OFN Assignment 1

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1. Maximise the volume of a rectangular prism with total surface area of $2m^2$ and total edge length of $12m$ Let x, y, z denote the sides of the prism. This gives the problem

$$\begin{aligned} &\max f = xyz \\ s.t. \quad &2xy + 2xz + 2yz = 2 \\ &4x + 4y + 4z = 12 \\ &x, y, z > 0 \end{aligned}$$

Using Lagrange multipliers the problem is:

$$\max \ h(x,y,z) = xyz + \lambda_1(xy + xz + yz - 1) + \lambda_2(x + y + z - 3)$$

Hence solve $\nabla h = \mathbf{0}$.

$$\frac{\partial h}{\partial x} = yz + \lambda_1(y+z) + \lambda_2 = 0 \tag{1}$$
$$\frac{\partial h}{\partial y} = xz + \lambda_1(x+z) + \lambda_2 = 0 \tag{2}$$
$$\frac{\partial h}{\partial z} = xy + \lambda_1(x+y) + \lambda_2 = 0 \tag{3}$$
$$\frac{\partial h}{\partial \lambda_1} = xy + xz + yz - 1 = 0 \tag{4}$$
$$\frac{\partial h}{\partial \lambda_2} = x + y + z - 3 = 0 \tag{5}$$

$(1) + (2) + (3) - (4)$:

$$\begin{aligned} \lambda_1(2y + 2x + 2z) + 3\lambda_2 - 1 = 0 \\ \implies 6\lambda_1 = 1 - 3\lambda_2, \quad (\text{using } x + y + z = 3) \\ \implies \lambda_2 = \frac{1 - 6\lambda_1}{3} \end{aligned}$$

So the system becomes:

$$\begin{aligned} yz + \lambda_1(y + z - 2) + \frac{1}{3} = 0 \\ xz + \lambda_1(x + z - 2) + \frac{1}{3} = 0 \\ xy + \lambda_1(x + y - 2) + \frac{1}{3} = 0 \\ xy + xz + yz - 1 = 0 \end{aligned}$$

Rearrange 3 to get

$$\lambda_1 = \frac{-\frac{1}{3} - xy}{x+y-2} \quad (3)$$

And hence the free equations are (the modified versions of) $(1), (2), (4)$

$$\begin{aligned}
yz + \frac{-\frac{1}{3} - xy}{x+y-2}y + \frac{-\frac{1}{3} - xy}{x+y-2}z - 2\frac{-\frac{1}{3} - xy}{x+y-2} + \frac{1}{3} &= 0 \\
xz + \frac{-\frac{1}{3} - xy}{x+y-2}x + \frac{-\frac{1}{3} - xy}{x+y-2}z - 2\frac{-\frac{1}{3} - xy}{x+y-2} + \frac{1}{3} &= 0 \\
xy + xz + yz - 1 &= 0
\end{aligned}$$

Expand (1):

$$\begin{aligned}
yz(x+y-2) - \frac{1}{3}y - xy^2 - \frac{1}{3}z - xyz + \frac{2}{3} + 2xy + \frac{1}{3}(x+y-2) &= 0 \\
xyz + y^2z - 2yz - \frac{1}{3}y - xy^2 - \frac{1}{3}z - xyz + \frac{2}{3} + 2xy + \frac{1}{3}x + \frac{1}{3}y - \frac{2}{3} &= 0 \\
xyz + y^2z - 2yz - xy^2 - \frac{1}{3}z - xyz + 2xy + \frac{1}{3}x &= 0
\end{aligned}$$

And similarly, (2) gives

$$\begin{aligned}
xz(x+y-2) - \frac{1}{3}x - x^2y - \frac{1}{3}z - xyz + \frac{2}{3} + 2xy + \frac{1}{3}(x+y-2) &= 0 \\
x^2z + xyz - 2xz - x^2y - \frac{1}{3}z - xyz + 2xy + \frac{1}{3}y &= 0
\end{aligned}$$

