LECTURE 23

Limits of functions

Last lecture we proved

Proposition 6.2: Let $S \subset \mathbb{R}$ be non-empty and let $x_0 \in \mathbb{R}$. Then x_0 is a limit point of S if and only if there exists a sequence $(x_n) \in S \setminus \{ x_0 \}$ such that $x_n \to x_0$.

This proposition explains the reason for the name 'limit point': x_0 is a limit point of S if and only if x_0 is a limit of a sequence in $S \setminus \{ x_0 \}$.

As an immediate corollary of this proposition we have

Corollary: If S is closed and x_0 is a limit point of S then $x_0 \in S$.

Proof: If x_0 is a limit point of S then there exists a sequence (x_n) in $S \setminus \{ x_0 \}$ such that $x_n \to x_0$. If S is closed then $x_0 \in S$ by Theorem 3.3.

Exercise: If $S \subset \mathbb{R}$ is non-empty let $S' = \{ y \in \mathbb{R} \mid y \text{ is a limit point of } S \}$ (the set S' is sometimes called the *derived set* of S). Then S is closed $\iff$ $S' \subset S$.

Exercise: Let S' be defined as above. Prove that $(S')' \subset S'$. Hence $S'' = (S')'$ is closed.

Definition 6.3: Let $f \colon S \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$ be a limit point of S. We say $f(x)$ *approaches* L *as* x *approaches* x_0, and we write $f(x) \to L$ as $x \to x_0$, if forall $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in S$, if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$.

Note: This definition makes precise the intution that $f(x)$ approaches L as x approaches x_0 if and only if the values of the function f get arbitrarily close to L if x is close to, but not equal to, x_0. The 'close to, but not equal to' part is captured in the statement $0 < |x - x_0| < \delta$; since $0 < |x - x_0|$, we cannot have $x = x_0$. We say that L is the *limit* of the function f as x approaches x_0.

Note: It is important in this definition that x_0 is a limit point. As we will see, without this requirement, we would not be able to prove many of the theorems below. Moreover, if x_0 was not a limit point, then the condition 'for all $x \in S$, $0 < |x - x_0| < \delta$' could be empty for all $\delta > 0$. In this case, $f(x) \to L$ as $x \to x_0$ for all $L \in \mathbb{R}$, which is obviously not desirable.

Note: Suppose $f \colon S \to \mathbb{R}$ is a function and that $x_0 \in S$ is a limit point of S. Then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Lemma 6.4: If $f(x) \to L$ and $f(x) \to M$ as $x \to x_0$ then $L = M$.

Proof: The proof of this proceeds as in the proof of Lemma 2.2. Suppose for a contradiction that $L \neq M$. Let $\epsilon = |L - M|/2 > 0$. Since $f(x) \to L$ as $x \to x_0$ there exists $\delta_1 > 0$ such that for all $x \in S$, if $0 < |x - x_0| < \delta_1$ then $|f(x) - L| < \epsilon$. Since $f(x) \to M$ as $x \to x_0$ there exists $\delta_2 > 0$ such that for all $x \in S$, if $0 < |x - x_0| < \delta_2$ then $|f(x) - M| < \epsilon$. Let $\delta = \min\{ \delta_1, \delta_2 \}$. Since x_0 is a limit point of S, there exists $x \in S$ such that $0 < |x - x_0| < \delta$. Therefore $|f(x) - L| < \epsilon$ and $|f(x) - M| < \epsilon$. Therefore, by the triangle inequality,

$$|L - M| \leq |f(x) - L| + |f(x) - M| < 2\epsilon = |L = M|,$$

a contradiction. Therefore $L = M$. ∎

Because of this lemma, we are justified in writing $\lim_{x \to x_0} f(x) = L$ instead of $f(x) \to L$ as $x \to x_0$.

Example 1: Let $f\colon [0,1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x+1, & \text{if } x \neq 0, \\ 4, & \text{if } x = 0. \end{cases}$$

We will prove that $\lim_{x\to 0} f(x) = 1$. Let $\epsilon > 0$ (all such proofs should start with this statement).

We pause to think about what we have to do: we have to prove that for this ϵ, there exists a $\delta > 0$ such that if $x \in [0,1)$ and $0 < |x| < \delta$, then $|f(x) - 1| < \epsilon$. If $x \in [0,1)$ then the inequality $0 < |x| < \delta$ becomes $0 < x < \delta$. Therefore, we have to prove that there exists a $\delta > 0$ such that if $0 < x < \delta$ and $x < 1$ then $|f(x) - 1| < \epsilon$. We investigate the expression $|f(x) - 1|$. This is equal to $|f(x) - 1| = |2x| = 2x$ if $x \in (0,1)$. If $x < \delta$ then $2x < 2\delta$ and so we see that the inequality $|f(x) - 1| < \epsilon$ will be satisfied if $2\delta < \epsilon$, i.e. if $\delta < \epsilon/2$. We now return to our proof.

