Specimen Solution

Question 1

$$F\{y\} = \int_0^1 xy^2y'^3\,dx.$$

(a)

$$\text{Assuming} \quad y = x^\epsilon \quad \Rightarrow \quad y' = \epsilon x^{\epsilon-1}.$$

$$\begin{aligned} \text{So} \quad xy^2y'^3 &= xx^{2\epsilon}\epsilon^3x^{3\epsilon-3} \\ &= \epsilon^3x^{5\epsilon-2}. \end{aligned}$$

Substituting into F we have

$$F(\epsilon) = \int_0^1 \epsilon^3x^{5\epsilon-2}\,dx = \left[\frac{\epsilon^3}{5\epsilon-1}x^{5\epsilon-1}\right]_0^1 = \frac{\epsilon^3}{5\epsilon-1}.$$

For an extremal $\dfrac{dF}{d\epsilon} = 0$.

$$\frac{dF}{d\epsilon} = \frac{(5\epsilon-1)3\epsilon^2 - 5\epsilon^3}{(5\epsilon-1)^2} = \frac{\epsilon^2(10\epsilon-5)}{(5\epsilon-1)^2}$$

So looking for zeros for $\epsilon > 1/5$ we conclude that the only term that will do that is

$$10\epsilon - 3 = 0 \quad \Rightarrow \quad \epsilon = \frac{3}{10}.$$

(b)

$$\begin{aligned} \text{From part (a)} \quad F(\epsilon) &= \frac{\epsilon^3}{5\epsilon-1} \\ F(3/10) &= \frac{(3/10)^3}{3/2-1} = \frac{27/1000}{1/2} = \frac{54}{1000} = 0.054. \end{aligned}$$

(c)

$$\text{From part (a)} \quad \frac{dF}{d\epsilon} = \frac{\epsilon^2(10\epsilon-5)}{(5\epsilon-1)^2}. \quad \text{So differentiating again}$$

$$\begin{aligned} \frac{d^2F}{d\epsilon^2} &= \frac{(5\epsilon-1)^2(30\epsilon^2-6\epsilon) - \epsilon^2(10\epsilon-3)2(5\epsilon-1)5}{(5\epsilon-1)^4} \\ &= \frac{6\epsilon(5\epsilon-1)^3 - 10\epsilon^2(10\epsilon-3)(5\epsilon-1)}{(5\epsilon-1)^4} \\ &= \frac{6\epsilon}{5\epsilon-1} - \frac{10\epsilon^2(10\epsilon-3)}{(5\epsilon-1)^3}. \end{aligned}$$

Now for $\epsilon = 3/10$ the second term is zero and so

$$\left.\frac{d^2F}{d\epsilon^2}\right|_{3/10} = \frac{18/10}{1/2} - 0 = \frac{36}{10} = 3.6 > 0.$$

Since $\left.\dfrac{d^2F}{d\epsilon^2}\right|_{3/10} > 0$ the extremal is a minimum.

Question 2

(a)

$$F\{y\} = \int_0^1 \left(y^2 - y'^2 - 2y\sin x\right)\, dx, \quad y(0) = 0, \quad y(1) = 1.$$

$$\text{So differentiating} \quad \frac{\partial f}{\partial y} = 2y - 2\sin x, \quad \frac{\partial f}{\partial y'} = -2y'.$$

$$\begin{aligned} \text{Euler–Lagrange} \quad \Rightarrow \quad 2y - 2\sin x - \frac{d}{dx}(-2y') &= 0 \\ y'' + y &= \sin x. \end{aligned}$$

$$\text{A.H.E.} \quad \Rightarrow \quad y_h'' + y_h = 0 \quad \Rightarrow \quad y_h = A\cos x + B\sin x.$$

$$\begin{aligned} \text{For } y_p \text{ try} \quad &\Rightarrow \quad y_p = Cx\cos x + Dx\sin x \\ \text{Differentiating} \quad &\Rightarrow \quad y_p' = C\cos x - Cx\sin x + D\sin x + Dx\cos x \\ \text{Differentiating} \quad &\Rightarrow \quad y_p'' = -2C\sin x - Cx\cos x + 2D\cos x - Dx\sin x. \end{aligned}$$

$$y_p'' + y_p = -2C\sin x + 2D\cos x = \sin x.$$

$$\text{So equating terms} \quad \Rightarrow \quad C = -\frac{1}{2}, \quad D = 0.$$

$$\text{General solution:} \quad y = A\cos x + B\sin x - \frac{1}{2}x\cos x.$$

Now applying the fixed end-point conditions

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad 0 = A. \\ y(1) = 1 \quad &\Rightarrow \quad 1 = B\sin 1 - \frac{1}{2}\cos 1 \\ &\qquad\; B = \frac{2+\cos 1}{2\sin 1}. \end{aligned}$$

