APP MTH 3001 Applied Probability III
Class Exercise 4, 2018
Due: 3pm, Monday 21 May 2018, via Canvas (PDF only)
(Note changed due date, for this exercise only!)

1. In a discrete time population branching process,

   the probability that an individual has $j = 0, 1, 2, 3$ offspring is given by $p_{1,j} = \dfrac{1}{4}$.

   (a) Find the probability of ultimate extinction of the line of descent from an individual.

   (b) Hence deduce whether the mean number of offspring per individual μ is greater than 1 or not, fully justifying your answer. Verify your conclusion by finding μ.

2. A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$\mathbb{P}_a = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

   (a) Is this Markov chain irreducible? Is the Markov chain recurrent or transient? Explain your answers.

   (b) What is the period of state 1? Hence deduce the period of the remaining states. Does this Markov chain have a limiting distribution?

   (c) Consider a general three-state Markov chain with transition matrix

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

   Give an example of a specific set of probabilities $p_{i,j}$ for which the Markov chain is *not* irreducible (there is no single right answer to this, of course !).

3. Consider a general irreducible Markov chain on the finite state space $\{1, 2, \ldots N\}$. . Show that one of the Global Balance equations

$$\pi_i = \sum_{j=1}^{N} \pi_j p_{j,i}, \quad i = 1, 2, \ldots, N,$$

   is always redundant.

4. Consider the random walk on the state space $\{0, 1, 2, \ldots\}$, with transition probabilities for $i = 1, 2, \ldots$, given by

$$p_{i,j} = \begin{cases} p & \text{if } \; j = i+1 \\ q & \text{if } \; j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_{01} = 1.$$

As usual, $p + q = 1$, and we assume that $p, q > 0$.

(a) What is the period of this Markov chain?

(b) For the case $p < q$, use the Global Balance Equations to show that the stationary distribution for this Markov chain is given by

$$\begin{aligned} \pi_i &= \frac{1}{2p}\left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^i, \quad i \geq 1, \\ \pi_0 &= \frac{1}{2}\left(1 - \frac{p}{q}\right). \end{aligned}$$

Hint: the state space is not finite. One way of approaching this situation is to solve the system of difference equations by trying a solution of the form $\pi_i = m^i$. For the general form of the solution, you will have two constant coefficients that need to be determined. In order to determine one of the coefficients, use the fact that $\sum_i \pi_i < \infty$, then use the normalization constraint $\sum_i \pi_i = 1$ to determine the other coefficient.

(c) For the case $p < q$, use the Partial Balance equations on the set $\mathcal{B} = \{0, 1, 2, \ldots, n\}$, for all n, to again find the stationary distribution, $\boldsymbol{\pi}$.

(d) For the case $p \geq q$, use the Partial Balance equations on the set $\mathcal{B} = \{0, 1, 2, \ldots, n\}$, for all n, to show that the stationary distribution does not exist for this Markov chain.