THE UNIVERSITY of ADELAIDE

Examination in the School of Mathematical Sciences

Semester 2, 2015

104831	**MATHS 2100**	**Real Analysis II**
104830	**MATHS 7100**	**Real Analysis**

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 60

Instructions

- Write your name in the tear-out section of your first answer booklet.
  Do NOT write your name anywhere else – only your student ID.
- Attempt all questions. Each is worth 10 marks.
- Begin each question on a new page.
- Examination materials may not be removed from the hall.

Materials

- One Blue Book is provided. You may request more if needed.
- Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Suppose $S \subseteq \mathbb{R}$.

   (a) What is a *limit point* x_0 of S?

   (b) Is $\sqrt{5}$ a limit point of $\mathbb{Q}$? Explain briefly.

   (c) Suppose that $F : S \to \mathbb{R}$ is a function and x_0 is a limit point of S. Give a precise definition of the expression $\lim\limits_{x \to x_0} F(x) = L$.

   (d) With F and x_0 as in (c), suppose that $\lim\limits_{n \to \infty} F(x_n) = L$ for every sequence $(x_n)_{n=1}^{\infty}$ in $S \backslash \{x_0\}$ converging to x_0. Use proof by contradiction to prove that $\lim\limits_{x \to x_0} F(x) = L$.

   [2+2+2+4 = 10 marks.]

2. Let $\mathcal{P}$ be a partition of the interval $[a, b]$ into N subintervals: $a = x_0 < x_1 < \cdots < x_N = b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function.

   (a) Define the lower and upper sums $L(f, \mathcal{P})$ and $U(f, \mathcal{P})$.

   (b) Carefully define what it means for f to be *integrable*. (You may quote any facts about upper and lower sums that you need.)

   (c) Let $C \in \mathbb{R}$ be a constant and let $f(x) = C$ for every $x \in [a, b]$. Apply your definition in (b) to show that f is integrable over $[a, b]$, and use your calculation to determine the value of $\int_a^b f(x)\, dx$.

   [2+3+5 = 10 marks.]

3. (a) Let $g : [a, b] \to \mathbb{R}$ be a bounded integrable function with $|g(t)| \leq M$ for every $t \in [a, b]$. If $c, d \in [a, b]$, show that $\left| \int_c^d g(t)\, dt \right| \leq M\,|c - d|$. [Note: c is not necessarily less than d.]

   (b) For $x \in [a, b]$, let $F(x) := \int_a^x f(t)\, dt$. Briefly explaining your calculation(s), show that $F(x) - F(x_0) - f(x_0)(x - x_0) = \int_{x_0}^x \big(f(t) - f(x_0)\big)\, dt$.

   (c) Let $I_x \subseteq [a, b]$ be the closed interval with endpoints x, x_0. Apply (a) to the integral on the right of (b) to show that $\big|F(x) - F(x_0) - f(x_0)(x - x_0)\big| \leq M_x\,|x - x_0|$, where $M_x := \sup_{t \in I_x} |f(t) - f(x_0)|$.

   (d) Hence show that if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

   [3+2+1+4 = 10 marks.]

Please turn over for Questions 4 to 6.

4. Let $I \subseteq \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be $n+1$ times differentiable.

   (a) If $x_0 \in I$, give the formula for the n-th Taylor polynomial p_n for f at x_0.

   (b) State the Lagrange Remainder theorem for f and p_n as presented in lectures.

   (c) Suppose that $f(x) = 1/x$ and $x_0 = 1$. Find $p_3(x)$ explicitly, and write down the explicit form of the remainder $f(x) - p_3(x)$ in this case. [Don't expand out $p_3(x)$.]

   (d) Use your calculations to approximate $\ln 2 = \int_1^2 (1/x)\, dx$, and also estimate the error between your approximation and the true value. Leave your answers as fractions.

[2+2+3+3 = 10 marks.]

5. Let $\sum\limits_{n=1}^{\infty} a_n$ be a series of real numbers.

   (a) What does it mean for the series to *converge*?

   (b) What does it mean for the series to converge *absolutely*?

   (c) Give an example (without proof) of a series that converges, but not absolutely.

   (d) Prove that if the series converges absolutely, then it converges.

[2+2+2+4 = 10 marks.]

6. Suppose that $\sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, and for $x \in (-R, R)$, let $f(x) := \sum\limits_{n=0}^{\infty} a_n\, x^n$. Let r be any number with $0 \leq r < R$.

   (a) Choose s with $r < s < R$. Briefly explain why $\lim\limits_{n\to\infty} a_n\, s^n \to 0$ as $n \to \infty$.

   (b) Briefly explain why $\sum\limits_{n=0}^{\infty} |a_n|\, r^n$ must therefore converge.

   (c) Hence show that the sequence of functions $(p_N)_{N=1}^{\infty}$ given by $p_N(x) = \sum\limits_{n=0}^{N} a_n\, x^n$ converges uniformly to f on $[-r, r]$.

[3+3+4 = 10 marks.]

End of examination