APP MTH 3001 Applied Probability III
Class Exercise 5, 2018
Due: 3pm, Friday 1 June 2018, via Canvas (PDF only)

1. In a random walk on the non-negative integers starting at the origin, the size X_n of the n^{th} step, $n \geq 1$ has distribution
$$\mathrm{P}\left(X_n = j\right) = \frac{e^{-1}}{j!}, \quad j \geq 0.$$
Define
$$\begin{aligned} S_0 &= 0, \\ S_n &= \sum_{i=1}^{n} X_i, \quad n \geq 1, \\ Y_n &= S_n - n, \quad n \geq 0. \end{aligned}$$
Show that $\{Y_n : n \in \mathbb{N}\}$ is a martingale wrt $\{X_n : n \in \mathbb{N}\}$.

2. If $\{X_n : n \in \mathbb{N}\}$ is a martingale wrt to itself, show that for any non-negative integers, $k \leq \ell < m$, the difference $X_m - X_\ell$ is uncorrelated with X_k. That is, show that
$$E\left[\left(X_m - X_\ell\right) X_k\right] = 0.$$

3. Let $\{X_n : n \in \mathbb{N}\}$ be a DTMC on the state space $\mathcal{S}$ with one-step transition probability matrix $\mathbb{P} = (p_{ij})$ and let $f : \mathcal{S} \to \mathbb{R}$ be a bounded function. Then, define
$$M_n = \sum_{m=1}^{n} f(X_m) - \sum_{m=0}^{n-1} \sum_{i \in \mathcal{S}} p_{X_m,i} f(i).$$
Show that $\{M_n : n \in \mathbb{N}\}$ is a martingale wrt $\{X_n : n \in \mathbb{N}\}$.

4. If $\{X_n : n \in \mathbb{N}\}$ is a sub-martingale wrt $\{Y_n : n \in \mathbb{N}\}$ and $Z \geq 0$ is a (measurable) function of $Y_0, \ldots, Y_n$, show that
$$E\left[X_n Z\right] \leq E\left[X_{n+1} Z\right]$$