# LECTURE 9

Recall from last time

**Lemma 2.2**: if $a_n \to L$ and $a_n \to M$ then $L = M$.

In other words, a limit of sequence (if it exists) is unique. Therefore the following notation make sense: we write $\lim_{n\to\infty} a_n = L$ if $a_n \to L$.

**Example**: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $L \in \mathbb{R}$. The following statements are all equivalent:

(1) $\forall\ \epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| < \epsilon$$

(2) $\forall\ m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| < 1/m$$

(3) $\forall\ \epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| \leq \epsilon$$

(4) $\forall\ \epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| < 2\epsilon.$$

It's clear that (1) $\implies$ (2), since if you know that (1) is true then for any $m \in \mathbb{N}$, you may take $\epsilon = 1/m$ in (1) and deduce that (2) is true for this $m$. Hence (2) is true for all $m \in \mathbb{N}$. Conversely, suppose you know that (2) is true. Let $\epsilon > 0$. By the Archimedean Property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $1/m < \epsilon$. Since (2) is true there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L| < 1/m$. Therefore, since $1/m < \epsilon$, if $n \geq N$ then $|a_n - L| < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows that (1) is true.

It's clear that (1) $\implies$ (3) since $|a_n - L| < \epsilon \implies |a_n - L| \leq \epsilon$. Conversely, suppose that (3) is true. Let $\epsilon > 0$. Since (3) is true, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L| \leq \epsilon/2$. Therefore, since $\epsilon/2 < \epsilon$, if $n \geq N$ then $|a_n - L| < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows that (1) is true.

The equivalence of (1) and (4) is left as an exercise.

**Definition 2.3**: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that

- $(a_n)_{n=1}^{\infty}$ is *bounded above* if the set $\{ a_n \mid n \in \mathbb{N} \}$ is bounded above.
- $(a_n)_{n=1}^{\infty}$ is *bounded below* if the set $\{ a_n \mid n \in \mathbb{N} \}$ is bounded below.
- $(a_n)_{n=1}^{\infty}$ is *bounded* if the set $\{ a_n \mid n \in \mathbb{N} \}$ is bounded.

**Remarks**: The following statements are true

(i) $(a_n)_{n=1}^{\infty}$ is bounded above if and only if there exists $K > 0$ such that $a_n < K$ for all $n \in \mathbb{N}$.

To see, this observe that if there exists such a $K > 0$ then the set $\{ a_n \mid n \in \mathbb{N} \}$ is bounded above (by $K$). Conversely, suppose that $\{ a_n \mid n \in \mathbb{N} \}$ is bounded above. Then there exists $L \in \mathbb{R}$ such that $a_n \leq L$ for all $n \in \mathbb{N}$. Choose a positive real number $K$ such that $L < K$ (exercise: there exists such a $K$). Then $a_n < K$ for all $n \in \mathbb{N}$.

(ii) $(a_n)_{n=1}^{\infty}$ is bounded below if and only if there exists $K > 0$ such that $-K < a_n$ for all $n \in \mathbb{N}$.

One way to see this is to observe that the set $\{ a_n \mid n \in \mathbb{N} \}$ is bounded below if and only if the set $\{ -a_n \mid n \in \mathbb{N} \}$ is bounded above, if and only if there exists a $K > 0$ such that $-a_n < K$ for all $n \in \mathbb{N}$, if and only if there exists $K > 0$ such that $-K < a_n$ for all $n \in \mathbb{N}$.

(iii) $(a_n)_{n=1}^{\infty}$ is bounded if and only if it is bounded above and below if and only if there exists $K > 0$ such that $|a_n| \leq K$.

Statement (iii) is left as an exercise.

**Example**: If $(a_n)$ is not bounded above, then for all $K > 0$ there exists $n \in \mathbb{N}$ such that $a_n > K$. Similarly, if $(a_n)$ is not bounded, then for all $K > 0$ there exists $n \in \mathbb{N}$ such that $|a_n| > K$.

**Theorem 2.4**: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. If $(a_n)_{n=1}^{\infty}$ is convergent then $(a_n)_{n=1}^{\infty}$ is bounded.

The idea behind the proof is simple: we want to show that if the sequence is convergent, then the terms of the sequence are bounded. If the sequence converges to $L$, then eventually all of the terms of the sequence must be close to $L$. Since there are only finitely many other terms, it follows that the set of terms of the sequence is bounded.

