Lecture 25: Point processes and renewal processes

Concepts checklist

At the end of this lecture, you should be able to:

- *define* a stationary Poisson process; and,
- *define* a Renewal Process.

1870 – Seidel considered the occurrence of thunderstorms.

1889 – Von Bortkiewicz gave a systematic account of phenomena that fit the Poisson distribution; the most famous example being that of the number of deaths from horse kicks in the Prussian cavalry.

1943 – Palm systematically described (as a generalisation of the Poisson process) the input to a service system. *A powerful concept:* the notion of a regeneration point or a time instant at which the system reverts to a specified state with the property that the future evolution is independent of how that state was reached.

⇒ Poisson process is characterised by the property that every instant is a regeneration point, which is different to other processes where the commencement of a new inter-event time is the only regeneration point.

Two distributions are necessary for describing a stationary point process:

1. the distribution of time to the next event from an arbitrary time, and
2. the distribution of time to the next event from an arbitrary event of the process.

Ideal examples of point processes: the Poisson process and the renewal processes.

By a point process, we refer to some method of randomly allocating points to intervals of the real line (this can easily be extended to hyper-rectangles in n-dimensional Euclidean space).

A basic approach to point processes: to consider counting the number of events in intervals or regions of various types.

Definition 20. *The stationary Poisson process on the line is completely defined by the following equation,*

$$\Pr\left(N(a_i, b_i] = n_i,\ i = 1, \dots, k\right) = \prod_{i=1}^{k} \frac{[\lambda(b_i - a_i)]^{n_i}}{n_i!} e^{-\lambda(b_i - a_i)}. \tag{30}$$

where $N(a_i, b_i]$ is the number of events in the open interval $(a_i, b_i]$ for $a_i < b_i \leq a_{i+1}$.

Properties:

1. the number of events in each finite interval $(a_i, b_i]$ is Poisson distributed.
2. the number of points in disjoint intervals are independent random variables.
3. the process is stationary as the distributions only depend on the lengths $b_i - a_i$ of the intervals, not the value of the actual end-points.

By (30), the mean $M(a,b]$ and variance $V(a,b]$ of the number of events in $(a,b]$ are given by

$$M(a,b] = \lambda(b-a) = V(a,b]. \tag{31}$$

The parameter λ can therefore be interpreted as the mean rate or mean density of points of the process.

Furthermore, $\Pr(N(0,\tau] = 0) = e^{-\lambda\tau}$, which is the probability of finding no points in the interval of length τ. This probability may also be interpreted as

1. the probability that the random interval extending from the origin to the first point to the right has a length exceeding τ, or
2. the survivor function for the length of this interval,

and shows that the interval under consideration has an exponential distribution.

From stationarity, the same result applies to

- the length of the interval to the next point immediately to the right from any arbitrarily chosen origin, and
- the length of the interval to the first point immediately to the left of any arbitrarily chosen origin.

In queueing terms, these are called the *forward and backward recurrence times.*

Hence, for a Poisson process the forward and backward recurrence times have an exponential distribution with parameter λ. Using the independence property, this distribution extends to the distribution of time between two consecutive points in the process, and it may be shown that successive intervals are independently distributed with exponential distributions.

Let t_k be the time from the origin $t_0 = 0$ until the kth point of the point process to the right of the origin, then the following events are equivalent: $\{t_k \geq x\}$ and $\{N(0,x] \leq k\}$, so that their respective probabilities are the same, with the probability of the latter being given by equation (30). That is,

$$\Pr(t_k > x) = \Pr(N(0,x] < k) = \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x},$$

$$\text{or} \quad \Pr(t_k \leq x) = 1 - \Pr(N(0,x] < k) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x}.$$

Differentiating this expression we get the corresponding density

$$\begin{aligned} f_k(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\Pr(t_k \le x) \\ &= \lambda \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} - \lambda \sum_{j=0}^{k-1} \frac{j(\lambda x)^{j-1}}{j!} e^{-\lambda x} \\ &= \lambda \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} - \lambda \sum_{j=0}^{k-1} \frac{(\lambda x)^{j-1}}{(j-1)!} e^{-\lambda x} \\ &= \lambda \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} - \lambda \sum_{j=0}^{k-2} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \\ &= \lambda \frac{(\lambda x)^{k-1}}{(k-1)!} e^{-\lambda x}. \end{aligned}$$

This is the Erlang density, which is therefore the sum of the lengths of the k random intervals $(t_0, t_1]$, $(t_1, t_2]$, $\ldots$, $(t_{k-1}, t_k]$, which are independently and identically distributed according to the exponential distribution with parameter λ. This gives an indirect proof that the result of the sum of k independent exponential random variables has the Erlang distribution.

