Topic C Assignment 5

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1.

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{2ixt(1-t^2/6+3it/4)}}{t^2+9} dt$$

(a) Identify saddle points and paths of steepest descent/ascent through them
We have $f(t) = \frac{1}{t^2+9}$ and $\phi(t) = 2it(1 - t^2/6 + 3it/4)$.

$$\phi(t) = i2t - it^3/3 - 3t^2/2$$

Saddle when:

$$\begin{aligned}
\phi'(t) &= 0 \\
\implies 2i - it^2 - 3t &= 0 \\
it^2 + 3t - 2i &= 0 \\
(t-i)(it+2) &= 0 \\
\implies t &= i,\, 2i
\end{aligned}$$

Let $t = \xi + i\eta$

$$\begin{aligned}
\phi(t) &= 2it(1 - t^2/6 + 3it/4) \\
\phi(\xi + i\eta) &= 2i(\xi + i\eta)\left(1 - (\xi + i\eta)^2/6 + 3i(\xi + i\eta)/4\right) \\
&= -\frac{\eta^3}{3} + i\eta^2\xi + \frac{3\eta^2}{2} + \eta\xi^2 - i3\eta\xi - 2\eta - \frac{i\xi^3}{3} - \frac{3\xi^2}{2} + i2\xi \\
&= -\frac{\eta^3}{3} + \frac{3\eta^2}{2} + \eta\xi^2 - 2\eta - \frac{3\xi^2}{2} + i\left(\eta^2\xi - 3\eta\xi - \frac{\xi^3}{3} + 2\xi\right)
\end{aligned}$$

For $t = i$, $\eta = 1$, $\xi = 0$. We require the complex part to be constant on lines through this point:

$$\begin{aligned}
Im(\phi) &= \eta^2\xi - 3\eta\xi - \frac{\xi^3}{3} + 2\xi \\
Im(\phi(t=i)) &= 0
\end{aligned}$$

For $t = i2$, $\eta = 2$, $\xi = 0$

$$\begin{aligned}
Im(\phi) &= \eta^2\xi - 3\eta\xi - \frac{\xi^3}{3} + 2\xi \\
Im(\phi(t=i2)) &= 0
\end{aligned}$$

So all paths go through $Im(\phi) = 0$:

$$\begin{aligned}
\implies \eta^2\xi - 3\eta\xi - \frac{\xi^3}{3} + 2\xi &= 0 \\
\eta^2 - 3\eta - \frac{\xi^2}{3} + 2 &= 0 \\
\xi = \pm\sqrt{3\eta^2 - 9\eta + 6},\ 0& \\
\xi = \pm\sqrt{3(\eta-1)(\eta-2)}&
\end{aligned}$$

Noting that the zero solution is contained in the other solution.
Identifying which is ascent and which is descent - look at the real part of ϕ on these paths, noting that all ξ terms are ξ^2.
For the square root ξ:

$$\begin{aligned}
Re(\phi) &= -\frac{\eta^3}{3} + \frac{3\eta^2}{2} + \eta\xi^2 - 2\eta - \frac{3\xi^2}{2} \\
&= -\frac{\eta^3}{3} + \frac{3\eta^2}{2} + \eta\left(3\eta^2 - 9\eta + 6\right) - 2\eta - \frac{3(3\eta^2 - 9\eta + 6)}{2} \\
&= -\frac{\eta^3}{3} + \frac{3\eta^2}{2} + 3\eta^3 - 9\eta^2 + 6\eta - 2\eta - \frac{9\eta^2 - 27\eta + 18)}{2} \\
&= \frac{8\eta^3}{3} - 12\eta^2 + \frac{35\eta}{2} - 9
\end{aligned}$$

Which is negative for small and negative η. I.e. the descent path will be that where η decreases (and hence ξ increases). And noting that the $\xi = 0$ solution gives vertical movement.
Direction of the path near the saddle

$$\begin{aligned}
\xi &= \pm\sqrt{3\eta^2 - 9\eta + 6} \\
\frac{\partial \xi}{\partial \eta} &= \pm\frac{(6\eta - 9)}{2\sqrt{\eta^2 - 3\eta + 2}} \\
\frac{\partial \xi}{\partial \eta}\Big|_{\eta=1} &= \pm\frac{(6-9)}{2\sqrt{1-3+2}} = \pm\infty \\
\frac{\partial \xi}{\partial \eta}\Big|_{\eta=2} &= \pm\frac{(8-9)}{2\sqrt{4-6+2}} = \pm\infty
\end{aligned}$$

