APP MTH 3020 Stochastic Decision Theory

Tutorial 3

Week 7, Friday, September 7

1. One morning, Bobby McGee wakes up and decides to toss 5 fair coins. Let N be a random variable representing the number of tails that Bobby sees.

   a. Calculate the pmf $\pi_N(n)$ of N, for $n = 0, 1, 2, 3, 4, 5$. What is the name of this distribution?
   The random variable N has the probability mass function:
   $$\pi_N(n) = \binom{5}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{5-n} = \binom{5}{n}\left(\frac{1}{2}\right)^5 \qquad \text{for } n = 0, 1, 2, 3, 4, 5.$$
   This is a binomial distribution $B(5, 1/2)$.

   b. Suppose Bobby loses \$1 (to himself!) for every tail that he sees, but wins \$1 for every head. Let M be the random variable representing the total amount that Bobby wins. What values can the random variable M take?
   $$\begin{aligned} M &= (5-N)1 + N(-1) \\ &= 5 - 2N \in \{5, 3, 1, -1, -3, -5\}. \end{aligned}$$

   c. Write down the pmf $\pi_M(m)$.
   It follows that
   $$\begin{aligned} \pi_M(5) = \pi_N(0), \quad \pi_M(3) = \pi_N(1), \quad \pi_M(1) = \pi_N(2), \\ \pi_M(-1) = \pi_N(3), \quad \pi_M(-3) = \pi_N(4), \quad \pi_M(-5) = \pi_N(5). \end{aligned}$$
   Alternatively, we could write
   $$\pi_M(m) = P(M = m) = P(N = (m-5)/2) = \binom{5}{(5-m)/2}\left(\frac{1}{2}\right)^5.$$

   Suppose Bobby adopts a different strategy, in which he keeps tossing a single fair coin until he sees a head and then stop.

   e. Write down the pmf $\pi_N(n)$, $n \geq 0$, for the number of tails that Bobby sees. What is the name of this distribution?
   The probability mass function of N is given by
   $$\pi_N(n) = \left(\frac{1}{2}\right)^{n+1} \qquad \text{for } n \geq 0,$$
   so N follows a geometric distribution with parameter $1/2$.

   g. Using the same betting rules as above, and assuming that Bobby has enough funds to finance this strategy, write down the pmf $\pi_M(m)$ for the amount of money that he wins.
   Here, $M = 1 - N$ and takes values $m = 1, 0, -1, -2, \ldots$ corresponding, respectively, to N taking the values $n = 0, 1, 2, 3, \ldots$ Consequently
   $$\pi_M(m) = P(M = m) = P(N = 1 - m) = \pi_N(1-m) = \left(\frac{1}{2}\right)^{2-m}.$$

2. Consider a butterfly who occupies one of four patches. The patches are located at co-ordinates

$$\{(2,2),(2,1),(1,3),(1,1)\}.$$

Given the butterfly is in patch i, it migrates overnight to patch $j \neq i$ with probability $\exp(-d_{ij})$ where d_{ij} is the distance from patch i to patch j, and remains in patch i otherwise.

a. Specify a discrete-time Markov chain $\{X_n\}$ to model the position of the butterfly. That is, define a state space $\mathcal{S}$ and transition probability matrix P.

Let $X_n \in \mathcal{S} = \{1,2,3,4\}$ be the state of the DTMC, which is the patch occupied by the butterfly at time $n = 0,1,2,\ldots$, where

- state 1 corresponds to coordinate $(2,2)$,
- state 2 to $(2,1)$,
- state 3 to $(1,3)$,
- state 4 to $(1,1)$.

