Random Processes Assignment 5

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Question 1. Consider a simple circuit-switched loss network consisting of 8 nodes (labelled A, B, C, D, E, F, G and H), and 9 links (labelled from 1 to 9), as shown in Figure 1, where the edges are labelled with ID:Capacity

(not including the graphics here)

There are four routes in the network; we assume that calls arrive to these routes as independent Poisson processes, and that all calls use 1 circuit on each link and have an exponentially distributed holding time with unit mean and use 1 circuit on each link it uses. Their arrival rates and links used are listed in the following table.

Note: for loss networks, we always assume an infinite number of servers.

Route Label	Route	Arrival Rate	Links Used
1	A-C	1	1,2
2	C-H	2	4,7,9
3	D-G	2	5,6,7
4	A-H	1	3,5,8

(a) By defining all necessary notation, write down an appropriate state space for a CTMC representation of this circuit-switched network.
Solution State space S:

$$S = \{\mathbf{n} : A\mathbf{n} \leq \mathbf{c}\}$$

Where A is the matrix of A_{ij} where A_{ij} is the number of circuits that route j uses on link i, and c is the vector of c_j such that c_j is the number of circuits on link j.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad c = (3, 3, 3, 6, 6, 3, 6, 3, 3)$$

As required.

(b) Write down an expression for the equilibrium distribution for this network.
Solution

$$\begin{aligned} \pi(\mathbf{n}) &= [G(\mathbf{c}, R)]^{-1} \prod_{r=1}^{R} \frac{\lambda_r^{n_r}}{n_r!} \\ &= [G(\mathbf{c}, 4)]^{-1} \frac{1}{n_1!} \frac{2^{n_2}}{n_2!} \frac{2^{n_3}}{n_3!} \frac{1}{n_4!} \\ &= [G(\mathbf{c}, 4)]^{-1} \frac{2^{n_2+n_3}}{n_1! n_2! n_3! n_4!} \end{aligned}$$

Where

$$G(\mathbf{c}, R) = \sum_{\mathbf{n}:A\mathbf{n}\leq\mathbf{c}} \prod_{r=1}^{R} \frac{\lambda_r^{n_r}}{n_r!}$$

As required.

(c) Write down an expression for the blocking probability of calls on Route 2 (that is between nodes C and H)

Solution Using theorem 13:

$$B_2 = 1 - \frac{G(\mathbf{c} - Ae_2, 4)}{G(\mathbf{c}, 4)}$$

Where $e_2 = (0, 1, 0, 0)^T$ **As required.**

(d) Write down the expressions required to define the Erlang Fixed Point approximation for a circuit-switched network, including an expression for the blocking probability on a route.

Note: You are not being asked to solve anything.

Solution

$$\alpha_j = B(c_j, y_j)$$

Where

$$y_j = \sum_{r:j\in r} a_r \prod_{\substack{i\in r\\ i\neq j}} (1 - \alpha_i)$$

Where

- a_r is the offered load on route r;
- $B()$ is the Erlang loss formula;

$$B(c_j, y_j) = \frac{y_j B(c_j - 1, y_j)}{c_j + y_j B(c_j - 1, y_j)}$$

- y_j is the offered load on link j;
- α_i the blocking probability for link i, and;
- c_j defined as above

As required.

(e) What assumption is made about the links on the network, in order to justify the Erlang Fixed Point approximation? Comment on the validity of this assumption, and whether it affects the approximation.

Solution Assuming the links are **independent**, and calls are only accepted if accepted on all links.

i. Independence: This is unreasonable - Burke's theorem states that for $\mu > \lambda$, then independence for downstream links holds. In this case, for all routes, $\mu \geq \Lambda$.
ii. accepted iff all links accepted - this is reasonable as a design assumption.

As required.

(f) Evaluate the approximate blocking probabilities of calls on Route 2 through use of the Erlang Fixed Point Method, as outlined at the end of Lecture 20. Provide code.

