# LECTURE 24

Recall from last time the following definition:

**Definition 6.9**: Suppose that $f\colon S \to \mathbb{R}$ is a function and that $x_0 \in S$ is a limit point of $S$. We say that $f$ is *differentiable* at $x_0 \in S$ if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exists. If $f$ is differentiable at $x_0$ then we write $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ and we call the number $f'(x_0)$ the *derivative* of $f$ at $x_0$. If every point of $S$ is a limit point of $S$ (for example if $S$ is an interval) then we say $f$ is *differentiable on* $S$ if $f$ is differentiable at $x_0$ for all $x_0 \in S$.

We give some examples:

**Example 1**: Let $c$ be a real number and let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$ for all $x \in \mathbb{R}$. Then $f$ is differentiable on $\mathbb{R}$ and $f'(x_0) = 0$ for all $x_0 \in \mathbb{R}$. This is extremely easy to prove: if $x \neq x_0$ then

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{0}{x-x_0} = 0,$$

hence

$$\frac{f(x)-f(x_0)}{x-x_0} \to 0 \quad \text{as } x \to x_0 0.$$

**Example 2**: Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$. Then $f$ is differentiable on $\mathbb{R}$ and $f'(x_0) = 1$ for all $x_0 \in \mathbb{R}$. Again, this is extremely easy to prove: if $x \neq x_0$ then

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{x-x_0}{x-x_0} = 1$$

and hence

$$\frac{f(x)-f(x_0)}{x-x_0} \to 1 \quad \text{as } x \to x_0.$$

**Example 3**: Let $f\colon [-1,1) \to \mathbb{R}$ be defined by $f(x) = |x|$ for $x \in [-1,1)$. Then $f$ is differentiable at $x_0$ for all $x_0 \in [0,1) \setminus \{0\}$. If $x_0$ then

$$\frac{f(x)-f(x_0)}{x-x_0} = \begin{cases} -1 & \text{if} -1 \leq x < 0, \\ 1 & \text{if } 0 < x < 1 \end{cases}$$

from which it follows that $(f(x)-f(0))/(x-0)$ does not have a limit as $x \to 0$. Hence $f$ is not differentiable at $x_0 = 0$. Note that $f$ is differentiable at $x_0 = 1$.

**Example 4**: Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } x \notin \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$. Then $f$ is differentiable at $x_0 = 0$, but it is not differentiable at any $x_0 \neq 0$. To see the differentiability at $x_0 = 0$, note that if $x \neq 0$, then

$$\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = \begin{cases} x & \text{if } x \notin \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$$

It is easy to see that in this case

$$\frac{f(x)-f(0)}{x-0} \to 0 \quad \text{as } x \to 0.$$

Hence $f$ is differentiable at $x_0 =$ with $f'(0) = 0$. But if $x_0 \neq 0$ then $f$ is not differentiable at $x_0$. Suppose for instance that $x_0 \in \mathbb{Q}$. Then

$$\frac{f(x) - f(x_0)}{x - x_0} = \begin{cases} \frac{x^2}{x - x_0} & \text{if } x \notin \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

It is clear that $\frac{f(x)-f(x_0)}{x-x_0}$ does not have a limit as $x \to x_0$. For example let $(a_n)$ be a sequence of irrational numbers such that $a_n \to 0$ (for example $a_n = \sqrt{2}/n$). Then $x_n = x_0 + a_n$ is a sequence of irrational numbers such that $x_n \to x_0$. But $\frac{f(x_n)-f(x_0)}{x_n - x_0} = a_n + 2x_0 + x_0/a_n^2$ which does not converge to any real number. The proof that $f$ is not differentiable at $x_0$ if $x_0 \notin \mathbb{Q}$ is similar.

**Proposition 6.10**: Let $f\colon S \to \mathbb{R}$ be a function and let $x_0 \in S$ be a limit point of $S$. If $f$ is differentiable at $x_0$ then $f$ is continuous at $x_0$.