The distances between patches determine the transition probabilities, which are symmetric (thus $P = P^\top$) and are given by

$$\begin{aligned}
p_{12} &= \exp(-d_{12}) = \exp(-1) = p_{21},\\
p_{13} &= \exp(-d_{13}) = \exp(-\sqrt{2}) = p_{31},\\
p_{14} &= \exp(-d_{14}) = \exp(-\sqrt{2}) = p_{41},\\
p_{11} &= 1 - \exp(-1) - 2\exp(-\sqrt{2}),\\
p_{23} &= \exp(-d_{23}) = \exp(-\sqrt{5}) = p_{32},\\
p_{24} &= \exp(-d_{24}) = \exp(-1) = p_{42},\\
p_{22} &= 1 - 2\exp(-1) - \exp(-\sqrt{5}),\\
p_{34} &= \exp(-d_{34}) = \exp(-2) = p_{43},\\
p_{33} &= 1 - \exp(-2) - \exp(-\sqrt{5}) - \exp(-\sqrt{2}),\\
p_{44} &= 1 - \exp(-2) - \exp(-1) - \exp(-\sqrt{2}).
\end{aligned}$$

b. Given the butterfly is in the patch located at $(1,1)$ now, what is the probability it will be in either of the patches located at positions $(1,3)$ and $(2,2)$ tomorrow, two days later, and seven days later?

We have

$$\begin{aligned}
P(X_1 = 2 \cup X_1 = 3 \mid X_0 = 4) &= p_{42} + p_{43} \approx 0.5033,\\
P(X_2 = 2 \cup X_2 = 3 \mid X_0 = 4) &= \left[P^2\right]_{42} + \left[P^2\right]_{43} \approx 0.4575,\\
P(X_7 = 2 \cup X_7 = 3 \mid X_0 = 4) &= \left[P^7\right]_{42} + \left[P^7\right]_{43} \approx 0.4998.
\end{aligned}$$

In fact,

$$\lim_{n\to\infty} P(X_n = 2 \cup X_n = 3 \mid X_0 = j) = 0.5 \quad \text{FOR ALL } j \in \{1,2,3,4\}.$$

c. How might you modify the Markov chain specified in part (a) to account for possible death of the butterfly?

We can introduce another state 0, which is absorbing such that there exist $p_{i0} > 0$ for all $i \in \{1,2,3,4\}$ as the probabilities of death for each patch. (These probabilities may be different for each patch.)

3. Ellie the Elephant wants to decide whether or not to go running each morning, over the course of a week, starting from Sunday (Day 0). Her health can be in one of the four states, $\{1, 2, 3, 4\}$, where 1 is when she is in her healthiest, 2 is slightly less, 3 is even less, and 4 is her worst. Let $f(i)$ be the cost of taking care of Ellie during a single stage (one day) when she is in state i:

$$f(1) = 1, f(2) = 3, f(3) = 5, f(4) = 7.$$

Two actions are possible each day, RUN, at a cost of $R = 4$, which takes place at the beginning of the day, or NOT. Each day, without running, Ellie can either remain the same, or deteriorate to a worse state of health according to $P(\text{NOT})$; if RUN is chosen, Ellie's health cannot always transition to the perfect state, but behaves according to $P(\text{RUN})$. The two transition matrices are:

$$P(\text{RUN}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 \\ 0.5 & 0.4 & 0.1 & 0 \end{bmatrix}, \quad P(\text{NOT}) = \begin{bmatrix} 0.6 & 0.3 & 0.1 & 0 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We assume the transition happens at the beginning of a stage/day and she remains in the new state until the end of the day. Also, assume that Ellie does not have to make a decision on the final day, Saturday, and that the cost associated with finishing in any particular state is 0, which implies $J_6(i) = 0$ for all $i = 1, 2, 3, 4$.

a. Find the expression for, and calculate, $J_{T-1}(i)$ for each of the states $i \in \{1, 2, 3, 4\}$. Show a detailed analysis similar to that in lectures for the calculation of these cost-to-go quantities.

For every state $i = 1, 2, 3, 4$, and for all stages $n = 0, \dots, 6$, $\mathcal{U}_n(i) = \{\text{RUN}, \text{NOT}\}$. Thus,

$$J_5(i) = \min_{u \in \{\text{RUN},\text{NOT}\}} \mathbb{E}\left[c(i, u_5) + \sum_{j=1}^{3} p_{ij}(u_5) J_6(j)\right].$$

