Assignment 3, Mathematical Statistics 3

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1. **Solution** Since X_1, X_2 are indep, the sum of indep MGFs will be the product of their MGFs For $X \sim B(n,p)$, $M_X(t) = (1 - p(1+e^t))^n$

$$\begin{aligned} M_Y(t) &= M_{X_1}(t)M_{X_2}(t) \\ &= (1 - p(1+e^t))^{n_1} \times (1 - p(1+e^t))^{n_2} \\ &= (1 - p(1+e^t))^{n_1+n_2} \end{aligned}$$

Which is the MGF for $B(n_1+n_2, p)$ I.e. $Y \sim B(n_1+n_2, p)$ **As Required**

2. X_1, X_2 iid $U(0,1)$ and let $U = X_1/(X_1+X_2)$ Show that U has PDF:

$$f_U(u) = \begin{cases} \frac{1}{2(1-u)^2} & \text{for } 0 < u \leq 1/2 \\ \frac{1}{2u^2} & \text{for } 1/2 < u < 1 \\ 0 & otherwise \end{cases}$$

Solution Using theorem 29

So define $h(X_1, X_2) = \begin{pmatrix} u = \frac{X_1}{(X_1+X_2)} \\ w = X_1 + X_2 \end{pmatrix}$ and $g(u,w) = \begin{pmatrix} uw \\ (1-u)w \end{pmatrix}$ Using this,

$$G = \begin{pmatrix} w & u \\ -w & 1-u \end{pmatrix}, \quad det(G) = (1-u)w + uw = w - uw + uw = w \geq 0, \text{ as } X_1 + X_2 \geq 0$$

Now note, $0 \leq u \leq 1$ and $0 \leq w \leq 2$

$$\begin{aligned} f_{U,W}(\mathbf{u}) &= f_{\mathbf{X}}(g(\mathbf{u}))|\det G(\mathbf{u})| \\ &= f_{X_1}(g(\mathbf{u}))f_{X_2}(g(\mathbf{u}))\det G(\mathbf{u}) \\ &= \begin{pmatrix} 1 & \text{if } 0 \leq uw \leq 1 \\ 0 & \text{otherwise} \end{pmatrix} \times \begin{pmatrix} 1 & \text{if } 0 \leq (1-u)w \leq 1 \\ 0 & \text{otherwise} \end{pmatrix} w \\ &= \begin{pmatrix} 1 & \text{if } 0 \leq w \leq 1/u \\ 0 & \text{otherwise} \end{pmatrix} \times \begin{pmatrix} 1 & \text{if } 0 \leq w \leq \frac{1}{(1-u)} \\ 0 & \text{otherwise} \end{pmatrix} w \end{aligned}$$

taking the composite of these

$$= \begin{pmatrix} w & \text{if } 0 \leq w \leq 1/u \\ w & \text{if } 0 \leq w \leq \frac{1}{(1-u)} \\ 0 & \text{otherwise} \end{pmatrix}$$

integrate to obtain f_U

$$f_U(u) = \begin{pmatrix} w & \text{if } 0 \leq w \leq 1/u \\ w & \text{if } 0 \leq w \leq \frac{1}{(1-u)} \\ 0 & \text{otherwise} \end{pmatrix}$$

$$= \begin{cases} w^2/2 \,\Big|_0^{1/u} \\ w^2/2 \,\Big|_0^{1/(1-u)} \\ 0 \end{cases}$$

$$= \begin{cases} (1/u)^2/2 \\ (1/(1-u))^2/2 \\ 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2u^2} \\ \frac{1}{2(1-u)^2} \\ 0 \end{cases}$$

With bounds: The first bound $0 < w < 1/u$ is achievable when $u \in (1/2, 1)$ as w$\in (0, 2)$.
The second bound $0 < w < \frac{1}{1-u}$ is achievable when $u \in (0, 1/2)$, for the same reason.
This gives:

$$f_U(u) = \begin{cases} \frac{1}{2u^2} & \frac{1}{2} < u < 1 \\ \frac{1}{2(1-u)^2} & 0 < u < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

