Fluid Mechanics III

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School of Mathematical Sciences

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No guarantee is made that these notes are free from error! Always check results against original sources and your own independent analysis.

1 Outline

1.1 General information

Course Information

Course Codes	:	APP MTH 3002 Fluid Mechanics III APP MTH 4102 Fluid Mechanics Hons APP MTH 7075 Fluid Mechanics PG
Pre-requisites	:	(MATHS 2101 and MATHS 2102) Multivariable & Complex Calculus II and Differential Equations II or (MATHS 2201 and MATHS 2202) Engineering Mathematics IIA and Engineering Mathematics IIB
Assumed Knowledge	:	MATHS 2104 Numerical Methods II

Learning Outcomes

Understand:

- Basic concepts of fluid mechanics.
- Mathematical description of fluid flow.
- Conservation principles governing fluid flows.

Be able to:

- Solve inviscid flow problems using streamfunctions and velocity potentials.
- Compute forces on bodies in fluid flows.
- Solve (analytical and numerical) viscous flow problems.
- Use mathematical software packages (MAPLE and MATLAB) in solution methods.

Staff

Lecturer / Course coordinator	:	Dr Trent Mattner
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Office Location	:	6.41, Ingkarni Wardli
Office Hours	:	2 pm, Tuesday

Course Organisation

Lectures (5 hours per fortnight):

- Tuesday 10am–11am in Napier, 209.
- Wednesday 2pm–3pm in Napier, G03.
- Friday 2pm–3pm in Napier, 209.
- Lectures will be recorded.

Tutorials (1 hour per fortnight):

- Every second Friday starting in week 2.
- Tutorials will not be recorded.

Resources

MyUni:

- New announcements to all students.
- Lecture materials for download.
- Access to tutorial question sheets and solutions.
- Access to assignment questions and solutions.
- Past exam papers will be available for download.
- Lecture recordings will be available for watching.

Suggested books:

- Introduction to Theoretical and Computational Fluid Dynamics, Pozrikidis, Oxford University Press.
- Elementary fluid dynamics, Acheson, Oxford University Press.
- An introduction to fluid mechanics, Batchelor, Cambridge University Press.

Assessment

Assignments:

- 30% of total assessment.
- Total 5 assignments of equal value.
- Submit electronically via MyUni.

Late policy:

- Late assignments submitted up to 24 hours late will receive 60% credit.
- Assignments will not be accepted more than 24 hours late.
- Any variation to this policy will require medical documentation.

Exam:

- 70% of total assessment.
- For timetables, alternative exam arrangements, and replacement and additional (R/AA) exam information, see:

  `https://www.adelaide.edu.au/student/exams/`

Academic Honesty

We encourage you to:

- Work together.
- Seek help from your lecturer.

However:

- All assignments submitted must be your own work.
- Substantially similar pieces of work from different students are not acceptable.

Therefore we recommend that you:

- Plan how to do the assignment in groups.
- Work on your written assignment separately.
- Be aware of the university academic honesty policy:

  `http://www.adelaide.edu.au/policies/230`

2 Introduction

2.1 Definitions

What is a fluid?

Materials are roughly classified as solid or fluid according to the ease with which they are deformed. There are no precise, universally-accepted definitions.

A simple fluid is a material that deforms continuously when acted on by a shear stress of any magnitude.

Fluids are further classified as:

Liquid A fluid that is difficult to compress. Volume is almost independent of pressure.

Gas A fluid that compresses readily. Volume is dependent on pressure.

What is mechanics?

Mechanics is the branch of applied mathematics that deals with the motion and equilibrium of bodies and the actions of forces (OED).

There are three branches of mechanics:

Statics The study of bodies in equilibrium.

Kinematics The study of bodies in motion without reference to forces.

Dynamics The study of forces that change or produce motion.

What is fluid mechanics?

Fluid Mechanics is the branch of applied mathematics that deals with the motion and equilibrium of fluids and the action of forces. Fluid mechanics has a remarkable diversity of applications including:

Industrial processes: Lubrication, coating processes, glass blowing, oil recovery.

Engineering: Hydraulics, heating and ventilation systems, aircraft design, ship design, traffic flow.

Biology and medicine: Blood flow, air flow in the lungs, swimming and flying.

Geophysical and environmental: Ocean currents, meteorology, lava flows, dispersion of pollutants in the atmosphere.

Astrophysical: Dust, stars, galaxies, accretion disks.

2.2 Assumptions

Continuum model

We shall treat the fluid mathematically as if it formed a complete continuous medium. That is, variables such as temperature, pressure, density and velocity are well-defined for infinitely small points in the fluid, and are continuous in space and time (except on certain surfaces).

In reality, fluids are composed of discrete molecules that are in constant motion. However, when observed at macroscopic scales, the average behaviour of clusters of these molecules appears to be smooth and continuous.

The continuum approximation allows us to create a model that is mathematically tractable.

The continuum approximation is justifiable if molecular variables are averaged over length and time scales that are:

1. Large compared with molecular scales, but
2. Small compared with scales of practical interest.

The approximation is harder to justify when:

1. Densities are tiny (rarefied gas dynamics — space vehicle re-entry).
2. Spatial scales are small (nanotechnology).
3. There are sharp interfaces (shocks in supersonic flow).

3 Notation

3.1 Suffix notation

Gibbs notation

In these notes, vectors will be denoted by bold face fonts. In this notation, a position vector is written as

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k},$$

where x, y and z are the components of $\boldsymbol{x}$ along the coordinate directions denoted by the unit-vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively. This is referred to as Gibbs notation.

The vector $\boldsymbol{x}$ can also be written as a column vector,

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

in which case $\boldsymbol{x}^T = (x, y, z)$. Although, formally, it is important to keep track of the transposition, the superscript T is often omitted for brevity.

Suffix notation

Suffix notation (also known as Cartesian index notation) provides an alternative way of writing vectors. It is widely used in fluid mechanics and other areas of mathematical physics. In general, it is easier to manipulate vectors using suffix notation (particularly when the manipulations involve differential operators).