Solution

The code is appended at the end of the document. It gives the output

```
BlockProb =

    0.1079
    0.2448
    0.2555
    0.1256
```

I.e. we find that the blocking probability for calls on Route 2 is

$$B_2 \approx 0.2448$$

As required.

Question 2. A CTMC model of a particular queue has states $n \in S = \{0, 1, 2, 3\}$, where n represents the number of customers in the queue. Assume that the arrival rate into the queue when it is in state n, λ_n, is given by

$$\lambda_n = (5 - n)\lambda, n = 0, 1, 2, 3$$

and that all arrivals to the queue when it is in state 0, 1 or 2 are accepted, and all arrivals to the queue when it is in state 3 are lost. Finally, note that the equilibrium distribution is given by $\pi = (2/5, 1/5, 1/5, 1/5)$

(a) What is the equilibrium probability that there are 3 customers in the queue? (Hint: this is easier than you think!)

Solution $\pi_3 = 1/5$

As required.

(b) What is the equilibrium probability that an arrival sees 3 customers in the queue, in other words, what is the blocking probability?

Solution

$$\begin{aligned}
\pi_3^{(arrival)} &= \frac{\gamma_j \pi_j}{\sum_{k \in S} \gamma_k \pi_k} \\
&= \frac{\gamma_3 \pi_3}{\sum_{n=0}^{3} \gamma_n \pi_n} \\
&= \frac{(5-3)\lambda \frac{1}{5}}{\sum_{n=0}^{3}(5-n)\lambda \pi_n} \\
&= \frac{2}{5\,(2 + 4/5 + 3/5 + 2/5)} \\
&= \frac{2}{10 + 4 + 3 + 2} \\
&= \frac{2}{19}
\end{aligned}$$

As required.

(c) What is the average length of the queue?

Solution

$$\bar{L} = E[length] = \sum_{n=0}^{3} \pi_n n = 2/5 * 0 + 1/5 * 1 + 1/5 * 2 + 1/5 * 3 = 6/5$$

As required.

(d) Determine the average time that a customer spends in the system.

Solution Littles law!

$$\bar{L}(\Gamma) = \bar{\lambda}(\Gamma)\bar{W}(\Gamma)$$

Where $\bar{\lambda}$ average arrival rate, $\bar{W}$ average time in queue. (Im just going to drop the gammas) We just got

$$\begin{aligned}
\bar{L} &= \bar{\lambda}\bar{W} \\
\bar{W} &= \frac{\bar{L}}{\bar{\lambda}} \\
&= \frac{6}{5\sum_{n=0}^{3} \lambda_n \pi_n} \\
\therefore \bar{W} &= \frac{6}{19\lambda}
\end{aligned}$$

I.e. the average time in queue is $\frac{6}{19\lambda}$ units of time

As required.

Question 3. Consider a counting process, $N(t)$, which counts the number of times a particular component is replaced. The lifetime of the component has a CDF of $F(t)$. Every time the component is replaced, there is a probability p that it will fail instantly; otherwise, the lifetime is assumed to be exponential with rate λ.

(a) Show that the CDF $F(t)$ is given by

$$F(t) = 1 - (1-p)e^{-\lambda t}$$

Solution Start with the complement of $F(t)$ i.e. $1 - F(t)$ - the CDF of failure.
If we first ignore the probability of the component failing instantly, we get

$$F(t|\textit{no fail at 0}) = 1 - e^{-\lambda t}$$

Then considering that with probability p, we fail instantly (i.e. the lifetime is 0), and with probability $1-p$ we continue. We can piecewise define as:

$$F(t) = \begin{cases} p & t = 0 \\ p + (1-p)(1-e^{-\lambda t}) & t > 0 \end{cases}$$

$$\begin{aligned} F(t)_{t>0} &= p + 1 - p + pe^{-\lambda t} - e^{-\lambda t} \\ &= 1 + (p-1)e^{-\lambda t} \\ &= 1 - (1-p)e^{-\lambda t} \end{aligned}$$

And note that when $t = 0$ we simply get p, so it gives:

$$F(t) = 1 - (1-p)e^{-\lambda t}$$

As required.

