Optimal Functions and Nanomechanics III

APP MTH 3022/7106

Barry Cox

Lecture 5

Last lecture

- Considered fixed point variational problems
- Derived the Euler-Lagrange equation for the first variation
- Calculated geodesics in the plane and found they were straight lines
- Analysed general functionals of the form $f(y')$ and found they were always straight lines
- Looked at the refraction of light, Snell's law and Fermat's principle

Special case 2

When f has no dependence on x we call this an autonomous problem, and we can replace the Euler-Lagrange equation with

$$H(y, y') = y' \frac{\partial f}{\partial y'} - f(y, y') = \text{const.}$$

We will see H again later – it often turns out to be a conserved quantity like energy, and so arises naturally in computing the shape of a catenary.
This is also called the Beltrami identity.

Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0}$$

Autonomous case

The autonomous (x-absent) case is where f has no explicit dependence on x, so $\partial f/\partial x = 0$.

Theorem 2.3.1: Let J be a functional of the form

$$J\{y\} = \int_{x_1}^{x_2} f(y, y')dx$$

and define the function H by

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f(y, y')$$

Then H is constant along any extremal of y.

Proof of Theorem 2.3.1

$$
\begin{aligned}
\frac{d}{dx}H(y,y') &= \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f(y,y')\right), \\
&= y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} - y'\frac{\partial f}{\partial y} - y''\frac{\partial f}{\partial y'} \\
&= y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right) \\
&= 0
\end{aligned}
$$

So

$$H(y,y') = \text{const.}$$

□

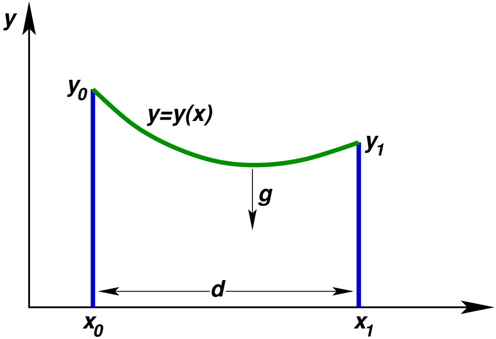
NB: this is a first order differential equation for the extremal y.

The Catenary

The potential energy of the cable is

$$W_p\{y\} = \int_0^L mgy(s)\, ds,$$

where L is the length of the cable, m is the mass per unit length and g is the gravitational constant.



The Catenary

Catenary is derived from the Latin word catena, which means "chain"
Examples: power-lines, hanging chains, spider web
The catenary is also called

- chainette (French)
- alysoid (the catenary is a special case of an alysoid)
  http://www.2dcurves.com/exponential/exponentiala.html
- funicular curve (a funicular polygon is formed by having a cord fastened at its ends, with weights at different points).
  http://dicoweb.gnu.org.ua/?q=Funicular&db=gcide&define=1
  A funicular rail (for instance) uses a chain to pull its cars up a steep slope.

The Catenary, reformulation

As with geodesic in the plane

$$ds = \sqrt{1 + y'^2}\, dx$$

So the functional of interest (the potential energy) is

$$W_p\{y\} = mg \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx$$

which does not contain x explicitly.

$$H(y, y') = y' \frac{\partial f}{\partial y'} - f = \text{const.}$$

where $f(y, y') = y\sqrt{1 + y'^2}$.

The Catenary (iii)

$$
\begin{aligned}
c_1 &= H(y, y') \\
&= y' \frac{\partial f}{\partial y'} - f \qquad\qquad \text{where } f(y, y') = y\sqrt{1 + y'^2} \\
&= y' \frac{yy'}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} \\
c_1\sqrt{1 + y'^2} &= yy'^2 - y(1 + y'^2) \\
c_1\sqrt{1 + y'^2} &= -y \\
c_1^2(1 + y'^2) &= y^2 \\
\frac{y^2}{1 + y'^2} &= c_1^2
\end{aligned}
$$

The Catenary (iv)

If $c_1 = 0$ the only solution is $y = 0$.
If $c_1 \neq 0$ then, rearrange to get

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{c_1^2} - 1}$$

$$dx = \frac{dy}{\sqrt{\frac{y^2}{c_1^2} - 1}}$$

$$\int dx = \int \frac{dy}{\sqrt{\frac{y^2}{c_1^2} - 1}}$$

$$x - c_2 = \int \frac{1}{\sqrt{\frac{y^2}{c_1^2} - 1}} dy$$

The Catenary (v)

Now

$$\frac{d}{dx}\left(\cosh^{-1} u\right) = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx},$$