If x_1, x_2 and x_3 are the components of $\boldsymbol{x}$ in the coordinate directions denoted by the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively, then

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \sum_{i=1}^{3} x_i\boldsymbol{e}_i.$$

In this notation, x_i is the i^{th} component of the vector $\boldsymbol{x}$ and is *scalar*. The vector $\boldsymbol{x}$ can also be written as $\{x_i\}$, and the scalar component x_i can be written as $[\boldsymbol{x}]_i$.

Consider the vector equation $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ in $\mathbb{R}^3$. This is really shorthand for three scalar equations,

$$\begin{aligned} x_1+y_1&=z_1,\\ x_2+y_2&=z_2,\\ x_3+y_3&=z_3. \end{aligned}$$

In suffix notation, we write

$$x_i+y_i=z_i,$$

which is understood to hold for each component i. The suffix i is called a free suffix as it appears exactly once in each term.

There is no need to write $i=1,2,3$. This is understood from the context. For physical problems, by default we have $i=1,2,3$. However, if we are working in a space of dimension N, we can have $i=1,2,\ldots,N$.

Einstein summation convention

A suffix may appear twice in a single term, in which case it is known as a repeated suffix or dummy suffix. Where this occurs the Einstein summation convention is that the index should be summed over all possible values of the index.

Using the summation convention,

$$\boldsymbol{x}=x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2+x_3\boldsymbol{e}_3=x_i\boldsymbol{e}_i.$$

The dot product of two vectors $\boldsymbol{a}=a_i\boldsymbol{e}_i$ and $\boldsymbol{b}=b_i\boldsymbol{e}_i$ is

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b}&=a_1b_1+a_2b_2+a_3b_3\\ &=\sum_{i=1}^{3}a_ib_i\\ &=a_ib_i \end{aligned}$$

Observe that there are no free suffixes on either side of the equation. The dummy suffix is ‘summed out’ of the equation.

Rules for suffix notation

1. Expressions with free suffixes hold for each possible value of the free suffix.
2. The free suffixes on each side of an equation should be the same
3. A repeated suffix (i.e., dummy suffix) implies summation over all values of that suffix.
4. We can rename dummy suffixes at will, hence

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_ib_i=a_kb_k=a_mb_m.$$

5. No suffix should appear more than twice in any term of an equation. If any suffix appears three or more times in any term, the equation is meaningless.

Suffix notation

Example 1. Write the following using suffix notation.

1. Newton's law, $\boldsymbol{F} = m\boldsymbol{a}$.
2. $\boldsymbol{u} + (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{v} = \|\boldsymbol{a}\|^2(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}$.
3. The gradient, $\nabla\phi$.
4. The divergence, $\nabla \cdot \boldsymbol{u}$.
5. The scalar quantity, $\boldsymbol{u} \cdot \nabla\phi$.
6. The Laplacian, $\nabla^2\phi$.

Example 2. Explain why the following equations cannot be correct:

1. $a_i = b_j$,
2. $a_i b_i = c_j d_k$,
3. $a_i = b_i b_j c_j d_j$.

Matrices in suffix notation

Suffix notation is also applicable to matrices. The elements of a matrix $\boldsymbol{A}$ are denoted by A_{ij}, where i is the row index and j is the column index.

For a 3×3 matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \{A_{ij}\},$$

where i, $j = 1, 2, 3$.

Using the Einstein summation convention, the trace of a 3×3 matrix $\boldsymbol{A}$ is

$$\mathrm{Tr}\,\boldsymbol{A} = A_{11} + A_{22} + A_{33} = A_{ii}.$$

Suffix notation provides a convenient way of denoting matrix-vector products. Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $\boldsymbol{A}$ is a 3×3 matrix. In suffix notation, the components of the product denoted by

$$\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u} \quad \text{are} \quad v_i = A_{ij} u_j.$$

This corresponds to multiplication of the matrix $\boldsymbol{A}$ by a column vector $\boldsymbol{u}$.

Example 3. Verify that $v_i = A_{ij} u_j$ corresponds to the matrix-vector product

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

The components of the product denoted by

$$\boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{A} \quad \text{are} \quad w_j = u_i A_{ij}.$$

This corresponds to multiplication of a row vector $\boldsymbol{u}$ by the matrix $\boldsymbol{A}$.

Example 4. Verify that $w_j = u_i A_{ij}$ corresponds to the matrix-vector product

$$(w_1, w_2, w_3) = (u_1, u_2, u_3) \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

The Kronecker delta

The Kronecker delta is

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

It corresponds to the identity matrix.

Example 5. Using the definition of the Kronecker delta δ_{ij}:

1. Evaluate δ_{ii} in three dimensions.
2. Show that $\delta_{ij} u_j = u_i$.

The last example illustrates the substitution rule for the Kronecker delta, whereby it swaps one of its suffixes for the other on anything it multiplies.

The alternating tensor

The alternating tensor, alternator or Levi-Civita symbol is

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases}$$

Even permutations are 123, 231, 312. Odd permutations are 321, 132, 213.

The alternating tensor is useful because it allows us to write equations involving vector (cross) products in suffix notation. The components of the vector product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ are

$$w_i = \epsilon_{ijk} u_j v_k.$$

The alternating tensor satisfies the following identities:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji},$$
$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

Example 6. Using the definition of the alternating tensor ϵ_{ijk}:

1. How could we write the curl, $\nabla \times \boldsymbol{u}$, in suffix notation?
2. Show that $\nabla \times \nabla\phi = \boldsymbol{0}$.
3. Show that $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

4 Kinematics

4.1 Eulerian and Lagrangian descriptions

Eulerian description

In the Eulerian specification of a flow field, flow quantities are regarded as functions of position $\boldsymbol{x}$ and time t.

For example, the Eulerian velocity field is written as $\boldsymbol{u}(\boldsymbol{x}, t)$.

