**Random Processes III 2018: Assignment 3,**

**to be submitted by 1pm on Friday $7^{\text{th}}$ September.**

[39 marks in total]

Question 0. [4 marks]

Make sure that in all your answers you:

(a) use full and complete sentences,

(b) include units where necessary,

(c) give ranges/constraints for parameters in formulae, and

(d) make sure your layout and mathematical arguments are clear and concise.

Question 1. [8 marks]

A CTMC $(Z(t), t \geq 0)$ has the following transition rate matrix:

$$Q = \begin{pmatrix} -5 & 2 & 0 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 2 & 1 & 0 \\ 0 & 4 & 2 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}.$$

(a) Write down the state space $S$ of $Z(t)$ and determine which pairs of states are said to communicate.

(b) Determine, with reasoning, whether each state is either transient or recurrent.

(c) Determine the communicating classes of $S$ and, with reasoning, correctly label each class as either transient or recurrent.

(d) Is $Z(t)$ reducible or irreducible? Why?

Question 2. [9 marks]

For Problem 1 from Tutorial 1,

(a) Evaluate analytically the equilibrium distribution of the CTMC.

(b) Explain what the equilibrium probabilities represent.

(c) Provide code (e.g., in MATLAB) which numerically evaluates the equilibrium distribution by using a function which solves systems of linear equations.

Question 3. [7 marks]

Consider a game of tennis when deuce is reached. If a player wins the next point, they are said to have advantage. On the following point, the player with advantage wins if they win the point, and the game returns to deuce if they lose the point. Each point tends to last for $1/\lambda_D$ minutes, on average, in deuce, and for $1/\lambda_A$ minutes, on average, if a player has advantage. Assume that the probability Player A wins any game is $p_A$. Let the state space be $S = \{1, 2, 3, 4, 5\}$, where $1$ : deuce; $2$ : advantage $A$; $3$ : advantage $B$; $4$ : $A$ wins; $5$ : $B$ wins.

(a) Determine the generator matrix if this were modelled using a CTMC.

(b) After the first time deuce has been reached in the game, determine the probability that Player A eventually wins (before Player B wins).

Question 4. [11 marks]

Packets arrive to a switch as a Poisson process at rate $\lambda$ and are stored in a buffer. The switch processes packets at rate $\mu$. The processing time for a packet in the switch is exponentially distributed.

After each packet has been processed, it is transmitted with probability $p$. However, with probability $1 - p$, a processed packet immediately returns to the buffer for re-processing by the switch, due to switch processing errors.

(a) Write down an appropriate state space for a CTMC model of the number of packets in the buffer.

(b) Write down the transition rates $q_{ij}$ for the CTMC.

(c) Let $f_j$ be the probability that the buffer ever empties, given that it starts with $j$ packets. Write down a set of equations and boundary conditions satisfied by the $f_j$. Give an explanation for your boundary conditions and explain what other conditions we need in order to distinguish which solution to these equations gives us the $f_j$?

(d) Solve these equations, and give the criterion for this CTMC to be recurrent.