Real Analysis II **Semester 2, 2017**

Class Exercise 4, due 5 pm Tuesday 3 October 2017

1. Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Suppose that $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$. Prove that $f([a,b]) = [m,M]$. (Hint: most likely you will need to use three facts, Theorem 4.7, Theorem 4.8, and the fact that the restriction of a continuous function to a subset of it's domain is continuous.)

[4 points]

2. Suppose that $f\colon [a,b] \to \mathbb{R}$ is an integrable function.

(a) Let $c \geq 0$. Let $\mathscr{P} = \{ x_0, \ldots, x_N \}$ be a partition of $[a,b]$. Prove that $m_i(cf) = cm_i(f)$ and $M_i(cf) = cM_i(f)$ for each $i = 1, \ldots, N$. (You may assume without proof that $\sup(cS) = c\sup(S)$ and $\inf(cS) = c\inf(S)$ for any non-empty bounded set $S \subset \mathbb{R}$.)

(b) Still under the assumption that $c \geq 0$, use part (a) to show that $L(cf) = cL(f)$ and $U(cf) = cU(f)$.

(c) Use Class Exercise 1 Question 2 (iii) to prove that $L(-f) = -U(f)$ and $U(-f) = -L(f)$.

(d) Using the previous parts of the question prove that cf is integrable on $[a,b]$ for any $c \in \mathbb{R}$ and that $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

[8 points]

3. Let $f\colon [a,b] \to \mathbb{R}$ be an increasing function (i.e. for all $x_1, x_2 \in [a,b]$, if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$).

(a) Prove that f is bounded.

(b) Calculate $U(f,\mathscr{P}) - L(f,\mathscr{P})$ for a *regular* partition $\mathscr{P}$ of $[a,b]$.

(c) Hence prove that f is integrable on $[a,b]$.

(d) Is the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - 1/n & \text{if } 1 - 1/n \leq x < 1 - 1/(n+1), \text{ where } n \in \mathbb{N}, \\ 1 & \text{if } x = 1 \end{cases}$$

integrable? Prove your answer.

[7 points]

4. Let $a < c < b$ be real numbers and suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable.

(a) Prove that the function $g\colon [a,b] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \neq c, \\ 1 & \text{if } x = c \end{cases}$$

is integrable on $[a,b]$ and that $\int_a^b g(x)\,dx = 0$.

(b) Let $k \neq 0$ be a real number. Use Theorem 5.5 and part (a) to prove that the function $F\colon [a, b] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \neq c, \\ f(c) + k & \text{if } x = c \end{cases}$$

is integrable on $[a, b]$ and $\int_a^b F(x)\,dx = \int_a^b f(x)\,dx$.

[6 points]

5. Prove that the function $f\colon [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

is integrable on $[0, 1]$. (You may assume that the function sin is continuous on $\mathbb{R}$ and that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$; these are all the facts about this function that you will need. Suggestion: use Theorem 5.3.)

[5 points]