(b) Determine the Laplace-Stieltjes transform, $\hat{F}(s)$ of $F(t)$

Solution

$$\begin{aligned} \hat{F}(s) &= \int_0^\infty e^{-st} d\left(1 - (1-p)e^{-\lambda t}\right) \\ &= p + \int_0^\infty \lambda(p-1)e^{-(\lambda+s)t} dt \\ &= p + \frac{\lambda(p-1)e^{-(\lambda+s)t}}{\lambda+s} \Big|_0^\infty \\ &= p - \frac{\lambda(p-1)}{\lambda+s} \\ &= p - \frac{\lambda(p-1)}{\lambda+s} \\ &= \frac{p(\lambda+s) - \lambda(p-1)}{\lambda+s} \\ \hat{F}(s) &= \frac{ps+\lambda}{\lambda+s} \end{aligned}$$

As required.

(c) Determine the Laplace-Stieltjes transform, $\hat{M}(s)$ of $M(t)$, and consequently determine $M(t)$.

Solution

$$\begin{aligned}
\hat{M}(s) &= \frac{\hat{F}(s)}{1-\hat{F}(s)} \\
&= \frac{\frac{ps+\lambda}{\lambda+s}}{1-\frac{ps+\lambda}{\lambda+s}} \\
&= \frac{ps+\lambda}{\lambda+s-ps-\lambda} \\
&= \frac{ps+\lambda}{s(1-p)} \\
&= \frac{\lambda}{s(1-p)} + \frac{p}{(1-p)} = \int_0^\infty e^{-st}M'(t)dt + M(0)
\end{aligned}$$

$$\implies M'(t) = \frac{\lambda}{1-p}, \quad M(0) = \frac{p}{1-p}$$

$$\therefore M(t) = \frac{\lambda t}{1-p} + \frac{p}{1-p}$$

As required.

The code for $1f$

```
%%Calculate the blocking probability on route 2 using EFPM
%%Andrew Martin
%%1704466
%%25/10/2018


%%declaring variables
alpha = zeros(9,1);
y = zeros(9,1);
c=[3;3;3;6;6;3;6;3;3]';
a=[1;2;2;1];
BlockProb=zeros(4,1);
numLinks=9;
numRoutes=4;
A =[1,0,0,0;
    1,0,0,0;
    0,0,0,1;
    0,1,0,0;
    0,0,1,1;
    0,0,1,0;
    0,1,1,0;
    0,0,0,1;
    0,1,0,0];
%Termination conditions for the while loop
tol=100*eps;
ticker=1;

%to enter the loop since matlab doesn't have do-while loops
alphatemp=inf*ones(9,1);
%while we are outside our tolerance
while (sum(abs(alphatemp - alpha)) > tol && ticker<=1000)
    alphatemp=alpha;
    for j=1:numLinks
         %Find r s.t. j in r
          r = find(A(j,:));

          temp=0;

          %r will give multiple values
          for index = 1:length(r)
              i = find(A(:,r(index)));
              %remove the i==j case
              i(i==j) = [];
              prodpart = prod(1- alpha(i));
              temp = temp + (a(r(index)).*  prodpart);


          end
          y(j) = temp;

        alpha(j) = B(c(j),y(j));
    end
    ticker= ticker +1;
end
```

```
for r=1:4
    BlockProb(r) = 1- prod((1-alpha).^A(:,r));
end

%just to print it lazily
BlockProb(2)


%Calcualtes the Erlang B formula recursively
function Bout = B(c,y)
if(c==0)
    Bout = 1;
else
    temp = B(c-1,y);

    Bout = y*temp/(c+y*temp);
end
end
```