As Required

3. Suppose X_1, X_2 have joint PDF $f_X(x_1, x_2)$

(a) If $Y_1 = X_1$ and $Y_2 = -X_2$ show the joint PDF of Y_1, Y_2 is

$$f_Y(y_1, y_2) = f_X(y_1, -y_2)$$

Solution Using theorem 29, defining $g(\mathbf{y}) = (y_1, -y_2)$, and $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\begin{aligned} f_Y(y_1, y_2) &= f_X(g(\mathbf{y}))|\det(G)| \\ &= f_X(y_1, -y_2)|1 \times -1| \\ &= f_X(y_1, -y_2)|-1| \\ &= f_X(y_1, -y_2) \end{aligned}$$

As Required

(b) Hence, show for $W = X_1 - X_2$ that

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x, x-w)dx$$

Hint: express W using Y_1, Y_2
Solution Write $W = Y_1 + Y_2$ Using theorem 27,

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(x, w-x)dx$$

Hence, using a)

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x, x-w)dx$$

As Required

(c) Suppose $X_1 \sim Exp(\lambda)$ and $X_2 \sim Exp(\lambda)$ indep and let $W = X_1 - X_2$ Find the PDF of W
Solution Using (b),
There are two cases to consider; $w \geq 0$ and $w < 0$.
For $w < 0$:

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x, x-w)dx \\ &= \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(x-w)dx \text{ due to independence} \\ &= \int_0^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(x-w)}dx \\ &= \lambda e^{\lambda w} \int_0^{\infty} \lambda e^{-2\lambda x}dx \\ &= \lambda e^{\lambda w} \frac{1}{2} \int_0^{\infty} 2\lambda e^{-2\lambda x}dx \text{ which is the cdf of the exp dist with param } 2\lambda \\ &= \frac{\lambda e^{\lambda w}}{2} \end{aligned}$$

And for $w \geq 0$, we have $0 \leq w < x$:

So $x \in (w, \infty)$, with $w \geq 0$

$$\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_X(x, x-w)dx \\
&= \int_{w}^{\infty} f_{X_1}(x) f_{X_2}(x-w)dx \\
&= \int_{w}^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(x-w)}dx \\
&= \lambda^2 e^{\lambda w} \int_{w}^{\infty} e^{-2\lambda x}dx \\
&= \lambda^2 e^{\lambda w} \frac{1}{-2\lambda} e^{-2\lambda x} \Big|_{w}^{\infty} \\
&= \frac{-1}{2} \lambda e^{\lambda w} \left(0 - e^{-2\lambda w}\right) \\
&= \frac{\lambda e^{-\lambda w}}{2}
\end{aligned}$$

I.e.

$$f_W(w) = \begin{cases} \frac{\lambda e^{\lambda w}}{2} & w < 0 \\ \frac{\lambda e^{-\lambda w}}{2} & w \geq 0 \end{cases}$$

As Required

4. $Z_1, Z_2 \sim N(0,1)$ IID, let $X_1 = Z_1 + Z_2$ and $X_2 = Z_1 - Z_2$. Find the joint distribution of X_1, X_2.

Solution

Note that Z_1, Z_2 have joint PDF

$$f_{(Z_1,Z_2)}(z_1, z_2) = \frac{1}{2\pi} e^{\frac{-(z_1^2 + z_2^2)}{2}}$$

Using theorem 29:
$h_1(Z_1, Z_2) = z_1 + z_2$ and $h_2(z_1, z_2) = z_1 - z_2$
Find g: $g(h_1, h_2) = (z_1, z_2)$ This gives: $g(h_1, h_2) = g(x_1, x_2) = (\frac{h_1+h_2}{2}, \frac{h_1-h_2}{2})$

$$G = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \quad \implies \quad \det G = -1/4 - 1/4 = -1/2$$

