Lecture 30: Renewal Theory – Renewal Theorems, and the Bus Paradox

Concepts checklist

At the end of this lecture, you should be able to:

- *State and use* the Basic Renewal Theorem and Blackwell's Renewal Theorem, and the Elementary Renewal Theorem.

Theorem 30 (Basic Renewal Theorem)**.** *Let $F(t)$ be the distribution function of a positive random variable with mean $\mu < \infty$, and assume that $F(t)$ is not lattice. (That is, it does not have all of its points of increase at multiples of some δ.)*

Suppose that $H(t)$ is a solution of the generalised renewal equation

$$H(t) = G(t) + \int_0^t H(t-y)\mathrm{d}F(y), \quad \textit{where } G(t) \textit{ is integrable,}$$

then

$$\lim_{t\to\infty} H(t) = \frac{1}{\mu}\int_0^\infty G(t)\mathrm{d}t. \tag{35}$$

Note: *If F is lattice, then equation (35) is valid for $t = n\delta$ with $n \in \mathbb{N}$.*

We now state another form of this theorem in terms of the renewal function $M(t) = \mathbb{E}\big[N(t)\big]$.

Theorem 31 (Blackwell's Renewal Theorem)**.** *Let F be the distribution function of a positive random variable with mean $\mu < \infty$, which is not lattice, then*

$$\lim_{t\to\infty} [M(t) - M(t-h)] = \frac{h}{\mu} \quad \textit{for } h > 0. \tag{36}$$

Proof:

$$\text{For } h > 0, \text{ if we let } G(y) = \begin{cases} 1 & \text{if } 0 \le y < h \\ 0 & \text{if } y \ge h \end{cases}$$

(which is integrable) and insert into the generalised renewal equation (34), then we can then use Theorem 28 to get for $t > h$ that

$$\begin{aligned} H(t) &= G(t) + \int_0^t G(t-y)dM(y) \\ &= 0 + \int_{t-h}^t 1dM(y) \\ &= M(t) - M(t-h)\ . \end{aligned}$$

Then by Theorem 30,

$$\begin{aligned}
\lim_{t\to\infty}\big[M(t)-M(t-h)\big] &= \lim_{t\to\infty} H(t) \\
&= \frac{1}{\mu}\int_0^\infty G(t)dt \\
&= \frac{1}{\mu}\int_0^h 1dt \\
&= \frac{h}{\mu}.
\end{aligned}$$

Corollary 5 (Elementary Renewal Theorem)**.** *If $F(t)$ is not lattice, then*

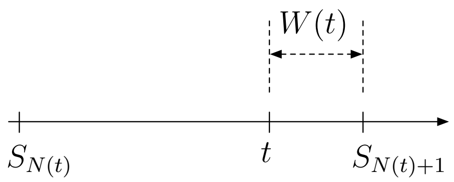
$$\lim_{t\to\infty}\frac{M(t)}{t}=\frac{1}{\mu}.$$

In words: *the rate of change of the renewal function $M(t)$ approaches $1/\mu$ as $t\to\infty$.*

Forward recurrence time & the Bus Paradox

Goal: Find the equilibrium distribution of the time until the next event.

We define $W(t)=S_{N(t)+1}-t$ and $H(z,t)=\Pr\{W(t)>z\}$.



To get an equation for $H(z,t)$, we again use the renewal argument and condition on the time of the first renewal. That is, we consider

$$\Pr(W(t)>z|X_1=x).$$

There are three cases:

Case 1: The first arrival occurs before time t, i.e., $0\le x\le t$:

$$\Pr\{W(t)>z|X_1=x\}=H(z,t-x),$$

because the process might as well have started at time x.

Case 2: The first arrival occurs after time t, but before time $t+z$, i.e., $t\le x\le t+z$:

$$\Pr\{W(t)>z|X_1=x\}=0,\qquad \text{because we know that } W(t)=x-t<z.$$

Case 3: The first arrival occurs after time $t+z$, i.e., $t+z<x$:

$$\Pr\{W(t)>z|X_1=x\}=1,\qquad \text{because we know that } W(t)=x-t>z.$$

Therefore, $H(z,t)$ is given by

$$
\begin{aligned}
H(z,t) &= \int_0^{\infty} \Pr(W(t) > z | X_1 = x) \mathrm{d}F(x) \\
&= \int_0^{t} H(z, t-x) \mathrm{d}F(x) + \int_{z+t}^{\infty} 1 dF(x) \\
&= 1 - F(z+t) + \int_0^{t} H(z, t-x) \mathrm{d}F(x)
\end{aligned}
$$

which is a generalised renewal equation with $G(t) = 1 - F(z+t)$ and $H(t) = H(z,t)$.

