School of Mathematical Sciences

Assignment Cover Sheet

Student Name	
Student ID	
Assessment Title	**Assignment 3**
Due Date	Thursday, 12 September, 2019 @ 12:00 noon
Course / Program	APP MTH 3022–Optimal Functions & Nanomechanics
Date Submitted	
OFFICE USE ONLY Date Received	

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OFN Assignment 3

Andrew Martin

September 11, 2019

1. Find the form of extremals to the following

   (a)
$$F\{y(x),z(x)\} = \int_{x_0}^{x_1} (8yz - 5y^2 + y'^2 - 4z'^2)dx$$

   Let $f = 8yz - 5y^2 + y'^2 - 4z'^2$
   The Euler-Lagrange equations are

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$
$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) - \frac{\partial f}{\partial z} = 0$$

   Where

$$\frac{\partial f}{\partial y'} = 2y', \quad \frac{\partial f}{\partial y} = 8z - 10y$$
$$\frac{\partial f}{\partial z'} = -8z', \quad \frac{\partial f}{\partial z} = 8y$$

   Plugging into EL

$$2y'' - 8z + 10y = 0$$
$$-8z'' - 8y = 0$$

   We get
$$y = -z''$$

   Use this in the first equation to get a fourth order ODE

$$-2z^{(4)} - 8z - 10z'' = 0 \implies z^{(4)} + 4z + 5z''$$

   Using the characteristic equation:

$$\lambda^4 + 5\lambda^2 + 4 = 0$$
$$\mu^2 + 5\mu + 4 = 0$$
$$(\mu+1)(\mu+4) = 0$$
$$\implies \mu = -1, \quad \mu = -4$$
$$\implies \lambda = \pm i, \quad \lambda = \pm 2i$$

Hence the z solution is

$$\boxed{z(x) = c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) + c_4 \cos(2x)}$$

And hence

$$\boxed{y(x) = -z'' = c_1 \sin(x) + c_2 \cos(x) + 4c_3 \sin(2x) + 4c_4 \cos(2x)}$$

Giving the extremal

$$F = \int_{x_0}^{x_1} (8yz - 5y^2 + y'^2 - 4z'^2)dx$$

(b)

$$F\{\mathbf{q}\} = \int_{t_0}^{t_1} \left(\dot{q}_1 q_2 + \dot{q}_2 q_3 + q_1 \dot{q}_3 - \dot{q}_1{}^2\right) dt$$

Where $\dot{q}_i := \frac{dq_i}{dt}$
EL is the set of equations

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} = 0$$

$$\begin{aligned}
\frac{\partial f}{\partial \dot{q}_1} &= q_2 - 2\dot{q}_1, & \frac{\partial f}{\partial q_1} &= \dot{q}_3 \\
\frac{\partial f}{\partial \dot{q}_2} &= q_3, & \frac{\partial f}{\partial q_2} &= \dot{q}_1 \\
\frac{\partial f}{\partial \dot{q}_3} &= q_1, & \frac{\partial f}{\partial q_3} &= \dot{q}_2
\end{aligned}$$

So the EL equations are:

$$\begin{aligned}
\dot{q}_2 - 2\ddot{q}_1 - \dot{q}_3 &= 0 \\
\dot{q}_3 - \dot{q}_1 &= 0 \\
\dot{q}_1 - \dot{q}_2 &= 0
\end{aligned}$$

$$\dot{q}_2 - 2\ddot{q}_1 - \dot{q}_3 = 0 \tag{1}$$
$$\dot{q}_1 = \dot{q}_3 \tag{2}$$
$$\dot{q}_1 = \dot{q}_2 \tag{3}$$

Using $(2), (3)$ shows that

$$\dot{q}_1 = \dot{q}_2 = \dot{q}_3$$

And hence (1) gives

$$-2\ddot{q}_1 = 0 \implies q_1 = at + c_1$$

And hence

$$\begin{aligned}
q_1 &= at + c_1 \\
q_2 &= at + c_2 \\
q_3 &= at + c_3
\end{aligned}$$

Hence the extremal has form

$$F = \int_{t_0}^{t_1} \left(\dot{q}_1 q_2 + \dot{q}_2 q_3 + q_1 \dot{q}_3 - \dot{q}_1{}^2\right) dt$$

2. Find the extremal to
$$F\{y\} = \int_0^1 \left(y''^2 - 360x^2y\right) dx$$
Subject to $y(0) = 0, y'(0) = 1, y(1) = 1$, and $y'(1) = 5/2$
$f = y''^2 - 360x^2y$
$$\frac{\partial f}{\partial y''} = 2y'', \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial f}{\partial y} = -360x^2$$

EL gives
$$\begin{aligned} 0 &= \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) \\ &= -360x^2 + 2y^{(4)} \\ y^{(4)} &= 180x^2 \\ y &= a + bx + cx^2 + dx^3 + \frac{x^6}{2} \end{aligned}$$

