Class Exercise 1: Applied Probability

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1. Prove the law of total probability: If $B_1, B_2, \ldots, B_n$ constitute a partition of Ω then:
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

   Solution Given in the lecture notes:
$$P(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} P(B_i)$$

   And, start with:
$$\begin{aligned} P(A) &= P(\bigcup_{i=1}^{n}(A \cap B_i)) \\ &= \sum_{i=1}^{n} P(A \cap B_i) \end{aligned}$$

   From the definition of conditional probability
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

   Rearrange:
$$P(A \cap B) = P(A|B)P(B)$$

   And using this in our equation, since we have a partition:
$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

   As required.

2. (a) For events $A_1, \ldots, A_n$ prove that
$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_n|A_1 \cap A_2 \cap \ldots, \cap A_{n-1})$$

   Whenever $P(A_n|A_1 \cap A_2 \cap \ldots, \cap A_{n-1}) > 0$

Solution Group A_1 up to A_{n-1} Using the definition of conditional probability from above (and since intersection is commutative)

$$P(A_n \cap \left(\cap_{i=1}^{n-1} A_i\right)) = P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))P(\left(\cap_{i=1}^{n-1} A_i\right))$$

Then, group up to A_{n-2}:

$$\begin{aligned}
P(A_n \cap \left(\cap_{i=1}^{n-1} A_i\right)) &= P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))P(\left(\cap_{i=1}^{n-1} A_i\right)) \\
&= P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))P(A_{n-1} \cap \left(\cap_{i=1}^{n-2} A_i\right)) \\
&= P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))P(A_{n-1} | \left(\cap_{i=1}^{n-2} A_i\right))P(\left(\cap_{i=1}^{n-2} A_i\right)) \\
&\vdots \\
&= P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))P(A_{n-1} | \left(\cap_{i=1}^{n-2} A_i\right)) \dots P(A_2 \cap A_1) \\
&= P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))P(A_{n-1} | \left(\cap_{i=1}^{n-2} A_i\right)) \dots P(A_2 | A_1)P(A_1) \\
&= P(A_1)P(A_2 | A_1) \dots P(A_n | \left(\cap_{i=1}^{n-1} A_i\right))
\end{aligned}$$

As required.

(b) Use the result from (a) to derive:

$$P(A \cap B | C) = P(A | B \cap C)P(B | C)$$

(hint: use n=3)

Solution For $n = 3$, (a) gives:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2)$$

So let $C = A_1$, $B = A_2$ and $A = A_3$:

$$P(C \cap B \cap A) = P(C)P(B | C)P(A | C \cap B)$$

Using conditional probability:

$$\begin{aligned}
P(A \cap B | C) &= \frac{P(A \cap B \cap C)}{P(C)} \\
&= \frac{P(C \cap B \cap A)}{P(C)} \\
&= \frac{P(C)P(B | C)P(A | C \cap B)}{P(C)} \\
&= P(B | C)P(A | C \cap B) \\
&= P(A | B \cap C)P(B | C)
\end{aligned}$$

As required.

3. Given Y is a binomial RV with parameters n and p, then write Y as the sum of appropriate indicator random variables and find $E[Y]$.

   Solution Given: $Y \sim Bin(n, p)$.
   A binomial counts the number of successes in n independent Bernoulli trials, each with success probability p. Denote: $B_i \sim Ber(p)$ with $i = 1, \dots n$, where B_i is the distribution of the ith event.
   The indicators are:

$$1_{B_i} = \begin{cases} 1 & \text{if } B_i \text{ occurs} \\ 0 & \text{if } B_i \text{ does not occur} \end{cases}$$

   So we can write $Y = \sum_{i=1}^{n} 1_{B_i}$.
   Finding the expectation:

$$\begin{aligned} E[Y] &= E[\sum_{i=1}^{n} 1_{B_i}] \\ &= \sum_{i=1}^{n} E[1_{B_i}] \\ &= \sum_{i=1}^{n} p \\ &= np \end{aligned}$$

   As required.

4. A coin is tossed repeatedly. For each toss, the probability the coin comes up heads is 1/2 (its a fair coin). Assume that each toss is independent of the previous tosses. Let E_n be the event that no heads come up in the first n tosses.