So around the saddle, the movement is strictly horizontal. It can be parameterised as

$$t = s + i$$

Figure 2 plots the chosen path. Where the paths are then vertically connected to $\eta = 0$ for $\xi = \pm\infty$. The assumption that this is negligible is made
Parameterise using the square root path in the integral, locally it is horizontal so $t = s + i$

$$I = \int_{-\infty}^{\infty} \frac{e^{2ixt(1-t^2/6+3it/4)}}{t^2+9} dt$$

Approximate the taylor series for $\frac{1}{t^2+9}$ to leading order:

$$\begin{aligned}
&\sim \int_{-\infty}^{\infty} \frac{e^{2ixt(1-t^2/6+3it/4)}}{9} dt \\
&= \frac{1}{9}\int_{-\epsilon}^{\epsilon} e^{-\frac{x\left(s^3 2i+3s^2+5\right)}{6}} ds \\
&= \frac{e^{-5x/6}}{9}\int_{-\epsilon}^{\epsilon} e^{-\frac{xs^2(2is+3)}{6}} ds \\
&= \frac{e^{-5x/6}}{9}\int_{-\epsilon}^{\epsilon} e^{-\frac{xs^2(2is+3)}{6}} ds \\
&\sim \frac{e^{-5x/6}}{9}\int_{-\epsilon}^{\epsilon} e^{-\frac{xs^2}{2}} ds \\
&= \frac{e^{-5x/6}}{9}\int_{-\infty}^{\infty} e^{-\frac{xs^2}{2}} ds \\
&= \frac{e^{-5x/6}}{9}\sqrt{\frac{2\pi}{x}}
\end{aligned}$$

To leading order.