BCs
$$\begin{aligned} y(0) = 0 &\implies a = 0 \\ y'(0) = 1 &\implies b = 1 \\ y(1) = 1 &\implies 1 + c + d + \frac{1}{2} = 1 \\ y'(1) = 5/2 &\implies 1 + 2c + 3d + 5/2 = 5/2 \end{aligned}$$

$$\begin{aligned} c + d &= -\frac{1}{2} \\ c + \frac{3}{2}d &= -\frac{1}{2} \end{aligned}$$

Hence $d = 0$ and $c = -\frac{1}{2}$ Hence
$$\boxed{y(x) = x - \frac{1}{2}x^2 + \frac{x^6}{2}}$$

3. (a) Consider the integral definition of the beta function
$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{4}$$
i. Write an integral expression for $B(z - \frac{1}{2}, z + \frac{1}{2})$
$$\begin{aligned} B(z - \frac{1}{2}, z + \frac{1}{2}) &= \int_0^1 t^{z-\frac{1}{2}-1}(1-t)^{z+\frac{1}{2}-1}dt \\ &= \int_0^1 t^{z-\frac{3}{2}}(1-t)^{z-\frac{1}{2}}dt \end{aligned}$$

ii. Substitute $2t = 1 + s$ into it (take care on the limits)
$t = 0$ gives $s = -1$ and $t = 1$ gives $s = 1$, and $dt = 1/2ds$

$$\begin{aligned}
B(z - \frac{1}{2}, z + \frac{1}{2}) &= \int_0^1 t^{z-\frac{3}{2}}(1-t)^{z-\frac{1}{2}}dt \\
&= \frac{1}{2}\int_{-1}^{1} \left(\frac{1+s}{2}\right)^{z-\frac{3}{2}} \left(1 - \left(\frac{1+s}{2}\right)\right)^{z-\frac{1}{2}} ds \\
&= \frac{1}{2}\int_{-1}^{1} \left(\frac{1+s}{2}\right)^{z-\frac{3}{2}} \left(\frac{1-s}{2}\right)^{z-\frac{1}{2}} ds \\
&= 2^{-1}\int_{-1}^{1} (1+s)^{z-\frac{3}{2}}\, 2^{\frac{3}{2}-z}\, (1-s)^{z-\frac{1}{2}}\, 2^{\frac{1}{2}-z} ds \\
&= 2^{1-2z}\int_{-1}^{1} (1+s)^{z-\frac{3}{2}}\, (1-s)^{z-\frac{1}{2}}\, ds \\
&= 2^{1-2z}\int_{-1}^{1} \left(1-s^2\right)^{z} (1+s)^{-\frac{3}{2}}\, (1-s)^{-\frac{1}{2}}\, ds
\end{aligned}$$

iii. Decompose into even and odd parts in s
To decompose a function $f(x)$ into even and odd parts:

$$f(x) = f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

So

$$f(s) = (1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}$$

$$\begin{aligned}
f_e(s) &= \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} \\
f_o(s) &= \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} - \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2}
\end{aligned}$$

So we have

$$\begin{aligned}
&= 2^{1-2z}\int_{-1}^{1} \left(1-s^2\right)^{z} (1+s)^{-\frac{3}{2}}\, (1-s)^{-\frac{1}{2}}\, ds \\
&= 2^{1-2z}\left(\int_{-1}^{1} f_e(s)ds + \int_{-1}^{1} f_o(s)ds\right) \\
&= 2^{1-2z}\left(\int_{-1}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds\right. \\
&\left. + \int_{-1}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} - \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds\right)
\end{aligned}$$

iv. Use parity to half the integration ranges in each integral. *Hint* bisect the interval and sub $\sigma = -s$ into the negative interval
Is this just to show that integrals over symmetric endpoints are
- Double one side for even;
- equal to zero for odd?

Even interval, F_e

$$F_e(z) = \int_{-1}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds$$

$$= \int_{0}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds$$
$$- \int_{1}^{0} \frac{(1-\sigma^2)^z(1-\sigma)^{-3/2}(1+\sigma)^{-1/2}}{2} + \frac{(1-\sigma^2)^z(1+\sigma)^{-3/2}(1-\sigma)^{-1/2}}{2} d\sigma$$

$$= \int_{0}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds$$
$$+ \int_{0}^{1} \frac{(1-\sigma^2)^z(1-\sigma)^{-3/2}(1+\sigma)^{-1/2}}{2} + \frac{(1-\sigma^2)^z(1+\sigma)^{-3/2}(1-\sigma)^{-1/2}}{2} d\sigma$$

$$= \int_{0}^{1} (1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2} + (1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2} ds$$

$$F_o(z) = \int_{-1}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} - \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds$$