So the extremal is

$$y = \left(\frac{2+\cos 1}{2\sin 1}\right)\sin x - \frac{1}{2}x\cos x.$$

(b)

$$F\{y\} = \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)\, dx, \quad y(0) = 0, \quad y(1) = \frac{3}{2}.$$

The integrand is x-absent and so we will investigate the conserved quantity H.

$$\begin{aligned} H &= y'\frac{\partial f}{\partial y'} - f \\ &= y'(y' + y + 1) - \left(\frac{1}{2}y'^2 + yy' + y' + y\right) \\ &= \frac{1}{2}y'^2 - y. \end{aligned}$$

Since H is conserved the first-order ODE is

$$\frac{1}{2}y'^2 - y = \alpha.$$

$$\begin{aligned}
\text{Solving the ODE} \quad \Rightarrow \quad \frac{dy}{dx} &= \sqrt{2y+\alpha} \\
\int \frac{dy}{\sqrt{2y+\alpha}} &= \int dx \\
\sqrt{2y+\alpha} &= x + \beta \\
2y + \alpha &= (x+\beta)^2 \\
y &= \frac{(x+\beta)^2 - \alpha}{2} \\
y &= \frac{x^2}{2} + \beta x + \frac{\beta^2 - \alpha}{2}.
\end{aligned}$$

Let's relabel $\gamma = (\beta^2 - \alpha)/2$ so that

$$\text{General solution} \quad \Rightarrow \quad y = \frac{x^2}{2} + \beta x + \gamma.$$

Now applying the fixed end-point conditions

$$\begin{aligned}
y(0) = 0 \quad &\Rightarrow \quad 0 = \gamma. \\
y(1) = \frac{3}{2} \quad &\Rightarrow \quad \frac{3}{2} = \frac{1}{2} + \beta \\
&\qquad\;\; \beta = 1.
\end{aligned}$$

So the extremal is

$$y = \frac{x^2}{2} + x.$$

(c)

$$F\{y\} = \int_0^1 \left(y''^2 - 360x^2 y\right)\, dx, \quad y(0) = 0, \quad y(1) = 1 \quad y'(0) = 1, \quad y'(1) = \frac{5}{2}.$$

$$\text{So differentiating} \quad \frac{\partial f}{\partial y} = -360x^2, \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial f}{\partial y''} = 2y''.$$

$$\begin{aligned}
\text{Euler–Poisson} \quad \Rightarrow \quad -360x^2 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') &= 0 \\
y'''' &= 180x^2.
\end{aligned}$$

This can be solved by direct integration

$$\begin{aligned}
\text{Integrating} \quad &\Rightarrow \quad y''' = 60x^3 + 6c_3. \\
\text{Integrating} \quad &\Rightarrow \quad y'' = 15x^4 + 6c_3 x + 2c_2. \\
\text{Integrating} \quad &\Rightarrow \quad y' = 3x^5 + 3c_3 x^2 + 2c_2 x + c_1. \\
\text{Integrating} \quad &\Rightarrow \quad y = \frac{x^6}{2} + c_3 x^3 + c_2 x^2 + c_1 x + c_0.
\end{aligned}$$

$$\text{General solution:} \quad y = \frac{x^6}{2} + c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

Now applying the fixed end-point conditions at $x = 0$

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad 0 = c_0. \\ y'(0) = 1 \quad &\Rightarrow \quad 1 = c_1. \end{aligned}$$

$$\text{So far we have} \quad y = \frac{x^6}{2} + c_3 x^3 + c_2 x^2 + x.$$

Now applying the fixed end-point conditions at $x = 1$

$$\begin{aligned} y(1) = 1 \quad &\Rightarrow \quad 1 = \frac{1}{2} + c_3 + c_2 + 1 \\ &\Rightarrow \quad c_3 + c_2 = -\frac{1}{2}. && (1) \\ y'(1) = \frac{5}{2} \quad &\Rightarrow \quad \frac{5}{2} = 3 + 3c_3 + 2c_2 + 1 \\ &\Rightarrow \quad 3c_3 + 2c_2 = -\frac{3}{2}. && (2) \\ 2 \times (1) \quad &\Rightarrow \quad 2c_3 + 2c_2 = -1. && (3) \\ (2) - (3) \quad &\Rightarrow \quad c_3 = -\frac{1}{2}. \\ \text{From } (1) \quad &\Rightarrow \quad c_2 = 0. \end{aligned}$$