Lagrangian description

In the Lagrangian specification of a flow field, we focus on material particles (points) as they move through the flow. Each particle is uniquely identified by a label $\boldsymbol{x}_0$, which is the position of the particle at some instant in time, usually $t = 0$.

Flow quantities are regarded as functions of the particle label $\boldsymbol{x}_0$ and time t.

The position of the particle is given by the function $\boldsymbol{X}(\boldsymbol{x}_0, t)$, where $\boldsymbol{x}_0 = \boldsymbol{X}(\boldsymbol{x}_0, 0)$.

The velocity and acceleration of the particle are

$$\boldsymbol{U}(\boldsymbol{x_0}, t) = \frac{\partial \boldsymbol{X}}{\partial t} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x_0}, t) = \frac{\partial^2 \boldsymbol{X}}{\partial t^2},$$

respectively.

Example 7. Consider $\boldsymbol{X}(\boldsymbol{x_0}, t) = x_0 e^t \boldsymbol{i} + y_0 e^{-t} \boldsymbol{j}$, where $\boldsymbol{x_0} = (x_0, y_0)$.

1. Verify that $\boldsymbol{x_0} = \boldsymbol{X}(\boldsymbol{x}_0, 0)$.
2. Find $\boldsymbol{x_0}$ for the particle that passes through (e^2, e^{-2}) at the time $t = 2$.
3. Determine the velocity and acceleration of that particle.

Transformation between descriptions

The transformation from a Lagrangian to an Eulerian description is accomplished by considering the particle that is at the Eulerian position $\boldsymbol{x}$ at time t, that is,

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t). \tag{1}$$

Given $\boldsymbol{x}$, t and the function $\boldsymbol{X}$, equation 1 can, in principle, be solved for $\boldsymbol{x}_0$.

Example 8. Consider $\boldsymbol{X}(\boldsymbol{x}_0, t) = x_0 e^t \boldsymbol{i} + y_0 e^{-t} \boldsymbol{j}$, as before.

1. Find $\boldsymbol{x}_0$ for the particle that passes through the arbitrary Eulerian point $\boldsymbol{x} = (x, y)$ at time t.
2. Find the Lagrangian description of the velocity $\boldsymbol{U}(\boldsymbol{x}_0, t)$ and acceleration $\boldsymbol{A}(\boldsymbol{x}_0, t)$.

3. Use the relationship between $\boldsymbol{x}_0$ and $\boldsymbol{x}$ found in part 1 to find the Eulerian description of the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$.
4. Use the expression for the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ to calculate $\partial \boldsymbol{u}/\partial t$. Is it the same as the acceleration?

Steady flow

A flow is steady when the Eulerian description of the velocity is independent of time, that is,

$$\frac{\partial \boldsymbol{u}}{\partial t} = \mathbf{0}.$$

Otherwise, the flow is unsteady.

Example 9. Is the flow in the previous example steady or not?

This concept only applies to the Eulerian description of the flow. A steady flow does not imply that the fluid particle velocity is independent of time.

Whether a flow is steady or not depends on the frame of reference. For example a wake trailing behind a moving boat would appear steady to an observer on the boat, and unsteady to an observer on the shore.

4.2 Flow Visualisation

Pathlines

A pathline is a curve traced out by a particle as it moves through a flow field. It is the curve generated by treating t as a parameter in $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t)$, hence is associated with Lagrangian coordinates.

Suppose instead that $\boldsymbol{u}(\boldsymbol{x}, t)$ is given, where $\boldsymbol{x}$ is the Eulerian position. Then

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x}_0, t), t) = \boldsymbol{U}(\boldsymbol{x}_0, t) = \frac{\partial \boldsymbol{X}}{\partial t},$$

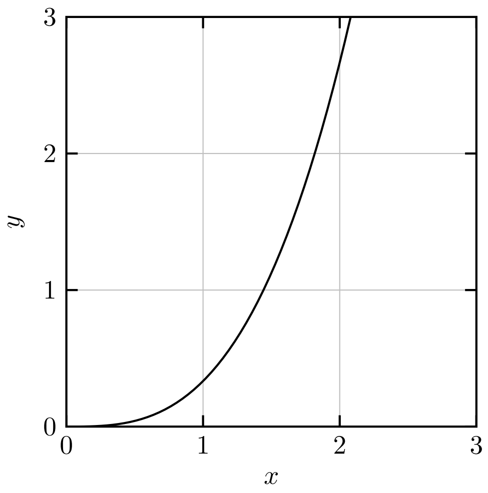
where $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t)$. For a given particle, this is simply

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}, t), \tag{2}$$

that is, a system of ODEs to solve for $\boldsymbol{X}$.

Example 10. Suppose $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} + xt\boldsymbol{j}$ and let $\boldsymbol{X} = X\boldsymbol{i} + Y\boldsymbol{j}$.

1. Write down a system of first-order ODEs for X and Y.
2. Solve this system and hence write down $\boldsymbol{X}(\boldsymbol{x}_0, t)$ and $\boldsymbol{x}_0$.
3. Write out the components of $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} = \boldsymbol{X}(\boldsymbol{x}_0, t)$, writing y in terms of x by eliminating t.
4. Draw the pathline of the particle that starts from the origin at time $t = 0$.



Streaklines

A streakline is a curve made up of all particles that have passed a particular point in space at some earlier time. It is what you would see if dye or smoke was introduced at some point in the flow. It is also associated with Lagrangian coordinates.