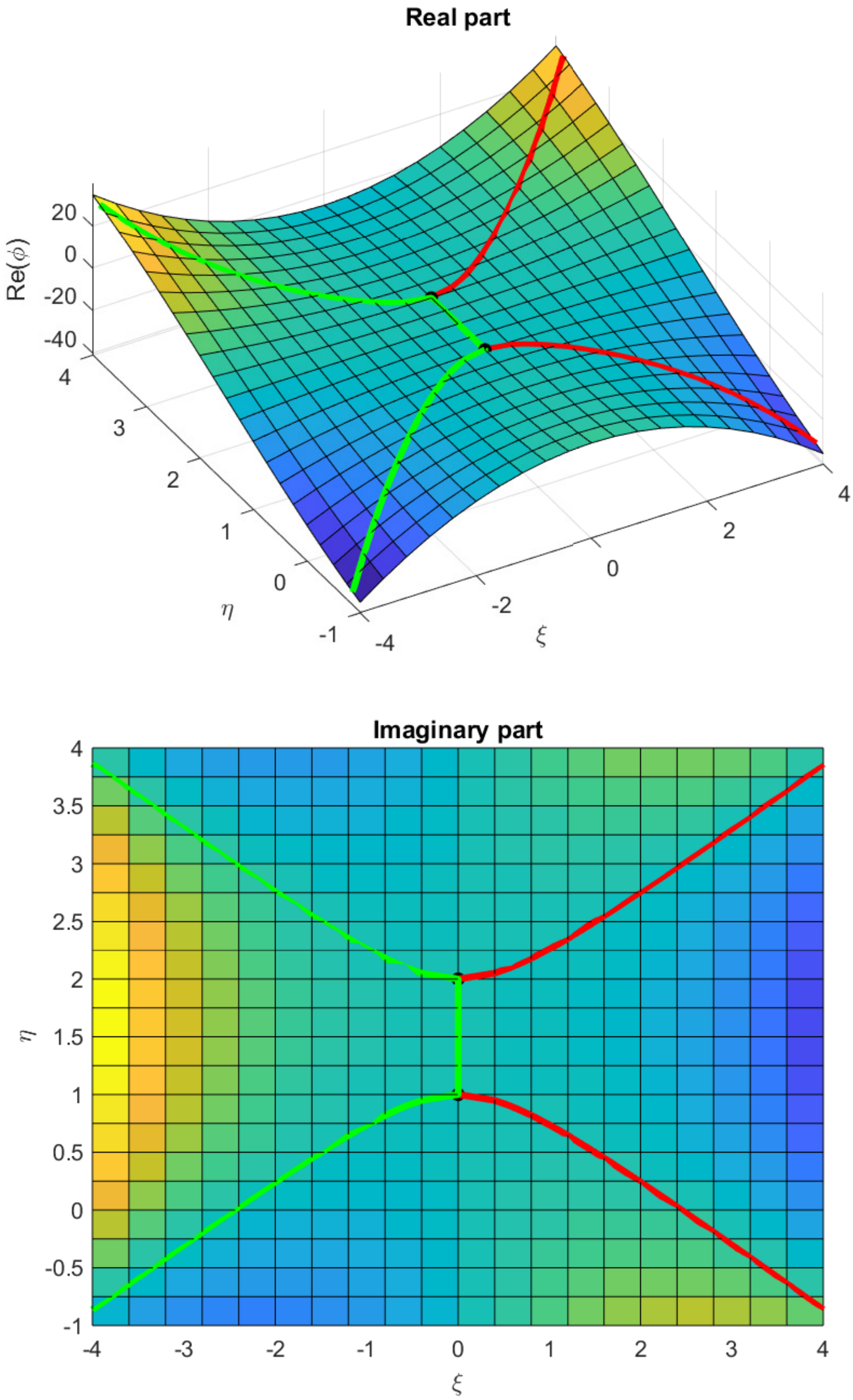

Figure 1: Plots of the real and imaginary parts of ϕ, with saddles and paths of steepest descent and ascent.

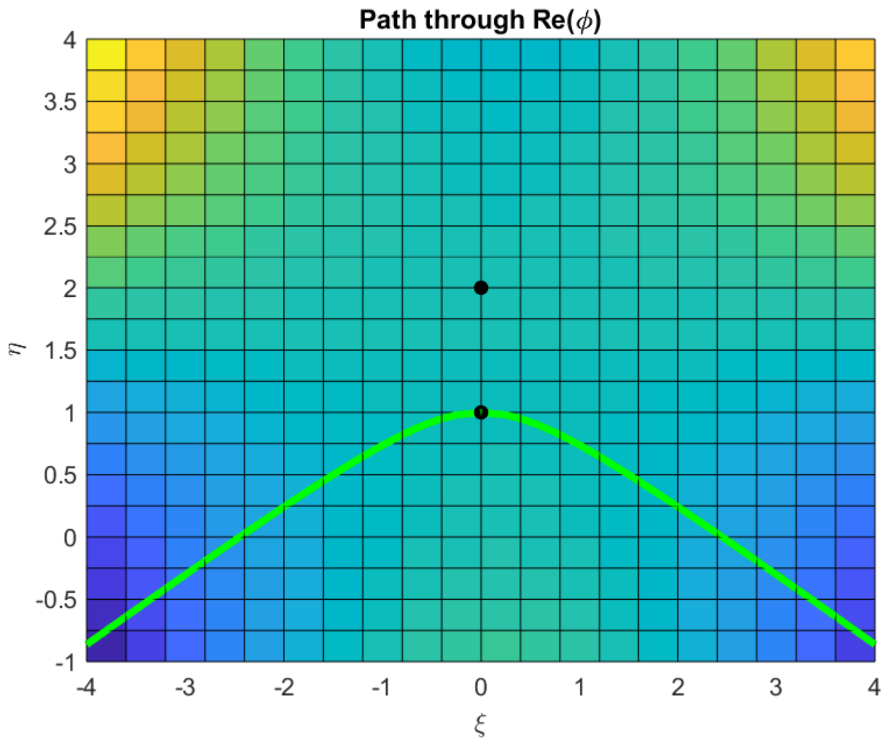

Figure 2: Plot of the imaginary part of ϕ, with the magenta line showing the deformed contour

2. Show that

$$I(x) = \frac{1}{2}\int_{-1}^{1} e^{-4xt^2+5ixt-ixt^3}dt \sim \frac{1}{2}e^{-2x}\sqrt{\pi/x}, \quad \text{as } x \to \infty$$