Using (3):

$$z = \frac{1-xy}{x+y}$$

Giving for (1):

$$\begin{aligned}
xy\frac{1-xy}{x+y} + y^2\frac{1-xy}{x+y} - 2y\frac{1-xy}{x+y} - xy^2 - \frac{1}{3}\frac{1-xy}{x+y} - xy\frac{1-xy}{x+y} + 2xy + \frac{1}{3}x &= 0 \\
xy - x^2y^2 + y^2 - xy^3 - 2y + 2xy^2 - xy^2 - \frac{1}{3} + \frac{1}{3}xy - xy + x^2y^2 + 2xy + \frac{1}{3}x &= 0 \\
y^2 - xy^3 - 2y + xy^2 - \frac{1}{3} + \frac{1}{3}xy + 2xy + \frac{1}{3}x &= 0
\end{aligned}$$

And for (2)

$$\begin{aligned}
x^2\frac{1-xy}{x+y} + xy\frac{1-xy}{x+y} - 2x\frac{1-xy}{x+y} - x^2y - \frac{1}{3}\frac{1-xy}{x+y} - xy\frac{1-xy}{x+y} + 2xy + \frac{1}{3}y &= 0 \\
x^2 - x^3y + xy - x^2y^2 - 2x + 2x^2y - x^2y - \frac{1}{3} + \frac{1}{3}xy - xy + x^2y^2 + 2xy + \frac{1}{3}y &= 0 \\
x^2 - x^3y - 2x + x^2y - \frac{1}{3} + \frac{1}{3}xy + 2xy + \frac{1}{3}y &= 0
\end{aligned}$$

Noting that these are equivocal with x and y swapped. Hence

$$x = y$$

Now using

$$\begin{aligned} x = y, \quad z &= \frac{1 - xy}{x + y} \\ x + y + z - 3 &= 0 \\ 2x + \frac{1 - x^2}{2x} - 3 &= 0 \\ 3x^2 - 6x - 1 &= 0 \\ \implies x &= \frac{6 \pm \sqrt{36 - 12}}{6} \\ x &= \frac{3 - \sqrt{6}}{3} \end{aligned}$$

And hence

$$\begin{aligned} x &= \frac{3 - \sqrt{6}}{3} \\ y &= \frac{3 - \sqrt{6}}{3} \\ z &= \frac{3 + 2\sqrt{6}}{3} \end{aligned}$$

Or any permutation of this. As a sanity check:

$$\begin{aligned} xy + xz + yz - 1 &= x^2 + xz + xz - 1 \\ &= x^2 + 2xz - 1 \\ &= \left(\frac{3 - \sqrt{6}}{3}\right)^2 + 2\left(\frac{3 - \sqrt{6}}{3}\right)\left(\frac{3 + 2\sqrt{6}}{3}\right) - 1 \\ &= \frac{9 - 6\sqrt{6} + 6 + 18 - 6\sqrt{6} + 12\sqrt{6} - 24 - 9}{9} \\ &= \frac{18 + 6 - 24}{9} = 0 \end{aligned}$$

Great!

We also get

$$\begin{aligned} \lambda_1 &= \frac{-\frac{1}{3} - xy}{x + y - 2} \\ &= \frac{-\frac{1}{3} - x^2}{2x - 2} \\ &= \frac{\sqrt{6} - 3}{3} \end{aligned}$$

And

$$\lambda_2 = \frac{1 - 6\lambda_1}{3} = \frac{5 - 2\sqrt{6}}{3}$$

This is also checked using symbolic `Matlab`:

```
syms x y z l1 l2 real
assume(x>0)
assume(y>0)
assume(z>0)
eqn1 = y*z + l1*(y+z) + l2 ==0;
eqn2 = x*z + l1*(x+z) + l2 ==0;
eqn3 = x*y + l1*(x+y) + l2 ==0;
eqn4 = x*y + x*z + y*z -1  ==0;
eqn5 = x + y + z -3        ==0;
sols = solve(eqn1,eqn2,eqn3,eqn4,eqn5)
sols.x(3)
sols.y(3)
sols.z(3)
```

Now to confirm this is a maximum. The hessian of h being positive definite is sufficient for it to be a maximum

