LECTURE 7

Example: True or False? Let S be a non-empty subset of $\mathbb{R}$ such that S does not contain any rational numbers. Then S does not have any subsets which are open intervals.

This statement is true. If $(a, b) \subset S$ where $a < b$ are real numbers, then by Theorem 1.10 there exists $q \in \mathbb{Q}$ such that $a < q < b$. But then $q \in (a, b)$ and hence $q \in S$ since $(a, b) \subset \mathbb{Q}$ by assumption. But then q is a rational number which belongs to S — contradiction. Therefore S does not contain any subsets which are open intervals.

Inequalities with absolute value

Recall that if $x \in RR$ then the *absolute value* $|x|$ of x is the real number

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

There are four important inequalities that we will need which are described in terms of absolute values.

1. Let $b \geq 0$. Then $|x| \leq b \iff -b \leq x \leq b$.

We will prove that $|x| \leq b \implies -b \leq x \leq b$. The reverse implication is left as an exercise. There are two possibilities: $x \geq 0$ or $x < 0$. If $x \geq 0$ then $x = |x| \leq b$. Since $b \geq 0$ we have $-b \leq 0$. Therefore $-b \leq 0 \leq x$ and hence $-b \leq x$. Combining these two inequalities we see that $-b \leq x \leq b$. On the other hand, if $x < 0$ then $-x = |x| \leq b$ and hence $-b \leq x$. From $x < 0$ and $0 \leq b$ we obtain the inequality $x \leq b$ and hence $-b \leq x \leq b$ in this case too.

Note that there is a version of 1 with $\leq$ replaced by $<$.

2. (The triangle inequality) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

To prove the triangle inequality we use 1. Since $|x| \leq |x|$ we have $-|x| \leq x \leq |x|$. Similarly we have $-|y| \leq y \leq |y|$. Adding these inequalities we find that $-|x| - |y| \leq x + y \leq |x| + |y|$. Hence $-(|x| + |y|) \leq x + y \leq |x| + |y|$. Therefore, applying 1 again, we find that $|x + y| \leq |x| + |y|$.

3. (The reversed triangle inequality) $|x| - |y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

If $x \in \mathbb{R}$ then $x = (x - y) + y$. Therefore, by 2, we have $|x| \leq |x - y| + |y|$. Rearranging this inequality gives us the reversed triangle inequality.

4. $||x| - |y|| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

From the reversed triangle inequality we have $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |y - x| = |x - y|$. Therefore $-|x - y| \leq |x| - |y| \leq |x - y|$. Now apply 1.

There are some other properties of the absolute value that are also useful:

- $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for all $x \in \mathbb{R}$, $y \in \mathbb{R} \setminus \{0\}$.

To prove these identities we use the definition of absolute value. For instance, to prove that $|xy| = |x| \cdot |y|$ we can argue as follows. Firstly, we see that it suffices to prove this identity in the case where $y > 0$, for if $y < 0$ then $y = -z$ for some $z > 0$ and hence $|xy| = |x(-z)| = |x| \cdot |z| = |x| \cdot |y|$. Similarly the identity is clear if $y = 0$. We have

$$|xy| = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$$

Since $y > 0$, $xy \geq 0 \iff x \geq 0$ and $xy < 0 \iff x < 0$. Therefore

$$|xy| = \begin{cases} x|y| & \text{if } x \geq 0, \\ (-x)|y| & \text{if } x < 0. \end{cases}$$

In other words, $|xy| = |x| \cdot |y|$, as required. The other identity is a recommended exercise.

Example: Describe the set $\{ x \in \mathbb{R} \mid |x - 2| < 0.1 \}$. Observe that $|x - 2| < 0.1 \iff -0.1 < x - 2 < 0.1 \iff 1.9 < x < 2.1$. Hence $\{ x \in \mathbb{R} \mid |x - 2| < 0.1 \} = (1.9, 2.1)$. In other words this is the set of all real numbers x such that the distance from x to 2 is less than 0.1.

We have the following very important relationship between absolute value and distance:

$$\boxed{|x - y| = \text{distance from } x \text{ to } y}$$

2. Sequences

A *sequence* of real numbers is a function

$$f \colon \mathbb{N} \to \mathbb{R}$$

Such a function is completely described by specifying the values $f(1), f(2), f(3), \ldots$. The number $f(n)$ is called the n-th *term* of the sequence. We usually write a_n, x_n, ... for $f(n)$.

