Lecture 2: An introduction to continuous-time Markov chains (CTMCs) – Problematic printers!

Concepts checklist

At the end of this lecture, you should be able to:

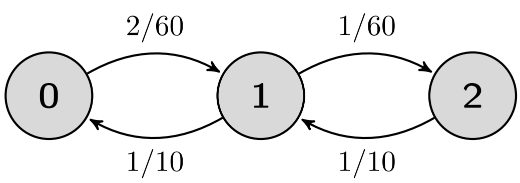
- Define the *state space* of a discrete-state random process;
- Define a *generator* from a *state transition diagram*, and vice-versa;
- State the *features/properties of a generator*;
- Define a continuous-time Markov chain (CTMC);
- Define a *time-homogeneous* CTMC; and,
- Define a *transition function* of a time-homogeneous CTMC and state its *assumed properties.*

Example 1. Repairman problem, with 2 machines

In the School of Mathematical Sciences we have 2 printers on level 6. From experience, a printer fails, independently of the other printer, on average, every 60 days. It takes, on average, 10 days to repair a single machine when it fails; but there is only one repairman.

We could model this process as a continuous-time Markov chain. If we assume that the *times of failures and repairs are independent and exponentially distributed*, then we can let $X(t)$ be the number of failed printers at time $t \geq 0$. Hence $X(t)$ can be either 0 (both printers working), 1 (1 printer working, 1 not) or 2 (both printers 'on the blink'). The *state space*, $\mathcal{S}$, of the process is then $\mathcal{S} = \{0, 1, 2\}$. **The state space is the set of possible values the process can adopt.**

Given the states, events, and rates, we can write down a *state transition diagram.*



Do you know where all the rates come from? What about the rate 2/60 per day of transitioning from State 0 to State 1? In State 0, we have 2 printers working, each failing independently at rate 1/60. Hence, the rate at which we see one fail can be determined by considering the distribution of $F = \min\{F_1, F_2\}$, where F_1 and F_2 are independent and identically distributed exponential random variables with rate $1/60$:

$$\begin{aligned}
\Pr(F > t) &= \Pr(\min\{F_1, F_2\} > t) \\
&= \Pr(F_1 > t \cap F_2 > t) \\
&= \Pr(F_1 > t)\Pr(F_2 > t) && \text{(independence)} \\
&= \exp(-(1/60)t)\exp(-(1/60)t) && \text{(identically exponential, rate } 1/60\text{)} \\
&= \exp(-(2/60)t).
\end{aligned}$$

Hence, the time to the first failure is exponentially distributed with rate $2/60$.

Useful for analysing the behaviour of processes such as this, is the *generator* (also called *transition rate matrix* or *Q-matrix*), Q. The generator, Q, corresponding to the process described, is

$$Q = \begin{bmatrix} -2/60 & 2/60 & 0 \\ 1/10 & -7/60 & 1/60 \\ 0 & 1/10 & -1/10 \end{bmatrix}.$$

Some important features to note about the generator are:

- Off-diagonal entries are non-negative, and correspond to the rates of events that change the state of the process;
- Each row sum is zero; and,
- Diagonal entries are non-positive (here negative), which follows from the first two observations.

Now, what defines a random process as being a continuous-time Markov chain?

Definition 1. *A continuous-time Markov chain is a random process which satisfies the* Markov property*:*

$$\Pr(X(t+s) = k \mid X(u) = i, X(s) = j, u < s) = \Pr(X(t+s) = k \mid X(s) = j),$$

for all $s, t \in [0, \infty)$ and all $i, j, k \in \mathcal{S}$.

This says that the future path depends on the history $\{X(u), u \leq s\}$ only through the present state $X(s)$. The process is memoryless.

Is the repairman process described a continuous-time Markov chain? Let's think about the behaviour of this process:

- If in State 0, we wait an exponentially-distributed time, T_0, with mean 30 days until we see a printer failure, and the process transitions to State 1;
- Similarly, if in State 2, we wait an exponentially-distributed time, T_2, with mean 10 days until we see a repair of a machine, and the process transitions to State 1; and,
- In State 1,

- (a) if we have entered from State 0, then one printer has not failed. But should the fact that it has not failed while sojourning in State 0 influence the time until it now fails, say, F_2?
- (b) we have two competing events: the repairman is working on fixing the broken printer, while the second printer might fail. Like the two competing events of failure of printers, these are independent and exponentially-distributed, but are not identical as they have different means. Let's say these random variables are $T_{1,0}$ and $T_{1,2}$, respectively, and then the event that occurs is whichever happens first; i.e., we are interested in $\min\{T_{1,0}, T_{1,2}\}$.