**Proof**: We will show that $\lim_{x\to x_0} f(x) = f(x_0)$. Since $x_0$ is a limit point of $S$ this suffices to prove that $f$ is continuous at $x_0$. If $x \neq x_0$ we have

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0).$$

Since $f$ is differentiable at $x_0$ we have $\frac{f(x)-f(x_0)}{x-x_0} \to f'(x_0)$ as $x \to x_0$. Therefore, by the Limit Laws,

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0) \to f'(x_0)(x_0 - x_0) + f(x_0) = f(x_0) \quad \text{as } x \to x_0.$$

Therefore $f$ is continuous at $x_0$. ■

**Note**: the *converse* to Proposition 6.10 is definitely not true! Continuity at a point does not imply differentiability at a point. The classic example is $f(x) = |x|$, this (as we have observed above) is not differentiable at $x_0 = 0$, but it is continuous at $x_0 = 0$.

**Proposition 6.11**: Suppose that $f, g\colon S \to \mathbb{R}$ are functions and that $x_0 \in S$ is a limit point of $S$. If $f$ and $g$ are differentiable at $x_0$ then

(1) for all $c, d \in \mathbb{R}$, $cf + dg$ is differentiable at $x_0$ with

$$(cf + dg)'(x_0) = cf'(x_0) + dg'(x_0),$$

(2) $f \cdot g$ is differentiable at $x_0$ with

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0),$$

(3) if $g(x) \neq 0$ for all $x \in S$ then $f/g$ is differentiable at $x_0$ and

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

**Proof**: The proofs of all of these statements are standard applications of the limit laws. For instance we prove (2): we have

$$\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \cdot g(x) + f(x_0) \cdot \left(\frac{g(x) - g(x_0)}{x - x_0}\right)$$
$$\to f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad \text{as } x \to x_0$$

by the limit laws (Proposition 6.6), where we have used the fact that $\frac{f(x)-f(x_0)}{x-x_0} \to f'(x_0)$, $g(x) \to g(x_0)$ and $\frac{g(x)-g(x_0)}{x-x_0} \to g'(x_0)$ as $x \to x_0$ (to conclude that $g(x) \to g(x_0)$ as $x \to x_0$ we need Proposition 6.10 — differentiability implies continuity). ■

Next we want to prove the Chain Rule. It will be convenient to reformulate what it means for a function to be differentiable at $x_0$ as follows.

**Lemma 6.12**: Suppose $f\colon S \to \mathbb{R}$ is a function and $x_0 \in S$ is a limit point of $S$. Then $f$ is differentiable at $x_0$ with $f'(x_0) = c$ if and only if there exists a function $R_f\colon S \to \mathbb{R}$ such that (i) $R_f(x_0) = 0$, (ii) $\lim_{x\to x_0} R_f(x) = 0$ and (iii) $f(x) = f(x_0) + c(x - x_0) + R_f(x)(x - x_0)$ for all $x \in S$.

**Proof**: ($\Rightarrow$) Suppose that $f$ is differentiable at $x_0$ and that $f'(x_0) = c$. Define $R_f\colon S \to \mathbb{R}$ by

$$R_f(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - c & \text{if } x \in S, x \neq x_0, \\ 0 & \text{if } x = c \end{cases}$$

Then $R_f(x_0) = 0$ and for $x \in S\backslash\{\, x_0 \,\}$, $R_f(x) \to 0$ since $f$ is differentiable at $x_0$ with $f'(x_0) = c$.

($\Leftarrow$) Suppose there exists a function $R_f\colon S \to \mathbb{R}$ satisfying (i), (ii) and (iii) above. If $x \in S\backslash\{\, x_0 \,\}$ then

$$\frac{f(x) - f(x_0)}{x - x_0} = c + R_f(x) \to c \qquad \text{as } x \to x_0$$

since $R_f(x) \to 0$ as $x \to x_0$ by hypothesis. Therefore $f$ is differentiable at $x_0$ with $f'(x_0) = c$. ■

**Note**: This lemma implies that close to $x_0$ the function $f$ approximated by the linear functin $f(x_0) + c(x - x_0)$.

**Exercise**: Prove that a function $f\colon S \to \mathbb{R}$ is differentiable at a limit point $x_0 \in S$ if and only if $f(x) - f(x_0) = \phi(x)(x - x_0)$ for all $x \in S$, where $\phi\colon S \to \mathbb{R}$ is continuous at $x_0$.

