2 Perturbation Methods

Much of the previous chapter we focussed on developing locally valid solutions to differential equations. In reality we are interested not just in how such problems behave near a point, but in their global behaviour. In the next chapter we will look at boundary-layer problems, for instance, where the solution behaviour in a thin region is different to the bulk of the domain. This chapter focusses on building techniques to consider such problems.

In the most general terms a perturbation problem is simply a problem that features a small parameter, often denoted ϵ. Such small parameters can be introduced to a modelling problem in a variety of ways. For instance,

- **initial conditions** a usually stable physical situation can be given a small 'kick', and we can look at its response.

  Example: Kelvin-Helmholtz instability

- **boundary conditions** similarly, a perturbation can be introduced at a boundary (inhomogeneity/boundary shape)

  Example: flow in a corrugated pipe

- **variable scaling** a modelling domain might, for instance, have different characteristic length scales in different directions (eg. long and thin) and therefore rescaling is appropriate.

  Example: fibre drawing

- **parameter scaling** assumptions around dominant physical processes simplify analysis; (eg. Reynolds number in fluids, Peclet number in heat tranfer) is large or small (physical scaling)

This is not an exhaustive list. It is also sometimes useful to arbitrarily introduce a small parameter into a problem to aid analysis.

In many examples in the previous chapter we were essentially already using perturbation methods! Let's revisit some definitions: regular and singular problems.

- **Regular perturbation problem:** The structure of the problem is unchanged in the limit $\epsilon \to 0$.

  Some quick examples

- **Singular perturbation problem:** The solution behaves differently in the limit $\epsilon \to 0$. In a singularly perturbed DE, the highest order derivative might be multiplied by ϵ. Boundary-layer problems are a much studied class of these.

Examples, including a boundary-layer problem

For many of our problems we seek solutions in the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon y_2(x) + \ldots$$

This is typically an iterative procedure. We find the leading-order solution $y_0(x)$, then solve for $y_1(x)$ which typically depends on $y_0(x)$ and can be thought of as a 'correction' term. In practice for regular perturbation problem we are interested in leading-order behaviour, sometimes interested in a first-order corrections and rarely in terms beyond that.

2.1 Ordinary differential equations

We have already used an expansion method to solve algebraic equations. Now let's use similar methods to solve some applied differential equation problems.

2.1.1 Flight of a projectile

Consider the flight of a projectile under the influence of gravity. Suitably non-dimensionalised (for details see Bowen & Witelski, Chapter 4) the height of the projectile $x(t)$ above the earth at time t is governed by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{1}{(1+\epsilon x)^2}, \quad x(0) = 1, \quad x'(0) = \alpha. \tag{2.1}$$

where $\epsilon \to 0$. The small parameter ϵ has a physical meaning: it is the ratio of the length scale of interest to that radius of the earth, so this approximation is valid as long as the projectile does not go too high.

We assume that $x(t)$ has the form of a perturbation series

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$$

The solution procedure involves substituting this series into (2.1) and solving for $x_0(t)$, $x_1(t)$, ... in turn.

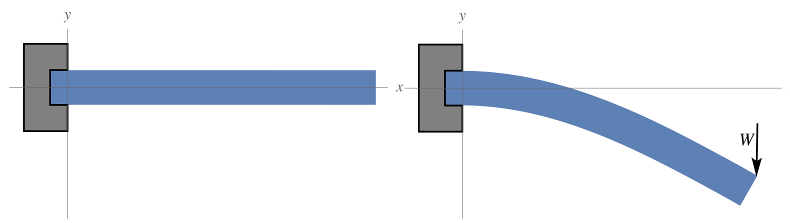
Derive perturbation series solution to the projectile problem.

A subtlety of this perturbation expansion solution (and all similar expansions) is that it is only valid when the asymptotic ordering of the terms is maintained.

Validity of the perturbation series approximation (discussion and MATLAB)

2.1.2 Shape of a diving board

Consider a diving board, as shown below.