So:

$$\begin{aligned}
f_{X_1,X_2}(x_1, x_2) &= f_{Z_1,Z_2}(g(\mathbf{x})) |\det G(\mathbf{x})| \\
&= f_{Z_1,Z_2}\left(\frac{x_1+x_2}{2}, \frac{x_1-x_2}{2}\right) |\frac{-1}{2}| \\
&= \frac{1}{2\pi} exp\{\frac{-((\frac{x_1+x_2}{2})^2 + (\frac{x_1-x_2}{2})^2)}{2}\} \frac{1}{2} \\
&= \frac{1}{4\pi} exp\{\frac{-(x_1^2 + 2x_1x_2 + x_2^2 + x_2^2 - 2x_1x_2 + x_1^2)}{8}\} \\
&= \frac{1}{4\pi} exp\{\frac{-(x_1^2 + x_2^2)}{4}\} \\
&= \frac{1}{2\pi\sqrt{2}\sqrt{2}} exp\{\frac{-(x_1^2 + x_2^2)}{2 \times \sqrt{2}\sqrt{2}}\}
\end{aligned}$$

Which is the Bivariate normal, i.e.

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2(\mathbf{0}, 2I_2)$$

As Required

5. (a) Suppose Σ is an $r \times r$ positive-definite symmetric matrix, with $\Sigma = E\Lambda E^T$ be the eigenvalue/eigenvector decomposition, where

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_r), \ with \ \lambda_i > 0$$

And E is an $r \times r$ orthogonal matrix, i.e. $E^T E = EE^T = E$. Define:

$$\Sigma^{-1/2} = E\Lambda^{-1/2}E^T$$

Where

$$\Lambda^{-\frac{1}{2}} = diag(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \ldots, \frac{1}{\sqrt{\lambda_r}})$$

Show that $\Sigma^{-1/2}$ is symmetric and satisfies $\Sigma^{-1/2}\Sigma\Sigma^{-1/2} = I$

Solution Two part question: Symmetric if $\Sigma^T = \Sigma$

$$\begin{aligned}
\Sigma^{-1/2} &= E\Lambda^{-1/2}E^T \\
(\Sigma^{-1/2})^T &= (E\Lambda^{-1/2}E^T)^T \\
&= E(\Lambda^{-1/2})^T E^T \\
&= E\Lambda^{-1/2}E^T \text{ as diagonal matrices are symmetric} \\
&= \Sigma^{-1/2}
\end{aligned}$$

Therefore it is symmetric. Second part:

$$\begin{aligned}
\Sigma^{-1/2}\Sigma\Sigma^{-1/2} &= E\Lambda^{-1/2}E^T E\Lambda E^T E\Lambda^{-1/2}E^T \\
&= E\Lambda^{-1/2}\Lambda\Lambda^{-1/2}E^T
\end{aligned}$$

Now since the Λ matrix is diagonal:

$$\begin{aligned}
[\Lambda^{-1/2}\Lambda]_{ii} &= \frac{\lambda_i}{\sqrt{\lambda_i}} \\
\text{And so: } [\Lambda^{-1/2}\Lambda\Lambda^{-1/2}]_{ii} &= \frac{\lambda_i}{\sqrt{\lambda_i}\sqrt{\lambda_i}} = \lambda_i/\lambda_i = 1 \\
\implies \Lambda^{-1/2}\Lambda\Lambda^{-1/2} &= I
\end{aligned}$$

$$\begin{aligned}
\implies \Sigma^{-1/2}\Sigma\Sigma^{-1/2} &= E\Lambda^{-1/2}\Lambda\Lambda^{-1/2}E^T \\
&= EIE^T \\
&= EE^T \\
&= I
\end{aligned}$$

As Required

(b) Suppose $\mathbf{Y} \sim N_r(\mu, \Sigma)$ and let $Z = \Sigma^{-1/2}(\mathbf{Y} - \mu)$. Find the distribution of Z

Solution

$$Z = \Sigma^{-1/2}Y - \Sigma^{-1/2}\mu$$

I.e. $A = \Sigma^{-1/2}$ which is $r \times r$ and $b = -\Sigma^{-1/2}\mu$. Using theorem 32:

$$Z \sim N_r(A\mu + b, A\Sigma A^T)$$

I.e.:

$$Z \sim N_r(\Sigma^{-1/2}\mu - \Sigma^{-1/2}\mu, \Sigma^{-1/2}\Sigma(\Sigma^{-1/2})^T)$$

Using (a) this gives:

$$Z \sim N_r(\mathbf{0}, I)$$

As Required

(c) Suppose $\mathbf{Y} \sim N_r(\mu, \Sigma)$, and let

$$V = (Y - \mu)^T\Sigma^{-1}(Y - \mu)$$

Show that $V \sim \chi^2_r$

Hint: Express $V = Z^T Z$ for Z found in (b).

