APP MTH 3020 Stochastic Decision Theory

Tutorial 5

Week 11, Friday, October 19

1. On Gallifrey, it is either sunny, rainy or foggy, and it is interesting to attempt to predict what the weather will do tomorrow. We have the following table of probabilities of what tomorrow's weather will be like given today's weather, based on historical data.

		Tomorrow		
		Sunny	Rainy	Foggy
Today	Sunny	0.8	0.05	0.15
	Rainy	0.2	0.6	0.2
	Foggy	0.2	0.3	0.5

,tionProbabilities of tomorrow's weather based on today's weather.

   a. Draw a state diagram of this process.

   b. Given that today is sunny, what is the probability that tomorrow is sunny and that the day after is rainy?

   Let S =Sunny, R =Rainy and F =Foggy. Then,

$$\begin{aligned} \mathrm{P}\left(X_1 = S, X_2 = R \mid X_0 = S\right) &= \mathrm{P}\left(X_1 = S \mid X_0 = S\right) \mathrm{P}\left(X_2 = R \mid X_1 = S\right) \\ &= 0.8 \times 0.05 = 0.04. \end{aligned}$$

   c. Given that today is foggy, what is the probability that it will be rainy in two days from now?

$$\begin{aligned} \mathrm{P}\left(X_2 = R \mid X_0 = F\right) &= \mathrm{P}\left(X_1 = R \mid X_0 = F\right) \mathrm{P}\left(X_2 = R \mid X_1 = R\right) \\ &\quad + \mathrm{P}\left(X_1 = F \mid X_0 = F\right) \mathrm{P}\left(X_2 = R \mid X_1 = F\right) \\ &\quad + \mathrm{P}\left(X_1 = S \mid X_0 = F\right) \mathrm{P}\left(X_2 = R \mid X_1 = S\right) \\ &= 0.3 \times 0.6 + 0.5 \times 0.3 + 0.2 \times 0.05 = 0.34. \end{aligned}$$

   You could also use the (F, R) entry of the transition matrix squared to get the result.

2. On Gallifrey people also use umbrellas. If you find yourself locked in a room, the only piece of evidence about the weather outside is if your keeper is carrying your daily meal enter that room with an umbrella or not. Suppose that the probability of seeing an umbrella based on the weather is as shown in the following table.

	Probability of umbrella
Sunny	0.1
Rainy	0.8
Foggy	0.3

Table 1: Probabilities of seeing an umbrella based on the weather.

   a. The day you were locked in the room, it was sunny and the next day the keeper carried an umbrella. What is the probability that the second day was rainy?

$$\begin{aligned}
\mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}\right) &= \frac{\mathrm{P}\left(w_2 = \mathrm{R}, w_1 = \mathrm{S}, u_2 = \mathrm{T}\right)}{\mathrm{P}\left(w_1 = \mathrm{S}, u_2 = T\right)} \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}, w_2 = \mathrm{R}\right) \mathrm{P}\left(w_1 = \mathrm{S}, w_2 = \mathrm{R}\right)}{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right) \mathrm{P}\left(w_1 = \mathrm{S}\right)} \\
\text{(conditional independence)} \qquad &= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}\right) \mathrm{P}\left(w_1 = \mathrm{S}\right)}{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right) \mathrm{P}\left(w_1 = \mathrm{S}\right)} \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}\right)}{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right)}.
\end{aligned}$$

As

$$\begin{aligned}
\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right) = \mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{S}\right) \mathrm{P}\left(w_2 = \mathrm{S} \mid w_1 = \mathrm{S}\right) \\
+ \mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}\right) \\
+ \mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{F}\right) \mathrm{P}\left(w_2 = \mathrm{F} \mid w_1 = \mathrm{S}\right) \\
= (0.3)(0.15) + (0.8)(0.05) + (0.1)(0.8) \; = \; 0.165.
\end{aligned}$$

Thus,

$$\mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}\right) = \frac{(0.8)(0.05)}{0.165} \; = \; 0.24\overline{24}$$

Note: Letting $\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right) = \mathrm{P}\left(u_2 = \mathrm{T}\right)$ may at first seem reasonable, but thinking more about it, u_2 is a function of w_2 which is dependent on w_1 as shown above. The correct assertion would be that $\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}, w_2\right) = \mathrm{P}\left(u_2 = \mathrm{T} \mid w_2\right)$, which has been done above.

b. Again, suppose that the day you were locked in the room, it was sunny and the next day the keeper carried an umbrella, but not on day 3. What is the probability that it is foggy on day 3?

