Test your understanding

Without looking at your notes, answer the following questions:

1. What is a *regular partition*? $\mathcal{P} = \{x_0, x_1, \ldots, x_N\}$
$\Delta x_i = (b-a)/N$.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Define $U(f)$. $U(f) = \inf\{U(f,\mathcal{P}) / \mathcal{P} \text{ part. of } [a,b]\}$.

3. True or False? Let $P = \{0, 0.2, 0.34, 0.51, 0.76, 0.9, 1\}$ and $Q = \{0, 0.1, 0.34, 0.8, 1\}$ be partitions of $[0,1]$. Then there is a bounded function f such that

$$L(f,P) = 5 \quad \text{and} \quad U(f,Q) = 4.$$

$$L(f,P) \leq U(f,Q)$$

4. Fill in the blanks to write down a mathematically correct statement:

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then f is not integrable $\iff$ there exists $\epsilon > 0$ such that $U(f,\mathcal{P}_\varepsilon) - L(f,\mathcal{P}_\varepsilon) \geq \epsilon$ for all partitions $\mathcal{P}_\varepsilon$ of $[a,b]$.

Last time:

Th^m 5.4: Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded. Then f is int. $\Longleftrightarrow \exists$ seq. of partitions $(\mathcal{P}_n)$ of $[a,b]$ s.th. $U(f, \mathcal{P}_n) \longrightarrow I$ & $L(f, \mathcal{P}_n) \longrightarrow I$ for some real number I. In this case $I = \int_a^b f(x)dx$.

Pf: $(\Leftarrow)$. last time

$(\Rightarrow)$ Suppose f is int. For every $n = 1, 2, 3, \ldots$

$\exists$ part. $\mathcal{P}_n'$ s.th. $U(f, \mathcal{P}_n') < U(f) + \frac{1}{n}$

$\exists$ part. $\mathcal{P}_n''$ s.th. $L(f, \mathcal{P}_n'') > L(f) - \frac{1}{n}$

Then $\mathcal{P}_n = \mathcal{P}_n' \cup \mathcal{P}_n''$ is a refinement of $\mathcal{P}_n'$, $\mathcal{P}_n''$

$$\therefore \quad U(f, \mathcal{P}_n) \leq U(f, \mathcal{P}_n') < U(f) + \frac{1}{n}$$

$$\& \quad L(f) - \frac{1}{n} < L(f, \mathcal{P}_n'') \leq L(f, \mathcal{P}_n)$$

$$\therefore \quad \underset{\downarrow \atop L(f)}{L(f) - \frac{1}{n}} < L(f, \mathcal{P}_n) \leq U(f, \mathcal{P}_n) < \underset{\downarrow \atop U(f)}{U(f) + \frac{1}{n}}.$$

f int. $\Longleftrightarrow U(f) = L(f)$.

$$\therefore \text{(Squeeze } Th^m\text{)} \quad U(f, \mathcal{P}_n) \longrightarrow U(f) = \int_a^b f(x)dx$$

$$L(f, \mathcal{P}_n) \longrightarrow L(f) = \int_a^b f(x)dx.$$

Th^m 5.5: Suppose $f, g : [a,b] \longrightarrow \mathbb{R}$ are bounded. If f, g int then

(i) $cf : [a,b] \longrightarrow \mathbb{R}$ is int $\forall c \in \mathbb{R}$.

& $\int_a^b cf(x)dx = c \int_a^b f(x)dx$ — exercise.

(ii) $f+g : [a,b]$ is int. &

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(iii) $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

(iv) if $f(x) \leq g(x)$ for all $x \in [a,b]$ then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

$h(x) \geq 0 \ \forall x$
$\Rightarrow \int_a^b h(x)\,dx \geq 0$.

(v) $|f| : [a,b] \to \mathbb{R}$ is int &
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Pf (ii): f int. $\Leftrightarrow$ $\exists$ seq. partitions (P_n^f)
s.th $U(f, P_n^f) \longrightarrow \int_a^b f(x)\,dx$
& $L(f, P_n^f) \longrightarrow \int_a^b f(x)\,dx$.

g int $\Rightarrow$ $\exists$ seq. partitions (P_n^g)
s.th. $U(g, P_n^g) \longrightarrow \int_a^b g(x)\,dx$
$L(g, P_n^g) \longrightarrow \int_a^b g(x)\,dx$ (Th^m 5.4).

