Stochastic Assignment 5

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Question 1. Marks for mathematical writing

Question 2. Squirrel and Koala live far apart from each other, and talk together daily over the telephone about what they have done that day. Squirrel is only interested in three activities: foraging, eating, and singing. What he does is determined entirely by his mood. Based on his daily activity, Koala tries to guess what Squirrels mood that day must have been. Koala believes that Squirrels mood operates as a DTMC $\{X_t\}_{t\geq 0}$. There are two states, Happy and Sad, but she cannot observe them directly. Let `HAPPY` =: 1 and `SAD`=: 2. Then we have the following transition probability matrix:

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

On Day 0, Squirrel is `HAPPY` with probability 0.9 and `SAD` with probability 0.1. We have

$$P(Foraging \mid Happy) = 0.2, \quad P(Eating \mid Happy) = 0.3, \quad P(Singing \mid Happy) = 0.5$$

$$P(Foraging \mid Sad) = 0.3, \quad P(Eating \mid Sad) = 0.6, \quad P(Singing \mid Sad) = 0.1.$$

Suppose over the period from Day 0 to Day 3 Squirrel tells Koala the following sequence of activities: {Foraging, Singing, Eating, Foraging}. Determine

(a) the marginal probability of this observation sequence,

Solution Want to find:

$$P(\underline{y}|\Lambda) = \sum_{\underline{x}} P(\underline{x}, \underline{y}|\Lambda)$$

Where $\underline{y}$ = {Foraging, Singing, Eating, Foraging} (We will drop the Λ term from here onward to save space). Abbreviate $H := Happy$ and $S := Sad$, and similarly $F := Foraging$, $E := Eating$ and $Si := Singing$

We use the forward algorithm: Define:

Step 1: For $i = 1, \ldots, N$, let

$$\alpha_0(i) = P(y_0, X_0 = i|\Lambda) = P(X_0 = i)P(y_0|X_0 = i, \Lambda))$$

Step 2: For $t = 1, 2, \ldots, T$ and $i = 1, 2, \ldots, N$ compute

$$\alpha_t(i) = \left(\sum_{j=1}^{N} \alpha_{t-1}(j)p_{ji}\right) P(y_t|X_t = i)$$

The probability of a sequence $\underline{y}_t$ given the parameters Λ is:

$$P(\underline{y}_t) = P(\underline{y}_t|\Lambda) = \sum_{i=1}^{N} \alpha_t(i)$$

So for this problem: $\{y_0, y_1, y_2, y_3\} = \{F, Si, E, F\}$

Step 1:

$$\alpha_0(H) = P(X_0 = H)P(F|H) = 0.9 \times 0.2 = 0.18$$
$$\alpha_0(S) = P(X_0 = S)P(F|S) = 0.1 \times 0.3 = 0.03$$

Step 2: For $t = 1, 2, 3$ and for $i = H, S$ compute $\alpha_t(i)$:

$t = 1$:

$$\alpha_1(H) = P(Si|X_1 = H)(\alpha_0(H)p_{HH} + \alpha_0(S)p_{SH}) = 0.5 \times (0.18 \times 0.8 + 0.03 \times 0.3) = 0.0765$$
$$\alpha_1(S) = P(Si|X_1 = S)(\alpha_0(H)p_{HS} + \alpha_0(S)p_{SS}) = 0.1 \times (0.18 \times 0.2 + 0.03 \times 0.7) = 0.0057$$

$t = 2$:

$$\alpha_2(H) = P(E|H)(\alpha_1(H)p_{HH} + \alpha_1(S)p_{SH}) = 0.3 \times (0.0765 \times 0.8 + 0.0057 \times 0.3) = 0.018873$$
$$\alpha_2(S) = P(E|S)(\alpha_1(H)p_{HS} + \alpha_1(S)p_{SS}) = 0.6 \times (0.0765 \times 0.2 + 0.0057 \times 0.7) = 0.011574$$

