Optimal Functions and Nanomechanics III

APP MTH 3022/7106

Barry Cox

Lecture 26

Last lecture

- Extended the natural boundary conditions to free x and y
- We found the additional constraint

$$\Big[p\,\delta y - H\,\delta x\Big]_{x_0}^{x_1} = 0,$$

where $p = f_{y'}$ and $H = y' f_{y'} - f$.

- If the end points are independent then the conditions separate to

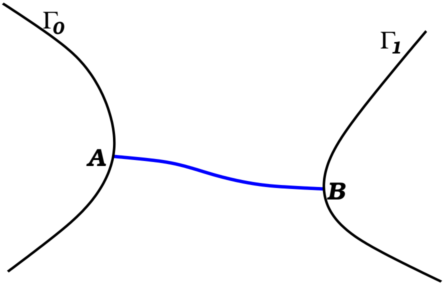
$$\Big[p\,\delta y - H\,\delta x\Big]_{x_0} = 0, \quad \Big[p\,\delta y - H\,\delta x\Big]_{x_1} = 0,$$

Traversals

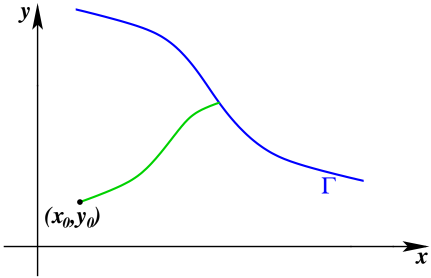
When we consider an extremal joining a curve to a point (or two curves) then we often call the extremal a transversal. The free-end-point condition simplifies in many such cases, for instance, in many situations we look for a transversal that joins the proscribed curve at right angles.

Transversal examples

Find the shortest path between two curves Γ_0 and Γ_1.



Find the shortest path from a point (x_0, y_0) to a curve Γ.



Transversals

Specify the curve Γ parametrically by $(x_\Gamma(\xi), y_\Gamma(\xi))$, then the end-points must lie on this line, and so we can write

$$\delta x = \delta\xi \, \frac{dx_\Gamma}{d\xi}, \quad \delta y = \delta\xi \, \frac{dy_\Gamma}{d\xi}$$

and then the condition is

$$\Big[p\delta y - H\delta x\Big]_{x_1} = 0$$

$$\left[p\frac{dy_\Gamma}{d\xi} - H\frac{dx_\Gamma}{d\xi}\right]_{x_1} = 0$$

Transversality condition

Note that the vector $\left(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}\right)$ is a tangent to the curve Γ.

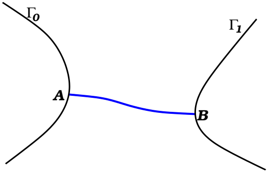
The **Transversality Condition** is that the vector $\boldsymbol{v} = (-H, p)$ is orthogonal to the tangent vector.

That is

$$\left(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}\right) \cdot (-H, p) = p\frac{dy_\Gamma}{d\xi} - H\frac{dx_\Gamma}{d\xi} = 0.$$

Transversals

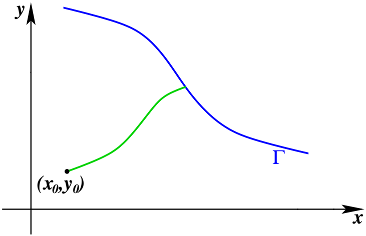
Find the shortest path between two curves Γ_0 and Γ_1.



$$\left[p\frac{dy_{\Gamma_0}}{d\xi} - H\frac{dx_{\Gamma_0}}{d\xi}\right]_{x_0} = 0,$$

$$\left[p\frac{dy_{\Gamma_1}}{d\xi} - H\frac{dx_{\Gamma_1}}{d\xi}\right]_{x_1} = 0.$$

Find the shortest path from a point (x_0, y_0) to a curve Γ.



$$\left[p\frac{dy_{\Gamma}}{d\xi} - H\frac{dx_{\Gamma}}{d\xi}\right]_{x_1} = 0.$$

Example 1

Shortest path from the origin to a curve $\boldsymbol{r}_\Gamma = (x_\Gamma(\xi), y_\Gamma(\xi))$. The path length is given by

$$F\{y\} = \int_0^{x_1} \sqrt{1 + y'^2}\, dx.$$

Then

$$p = \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$H = y' \frac{\partial f}{\partial y'} - f = \frac{y'^2}{\sqrt{1+y'^2}} - \sqrt{1+y'^2} = \frac{-1}{\sqrt{1+y'^2}}$$

Example 1

Thus the transversality condition becomes

$$\left[p\frac{dy_\Gamma}{d\xi} - H\frac{dx_\Gamma}{d\xi}\right]_{x_1} = 0$$

$$\left[\frac{y'}{\sqrt{1+y'^2}}\frac{dy_\Gamma}{d\xi} + \frac{1}{\sqrt{1+y'^2}}\frac{dx_\Gamma}{d\xi}\right]_{x_1} = 0.$$

