$$\omega_z = \frac{1}{r}\frac{\partial rv}{\partial r}, \quad \omega_r = -\frac{\partial rv}{\partial z}, \quad \omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$
$$w = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad u = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$$

$$\begin{aligned}\omega_\theta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ &= -\frac{1}{r}\frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \Psi}{\partial r} \\ &= -\frac{1}{r}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r}\right)\end{aligned}$$

$$w \times v - \frac{\partial w}{\partial t} = \nabla H$$
$$H = \frac{1}{2}(w^2 + u^2 + v^2) + \frac{p}{\rho}$$

$$\begin{aligned}u\omega_\theta - v\omega_r - \frac{\partial w}{\partial t} &= \frac{\partial H}{\partial x} \\ v\omega_z - w\omega_\theta - \frac{\partial u}{\partial t} &= \frac{\partial H}{\partial r} \\ w\omega_r - u\omega_z - \frac{\partial v}{\partial t} &= 0\end{aligned}$$

$$\frac{D(rv)}{Dt} = 0$$

$$rv = C(\Psi)$$
$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}|\vec{w}|^2 + \frac{p}{\rho} = H(\Psi)$$

$$\omega_z = w\frac{dC}{d\Psi}, \quad \omega_r = u\frac{dC}{d\Psi}$$

$$\frac{\omega_\theta}{r} = \frac{v\omega_r}{ru} + \frac{1}{ru}\frac{dH}{d\Psi}\frac{\partial \Psi}{\partial x} = \frac{C}{r^2}\frac{dC}{d\Psi} - \frac{dH}{d\Psi}$$
$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = r^2\frac{dH}{d\Psi} - C\frac{dC}{d\Psi}$$