Lastly, $t = 3$:

$$\alpha_3(H) = P(F|H)(\alpha_2(H)p_{HH} + \alpha_2(S)p_{SH}) = 0.2 \times (0.018873 \times 0.8 + 0.011574 \times 0.3) = 0.00371412$$
$$\alpha_3(S) = P(F|S)(\alpha_2(H)p_{HS} + \alpha_2(S)p_{SS}) = 0.3 \times (0.018873 \times 0.2 + 0.011574 \times 0.7) = 0.00356292$$

Step 3: And sum the α_3:

$$P(\underline{y}_t) = P(\underline{y}_t|\Lambda) = \alpha_3(H) + \alpha_3(S) = 0.00371412 + 0.00356292 = 0.00727704$$

I.e. the probability of the sequence $\underline{y} = \{\text{Foraging, Singing, Eating, Foraging}\}$ is 0.00727704

As required.

(b) the most likely underlying mood on Day 2,

Solution We want to find the most likely X_2 - use the forward-backwards algorithm. We have already used the forward algorithm - have to use the backwards algorithm:

Step 1: Let $\beta_T(i) = 1$ for $i = 1, \dots, N$

Step 2: for $t = T-1, T-2, \dots, 0$, and $i = 1, \dots, N$ compute:

$$\beta_t(i) = \sum_{j=1}^{N} P(y_{t+1}|X_{t+1} = j, \lambda)\beta_{t+1}(j)$$

Then to use the forward-backward algorithm: For $t = 0, 1, \dots, T-1$ and $i = 1, \dots, N$

$$\gamma_t(i) := P(X_t = i|y_t, \Lambda) = \frac{\alpha_t(i)\beta_t(i)}{P(y|\Lambda)} = \frac{P(X_t = i, \underline{y}_T|\Lambda)}{P(\underline{y}_T|\Lambda)}$$

Such that the most likely state at time t, given $\underline{y}$ is the state which $\gamma_T(i)$ is maximised over i.

Start with the backward algorithm:

Step 1: $\beta_3(H) = 1$ and $\beta_3(S) = 1$

Step 2:

$$\beta_2(H) = p_{HH}P(F|H)\beta_3(H) + p_{HS}P(F|S)\beta_3(S) = 0.8 * 0.2 + 0.2 * 0.3 = 0.22$$
$$\beta_2(S) = p_{SH}P(F|H)\beta_3(H) + p_{SS}P(F|S)\beta_3(S) = 0.3 * 0.2 + 0.7 * 0.3 = 0.27$$

We don't need β_1 or β_0 for this particular problem.

Now apply the forward-backward algorithm:

$$\gamma_2(H) = \frac{\alpha_2(S)\beta_2(S)}{P(\underline{y}_T|\Lambda)} = \frac{0.018873 * 0.22}{0.00727704} = 0.570569902048085$$
$$\gamma_2(S) = \frac{\alpha_2(S)\beta_2(S)}{P(\underline{y}_T|\Lambda)} = \frac{0.011574 * 0.27}{0.00727704} = 0.429430097951915$$

(where these values are taken directly from `matlab`) Since $\gamma_2(H) > \gamma_2(S)$, the more likely mood on day 2 is that Squirrel is happy!

As required.

(c) the most likely underlying mood sequence from Day 0 to Day 3

Solution Want to find:

$$\underset{\hat{X}=\{x_0,x_1,x_2,x_3\}}{\arg\max} \quad P(X=\hat{X}|y)$$

I.e. the most likely mood sequence given the observed actions.
The way to calculate this is by using the Viterbi algorithm:

Step 1: For $i = 1, 2, \ldots, N$

$$\delta_0(i) = p_0(i)P(y_0|X_0 = i)$$

Step 2: For $t = 1, \ldots, T$ and $i = 1, \ldots, N$ compute:

$$\delta_t(i) = \max_{j=1,\ldots,N} (\delta_{t-1}(j)p_{ji}P(y_t|X_t = i))$$

Applying the algorithm:

Step 1:

$$\begin{aligned}\delta_0(H) &= p_0(H)P(F|H) = 0.9 * 0.2 = 0.18\\ \delta_0(S) &= p_0(S)P(F|S) = 0.1 * 0.3 = 0.03\end{aligned}$$

