Lecture 18: Queueing Systems - Loss Networks

Concepts checklist

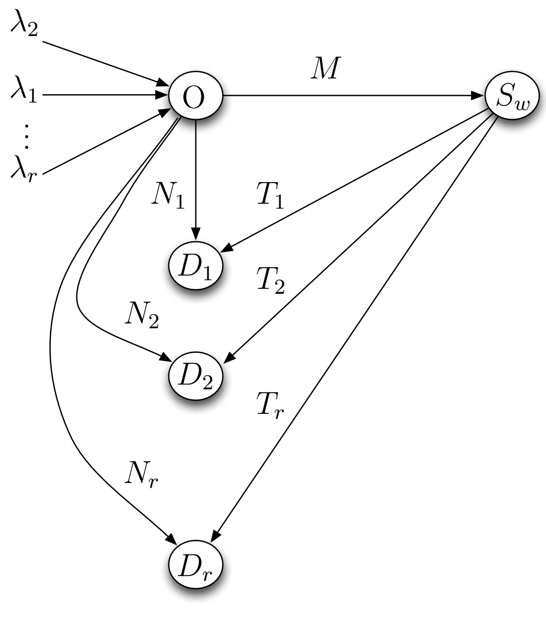
At the end of this lecture, you should be able to:

- *model* and *specify equilibrium distributions* for alternative routing with call packing, and circuit-switched networks; and,
- *state a theorem* and *specify exact* route blocking probabilities for circuit-switched networks.

Example 10. Alternative Routing with Call Packing

There are r Poisson streams of calls sharing a common origin O, which are routed to their destinations D_i for $i \in \{1, 2, \ldots, r\}$ via N_i direct circuits before overflowing onto M common circuits through the switch S_w.

From S_w, there are T_i circuits available to each of the required destinations D_i. We assume arrival rates λ_i and mean call holding times $1/\mu_i$ for each $i \in \{1, 2, \ldots, r\}$.



Let n_i be the number of customers in the system from stream i and let $\boldsymbol{n} = (n_1, n_2, \ldots, n_r)$.

We assume call packing, which means that calls are packed back onto the direct circuits from a common (overflow) link whenever a direct circuit becomes available.

Initially, we consider an infinite number of available circuits for each stream. Hence, every caller gets a circuit and essentially sees an infinite server queue, so the invariant measure is

$$\prod_{i=1}^{r} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \frac{1}{n_i!}.$$

The number of overflow circuits from the ith direct route is $[n_i - N_i]^+$. Therefore, the state space $\mathcal{A}$ is restricted by

1. $[n_i - N_i]^+ \le T_i$ and

2. $\displaystyle\sum_{i=1}^{r}[n_i - N_i]^+ \le M.$

Truncating the state space to $\mathcal{A}$ gives us the equilibrium distribution

$$\pi(n_1, n_2, \ldots, n_r) = C\prod_{i=1}^{r}\left(\frac{\lambda_i}{\mu_i}\right)^{n_i}\frac{1}{n_i!}, \quad \text{where } C = \left(\sum_{\mathcal{A}}\prod_{i=1}^{r}\left(\frac{\lambda_i}{\mu_i}\right)^{n_i}\frac{1}{n_i!}\right)^{-1}.$$

Example 11. Circuit-Switched Network

Assumptions:

- fixed routing is used (that is, no overflow or alternative routing),
- there are J links in total – link j has c_j circuits, for $1 \le j \le J$, and $\boldsymbol{c} = (c_1, c_2, \ldots, c_J)$,
- there are R routes in total – calls requesting route r arrive in a Poisson stream of rate λ_r, for $1 \le r \le R$,
- without loss of generality (wlog), call holding times have unit mean,
- route r calls use $A_{j,r}$ circuits on link j.

