

# School of Mathematical Sciences

## APP MTH 3022/7106 - Optimal Functions and Nanomechanics

### Assignment 4 question sheet

*Due: Thursday, 10 October, at 12 noon (in the hand-in box on level 6)*

*When presenting your solutions to the assignment, please include some explanation in words to accompany your calculations. It is not necessary to write a lengthy description, just a few sentences to link the steps in your calculation. Messy, illegible or inadequately explained solutions may be penalised. The marks awarded for each part are indicated in brackets.*

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*All students are to attempt Questions 1–3.*

1. A spheroidal surface  $\mathcal{P}$  is given parametrically by the position vector  $\mathbf{r}(\theta, \phi)$  as

$$\mathbf{r}(\theta, \phi) = (b \cos \theta \sin \phi, b \sin \theta \sin \phi, c \cos \phi),$$

where  $-\pi < \theta \leq \pi$ , and  $0 \leq \phi \leq \pi$  and the constant  $b$  is the minor semi-axis length and  $c$  is the major semi-axis length.

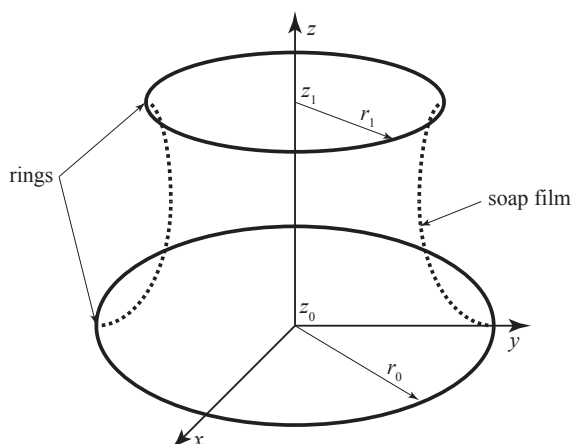
- (a) Derive an expression for the scalar surface element  $dA$  for  $\mathcal{P}$ .
- (b) Integrate your answer from part (a) to find the surface area of  $\mathcal{P}$  as a function of  $b$  and  $c$ .  
*Hint: it is easiest to express this in terms of the usual hypergeometric function.*
- (c) One isomer of the  $C_{70}$  fullerene may be modelled as a spheroid with a minor semi-axis length of  $b = 3.59 \text{ \AA}$  and major semi-axis length of  $c = 4.17 \text{ \AA}$ . Use your answer from part (b) to derive a reasonable approximation (to four decimal places) of the mean surface density of carbon atoms for this molecule.

[8 marks]

2. Consider a soap film suspended between two parallel concentric, but displaced rings of radius  $r_0$  and  $r_1$  at an offset in the  $z$ -direction of  $z_0$  and  $z_1$  (see the figure for a clearer view). Ignoring gravity and other external forces, the shape of the soap film will minimise the surface area.

- (a) Use the Calculus of Variations to determine the profile that the soap film will adopt.
- (b) Plot the resulting profile for  $(z_0, r_0) = (0, 9)$  and  $(z_1, r_1) = (10, 10)$ .

[8 marks]



3. Consider the problem of finding the extremal of the functional

$$F\{y\} = \int_0^1 x(y'^2 - \lambda y^2) dx,$$

subject to  $y(0)$  being finite and  $y(1) = 0$ . Assuming that the extremal will be an even function of  $x$ , we may choose as basis functions

$$\phi_n(x) = 1 - x^{2n},$$

and propose an approximate series solution of the form

$$y(x) \approx \bar{y}(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots.$$

Use Ritz's method with  $\bar{y}(x) = c_1\phi_1(x) + c_2\phi_2(x)$ , to derive a constraint on  $\lambda$  that must be satisfied for a non-trivial solution to exist. Give the numerical value (to four significant digits) of the smallest  $\lambda$  that satisfies the constraint.

[8 marks]

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