Lecture 12: Hitting Probabilities – Our first performance measure

Concepts checklist

At the end of this lecture, you should be able to:

- *derive a system of linear equations for the hitting probability of a particular state* for simple CTMCs; and,
- *solve homogeneous second-order difference equations with constant coefficients*, in order to solve simple hitting probabilities equations.

Hitting Probabilities

So far, we've seen how to calculate

1. time-dependent probabilities for some simple CMTCs, and
2. equilibrium probabilities for some more complex CTMCs.

Now, we would like to calculate the probability that a CTMC ever reaches a given state. This is useful in answering questions we might have about a particular process, and can be used as a *performance measure*, but it is also useful for determining whether a state is recurrent or not.

Example 7. Finite-capacity single-server queue.

A single-server queue with finite capacity N has arrival rate λ and service rate μ. Without loss of generality we assume that N is even and that the system starts half full.

Question: What is the probability that the system empties before it fills up?

We use a slight *modification* of the earlier single-server queue continuous-time Markov chain. We put $\mathcal{S} = \{0, 1, 2, \ldots, N\}$ and the non-zero rates are

$$\begin{aligned} q_{n,n+1} &= \lambda, \qquad \text{for } n = 1, 2, \ldots, N-1, \\ q_{n,n-1} &= \mu, \qquad \text{for } n = 1, 2, \ldots, N-1, \\ q_{nn} &= -(\lambda + \mu), \qquad \text{for } n = 1, 2, \ldots, N-1. \end{aligned}$$

Note: States 0 and N have no transition rate out and hence are now absorbing states.

This new Markov chain is not irreducible, as it has (a) two recurrent communicating classes $\{0\}$ and $\{N\}$ and, (b) one transient communicating class $\{1, 2, \ldots, N-1\}$.

Since the CTMC will be absorbed into one of the states 0 or N,

$$\Pr(\text{the chain visits state 0 before state } N) = \Pr(\text{it ever visits state 0}).$$

We want to know, for states $1, 2, \ldots, N-1$, the probability that the CTMC is absorbed in state 0 rather than state N.

Let f_i be the probability that the chain is absorbed in state 0 given the initial state i:

$$f_i = \Pr(\text{absorbed in state } 0 \mid \text{starts in state } i).$$

Assuming that $i \neq \{0, N\}$ we use a *first step analysis*:

$$\begin{aligned}
f_i &= \Pr(\text{absorbed in state } 0 \mid \text{starts in state } i) \\
&= \Pr(\text{goes to } i+1 \text{ in the next step, gets absorbed in state } 0 \mid \text{ starts in state } i) \\
&\quad + \Pr(\text{goes to } i-1 \text{ in the next step, absorbed in state } 0 \mid \text{ starts in state } i) \\
&= \Pr(X(t_1) = i+1 \cap \text{absorbed in state } 0 \mid X(0) = i) \\
&\quad + \Pr(X(t_1) = i-1 \cap \text{absorbed in state } 0 \mid X(0) = i)
\end{aligned}$$

where t_1 is the first time the Markov chain makes a transition out of i,

$$\begin{aligned}
&= \Pr(\text{absorbed in state } 0 \mid X(0) = i, X(t_1) = i+1) \Pr(X(t_1) = i+1 \mid X(0) = i) \\
&\quad + \Pr(\text{absorbed in state } 0 \mid X(0) = i, X(t_1) = i-1) \Pr(X(t_1) = i-1 \mid X(0) = i) \\
&= \Pr(\text{absorbed in state } 0 \mid X(t_1) = i+1) \frac{\lambda}{\lambda+\mu} + \Pr(\text{absorbed in state } 0 \mid X(t_1) = i-1) \frac{\mu}{\lambda+\mu}
\end{aligned}$$

by the Markov property, and by the facts that $\lambda/(\lambda+\mu)$ is the probability of a transition from i to $i+1$, and $\mu/(\lambda+\mu)$ is the probability of a transition from i to $i-1$,