Suppose $\boldsymbol{x}^*$ is the point where the dye is being introduced. We want to find all the points $\boldsymbol{x}_0$ such that

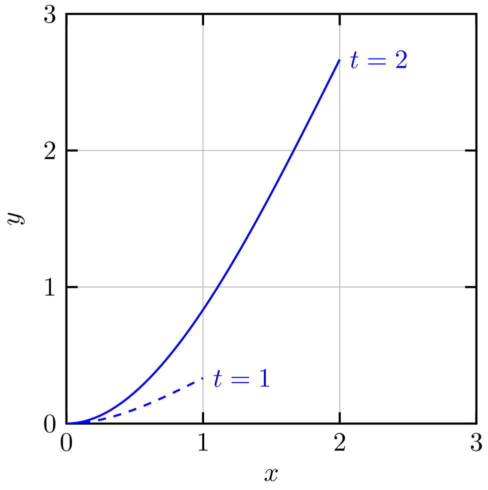
$$\boldsymbol{x}^* = \boldsymbol{X}(\boldsymbol{x}_0, \tau), \tag{3}$$

where τ is any time before the present (that is, $0 \le \tau \le t$). If the function $\boldsymbol{X}$ is known, then equation (3) can, in principle, be used to find $\boldsymbol{x}_0$ in terms of $\boldsymbol{x}^*$ and τ, that is, $\boldsymbol{x}_0(\boldsymbol{x}^*, \tau)$. The streakline is the curve given by the set of points

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0(\boldsymbol{x}^*, \tau), t), \quad 0 \le \tau \le t. \tag{4}$$

Example 11. Consider the previous example $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} + xt\boldsymbol{j}$, again. Suppose $\boldsymbol{x}^* = x^*\boldsymbol{i} + y^*\boldsymbol{j}$.

1. Using the results from the previous example, find $\boldsymbol{x}_0$ for the particle that passes through $\boldsymbol{x}^*$ at time τ, that is find $\boldsymbol{x}_0(\boldsymbol{x}^*, \tau)$.
2. Write out the components of $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0(\boldsymbol{x}^*, \tau), t)$. Write y in terms of x by eliminating τ.
3. Draw the streakline emanating from the origin at $t = 1$ and $t = 2$.



Streamlines

A streamline is a curve that is everywhere tangent to the velocity field. It is associated with Eulerian coordinates.

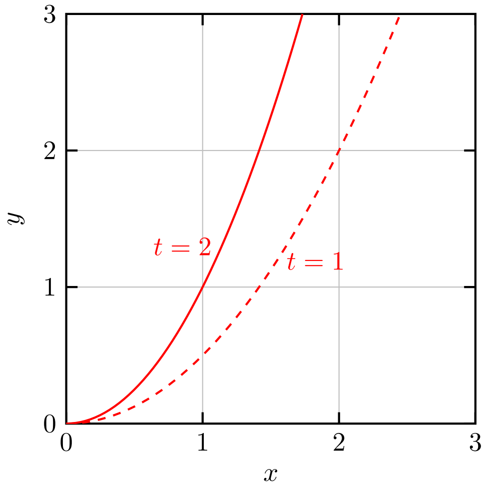
Recall from vector calculus that the tangent to a curve $\boldsymbol{x}(s)$ is given by $\mathrm{d}\boldsymbol{x}/\mathrm{d}s$, where s is a parameter. This must be parallel to the local velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. This is accomplished by setting

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}(\boldsymbol{x}, t), \tag{5}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is expressed in Eulerian coordinates. This is a system of ODEs that can, in principle, be solved for $\boldsymbol{x}$ in terms of s at any instant in time t.

Example 12. Consider the previous example $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} + xt\boldsymbol{j}$, yet again. Suppose $\boldsymbol{x}(s, t) = x(s, t)\boldsymbol{i} + y(s, t)\boldsymbol{j}$.

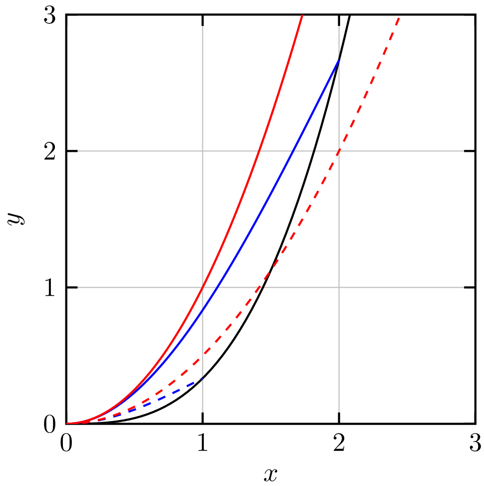
1. Write down a system of first-order ODEs for the dependant variables x and y.
2. Solve this system of ODEs and hence write down $\boldsymbol{x}(s, t)$
3. Find y in terms of x by eliminating s.
4. Draw a streamline that passes through the origin for times $t = 1$ and $t = 2$.



Pathlines, streaklines and streamlines

Pathlines, streamlines and pathlines are coincident when the flow is steady.

Examples 10–12



4.3 The material derivative

The material derivative

The material or convective or substantial derivative is an expression for the rate of change of quantity following a particle (Lagrangian time derivative), written in terms of its Eulerian description.

Consider a particle labelled $\boldsymbol{x}_0$ that passes through the Eulerian position $\boldsymbol{x}$ at time t, so that $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t)$. Suppose that a quantity is expressed in Eulerian coordinates by $f(\boldsymbol{x},t)$ and in Lagrangian coordinates by $F(\boldsymbol{x}_0,t)$. The two functions are related by

$$F(\boldsymbol{x}_0,t) = f(\boldsymbol{X}(\boldsymbol{x}_0,t),t)$$

The rate of change of the quantity following the particle is $\partial F/\partial t$. Using the chain rule,

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial X_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial X_2}{\partial t} + \frac{\partial f}{\partial x_3}\frac{\partial X_3}{\partial t} + \frac{\partial f}{\partial t}.$$

But $\partial \boldsymbol{X}/\partial t = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x}_0,t),t)$, hence

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial t} + u_1\frac{\partial f}{\partial x_1} + u_2\frac{\partial f}{\partial x_2} + u_3\frac{\partial f}{\partial x_3}.$$

The right-hand-side of the above is the material, convective or substantial derivative of f. It is denoted by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f = \frac{\partial f}{\partial t} + u_i\frac{\partial f}{\partial x_i}. \tag{6}$$

Hence, in Eulerian coordinates, the rate of change of a quantity following a fluid particle consists of two components:

1. The local rate of change, $\partial f/\partial t$, due to temporal variation.
2. The convective rate of change, $\boldsymbol{u}\cdot\nabla f$, due to movement of fluid particles through spatial gradients.

