LECTURE 27

We begin by explaining how to define x^r if r is a rational number and $x \geq 0$ is a real number. Write $r = m/n$ where m and n are integers and $n > 0$. Define

$$x^r := (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}.$$

Exercise: show that $(x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$.

Thus we have a function which sends x to x^r (a priori this function is only defined for non-negative real numbers). We investigate the differentiability of this function. First we investigate the differentiablity of the n-th root function $g\colon [0,\infty) \to [0,\infty)$ defined by $g(x) = x^{\frac{1}{n}}$. This function is inverse to the function $f\colon [0,\infty) \to [0,\infty)$ defined by $f(x) = x^n$. Observe that if $x > 0$ then $f'(x) = nx^{n-1} > 0$. Therefore the function $g\colon (0,\infty) \to (0,\infty)$ is differentiable, with

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

by the Inverse Function Theorem. It follows by the chain rule that the function $x \mapsto x^r$ is differentiable on $(0,\infty)$ with derivative

$$\frac{d}{dx}x^r = rx^{r-1}.$$

We return to our investigation of the natural logarithm function $\ln(x)\colon (0,\infty) \to \mathbb{R}$. By the Chain Rule, we have

$$\frac{d}{dx}\ln(x^r) = \frac{1}{x^r}rx^{r-1} = \frac{r}{x}.$$

On the other hand

$$\frac{d}{dx}r\ln(x) = \frac{r}{x}.$$

Therefore by the first Corollary to the Mean Value Theorem, there is a constant c such that

$$\ln(x^r) - r\ln(x) = c$$

for all $x \in (0,\infty)$. Setting $x = 1$ we see that $c = 0$ and hence that

$$\ln(x^r) = r\ln(x).$$

Exercise: show that $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y > 0$. (Let y be fixed and differentiate $\ln(xy)$.)

In particular we see that $\ln(2^n) = n\ln(2)$. We have $\ln(2) = \int_1^2 1/t\,dt \geq 1/2$ since $1/t \geq 1/2$ for $t \in [1,2]$. Hence

$$\ln(2^n) \geq \frac{n}{2}.$$

Hence

$$\ln(2^{-n}) = -n\ln(2) \leq -\frac{n}{2}.$$

We have seen that $\ln(x)$ is strictly increasing and continuous on $(0,\infty)$. Therefore the range of $\ln(x)$ is an open interval. Since the range is unbounded above and below we must have that $\ln((0,\infty)) = \mathbb{R}$. Thus

$$\ln\colon (0,\infty) \to \mathbb{R}$$

is 1-1, onto and continuous. Therefore the inverse function

$$\ln^{-1}\colon \mathbb{R} \to (0, \infty)$$

exists and is continuous. We denote

$$\exp(x) := \ln^{-1}(x).$$

Thus $\exp\colon \mathbb{R} \to (0, \infty)$. Since $\ln(x) > 0$ for all x, we see, by the Inverse Function Theorem that $\exp(x)$ is differentiable for all x with

$$\frac{d}{dx}\exp(x) = (\ln^{-1})'(x) = \frac{1}{\ln'(\ln^{-1}(x))} = \ln^{-1}(x) = \exp(x).$$

We make some easy observations about the function $\exp(x)$:

- $\exp(x + y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$ (let $x = \ln(a)$, $y = \ln(b)$ for $a, b > 0$, then $\exp(x + y) = \exp(\ln(a) + \ln(b)) = \exp(\ln(ab)) = ab = \exp(x)\exp(y)$).
- $\exp(x^r) = r\exp(x)$ for all $x \in \mathbb{R}$, $r \in \mathbb{Q}$ (exercise).
- $\exp(0) = 1$.

Define $e := \exp(1)$. Let $b > 0$ and write $b = \exp(a)$ for some $a \in \mathbb{R}$. Then

$$b^r = \exp(a)^r = \exp(ra) = \exp(r\ln(b)).$$

Notice that the right hand side is well defined even if r is irrational. This leads to the following definition: if $b > 0$ and $x \in \mathbb{R}$ we define

$$b^x = \exp(x\ln(b)).$$

In particular

$$e^x = \exp(x).$$

By the Chain Rule, we see that

$$\frac{d}{dx}b^x = b^x\ln(b).$$

Taylor Polynomials

Suppose f is an n-times differentiable function, differentiable on an open interval I containing x_0. The statement that f is n-times differentiable means that

$$f'(x), f''(x), f^{(3)}(x) := f'''(x), \ldots, f^{(n)}(x)$$

all exist for $x \in I$.

The n-th *Taylor polynomial* of f at x_0 is defined to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Example: Let $f(x) = 1/x$ and let $x_0 = 1$. Then f is differentiable to arbitraruly high orders on an open interval containing x_0. We have

$$f'(x) = -x^{-2}, f''(x) = 2x^{-3}, f'''(x) = -3!x^{-4}, \ldots, f^{(n)}(x) = (-1)^n n!x^{-n-1}.$$

Therefore the n-th Taylor polynomial for f at $x_0 = 1$ is

$$p_n(x) = 1 - (x - 1) + (x - 1)^2 - + \cdots + (-1)^n(x - 1)^n.$$