Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 21

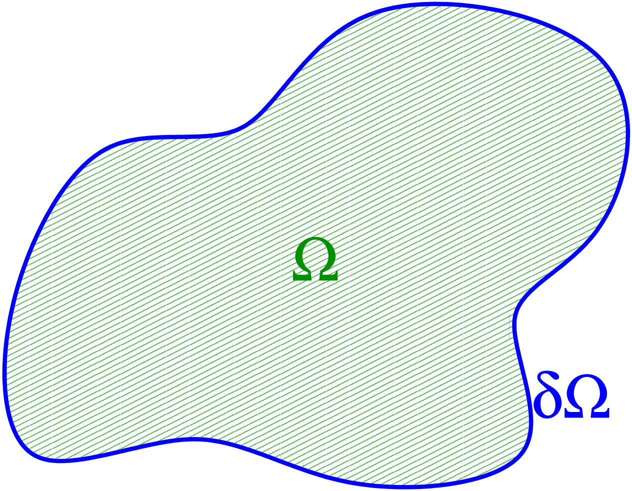
Last lecture

- Introduced isoperimetric problems
- We saw how Lagrange multipliers could be applied to handle integral constraints
- Solved Dido's problem (maximal area) with a coastline and
- Finally solved the problem of a catenary of fixed length!

What now?

We solve the more general case of Dido's problem: a general shape, without a coast, so that the perimeter must be parametrically described.

Isoperimetric problems



Dido's problem - traditional

Dido's problem is usually posed as follows

Find the curve of length L which encloses the largest possible area, i.e. maximize

$$\text{Area} = \iint_{\Omega} 1\, dA$$

subject to the constraint

$$\oint_{\delta\Omega} 1\, ds = L$$

Of course the problem is not yet in a convenient form.

Green's theorem

Green's theorem converts an integral over the area Ω to a contour integral around the boundary $\delta\Omega$.

$$\iint_{\Omega} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) dx\, dy = \oint_{\delta\Omega} \phi\, dy - \psi\, dx$$

for $\phi, \psi : \bar{\Omega} \to \mathbb{R}$ such that ϕ, ψ, ϕ_x and ψ_y are continuous.

This converts an area integral over a region into a line integral around the boundary.

Geometric representation of area

The area of a region is given by

$$\text{Area} = \iint_\Omega 1\, dA$$

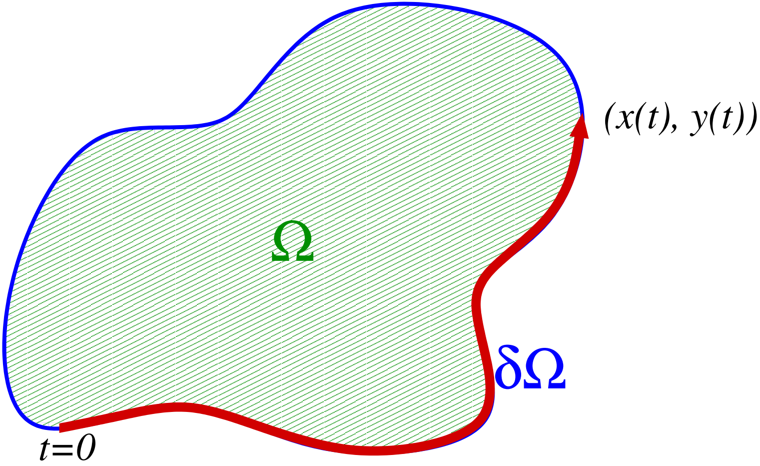
In Green's theorem choose $\phi = x/2$ and $\psi = y/2$, so that we get

$$\text{Area} = \iint_\Omega 1\, dx\, dy = \frac{1}{2} \oint_{\delta\Omega} x\, dy - y\, dx$$

Previous approach to Dido, was to use $y = y(x)$, but in more general case where the boundary must be closed, we can't define y as a function of x (or visa versa). So we write the boundary curve parametrically as $(x(t), y(t))$.

Parametric description of boundary

Boundary $\delta\Omega$ represented parametrically by $(x(t), y(t))$



Dido's problem

If the boundary $\delta\Omega$ is represented parametrically by $(x(t), y(t))$ then

$$
\begin{aligned}
F\{x, y\} = \text{Area} &= \iint_{\Omega} dx\, dy \\
&= \frac{1}{2} \oint_{\delta\Omega} x\, dy - y\, dx \\
&= \frac{1}{2} \oint_{\delta\Omega} (x\dot{y} - y\dot{x})\, dt
\end{aligned}
$$

So now the problem is written in terms of

$$
\begin{aligned}
\text{one independent variable} \quad &\Rightarrow \quad t \\
\text{two dependent variables} \quad &\Rightarrow \quad (x, y)
\end{aligned}
$$

Isoperimetric constraint

Previously we wrote the isoperimetric constraint as

$$G\{y\} = \int 1\, ds = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx = L$$

but now we must also modify this using

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

to get

$$G\{x, y\} = \oint 1\, ds = \oint \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = L$$

