Random Processes III 2018: Tutorial 3,

please come to the tutorial on Friday 31^{st} September having attempted this sheet. Solutions to these questions will not be uploaded to MyUni.

Problem 1

We didn't attempt to calculate $P_{Nj}(t)$ for the linear pure-death process (Example 6) in lectures because it is tedious. This question takes you through an alternative (and much more efficient) approach. It considers the system of N individuals/components subject to death/breakdown as N *independent* individuals/components.

(a) Write down the transition matrix Q for the model in which there is only one individual/component.

(b) Write down and solve the Kolmogorov forward differential equations for $P_{10}(t)$ and $P_{11}(t)$ in this model.

(c) State physically the meaning of $P_{10}(t)$ and $P_{11}(t)$.

(d) Now consider a system of N such individuals/components. Calling a component a "success" if it is still working at time t and a "failure" otherwise, derive the appropriate expression for $P_{Nj}(t)$ using a binomial model.

Problem 2

An automatic switch in a control system can either be on, off, or stand-by. When the switch is on, it changes to off after an exponentially distributed time with expected value 5 seconds. If the switch is off, it either turns on with intensity 0.2 per second, or changes to stand-by with intensity 0.4 per second. Finally, when stand-by the switch turns to on after an exponential time with expected value 2 seconds.

(a) Write the state space $\mathcal{S}$, and the generator Q.

(b) In equilibrium, what fraction of time is the switch in each of the different states?

Problem 3

The SIS epidemic model in a population of constant size N is a CTMC $(X(t), t \geq 0)$ such that $X(t)$ is the number of infectious individuals at time t. Assume new infections happen at rate $q_{i,i+1} = \beta i(N-i)/(N-1)$, and recoveries at rate $q_{i,i-1} = \gamma i$ for $i \in S = \{0, 1, \ldots, N\}$.

(a) Characterise the states of the CTMC, into communicating classes etc., and hence the CTMC itself. What does this mean in terms of limiting distribution(s) and the relationship to equilibrium distribution(s)?

(b) Write down a system of linear equations which allows us to calculate the probability of the epidemic having at least j $(0 < j \leq N)$ infectious individuals at some stage over the course of the epidemic, having started with $i \leq j$ infectious individuals. Evaluate this probability for $i = 1$ and $j = 1, 2, 3$.