If a weight is applied to the end of the board the deformation is described by

$$B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = W\cos\theta, \quad \theta(0) = \frac{\mathrm{d}\theta}{\mathrm{d}s}(L) = 0.$$

where θ is the angle of deflection along the board and s is an arc-length variable. This is a dimensional equation where B is the bending stiffness of the board and W is the weight applied at the end of the board. The centre-line of the board $(x(s), y(s))$ is calculated via

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta.$$

We non-dimensionalise by scaling s with the length of the board L, so that $s = L\hat{s}$. Making this substitution (and dropping the hats) gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \delta\cos\theta, \quad \theta(0) = \frac{\mathrm{d}\theta}{\mathrm{d}s}(1) = 0.$$

Here the small parameter δ appears in the equation.

where $\delta = WL^2/B$ is now the single dimensionless parameter that in this problem. When this parameter is small we can seek an asymptotic solution to this problem.

Asymptotic solution to the diving board problem

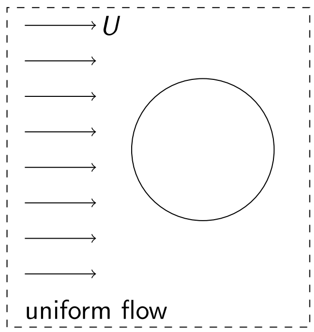
Comparison with numerical solution (MATLAB)

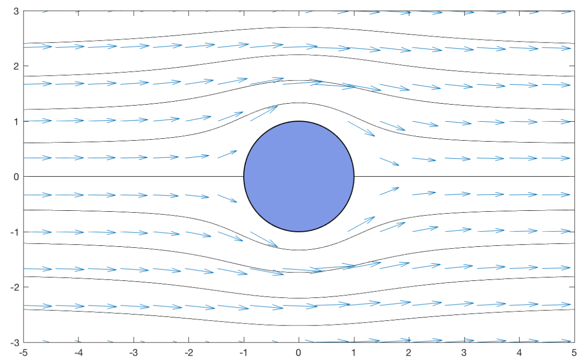
2.2 Partial differential equations

We now use perturbation methods to solve some partial differential equation models. The examples here are from fluid mechanics, a popular application area for asymptotic and perturbation methods.

2.2.1 Flow around an almost cylindrical obstacle

A very well-known problem from classical fluids mechanics is the inviscid flow around a cylinder. This problem is something of a rarity because it has an exact analytic solution!





First, a (very) brief review of some fluid mechanics ideas. We want to model the velocity (vector) field $\mathbf{u}$ for the flow of an inviscid, incompressible fluid. To do this we define a velocity potential ϕ such that $\mathbf{u} = \nabla\phi$. This velocity potential satisfies Laplace's equation:

$$\nabla^2\phi = 0.$$

A simple example is that uniform horizontal flow at speed U is given by $\phi = Ux$. In cartesian coordinates the velocity components in the x and y directions are just

$$u_x = \frac{\partial\phi}{\partial x}, \qquad u_y = \frac{\partial\phi}{\partial y}.$$

Recall that $x = r\cos\theta$ and $y = r\sin\theta$

In cylindrical polars (which we'll use for the next example) the radial and azimuthal velocity components are

$$u_r = \frac{\partial\phi}{\partial r}, \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}.$$

A related quantity is the streamfunction ψ, the Laplacian of which is vorticity ω (a measure of fluid rotation),

$$\nabla^2\psi = -\omega,$$

and from which the velocity components are be calculated via:

$$\begin{array}{lll} \text{(cartesian)} & u_x = \dfrac{\partial\psi}{\partial y}, & u_y = -\dfrac{\partial\psi}{\partial x}, \\ \text{(cylindrical)} & u_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}, & u_\theta = -\dfrac{\partial\psi}{\partial r}. \end{array}$$

There's also a complex potential $F(z) = \phi + i\psi$.

