APP MTH 3001 Applied Probability III
Class Exercise 1, 2018

To be submitted electronically via Canvas (PDF only!) by 3pm on the 9^{th} of March, 2018.

This is a revision assignment based on assumed knowledge.

1. Prove the law of total probability, which states that if the events $B_1, B_2, \ldots, B_n$ constitute a partition of Ω, then

$$\mathrm{P}(A) = \sum_{i=1}^{n} \mathrm{P}(A \mid B_i)\,\mathrm{P}(B_i).$$

Hint: recall that $\mathrm{P}(A) = \mathrm{P}\left(\bigcup_{i=1}^{n}(A \cap B_i)\right)$.

2. (a) For events $A_1, A_2, \ldots, A_n$, prove that

$$\begin{aligned}&\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_n)\\ &\quad= \mathrm{P}(A_1)\,\mathrm{P}(A_2 \mid A_1)\,\mathrm{P}(A_3 \mid A_1 \cap A_2) \cdots \mathrm{P}(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})\ ,\end{aligned}$$

whenever $\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$.

(b) Use the result of part (a) to derive the expression

$$\mathrm{P}(A \cap B \mid C) = \mathrm{P}(A \mid B \cap C)\,\mathrm{P}(B \mid C)$$

for events A, B and C

> Hint: Use $n = 3$ in part (a), and determine which correspondences of A, B and C with A_1, A_2 and A_3 give the required result).

Note that this expression will play a fundamental role in several derivations and proofs in this course and is therefore worth understanding well.

3. If Y is a binomial random variable having parameters n (the number of independent trials) and p (the probability of success on any single trial), then write Y as the sum of appropriate indicator random variables and find $E[Y]$.

Hint: See slide 26 of Section 00 (the assumed knowledge) .

4. A coin is tossed repeatedly. For each toss, the probability that the coin comes up heads is $\frac{1}{2}$ (the coin is fair). Assume that the outcome of each toss is independent of the previous outcomes. Let E_n be the event that no heads come up in the first n tosses.

   (a) Find $\mathrm{P}(E_n)$

   (b) How can we interpret the quantity

   $$\lim_{n\to\infty} \mathrm{P}(E_n),$$

   and what is its value ?

   (c) Show that a head is bound to turn up eventually. That is, you need to show that

   $$\mathrm{P}(\text{head turns up eventually}) = 1,$$

   Hint: Use (b).

   (d) Harder question: Show that any given finite sequence of heads and tails occurs eventually with probability one.

   Hint: Consider a specific sequence S of heads and tails with length K. Now consider N disjoint lots of sequences of length K (the result of a total of NK tosses). To help visualize this description, draw a diagram consisting of NK "slots" (where each slot contains the outcome of a toss), and divide the slots into N lots of K slots. Each of these N disjoint sequences has probability 2^{-K} of being the particular sequence S that we are interested in. With a little thought, we recognize that the event

   $$\{\text{one of the } N \text{ groups is } S\} \subset \{S \text{ occurs somewhere in the } NK \text{ tosses}\},$$

   that is, the event that one of the N groups is S is a (proper) subset of the event that S occurs somewhere in the NK tosses (note: in the latter event, the sequence S does not need to fall exactly within one of the N groups, and thus may overlap two of these groups). Use the fact that

   $$\mathrm{P}(S \text{ occurs somewhere in the } NK \text{ tosses}) \geq \mathrm{P}(\text{at least one of the } N \text{ groups is } S).$$