Using the law of total probability and the definition of conditional probability,

$$\begin{aligned}
&\mathrm{P}\left(w_3 = \mathrm{F} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}, u_3 = \mathrm{F}\right) \\
&\quad = \mathrm{P}\left(w_2 = \mathrm{R}, w_3 = \mathrm{F} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}, u_3 = \mathrm{F}\right) \\
&\qquad + \mathrm{P}\left(w_2 = \mathrm{S}, w_3 = \mathrm{F} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}, u_3 = \mathrm{F}\right) \\
&\qquad\quad + \mathrm{P}\left(w_2 = \mathrm{F}, w_3 = \mathrm{F} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}, u_3 = \mathrm{F}\right).
\end{aligned}$$

Consider the first term

$$\begin{aligned}
&\mathrm{P}\left(w_2 = \mathrm{R}, w_3 = \mathrm{F} \mid w_1 = \mathrm{S}, u_2 = \mathrm{T}, u_3 = \mathrm{F}\right) \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_1 = \mathrm{S}, w_2 = \mathrm{R}, w_3 = \mathrm{F}, u_3 = \mathrm{F}\right) \mathrm{P}\left(w_1 = \mathrm{S}, w_2 = \mathrm{R}, w_3 = \mathrm{F}, u_3 = \mathrm{F}\right)}{\mathrm{P}\left(w_1 = \mathrm{S}, u_3 = \mathrm{F}, u_2 = \mathrm{T}\right)} \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(w_1 = \mathrm{S}, w_2 = \mathrm{R}, w_3 = \mathrm{F}, u_3 = \mathrm{F}\right)}{\mathrm{P}\left(w_1 = \mathrm{S}, u_3 = \mathrm{F}, u_2 = \mathrm{T}\right)} \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(u_3 = \mathrm{F} \mid w_3 = \mathrm{F}\right) \mathrm{P}\left(w_1 = \mathrm{S}, w_2 = \mathrm{R}, w_3 = \mathrm{F}\right)}{\mathrm{P}\left(w_1 = \mathrm{S}, u_3 = \mathrm{F}, u_2 = \mathrm{T}\right)} \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(u_3 = \mathrm{F} \mid w_3 = \mathrm{F}\right) \mathrm{P}\left(w_3 = \mathrm{F} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}\right) \mathrm{P}\left(w_1 = \mathrm{S}\right)}{\mathrm{P}\left(u_3 = \mathrm{F}, u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right) \mathrm{P}\left(w_1 = \mathrm{S}\right)} \\
&= \frac{\mathrm{P}\left(u_2 = \mathrm{T} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(u_3 = \mathrm{F} \mid w_3 = \mathrm{F}\right) \mathrm{P}\left(w_3 = \mathrm{F} \mid w_2 = \mathrm{R}\right) \mathrm{P}\left(w_2 = \mathrm{R} \mid w_1 = \mathrm{S}\right)}{\mathrm{P}\left(u_3 = \mathrm{F}, u_2 = \mathrm{T} \mid w_1 = \mathrm{S}\right)}.
\end{aligned}$$

Consider now the denominator

$$
\begin{aligned}
&\mathrm{P}\left(u_3=\mathrm{F}, u_2=\mathrm{T} \mid w_1=\mathrm{S}\right) \\
&=\sum_{i \in\{\mathrm{F}, \mathrm{R}, \mathrm{S}\}} \sum_{j \in\{\mathrm{F}, \mathrm{R}, \mathrm{S}\}} \mathrm{P}\left(u_2=\mathrm{T} \mid w_2=i\right) \mathrm{P}\left(u_3=\mathrm{F} \mid w_3=j\right) \mathrm{P}\left(w_3=j \mid w_2=i\right) \mathrm{P}\left(w_2=i \mid w_1=\mathrm{S}\right) \\
&=0.0056+0.01575+0.0084+0.0048+0.0027+0.0008+0.0072+0.0081+0.0576 \\
&=0.11095 .
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\mathrm{P}\left(w_3=\mathrm{F} \mid w_1=\mathrm{S}, u_2=\mathrm{T}, u_3=\mathrm{F}\right) \\
&\qquad=\frac{\mathrm{P}\left(u_2=\mathrm{T} \mid w_2=\mathrm{F}\right) \mathrm{P}\left(u_3=\mathrm{F} \mid w_3=\mathrm{F}\right) \mathrm{P}\left(w_3=\mathrm{F} \mid w_2=\mathrm{F}\right) \mathrm{P}\left(w_2=\mathrm{F} \mid w_1=\mathrm{S}\right)}{\mathrm{P}\left(u_3=\mathrm{F}, u_2=\mathrm{T} \mid w_1=\mathrm{S}\right)} \\
&\qquad+\frac{\mathrm{P}\left(u_2=\mathrm{T} \mid w_2=\mathrm{R}\right) \mathrm{P}\left(u_3=\mathrm{F} \mid w_3=\mathrm{F}\right) \mathrm{P}\left(w_3=\mathrm{F} \mid w_2=\mathrm{R}\right) \mathrm{P}\left(w_2=\mathrm{R} \mid w_1=\mathrm{S}\right)}{\mathrm{P}\left(u_3=\mathrm{F}, u_2=\mathrm{T} \mid w_1=\mathrm{S}\right)} \\
&\qquad+\frac{\mathrm{P}\left(u_2=\mathrm{T} \mid w_2=\mathrm{S}\right) \mathrm{P}\left(u_3=\mathrm{F} \mid w_3=\mathrm{S}\right) \mathrm{P}\left(w_3=\mathrm{S} \mid w_2=\mathrm{S}\right) \mathrm{P}\left(w_2=\mathrm{S} \mid w_1=\mathrm{S}\right)}{\mathrm{P}\left(u_3=\mathrm{F}, u_2=\mathrm{T} \mid w_1=\mathrm{S}\right)} \\
&\qquad=\frac{0.01575+0.00560+0.00840}{0.11095}=0.268138801261830 .
\end{aligned}
$$