The context (eg. geometry) of the problem determines choice of co-ordinate system, with velocity potential and streamfunction used interchangeably (more or less).

Flow around cylinder and exact solution: Working in cylindrical co-ordinates, the uniform flow of speed U around a cylinder of radius a is described by

$$\nabla^2\phi = 0, \quad \text{in } r > a,$$

with boundary conditions

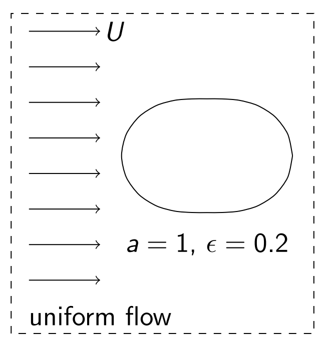
$$\begin{aligned} \mathbf{n}\cdot\nabla\phi &= 0, \quad \text{at } r = a, \\ \phi &\sim Ur\cos\theta, \quad \text{as } r \to \infty. \end{aligned}$$

Written in terms of streamfunction the first condition would be $\psi = 0$ at $r = a$.

The first says there is the component of velocity normal (perpendicular) to the cylinder surface is zero; that is the fluid goes around, rather than passing through the cylinder. The second is that far from the cylinder there is uniform flow with speed U. The exact solution is

$$\phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta.$$

Flow around an approximate ellipse: We now introduce a perturbation by perturb changing the shape of the obstacle. Instead of being circular, let's try an approximate ellipse given by $r = a(1+\epsilon\cos 2\theta)$.



The problem is now

$$\nabla^2\phi = 0, \quad \text{in } r > a(1+\epsilon\cos 2\theta),$$

with boundary conditions

$$\begin{aligned} \mathbf{n}\cdot\nabla\phi &= 0, \quad \text{at } r = a(1+\epsilon\cos 2\theta), \\ \phi &\sim Ur\cos\theta, \quad \text{as } r \to \infty. \end{aligned}$$

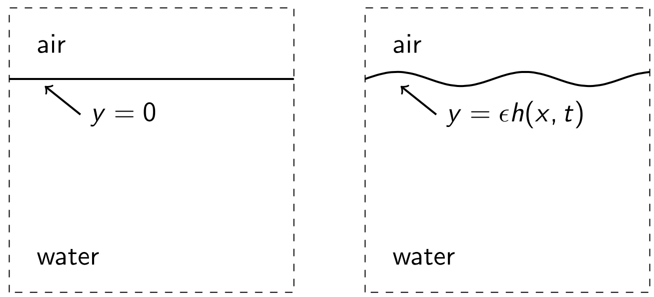
This is different in character than the problem we previously considered because the small parameter ϵ appears in the boundary condition. Our approach relies on perturbing about the circular solution, which in this case involves replacing the problem in $r > a(1+\epsilon\cos 2\theta)$ with an equivalent problem in $r > a$.

Solution of the flow around the approximate ellipse.

MATLAB: plots of this solution

2.2.2 Deep water waves

Another classic fluid mechanics problem is to look at the behaviour of small amplitude waves on the surface of water. This is a prototypical example of a 'free-surface problem', where the shape of the boundary is unknown and must be determined as part of the solution.



unperturbed perturbed

Assume surface of the waters is $y = \epsilon h(x, t)$, with $\epsilon \ll 1$ and $h = \mathcal{O}(1)$. The velocity is described by Laplaces equation in the water,

$$\nabla^2 \phi = 0, \qquad \text{for } y < \epsilon h(x, t).$$

The boundary conditions on the free-surface are a kinematic condition,

$$\frac{\partial \phi}{\partial y} = \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x, t),$$

which state that the normal velocity of the water is equation to the normal velocity of the surface. There is also a Bernoulli condition namely,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + gy = 0, \quad \text{on } y = \epsilon h(x, t),$$