Solution Rewrite

$$V = (Y - \mu)^T\Sigma^{-1/2}\Sigma^{-1/2}(Y - \mu)$$

And since $\Sigma^{-1/2}$ is symmetric (from (a)):

$$V = (Y - \mu)^T(\Sigma^{-1/2})^T\Sigma^{-1/2}(Y - \mu)$$

$$V = \left(\Sigma^{-1/2}(Y - \mu)\right)^T \Sigma^{-1/2}(Y - \mu)$$

I.e.

$$V = Z^T Z = \sum_{i=1}^{r} Z_i Z_i = \sum_{i=1}^{r} Z_i^2$$

Where $Z_i \sim N(0,1)$ and $cov(Z_i, Z_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ And since $\sum_{i=1}^{k} Z_i^2 \sim \chi^2_k$, if $Z \sim N(0,1)$. This means that

$$V \sim \chi^2_r$$

As Required

Note that question 6 doesn't exist

Honours

7. X_1, X_2 continuous RVs with joint PDF $f(x_1, x_2)$ and let $Y = X_1X_2$. Obtain an expression for the PDF $f_Y(y)$
Hint: the construction is similar to that for the ratio.

Solution

$$\begin{aligned}
F_Y(y) &= P(\{Y \leq y\}) \\
&= P(\{X_1X_2 \leq y\}) \\
&= P(\{X_1X_2 \leq y \cap X_1 \leq 0\}) + P(\{X_1X_2 \leq y \cap X_1 \geq 0\}) + P(\{X_1X_2 \leq y \cap X_1 = 0\}) \\
&= P(\{X_1X_2 \leq y \cap X_1 \leq 0\}) + P(\{X_1X_2 \leq y \cap X_1 \geq 0\}) + 0 \\
&= P(\{X_2 \geq y/X_1 \cap X_1 \leq 0\}) + P(\{X_2 \leq y/X_1 \cap X_1 \geq 0\}) \\
&= \int_{-\infty}^{0} \int_{y/x_1}^{\infty} f(x_1, x_2) dx_2 dx_1 + \int_{0}^{\infty} \int_{-\infty}^{y/x_1} f(x_1, x_2) dx_2 dx_1
\end{aligned}$$

Now make the substitution: $x_2 = t/x_1 \implies dx_2 = \frac{1}{x_1} dt$

$$\begin{aligned}
&= \int_{-\infty}^{0} \int_{y}^{\infty} \frac{1}{x_1} f(x_1, t/x_1) dt dx_1 + \int_{0}^{\infty} \int_{-\infty}^{y} \frac{1}{x_1} f(x_1, t/x_1) dt dx_1 \\
&= \int_{-\infty}^{0} \int_{\infty}^{y} \frac{1}{-x_1} f(x_1, t/x_1) dt dx_1 + \int_{0}^{\infty} \int_{-\infty}^{y} \frac{1}{x_1} f(x_1, t/x_1) dt dx_1 \\
&= \int_{-\infty}^{\infty} \int_{\infty}^{y} \frac{1}{|x_1|} f(x_1, t/x_1) dt dx_1 \\
\implies F_Y(y) &= \int_{\infty}^{y} \int_{-\infty}^{\infty} \frac{1}{|x_1|} f(x_1, t/x_1) dx_1 dt
\end{aligned}$$

Using the fundamental theorem of calculus gives:

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{|x_1|} f(x_1, y/x_1) dx_1$$

As Required

8. Suppose $X_1 \sim U(0, 1)$ and $X_2 \sim U(0, 1)$ and let $Y = \sqrt{X_1}X_2$. Find the PDF, $f_Y(y)$ and perform a simulation in R to illustrate that this answer is correct.

