usual

$$L = (L_i, i \in B)$$

$$F = \begin{pmatrix} f(1) & & & \\ & f(2) & & \\ & & \dots & \\ & & & f(|B|) \end{pmatrix}$$