Examination in School of Mathematical Sciences

Semester 2, 2015

107352 APP MTH 3022 Optimal Functions and Nano III

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 50

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Formulae sheets are provided.
- Calculators without remote communications or CAS capability are allowed.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. (a) Determine the extremal of the functional

$$F\{y\} = \int_0^{\pi/4} \left(y^2 + y'^2\right) dx,$$

with fixed end-points $y(0) = 0$ and $y(\pi/4) = \sqrt{2}$.

(b) Find the general form of the extremal for the functional

$$G\{y\} = \int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx,$$

subject to fixed end-point boundary conditions. Your final answer will be in terms of arbitrary constants.

[4+4 marks]

2. (a) Using the integral definition of the gamma function, show that

$$\Gamma(1/2) = \sqrt{\pi}.$$

(b) The Legendre polynomials $P_n(x)$ may be defined in terms of hypergeometric functions (F given in the formula sheet) by the relation

$$P_n(x) = F(-n, n+1; 1; (1-x)/2).$$

Use this relation to produce the Legendre polynomial of second order $P_2(x)$ as a polynomial in x.

[6+4 marks]

3. Consider the functional

$$J\{x, y\} = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt,$$

where x and y are functions of the independent variable t, and dots denote differentiation with respect to t.

(a) Use a system of Euler-Lagrange equations to determine the shape of extremal curves for $J\{x, y\}$.

(b) Use your answer from part (a) and the traversality condition to find the curve which has the shortest path from the origin to the line $y = -2 - x$.

(c) Make a sketch of your solution to part (b).

[6+4+2 marks]

4. Consider the functional

$$K\{y\} = \int_{-1}^{2} y'(x^2 y' - 1)\, dx,$$

subject to the end-point constraints $y(-1) = 1$, $y(2) = 4$.

(a) Considering possible solutions $y \in C^2$, write down the Euler-Lagrange equation for this problem and find an expression for y'(x).

(b) Find the general solution of the Euler-Lagrange problem for part (a).

(c) By considering the end-point conditions, determine the particular y which is an extremal for this problem.

(d) Sketch your solution from part (c) for $-1 \leqslant x \leqslant 2$.

(e) Have you found a continuous curve in the plane, connecting $(-1, 1)$ to $(2, 4)$? Discuss.

[2+2+2+2+2 marks]

5. Consider the functional

$$L\{y\} = \int_{0}^{1} (x^3 y''^2 + x y^2 + 5xy)\, dx, \quad \text{subject to} \quad y(1) = y'(1) = 0.$$

Using Ritz's method with a trial function of the form

$$\hat{y} = c_0 \phi_0, \quad \text{where} \quad \phi_n = x^n \left(\frac{x}{2} - 1\right)^2,$$

find an approximate extremal for $L\{y\}$.

[10 marks]

Formula Sheet, Special Functions

Gamma function, definition	$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \text{for } \Re(z) > 0.$
Gamma function, duplication	$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2).$
Beta function, definition	$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \text{for } \Re(x), \Re(y) > 0.$
Beta function, gamma relation	$B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$
Pochhammer symbol, definition	$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}.$
Hypergeometric function, series	$F(a,b;c;z) = \sum_{n=0}^{\infty} \dfrac{(a)_n(b)_n}{(c)_n n!} z^n.$
Hypergeometric function, integral	$F(a,b;c;z) = \dfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt.$
Hypergeometric function, derivative	$\dfrac{d^n}{dz^n} F(a,b;c;z) = \dfrac{(a)_n(b)_n}{(c)_n} F(a+n, b+n; c+n; z).$
Elliptic integral, first kind	$F(\varphi, k) = \int_0^\varphi \dfrac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}.$
Elliptic integral, second kind	$E(\varphi, k) = \int_0^\varphi \sqrt{1-k^2\sin^2\vartheta}\, d\vartheta.$
Complete elliptic integrals	$K(k) = F\left(\dfrac{\pi}{2}, k\right), \quad E(k) = E\left(\dfrac{\pi}{2}, k\right).$
Lennard-Jones potential	$\Phi(\rho) = -\dfrac{A}{\rho^6} + \dfrac{B}{\rho^{12}}.$

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Formula Sheet, Variational

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0, y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0} \qquad \text{The Euler-Lagrange equation}$$

Theorem 2.3.1: Let J be a functional of the form

$$J\{y\} = \int_{x_1}^{x_2} f(y, y')dx$$

and define the function H by

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f(y, y')$$

Then H is constant along any extremal of y.

Generalisation: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\, dx,$$

where f has continuous partial derivatives of second order with respect to $x, y, y', \ldots, y^{(n)}$, and $x_0 < x_1$, and the values of $y, y', \ldots, y^{(n-1)}$ are fixed at the end-points, then the extremals satisfy the Euler-Poisson equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} = 0$$

Natural boundary condition: When we extend the theory to allow a free x and y, we find the additional constraint

$$\Big[p\,\delta y - H\,\delta x\Big]_{x_0}^{x_1} = 0,$$

where $p = f_{y'}$ and $H = y'f_{y'} - f$.

Weierstrass-Erdman corner conditions: For a broken extremal

$$p\Big|_{x^{\star}-} = p\Big|_{x^{\star}+}, \quad H\Big|_{x^{\star}-} = H\Big|_{x^{\star}+},$$

must hold at any "corner".