So taking $u = y/c_1$ we get

$$\frac{d}{dx}\cosh^{-1}\left(y/c_1\right) = \frac{1}{\sqrt{y^2/c_1^2 - 1}}\frac{1}{c_1},$$

So, the integral above results in

$$x - c_2 = c_1 \cosh^{-1}\left(y/c_1\right).$$

The Catenary (vi)

The extremals are thus given by

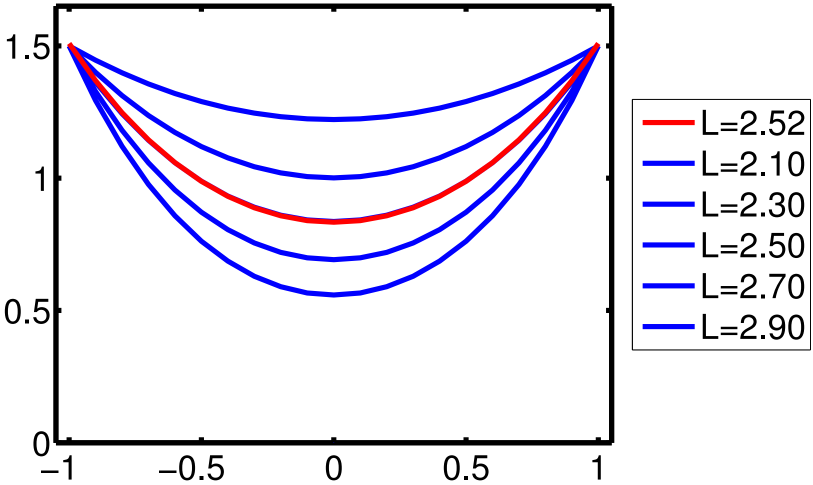
$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$$

In particular, the minimal potential energy occurs when y takes this form, a **catenary**.

The constants c_1 and c_2 are determined by the end conditions, the heights of the poles, e.g. $y(x_0) = x_0$ and $y(x_1) = x_1$.

Notice that we didn't specify L anywhere here.

Catenaries of different L



Finding the constants

$\cosh(\cdot)$ is an even function so if $x_0 = -1$ and $x_1 = 1$, and $y_1 = y_2$ then the constant $c_2 = 0$. So we can rewrite this as

$$y(x) = c_1 \cosh\left(\frac{x}{c_1}\right)$$

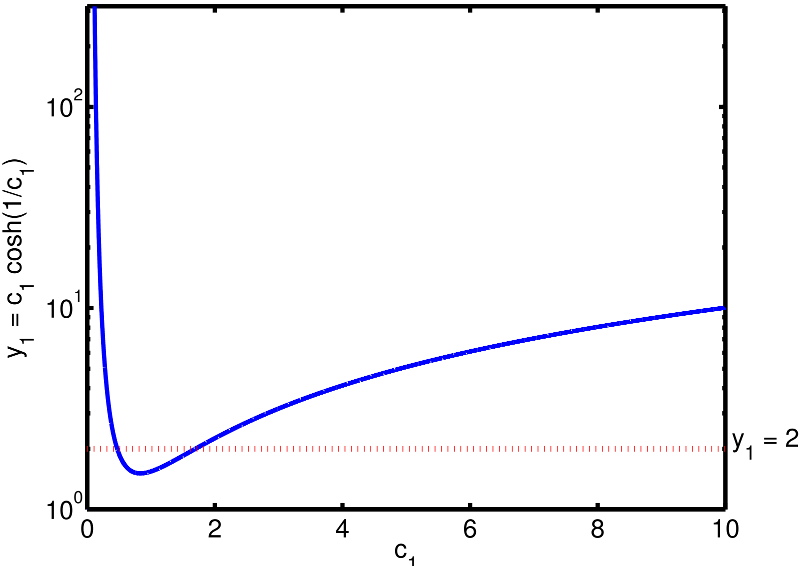
which we solve for $y(1) = c_1 \cosh(1/c_1) = y_1$ to get c_1.