The material derivative also applies to vectors. For example, the j-th component of the acceleration of a fluid particle is

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = \frac{\partial u_j}{\partial t} + \boldsymbol{u}\cdot\nabla u_j = \frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i} \tag{7a}$$

which is the j-th component of the acceleration. In vector notation, this is written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}. \tag{7b}$$

Example 13. Let $\boldsymbol{u} = (u,v,w)$ and $\boldsymbol{x} = (x,y,z)$. Write out all three components of the acceleration.

Example 14. In example 8, we found that $\boldsymbol{u}(\boldsymbol{x},t) = u\,\boldsymbol{i} + v\,\boldsymbol{j} = x\,\boldsymbol{i} - y\,\boldsymbol{j}$ and $\boldsymbol{a}(\boldsymbol{x},t) = x\,\boldsymbol{i} + y\,\boldsymbol{j} \neq \partial\boldsymbol{u}/\partial t$, where $\boldsymbol{x} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$. Use the material derivative to verify that

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{a}.$$

4.4 Decomposition of local fluid motion

Decomposition of local fluid motion

Consider the fluid velocity in the vicinity of the point $\boldsymbol{x}$ in Eulerian coordinates. As the velocity is a continuously differentiable function of space and time, it can be expressed using a multivariable Taylor series. The j-th component of the velocity at a nearby point $\boldsymbol{x} + \delta\boldsymbol{x}$ is

$$\begin{aligned} u_j(\boldsymbol{x} + \delta\boldsymbol{x}, t) &= u_j(\boldsymbol{x}, t) + \delta x_1 \left.\frac{\partial u_j}{\partial x_1}\right|_{\boldsymbol{x},t} + \delta x_2 \left.\frac{\partial u_j}{\partial x_2}\right|_{\boldsymbol{x},t} + \delta x_3 \left.\frac{\partial u_j}{\partial x_3}\right|_{\boldsymbol{x},t} + \text{h.o.t.} \\ &= u_j(\boldsymbol{x}, t) + \delta x_i \left.\frac{\partial u_j}{\partial x_i}\right|_{\boldsymbol{x},t} + \text{h.o.t.}, \end{aligned}$$

where 'h.o.t.' stands for 'higher order terms', which involve products of δx_i.

In vector form, this is written as

$$\boldsymbol{u}(\boldsymbol{x} + \delta\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) + \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u}(\boldsymbol{x}, t) + \text{h.o.t.},$$

where $\nabla\boldsymbol{u}$ is the velocity gradient matrix (or tensor) with components

$$[\nabla\boldsymbol{u}]_{ij} = \frac{\partial u_j}{\partial x_i}.$$

Example 15. Let $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{u} = (u, v, w)$. Write out the velocity gradient tensor.

If $\|\delta\boldsymbol{x}\|$ is sufficiently small, the higher order terms can be neglected. Then the velocity relative to that at $\boldsymbol{x}$ is

$$\boldsymbol{\Delta u} = \boldsymbol{u}(\boldsymbol{x} + \delta\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{x}, t) = \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u},$$

or in suffix notation,

$$\Delta u_j = \delta x_i \frac{\partial u_j}{\partial x_i}.$$

This is the velocity field that we would observe near the point $\boldsymbol{x}$ if we were to travel with the flow at the point $\boldsymbol{x}$. This local relative motion depends on the velocity gradient tensor.

The velocity gradient can be decomposed into the sum of a symmetric matrix $\boldsymbol{E}$ and an antisymmetric matrix $\boldsymbol{\Omega}$, such that

$$\nabla\boldsymbol{u} = \boldsymbol{E} + \boldsymbol{\Omega},$$

where

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right], \\ \boldsymbol{\Omega} &= \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^T\right]. \end{aligned}$$

In suffix notation, this is

$$\frac{\partial u_j}{\partial x_i} = E_{ij} + \Omega_{ij}$$

where

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \\ \Omega_{ij} &= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right). \end{aligned}$$

A symmetric matrix $\boldsymbol{E}$ satisfies $E_{ij} = E_{ji}$. An antisymmetric matrix $\boldsymbol{\Omega}$ satisfies $\Omega_{ij} = -\Omega_{ji}$.

$\boldsymbol{E}$ is responsible for deformation of fluid elements and is called the rate-of-strain or rate-of-deformation tensor.

$\boldsymbol{\Omega}$ is responsible for rotation of fluid elements and is called the rate-of-rotation tensor.

Example 16. Use suffix notation to show that:

1. $\boldsymbol{E}$ is symmetric, and
2. $\boldsymbol{\Omega}$ is antisymmetric.

Example 17. Let $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{u} = (u, v, w)$. Write out in matrix form the components of:

1. the rate-of-strain tensor $\boldsymbol{E}$, and
2. the rate-of-rotation tensor $\boldsymbol{\Omega}$.