Dido's problem: Lagrange multiplier

Hence, we look for extremals of

$$H\{x,y\} = \oint \left(\frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2} \right) dt$$

So $h(t,x,y,\dot{x},\dot{y}) = \frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$, and there are two dependent variables, with derivatives

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{1}{2}\dot{y}, & \frac{\partial h}{\partial \dot{x}} &= -\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \\ \frac{\partial h}{\partial y} &= -\frac{1}{2}\dot{x}, & \frac{\partial h}{\partial \dot{y}} &= \frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}. \end{aligned}$$

Dido's problem: EL equations

Leading to the 2 Euler-Lagrange equations

$$\frac{d}{dt}\left(-\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = \frac{1}{2}\dot{y},$$

$$\frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = -\frac{1}{2}\dot{x},$$

Integrating

$$-\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{1}{2}y - A,$$

$$\frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -\frac{1}{2}x + B.$$

Dido's problem: solution

$$\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = y - A, \qquad \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -x + B$$

Now square the two, and add them to get

$$\lambda^2 \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2 + \dot{y}^2} = (y - A)^2 + (x - B)^2$$

or, more simply $(y - A)^2 + (x - B)^2 = \lambda^2$, the equation of a circle with centre (A, B), and radius $|\lambda|$.

End-conditions

Note, we can't set value at end points arbitrarily.

- if $x(t_0) = x(t_1)$, and $y(t_0) = y(t_1)$, then we get a closed curve, obviously a circle.
  - these conditions only amount to setting one constant, λ
  - there are many valid circles through (x_0, y_0), with centered along a circle of radius $|\lambda|$ about (x_0, y_0).
- on the other hand, if we specify different end-points, we are really solving a problem such as the simplified problem considered last week.
  - solutions need an extra edge to enclose the region
  - solutions are still arcs of circles

Why does it work?

Why does the Lagrange multiplier approach work here?

Consider Euler's finite difference method on a uniform grid for approximation of the functional

$$F\{y\} = \int_a^b f(x, y, y')\, dx \simeq \sum_{i=1}^{n} f\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \Delta x = \bar{F}(\boldsymbol{y})$$

where $\Delta x = (b-a)/n$, and $\Delta y_i = y_i - y_{i-1}$. The problem of finding an extremal curve now becomes one of finding stationary points of the function $\bar{F}(y_1, y_2, \ldots, y_n)$.

- we solve this by looking for $\partial \bar{F} / \partial y_i = 0$ for all $i = 1, 2, \ldots, n$.

Why does it work?

The constraint can be likewise approximated to give

$$G\{y\} \simeq \sum_{i=1}^{n} g\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \Delta x = \bar{G}(\boldsymbol{y}) = L$$

Under our usual conditions on F and G, the limit as $n \to \infty$ gives

$$\begin{aligned} \bar{F}(\boldsymbol{y}) &\to F\{y\} \\ \bar{G}(\boldsymbol{y}) &\to G\{y\} \end{aligned}$$

That is, the **functions** of the approximation $\boldsymbol{y}$ converge to the **functionals** of the curve $y(x)$.

Why does it work?

In the finite dimensional case the constraint is

$$\bar{G}(y_1, y_2, \ldots, y_n) - L = 0$$

we use a standard Lagrange multiplier

$$\bar{H}(y_1, y_2, \ldots, y_n, \lambda) = \bar{F}(y_1, y_2, \ldots, y_n) + \lambda \left[\bar{G}(y_1, y_2, \ldots, y_n) - L\right]$$

- we solve this by looking for

$$\frac{\partial \bar{H}}{\partial y_i} = 0, \quad \forall i = 1, 2, \ldots, n, \qquad \text{and} \qquad \frac{\partial \bar{H}}{\partial \lambda} = 0$$

- last equation just gives you back your constraint

Why does it work?

In our formulation of the isoperimetric problem we take

$$H\{y\} = F\{y\} + \lambda G\{y\}$$

and we also have

$$\bar{H}(\boldsymbol{y}, \lambda) = \bar{F}(\boldsymbol{y}) + \lambda \left[\bar{G}(\mathbf{y}) - L\right]$$

In the limit as $n \to \infty$ we find that

$$\bar{H}(\boldsymbol{y}, \lambda) \to H\{y\} - \lambda L$$

The E-L equations for $H\{y\} - \lambda L$ and $H\{y\}$ are the same, so they have the same extremals!

Multiple constraints

We can also handle multiple constraints via multiple Lagrange multipliers. For instance, given we wish to find extremals of

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

with the m constraints

$$G_k\{y\} = \int_{x_0}^{x_1} g_k(x, y, y')\, dx = L_k$$

we would look for extremals of

$$H\{y\} = \int_{x_0}^{x_1} h(x, y, y')\, dx = \int_{x_0}^{x_1} \left[f(x, y, y') + \sum_{k=1}^{m} \lambda_k g_k(x, y, y') \right] dx$$