Solution From assignment 1,

$$X_3 = \sqrt{X_1} = \begin{cases} 2x_3 & if\ 0 < x_3 < 1 \\ 0 & otherwise \end{cases}$$

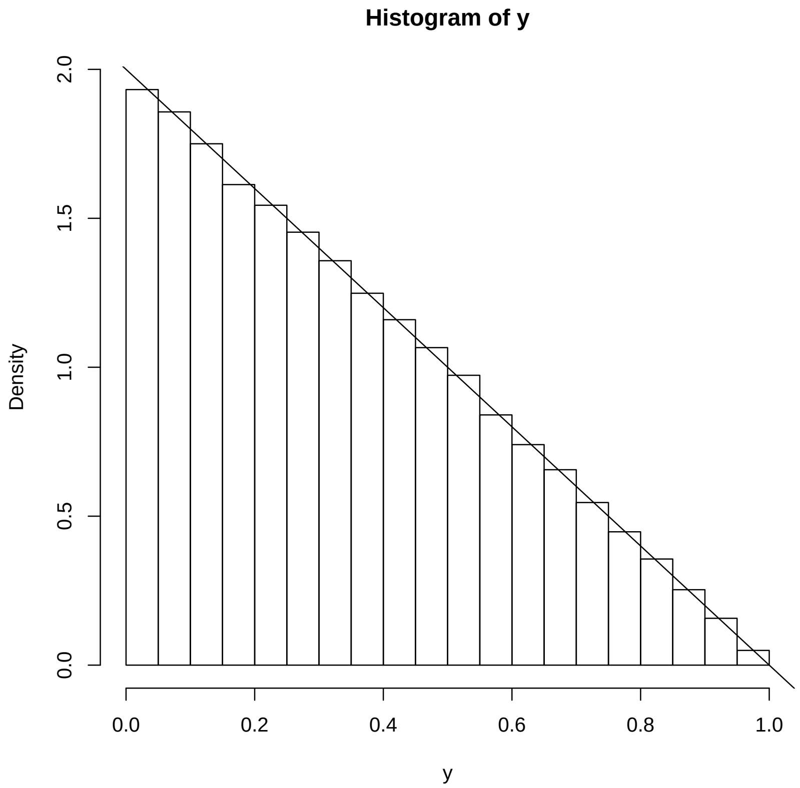
So write $Y = X_2X_3$. Using this and the formula found in (7) gives:

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} \frac{1}{|x_2|} f(x_2, y/x_2) dx_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{|x_2|} f_{x_2}(x_2) f_{x_3}(y/x_2) dx_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{|x_2|} f_{x_2}(x_2) f_{x_3}(y/x_2) dx_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{|x_2|} \begin{cases} 1 & if\ 0 < x_2 < 1 \\ 0 & otherwise \end{cases} \times \begin{cases} 2y/x_2 & if\ 0 < y/x_2 < 1 \\ 0 & otherwise \end{cases} dx_2
\end{aligned}$$

Find the bounds for the integral: $0 < y/x_2 < 1 \implies 1 < x_2/y < \infty$, so $0 < y < x_2 < \infty$ become the bounds for the integral. Note that $y > 0$ always. And $x_2 < 1$, so the bounds become $y < x_2 < 1$

$$\begin{aligned}
\implies f_Y(y) &= \int_{y}^{\infty} 2y/x_2^2 dx_2 \\
&= 2y \int_{y}^{\infty} 1/x_2^2 dx_2 \\
&= 2y \left(\frac{-1}{x_2}\right) \Big|_{y}^{1} \\
&= 2y(-1 + \frac{1}{y}) \\
&= 2 - 2y
\end{aligned}$$

Figure 1: Histogram of $Y = \sqrt{X_1}X_2$ with the line $Y = 2 - 2x$ superimposed



This is shown (verified) in figure 1; Y was generated by uniformly generating x_1 and x_2 and setting $y = \sqrt{x_1} * x_2$. The `R` code to produce this figure is below:

```
x1 = runif(100000)
x2 = runif(100000)
y = sqrt(x1) * x2
pdf("A3Stats.pdf")
hist(y,freq=FALSE)
##line corresponding to y = 2-2x
abline(2,-2)
dev.off()
```

As Required