Note

- The deformed steepest descent path should have the same endpoints as the original contour
- The contributions from the end points are negligible compared to that from a saddle point (show this)

Rewrite as

$$I(x) = \frac{1}{2}\int_{-1}^{1} e^{x\left(-4t^2+5it-it^3\right)}$$
$$\phi(t) = -4t^2 + i5t - it^3$$

Find any saddle points:

$$\begin{aligned}\phi'(t) = -8t + i5 - i3t^2 = 0\\ (it-1)(3t-5i) = 0\\ t = i,\, i5/3\end{aligned}$$

$t = \xi + i\eta$

$$\begin{aligned}\phi(t) &= -4(\xi+i\eta)^2 + i5(\xi+i\eta) - i(\xi+i\eta)^3\\ &= -4(\xi^2-\eta^2+i2\xi\eta) + i5(\xi+i\eta) - i(\xi^3+i3\xi^2\eta-3\xi\eta^2-i\eta^3)\\ &= -4\xi^2+4\eta^2-i8\xi\eta+i5\xi-5\eta-i\xi^3+3\xi^2\eta+i3\xi\eta^2-\eta^3\\ &= -4\xi^2+4\eta^2-5\eta+3\xi^2\eta-\eta^3+i\left(-8\xi\eta+5\xi-\xi^3+3\xi\eta^2\right)\end{aligned}$$

Steepest paths for $t = 0 + in$, $\xi = 0$, $\eta = n$

$$\begin{aligned}Im(\phi) = -8\xi\eta + 5\xi - \xi^3 + 3\xi\eta^2\\ Im(\phi(t=i)) = 0 = Im(\phi(t=i5/3))\end{aligned}$$

$$\begin{aligned}-8\xi\eta + 5\xi - \xi^3 + 3\xi\eta^2 = 0\\ -8\eta + 5 - \xi^2 + 3\eta^2 = 0\\ \xi = \sqrt{-8\eta+5+3\eta^2},\, 0\\ \xi = \sqrt{(3\eta-5)(\eta-1)},\, 0\end{aligned}$$

With local direction(s)

$$\frac{\partial\xi}{\partial\eta} = \frac{6\eta-8}{2\sqrt{(3\eta-5)\,(\eta-1)}}$$
$$\frac{\partial\xi}{\partial\eta}\bigg|_{\eta=1,5/3} = \infty$$

I.e. locally $t = i + s$

Considering the end points:

Steepest path around $t = \pm 1$ so $\xi = \pm 1, \eta = 0$.

$$\begin{aligned} Im(\phi) &= -8\xi\eta + 5\xi - \xi^3 + 3\xi\eta^2 \\ Im(\phi(t = \pm 1)) &= \pm 5 \mp 1 = \pm 4 \end{aligned}$$

For $t = -1$ want $Im(\phi) = -4$

$$\begin{aligned} -8\xi\eta + 5\xi - \xi^3 + 3\xi\eta^2 &= -4 \\ 3\xi\eta^2 - 8\xi\eta - \xi^3 + 5\xi + 4 &= 0 \\ \eta &= \frac{8\xi \pm \sqrt{64\xi^2 - 4(3\xi)(-\xi^3 + 5\xi + 4)}}{6\xi} \\ \eta &= \frac{4\xi \pm \sqrt{\xi(16\xi + 3\xi^3 - 15\xi - 12))}}{6\xi} \\ \eta &= \frac{4\xi \pm \sqrt{\xi(3\xi^3 + \xi - 12)}}{3\xi} \end{aligned}$$

With direction:

$$\begin{aligned} \frac{\partial\eta}{\partial\xi} &= \pm\frac{\xi^3 + 2}{\xi\sqrt{\xi\left(3\xi^3 + \xi - 12\right)}} \\ \frac{\partial\eta}{\partial\xi}\bigg|_{\xi=-1} &= \pm\frac{-1 + 2}{-1\sqrt{-1\left(-3 - 1 - 12\right)}} = \frac{\pm 1}{4} \end{aligned}$$