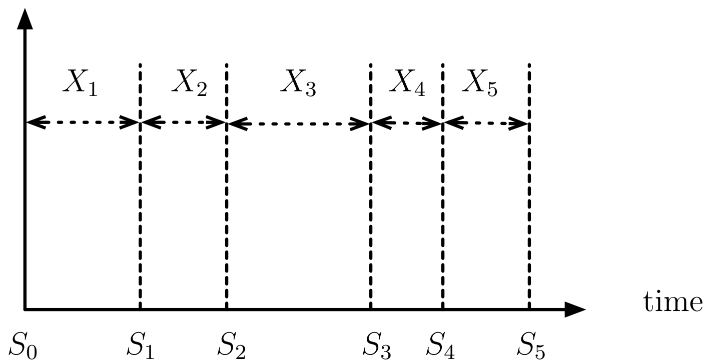
We have defined and analysed the Poisson process, which assumes that points arrive in a memoryless stream. This assumption is reasonable in cases where a large population all have a small chance of generating a point. For example, in communications, this might be requesting a particular route in a communications network. No matter how many requests have accrued in the recent past there are still so many potential callers in the population that future calls arrive independently of the past history.

However, the Poisson process is not ideal for many cases, where the inter-event time distribution is not exponential.

Renewal Processes

The class of renewal processes can be viewed as a natural generalisation of the Poisson process:

- In a Poisson process, the state moves from i to $i+1$ at a jump and the times X_i between jumps are independent and exponentially distributed.
- A renewal process is similar, as the times X_i between jumps are independent and identically distributed, but they are not necessarily exponentially distributed.



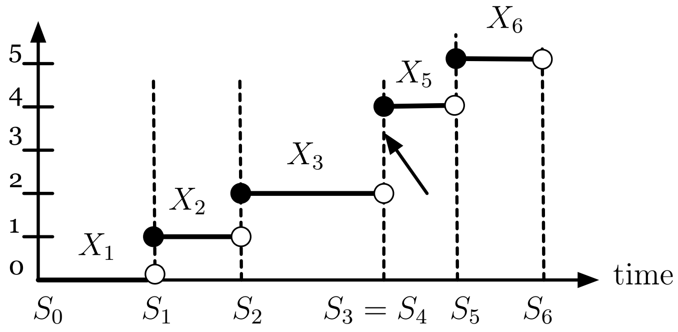


Figure 5: Counting and waiting time processes

Definition 21 (Renewal Processes.). *Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) non-negative random variables with*

$$\Pr\{X_i \le x\} = F(x) \quad \text{if } x \ge 0.$$

We assume that $F(0) < 1$ and that $\mathbb{E}[X_i] = \mu < \infty$. Let

$$\begin{aligned} S_n &= X_1 + X_2 + \cdots + X_n \quad (\textit{with } S_0 = 0) \\ &= \textit{waiting time to the } n\textit{th event} \\ \textit{and} \quad N(t) &= \sup\{n : S_n \le t\} \\ &= \textit{number of events before time } t. \end{aligned}$$

Then, $\{N(t), t > 0\}$ is called the counting process and $\{S_n, n \ge 1\}$ is called the waiting time process. The process $\{N(t)\}$ is referred to as a renewal process.

Note: If $F(0) > 0$, then two events can occur simultaneously. This is reflected in Figure (5).

Examples of renewal processes.

1. Poisson process (X_i's are exponential).
2. Replacement models: where a component with lifetime distribution $F(x)$ is replaced every time it breaks down (e.g., lightbulbs, or Professors!). Then the number of replacements before time t gives the counting process and the time until the nth replacement gives the waiting time process.
3. Breakdown and repair models: where the event times could be either
   (a) times of breakdown
   (b) times of repair.

   In both cases, the inter-event time random variable is the sum of a breakdown random variable and a repair random variable.
4. Sometimes renewal processes can be "embedded" in more complex stochastic processes. For example, consider a queueing system with exponential inter-arrival times, but general (non-memoryless) service times. Then, consider the points at which successive idle periods begin. The system is memoryless at these points and the times between them are iid, even though this distribution may be very complicated. These are the regeneration points mentioned at the start of the lecture.