- non-linear, so solve numerically

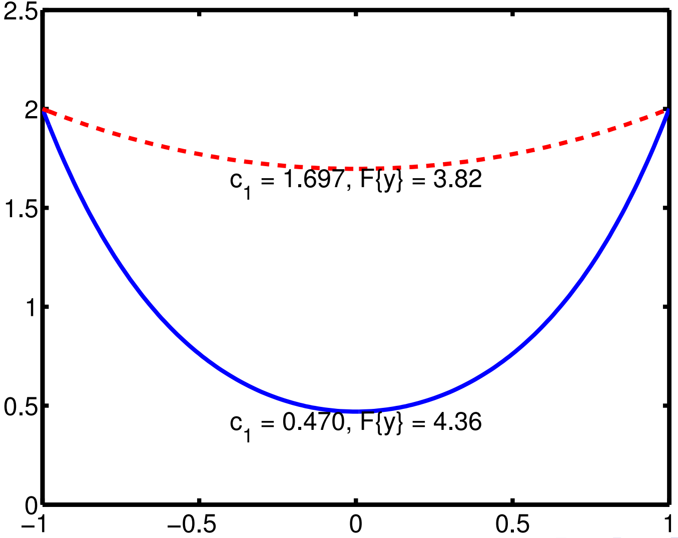
For instance $y(1) = 2$ we get two possible values $c_1 = 0.47$ and $c_1 = 1.697$

- they don't have to both be minima
- one could be a maxima, or a stationary point

Finding the constants



Finding the constants



Existence of a solution

In the above solution, note that for some values of y_0 and y_1, we can get multiple solution, but in some cases there may be a unique solution, or no solutions!!!

Calculating the functional

Once we know y, it is (in principle) easy to calculate $F\{y\}$, e.g., for the catenary note the following identities

$$\begin{aligned}\frac{d}{dx} c_1 \cosh(x/c_1) &= \sinh(x/c_1)\\ 1 + \sinh^2(x/c_1) &= \cosh^2(x/c_1)\end{aligned}$$

and so

$$\begin{aligned}F\{y\} &= \int_{-1}^{1} y\sqrt{1+y'^2}\, dx\\ &= \int_{-1}^{1} c_1 \cosh(x/c_1)\sqrt{1+\sinh^2(x/c_1)}\, dx\\ &= \int_{-1}^{1} c_1 \cosh^2(x/c_1)\, dx\end{aligned}$$

Calculating the functional

Now note that

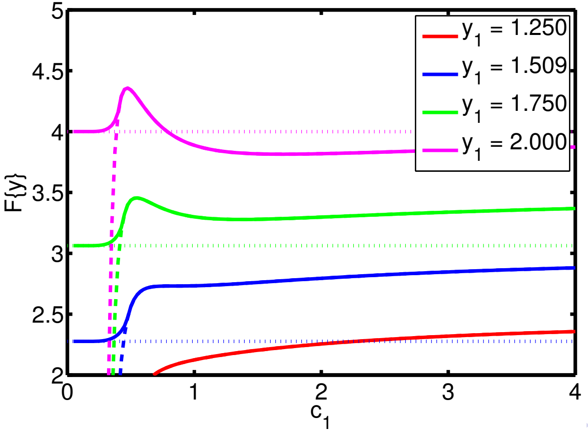
$$\cosh^2(x) = (\cosh(2x) + 1)/2$$

so that

$$\begin{aligned} F\{y\} &= \frac{c_1}{2} \int_{-1}^{1} (\cosh(2x/c_1) + 1)\, dx \\ &= \frac{c_1}{2} \int_{-1}^{1} dx + \frac{c_1}{2} \int_{-1}^{1} \cosh(2x/c_1)\, dx \\ &= c_1 + \frac{c_1^2}{4} \left[\sinh(2x/c_1)\right]_{-1}^{1} \\ &= c_1 + \frac{c_1^2}{2} \sinh(2/c_1) \end{aligned}$$

Calculating the functional

You can think of the length as changing slowly, so at each point in time, the shape is a catenary with constant c_1, where this varies over time, i.e., optimise with respect to c_1.



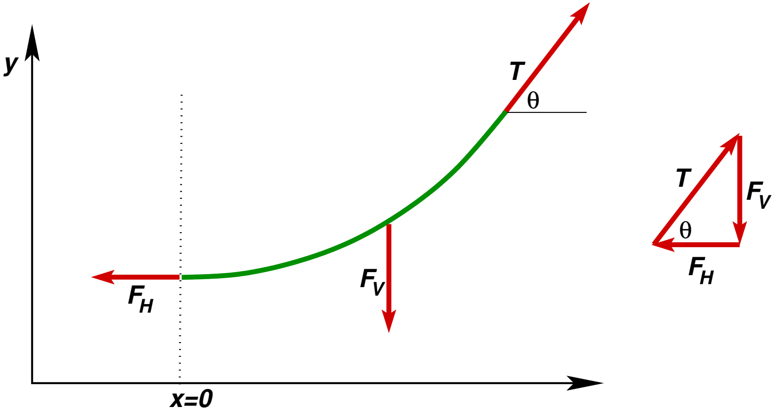
The length of the Catenary

$$
\begin{aligned}
L\{y\} &= \int_{-1}^{1} \sqrt{1 + y'^2}\, dx \\
&= \int_{-1}^{1} \cosh(x/c_1)\, dx \\
&= c_1 \left[\sinh(x/c_1)\right]_{-1}^{1} \\
&= 2c_1 \sinh(1/c_1)
\end{aligned}
$$

But note that in this version of the problem we can't **set** the length, it is an output. Later on we will constrain the length so it is an input to the problem.