Let $\delta = \epsilon/2$. Then if $0 < x < \delta$ and $x \in [0,1)$ then $|f(x) - 1| = 2|x| < 2\delta < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows that $\lim_{x\to 0} f(x) = 1$.

Example 2: Suppose $f\colon S \to \mathbb{R}$, $T \subset S \subset \mathbb{R}$ and x_0 is a limit point of T. Then x_0 is also a limit point of S (for any $\epsilon > 0$ there exists $x \in T \setminus \{\, x_0 \,\}$ such that $x \in I_\epsilon(x_0)$, but then $x \in S \setminus \{\, x_0 \,\}$ from which it follows that x_0 is also a limit point of S). Suppose that $\lim_{x\to x_0} f(x) = L$. Since $T \subset S$ we can consider the *restriction* of f to T, i.e. the function $f|_T\colon T \to \mathbb{R}$. This function satisfies

$$\lim_{x\to x_0} (f|_T)(x) = \lim_{x\to x_0} f(x) = L.$$

To see this, let $\epsilon > 0$. Then there exists $\delta > 0$ such that for all $x \in S$, if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \epsilon$. Therefore, if $x \in T$ and $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$. It follows that $\lim_{x\to x_0} (f|_T)(x) = L$.

Proposition 6.5: Suppose $f\colon S \to \mathbb{R}$ is a function and $x_0 \in \mathbb{R}$ is a limit point of S. Then

$$\lim_{x\to x_0} f(x) = L \iff \lim_{n\to\infty} f(x_n) = L \text{ for every sequence } (x_n) \text{ in } S\setminus\{\, x_0 \,\} \text{ such that } \lim_{n\to\infty} x_n = x_0.$$

Proof: The proof is analogous to the proof of Theorem 4.3 and is left as a highly recommended exercise. ■

Example: Let $f\colon \mathbb{R} \setminus \{\, 0 \,\} \to \mathbb{R}$ be defined by $f(x) = \cos(1/x)$. Then f does not have a limit as $x \to 0$. We use Proposition 6.5. Clearly 0 is a limit point of $\mathbb{R} \setminus \{\, 0 \,\}$. We will find two sequences (x_n) and (y_n) in $\mathbb{R} \setminus \{\, 0 \,\}$ such that $x_n \to 0$, $y_n \to 0$ but the sequences $(f(x_n))$ and $(f(y_n))$ do not converge to the same value.

For example we may take $x_n = 1/2n\pi$ for $n \in \mathbb{N}$ and $y_n = 1/(2n\pi + \pi/2)$. Then $x_n \to 0$, $y_n \to 0$. We have $f(x_n) = \cos(2n\pi) = 1$ for all n. Hence $f(x_n) \to 1$. But $f(y_n) = \cos(2n\pi + \pi/2) = \cos(\pi/2) = 0$ for all n. Hence $f(y_n) \to 0$. It follows from Proposition 6.5 that there is no real number L such that $\lim_{x\to x_0} f(x) = L$, as otherwise we would have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = L$.

Proposition 6.6 (Limit Laws): Suppose $S \subset \mathbb{R}$, $x_0 \in \mathbb{R}$ is a limit point of S and $f, g\colon S \to \mathbb{R}$. Suppose further that $\lim_{x\to x_0} f(x) = L$, $\lim_{x\to x_0} g(x) = M$. Then

(1) $\lim_{x\to x_0} cf(x) = cL$ for all $c \in \mathbb{R}$

(2) $\lim_{x\to x_0} (f(x) + g(x)) = L + M$

(3) $\lim_{x\to x_0} f(x)g(x) = LM$

(4) $\lim_{x\to x_0} f(x)/g(x) = L/M$ provided $g(x) \neq 0$ for all $x \in S$ and $M \neq 0$.

Proof: This is a straightforward application of Proposition 6.5 with the Algebraic Limit Theorem (Theorem 2.5). Again, this comes as a highly recommended exercise. You should attempt this exercise yourself before reading the proof.

We prove (1). (Idea: By Proposition 6.5, it suffices to show that for any sequence (x_n) in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} cf(x_n) = cL$.) Let (x_n) be a sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$. (All the proofs should start with this statement.) Since $\lim_{x\to x_0} f(x) = L$, Proposition 6.5 implies that $\lim_{n\to\infty} f(x_n) = L$. Therefore, by (1) of the Algebraic Limit Theorem (Theorem 2.2) we see that

$$\lim_{n\to\infty} cf(x_n) = cL.$$

Therefore, since we have shown that this is true for *any* sequence $(x_n) \in S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, it follows from Proposition 6.5 that $\lim_{x\to x_0} cf(x) = cL$.

