APP MTH 3001 Applied Probability III
Class Exercise 2 Solutions

1. Proceeding as in lectures; we have the same system of difference equations, but this time with boundary conditions $v_0 = 0, v_N = 1$. Try a solution of the form $v_i = Aw^i$. In the case $p \neq q$, we have the general form of the solution

$$v_i = A_1 + A_2 \left(\frac{q}{p}\right)^i .$$

In the case $p = q$, the general form of the solution is $v_i = A_1 + A_2 i$.

Using the boundary conditions and some algebra, we obtain

$$v_i = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1} & p \neq q \\[2ex] \dfrac{i}{N} & p = q. \end{cases} \qquad \Rightarrow u_i + v_i = 1, \ \forall i \in \mathcal{S} ,$$

which means the process will eventually be absorbed with probability 1 in either $i = 0$ or $i = N$.

Alternate solution:

Note that the same result can be obtained by exploiting the symmetry of the problem as follows. Since v_i can equivalently be thought of as the probability that Player B loses the contest, given that he has $N - i$ dollars, we conclude that we can arrive at the same conclusion by making the following transformations in the expression for u_i

$$i \to (N - i) \quad \text{and} \quad \left(\frac{q}{p}\right) \to \left(\frac{p}{q}\right) .$$

Making the above transformations, for $p \neq q$,

$$\begin{aligned} v_i &= \frac{\left(\frac{q}{p}\right)^{-N} - \left(\frac{q}{p}\right)^{-(N-i)}}{\left(\frac{q}{p}\right)^{-N} - 1} \\ &= \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N} \times \frac{\left(\frac{q}{p}\right)^N}{\left(\frac{q}{p}\right)^N} \\ &= \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1} \end{aligned}$$

and then for $p = q$,

$$v_i = u_{N-i} = \frac{N - (N - i)}{N} = \frac{i}{N},$$

2. (a)

$$\begin{aligned}
E(XW) &= \sum_i \sum_j x_i w_j \mathrm{P}\left(X = x_i, W = w_j\right) \\
&= \sum_i \sum_j x_i w_j \mathrm{P}\left(X = x_i\right) \mathrm{P}\left(W = w_j\right) \\
& \quad \text{(independence)} \\
&= \sum_i x_i \mathrm{P}\left(X = x_i\right) \sum_j w_j \mathrm{P}\left(W = w_j\right) \\
&= E(X)E(W).
\end{aligned}$$

(b)

$$\begin{aligned}
Var(X+W) &= E\Big(\big(X + W - E(X+W)\big)^2\Big) \\
&= E\Big(\big(X - E(X)\big)^2 + 2\big(XW - E(X)E(W)\big) + \big(W - E(W)\big)^2\Big) \\
&= Var(X) + 2\big(E(XW) - E(X)E(W)\big) + Var(W) \\
&= Var(X) + Var(W) \quad \text{(by independence).}
\end{aligned}$$

3. Write the binomial distribution in the form

$$\begin{aligned}
\mathrm{P}\left(Y = k\right) &= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)\cdots(n-k+1) \frac{p^k (1-p)^n}{k!(1-p)^k},
\end{aligned}$$

then substitute $p = \frac{\lambda}{n}$ to get

$$\begin{aligned}
\mathrm{P}\left(Y = k\right) &= n(n-1)\cdots(n-k+1) \frac{\left(\frac{\lambda}{n}\right)^k (1-\frac{\lambda}{n})^n}{k!(1-\frac{\lambda}{n})^k} \\
&= 1\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \frac{\lambda^k (1-\frac{\lambda}{n})^n}{k!(1-\frac{\lambda}{n})^k}.
\end{aligned}$$

Now let $n \to \infty$, and use the fact that

$$\begin{aligned}
1\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) &\to 1 \quad \text{as } n \to \infty \\
\left(1 - \frac{\lambda}{n}\right)^k &\to 1 \quad \text{as } n \to \infty,
\end{aligned}$$

as well as the identity

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}, \tag{1}$$

to obtain the Poisson distribution

$$\mathrm{P}\left(Y = k\right) = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad k = 0, 1, \ldots$$

4. Any reasonable-looking solution is acceptable here! The aim of this question was to make sure that the "infrastructure" for completing the group project is set up. Note to students: you are allowed to reuse the figures you made for this question in your final group project report.