Lecture 7: Kolmogorov differential equations – Keep on solving

Concepts checklist

At the end of this lecture, you should be able to:

- Understand how *generating functions may sometimes assist in the solution of the KFDEs*;
- Solve the KFDEs for the Poisson process using generating functions; and,
- Appreciate that it is too difficult to be able to typically solve the KFDEs.

Example 2. Poisson process as a CTMC (continued)

Recall that

$$q_{n,n+1} = \lambda \quad \text{for } n \geq 0,$$
$$q_{nn} = \sum_{m \neq n} q_{nm} = -\lambda \quad \text{for } n \geq 0,$$

and the KFDEs are

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{ij}(t) = \sum_{k \in \mathcal{S}} P_{ik}(t) q_{kj}.$$

For $i = 0$ and $n > 0$, we have

$$\frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} = \sum_{k \in \mathcal{S}} P_{0k}(t) q_{kn} = P_{0,n-1}(t) q_{n-1,n} + P_{0n}(t) q_{nn}.$$

Hence,

$$\frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} = \lambda P_{0,n-1}(t) - \lambda P_{0n}(t) \quad \text{for} \quad n > 0,$$
$$\text{and} \quad \frac{\mathrm{d}P_{00}(t)}{\mathrm{d}t} = -\lambda P_{00}(t).$$

We shall now consider solving these equations using a generating function approach.

Definition 6. *The generating function $P(z,t)$ for a process with transition probabilities $P_{0n}(t)$ for $n = 0, 1, 2, \dots,$ is given by*

$$P(z,t) = \sum_{n=0}^{\infty} P_{0n}(t) z^n.$$

By the triangle inequality, for $|z| \leq 1$, we have

$$\left| \sum_{n=0}^{\infty} P_{0n}(t) z^n \right| \leq \sum_{n=0}^{\infty} P_{0n}(t) \left| z^n \right| \leq \sum_{n=0}^{\infty} P_{0n}(t) = 1.$$

Therefore, the generating function $P(z,t)$ is well defined for $|z| \leq 1$.

If we multiply both sides of the equation

$$\frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} = \lambda P_{0,n-1}(t) - \lambda P_{0n}(t)$$

by z^n and sum from $n = 1$ to ∞, we have

$$\sum_{n=1}^{\infty} \frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} z^n = -\lambda \sum_{n=1}^{\infty} P_{0n}(t) z^n + \lambda \sum_{n=1}^{\infty} P_{0,n-1}(t) z^n. \tag{2}$$

We then add $\frac{\mathrm{d}P_{00}(t)}{\mathrm{d}t} z^0 = -\lambda P_{00}(t) z^0$ to Equation (2) to get

$$\sum_{n=0}^{\infty} \frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} z^n = -\lambda \sum_{n=0}^{\infty} P_{0n}(t) z^n + \lambda \sum_{n=1}^{\infty} P_{0,n-1}(t) z^n.$$

This gives us

$$\begin{aligned}
\frac{\mathrm{d}P(z,t)}{\mathrm{d}t} &= -\lambda P(z,t) + \lambda z \sum_{n=1}^{\infty} P_{0,n-1}(t) z^{n-1} \\
&= -\lambda P(z,t) + \lambda z \sum_{n=0}^{\infty} P_{0n}(t) z^n \\
&= -\lambda P(z,t) + \lambda z P(z,t) \\
&= -(\lambda - \lambda z) P(z,t).
\end{aligned}$$

The solution to this linear ordinary differential equation is

$$P(z,t) = c(z) e^{-(\lambda - \lambda z)t}. \tag{3}$$

To find $c(z)$, we need to know the value $P(z,t)$ at $t = 0$. That is,

$$P(z,0) = \sum_{n=0}^{\infty} P_{0n}(0) z^n = 1(z^0) + 0(z^1) + 0(z^2) + \cdots = 1. \tag{4}$$

Using (4) and substituting $t = 0$ into equation (3) yields

$$\begin{aligned}
1 &= c(z) e^{-(\lambda - \lambda z) \times 0} \\
\Rightarrow \quad c(z) &= 1 \\
\Rightarrow \quad P(z,t) &= e^{-(\lambda - \lambda z)t}.
\end{aligned}$$

Note that

$$P(z,t) = e^{-(\lambda - \lambda z)t} = e^{-\lambda t} e^{\lambda z t} = e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda z t)^j}{j!} = \sum_{j=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^j}{j!} z^j. \tag{5}$$

Comparing (5) to the generating function,

$$P(z,t) = \sum_{j=0}^{\infty} P_{0j}(t) z^j, \quad \text{which is valid for } |z| \le 1,$$

we must have that for each j,

$$P_{0j}(t) = \frac{e^{-\lambda t}(\lambda t)^j}{j!}.$$

Thus, the probability that the system is in state j by time t, given that it starts in state 0 at time 0, follows a Poisson distribution with parameter λt. In other words, the number of events that have happened by time t follows a Poisson distribution with parameter λt.

However, taking this just a little bit further, and considering a still relatively simple model, evaluating the transition function explicitly becomes difficult (and quickly becomes infeasible).

Example 3. The M/M/1 queue (continued)

The KFDEs for the M/M/1 queue are

$$\begin{aligned} \frac{d\,P_{00}(t)}{dt} &= -\lambda P_{00}(t) + \mu P_{01}(t), \text{ and} \\ \frac{d\,P_{0j}(t)}{dt} &= \lambda P_{0,j-1}(t) - (\lambda+\mu)P_{0j}(t) + \mu P_{0,j+1}(t) \quad \text{for } j \geq 1. \end{aligned}$$

The solution $P_{0j}(t)$ to these equations is given by

$$e^{-(\lambda+\mu)t}\left[\left(\frac{\lambda}{\mu}\right)^{-\frac{j}{2}} I_j\left(2\sqrt{\lambda\mu}t\right) + \left(\frac{\lambda}{\mu}\right)^{-\frac{(j+1)}{2}} I_{j-1}\left(2\sqrt{\lambda\mu}t\right) \right.$$
$$\left. + \left(1-\frac{\lambda}{\mu}\right)\sum_{\ell=j+2}^{\infty}\left(\frac{\lambda}{\mu}\right)^{-\frac{\ell}{2}} I_\ell\left(2\sqrt{\lambda\mu}t\right)\right],$$

where $I_j(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{j+2m}}{(j+m)!m!}$ is a Bessel Function of order j.

For most models it is too much to ask for us to calculate the $P_{i,j}(t)$ and so, **we need to look for simpler measures of our CTMC.**
