LECTURE 32

Last lecture we introduced the concepts of *pointwise* convergence and *uniform* convergence for a sequence of functions $f_n : S \to \mathbb{R}$.

Example: For each $n \in \mathbb{N}$ let $f_n : [0,1] \to \mathbb{R}$ be the function defined by $f_n(x) = \frac{x}{nx^2+1}$. To investigate the pointwise convergence of the sequence of functions (f_n), let $x \in [0,1]$ and consider the sequence of real numbers $(f_n(x))$, i.e. the sequence $(x/(nx^2+1))$. We have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{nx^2+1} = 0$$

for all $x \in [0,1]$ (hopefully this is easy to justify rigorously by this stage of the course; one option is to observe that if $x \neq 0$ then $|\frac{x}{nx^2+1}| \leq \frac{1}{nx} \to 0$ since $\frac{1}{n} \to 0$ and hence the limit is zero by the Sandwich Theorem). Therefore we have shown that $f_n \to 0$ pointwise on $[0,1]$, where 0 denotes the constant function which is zero everywhere.

Is this convergence uniform? In other words, does (f_n) converge uniformly on $[0,1]$ to the constant function 0? To investigate this we investigate the sequence (M_n), where $M_n = \sup_{x\in[0,1]} |f_n(x) - 0|$. We have

$$|f_n(x) - 0| = \frac{x}{nx^2+1} \leq \frac{1}{2\sqrt{n}}$$

since $nx^2 + 1 \geq 2\sqrt{n}x$ for all $x \in [0,1]$ and hence $\frac{x}{nx^2+1} \leq \frac{x}{2\sqrt{n}x} = \frac{1}{2\sqrt{n}}$ if $x \neq 0$ (if $x = 0$ this inequality is trivially true). Therefore $M_n \leq \frac{1}{2\sqrt{n}}$. Since $\frac{1}{2\sqrt{n}} \to 0$, it follows by the Sandwich Theorem that $M_n \to 0$. Hence the convergence is uniform.

Example: For each $n \in \mathbb{N}$ let $f_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$. The sequence (f_n) converges pointwise on $\mathbb{R}$ to the constant function $f(x) = 0$, since

$$|f_n(x)| = \left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \to 0.$$

Hence $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$. In fact $f_n \to f$ uniformly on $\mathbb{R}$, since as we have just observed, $|f_n(x)| \leq \frac{1}{\sqrt{n}}$ for all $x \in \mathbb{R}$ and hence $M_n = \sup_{x\in\mathbb{R}} |f_n(x)| \leq \frac{1}{\sqrt{n}}$ and so $M_n \to 0$.

This example is interesting, since each function f_n is differentiable with $f_n'(x) = \sqrt{n}\cos(nx)$. But $f_n'(0) \not\to f'(0) = 0$, since $f_n'(0) = \sqrt{n}$. This example shows that if a sequence of differentiable functions (f_n) converges to a differentiable function f, it is not necessarily true that $f_n' \to f'$ pointwise.

Theorem 8.3: Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of functions $f_n : S \to \mathbb{R}$ which converges unifornly on S to a function $f : S \to \mathbb{R}$. If $f_n : S \to \mathbb{R}$ is continuous on S for all n, then $f : S \to \mathbb{R}$ is continuous on S.

Proof: To show that f is continuous on S we have to show that f is continuous at x_0 for all $x_0 \in S$. So let $x_0 \in S$. Let $\epsilon > 0$. To show that f is continuous at x_0, we need to find $\delta > 0$ such that for all $x \in S$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Since $f_n \to f$ uniformly on S, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon/3$ for all $x \in S$ (the reason for the choice of $\epsilon/3$ will be clearer soon). In particular, note that $|f_N(x) - f(x)| < \epsilon/3$ for all $x \in S$.

Since the function f_N is continuous on S, it is continuous at x_0. Therefore there is a $\delta > 0$ such that for all $x \in S$, if $|x - x_0| < \delta$, then $|f_N(x) - f_N(x_0)| < \epsilon/3$.

Suppose that $x \in S$ and $|x - x_0| < \delta$. By the triangle inequality, we have

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|.$$

We have the following three inequalities:

1. $|f(x) - f_N(x)| < \epsilon/3$ (by the choice of N)
2. $|f_N(x_0) - f(x_0)| < \epsilon/3$ (again by the choice of N)
3. $|f_N(x) - f_N(x_0)| < \epsilon/3$ (by the choice of δ).