**Theorem 6.13 (Chain Rule)**: Suppose that $f\colon S \to \mathbb{R}$ is differentiable at a limit point $x_0 \in S$, $f(S) \subset T$, $g\colon T \to \mathbb{R}$ is differentiable at the limit point $f(x_0) \in T$. Then the composite function $g \circ f\colon S \to \mathbb{R}$ is differentiable at $x_0$ and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

**Proof**: Since $g$ is differentiable at $f(x_0)$ there exists a function $R_g\colon T \to \mathbb{R}$ satisfying $R_g(f(x_0)) = 0$, $\lim_{y\to f(x_0)} R_g(y) = 0$, and

$$g(y) = g(f(x_0)) + g'(f(x_0))(y - f(x_0)) + R_g(y)(y - f(x_0))$$

for all $y \in T$. Since $f$ is differentiable at $x_0$ there exists a function $R_f\colon S \to \mathbb{R}$ such that $R_f(x_0) = 0$, $\lim_{x\to x_0} R_f(x) = 0$ and

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_f(x)(x - x_0)$$

for all $x \in S$. Therefore,

$$\begin{aligned} g(f(x)) = g(f(x_0)) + g'(f(x_0))(f'(x_0)(x - x_0) - R_f(x)(x - x_0)) \\ + R_g(f(x))(f'(x_0)(x - x_0) + R_f(x)(x - x_0)) \end{aligned}$$

and so

$$g(f(x)) = g(f(x_0)) + g'(f(x_0))f'(x_0)(x - x_0) + R_{g\circ f}(x)(x - x_0)$$

where we have defined

$$R_{g\circ f}(x) = R_g(f(x))f'(x_0) - g'(f(x_0))R_f(x) + R_g(f(x))R_f(x))$$

for all $x \in S$. Then $R_{g\circ f}(x_0) = R_g(f(x_0))f'(x_0) - g'(f(x_0))R_f(x_0) + R_g(f(x_0))R_f(x_0) = 0$ and

$$R_{g\circ f}(x) \to R_g(f(x_0))f'(x_0) - g'(f(x_0))R_f(x_0) + R_g(f(x_0))R_f(x_0) \quad \text{as } x \to x_0$$

where we have used the limit laws (Proposition 6.6), the fact that $f$ is continuous at $x_0$ (Proposition 6.10), the fact that $R_f$ is continuous at $x_0$, the fact that $R_g$ is continuous at $f(x_0)$ and the fact that the composite function $R_g \circ f$ is continuous at $x_0$ (Theorem 4.5). Therefore, the function $R_{g\circ f}\colon S \to \mathbb{R}$ satisfies (i), (ii) and (iii) of Lemma 6.12. Therefore, by Lemma 6.12, the function $g \circ f$ is differentiable at $x_0$ and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$. ■

Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous. Then $f([a, b]) = [c, d]$ for some real numbers $c$ and $d$ (this is an exercise using the Intermediate Value Theorem, the fact that $f$ attains it maximum and minimum on $[a, b]$ and the fact that $f$ restricts to a continuous function on any subinterval of $[a, b]$).

**Note**: in fact if $I$ is an interval of one of the following forms: $(a, b)$, $(a, \infty)$, $(-\infty, b)$ or $\mathbb{R}$, and $f$ is continuous *and* 1-1, then $f(I)$ is an interval of one of these same forms (not necessarily of the same form as $I$ — for example the exponential function $\exp\colon \mathbb{R} \to (0, \infty)$ sends the interval $\mathbb{R}$ to the interval $(0, \infty)$). The assumption that $f$ is 1-1 cannot be relaxed here.