We prove (2). (Idea: By Proposition 6.5, it suffices to show that for any sequence (x_n) in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty}(f(x_n) + g(x_n)) = L + M$.) Let (x_n) be a sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$. As an aside, a very good question is 'Why does there exist such a sequence'? This is exactly the kind of question you should ask yourself when trying to understand these proofs. The answer is, because x_0 is a limit point of S (Proposition 6.2).

Since $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, Proposition 6.5 implies that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. Therefore, by (2) of the Algebraic Limit Theorem (Theorem 2.2) we see that

$$\lim_{n\to\infty} (f(x_n) + g(x_n)) = L + M.$$

Therefore, since we have shown that this is true for *any* sequence $(x_n) \in S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, it follows from Proposition 6.5 that $\lim_{x\to x_0}(f(x) + g(x)) = L + M$.

Now, you're likely getting sick of this (and my fingers are getting sore), but I'll continue. We prove (3). The idea is exactly the same as in the previous two cases. Let (x_n) be a sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$.

Since $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, Proposition 6.5 implies that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. Therefore, by (3) of the Algebraic Limit Theorem (Theorem 2.2) we see that

$$\lim_{n\to\infty} (f(x_n)g(x_n)) = LM.$$

Therefore, since we have shown that this is true for *any* sequence $(x_n) \in S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, it follows from Proposition 6.5 that $\lim_{x\to x_0}(f(x)g(x)) = LM$.

Ok, probably your stomach is churning by now, but there is one last statement to go. Let (x_n) be a sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$.

Since $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, Proposition 6.5 implies that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. We want to apply (4) of the Algebraic Limit Theorem (Theorem 2.2). There is a smidgin more that needs to be said here. By assumption, $g(x) \neq 0$ for all $x \in S$. Therefore, $g(x_n) \neq 0$ for all $n \in \mathbb{N}$, since (x_n) is a sequence in S. Therefore, since $M \neq 0$, we may apply (4) of the Algebraic Limit Theorem (Theorem 2.2) to see that

$$\lim_{n\to\infty} (f(x_n)/g(x_n)) = L/M.$$

Therefore, since we have shown that this is true for *any* sequence $(x_n) \in S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, it follows from Proposition 6.5 that $\lim_{x\to x_0}(f(x)/g(x)) = L/M$. ■

Note: Of course, there is nothing to stop us from proving the statements (1)–(4) directly from Definition 6.3 in terms of ϵ's and δ's, and indeed we should be able to do that. But it is more efficient to prove these statements in the way that we have just done.

Proposition 6.7 (Squeeze Theorem): Suppose that $f, g, h\colon S \to \mathbb{R}$ are functions and that x_0 is a limit point of S. Suppose that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in S \setminus \{ x_0 \}.$$

If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} h(x) = L$ then $\lim_{x\to x_0} g(x) = L$.

Proof: Again, this is a straightforward application of Proposition 6.5. Let (x_n) be a sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$. Then

$$f(x_n) \leq g(x_n) \leq h(x_n) \text{ for all } n \in \mathbb{N}.$$

Since $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} h(x) = L$, Proposition 6.5 implies that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n) = L$. Therefore, by the Squeeze Theorem for sequences (Theorem 2.6), we see that $\lim_{n\to\infty} g(x_n) = L$. Since the sequence (x_n) was an arbtitrary sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$, it follows from Proposition 6.5 again that $\lim_{x\to x_0} g(x) = L$. ■

Proposition 6.8 (Preservation of Inequalities): Suppose that $f, g\colon S \to \mathbb{R}$ are functions and x_0 is a limit point of S. Suppose that

$$f(x) \leq g(x) \text{ for all } x \in S \setminus \{ x_0 \}.$$

If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$ then $L \leq M$.

Proof: Let (x_n) be a sequence in $S \setminus \{ x_0 \}$ such that $\lim_{n\to\infty} x_n = x_0$. Since $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, it follows from Proposition 6.5 that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. We have $f(x_n) \leq g(x_n)$ for all $n \in \mathbb{N}$, since (x_n) is a sequence in S and $f(x) \leq g(x)$ for all $x \in S \setminus \{ x_0 \}$. Therefore, by Preservation of Inequalities for sequences (Theorem 2.7), it follows that $L \leq M$.

Differentiability

Definition 6.9: Suppose that $f\colon S \to \mathbb{R}$ is a function and that $x_0 \in S$ is a limit point of S. We say that f is *differentiable* at $x_0 \in S$ if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. If f is differentiable at x_0 then we write $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ and we call the number $f'(x_0)$ the *derivative* of f at x_0. If every point of S is a limit point of S (for example if S is an interval) then we say f is *differentiable on* S if f is differentiable at x_0 for all $x_0 \in S$.