Now the reason for the choice $\epsilon/3$ becomes clear: we are estimating $|f(x) - f(x_0)|$ by the sum of three numbers, and we want the sum to be less than ϵ — we can achieve this if each of the numbers is less than $\epsilon/3$.

So, putting all of these facts together, we see that

$$|f(x) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that f is continuous at x_0. Since $x_0 \in S$ was arbitrary it follows that f is continuous on S. ■

Example: We revisit an example from last lecture. Recall that the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$ converges pointwise on $[0,1]$ to the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Since each of the functions $f_n(x)$ is continuous on $[0,1]$ but the function $f(x)$ is not continuous on $[0,1]$ (it is not continuous at $x_0 = 1$), the convergence cannot be uniform (if it were then f would be continuous).

We would now like to investigate the behaviour of sequences of integrable functions under pointwise and uniform convergence. To be integrable a function needs to be bounded. Therefore, before we undertake this investigation, we first investigate sequences of bounded functions. The following example shows that a sequence of bounded functions which converges pointwise might not converge to a bounded function.

Example: Let (f_n) be the sequence of functions $f_n\colon (0,1) \to \mathbb{R}$ defined by $f_n(x) = \frac{n}{nx+1}$. Each function $f_n(x)$ is bounded: we have $|f_n(x)| = \frac{n}{nx+1} \leq n$ for all $x \in (0,1)$. If $x \in (0,1)$ then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{nx+1} = \lim_{n\to\infty} \frac{1}{x + \frac{1}{n}} = \frac{1}{x}.$$

Therefore $f_n \to f$ pointwise on $(0,1)$ where $f\colon (0,1) \to \mathbb{R}$ is the function defined by $f(x) = \frac{1}{x}$. The function f is not bounded on $(0,1)$, in fact $f(x) \to \infty$ as $x \to 0^+$.

Theorem 8.4: Suppose that $f_n\colon S \to \mathbb{R}$ is bounded for all $n \in \mathbb{N}$. If $f_n \to f$ uniformly on S to a function $f\colon S \to \mathbb{R}$ then f is bounded.

Proof: Since $f_n \to f$ uniformly on S, then (taking $\epsilon = 1$ in the definition of uniform convergence) there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|f_n(x) - f(x)| < 1$ for all $x \in S$. In particular $|f_N(x) - f(x)| < 1$ for all $x \in S$. Since the function $f_N : S \to \mathbb{R}$ is bounded, there exists $M > 0$ such that $|f_N(x)| \leq M$ for all $x \in S$. Now observe that, by the triangle inequality, if $x \in S$ then

$$|f(x)| = |(f(x) - f_N(x)) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M.$$

Therefore $|f(x)| \leq M + 1$ for all $x \in S$. Hence f is bounded. ■

With these preparations out of the way we investigate the behaviour of sequences of integrable functions. The following example shows that pointwise convergence does not lead to good behavious.

Example: Recall that the set $\mathbb{Q}$ is countable. Therefore every subset of $\mathbb{Q}$ is countable. In particular the set $\mathbb{Q} \cap [0,1]$ of rational numbers belonging to the interval $[0,1]$ is countable. Recall that if a set is countable then we can list out its elements as a sequence. Therefore there is a sequence (r_n) of rational numbers such that $0 \leq r_n \leq 1$ for all n and $\mathbb{Q} \cap [0,1] = \{ r_n \mid n \in \mathbb{N} \}$. For each $n \in \mathbb{N}$ define a function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, \ldots, r_n \\ 0 & \text{otherwise.} \end{cases}$$

Each function $f_n(x)$ is integrable, since it differs from the constant function $f(x) = 0$ at finitely many points. If $x_0 \in [0,1]$ then either x_0 is rational or it is not. If it is not rational then $f_n(x_0) = 0$ for all n and hence $\lim_{n\to\infty} f_n(x_0) = 0$ in this case. If x_0 is rational, then $x_0 = r_m$ for some m. Hence $f_m(x_0) = 1$. If $n \geq m$ then $f_n(x_0) = 1$ also, since $f_n(x) = 1$ if $x = r_1, \ldots, r_n$ and the list of numbers $r_1, \ldots, r_n$ will always include r_m since $n \geq m$. Therefore $\lim_{n\to\infty} f_n(x_0) = 1$. Therefore $f_n \to f$ pointwise on $[0,1]$ where $f : [0,1] \to \mathbb{R}$ is the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We have seen before that the function f is not integrable on $[0,1]$. Thus we have a sequence of functions (f_n), such that every function f_n is integrable, and which converges pointwise to a function f, but the function f is not integrable.

