Lecture 27: Laplace-Stieltjes Transforms, Convolution Theorem and the Renewal Function

Concepts checklist

At the end of this lecture, you should be able to:

- *Define and evaluate* Laplace-Stieltjes transforms;
- *State* the Convolution Theorem and *apply* it appropriately;
- *Define* the renewal function (or mean-value function).

Laplace-Stieltjes Transform

In renewal theory, we make extensive use of Laplace-Stieltjes transforms.

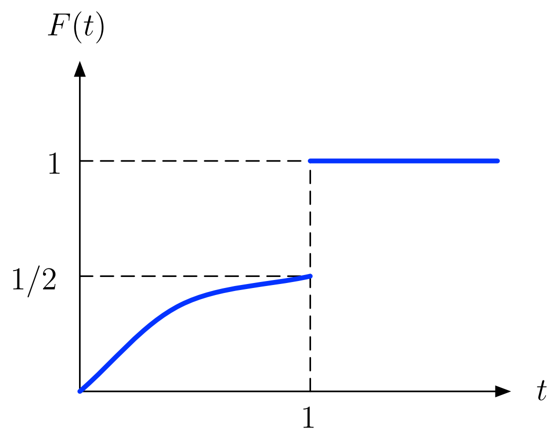
Definition 27. *Let X be a non-negative random variable with distribution function $F(x)$. Then, the Laplace-Stieltjes transform of X is given by*

$$\widehat{F}(s) = \int_0^\infty e^{-sx} \mathrm{d}F(x).$$

Note: $\widehat{F}(s) = \mathbb{E}(e^{-sX})$.

Example 25.

$$F(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \frac{1}{2}(1 - e^{-t}) & \text{if} \quad 0 \leq t < 1 \\ 1 & \text{if} \quad t \geq 1 \end{cases}$$



Then,

$$\begin{aligned}
\widehat{F}(s) &= \int_0^\infty e^{-st}\mathrm{d}F(t) \\
&= \int_0^1 e^{-st}\left(\frac{1}{2}e^{-t}\mathrm{d}t\right) + \left[1 - \frac{1}{2}(1-e^{-1})\right]e^{-s} + \int_1^\infty e^{-st}(0\mathrm{d}t) \\
&= -\frac{1}{2}\frac{1}{s+1}e^{-(s+1)t}\Big|_0^1 + \left[\frac{1}{2}+\frac{1}{2}e^{-1}\right]e^{-s} \\
&= \frac{1}{s+1}\left(\frac{1}{2}-\frac{1}{2}e^{-(s+1)}\right) + \left[\frac{1}{2}e^{-s}+\frac{1}{2}e^{-(s+1)}\right].
\end{aligned}$$

Convolution Theorem

The most important result on Laplace-Stieltjes transforms for renewal theory is the Convolution Theorem.

Definition 28. *If X and Y are independent random variables then the random variable $Z = X+Y$ is known as the convolution of X and Y. By conditioning on the value of Y, we can see that Z has the distribution function*

$$F_Z(z) = \int_0^z F_X(z-y)\mathrm{d}F_Y(y),$$

where $F_W(\cdot)$ is the distribution function for the random variable W, for $W = X, Y, Z$.

Theorem 25 (Convolution Theorem)**.** *For independent random variables X and Y, the random variable $Z = X+Y$ has the Laplace-Stieltjes transform*

$$\widehat{F}_Z(s) = \widehat{F}_X(s)\widehat{F}_Y(s).$$

More generally, for independent random variables X_i where $i \in \{1, 2, \ldots, n\}$, the random variable $Z := \sum_{i=1}^{n} X_i$, has the Laplace-Stieltjes transform

$$\widehat{F}_Z(s) = \prod_{i=1}^{n}\widehat{F}_{X_i}(s).$$

Example 26. Distribution of waiting time

If we define S_n to be the time until the nth event in a stochastic process, $S_n = X_1+X_2+\cdots+X_n$, where all the X_i are i.i.d., then

$$F_n(t) = \Pr(S_n \leq t) \quad \text{is the waiting time distribution function.}$$

Then, using the Convolution Theorem we have

$$\widehat{F}_n(s) = \prod_{i=1}^{n}\widehat{F}_{X_i}(s) = \left(\widehat{F}(s)\right)^n,$$

where $\widehat{F}(s)$ is the Laplace-Stieltjes transform of the distribution function of each of the random variables X_i.

Remark: The main purpose of renewal theory is to derive information about the counting process and the waiting time process from the inter-event time distribution $F(t)$. The above result is an example of this for the waiting time process $F_n(t)$.

Letting $P_n(t) = \Pr(N(t) = n)$, and since the events $\{N(t) < n\}$ and $\{S_n > t\}$ are equivalent, we have

$$\begin{aligned} P_n(t) &= \Pr(N(t) \geq n) - \Pr(N(t) \geq n+1) \\ &= \Pr(S_n \leq t) - \Pr(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t). \end{aligned}$$

Definition 29. *The renewal function (or mean-value function) $M(t)$ is defined as*

$$M(t) = \mathbb{E}[N(t)] = \sum_{n=0}^{\infty} n P_n(t).$$

$\equiv$ *$M(t)$ is the expected number of events that have occurred by time t.*

Note: It can also be shown (not trivial, and omitted) that if $F(0) < 1$, then $M(t) < \infty$ for $t > 0$.

Letting $\widehat{M}(s) = \int_0^\infty e^{-st} \mathrm{d}M(t)$, we have

$$\widehat{M}(s) = \sum_{n=1}^{\infty} \left(\widehat{F}(s)\right)^n,$$

and since $\hat{F}(s) < 1$ because

$$\widehat{F}(s) = \int_0^\infty e^{-st} \mathrm{d}F(t) < \int_0^\infty 1 \mathrm{d}F(t) = 1,$$

and $F(0) < 1$ means that there must be some contribution to both of the above integrals for a positive value of t, for which $e^{-st} < 1$ for $s > 0$, we have

$$\widehat{M}(s) = \frac{\widehat{F}(s)}{1 - \widehat{F}(s)}.$$

Example 27. Poisson process

$$\begin{aligned} F(t) = 1 - \mathrm{e}^{-\lambda t} \quad &\Rightarrow \widehat{F}(s) = \frac{\lambda}{\lambda + s} \\ &\Rightarrow \widehat{M}(s) = \frac{\frac{\lambda}{\lambda+s}}{1 - \frac{\lambda}{\lambda+s}} = \frac{\lambda}{s} \\ &\Rightarrow M(t) = \lambda t, \end{aligned}$$

which is exactly what we expect, as $N(t)$ is Poisson distributed with parameter λt.