Strain

To study straining motion, put $\boldsymbol{\Omega} = \mathbf{0}$. Then the relative velocity is

$$\Delta\boldsymbol{u} = \delta\boldsymbol{x} \cdot \boldsymbol{E} \quad \text{or} \quad \Delta u_j = \delta x_i E_{ij}.$$

We consider two cases:

1. Normal strain (given by the diagonal terms).
2. Shear strain (the non-diagonal terms).

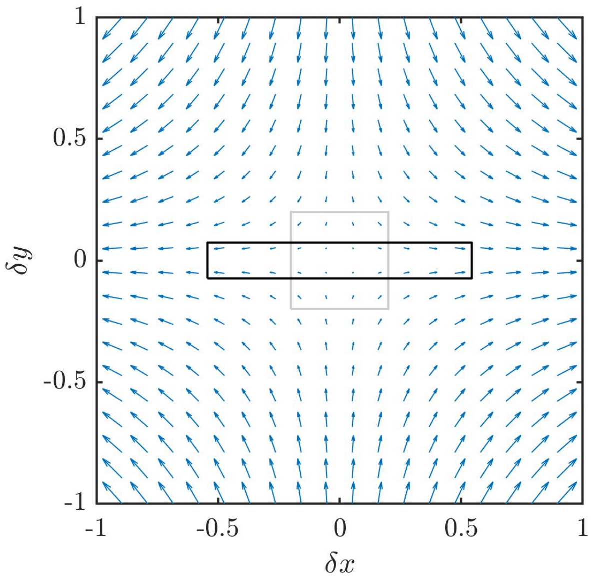
Normal strain

Consider the two-dimensional strain,

$$\boldsymbol{E} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha > 0.$$

The components of the relative velocity are

$$\Delta\boldsymbol{u} = \delta\boldsymbol{x} \cdot \boldsymbol{E} = (\delta x, \delta y, \delta z) \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\alpha\,\delta x, -\alpha\,\delta y, 0)$$



In this example, $E_{11} = \alpha > 0$ and $E_{22} = -\alpha < 0$. Fluid elements are stretched apart in the x-direction (extension) and squeezed together in the y direction (contraction).

Notice that

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_i}{\partial x_i} = E_{ii} = 0$$

in this example. This is a common situation in incompressible flows, as we shall see later.

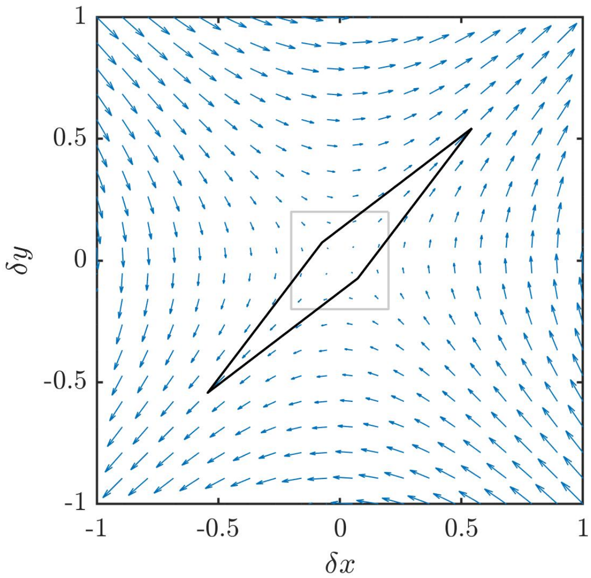
Shear strain

Suppose that

$$\boldsymbol{E} = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma > 0.$$

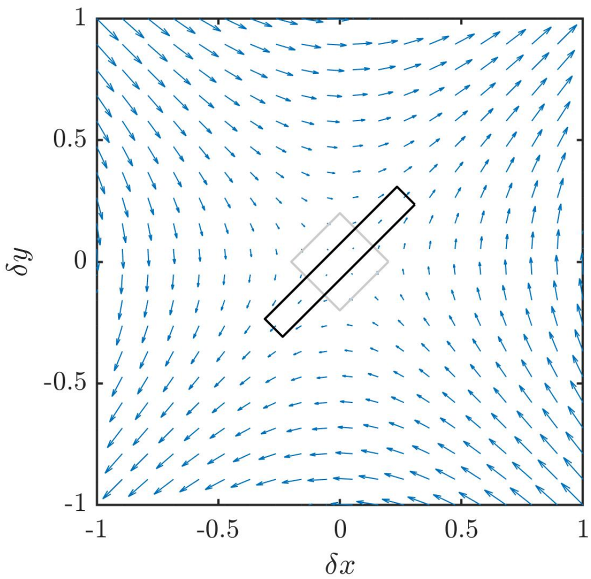
The components of the relative velocity are

$$\boldsymbol{\varDelta u} = \delta \boldsymbol{x} \cdot \boldsymbol{E} = (\delta x, \delta y, \delta z) \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\gamma\, \delta y, \gamma\, \delta x, 0).$$



It is always possible to rotate the coordinate system so that the rate-of-strain tensor only has nonzero entries on its diagonal. The axes of that coordinate system are known as principal axes. The diagonal entries are the eigenvalues of $\boldsymbol{E}$ and the directions of the principal axes are the eigenvectors.

Indeed, the last two examples are the same when one is rotated by 45 degrees with respect to the other!



Rotation

To study rotation, put $\boldsymbol{E} = \mathbf{0}$. Then the relative velocity is

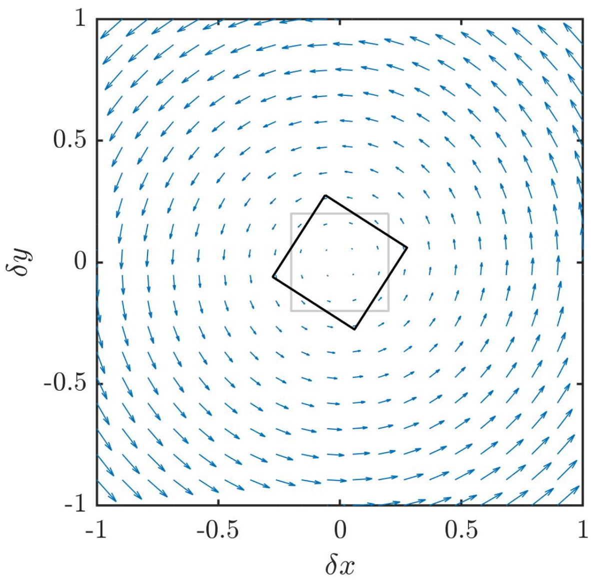
$$\Delta\boldsymbol{u} = \delta\boldsymbol{x} \cdot \boldsymbol{\Omega} \quad \text{or} \quad \Delta u_j = \delta x_i \Omega_{ij}.$$

Suppose that

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega > 0.$$

The components of the relative velocity $\Delta\boldsymbol{u}$ are

$$\Delta\boldsymbol{u} = \delta\boldsymbol{x} \cdot \boldsymbol{\Omega} = (\delta x, \delta y, \delta z) \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (-\omega\,\delta y, \omega\,\delta x, 0).$$



This results in rotation about the z-axis.