Catenary addendum

The usual explanation for the shape of the catenary is based on a simple physical argument: **forces must be balanced in equilibrium.**



Catenary addendum

forces must be balanced in equilibrium so tension in the cable (which must be in the direction of the cable) must balance the horizontal force F_H at the lowest point, and the downwards force F_V. The results is

$$\tan\theta = \frac{F_V}{F_H}$$
$$\frac{dy}{dx} = \frac{gms}{F_H}$$

where ms is the mass of the cable integrated from $[0, s]$ along the cable, and F_H is constant.

Catenary addendum

Taking derivatives with respect to x we get

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{m(x)g}{F_H}$$
$$y'' = \frac{mg}{F_H}\frac{ds}{dx}$$

where we know that $ds/dx = \sqrt{1+y'^2}$ so

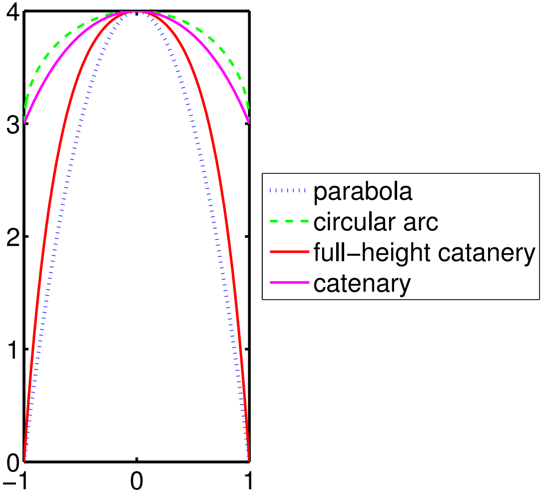
$$\frac{y''}{\sqrt{1+y'^2}} = \frac{mg}{F_H}$$

which has the same solution, but now c_1 has a meaning

$$y(x) = \frac{F_H}{mg}\cosh\left(\frac{mg}{F_H}x\right).$$

The shape of an arch

Flip a catenary upside down, and the above argument shows simply that the strongest form of an arch is an inverted catenary. This balances the forces at each point, so that the arch is under the least possible stress.



Note that F_H must be applied to the edges or the arch will collapse outwards.

The shape of an arch

However, this argument assumes that the arch's own weight is all that matters. Commonly, an arch supports a wall above, and so the forces are not so simply described. The shape that is optimal is closer to the shape of a suspension bridge, which we shall see in tutorials is a parabola.

- BTW, the Gateway Arch in St Louis isn't strictly a catenary as is sometimes claimed.
  `http://www.springerlink.com/content/u7734w06700776x0/`
- the optimal form changes if the "arch" isn't a pure curve, but has shape.

Other arches

- Sheffield Winter Garden
  http://en.wikipedia.org/wiki/Sheffield_Winter_Garden
  http://algebraproject07.wikispaces.com/Mathematical+Information+of+Sheffield+Winter+Garden
- Arches under Gaudi's Cassa Milà
  http://en.wikipedia.org/wiki/Casa_Mil%C3%A0
- Dome in St Paul's Catherdal
  http://en.wikipedia.org/wiki/St_Paul%27s_Cathedral

There are others but they often aren't exact catenaries – sometimes they are parabolas, which is also the shape of a suspension bridge. (By the way, the difference is tiny for such cases.)

Some history

- 1638, Galileo, a hanging cord is an approximate parabola, and the approximation improves as the curvature gets smaller
- Joachim Jungius showed it wasn't a parabola (published posthumously in 1669)
- Hooke discovered optimal shape of arch in 1671 published it as a Latin Anagram
  - Published posthumously in 1705 as "Ut pendet continuum flexile, sic stabit contiguum rigidum inversum", meaning "as hangs a flexible cable so, inverted, stand the touching pieces of an arch."
- Derived by Leibniz, Huygens and Johann Bernoulli in 1691
- Euler worked on related problems in 18th century