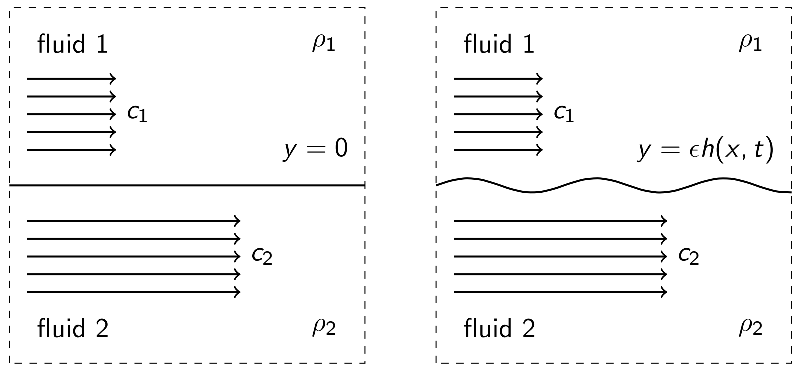
which is essentially Newton's Second Law. This simply stated problem is an incredibly rich field of study.

Perturbation series solution of deep water waves

2.2.3 Kelvin-Helmholtz instability

One important use of perturbation methods is to determine stability of a system. An example is analysis of critical points in dynamical systems (unstable versus stable nodes, and so on). Analogous considerations of stability are ubiquitous in many application areas.

Here we consider an example from fluid mechanics, to determine the stability of the interface between two fluid layers moving in parallel.



unperturbed perturbed

The two fluid layers are as shown above. The upper layer is of density ρ_1 and the lower is of density ρ_2. In the unperturbed state, the upper layer moves with speed c_1 and the lower moves with speed c_2. As with the deep water waves, assume the interface between the two fluid layers is small in amplitude by writing $y = \epsilon h(x, t)$, where $\epsilon \ll 1$ and $h = \mathcal{O}(1)$.

The flow of the two layers is described by

$$\begin{aligned} \nabla^2 \phi_1 &= 0, \qquad \text{for } y > \epsilon h(x, t). \\ \nabla^2 \phi_2 &= 0, \qquad \text{for } y < \epsilon h(x, t), \end{aligned}$$

where it is assumed that far from the interface the layers are at their unperturbed speeds, namely

$$\begin{aligned} \phi_1 &= c_1 x, \quad y \to \infty, \\ \phi_2 &= c_2 x, \quad y \to -\infty. \end{aligned}$$

The kinematic conditions in each fluid are

$$\begin{aligned} \frac{\partial \phi_1}{\partial y} &= \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi_1}{\partial x} \frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x, t), \\ \frac{\partial \phi_2}{\partial y} &= \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi_2}{\partial x} \frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x, t). \end{aligned}$$

Finally, the Bernoulli condition is

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} - \frac{1}{2} c_1^2 + \frac{1}{2} |\nabla \phi_1|^2 + g y \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} - \frac{1}{2} c_2^2 + \frac{1}{2} |\nabla \phi_2|^2 + g y \right),$$

on $y = \epsilon h(x, t)$.

Perturbation series solution (derivation of stability criterion)

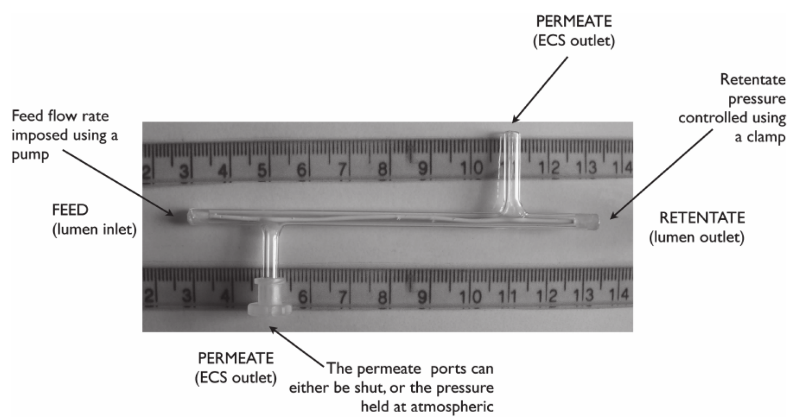
2.3 Case study: hollow fibre bioreactor

An important application area in mathematical biology is tissue engineering, the sub-field of bio-engineering which deals with growing artificial tissues. This typically involves seeding an engineered scaffold with a few cells and then culturing it in a bioreactor; after a culture period the goal is for the scaffold to have developed into a tissue (which might then be implanted).