The three independent components of the rate-of-rotation tensor are associated with a vector known as the vorticity.

Vorticity

The vorticity is defined as the curl of the velocity, that is,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{8a}$$

In suffix notation, the components of the vorticity are

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \tag{8b}$$

Example 18. Use (8a) and (8b) to write out the components of the vorticity $\boldsymbol{\omega}$.

The rate-of-rotation tensor is related to the vorticity $\boldsymbol{\omega} = \omega_k \boldsymbol{e}_k$ by

$$\Omega_{ij} = \frac{1}{2} \epsilon_{ijk} \omega_k.$$

Explicitly,

$$\boldsymbol{\Omega} = \frac{1}{2} \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

Example 19. Use the definition of the vorticity $\omega_k = \epsilon_{klm} \partial u_m / \partial x_l$ and the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ to show that

$$\frac{1}{2} \epsilon_{ijk} \omega_k = \Omega_{ij}.$$

In suffix notation, the local relative velocity associated with the rate-of-rotation tensor is

$$\Delta u_j = x_i \Omega_{ij} = x_i \left(\frac{1}{2} \epsilon_{ijk} \omega_k \right) = \frac{1}{2} \epsilon_{jki} \omega_k x_i.$$

which is just $\Delta \boldsymbol{u} = \frac{1}{2} \boldsymbol{\omega} \times \boldsymbol{x}$. This corresponds to rotation about the vector $\boldsymbol{\omega}$ with angular velocity $\frac{1}{2} \|\boldsymbol{\omega}\|$.

Example 20. Consider the two-dimensional flow

$$\boldsymbol{u}(\boldsymbol{x}) = \omega y \, \boldsymbol{i} - \omega x \, \boldsymbol{j}.$$

where $\omega > 0$ is a constant.

1. Determine streamlines for the flow.
2. Calculate the components of the rate-of-strain tensor.
3. Find the vorticity vector $\boldsymbol{\omega}$.

In this example, the fluid elements are not deformed. The fluid moves like a rigid body with the same angular velocity everywhere.

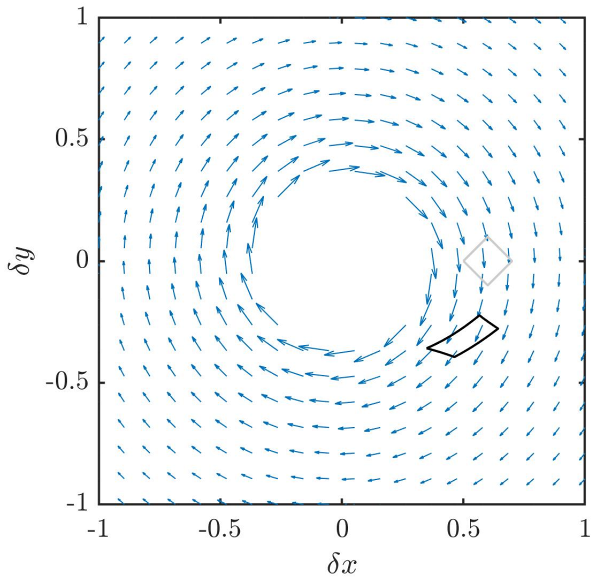
Example 21. Consider the two-dimensional flow

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{y}{x^2 + y^2} \, \boldsymbol{i} - \frac{x}{x^2 + y^2} \, \boldsymbol{j}, \quad x, y \neq 0.$$

1. Determine streamlines for the flow.
2. Calculate the components of the rate-of-strain tensor $\boldsymbol{E}$ at $\boldsymbol{x} = (1, 0)$, $(1, 1)$ and $(0, 1)$.

3. Find the vorticity $\boldsymbol{\omega}$.

In this example, small fluid elements are distorted (strained) but do not rotate, despite the fact that the streamlines are circular. This is an example of an irrotational flow called a potential vortex, which we will see again later.



5 Conservation of mass

5.1 Mass conservation equation

Mass conservation equation

Let $\mathcal{V}$ be an *arbitrary* fixed volume of fluid enclosed by a surface $\mathcal{S}$. Suppose that mass is neither created nor destroyed in $\mathcal{V}$. Then the rate of increase in mass in $\mathcal{V}$ equals the rate at which mass flows into $\mathcal{V}$ through $\mathcal{S}$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{S}} \rho \boldsymbol{u} \cdot \hat{\boldsymbol{n}} \,\mathrm{d}\mathcal{S},$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to $\mathcal{S}$.

Recall that the divergence theorem states that

$$\int_{\mathcal{V}} \nabla \cdot \boldsymbol{F} \,\mathrm{d}\mathcal{V} = \int_{\mathcal{S}} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \,\mathrm{d}\mathcal{S}.$$

Using the divergence theorem to convert the surface integral to a volume integral gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} \nabla \cdot (\rho \boldsymbol{u}) \,\mathrm{d}\mathcal{V}.$$

Since the volume $\mathcal{V}$ is fixed, we can differentiate under the integral sign on the LHS to obtain

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \, \mathrm{d}\mathcal{V} = 0.$$

This must hold for *any* fixed volume $\mathcal{V}$, hence the integrand must be identically zero. Therefore,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{9}$$

Equation (9) is referred to as the mass conservation equation.

5.2 Incompressible flow

Incompressible flow

A fluid is incompressible if the density of every fluid particle is constant. This means that

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0.$$

The mass conservation equation (9) gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u}.$$

Substituting the first equation into the second, we see that for an incompressible fluid

$$\nabla \cdot \boldsymbol{u} = 0. \tag{10}$$

Equation (10) is known as the continuity equation.