$$H(h) = \begin{pmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial h}{\partial x \partial y} & \frac{\partial h}{\partial x \partial z} & \frac{\partial h}{\partial x \partial \lambda_1} & \frac{\partial h}{\partial x \partial \lambda_2} \\ \frac{\partial h}{\partial y \partial x} & \frac{\partial^2 h}{\partial y^2} & \frac{\partial h}{\partial y \partial z} & \frac{\partial h}{\partial y \partial \lambda_1} & \frac{\partial h}{\partial y \partial \lambda_2} \\ \frac{\partial h}{\partial z \partial x} & \frac{\partial h}{\partial z \partial y} & \frac{\partial^2 h}{\partial z^2} & \frac{\partial h}{\partial z \partial \lambda_1} & \frac{\partial h}{\partial z \partial \lambda_2} \\ \frac{\partial h}{\partial \lambda_1 \partial x} & \frac{\partial h}{\partial \lambda_1 \partial y} & \frac{\partial h}{\partial \lambda_1 \partial z} & \frac{\partial^2 h}{\partial \lambda_1^2} & \frac{\partial h}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial h}{\partial \lambda_2 \partial x} & \frac{\partial h}{\partial \lambda_2 \partial y} & \frac{\partial h}{\partial \lambda_2 \partial z} & \frac{\partial h}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 h}{\partial \lambda_2^2} \end{pmatrix}$$

$$H(x) = \begin{pmatrix} 0 & z+\lambda_1 & y+\lambda_1 & y+z & 1 \\ z+\lambda_1 & 0 & x+\lambda_1 & x+z & 1 \\ y+\lambda_1 & x+\lambda_1 & 0 & x+y & 1 \\ y+z & x+z & x+y & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

And using `Matlab` to obtain the eigenvalues

```
>> eig(H)

ans(x, y, z, l1, l2) =

  -0.77249408774975732627028770768242
   0.83919728731250742923420535039793
  -2.4494897427831780981972840747059
  -3.2729188222113425599830987927475
   5.6557053654317705552164652247378
```

And clearly some of these are non-positive so it is not a positive definite matrix.

2.

$$F\{y\} = \int_0^1 xy^2 y'^3 dx$$

(a) Letting $y(x) = x^\epsilon$, and $\epsilon > 1/5$, what ϵ gives an extremum for F? This gives

$$\begin{aligned}
F\{x\} &= \int_0^1 x\,(x^\epsilon)^2\left(\epsilon x^{\epsilon-1}\right)^3 dx \\
&= \epsilon^3 \int_0^1 x^{5\epsilon-2} dx \\
&= \epsilon^3 \frac{x^{5\epsilon-1}}{5\epsilon-1}\Big|_{x=0}^{x=1} \\
&= \frac{\epsilon^3}{5\epsilon-1}
\end{aligned}$$

Extremum for

$$\begin{aligned}
\frac{\partial F}{\partial \epsilon} &= 0 \\
\frac{\partial F}{\partial \epsilon} &= \frac{3\epsilon^2(5\epsilon-1) - 5\epsilon^3}{(5\epsilon-1)^2} \\
&= \frac{\epsilon^2(10\epsilon-3)}{(5\epsilon-1)^2}
\end{aligned}$$

And set to 0

$$\begin{aligned}
\frac{\epsilon^2(10\epsilon-3)}{(5\epsilon-1)^2} &= 0 \\
\epsilon^2(10\epsilon-3) &= 0 \\
10\epsilon-3 &= 0 \\
\epsilon &= \frac{3}{10}
\end{aligned}$$

Ignoring the $\epsilon = 0$ solution as we have assumed $\epsilon > 1/5$. Hence there is an extremum at $\epsilon = \frac{3}{10}$

(b) What is the value of F for the extremum

$$\begin{aligned}
F &= \epsilon^3 \int_0^1 x^{5\epsilon-2} dx \\
&= \epsilon^3 \frac{1}{5\epsilon-1} \\
&= \frac{27}{1000}\frac{1}{5\frac{3}{10}-1} \\
&= 2\frac{27}{1000} = \frac{54}{1000}
\end{aligned}$$

(c) Is it a maximum or a minimum? Look at

$$\begin{aligned}
\frac{\partial^2 F}{\partial \epsilon^2} &= \frac{\partial}{\partial \epsilon}\left(\frac{\epsilon^2(10\epsilon-3)}{(5\epsilon-1)^2}\right) \\
&= \frac{(30\epsilon^2-6\epsilon)(5\epsilon-1)^2-\epsilon^2(10\epsilon-3)10(5\epsilon-1)}{(5\epsilon-1)^4} \\
&= \frac{(30\epsilon^2-6\epsilon)(5\epsilon-1)-10\epsilon^2(10\epsilon-3)}{(5\epsilon-1)^3} \\
&= \frac{(150\epsilon^3-30\epsilon^2-30\epsilon^2+6\epsilon)-(100\epsilon^3-30\epsilon^2)}{(5\epsilon-1)^3} \\
&= \frac{50\epsilon^3-30\epsilon^2+6\epsilon}{(5\epsilon-1)^3} \\
&= \frac{2\epsilon(25\epsilon^2-15\epsilon+3)}{(5\epsilon-1)^3}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial^2 F}{\partial \epsilon^2}\bigg|_{\epsilon=3/10}\right) &= \left(\frac{2\frac{3}{10}(25\frac{9}{100}-15\frac{3}{10}+3)}{(5\frac{3}{10}-1)^3}\right) \\
&\approx 3.6
\end{aligned}$$