By Theorem 30, the solution is therefore given by

$$
H(z,t) = (1 - F(z+t)) + \int_0^{t} (1 - F(z+t-y)) \mathrm{d}M(y).
$$

• Now, as we are looking for the equilibrium distribution, let $t \to \infty$. It is clear that $\big(1 - F(z+t)\big) \to 0$, but what about the second term? This is much more difficult to directly evaluate.

However, the Basic Renewal Theorem tells us that

$$
\lim_{t \to \infty} H(z,t) = \frac{1}{\mu} \int_0^{\infty} G(t) \mathrm{d}t = \frac{1}{\mu} \int_0^{\infty} (1 - F(z+t)) \mathrm{d}t,
$$

as long as

$$
\int_0^{\infty} G(t) \mathrm{d}t = \int_0^{\infty} (1 - F(z+t)) \mathrm{d}t < \infty.
$$

Let us start by considering

$$
\begin{aligned}
\int_0^{\infty} \big(1 - F(u)\big) \mathrm{d}u &= \int_0^{\infty} \left[\int_u^{\infty} dF(y) \right] \mathrm{d}u \\
&= \int_0^{\infty} \left[\int_0^{y} 1 \mathrm{d}u \right] \mathrm{d}F(y) \\
&= \int_0^{\infty} y \, dF(y) = \mu \qquad \text{(the mean).}
\end{aligned}
\tag{37}
$$

Therefore,

$$
\begin{aligned}
\int_0^{\infty} G(t) \mathrm{d}t &= \int_0^{\infty} \big(1 - F(z+t)\big) \mathrm{d}t \\
&\le \int_0^{\infty} \big(1 - F(t)\big) \mathrm{d}t \quad \text{since } F(t+z) \ge F(t) \text{ for } z \ge 0, \\
&= \mu < \infty.
\end{aligned}
$$

Hence,

$$
\lim_{t \to \infty} H(z,t) = \frac{1}{\mu} \int_0^{\infty} (1 - F(z+t)) \mathrm{d}t = \frac{1}{\mu} \int_z^{\infty} (1 - F(u)) \mathrm{d}u.
$$

Therefore, by equation (37),

$$\Pr\{W \leq z\} = 1 - \frac{1}{\mu}\int_z^\infty \big(1 - F(u)\big)\mathrm{d}u = \frac{1}{\mu}\int_0^z \big(1 - F(u)\big)\mathrm{d}u.$$

- So,

$$\begin{aligned}
\mathbb{E}[W] &= \int_0^\infty z \,\mathrm{d}\Pr\{W \leq z\} \\
&= \frac{1}{\mu}\int_0^\infty z[1 - F(z)]\mathrm{d}z \\
&= \frac{1}{\mu}\int_0^\infty z\left[\int_z^\infty 1\mathrm{d}F(u)\right]\mathrm{d}z \\
&= \frac{1}{\mu}\int_0^\infty \left[\int_0^u z\mathrm{d}z\right]\mathrm{d}F(u) \\
&= \frac{1}{\mu}\int_0^\infty \frac{u^2}{2}\mathrm{d}F(u) \\
&= \frac{1}{2\mu}\int_0^\infty u^2\mathrm{d}F(u) \\
&= \frac{1}{2\mu}\left[\mu^2 + \sigma^2\right],
\end{aligned}$$

where σ^2 is the variance of the waiting time distribution.

Hence, we have shown that the expected time to the next renewal is given by $\dfrac{\mu^2 + \sigma^2}{2\mu}$.

If the variance, σ^2, is large, it is possible for this to be bigger than μ, so that the average wait until the next arrival can be greater than the average time between arrivals.

This is known as the bus paradox, which is caused by the fact that an arbitrary time point is more likely to fall into a large interval.