LECTURE 13

Cauchy sequences

Roughly speaking, a Cauchy sequence is a sequence in which all of the terms of the sequence eventually get close to one another. Intuitively, all of the terms of the sequence must get close to *something* and therefore one is lead to suspect that every Cauchy sequence is convergent. Later we shall see that this intuition is indeed true.

Definition 2.15: A sequence (a_n) is said to be *Cauchy* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m \geq N$ and $n \geq N$ then $|a_m - a_n| < \epsilon$.

Example: Let (a_n) be the sequence defined by $a_n = n/(n+1)$. We will show that (a_n) is a Cauchy sequence. Let $\epsilon > 0$. We have

$$\begin{aligned} |a_m - a_n| &= \left| \frac{m}{m+1} - \frac{n}{n+1} \right| \\ &= \left| \frac{m-n}{(m+1)(n+1)} \right| \\ &\leq \frac{m}{(m+1)(n+1)} + \frac{n}{(m+1)(n+1)} \end{aligned}$$

where in the last step we have used the triangle inequality. Notice that $m/(m+1) < 1$ and $n/(n+1) < 1$ for all natural numbers m and n. Therefore,

$$|a_m - a_n| < \frac{1}{n+1} + \frac{1}{m+1} < \frac{1}{n} + \frac{1}{m}.$$

To show that (a_n) is Cauchy, it suffices to show that there is an $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $1/m + 1/n < \epsilon$. The inequality $1/m + 1/n < \epsilon$ will be satisfied if $1/m < \epsilon/2$ and $1/n < \epsilon/2$ for example. Choose the natural number N large enough so that $1/N < \epsilon/2$, for example by choosing N larger than $2/\epsilon$. If $m \geq N$ and $n \geq N$ then $1/m \leq 1/N$ and $1/n \leq 1/N$. Hence, if $m, n \geq N$ then

$$|a_m - a_n| < \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that (a_n) is Cauchy.

Example: Consider the sequence (a_n) defined by $a_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. Observe that

$$|a_{n+1} - a_n| = \frac{1}{n+1}.$$

Therefore for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_{n+1} - a_n| < \epsilon$. In other words *consecutive* terms of the sequence become arbitrarily close to one another. This is *not* the same as having eventually all terms of the sequence becoming arbitrarily close to one another. Indeed, this sequence is not a Cauchy sequence. For example

$$|a_{2n} - a_n| = \frac{1}{2n} + \frac{1}{2n-1} + \cdots + \frac{1}{n+1} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

Note: For a sequence of real numbers (a_n), the following statements are equivalent:

(1) (a_n) is a Cauchy sequence,

(2) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|a_{n+k} - a_n| < \epsilon$ for all $k \in \mathbb{N}$.

To see the equivalence of these two statements, suppose that statement (1) is true, i.e. (a_n) is a Cauchy sequence. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n \geq N$ then $|a_m - a_n| < \epsilon$. In particular, taking $m = n + k$ for $k \in \mathbb{N}$, we see that $n \geq N \implies |a_{n+k} - a_n| < \epsilon$. Conversely, suppose that statement (2) is satisfied. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N \implies |a_{n+k} - a_n| < \epsilon$ for all $k \in \mathbb{N}$. If $m, n \geq N$ then either $m \leq n$ or $m > n$. If $m > n$ then $m = n + k$ for some $k \in \mathbb{N}$ and hence $|a_m - a_n| < \epsilon$. If $m \leq n$ then either $m < n$ in which case $n = m + k$ for some $k \in \mathbb{N}$ and hence $|a_m - a_n| < \epsilon$, or $m = n$ in which case $|a_m - a_n| = 0 < \epsilon$.

Proposition 2.16: Let (a_n) be a sequence of real numbers. If (a_n) is convergent then (a_n) is Cauchy.

Proof: Suppose $a_n \to L$. Let $\epsilon > 0$. By the triangle inequality,

$$|a_m - a_n| \leq |a_m - L| + |a_n - L|.$$

Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L| < \epsilon/2$. Therefore, if $m \geq N$ and $n \geq N$ then

$$|a_m - a_n| \leq |a_m - L| + |a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that (a_n) is Cauchy. ■

Proposition 2.17: Let (a_n) be a sequence of real numbers. If (a_n) is Cauchy then (a_n) is bounded.

Proof: Since (a_n) is Cauchy, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|a_m - a_n| < 1$. In particular, $|a_n - a_N| < 1$ if $n \geq N$. Therefore, by the reversed triangle inequality, if $n \geq N$ then $|a_n| - |a_N| \leq |a_n - a_N| < 1$. Hence $n \geq N \implies |a_n| < |a_N| + 1$. If $n < N$ then $|a_n| \leq \max\{\,|a_1|, \ldots, |a_{N-1}|\,\}$. Therefore, for all $n \in \mathbb{N}$,

$$|a_n| \leq \max\{\,|a_1|, \ldots, |a_{N-1}|, |a_N| + 1\,\}$$

Hence (a_n) is bounded. ■

Proposition 2.18: Suppose (a_n) is a Cauchy sequence of real numbers. If (a_{n_k}) is a subsequence of (a_n) such that $a_{n_k} \to L$ then $a_n \to L$.