Note that:

1. There are some incompressible flows for which $\nabla \cdot \boldsymbol{u} \neq 0$ (such as variable-density and chemically reacting flows, for example). We will not consider such flows in this course. f a flow is incompressible, then you may assume that $\nabla \cdot \boldsymbol{u} = 0$.
2. In an incompressible flow, it is still possible for the density to vary from fluid particle to fluid particle, but it cannot change on a given particle.
3. If the density is uniform and constant throughout the fluid, then $\nabla \cdot \boldsymbol{u} = 0$ follows immediately from the mass conservation equation.
4. No fluid is perfectly incompressible, but it is an excellent approximation when flow velocities are much smaller than the speed of sound.

5.3 Stream function

The stream function

Recall from vector calculus that the divergence of a curl is always zero

$$\nabla \cdot (\nabla \times \boldsymbol{f}) = 0.$$

Hence, we can automatically satisfy the continuity equation (10) by setting

$$\boldsymbol{u} = \nabla \times \boldsymbol{\Psi},$$

for some vector function $\boldsymbol{\Psi}$. This does not, at first sight, seem to make things much easier. However, for certain types of flow, $\boldsymbol{\Psi}$ takes a particularly simple form.

Consider a *two-dimensional* incompressible flow in the (x, y)-plane. The continuity equation is

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

This can be satisfied by defining a scalar function $\psi(x, y, t)$ such that

$$\boldsymbol{\Psi} = \psi(x, y, t)\, \boldsymbol{k}.$$

Upon taking the curl, we obtain

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{11}$$

The function ψ is called the stream function.

Example 22. Verify that ψ automatically satisfies $\nabla \cdot \boldsymbol{u} = 0$ for the two-dimensional flow $\boldsymbol{u} = (u, v)$.

Example 23. Consider the streamline

$$\boldsymbol{x}(s) = x(s)\, \boldsymbol{i} + y(s)\, \boldsymbol{j},$$

parameterised by s. The rate of change of ψ along the streamline is

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{\partial \psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial \psi}{\partial y}\frac{y}{s}.$$

Show that ψ is constant along a streamline.

Example 24. In a two-dimensional flow, the volume flux (per unit width) Q across any curve $\mathcal{C}$ between two points is

$$Q = \int_{\mathcal{C}} \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}s,$$

wher s is the arc length. Show that the volume flux is equal to the difference in ψ between the two points, that is,

$$Q = \int_{\mathcal{C}} u\, \mathrm{d}y - v\, \mathrm{d}x = \psi_2 - \psi_1.$$

Notes:

1. Lines of constant ψ are tangential to the fluid velocity, hence fluid cannot cross a streamline. Lines of constant ψ *may* represent a fixed impermeable boundary.

2. It will be shown (see tutorial) that ψ is related to the vorticity $\boldsymbol{\omega} = \omega\,\boldsymbol{k}$ by
$$\nabla^2\psi = -\omega.$$

Example 25. Find the stream function of:

1. the parallel flow $\boldsymbol{u} = U\,\boldsymbol{i}$, for some constant U.
2. the rotating flow $\boldsymbol{u} = \omega y\,\boldsymbol{i} - \omega x\,\boldsymbol{j}$, for some constant ω.

Stream functions in other coordinate systems

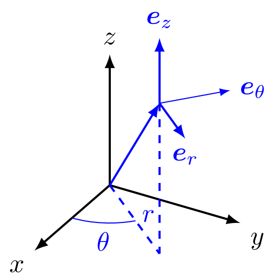
There is no equivalent to the scalar stream function in a general three-dimensional flow. But there are some other special cases with a high degree of symmetry, where a stream function can be defined.

Here we will consider:

1. Cylindrical coordinates: (r, θ, z)
2. Spherical coordinates: (r, θ, ϕ)

Cylindrical coordinates

Consider the cylindrical coordinate system with radial, azimuthal and axial coordinates (r, θ, z) and associated unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$, respectively. These are related to Cartesian coordinates by $x = r\cos\theta$ and $y = r\sin\theta$.



For an incompressible flow,

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0, \tag{12}$$

where $\boldsymbol{u} = u_r\,\boldsymbol{e}_r + u_\theta\,\boldsymbol{e}_\theta + u_z\,\boldsymbol{e}_z$.

Plane flow in cylindrical coordinates

For two-dimensional flow in the r-θ plane, $\partial/\partial z = 0$, hence

$$\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta} = 0.$$

The stream function is then defined by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}. \tag{13}$$

Axisymmetric flow in cylindrical coordinates

For axisymmetric flow, $\partial/\partial\theta = 0$, hence

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0.$$

The stream function is then defined by

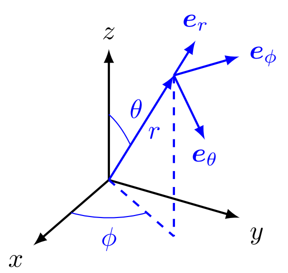
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{14}$$

Lines of constant ψ in the r-z plane correspond to stream surfaces in three-dimensions. The volume flux across any surface connecting two stream surfaces is constant.

Example 26.
1. Verify that this stream function automatically satisfies the continuity equation if the flow is axisymmetric.
2. Find the stream function of the parallel flow $\boldsymbol{u} = U\,\boldsymbol{e}_z$.

Spherical coordinates

Consider the spherical coordinate system with radial, latitudinal and longitudinal coordinates (r, θ, ϕ) and associated unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$, respectively. These are related to Cartesian coordinates by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$.



For an incompressible flow,

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0, \tag{15}$$

where $\boldsymbol{u} = u_r\,\boldsymbol{e}_r + u_\theta\,\boldsymbol{e}_\theta + u_\phi\,\boldsymbol{e}_\phi$.

Axisymmetric flow in spherical coordinates

For axisymmetric flow, $\partial/\partial\phi = 0$, hence

$$\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) = 0,$$

The stream function is then defined by

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{16}$$