STATS 3006 Mathematical Statistics III
Assignment 4
2018

Assignment 4 is due by 23:59 Tuesday 15^{th} May 2018.

Assignments are to be submitted as a single pdf file online on MyUni.

1. Consider a sequence of independent random variables, $X_1,\ X_2,\ X_3,\ \ldots,$ with $E(X_i) = \mu$ and $\text{var}(X_i) = \sigma_i^2$. Let

$$\tilde{X}_n = \sum_{i=1}^{n} w_{in} X_i \text{ where } w_{in} = \frac{1/\sigma_i^2}{\sum_{i=1}^{n} 1/\sigma_i^2}.$$

   (a) Show for each n that $\tilde{X}_n$ is an unbiased estimator for μ.

   (b) Find $\text{var}(\tilde{X}_n)$.

   (c) Show that $\tilde{X}_n \to \mu$ in probability if

$$\sum_{i=1}^{n} 1/\sigma_i^2 \to \infty \text{ as } n \to \infty.$$

   (d) Construct a counter example to demonstrate that

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

   will not necessarily converge to μ in quadratic mean under the assumption

$$\sum_{i=1}^{n} 1/\sigma_i^2 \to \infty \text{ as } n \to \infty.$$

2. Consider a Poisson random variable $X \sim \text{Po}(\mu)$ and let

$$Z = \mu^{-1/2}(X - \mu).$$

   Use moment generating functions to show that

$$\mathcal{L}(Z) \to N(0,1) \text{ as } \mu \to \infty.$$

   Hint: Use the fact that

$$e^a = 1 + a + a^2/2 + r(a) \text{ where } \lim_{h \to 0} \frac{r(h)}{h^2} = 0.$$

3. Suppose T is an unbiased estimator for the scalar parameter, θ.
   (a) Show that $E(T^2) \geq \theta^2$.
   (b) Show that $E(T^2) = \theta^2$ can occur only if $P(T = \theta) = 1$.
   (c) Suppose $X_1, X_2, \ldots, X_n$ are IID $N(\mu, \sigma^2)$ and let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.
      i. Deduce that the sample standard deviation $S = \sqrt{S^2}$ is a biased estimator for σ.
      ii. Is the bias positive or negative?

4. Suppose $X_1,\ X_2,\ \ldots, X_n$ are independent Poisson observations with probability function

$$p(x; \mu) = \frac{\mu^x e^{-\mu}}{x!} \text{ for } x = 0, 1, 2, 3, \ldots$$

   (a) Find $E(X)$ and $\text{var}(X)$.
   (b) Write down the log-likelihood function, the score and the Fisher information.
   (c) Verify that

$$\mathcal{I}(\mu) = E\left(-\frac{\partial^2 \ell}{\partial \mu^2}\right)$$

   in this case.
   (d) Prove that $\bar{X}$ is the MVUE for θ.

5. Suppose $X_1,\ X_2,\ \ldots, X_n$ are IID exponential random variables with common PDF

$$f(x) = e^{-x/\theta} \text{ for } x > 0$$

and let $T = \sum_{i=1}^{n} X_i$. Note that the exponential distribution was defined in lectures by the PDF

$$f(x) = \lambda e^{\lambda x}$$

so, for this question, the parameter is $\theta = 1/\lambda$.
   (a) Prove that T is sufficient for θ.
   (b) Prove that X_1 is an unbiased estimate for θ and find its variance.
   (c) i. Let $Y = X_2 + X_3 + \ldots + X_n$. State the distribution of Y and the joint PDF, $f_{X_1,Y}(x_1, y)$, of (X_1, Y).
      ii. State the distribution of T and its PDF.
      iii. Find the conditional density $f_{X_1|T}(x_1|t)$.
      Hint: You may use the fact that, since $T = X_1 + Y$, the joint pdf is

$$f_{X_1,T}(x_1, t) = f_{X_1,Y}(x_1, t - x_1) \text{ for } 0 < x_1 < t.$$

   (d) Use the Rao-Blackwell theorem to obtain an unbiased estimator, T^*, with smaller variance than X_1.

Honours Questions

The following questions are compulsory for Honours and Masters students and may be attempted for bonus marks by all students.

6. Extend the counter-example from question 1(d) to show that $\bar{X}_n$ need not converge to μ in probability under the assumption

$$\sum_{i=1}^{n} 1/\sigma_i^2 \to \infty \text{ as } n \to \infty.$$

7. Justify the hint in question 5c(iii).
8. Suppose $X \sim B(n, p)$ and let

$$\theta = \log \frac{p}{1-p}.$$

Show that no unbiased estimator for θ exists.

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