Step 2:

$$\begin{aligned}\delta_1(H) &= P(Si|H)\max_{j\in\{H,S\}}(\delta_0(j)p_{jH}) = 0.5\max(0.18*0.8, 0.03*0.3) = 0.5\max(0.144, 0.009) = 0.072\\ \delta_1(S) &= P(Si|S)\max_{j\in\{H,S\}}(\delta_0(j)p_{jS}) = 0.1\max(0.18*0.2, 0.03*0.7) = 0.1\max(0.036, 0.021) = 0.0036\end{aligned}$$

$$\begin{aligned}\delta_2(H) &= P(E|H)\max_{j\in\{H,S\}}(\delta_1(j)p_{jH}) = 0.3\max(0.072*0.8, 0.0036*0.3) = 0.3\max(0.0576, 0.00108) = 0.01728\\ \delta_2(S) &= P(E|S)\max_{j\in\{H,S\}}(\delta_1(j)p_{jS}) = 0.6\max(0.072*0.2, 0.0036*0.7) = 0.6\max(0.0144, 0.00252) = 0.00864\end{aligned}$$

$$\begin{aligned}\delta_3(H) &= P(F|H)\max_{j\in\{H,S\}}(\delta_2(j)p_{jH}) = 0.2\max(0.01728*0.8, 0.00864*0.3)\\ &= 0.2\max(0.013824, 0.002592) = 0.0027648\\ \delta_3(S) &= P(F|S)\max_{j\in\{H,S\}}(\delta_2(j)p_{jS}) = 0.3\max(0.01728*0.2, 0.00864*0.7)\\ &= 0.3\max(0.003456, 0.006048) = 0.0018144\end{aligned}$$

So we get that the most likely mood sequence is {Happy, Happy, Happy, Happy}

As required.

(d) Is $P(X_1 = Happy \mid X_0 = Sad) = P(X_1 = Happy \mid X_0 = Sad,\ Y_0 = Eating)$? As always, justify your answer.

Solution

$$\begin{aligned}P(X_1|X_0, Y_0) &= \frac{P(X_1, Y_0|X_0)}{P(Y_0|X_0)} \quad \text{(conditional definition)}\\ &= \frac{P(Y_0|X_0)P(X_1|X_0)}{P(Y_0|X_0)} \quad \text{(conditional independence)}\\ &= P(X_1|X_0)\end{aligned}$$

Yes they are the same. The information contained in Y_0 could be interpreted as a subset of the information given by X_0. For this example, the two terms come out as $p_{SH} = 0.3$. This is an assumption made for the Hidden Markov Model.

As required.

Question 3. Consider a Hidden Markov chain $\{X_t, Y_t\}_{t\geq 0}$ with model parameters Λ and an observed sequence $y = (y_0, \ldots, y_T)$.

(a) Show that, for $i = 1, \ldots, N$ and for $t = 0, ..., T-1$,

$$P(X_t = i, \underline{y}|\Lambda) = \alpha_t(t)\beta_t(i)$$

where $\alpha_t(i) := P((y_0, \ldots, y_t), X_t = i|\Lambda)$ and $\beta_t(i) := P((y_{t+1}, \ldots, y_T)|X_t = i, \Lambda)$

Solution Start by partitioning $\underline{y}$ into $y_{0:t} := (y_0, \ldots, y_t)$ and $y_{t+1:T} := (y_{t+1}, \ldots, T)$. And condition on X_t and $y_{0:t}$:

$$\begin{aligned} P(X_t = i, \underline{y}|\Lambda) &= P(y_{0:t}, y_{t+1:T}, X_t = i|\Lambda)) \\ &= P(y_{0:t}, X_t = i|\Lambda)P(y_{t+1:T}|X_t = i, y_{0:t}, \Lambda) \\ &= P(y_{0:t}, X_t = i|\Lambda)P(y_{t+1:T}|X_t = i, \Lambda) \text{ (conditional independence)} \\ &= \alpha_t(t)\beta_t(i) \end{aligned}$$

As required.