Now $\sqrt{1+y'^2} \neq 0$, so we can multiply through by $\sqrt{1+y'^2}$ to give

$$\left[\frac{dx_\Gamma}{d\xi} + y'\frac{dy_\Gamma}{d\xi}\right]_{x_1} = 0$$

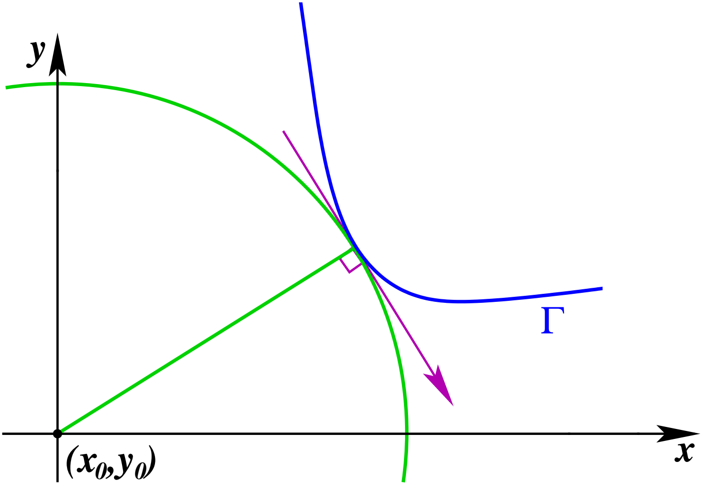
Alternatively stated, $\left(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}\right) \cdot (1, y') = 0$

Example 1

We can interpret $\left(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}\right) \cdot (1, y') = 0$ geometrically

- the condition means that the tangent to the extremal must be orthogonal to the tangent to the curve Γ where they connect.
- Euler-Lagrange equations still imply $y(x)$ will be straight line
- this makes perfect sense!
  - find the distance of curve Γ from the origin.
  - do this by creating expanding circles, and the one that touched the curve would give us the distance.
  - it would touch so the circle was tangent
  - the (straight line) radius would be perpendicular to the tangent.

Example 1



Example 1

Sometimes, there will be many possible solutions, for instance if the curve Γ was a circle around the origin!.

But now we know how to find them, it would be easy.

Example 2

Consider the general functional

$$F\{y\} = \int_0^{x_1} K(x,y)\sqrt{1+y'^2}\,dx,$$

for which we wish to find stationary paths between two curves $\boldsymbol{r}_{\Gamma_0} = (x_{\Gamma_0}(\xi), y_{\Gamma_0}(\xi))$, and $\boldsymbol{r}_{\Gamma_1} = (x_{\Gamma_1}(\xi), y_{\Gamma_1}(\xi))$.
Then

$$p = \frac{\partial f}{\partial y'} = \frac{y' K(x,y)}{\sqrt{1+y'^2}},$$

$$H = y'\frac{\partial f}{\partial y'} - f = \frac{y'^2 K(x,y)}{\sqrt{1+y'^2}} - K(x,y)\sqrt{1+y'^2} = \frac{-K(x,y)}{\sqrt{1+y'^2}}.$$

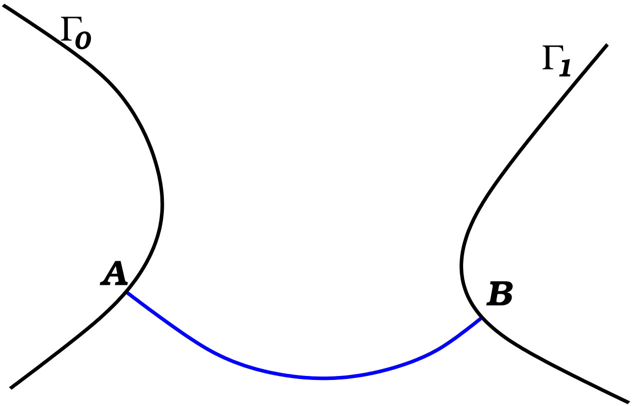
Example 2

Once again the transversality conditions at each end will revert to the extremal being orthogonal to the tangent to the curves at either end.

However, in this case, the curve joining the two could be distorted by the factor of $K(x, y)$ so that it is no longer a straight line. Its shape can be determined from the Euler-Lagrange equations.

Example 3

Find the shape of a fixed length chain hanging between two curves (similar to catenary problem, but end-points can move freely along two curves).



Example 3

This problem is a special case of Example 2, and so

- from transversality constraints that the chain will join the two curves at a right angle.
- Euler-Lagrange equations imply the curve will be a catenary (see earlier lectures for the derivation of the catenary)

$$y(x) + \lambda = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$

- we need simply to use the perpendicularity (and fixed length) constraints to derive the values of the constant of integration, and the Lagrange multiplier.

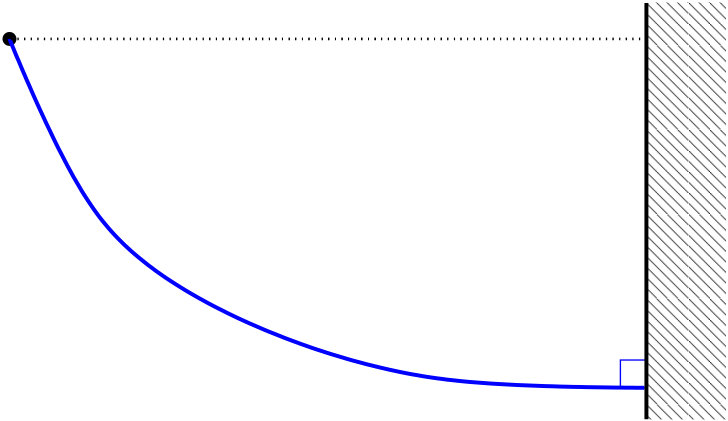
Example 4

A variant of the Brachistochrone: find the curve of fastest descent from a point to line.

$$T\{y\} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2E/m - 2gy(x)}}\,dx = \int_{x_0}^{x_1} K(y)\sqrt{1+y'^2}\,dx.$$

- Euler-Lagrange equations show that the curve must be a cycloid.
- Transversality constraints (see Example 2) show that, at the point of contact, the extremal will be perpendicular to the line.

Example 4



Example 5

Shortest path from a point to a surface.

- E-L equations show that the curve must be a straight line
- Transversality constraints show that, at the point of contact, the extremal will be normal to the surface.
  (see tutorials for an example that shows this).