Let n_r be the number of calls using route r and let $\boldsymbol{n} = (n_1, n_2, \ldots, n_R)$ be the state of the network. Furthermore, if we let $A = \{A_{j,r}\}$ be the $J \times R$ matrix such that $A_{j,r}$ represents the circuit usage on link j for route r, then we can write

$$\begin{aligned} S(\boldsymbol{c}) &= \{\boldsymbol{n} : \sum_r A_{j,r}n_r \le c_j, \quad 1 \le j \le J\} \\ &= \{\boldsymbol{n} : A\boldsymbol{n} \le \boldsymbol{c}\}. \end{aligned}$$

If we let $c_j \longrightarrow \infty$ for all j, then the process is an R-dimensional Birth-and-Death process, which is reversible. Then, by truncating the process by making each c_j finite, we can use Theorem 17 to find the equilibrium probability distribution, which is given by

$$\pi(\boldsymbol{n}) = [G(\boldsymbol{c}, R)]^{-1}\prod_{r=1}^{R}\frac{\lambda_r^{n_r}}{n_r!} \quad \text{for } \boldsymbol{n} \in \mathcal{S}(\boldsymbol{c}),$$

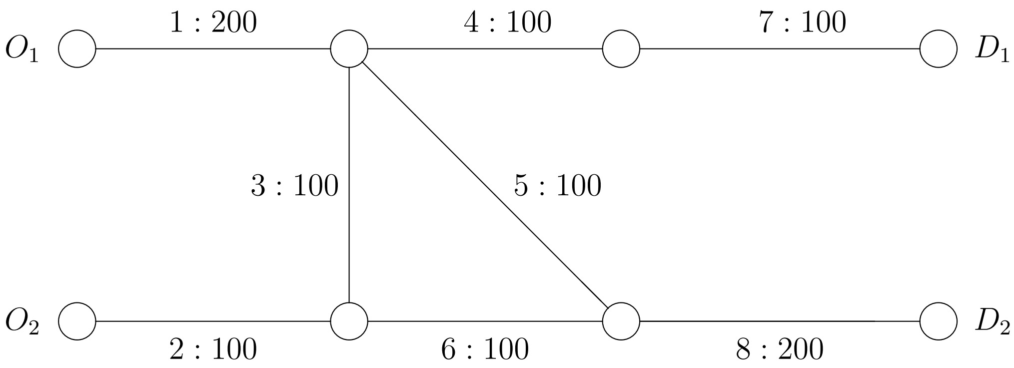
where $\quad \boldsymbol{c} = (c_1, c_2, \ldots, c_J)^\top$ and $\mathcal{S}(\boldsymbol{c}) = \{\boldsymbol{n} : \sum_r A_{j,r}n_r \le c_j, \quad 1 \le j \le J\}$.

Here, $[G(\boldsymbol{c}, R)]^{-1}$ is the normalising constant, which is dependent on the state space $\mathcal{S}(\boldsymbol{c})$ defined by $\boldsymbol{c}$. Note that, summing the invariant measure over $\boldsymbol{n} \in \mathcal{S}(\boldsymbol{c})$ gives

$$G(\boldsymbol{c}, R) = \sum_{\boldsymbol{n}:A\boldsymbol{n}\le\boldsymbol{c}}\prod_{r=1}^{R}\frac{\lambda_r^{n_r}}{n_r!}.$$

An instance

Consider the following (simple, circuit-switched) loss network:



A label on the link is given on the diagram using the legend $a : b$, where a is the link number and b is the number of circuits on link a.

It is required to route traffic of offered load 60, 70, 50, and 80 (Erlangs) respectively between the four origin-destination pairs ($O_1 - D_1$, $O_1 - D_2$, $O_2 - D_1$ and $O_2 - D_2$). This means $\lambda_1/\mu_1 = 60$ Erlangs, $\lambda_2/\mu_2 = 70$ Erlangs, and so on. Note that as stated above we assume $\mu_i = 1$ for $i = 1, 2, 3, 4$.