$$= \Pr(\text{absorbed in state } 0 \mid X(0) = i+1) \frac{\lambda}{\lambda+\mu} + \Pr(\text{absorbed in state } 0 \mid X(0) = i-1) \frac{\mu}{\lambda+\mu}$$

by the time-homogeneity property,

$$= \frac{\lambda}{\lambda+\mu} f_{i+1} + \frac{\mu}{\lambda+\mu} f_{i-1}.$$

In summary, we have

$$f_i = \frac{\lambda}{\lambda+\mu} f_{i+1} + \frac{\mu}{\lambda+\mu} f_{i-1} \quad \text{for } i = 1, \ldots, N-1, \tag{14}$$

with the following boundary conditions $f_0 = 1$ and $f_N = 0$.

Equations (14) are simply a system of linear equations. Try to solve these equations using a standard approach. See if you can write it in a matrix-vector form.

Equations (14) can also be viewed as a homogeneous second-order difference equation with constant coefficients[2]. One way to solve these is to try a solution of the form $f_i = m^i$, which

[2]The *homogeneous* part refers to the property that we can rearrange (14) as

$$\frac{\lambda}{\lambda+\mu} f_{i+1} - f_i + \frac{\mu}{\lambda+\mu} f_{i-1} = 0 \quad \text{for } i = 1, \ldots, N-1,$$

with 0 on the right-hand side, as opposed to an *inhomogeneous* equation

$$\frac{\lambda}{\lambda+\mu} f_{i+1} - f_i + \frac{\mu}{\lambda+\mu} f_{i-1} = c \quad \text{for } i = 1, \ldots, N-1,$$

for some constant $c \neq 0$.

we substitute into (14) to get

$$m^i = \frac{\lambda}{\lambda+\mu}m^{i+1} + \frac{\mu}{\lambda+\mu}m^{i-1}.$$

Thus,

$$\lambda m^2 - (\lambda+\mu)m + \mu = 0 \quad \Leftrightarrow \quad (\lambda m - \mu)(m-1) = 0 \quad \Leftrightarrow \quad m = \frac{\mu}{\lambda}, 1.$$

• If $\mu \neq \lambda$, then the general solution is of the form $f_i = A\left(\frac{\mu}{\lambda}\right)^i + B(1)^i = A\left(\frac{\mu}{\lambda}\right)^i + B$.

Now, we use the boundary conditions see that

$$\begin{aligned}
f_0 = 1 \quad &\Rightarrow A + B = 1, \quad \text{ so that } B = 1 - A, \\
f_N = 0 \quad &\Rightarrow A\left(\frac{\mu}{\lambda}\right)^N + 1 - A = 0 \quad \Rightarrow A = \frac{1}{1-\left(\frac{\mu}{\lambda}\right)^N} \\
&\Rightarrow f_i = \frac{\left(\frac{\mu}{\lambda}\right)^i - \left(\frac{\mu}{\lambda}\right)^N}{1-\left(\frac{\mu}{\lambda}\right)^N} \\
&\Rightarrow f_{N/2} = \frac{\left(\frac{\mu}{\lambda}\right)^{N/2} - \left(\frac{\mu}{\lambda}\right)^N}{1-\left(\frac{\mu}{\lambda}\right)^N}.
\end{aligned}$$

• If $\mu = \lambda$, we have repeated roots of $m = 1$, which implies that the general solution is of the form $f_i = Ai(1) + B(1) = Ai + B$.

Using the boundary conditions:

$$\begin{aligned}
f_0 = 1 &\Rightarrow B = 1 \\
f_N = 0 &\Rightarrow AN + 1 = 0 \\
&\Rightarrow \quad A = -\frac{1}{N} \\
&\Rightarrow \quad f_i = 1 - \frac{i}{N}.
\end{aligned}$$

In this case, $f_{N/2} = \frac{N/2}{N} = \frac{1}{2}$, which fits with intuition.