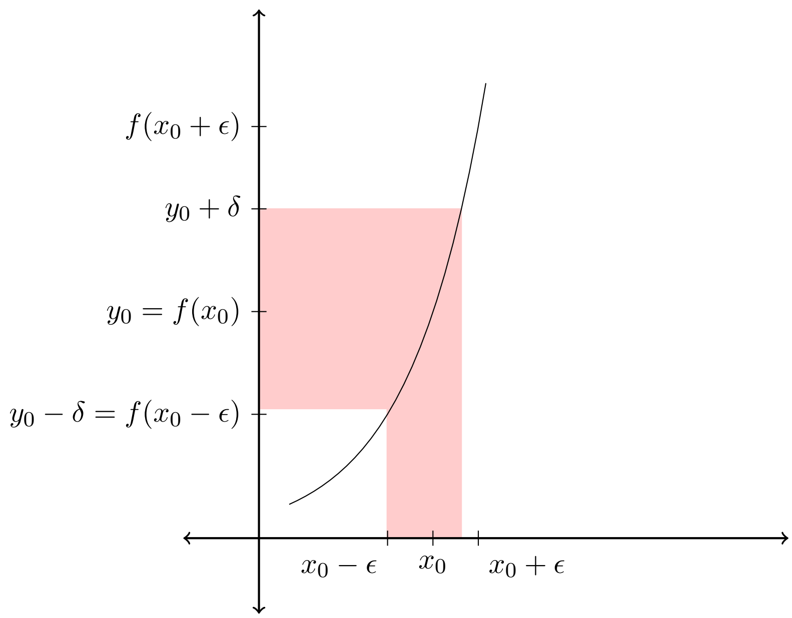
If $f\colon [a, b] \to \mathbb{R}$ is continuous and 1-1, then $f$ is either strictly increasing or strictly decreasing as follows easily from the Intermediate Value Theorem. We see this as follows. Since $f$ is 1-1 we must have $f(a) < f(b)$ or $f(a) > f(b)$. Without loss of generality we may suppose that $f(a) < f(b)$ (otherwise we apply the following argument to $-f$). If $a < x < y < b$ then a moments reflection shows that $f(a) < f(x) < f(y) < f(b)$ (why?); hence $f$ is strictly increasing.

Thus for a continuous, 1-1 function $f\colon [a, b] \to \mathbb{R}$, we have that $f$ is either strictly increasing or strictly decreasing, $f([a, b]) = [c, d]$, and hence the inverse function $f^{-1}\colon [c, d] \to [a, b]$ is defined.

**Proposition 6.14**: Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous and 1-1 with image $f([a, b]) = [c, d]$. Then $f^{-1}\colon [c, d] \to \mathbb{R}$ is continuous.

**Proof**: Suppose without loss of generality that $f$ is strictly increasing on $[a, b]$ (otherwise we can apply the following argument to $-f$ to conclude that $-f$ is continuous and hence that $f$ is continuous). Let $y_0 \in [c, d]$ so that $y_0 = f(x_0)$ for a unique $x_0 \in [a, b]$, since $f$ is 1-1 with $f([a, b]) = [c, d]$. We will show that $f^{-1}$ is continuous at $y_0$. Let $\epsilon > 0$. Suppose that $y_0 \in (c, d)$; it follows that $x_0 \in (a, b)$. Without loss of generality we may suppose that $x_0 - \epsilon, x_0 + \epsilon \in (a, b)$ (if not, then we can make $\epsilon$ smaller).

Consider the following graph of a function:




Since $f$ is strictly increasing we have $f(x_0-\epsilon) < f(x_0) < f(x_0+\epsilon)$. Let

$$\delta = \min\{\,(f(x_0) - f(x_0-\epsilon), f(x_0+\epsilon) - f(x_0)\,\}$$

From looking at the picture, it is clear that if $y$ belongs to the interval $(y_0-\delta, y_0+\delta)$, then $f^{-1}(y)$ will belong to the interval $(x_0-\epsilon, x_0+\epsilon)$. More precisely,

$$|y-y_0| < \delta \iff f(x_0) - \delta < y < f(x_0) + \delta$$

Therefore, by the choice of $\delta$, if $|y-y_0| < \delta$ then

$$f(x_0-\epsilon) \leq f(x_0) - \delta < y < f(x_0) + \delta \leq f(x_0+\epsilon).$$

Since $f$ is strictly increasing on $[a,b]$, $f^{-1}$ is strictly increasing on $[c,d]$ (why?) and hence

$$|y-y_0| < \delta \implies x_0 - \epsilon < f^{-1}(y) < x_0 + \epsilon.$$

Therefore

$$|y-y_0| < \delta \implies |f^{-1}(y) - f^{-1}(y_0)| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $f^{-1}$ is continuous at $y_0$. The case where $y_0 = c$ or $y_0 = d$ is done similarly and is left as an exercise. ■