**Proof**: We need to find $K > 0$ such that $|a_n| \leq K$ for all $n \in \mathbb{N}$. Let $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ if $n \geq N$. Therefore, if $n \geq N$ then $|a_n| - |L| \leq |a_n - L| < 1$. Hence $|a_n| \leq |L| + 1$ for all $n \geq N$. Let $K = \max\{ \, |a_1|, \ldots, |a_{N-1}|, |L| + 1 \}$. Then $K > 0$ and $|a_n| \leq K$ for all $n \in \mathbb{N}$. ■

**Example**: The converse of Theorem 2.4 is the statement that if $(a_n)_{n=1}^{\infty}$ is bounded, then $(a_n)_{n=1}^{\infty}$ is convergent. Note that the converse statement need not be true: there exist bounded sequences that are not convergent, for instance the sequence $(a_n)_{n=1}^{\infty}$ defined by $a_n = (-1)^n$. This is bounded, but it does not converge to any $L \in \mathbb{R}$ (later we shall give a proof of this fact).

Consider the following problem: prove that

$$\lim_{n \to \infty} \frac{2n^5 + \sqrt{2}n^4 - 103n^3 + 27n^2 - 119}{6n^5 + 1024n^3 + 6n + 4} = \frac{1}{3}.$$

It is not practical to prove every statement such as this one from the definition of limit. Instead, we would like to prove some general theorems which will make proving statements such as this one a fairly trivial affair. For instance, the way we would like to prove this statement is by performing manipulations like

$$\frac{2n^5 + \sqrt{2}n^4 + \cdots}{6n^5 + 1024n^3 + \cdots} = \frac{2 + \sqrt{2}/n + \cdots}{6 + 1024/n^2 + \cdots} \to \frac{1}{3}$$

To justify these manipulations we need the following theorem.

**Theorem 2.5: (Algebraic Limit Theorem)** Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. If $a_n \to L$ and $b_n \to M$ then the following statements are true:

(i) $ca_n \to cL$ for all $c \in \mathbb{R}$

(ii) $a_n + b_n \to L + M$

(iii) $a_n b_n \to LM$

(iv) if $b_n \neq 0$ for all $n$ and $M \neq 0$ then $a_n/b_n \to L/M$.

**Proof**: We first prove (i). Suppose $a_n \to L$ and $c \in \mathbb{R}$. If $c = 0$ then $ca_n$ is the constant sequence 0, which converges to $0 = cL$. Therefore, we may suppose without loss of generality that $c \neq 0$. Therefore $|c| > 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - L| < \epsilon/|c|$. Therefore, if $n \geq N$ then

$$|ca_n - cL| = |c| \cdot |a_n - L| < |c| \cdot (\epsilon/|c|) = \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, it follows that $ca_n \to cL$.

We prove statement (ii). Let $\epsilon > 0$. We want to show that $|(a_n + b_n) - (L + M)| < \epsilon$ for all $n$ sufficiently large. We begin by estimating $|(a_n + b_n) - (L + M)|$ using the triangle inequality. We have

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M|.$$

Since $a_n \to L$, $|a_n - L|$ is small if $n$ is sufficiently big. Since $b_n \to M$, $|b_n - M|$ is also small if $n$ is sufficiently big. Therefore $|a_n - L| + |b_n - M|$ is small if $n$ is sufficiently big.

We need to make this idea mathematically precise. Remember, we want to prove that if $n$ is sufficiently big then $|(a_n + b_n) - (L + M)| < \epsilon$. We observe that we will know this provided that we know that both $|a_n - L| < \epsilon/2$ and $|b_n - M| < \epsilon/2$, since $\epsilon/2 + \epsilon/2 = \epsilon$.

Since $a_n \to L$, we may choose $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - L| < \epsilon/2$. Since $b_n \to M$, we may choose $N_2 \in \mathbb{N}$ such that $n \geq N_2 \implies |b_n - M| < \epsilon/2$. We want to ensure that both inequalities are satisfied if $n$ is sufficiently big. To this end, let $N = \max\{ N_1, N_2 \}$. Notice that if $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. Therefore, if $n \geq N$ then $|a_n - L| < \epsilon/2$ *and* $|b_n - M|\epsilon/2$.

Suppose that $n \geq N$. Then

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that $a_n + b_n \to L + M$.

What are the take away points from this proof?

- The triangle inequality is very useful in this situation: it implies that the distance between $a_n + b_n$ and $L + M$ is $\leq$ the sum of the distance from $a_n$ to $L$ and the distance from $b_n$ to $M$.
- If we want both inequalities $|a_n - L| < \epsilon/2$ and $|b_n - M| < \epsilon/2$ to be satisfied, then we need $n \geq N_1$ and $n \geq N_2$. A simple way to arrange for this is to take $n \geq N$, where $N = \max\{ N_1, N_2 \}$.

We will use both of those tricks constantly throughout the course.