For $t = 1$ want $Im(\phi) = 4$

$$\begin{aligned} -8\xi\eta + 5\xi - \xi^3 + 3\xi\eta^2 &= 4 \\ 3\xi\eta^2 - 8\xi\eta - \xi^3 + 5\xi - 4 &= 0 \\ \eta &= \frac{8\xi \pm \sqrt{64\xi^2 - 4(3\xi)(-\xi^3 + 5\xi - 4)}}{6\xi} \\ \eta &= \frac{4\xi \pm \sqrt{\xi(16\xi + 3\xi^3 - 15\xi + 12))}}{6\xi} \\ \eta &= \frac{4\xi \pm \sqrt{\xi(3\xi^3 + \xi + 12)}}{3\xi} \end{aligned}$$

With direction:

$$\begin{aligned} \frac{\partial\eta}{\partial\xi} &= \pm\frac{\xi^3 - 2}{\xi\sqrt{\xi\left(3\xi^3 + \xi + 12\right)}} \\ \frac{\partial\eta}{\partial\xi}\bigg|_{\xi=1} &= \frac{\pm 1}{4} \end{aligned}$$

I.e. parameterise as $t = 1 + s(1 + \frac{i}{4})$

The chosen path is plotted in figure 4.

Path from the end point leftward:

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{C_1} e^{x\left(-4t^2+5it-it^3\right)} \\
&= \frac{1}{2}\int_{C_1} e^{x\left(-4(-1+s(1+\frac{i}{4}))^2+5i(-1+s(1+\frac{i}{4}))-i(-1+s(1+\frac{i}{4}))^3\right)}ds \\
&= \frac{1}{2}\int_0^{-\infty} e^{x\left(s\left(\frac{15}{2}+4i\right)+s^2\left(-\frac{21}{4}+\frac{13}{16}i\right)+s^3\left(\frac{47}{64}-\frac{13}{16}i\right)-4-4i\right)}ds
\end{aligned}$$

Noting that all of the real parts go to 0 as $s \to -\infty$. Which is negligible, and the same occurs for the right point to ∞.

Only case about local behaviour around $t = i$ so parameterise as $t = s + i$.

$$\begin{aligned}
I(x) &= \frac{1}{2}\int_{-1}^{1} e^{x\left(-4t^2+5it-it^3\right)} \\
&= \frac{1}{2}\int_{-\epsilon}^{\epsilon} e^{x\left(-4(s+i)^2+5i(s+i)-i(s+i)^3\right)}ds \\
&= \frac{1}{2}\int_{-\epsilon}^{\epsilon} e^{x(-is^3-s^2-2)}ds \\
&= \frac{1}{2}e^{-2x}\int_{-\epsilon}^{\epsilon} e^{-x(is^3+s^2)}ds \\
&= \frac{1}{2}e^{-2x}\int_{-\epsilon}^{\epsilon} e^{-xs^2(is+1)}ds \\
&\sim \frac{1}{2}e^{-2x}\int_{-\epsilon}^{\epsilon} e^{-xs^2}ds \\
&\sim \frac{1}{2}e^{-2x}\int_{-\infty}^{\infty} e^{-xs^2}ds \\
&\sim \frac{1}{2}e^{-2x}\sqrt{\pi/x}
\end{aligned}$$

To leading order, as required.

