thomas.leistner@adelaide.edu.au

SCHOOL OF MATHEMATICAL SCIENCES
THE UNIVERSITY of ADELAIDE

Assignment 5 — due on 31/05/2017 (note the extended deadline)

- Please hand up solutions in the Hand-In Box by the above due date, **at 4pm**.

5.1 Consider the following curve, a *helix*, in $\mathbb{R}^3$:

$$C := \{(\cos(t), \sin(t), t) \in \mathbb{R}^3 \mid 0 \leq t \leq 2\pi\},$$

a) Compute the length of C.
b) For the function $f(x, y, z) = x^2 + y^2 + z^2$ compute the integral $\int_C f ds$ of f along C.
c) For the vector field $\mathbf{F}(x, y, z) = (-yz, xz, z)$ compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ of $\mathbf{F}$ along C.

5.2 Consider the following vector field on $\mathbb{R}^2$,

$$\mathbf{F}(x, y) = (P(x, y), Q(x, y)) = \left(y(x^2 + 1), (y + 1)(x^2 - 1)\right),$$

Confirm Green's Theorem for $\mathbf{F}$ and the region R by calculating both sides of

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial R} \mathbf{F}(x, y) \cdot d\boldsymbol{s},$$

where R is

a) the interior of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$;
b) $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 - 1 \leq y \leq 0, x \geq 0\}$.

You may want to sketch the regions and their boundaries first in order to find the parametrisation. Remember that ∂R is oriented such that the interior of the region R is on the left.

5.3 Consider the following parametrisation ϕ of the parametric surface $S = \mathrm{Im}(\phi) = \phi(R)$ with

$$\begin{aligned} \phi \ : \ R = [0, 2\pi] \times [0, 2] \subset \mathbb{R}^2 \quad &\to \quad \mathbb{R}^3 \\ (u, v) \quad &\mapsto \quad (x(u, v), y(u, v), z(u, v)) = (v\cos(u), v\sin(u), v^2), \end{aligned}$$

and the vector field on $\mathbb{R}^3$,

$$\mathbf{F}(x, y, z) = (x, y, x + y + z).$$

a) Find a unit normal $\boldsymbol{n}$ on S from the given parametrisation.
b) With the orientation given by $\boldsymbol{n}$, compute both sides of Stokes' Theorem for the vector field $\mathbf{F}$, i.e., verify that

$$\iint_S \mathrm{curl}(F) dS = \oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{s}.$$

Remember that the orientation of ∂S is given by the condition that $\boldsymbol{n} \times T$ points towards the surface, where T is the unit tangent vector of ∂S defining its orientation.
c) Let $W = \{(x, y, z) \mid x^2 + y^2 \leq z, z \in [0, 2]\}$ be the volume enclosd by S and the disk $D = \{(x, y, 4) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4\}$. For the vector field $\mathbf{F}$ and the volume W verify Gauss' divergence theorem, i.e., verify that

$$\iiint_W \mathrm{div}(\mathbf{F}) dV = \iint_{\partial W} \mathbf{F} dS.$$

(*Hint:* You may compute the left-hand-side using the change of variables formula for triple integrals and for the right-hand-side you need a parametrisation of the disk. Remember that ∂W is oriented by the *outward* normal.

(2+2+2) + (5+5) + (2+5+7) = 30 marks