Since $\frac{\partial^2 F}{\partial \epsilon^2} > 0$ everywhere for $\epsilon > 0$ - this must be a local (and possibly global) minimum.

3.

$$f(x_1, x_2, x_3) = \cosh(x_1)\cos(x_2)e^{x_2 x_3}$$

(a) Taylor expansion around $\mathbf{x} = \mathbf{0}$ In nD

$$f(\mathbf{x}+\delta\mathbf{x}) = f(\mathbf{x}) + \delta\mathbf{x}^T\nabla f(\mathbf{x}) + \frac{1}{2}\delta\mathbf{x}^T H(\mathbf{x})\delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x}^3)$$

Obtain the Jacobian and Hessian

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \sinh(x_1)\cos(x_2)e^{x_2x_3} \\ -\cosh(x_1)\sin(x_2)e^{x_2x_3} + \cosh(x_1)\cos(x_2)e^{x_2x_3}x_3 \\ \cosh(x_1)\cos(x_2)e^{x_2x_3}x_2 \end{pmatrix}$$

$$\begin{aligned}
H_{11} &= \cosh(x_1)\cos(x_2)e^{x_2x_3} \\
H_{12} = H_{21} &= -\sinh(x_1)\sin(x_2)e^{x_2x_3} + \sinh(x_1)\cos(x_2)e^{x_2x_3}x_3 \\
H_{13} = H_{31} &= \sinh(x_1)\cos(x_2)e^{x_2x_3}x_2 \\
H_{22} &= -\cosh(x_1)\cos(x_2)e^{x_2x_3} - 2\cosh(x_1)\sin(x_2)e^{x_2x_3}x_3 + \cosh(x_1)\cos(x_2)e^{x_2x_3}x_3^2 \\
H_{23} = H_{32} &= -\cosh(x_1)\sin(x_2)e^{x_2x_3}x_2 + \cosh(x_1)\cos(x_2)e^{x_2x_3} + \cosh(x_1)\cos(x_2)e^{x_2x_3}x_2x_3 \\
H_{33} &= \cosh(x_1)\cos(x_2)e^{x_2x_3}x_2^2
\end{aligned}$$

At $\mathbf{x} = \mathbf{0}$:

$$f(\mathbf{0}) = 1$$

$$\nabla f(\mathbf{x})|_{\mathbf{x}=\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H(\mathbf{x})|_{\mathbf{x}=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hence the expansion around $\mathbf{x} = \mathbf{0}$ will be:

$$\begin{aligned} f(\mathbf{0} + \delta\mathbf{x}) &= f(\mathbf{0}) + \delta\mathbf{x}^T \nabla f(\mathbf{0}) + \frac{1}{2}\delta\mathbf{x}^T H(\mathbf{0})\delta\mathbf{x} + \mathcal{O}(3) \\ &= 1 + \delta\mathbf{x}^T \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \delta\mathbf{x}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \delta x + \mathcal{O}(\mathbf{x}^3) \\ &= 1 + x_1^2/2 - x_2^2/2 + x_2x_3 + 1 + \mathcal{O}(\mathbf{x}^3) \end{aligned}$$

Where $\delta x = (x_1, x_2, x_3)'$

(b) Would there be any terms of order 3 if we were to continue expanding? No there wouldn't. Odd derivatives of $\cosh x, \cos x$ will give $\sinh x, \sin x$ terms respectively - Both of which give 0 at $x = 0$. As for derivatives of $e^{x_2 x_3}$ this will always be zero for $x_2 = x_3 = 0$.

So a more precise form of the expansion would be

$$f(\mathbf{x}) = 1 + x_1^2/2 - x_2^2/2 + x_2x_3 + 1 + \mathcal{O}(\mathbf{x}^4)$$

Around $\mathbf{x} = 0$.