To correct this behaviour we need uniform convergence.

Theorem 8.5: Suppose $f_n : [a,b] \to \mathbb{R}$ is bounded for all $n \in \mathbb{N}$ and that $f_n \to f$ uniformly on $[a,b]$ for some function $f : [a,b] \to \mathbb{R}$. If f_n is integrable on $[a,b]$ for all n, then f is integrable on $[a,b]$.

Proof: To prove that f is integrable on $[a,b]$ we use Theorem 5.3. Without loss of generality $a < b$. Let $\epsilon > 0$. We will show that there is a partition $\mathscr{P}$ of $[a,b]$ such that $U(f, \mathscr{P}) - L(f, \mathscr{P}) < \epsilon$.

Since $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon/3(b-a)$ for all $x \in [a,b]$. Choose an $n \geq N$.

Since the function f_n is integrable on $[a,b]$, by Theorem 5.3 there exists a partition $\mathscr{P}$ of $[a,b]$ such that $U(f_n, \mathscr{P}) - L(f_n, \mathscr{P}) < \epsilon/3$.

Recall that for any real number a we have $-|a| \leq a \leq |a|$. Therefore for every $x \in [a,b]$, we have

$$-|f(x) - f_n(x)| \leq f(x) - f_n(x) \leq |f(x) - f_n(x)|$$

Adding $f_n(x)$ implies that

$$f_n(x) - |f(x) - f_n(x)| \leq f(x) \leq f_n(x) + |f(x) - f_n(x)|$$

for all $x \in [a, b]$. Suppose $\mathscr{P} = \{ x_0, x_1, \ldots, x_N \}$. Consider the i-th subinterval $[x_{i-1}, x_i]$ of this partition. Then, using the inequality above, and the fact that $|f_n(x) - f(x)| < \epsilon/3(b-a)$ for all $x \in [a, b]$, we have

$$m_i(f_n) - \frac{\epsilon}{3(b-a)} \leq f(x) \leq M_i(f_n) + \frac{\epsilon}{3(b-a)}$$

for all $x \in [x_{i-1}, x_i]$. Therefore $M_i(f_n) + \epsilon/3(b-a)$ is an upper bound for the set $\{ f(x) \mid x \in [x_{i-1}, x_i] \}$ and $m_i(f_n) - \epsilon/3(b-a)$ is a lower bound for the set $\{ f(x) \mid x \in [x_{i-1}, x_i] \}$. Therefore

$$m_i(f_n) - \frac{\epsilon}{3(b-a)} \leq m_i(f) \leq M_i(f) \leq M_i(f_n) + \frac{\epsilon}{3(b-a)}$$

Multiplying these inequalities by $\Delta_i(x) = x_i - x_{i-1} > 0$ and summing from $i = 1$ to $i = N$, we see that

$$L(f_n, \mathscr{P}) - \frac{\epsilon}{3} \leq L(f, \mathscr{P}) \leq U(f, \mathscr{P}) \leq U(f_n, \mathscr{P}) + \frac{\epsilon}{3}$$

since $\sum_{i=1}^{N} \Delta_i(x) = b - a$. Therefore

$$\begin{aligned} U(f, \mathscr{P}) - L(f, \mathscr{P}) &\leq (U(f_n, \mathscr{P}) + \frac{\epsilon}{3}) - (L(f_n, \mathscr{P}) - \frac{\epsilon}{3}) \\ &= U(f_n, \mathscr{P}) - L(f_n, \mathscr{P}) + \frac{2\epsilon}{3} \\ &< \frac{\epsilon}{3} + \frac{2\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Hence $U(f, \mathscr{P}) - L(f, \mathscr{P}) < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows by Theorem 5.3 that f is integrable on $[a, b]$. ■