In particular,

$$J_5(1) = \min_{u \in \{\text{RUN},\text{NOT}\}} \left[\overbrace{\mathbb{E}[c(1, u)]}^{\text{EXPECTED COST OF ACTION } u} + \overbrace{\sum_{j=1}^{3} p_{1j}(u) J_6(j)}^{\text{EXPECTED NEXT STAGE COSTS}} \right]$$
$$= \min \left\{ \overbrace{4 + 1 \times 1}^{\mathbb{E}[\text{COST TO RUN}]} + \overbrace{0}^{\mathbb{E}[\text{NEXT STAGE COST}]}, \quad \overbrace{0 + 0.6 \times 1 + 0.3 \times 3 + 0.1 \times 5}^{\mathbb{E}[\text{COST OF NOT RUNNING}]} + \overbrace{0}^{\mathbb{E}[\text{NEXT STAGE COST}]} \right\}$$
$$= \min\{5, 2\} = 2, \quad \text{resulting from not running.}$$

Similarly,

$$\begin{aligned} J_5(2) &= \min\{4 + 1 \times 1 + 0, \quad 0 + 0.7 \times 3 + 0.2 \times 5 + 0.1 \times 7 + 0\} \\ &= \min\{5, 3.8\} = 3.8, \quad \text{resulting from not running.} \\ J_5(3) &= \min[4 + 0.8 \times 1 + 0.2 \times 3 + 0], \quad 0 + 0.8 \times 5 + 0.2 \times 7 + 0] \\ &= \min\{5.4, 5.4\} = 5.4, \quad \text{both actions are optimal!} \\ J_5(4) &= \min[4 + 0.5 \times 1 + 0.4 \times 3 + 0.1 \times 5 + 0, \quad 0 + 1 \times 7 + 0] \\ &= \min\{6.2, 7\} = 6.2, \quad \text{given that we choose to run.} \end{aligned}$$

b. Calculate, either by hand or with the help of MATLAB, the remaining $J_t(i)$ for each state $i \in \{1, 2, 3, 4\}$ and for all stages t over the finite horizon $T = 6$. Generate a table showing the optimal policy and costs for this problem.

For all other stages $t = 0, \ldots, 4$, we solve for $i = 1, \ldots, 4$

$$J_t(i) = \min_{u \in \{\text{RUN},\text{NOT}\}} \mathbb{E}\left[c(i, u_t) + \sum_{j=1}^{3} p_{ij}(u_t) J_{t+1}(j)\right].$$

state i	$J_6(i)$	$J_5(i)$	action	$J_4(i)$	action	$J_3(i)$	action
1	0	2.0	NOT RUN	4.88	NOT RUN	7.804	NOT RUN
2	0	3.8	NOT RUN	7.00	RUN	9.880	RUN
3	0	5.4	NOT RUN or RUN	7.76	RUN	10.704	RUN
4	0	6.2	RUN	9.26	RUN	12.216	RUN

state i	$J_2(i)$	ACTION	$J_1(i)$	ACTION	$J_0(i)$	ACTION
1	10.7168	NOT RUN	13.6332	NOT RUN	16.5484	NOT RUN
2	12.8040	RUN	15.7168	RUN	18.6332	RUN
3	13.6192	RUN	16.5342	RUN	19.4499	RUN
4	15.1244	RUN	18.0419	RUN	20.9567	RUN

c. Interpret this policy in words for Ellie.

Exercise rules over one week:

- Don't run if you are perffectly healthy.
- If feeling a bit unhealthy (states 2 or 3), run if you are in the first five days of the week (Sunday, Monday, Tuesday, Wednesday, Thursday), and don't run if it is Friday.
- If feeling like your health is at the worst stage possible (state 4), run every day ($n = 0, \ldots, 5$).

d. If Ellie is considering her exercise routine over a much larger time horizon, $T \gg 6$, what would you suggest as the optimal policy for each state as a measure of time-to-go (that is, of the number of remaining time periods)?

We can solve the above problem using MATLAB for a much larger value of T (say, $T = 100$).

Exercise rules for life:

- Never run if you feel perfectly healthy.
- If feeling a bit unhealthy (states 2 or 3), run if there is more than one day until the end of your life. (Granted, this is a little hard for Ellie to figure this out, but there is a limit to how helpful mathematics can be for an elephant.)
- Always run if you feel like your health is at the worst state.