We prove statement (iii). This requires a new idea. Let $\epsilon > 0$. We estimate the distance between $a_n b_n$ and $LM$ using the triangle inequality in the following way.

$$|a_n b_n - LM| = |(a_n - L)b_n + L(b_n - M)| \leq |(a_n - L| \cdot |b_n| + |L| \cdot |b_n - M|.$$

Since $b_n \to M$ we know that $|b_n - M|$ is small if $n$ is sufficiently big. Therefore, since $|L|$ is a constant, we can arrange for $|L| \cdot |b_n - M|$ to be small by taking $|b_n - M|$ even smaller.

The term $|a_n - L| \cdot |b_n|$ is more problematic. By taking $n$ sufficently big, we can arrange for $|a_n - L|$ to be small. But then we have to worry that $|b_n|$ might be getting big while $|a_n - L|$ is getting small. This might have the net effect that $|a_n - L| \cdot |b_n|$ is not as small as we wanted.

Fortunately, this is not a problem. By Theorem 2.4, the sequence $(b_n)$ is bounded, since it is convergent. Therefore, there is a constant $K > 0$ such that if $|b_n| \leq K$ for all $n \in \mathbb{N}$. Therefore

$$|a_n - L| \cdot |b_n| \leq K|a_n - L|.$$

We would like to ensure that $K|a_n - L| < \epsilon/2$ provided that $n$ is sufficiently big. Since $a_n \to L$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - L| < \epsilon/(2K)$.

At the same time, we would like to ensure that

$$|L| \cdot |b_n - M| < \epsilon/2$$

if $n$ is sufficiently big. At first we are tempted to choose $N_2 \in \mathbb{N}$ such that $n \geq N_2 \implies |b_n - M| < \epsilon/(2|L|)$. There is a problem with this idea however — we could have $L = 0$. What we do instead is to choose $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \implies |b_n - M| < \frac{\epsilon}{2(|L|+1)}$$

Then, if $n \geq N_2$ then

$$|L| \cdot |b_n - M| < |L| \cdot \frac{\epsilon}{2(|L|+1)} = \frac{\epsilon}{2} \cdot \frac{|L|}{|L|+1} < \frac{\epsilon}{2}$$

since $|L|/(|L|+1) < 1$.

We now proceed as in the proof of statement (ii). Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$ we have

$$|a_n b_n - LM| \leq |a_n - L| \cdot |b_n| + |L| \cdot |b_n - M| < \frac{\epsilon}{2K} \cdot K + \frac{\epsilon|L|}{2(|L|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since $\epsilon > 0$ was arbitrary it follows that $a_n b_n \to LM$.

We prove (iv). We try to make life easier for ourselves by using things that we have already proven. Observe that if we can prove statement (iv) in the special case when $a_n = 1$ then we will be able to conclude that statement (iv) is true in general. This is because if $b_n \to M$ then applying (iv) in the special case where $a_n = 1$ we conclude that $1/b_n \to 1/M$. Therefore, applying (iii), we conclude that

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to L \cdot \frac{1}{M} = \frac{L}{M}.$$

So we need to prove that (iv) is true in the special case when $a_n = 1$. This will require some new tricks.

Let $\epsilon > 0$. We begin by estimating the distance from $1/b_n$ to $1/M$. We have

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \left|\frac{M - b_n}{Mb_n}\right| = \frac{|b_n - M|}{|M| \cdot |b_n|}.$$

Since $b_n \to M$, we can ensure that $|b_n - M|$ is small by choosing $n$ sufficiently big. But now we have to worry that $b_n$ might be getting small at the same time. However, we know that when $n$ is big, $|b_n|$ is close to $|M|$. Since $M \neq 0$, $|M| > 0$. Therefore, if we choose $n$ big enough, we see that $|b_n|$ will be close to $|M|$ and hence will not be getting arbitrarily small. We need to make this idea mathematically precise.

Since $b_n \to M$ and $|M| > 0$, we can choose $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|b_n - M| < |M|/2$. Applying the reversed triangle inequality, we see that if $n \geq N_1$ then

$$|M| - |b_n| \leq |M - b_n| = |b_n - M| < \frac{|M|}{2}$$

Therefore, if $n \geq N_1$ then $|b_n| > |M|/2$ and hence $1/|b_n| < 2/|M|$.

Now choose $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|b_n - M| < \epsilon/2|M|^2$. Let $N = \max\{\, N_1, N_2 \,\}$. Then if $n \geq N$ we have

$$\left| \frac{1}{b_n} - \frac{1}{M} \right| = \frac{|b_n - M|}{|M| \cdot |b_n|} < \frac{2\epsilon |M|^2}{2|M|^2} = \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, it follows that $1/b_n \to 1/M$. ■