$$= \int_{0}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds$$
$$+ \int_{1}^{0} \frac{(1-\sigma^2)^z(1-\sigma)^{-3/2}(1+\sigma)^{-1/2}}{2} + \frac{(1-\sigma^2)^z(1+\sigma)^{-3/2}(1-\sigma)^{-1/2}}{2} d\sigma$$

$$= \int_{0}^{1} \frac{(1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2}}{2} + \frac{(1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}}{2} ds$$
$$- \int_{0}^{1} \frac{(1-\sigma^2)^z(1-\sigma)^{-3/2}(1+\sigma)^{-1/2}}{2} + \frac{(1-\sigma^2)^z(1+\sigma)^{-3/2}(1-\sigma)^{-1/2}}{2} d\sigma$$

$$= 0$$

Hence the full integral is

$$2^{1-2z}(F_e+F_o) = 2^{1-2z} \int_{0}^{1} (1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2} + (1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2} ds$$

v. Use $\tau = s^2$ to get the form from eqn 4

$\tau = s^2$ gives $ds = \frac{1}{2\sqrt{\tau}}d\tau = 2^{-1}\tau^{-1/2}d\tau$ and bounds stay the same

$$\begin{aligned}
&= 2^{1-2z}\int_0^1 (1-s^2)^z(1+s)^{-3/2}(1-s)^{-1/2} + (1-s^2)^z(1-s)^{-3/2}(1+s)^{-1/2}ds \\
&= 2^{1-2z}\int_0^1 ((1-s)+(1+s))(1-s^2)^z(1-s)^{-3/2}(1+s)^{-3/2}ds *** \\
&= 2^{1-2z}\int_0^1 2(1-s^2)^z(1-s)^{-3/2}(1+s)^{-3/2}ds \\
&= 2^{1-2z}\int_0^1 2(1-s^2)^z((1-s)(1+s))^{-3/2}ds \\
&= 2^{1-2z}\int_0^1 2(1-s^2)^z(1-s^2)^{-3/2}ds \\
&= 2^{1-2z}\int_0^1 (1-\tau)^{z-3/2}\tau^{-1/2}d\tau \\
&= 2^{1-2z}\int_0^1 \tau^{-1/2}(1-\tau)^{z-3/2}d\tau \\
&= 2^{1-2z}B(1/2, z-1/2)
\end{aligned}$$

$***$ putting the two parts even, I have multiplied the left part by $(1-s)^{2/2}(1-s)^{-2/2}$ and the right part by $(1+s)^{2/2}(1+s)^{-2/2}$, i.e. multiplying by 1.

vi. Using all the previous parts, and the relationship between the beta and gamma functions, derive

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2)$$

We have the beta/gamma relationship:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\begin{aligned}
B(z-\frac{1}{2}, z+\frac{1}{2}) &= 2^{1-2z}B(\frac{1}{2}, z-\frac{1}{2}) \\
\frac{\Gamma(z-\frac{1}{2})\Gamma(z+\frac{1}{2})}{\Gamma(z-\frac{1}{2}+z+\frac{1}{2})} &= 2^{1-2z}\frac{\Gamma(\frac{1}{2})\Gamma(z-\frac{1}{2})}{\Gamma(z)} \\
\frac{\Gamma(z+\frac{1}{2})}{\Gamma(2z)} &= 2^{1-2z}\frac{\pi^{-1/2}}{\Gamma(z)} \\
\implies \Gamma(2z) &= 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2)
\end{aligned}$$

(b) From the integral definitions given in class, show

$$K(k) = \frac{\pi}{2}F(1/2, 1/2; 1; k^2)$$

Note that, using Euler's integral formula

$$\begin{aligned}
F(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \\
F(1/2,1/2;1;k^2) &= \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-tk^2)^{-1/2}dt \\
&= \frac{1}{\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-tk^2)^{-1/2}dt
\end{aligned}$$

Sub $t = \sin^2 \vartheta$, $dt = 2\sin\cos\vartheta d\theta$, and $t = 1$ gives $\vartheta = \pi/2$

$$
\begin{aligned}
F(1/2, 1/2; 1; k^2) &= \frac{1}{\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-tk^2)^{-1/2}dt \\
&= \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{\sin\vartheta}\frac{1}{\sqrt{1-\sin^2\vartheta}}\frac{1}{\sqrt{1-k^2\sin^2\vartheta}}2\sin\cos\vartheta d\vartheta \\
&= \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{\cos\vartheta}\frac{1}{\sqrt{1-k^2\sin^2\vartheta}}\cos\vartheta d\vartheta \\
&= \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\vartheta}}d\vartheta
\end{aligned}
$$

Where $F(\varphi, k)$ denotes the elliptic integral of the first kind, we get:

$$
\begin{aligned}
K(k) &= F(\pi/2, k) \\
&= \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \\
&= \frac{\pi}{2}\frac{2}{\pi}\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\vartheta}}d\vartheta \\
&= \frac{\pi}{2}F\left(1/2, 1/2; 1; k^2\right)
\end{aligned}
$$

As required