   (a) Find $P(E_n)$
   Solution Let $X \sim Bin(n, 1/2)$
   Noting that the binomial distribution has PDF

$$P(X = x) = f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x \in N$$

$$
\begin{aligned}
P(E_n) &= P(\text{no heads in n tosses}) \\
&= P(X=0) \\
&= \binom{n}{0}\left(\frac{1}{2}^0\right)(1-\frac{1}{2})^{n-0} \\
&= 1*1*(1-\frac{1}{2})^n \\
&= (1-\frac{1}{2})^n \\
&= \frac{1}{2^n}
\end{aligned}
$$

As required.

(b) How can we interpret the quantity

$$\lim_{n\to\infty} P(E_n)$$

and what is its value?

Solution This limit (read directly) is the probability of getting no heads, in n throws, as n goes to ∞. I.e. The probability of *never* getting a head.

Its value is:

$$
\begin{aligned}
\lim_{n\to\infty} P(E_n) &= \lim_{n\to\infty} \frac{1}{2^n} \to 0 \text{ as } n \to \infty \\
&= 0
\end{aligned}
$$

As required.

(c) Show that a head is bound to turn up eventually. I.e. show that

$$P(\text{head turns up eventually}) = 1$$

(hint: use b)

Solution

$$
\begin{aligned}
P(\text{head turns up eventually}) &= 1 - P(\text{head turns up eventually})^C \\
&= 1 - P(\text{never get a head}) \\
&= 1 - \lim_{n\to\infty} \frac{1}{2^n} \text{ (from above)} \\
&= 1 - 0 \\
&= 1
\end{aligned}
$$

As required.

(d) Show that any given finite sequence of heads and tails occurs eventually with probability one.

Hint: Consider a specific sequence S of heads and tails with length K. Now consider N disjoint lots of sequences off length K (the result of a total of NK tosses). Draw a diagram consisting of NK "slots" (where each slot contains the outcome of a toss), and divide the slots into N groups of K slots. Each of these N disjoint sequences has probability 2^{-K} of being the particular sequence S that we are interested in. With a little thought, we recognise that the event

{one of the N groups is S} $\subset$ {S occurs somewhere in the NK tosses}

that is, the event that one of the N groups is S is a (proper) subset of the event that S occurs somewhere in the NK tosses (note: in the latter event, the sequence S does not need to fall exactly within one of the N groups, and thus may overlap two of these groups). Use the fact that:

$P(S$ occurs somewhere in the NK tosses$) \geq P($at least one of the N groups *is* S)

Solution So if we show that the probability on the right is 1 then they are both 1. I.e. show $P($at least one of the N groups *is* S).
As above, split the NK tosses into N groups of K tosses (given the sequence we care about, S, has fixed length K).
We can consider the ith sequence of K tosses as a random variable: X_i. Since this split still leaves $X_i \sim Bin(1/2, K)$ and as the state of each X_i is not affected by X_j, i.e. $Cov(X_i, X_j) = 0 \quad j \neq i$, we have N independent and identically distributed random variables.
After the first K tosses (label this K_1), the probability of observing our particular sequence S is:

$$P(K_1 = S) = \frac{1}{2^K} = 2^{-K}$$

So if we repeat the process of these K tosses N times, the probability of observing the particular sequence S is:

$$P(K_1 = S \cup K_2 = S \cup \ldots \cup K_N = S)$$

Since the events are disjoint ($K_1 \cap K_2 = 0$), the following conversion can be made:

$$= \sum_{i=1}^{N} \frac{1}{2^{-K}}$$

$$= N2^{-K}$$

It is necessary to note that no matter which K you choose, there is always an N larger than it, i.e.

$$\forall K,\ \exists N \text{ such that } N > K$$

So if we choose N sufficiently large, $2^{-K} \to 0$ as $K \to \infty$, $N2^{-K} \to 1$. So as we take the limit of this with $N \to \infty$

$$\lim_{N \to \infty} N2^{-K} \to \infty \quad \text{as } N \to \infty$$

So this means:

$$\lim_{N \to \infty} P(K_1 = S \cup K_2 = S \cup \ldots \cup K_N = S) \to 1$$

This implies that there is a point where S will be obtained in N tosses, i.e.

$$\lim_{N \to \infty} P(\text{at least one of the N groups is S}) = 1$$

$$\implies P(S \text{ occurs somewhere in the NK tosses}) = 1$$

As required.