LECTURE 30

Absolute Convergence

Given a series $\sum_{n=1}^{\infty} a_n$ we may form a new series $\sum_{n=1}^{\infty} |a_n|$ by replacing each term a_n of the original series with its absolute value $|a_n|$. Note that the new series $\sum_{n=1}^{\infty} |a_n|$ is a series of non-negative terms.

Definition 7.6: A series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 7.7: If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely then it converges.

Proof: Let $\epsilon > 0$. Let $S = \sum_{n=1}^{\infty} |a_n|$. Choose a natural number N such that $M \geq N$ implies $|S - \sum_{n=1}^{M} |a_n|| < \epsilon$. Observe that $S - \sum_{n=1}^{M} |a_n| \geq 0$ since $\sum_{n=1}^{\infty} |a_n|$ is a series of non-negative terms and hence its sequence of partial sums is increasing:

$$|a_1| \leq |a_1| + |a_2| \leq |a_1| + |a_2| + |a_3| \leq \cdots \leq S.$$

Hence

$$M \geq N \implies S - \sum_{n=1}^{M} |a_n| < \epsilon$$

We will prove that the sequence $(\sum_{n=1}^{M} a_n)_{M=1}^{\infty}$ of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is a Cauchy sequence — since every Cauchy sequence is convergent this will imply that the series $\sum_{n=1}^{\infty} a_n$ converges.

Suppose that $M_1, M_2 \geq N$. We will prove that

$$\left| \sum_{n=1}^{M_1} a_n - \sum_{n=1}^{M_2} a_n \right| < \epsilon.$$

Without loss of generality we may suppose that $M_1 > M_2$. Therefore

$$\left| \sum_{n=1}^{M_1} a_n - \sum_{n=1}^{M_2} a_n \right| = \left| \sum_{n=M_2+1}^{M_1} a_n \right| \leq \sum_{n=M_2+1}^{M_1} |a_n|$$

using the triangle inequality. We have

$$\begin{aligned} \sum_{n=M_2+1}^{M_1} |a_n| &= \sum_{n=1}^{M_1} |a_n| - \sum_{n=1}^{M_2} |a_n| \\ &= \left(S - \sum_{n=1}^{M_2} |a_n| \right) + \left(\sum_{n=1}^{M_1} |a_n| - S \right) \\ &< \epsilon \end{aligned}$$

since $S - \sum_{n=1}^{M_2} |a_n| < \epsilon$ (because $M_2 \geq N$) and since the partial sums of $\sum_{n=1}^{\infty} |a_n|$ are bounded above by S, so that $\sum_{n=1}^{M_1} |a_n| - S \leq 0$. Since $\epsilon > 0$ was arbitrary it follows that $(\sum_{n=1}^{M} a_n)_{M=1}^{\infty}$ is a Cauchy sequence. ■

Remark: We can summarize Theorem 7.7 in the slogan:

"Absolute convergence $\implies$ convergence"

Note that the converse

$$\text{“Convergence} \implies \text{absolute convergence”}$$

is **not** true in general. For example, the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges, but it does not converge absolutely (a series which converges, but does not converge absolutely is said to converge *conditionally*). It is clear that this series does not converge absolutely, since the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. It is not so clear that the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges.

Alternating Series

The series $\sum_{n=1}^{\infty}(-1)^{n+1}1/n$ above is an example of what is called an *alternating series* — this is a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where $a_n \geq 0$. Thus the signs of the terms of an alternating series alternate in a $+-+-+-\cdots$ pattern.

There is the following very useful test for convergence of alternating series.

Theorem 7.8 (Alternating Series Test): Suppose that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ (i.e (a_n) is a decreasing sequence of non-negative terms). Then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

Proof: ($\Rightarrow$) Suppose $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Then $\lim_{n\to\infty}(-1)^{n+1}a_n = 0$. Therefore $\lim_{n\to\infty}|(-1)^{n+1}a_n| = \lim_{n\to\infty} a_n = 0$ (recall that $a_n \geq 0$ for all n).

($\Leftarrow$) Suppose that $\lim_{n\to\infty} a_n = 0$. We will first prove that the sequence of even partial sums (s_{2N}) of the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. We have

$$s_{2N} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2N-1} - a_{2N}).$$

Since (a_n) is a decreasing sequence it follows that $s_{2N} \geq 0$ for all N (since $a_i - a_{i+1} \geq 0$ for all i). Therefore (s_{2N}) is an increasing sequence since $s_{2N+2} = s_{2N} + (a_{2N+1} - a_{2N+2}) \geq s_{2N}$. On the other hand, re-grouping terms in a different way, we have

$$s_{2N} = a_1 + (-a_2 + a_3) + (-a_4 + a_5) + \cdots + (-a_{2N-2} + a_{2N-1}) - a_{2N} \leq a_1.$$

Therefore (s_{2N}) is an increasing sequence which is bounded above by a_1. Hence $s_{2N} \to L$ for some real number L.

Now observe that

$$s_{2N+1} = s_{2N} + a_{2N+1} \to L + 0 = L$$

since $a_{2N+1} \to 0$ (the sequence (a_{2N+1}) is a subsequence of (a_N) and hence converges to 0 by hypothesis). Therefore

$$s_{2N} \to L \quad \text{and} \quad s_{2N+1} \to L.$$

It follows that $s_N \to L$ (see Question 6 of Tutorial 3). Hence $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. ■

Example: The convergence of $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is now an easy corollary of the Alternating Series Test, since this series is of the from $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where $a_n = 1/n$. Since $(1/n)$ is a decreasing sequence of non-negative numbers such that $1/n \to 0$, it follows that $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges. In fact,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln(2).$$

Power Series

A *power series* is an expression of the form $\sum_{n=0}^{\infty} a_n x^n$, where (a_n) is a sequence of real numbers. As with series, it is sometimes convenient to consider power series of the form $\sum_{n=k}^{\infty} a_n x^n$, where k is a natural number.

Our first task is try and understand the set

$$S = \{\, x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \,\}$$

of real numbers x for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges. Observe that if $x = 0$ then this series converges, hence $0 \in S$, in particular S is non-empty.

Proposition 7.9: Suppose $\sum_{n=0}^{\infty} a_n x_0^n$ converges for some $x_0 \in \mathbb{R}$. If $|x| < |x_0|$ then the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

Proof: Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges, $\lim_{n\to\infty} a_n x_0^n = 0$. In particular there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n x_0^n| < 1$. If $x_0 = 0$ then there is nothing to prove, so suppose that $|x_0| > 0$. Suppose that $|x| < |x_0|$. If $n \geq N$ then (since $|x_0| > 0$)

$$|a_n x^n| = |a_n x_0^n| \left(\frac{|x|}{|x_0|} \right)^n < r^n,$$

since $|a_n x_0^n| < 1$ and where we have defined $r := |x|/|x_0|$. Note that $0 \leq r < 1$ and hence the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges by comparison with the geometric series $\sum_{n=0}^{\infty} r^n$. Hence the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. ∎