Practical Asymptotics (APP MTH 4048/7044)
Assignment 1 (5%)

Due 22 March 2019

1. Consider the transcendental equation
$$x = \tanh\left(\frac{x}{\epsilon}\right).$$
   (a) How many (non-zero) real solutions are there to the above equation for $\epsilon \to 0$? [Hint: Sketch x and $\tanh(x/\epsilon)$.]
   (b) For each of the non-zero solutions find three terms in an asymptotic expansion as $\epsilon \to 0$.
   (c) Compare your expansion with a numerical solution.

2. Consider the differential equation
$$x^4 y'' - x^2 y' + \tfrac{1}{4} y = 0, \quad \text{as } x \to 0.$$
   (a) Classify the ordinary, regular singular and irregular singular points of this equation.
   (b) Use the method of dominant balance to find the leading behaviours as $x \to 0$.
   [Hint: it is possible to have a balance between three terms]
   (c) Solve the differential equation numerically over a suitable range, subject to initial conditions of your choice. Discuss how this numerical solution relates to the behaviours you found in part (b).

3. Consider the integral representation of the Stieljes function:
$$S(\epsilon) = \int_0^\infty \frac{\mathrm{e}^{-t}}{1 + \epsilon t}\mathrm{d}t.$$
   (a) Develop a series representation of this function using the definition of a geometric series,
$$\frac{1}{1+z} = \sum_{j=0}^{N} (-z)^j + \frac{(-z)^{N+1}}{1+z}.$$
   What is the error if the resulting series is truncated after N terms? This is a divergent series, how many terms are required for an optimal truncation for a given value of ϵ?
   (b) Having optimally truncated the series, let's now try and improve this by examining the error term. To do this, develop a series representation of the error from part (a) using the fact that
$$\frac{1}{1+y} = \frac{1}{2[1 + \frac{1}{2}(y-1)]}.$$
   Find a series expression for the error term of this new representation (it will depend on the number of terms included in both series).
   (c) Use MATLAB to plot the approximate errors found in parts (a) and (b) for a fixed value of ϵ.