So the extremal is

$$y = \frac{x^6}{2} - \frac{x^3}{2} + x.$$

Question 3

(a)

$$K(k) = F(\pi/2, k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

Let $t = \sin^2\theta$ so that

$$dt = 2\sin\theta\cos\theta\, d\theta \quad \Rightarrow \quad d\theta = \frac{1}{2} t^{-1/2}(1-t)^{-1/2}\, dt$$

$$\text{So} \quad K(k) = \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-k^2 t)^{-1/2}\, dt.$$

This is in Euler form with

$$a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = 1, \quad z = k^2,$$

$$\text{So} \quad K(k) = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} F(1, 2, 1/2; 1; k^2).$$

Now $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$ and so

$$K(k) = \frac{\pi}{2} F(1/2, 1/2; 1; k^2).$$

(b)

$$\boldsymbol{x} = (b\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi).$$

(i)

$$\frac{\partial \boldsymbol{x}}{\partial \theta} = (-b\sin\theta\sin\phi, b\cos\theta\sin\phi, 0),$$
$$\frac{\partial \boldsymbol{x}}{\partial \phi} = (b\cos\theta\cos\phi, b\sin\theta\cos\phi, -c\sin\phi).$$

Now

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial \phi} \times \frac{\partial \boldsymbol{x}}{\partial \theta} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b\cos\theta\cos\phi & b\sin\theta\cos\phi & -c\sin\phi \\ -b\sin\theta\sin\phi & b\cos\theta\sin\phi & 0 \end{vmatrix} \\
&= (bc\cos\theta\sin^2\phi, bc\sin\theta\sin^2\phi, b^2\sin\phi\cos\phi) \\
&= b\sin\phi(c\cos\theta\sin\phi, c\sin\theta\sin\phi, b\cos\phi).
\end{aligned}$$

So

$$\left|\frac{\partial \boldsymbol{x}}{\partial \phi} \times \frac{\partial \boldsymbol{x}}{\partial \theta}\right| = b\sin\phi\sqrt{c^2\sin^2\phi + b^2\cos^2\phi}.$$

So

$$dA = b\sin\phi\sqrt{c^2\sin^2\phi + b^2\cos^2\phi}\, d\theta\, d\phi.$$

(ii)

$$\begin{aligned}
\text{Area} &= \iint_S dA \\
&= \int_0^\pi \int_{-\pi}^\pi b \sin\phi \sqrt{c^2 \sin^2\phi + b^2 \cos^2\phi}\, d\theta\, d\phi \\
&= 2\pi b \int_0^\pi \sin\phi \sqrt{c^2 \sin^2\phi + b^2 \cos^2\phi}\, d\phi \\
&= 2\pi b \int_0^\pi \sin\phi \sqrt{c^2 \sin^2\phi + b^2 (1 - \sin^2\phi)}\, d\phi \\
&= 2\pi b \int_0^\pi \sin\phi \sqrt{b^2 - (b^2 - c^2) \sin^2\phi}\, d\phi \\
&= 2\pi b^2 \int_0^\pi \sin\phi \sqrt{1 - (1 - c^2/b^2) \sin^2\phi}\, d\phi.
\end{aligned}$$

No $\sin x$ for $x \in [0, \pi/2]$ is equiv to $\sin x$ for $x \in [\pi/2, \pi]$ so

$$\text{Area} = 4\pi b^2 \int_0^{\pi/2} \sin\phi \sqrt{1 - (1 - c^2/b^2) \sin^2\phi}\, d\phi.$$

Now let $t = \sin^2\phi$ so $d\phi = t^{-1/2}(1-t)^{-1/2} dt/2$, so

$$\text{Area} = 2\pi b^2 \int_0^1 (1-t)^{-1/2} [1 - (1 - c^2/b^2)t]^{1/2}\, dt.$$

This is in Euler form with

$$a = -\frac{1}{2}, \quad b = 1, \quad c = \frac{3}{2}, \quad z = 1 - \frac{c^2}{b^2},$$

So

$$\text{Area} = 2\pi b^2 \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)} F(-1/2, 1; 3/2; 1 - c^2/b^2).$$

Also $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(3/2) = \Gamma(1/2)/2 = \sqrt{\pi}/2$ so

$$\text{Area} = 4\pi b^2 F(-1/2, 1; 3/2; 1 - c^2/b^2).$$

Question 4

$$F\{y\} = \int_0^1 \left(\frac{1}{2}y' + \frac{1}{2}y^2 - y\right)\,dx, \quad y(0) = 0, \quad y(1) = 0.$$