Hence, the behaviour of the process is memoryless when in States 0 and 2, but we need to check that it is memoryless when in State 1. Let's consider (a) first.

Since the time to failure is assumed exponentially-distributed, then assuming that the process sojourned in State 0 for t units of time, we have

$$\begin{aligned}
\Pr(F_2 > t+s | F_2 > t) &= \frac{\Pr(F_2 > t+s, F_2 > t)}{\Pr(F_2 > t)} \\
&= \frac{\Pr(F_2 > t+s)}{\Pr(F_2 > t)} \\
&= \frac{\exp(-(t+s)/60)}{\exp(-t/60)} \\
&= \exp(-s/60) \\
&= \Pr(F_2 > s).
\end{aligned}$$

Hence, the distribution of the time until failure remains exponential, with the time elapsed in State 1, regardless of its history. It is said to be memoryless – a property for which the exponential distribution is the only continuous distribution to possess.

Definition 2. *A continuous random variable T is said to have a memoryless property if*

$$\Pr(T > t+s | T > t) = \Pr(T > s)$$

for $s, t \geq 0$.

Now, let's return to (b). As before, considering the distribution of $M = \min\{T_{1,0}, T_{1,2}\}$ we have:

$$\begin{aligned}
\Pr(M > t) &= \Pr(\min\{T_{1,0}, T_{1,2}\} > t) \\
&= \exp(-(7/60)t).
\end{aligned}$$

Hence, the time to the next event is exponentially distributed with rate 7/60. This means that the memoryless property is preserved with respect to the time to the next event.

Let us now consider the probabilities of transitioning to each of the states, given that a jump occurs. We have, for some $t > 0$,

$$\begin{aligned}
\Pr(\text{moves to State 0}|\text{moves out of State 1 at time } t) &= \Pr(T_{1,0} < T_{1,2} | M \in [t, t+dt)) \\
&= \frac{\Pr(T_{1,0} \in [t, t+dt) \cap T_{1,2} > t)}{\Pr(M \in [t, t+dt))} \\
&= \frac{(1/10)e^{-t/10}e^{-t/60}}{(7/60)e^{-7t/60}} \text{(independence)} \\
&= 6/7.
\end{aligned}$$

This probability is independent of t. In other words, the time of the next event, and which event occurs first, are independent random variables. Hence, this process is memoryless; i.e., it is a continuous-time Markov chain!

For this particular process, we can write down its *transition function*, $P(s, t+s)$. This can be written as matrix with entries corresponding to the probabilities of transitioning between states from time s to time $t+s$. The transition function is [I got this from Q, and will tell you how soon (like, next week!)]:

$$P(s, t+s) = \frac{1}{25}\begin{bmatrix} 18 + 3e^{-t/6} + 4e^{-t/12} & 6 - 4e^{-t/6} - 2e^{-t/12} & 1 + e^{-t/6} - 2e^{-t/12} \\ 18 - 12e^{-t/6} - 6e^{-t/12} & 6 + 16e^{-t/6} + 3e^{-t/12} & 1 - 4e^{-t/6} + 3e^{-t/12} \\ 18 + 18e^{-t/6} - 36e^{-t/12} & 6 - 24e^{-t/6} + 18e^{-t/12} & 1 + 6e^{-t/6} + 18e^{-t/12} \end{bmatrix}.$$

Note that this transition function is actually only dependent upon the elapsed time t (and not on the precise times s and $t+s$). It is said to be *time-homogeneous.*

Definition 3. *A continuous-time Markov chain is* time-homogeneous *if*

$$\Pr(X(t+s) = j | X(s) = i) = \Pr(X(t) = j | X(0) = i)$$

for all $i, j \in \mathcal{S}$ *and* $s, t \geq 0$.

The *transition function* of a time-homogeneous CTMC is $P(t) = (P_{ij}(t))_{i,j\in\mathcal{S},t\geq 0}$.

For such a time-homogeneous CTMC, knowledge of the transition function

$$P_{i,j}(t) = \Pr(X(t) = j | X(0) = i)$$

for all $i, j \in \mathcal{S}$ and $t \geq 0$ would allow to answer any question we'd like about the process! So it would be good to be able to know this for any continuous-time Markov chain; you'll find out how in this course! But for now some *assumed properties*:

$$\sum_{j\in\mathcal{S}} P_{i,j}(t) = 1$$

for each $i \in \mathcal{S}$ and for all $t \geq 0$ (the probabilities across states always sum to one); we then say that the transition function is *honest.* And the probabilities are non-negative,

$$P_{i,j}(t) \geq 0$$

for all $i, j \in \mathcal{S}$ and $t \geq 0$.