Let $P_n = P_n^f \cup P_n^g$. Then.

$$\begin{array}{ccccccc} L(f, P_n^f) & \leq & L(f, P_n) & \leq & U(f, P_n) & \leq & U(f, P_n^f) \\ \downarrow & & & & & & \downarrow \\ \int_a^b f(x)\,dx & & & & & & \int_a^b f(x)\,dx. \end{array}$$

(Squeeze Th^m) $\Rightarrow$ $U(f, P_n) \longrightarrow \int_a^b f(x)\,dx$
& $L(f, P_n) \longrightarrow \int_a^b f(x)\,dx$.

Similarly, $U(g, P_n) \longrightarrow \int_a^b g(x)\,dx$
$L(g, P_n) \longrightarrow \int_a^b g(x)\,dx$.

$\left(\text{Aim: show } U(f+g, P_n) \to \int_a^b f + \int_a^b g \right.$

$\left. \quad \& \ L(f+g, P_n) \to \int_a^b f + \int_a^b g. \right)$.

$$\sum_{i=1}^N (m_i(f)+m_i(g))\Delta x_i \le \sum_{i=1}^N m_i(f+g)\Delta x_i \le \sum_{i=1}^N M_i(f+g)\Delta x_i \le \sum_{i=1}^N (M_i(f)+M_i(g))\Delta x_i$$

Conclude:

$$L(f,P_n)+L(g,P_n) \le L(f+g,P_n) \le U(f+g,P_n) \le U(f,P_n)+U(g,P_n).$$

$$L(f,P_n)+L(g,P_n) \to \int_a^b f + \int_a^b g \qquad U(f,P_n)+U(g,P_n) \to \int_a^b f + \int_a^b g$$

Squeeze $Th^m \Rightarrow L(f+g, P_n) \to \int_a^b f + \int_a^b g$

$U(f+g, P_n) \to \int_a^b f + \int_a^b g$.

Th^m 5.4 $\Rightarrow f+g$ is int. & $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(v) $|f|$ int & $\left|\int_a^b f\right| \le \int_a^b |f|$.

Suppose $|f|$ int. Then

$$-|f(x)| \le f(x) \le |f(x)| \quad \forall x \in [a,b].$$

$\therefore$ (i) & (iv) $\Rightarrow$

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$$

$$\Rightarrow \left|\int_a^b f\right| \le \int_a^b |f|.$$

Let $\varepsilon > 0$. (We'll show $\exists$ part. P_ε s.th.

$U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \varepsilon$).

Since f is int, $\exists$ part. P_ε s.th.

$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

Let $I = [x_{i-1}, x_i]$ be the i^{th} sub-interval of P_ε.

We'll show that

$$\sup_I |f| - \inf_I |f| \leq \sup_I f - \inf_I f.$$

(i.e. $M_i(|f|) - m_i|f| \leq M_i(f) - m_i(f)$).

$x_1, x_2 \in I$.

$$|f(x_2)| - |f(x_1)| \leq |f(x_2) - f(x_1)|.$$

$$= \max\{f(x_2) - f(x_1), f(x_1) - f(x_2)\}.$$

$$\leq \sup_I f - \inf_I f$$

$\therefore$ $|f(x_2)| - |f(x_1)| \leq \sup_I f - \inf_I f \quad \forall x_1, x_2 \in I.$

$\therefore$ $\sup_I |f| - |f(x_1)| \leq \sup_I f - \inf_I f. \quad \forall x_1 \in I.$

$\therefore$ $\sup_I |f| - \inf_I |f| \leq \sup_I f - \inf_I f.$

i.e. $M_i|f| - m_i|f| \leq M_i(f) - m_i(f).$

multiply by $\Delta x_i > 0$ & sum from $i = 1, .., N$

$$\Rightarrow U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon)$$
$$< \varepsilon$$

$\therefore$ $|f|$ int.

<u>Note</u>: f int $\Rightarrow$ $|f|$ int.

f int. $\not\Leftarrow$ $|f|$ int. eg $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$

f not. int but $|f|$ int.

$\underline{\text{Th}^{\text{m}}\ 5.6}$: Suppose $f : [a,b] \to \mathbb{R}$. is bounded.

Then f is int. on $[a,b] \iff \forall\, c \in [a,b]$, $f|_{[a,c]}$ is int on $[a,c]$ & $f|_{[c,b]}$ is int on $[c,b]$

Moreover, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

[Define: $\int_a^a f(x)dx = 0$, if $a<b$ define $\int_b^a f(x)dx = -\int_a^b f(x)dx$].