Sequences are typically denoted by (a_n) or $(a_n)_{n=1}^{\infty}$ or $(a_n)_{n \in \mathbb{N}}$ or $a_1, a_2, a_3, \ldots$. We will use a variety of these notations in the course.

One notation that we will not use is $\{ a_n \}_{n=1}^{\infty}$. It is important to realize that **a sequence is not a set, it is a function**; the notation $\{ a_n \}_{n=1}^{\infty}$ might perhaps suggest that this is not the case. There is an important distinction between a sequence $(a_n)_{n=1}^{\infty}$ and the set of values

$$\{ a_n \mid n \in \mathbb{N} \}.$$

The latter is a set, it is not a function. The distinction becomes stark if we consider an example such as the sequence $a_n = 1$ for all n. This is the sequence $1, 1, 1, \ldots$ (we need to describe the value $f(n)$ of the function $f \colon \mathbb{N} \to \mathbb{R}$ at every $n \in \mathbb{N}$). On the other hand, the set of values of the sequence is just the set $\{ 1 \}$.

Sequences can be defined in many ways. For example, we might describe a sequence by

- listing, eg. 1,0,1,0,0,1,0,0,0,1,0,0,0,0,1,... (in this case it should be clear what the 'pattern' is; with a little work you could say what the n-th term of the sequence is for any n.)
- a formula, eg. $a_n = \sqrt{n}/(n + 1)$,
- a recursion relation, eg. $a_1 = 2/3$, $a_{n+1} = 1/(1 + a_n)$ for $n \geq 1$.

The last sequence is interesting; it can be shown that it converges to $\sqrt{2}$.

It is important to understand that 'a sequence has a beginning', for example

$$1, 2, 3, \ldots$$

is a sequence but

$$\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots$$

is not a sequence since it does not have a beginning — it does not correspond to a function $f \colon \mathbb{N} \to \mathbb{R}$ (instead it corresponds to a function $f \colon \mathbb{Z} \to \mathbb{R}$).

Sometimes you might start a sequence at different values of n, for example $(a_n)_{n=0}^{\infty}$, $(a_n)_{n=6}^{\infty}$. We will sometimes do this during the course.

The next topic is perhaps the most crucial topic for the course.

Convergence of sequences

The following definition is probably the most important definition in the course.

Definition 2.1 A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is said to *converge* to a real number L if

$$\forall\, \epsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that for all } n \in \mathbb{N},\ \text{if } n \geq N,\ \text{then } |a_n - L| < \epsilon.$$

We write $a_n \to L$ if (a_n) converges to L.

This definition is making precise the idea that a sequence (a_n) converges to a real number L if a_n gets arbitrarily close to L provided that n is sufficiently large. What does 'close to' mean? It means that the distance is small. Therefore, we can re-phrase this as '(a_n) converges to L if the distance from a_n to L can be made arbitrarily small provided that n is sufficiently large'.

One way to think about this definition is as follows. Somebody challenges you to show that eventually all of the terms of the sequence are within 0.01 of the number L. You could do that by showing that from some point onwards in the sequence, say past the N-th term of the sequence, the distance between the terms of the sequence and L is less than 0.01. Somebody else may challenge you to show that eventually all of the terms of the sequence are within 0.001 of the number L. Then you could do that by showing that there is another number N (possibly different to the first) such that once you're past the N-th term of the sequence the distance from L is less than 0.001.

Clearly you could be challenged over an over again with different positive numbers ϵ. By proving that you can find such an N for **any** $\epsilon > 0$ you are able to point any would-be challenger to your proof and give them an N which would work for their ϵ. So the ϵ is like a 'challenge' and the natural number N is like a 'response' to this challenge — it tells you how far out you have to go in your sequence to ensure that the challenge is met, i.e. the distance from L is less than ϵ.

From another point of view, Definition 2.1 is saying that for any $\epsilon > 0$, all but finitely many terms of the sequence belong to the open interval $(L - \epsilon, L + \epsilon)$. For, if $n \geq N$, then $a_N, a_{N+1}, a_{N+2}, a_{N+3}, \ldots \in (L - \epsilon, L + \epsilon)$. Hence there are only finitely many terms, namely $a_1, a_2, \ldots, a_{N-1}$ which potentially might not belong to the open interval $(L - \epsilon, L + \epsilon)$.