Proof: Let $\epsilon > 0$. Since $a_{n_k} \to L$ there exists $N_1 \in \mathbb{N}$ such that $k \geq N_1 \implies |a_{n_k} - L| < \epsilon/2$. Since (a_n) is Cauchy there exists $N_2 \in \mathbb{N}$ such that $m, n \geq N_2 \implies |a_m - a_n| < \epsilon/2$. Choose $k \geq \max\{N_1, N_2\}$. If $n \geq N_2$ then

$$|a_n - L| \leq |a_{n_k} - L| + |a_{n_k} - a_n|$$

by the triangle inequality. Since $k \geq N_1$, $|a_{n_k} - L| < \epsilon/2$. Since $k \geq N_2$, $n_k \geq N_2$ and hence $|a_{n_k} - a_n| < \epsilon/2$ since $n \geq N_2$. Therefore, if $n \geq N_2$ then

$$|a_n - L| \leq |a_{n_k} - L| + |a_{n_k} - a_n| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that $a_n \to L$. ■.

Theorem 2.19: If (a_n) is a Cauchy sequence of real numbers then (a_n) converges (in $\mathbb{R}$). Hence a sequence of real numbers is convergent if and only if it is Cauchy.

Proof: Since (a_n) is Cauchy, (a_n) is bounded. Therefore, (a_n) has a convergent subsequence (a_{n_k}) by the Bolzano-Weierstrass Theorem. Suppose $a_{n_k} \to L$. Then $a_n \to L$ by Proposition 2.18. ■

Note: The same statement is not true for sequences in $\mathbb{Q}$. The notion of Cauchy sequence makes sense in $\mathbb{Q}$, and more generally in any ordered field F. Similarly, the notion of convergent sequence makes sense in $\mathbb{Q}$ and more generally in any orderded field F. But it is not true that a Cauchy sequence in $\mathbb{Q}$ is a convergent sequence in $\mathbb{Q}$. For example, the sequence (a_n) defined recursively by $a_1 = 2$ and $a_{n+1} = (a_n + 2/a_n)/2$ is a Cauchy sequence in $\mathbb{Q}$, but it does not converge in $\mathbb{Q}$.

Countable and uncountable sets

Consider the sequence $0, 1, -1, 2, -2, 3, -3, \ldots$ Since the terms of this sequence belong to $\mathbb{Z}$, the sequence is defined by a function $f\colon \mathbb{N} \to \mathbb{Z}$. Notice that every integer appears exactly once as a term of this sequence. Therefore the function f is 1-1 and onto.

Definition: We say a set S is *countably infinite* if there exists a bijection $f\colon \mathbb{N} \to S$. We say S is *countable* if it is countably infinite or if it is *finite* in the sense that either $S = \emptyset$, or there is a bijection $S \to \{1, 2, \ldots, n\}$ for some natural number n. If S is not countable then we say that S is *uncountable.*

Lemma: Let S be a set. The following statements are equivalent.

(1) S is countable.

(2) S is empty or there exists an onto function $f\colon \mathbb{N} \to S$.

(3) There exists a 1-1 function $g\colon S \to \mathbb{N}$.

Proof: Suppose that (1) is true. If S is countably infinite then there exists a bijection $f\colon \mathbb{N} \to S$, in particular f is onto. If there is a bijection $f\colon S \to \{1, 2, \ldots, n\}$ then the function $g\colon \mathbb{N} \to S$ defined by $g(i) = f^{-1}(i)$ if $1 \le i \le n$ and $g(i) = f^{-1}(n)$ if $i > n$ is an onto function. Therefore (1) $\implies$ (2).

Suppose that (2) is true. If S is not empty let $f\colon \mathbb{N} \to S$ be an onto function. Fo each $s \in S$, choose $g(s) \in f^{-1}(s)$. This defines a function $g\colon S \to \mathbb{N}$. Suppose that $g(s_1) = g(s_2)$. Then $s_1 = f(g(s_1)) = f(g(s_2)) = s_2$. Hence g is 1-1. Therefore (2) $\implies$ (3).

Suppose that (3) is true. Suppose that S is not empty. Then $g(S)$ is not empty. Let n_1 be the minimum element of $g(S)$. If $g(S) \setminus \{n_1\}$ is not empty, let n_2 be the minimum element of $g(S) \setminus \{n_1\}$. If $g(S) \setminus \{n_1, n_2\}$ is not empty, let n_3 be the minimum element of $g(S) \setminus \{n_1, n_2\}$. Continue, in this way. Either this process terminates, in which case there is a bijection $S \to \{1, 2, \ldots, n\}$ for some natural number n, or it continues indefinitely, in which case we have defined a bijection $f\colon \mathbb{N} \to S$ which sends $f(k) = n_k$. Hence S is countable. ■

Remark: (2) of the Lemma above is the statement that either S is empty, or the elements of S can be listed as the terms of a sequence, i.e. $S = \{ s_n \mid n \in \mathbb{N} \}$.