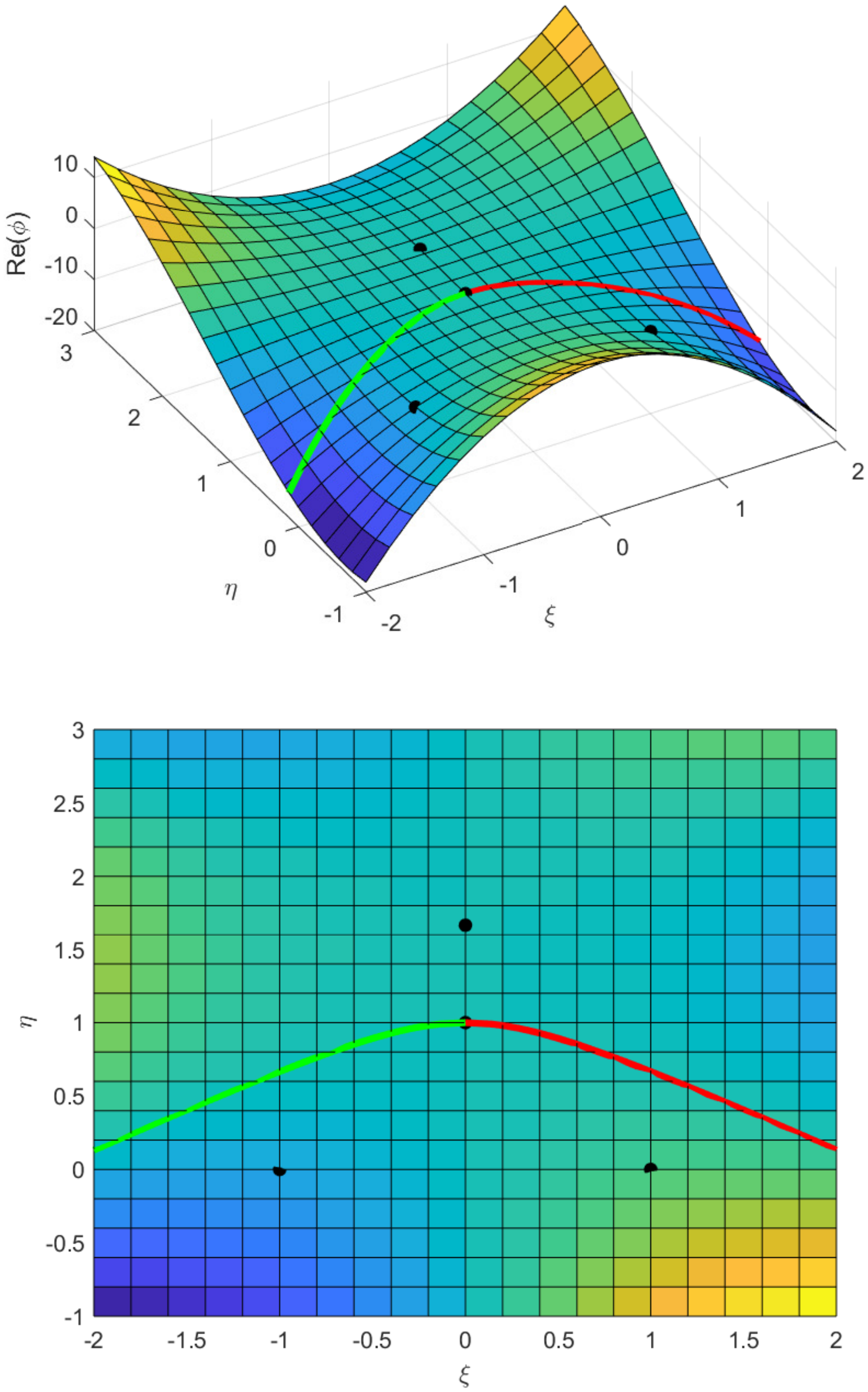

Figure 3: Plots of the real and imaginary parts of ϕ, with saddles and paths of steepest descent through the chosen saddle, $\eta = 1$.