(a)

$$y_N = \phi_0 + \sum_{i=1}^{N} c_i\phi_i, \quad \phi_0 = 0, \quad \phi_i = x^i(1-x)^i.$$

$$\text{So} \quad y_1 = c_1 x(1-x).$$

(b)

$$\text{Differentiating} \quad y_1' = c_1(1-2x).$$

So

$$\begin{aligned}
F_1(c_1) &= \int_0^1 \left[\frac{1}{2}c_1(1-2x) + \frac{1}{2}c_1^2x^2(1-x)^2 - c_1x(1-x)\right]\,dx \\
&= \int_0^1 \left[c_1\left(\frac{1}{2} - 2x + x^2\right) + c_1^2\left(\frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4\right)\right]\,dx \\
&= c_1\left(\frac{1}{2} - 1 + \frac{1}{3}\right) + c_1^2\left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right) \\
&= \frac{c_1^2}{60} - \frac{c_1}{6}.
\end{aligned}$$

(c) Solving $dF_1/dc_1 = 0$ we find

$$\begin{aligned}
\frac{dF_1}{dc_1} = \frac{c_1}{30} - \frac{1}{6} &= 0 \\
\frac{c_1}{30} &= \frac{1}{6} \\
c_1 &= 5.
\end{aligned}$$

$$\text{So} \quad y_1 = 5x(1-x).$$

(d)

$$\frac{d^2F_1}{dc_1^2} = \frac{1}{30}.$$

Since $d^2F_1/dc_1^2 > 0$ this extremum is a minimum.

Question 5

$$F\{y\} = \int_0^1 (y'^2 + x^2)\,dx, \quad y(0) = y(1) = 0, \quad G\{y\} = \int_0^1 y^2\,dx = 2.$$

(a)

$$H\{y\} = F\{y\} + \lambda G\{y\} = \int_0^1 (y'^2 + x^2 + \lambda y^2)\,dx.$$

(b)

$$\text{So differentiating} \quad \frac{\partial h}{\partial y} = 2\lambda y, \quad \frac{\partial h}{\partial y'} = 2y'.$$

$$\begin{aligned} \text{Euler–Lagrange} \quad \Rightarrow \quad 2\lambda y - \frac{d}{dx}(2y') &= 0 \\ y'' - \lambda y &= 0. \end{aligned}$$

(c) There are three cases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$,

Case 1: $\lambda > 0$. Let $\lambda = \mu^2$

$$y'' - \mu^2 y = 0$$

$$\text{General solution:} \quad y = A\cosh\mu x + B\sinh\mu x.$$

Now applying the fixed end-point conditions

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad 0 = A. \\ y(1) = 0 \quad &\Rightarrow \quad 0 = B\sinh\mu \\ & \qquad B = 0. \end{aligned}$$

Only trivial solution.

Case 2: $\lambda = 0$.

$$y'' = 0$$

$$\text{General solution:} \quad y = Ax + B.$$

Now applying the fixed end-point conditions

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad 0 = B. \\ y(1) = 0 \quad &\Rightarrow \quad 0 = A. \end{aligned}$$

Only trivial solution.

Case 3: $\lambda < 0$. Let $\lambda = -\mu^2$

$$y'' + \mu^2 y = 0$$

$$\text{General solution:} \quad y = A\cos\mu x + B\sin\mu x.$$

Now applying the fixed end-point conditions

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad 0 = A. \\ y(1) = 0 \quad &\Rightarrow \quad 0 = B\sin\mu. \end{aligned}$$

Non trivial solutions need $B \neq 0$ and so we require $\sin\mu = 0$.

$$\begin{aligned} \text{So} \quad \mu &= n\pi, \quad n = 1, 2, 3, \cdots, \\ \lambda_n &= -n^2\pi^2. \end{aligned}$$

Thus from all three cases, the only non-trivial solution are $y = B_n \sin(n\pi x)$ for the discrete spectrum $\lambda_n = -n^2\pi^2$.

(d)

$$G\{y\} = \int_0^1 B_n^2 \sin^2(n\pi x)\, dx = 2.$$

Using the hint

$$\frac{B_n^2}{2} = 2 \quad \Rightarrow \quad B_n = \pm 2.$$

$$\begin{aligned} \text{So} \quad y &= \pm 2\sin(n\pi x) \\ y' &= \pm 2n\pi \cos(n\pi x). \end{aligned}$$

Now substituting in F

$$\begin{aligned} F\{y\} &= \int_0^1 (4n^2\pi^2 \cos^2(n\pi x) + x^2)\, dx \\ &= 4n^2\pi^2 \frac{1}{2} + \frac{1}{3} \\ &= 2n^2\pi^2 + \frac{1}{3}. \end{aligned}$$

By inspection minimum occurs for $n = 1$ so the minimum is

$$y = \pm 2\sin(\pi x), \quad \text{with} \quad F = 2\pi^2 + \frac{1}{3}.$$

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