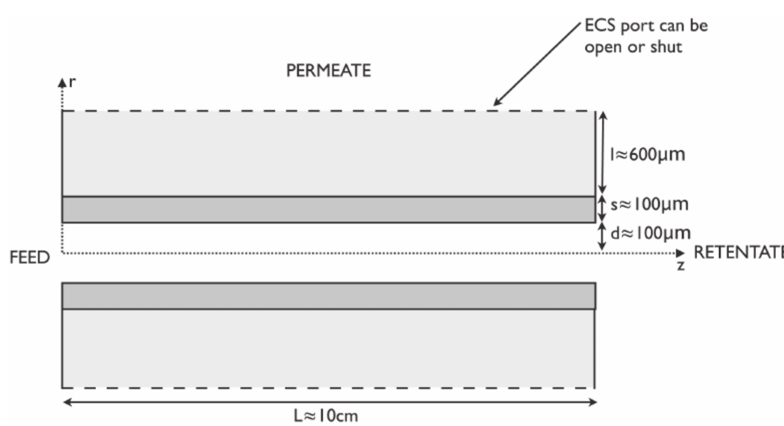
One common type of bioreactor is described in:

Shipley, R.J., Waters, S.L., *Fluid and mass transport modelling to drive the design of cell-packed hollow fibre bioreactors for tissue engineering applications*, **29**:329–359, Math. Med. Biol., 2012.

This treatment is of interest since it takes advantage of the slender geometry of this type of bioreactor (see below), which permits the introduction of a small parameter.



A hollow fibre bioreactor consists of a hollow outer tube and a coaxial lumen (permeable tube). Cells are seeded around the outside of the lumen (ECS, extra-capillary space). Fluid containing nutrients is pumped along the lumen; the idea being that nutrients pass through the permeable tube wall into the space containing to the cells.



Summary:

The approach taken in Shipley and Waters is:

- Develop a model for fluid flow in the lumen and couple this to a model for nutrient transport to the ECS (sections 3 and 4).
- Assumption of slenderness; aspect ratio of the lumen is small $\epsilon = d/L \approx 2 \times 10^{-3} \ll 1$, see Eq. (1).
- Derivation of reduced, leading-order models (sections 3.1 and 4.1).
- Application of model to investigate effect of opening ECS port on nutrient concentration and so effect on different cell types (section 5).

Use of asymptotics/perturbation methods:

The key step in the model reduction involves using lubrication theory, where in addition to the assumption of slender geometry it is further assumed that the radial flow is small compared to the axial flow. An interesting subtlety is in the choice of pressure scaling, which is different in the lumen (large), the membrane and the ECS (both small), see Eqs. (15) and (16).

An advantage to this approach is that the reduced model permits analytic solutions to the leading-order equations.

Repeat derivation of reduced model via lubrication theory

Case study project (5% of total mark)

This case study is a guide to the form of the short written project.

You should find a research paper that makes use of asymptotic and/or perturbation methods (ideally in an application area that you're interested in) and write a brief report.

A good way to structure this is as above, namely:

- **Background:** Provide some background to the application;
- **Summary:** Briefly summarise the mathematical approach used;
- **Use of asymptotics/perturbation methods:** Give some detail on the use of asymptotics/perturbation methods ...
- **Redo some working:** ... and repeat a few steps of working from the paper (eg. filling in the gaps between steps).

This should be a **short** report (2–3 pages).