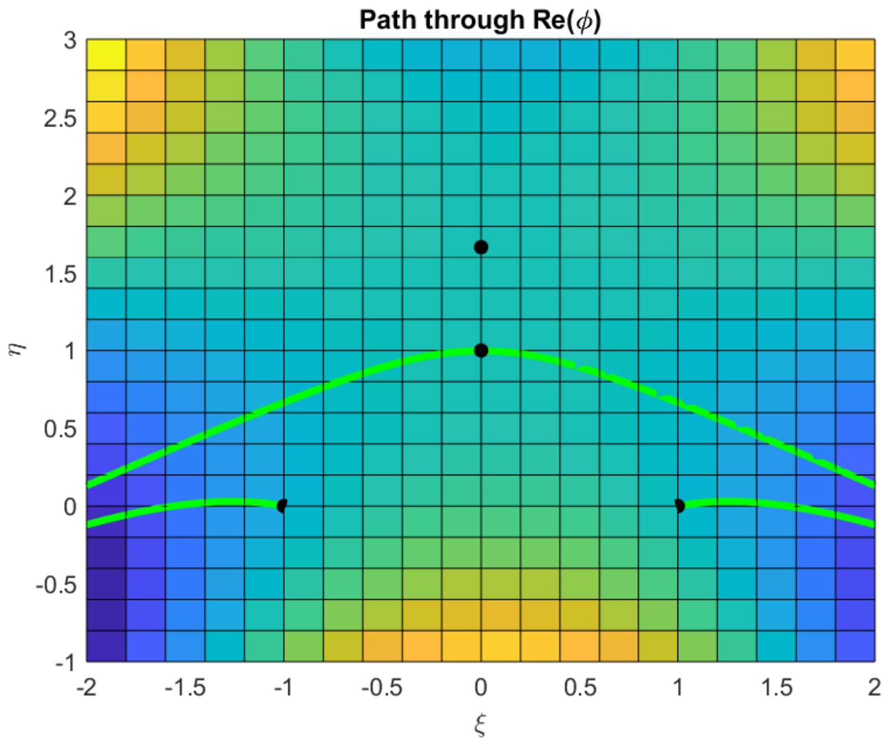

Figure 4: Chosen path, going left from $\xi = -1$ to $\xi = -\infty$, back to $\xi = 0$, through $\xi = 0$ to $\xi = \infty$ then from $\xi = \infty$ to $\xi = 1$

Practical Asymptotics (APP MTH 4051/7087)
Assignment 5 (5%)

Due 14 June 2019

1. Consider the integral

$$I(x) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{x2\mathrm{i}t(1-t^2/6+3\mathrm{i}t/4)}}{t^2+9}\mathrm{d}t$$

   (a) Identify any saddle points, then find the paths of steepest descent and ascent through each of these saddle(s).

   (b) Sketch (or plot) the saddles and paths of steepest descent/ascent.

   (c) Sketch a deformed contour that passes through a saddle point in a direction that will permit $I(x)$ to be evaluated by the method of steepest descent.

   (d) Use the deformed contour from part (c) to approximate $I(x)$ to leading-order as $x \to \infty$.

2. Use the method of steepest descents to show that

$$I(x) = \frac{1}{2}\int_{-1}^{1} \mathrm{e}^{-4xt^2+5\mathrm{i}xt-\mathrm{i}xt^3}\mathrm{d}t \sim \frac{1}{2}\mathrm{e}^{-2x}\sqrt{\pi/x}, \quad \text{as } x \to \infty.$$

   A complete solution should go through similar steps to Question 1, but will require a few extra details. [Hints:

   - The deformed steepest descent path should have the same endpoints as the original contour (this might look a bit weird).
   - The contributions from the end points are negligible compared to that from a saddle point (you need to show this).]

The following is an extension question which you may do as an alternative the short project.

3. Continue the analysis of the Airy function $\mathrm{Ai}(x)$. Recall that the integral representation was

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{C'} \mathrm{e}^{\mathrm{i}(t^3/3+xt)}\mathrm{d}t,$$

   where the contour C' starts at infinity with $2\pi/3 < \arg(t) < \pi$ and ends at infinity with $0 < \arg(t) < \pi/3$.

   (a) Use the method of steepest descents to find $\mathrm{Ai}(x)$ to leading-order as $x \to \infty$.

   (b) Now investigate **Stokes phenomenon**, the idea that the behaviour of these integrals depends on the direction x approaches infinity. We have already seen this is the case along the real axis, and will now extend this to the complex plane. Consider the integral:

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{C'} \mathrm{e}^{\mathrm{i}(t^3/3+zt)}\mathrm{d}t, \quad \text{as } |z| \to \infty.$$

   where C' is as above and $z = \mathrm{e}^{\mathrm{i}\theta}x$, that is $\arg(z) = \theta$.

i. Determine the location of the two saddle points, which will now vary with θ.
ii. Find expressions for the steepest descent paths through each saddle (these will now also depend on θ).
iii. Write a MATLAB code to plot the saddles and steepest descent paths for any value of θ.
iv. With reference to the original contour (thinking about how it can be deformed) and the above analysis, discuss why and for what value of θ there is a qualitative change in leading-order behaviour.

(c) The following paper (available on MyUni) extends the above analysis:

Berry, M.V., **Asymptotics, superasymptotics, hyperasymptotics**, *in Asymptotics Beyond All Orders, 1991.*

Briefly summarise the contents of this paper, and discuss how it relates to part (b). You may be particularly interested in Figure 6, which will (hopefully) look familiar.