Assuming that traffic between origin O_i and destination D_j is routed along the shortest possible path and requires a single circuit from each such link, the routes of the loss network are:

Route ID	Route	Links used	Offered loads
1	$O_1 - D_1$	1, 4, 7	60
2	$O_1 - D_2$	1, 5, 8	70
3	$O_2 - D_1$	2, 3, 4, 7	50
4	$O_2 - D_2$	2, 6, 8	80

The state space $\mathcal{S}$ of the network, in the form $\{\boldsymbol{n} : A\boldsymbol{n} \le \boldsymbol{c}\}$, where A is the matrix with components A_{jr} being the number of circuits that route r calls use on link j, and $\boldsymbol{c} = (c_1, c_2, \ldots)$ is the vector containing the number c_j of circuits on link j, is

$$\mathcal{S} = \{\boldsymbol{n} : A\boldsymbol{n} \le \boldsymbol{c}\}$$

where

$$\boldsymbol{c} = (200, 100, 100, 100, 100, 100, 100, 200)^\top$$

and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Exact Route Blocking Probabilities

The equilibrium distribution can be used to give the exact route blocking probabilities. However, it is not clear which states block a route r call and therefore over which set of states we need to sum to calculate the blocking probability. Fortunately, there is a much simpler formula which requires calculating only the normalising constants for two networks.

Theorem 18. *If $B_r := \Pr$(call on route r is blocked) and $\boldsymbol{e}_r := (0,0,\ldots,0,1,0,\ldots,0)^\top$, with 1 in the rth position, then*

$$B_r = 1 - \frac{G(\mathbf{c} - A\boldsymbol{e}_r, R)}{G(\mathbf{c}, R)}.$$

Proof:

$$\begin{aligned}
&\Pr(\text{call is accepted on route } r)\\
&\qquad = \Pr(\text{there are at least } A_{j,r} \text{ circuits available on link } j, \text{ for all } 1 \le j \le J)\\
&\qquad = \Pr(\text{the number of circuits in use on link } j \le c_j - A_{j,r}, \text{ for all } 1 \le j \le J)\\
&\qquad = \Pr(A\boldsymbol{n} \le \mathbf{c} - A\boldsymbol{e}_r)\\
&\qquad = \sum_{\{\mathbf{n}:A\mathbf{n} \le \mathbf{c} - A\mathbf{e}_r\}} [G(\mathbf{c}, R)]^{-1} \prod_{\ell=1}^{R} \frac{\lambda_\ell^{n_\ell}}{n_\ell!}\\
&\qquad = \frac{1}{G(\mathbf{c}, R)} \sum_{\{\mathbf{n}:A\mathbf{n} \le \mathbf{c} - A\mathbf{e}_r\}} \prod_{\ell=1}^{R} \frac{\lambda_\ell^{n_\ell}}{n_\ell!}\\
&\qquad = \frac{G(\mathbf{c} - A\boldsymbol{e}_r, R)}{G(\mathbf{c}, R)}.
\end{aligned}$$

Therefore,

$$B_r = \Pr\left(\text{call on route } r \text{ is blocked}\right) = 1 - \frac{G(\mathbf{c} - A\boldsymbol{e}_r, R)}{G(\mathbf{c}, R)}. \qquad \square$$

Example 11: Instance of a circuit-switched network

What is the exact expression for the probability that a call attempting to access D_1 from O_1 is accepted?

The set of states $\mathcal{S}_1$ in which a call attempting to access D_1 from O_1 is accepted is

$$\mathcal{S}_1 = \{\boldsymbol{n} : A\boldsymbol{n} \le \boldsymbol{c} - A\boldsymbol{e}_1\}.$$

Hence the probability that a call attempting to access D_1 from O_1 is accepted is given by

$$\frac{G(\boldsymbol{c} - A\boldsymbol{e}_1, R)}{G(\boldsymbol{c}, R)}, \text{ where } G(\boldsymbol{c}, R) = \sum_{\boldsymbol{n}:A\boldsymbol{n} \le \boldsymbol{c}} \frac{60^{n_1}}{n_1!} \frac{70^{n_2}}{n_2!} \frac{50^{n_3}}{n_3!} \frac{80^{n_4}}{n_4!}.$$