Lecture 5: Transition rates to transition functions – The journey begins

Concepts checklist

At the end of this lecture, you should be able to:

- Define the *generator* of a CTMC as *limit of the transition function*;
- Define a *conservative* generator;
- Have an intuitive / *physical interpretation of the transition rates* of a CTMC; and,
- *Specify, and derive, the Chapman-Kolmogorov equation* of a CTMC.

As discussed, the transition function is a powerful tool that we'd like to evaluate, but our specification of models, and the *sample-path behaviour* has been in terms of rates of events/transitions. Here we begin our journey of going from transition rates to transition functions.

Definition 5. *The generator Q of a continuous-time Markov chain $\mathcal{X}$ has entries (when the limits exist)*

$$q_{ij} := \lim_{h\to 0^+} \frac{P_{ij}(h) - P_{ij}(0)}{h} = \lim_{h\to 0^+} \frac{P_{ij}(h)}{h} \quad (\geq 0) \qquad \textit{for } j \in \mathcal{S}, j \neq i,$$

$$q_{ii} := \lim_{h\to 0^+} \frac{P_{ii}(h) - P_{ii}(0)}{h} = \lim_{h\to 0^+} \frac{P_{ii}(h) - 1}{h} \quad (\leq 0)\,,$$

where $P_{ij}(h) := \Pr(X(t+h) = j \mid X(t) = i)$ is the conditional probability that the system is in state j by the end of the time interval h.

Loosely speaking, $Q = (q_{ij})_{i,j\in\mathcal{S}}$ is the *right-derivative* of the matrix $P(t)$ at the point $t = 0$. Q is sometimes referred to as the infinitesimal generator, or the (instantaneous) transition rate matrix (in particular the latter if $\mathcal{S}$ is finite).

In matrix notation:

$$Q = \lim_{h\to 0^+} \frac{P(h) - I}{h},$$

where $P(h) = (P_{ij}(h))_{i,j\in\mathcal{S}}$ and I is an identity matrix.

Properties

(i) Non-negative off diagonal elements:

$$q_{ij} \geq 0 \text{ for } j \in \mathcal{S}, j \neq i.$$

(ii) Non-positive diagonal elements:
Since we have
$$\sum_{j\in\mathcal{S}} P_{ij}(h) = 1 \text{ for } i\in\mathcal{S} \text{ and } h\in[0,\infty),$$
$$\begin{aligned}
1 - P_{ii}(h) &= \sum_{\substack{j\neq i\\ j\in\mathcal{S}}} P_{ij}(h)\\
\therefore \lim_{h\to 0^+} \frac{1-P_{ii}(h)}{h} &= \lim_{h\to 0^+} \sum_{\substack{j\neq i\\ j\in\mathcal{S}}} \frac{P_{ij}(h)}{h}\\
\Rightarrow \quad -q_{ii} &\geq \sum_{\substack{j\neq i\\ j\in\mathcal{S}}} \lim_{h\to 0^+} \frac{P_{ij}(h)}{h}\\
\Rightarrow \quad -q_{ii} &\geq \sum_{\substack{j\neq i\\ j\in\mathcal{S}}} q_{ij}.
\end{aligned}$$

Note, that if every row sum is zero,
$$\sum_{j\in\mathcal{S}} q_{ij} = 0 \text{ for all } i\in\mathcal{S},$$
then we say that Q is *conservative*. We will deal with conservative generators only.

The **input to our model** are the q_{ij}. So, it will be useful to get some feeling about what these mean physically. Recall that we define for $i,j\in\mathcal{S}$ and $s,t\in[0,\infty)$
$$P_{ij}(t) = \mathbb{P}(X(t+s) = j | X(s) = i)$$
to be the probability of being in j at $t\geq 0$, given the system starts in i.

Physical Meaning

(i) For small h and for $i\neq j$,
$$P_{ij}(h) = q_{ij}h + o(h),$$
where $o(h)$ denotes a function $f(h)$ that satisfies $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

$\equiv$ Pr(the chain moving out of state $i \to j$ in some small time h) $\approx q_{ij}h$.

$\equiv q_{ij}$ is the *instantaneous rate* (in a probabilistic sense) that the chain moves from $i\to j$.

(ii) For small h and $i\in\mathcal{S}$, we have
$$1 - P_{ii}(h) = -hq_{ii} + o(h).$$

$\equiv$ Pr(the chain moving out of state i in some small time h) $\approx (-q_{ii})h$.

$\equiv -q_{ii}$ is the instantaneous rate that the chain moves out of state i.

We now have an intuitive feel for what the entries of the generator represent, and how that relates, loosely, to derivatives of the transition function at time $t=0$. However, how do we extend this to information about the transition function at any time t. Important to this translation are the Chapman-Kolmogorov equations.

Chapman-Kolmogorov Equation

Theorem 3. *For a continuous-time Markov chain* $(X(t) : t \geq 0)$ *on a state space* $\mathcal{S}$ *and for* $i, j \in \mathcal{S}$*, we have*

$$P_{ij}(t) = \sum_{k \in \mathcal{S}} P_{ik}(u) P_{kj}(t-u) \quad \textit{for} \quad 0 < u \leq t.$$

This is known as the Chapman-Kolmogorov equation.

In other words, the probability of going from $i \to j$ in time t, is the probability of going from $i \to k$ in time u multiplied by the probability of going from $k \to j$ in time $(t-u)$, summed over all possible states k.

Proof. Consider the CTMC at some time $s+u$ which is such that

$$s < s+u \leq s+t \quad \text{for} \quad s, u, t \in [0, \infty).$$

The chain must be in some state $k \in \mathcal{S}$ at time $s+u$, thus,

$$\begin{aligned}
P_{ij}(t) &= \Pr(X(s+t) = j | X(s) = i) \\
&= \sum_{k \in \mathcal{S}} \Pr(X(s+t) = j, X(s+u) = k | X(s) = i) \\
&= \sum_{k \in \mathcal{S}} \Pr(X(s+t) = j | X(s+u) = k, X(s) = i) \Pr(X(s+u) = k | X(s) = i) \;\; \text{(L.T.P.)} \\
&= \sum_{k \in \mathcal{S}} \Pr(X(t) = j | X(u) = k) \Pr(X(u) = k | X(0) = i) \; \text{(Markov property; time homogeneity)} \\
&= \sum_{k \in \mathcal{S}} P_{kj}(t-u) P_{ik}(u).
\end{aligned}$$

□

In matrix form: $P(t) = P(u)P(t-u)$, where the time-dependent matrix $P(t)$ is given by

$$P(t) = \begin{bmatrix} P_{i_0,i_0}(t) & P_{i_0,i_1}(t) & P_{i_0,i_2}(t) & \cdots \\ P_{i_1,i_0}(t) & P_{i_1,i_1}(t) & P_{i_1,i_2}(t) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \quad \text{with } i_0, i_1, i_2, \ldots \in \mathcal{S}.$$
