APP MTH 3002 Fluid Mechanics III
Tutorial 5 (Week 10)

1. Consider an inviscid flow in which there is no external force.

   (a) Use Euler's equation,
   $$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega} = -\frac{\nabla p}{\rho} - \frac{1}{2}\nabla|\boldsymbol{u}|^2,$$
   to obtain Bernoulli's equation for unsteady incompressible irrotational flow.

   Solution: For irrotational flow $\boldsymbol{\omega} = \boldsymbol{0}$ and $\boldsymbol{u} = \nabla\phi$. In the absence of external forces, Euler's equation is then
   $$\frac{\partial \nabla\phi}{\partial t} = -\frac{\nabla p}{\rho} - \frac{1}{2}\nabla|\boldsymbol{u}|^2 \Rightarrow \nabla \underbrace{\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2\right)}_{f} = \boldsymbol{0}.$$
   Since $\nabla f = \boldsymbol{0}$,
   $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$
   This means that f can only depend in t, not on spatial coordinates, that is,
   $$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 = f(t).$$

   (b) Suppose that the pressure $p \to p_\infty$, the fluid speed $|\boldsymbol{u}| \to 0$, and $\partial\phi/\partial t \to 0$ as $|\boldsymbol{x}| \to \infty$. Write down an expression for the pressure.

   Solution: Far from the origin, $p \to p_\infty$, $|\boldsymbol{u}| \to 0$ and $\partial\phi/\partial t \to 0$, hence
   $$\frac{p_\infty}{\rho} = f(t).$$
   The pressure at any point in the flow is
   $$p = p_\infty - \rho\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2\right).$$

2. The velocity potential of an irrotational flow in plane polar coordinates is
   $$\phi = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi}\theta$$
   where U, a, and κ are positive constants.

   (a) Show that this velocity potential represents a flow past a fixed cylinder of radius a and that it approaches uniform flow at infinity.

   Solution: The velocity components are
   $$u_r = \frac{\partial \phi}{\partial r} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$
   $$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r}$$

For $r = a$ to represent an impermeable boundary boundary, $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \boldsymbol{u} \cdot \boldsymbol{e}_r = u_r$ at $r = a$. We find that

$$u_r(a, \theta) = U \cos\theta \left(1 - \frac{a^2}{a^2}\right) = 0,$$

hence $r = a$ is indeed an impermeable boundary.

As $r \to \infty$, $u_r \to U \cos\theta$ and $u_\theta \to -U \sin\theta$. Now

$$\boldsymbol{e}_r = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}, \quad \boldsymbol{e}_\theta = -\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j},$$

hence

$$\begin{aligned}
\boldsymbol{u} &= U \cos\theta\, \boldsymbol{e}_r - U \sin\theta\, \boldsymbol{e}_\theta \\
&= U \cos\theta(\cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}) - U \sin\theta(-\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}) \\
&= U(\cos^2\theta + \sin^2\theta)\boldsymbol{i} + U(\cos\theta \sin\theta - \cos\theta \sin\theta)\boldsymbol{j} \\
&= U\, \boldsymbol{i}.
\end{aligned}$$

So the flow far from the cylinder is uniform flow in the x-direction at speed U.

(b) Calculate the velocity and locate the stagnation points for the three cases $\Omega < 1$, $\Omega = 1$, and $\Omega > 1$, where $\Omega = \kappa/(4\pi U a)$.

Solution: For stagnation points,

$$\begin{aligned}
u_r &= U \cos\theta \left(1 - \frac{a^2}{r^2}\right) = 0, \\
u_\theta &= -U \sin\theta \left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r} = 0.
\end{aligned}$$

From the first equation,

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \quad 1 - \frac{a^2}{r^2} \Rightarrow r = a.$$

Substituting $\theta = \frac{\pi}{2}$ in the $u_\theta = 0$ equation,

$$-U\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r} = 0.$$

Multiplying by r^2 and rearranging,

$$-Ur^2 + \frac{\kappa}{2\pi} r - Ua^2 = 0.$$

Solving this quadratic equation,

$$\begin{aligned}
r &= \frac{-\frac{\kappa}{2\pi} \pm \sqrt{\left(\frac{\kappa}{2\pi}\right)^2 - 4(-U)(-Ua^2)}}{-2U} \\
\Rightarrow \frac{r}{a} &= \frac{K}{4\pi U a} \pm \sqrt{\left(\frac{K}{4\pi U a}\right)^2 - 1}
\end{aligned}$$

The solutions are real when

$$\frac{K}{4\pi U a} \geq 1.$$

Following a similar analysis for $\theta = \frac{3\pi}{2}$ you should find that although you can get real solutions for r, they are negative, hence discarded. Substituting $r = a$ in the $u_\theta = 0$ equation,

$$-U \sin\theta \left(1 + \frac{a^2}{a^2}\right) + \frac{\kappa}{2\pi a} = -2U \sin\theta + \frac{\kappa}{2\pi a} = 0,$$

hence

$$\sin\theta = \frac{\kappa}{4\pi U a}.$$

For

$$0 \le \frac{\kappa}{4\pi U a} < 1,$$

this has two solutions

$$\theta_1 = \arcsin\left(\frac{\kappa}{4\pi U a}\right), \quad \theta_2 = \pi - \theta_1.$$

In summary, for $0 \le \Omega < 1$, there are two stagnation points on the surface of the cylinder at the points $(r, \theta) = (a, \theta_1)$ and (a, θ_2). When $\Omega = 1$, there is a single stagnation point on the surface of the cylinder at $(a, \pi/2)$. For $\Omega > 1$, there are two stagnation points on the y-axis, one inside the cylinder and one outside the cylinder, at the points

$$\left(\frac{\kappa}{4\pi U} + \sqrt{\left(\frac{\kappa}{4\pi U}\right)^2 - a^2}, \frac{\pi}{2}\right), \quad \left(\frac{\kappa}{4\pi U} - \sqrt{\left(\frac{\kappa}{4\pi U}\right)^2 - a^2}, \frac{\pi}{2}\right).$$

(c) Find the stream function.

Solution: The stream function satisfies

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{\partial\psi}{\partial r},$$

hence

$$\frac{\partial\psi}{\partial\theta} = Ur\cos\theta\left(1 - \frac{a^2}{r^2}\right),$$

$$\frac{\partial\psi}{\partial r} = U\sin\theta\left(1 + \frac{a^2}{r^2}\right) - \frac{\kappa}{2\pi r}.$$

Integrating the first of these,

$$\psi = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right) + f(r).$$

Differentiating the result,

$$\frac{\partial\psi}{\partial r} = U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + f'(r).$$

Equating this with the second equation

$$f'(r) = -\frac{\kappa}{2\pi r}.$$

Integrating,

$$f(r) = -\frac{\kappa}{2\pi}\ln r + C.$$

Choosing $C = 0$, the stream function is

$$\psi = Ur \sin\theta \left(1 - \frac{a^2}{r^2}\right) - \frac{\kappa}{2\pi} \ln r.$$

(d) Sketch or plot the streamlines for $\Omega = 0.5, 1, 1.5$. Mark the location of the stagnation points on this plot.

Solution:

```
% Parameters

Omega = 0.5;
a = 1;
U = 1;
K = 4*pi*U*a*Omega;
filename = 'Figures/t5q2_0p5.eps';

% Create grid points in spherical coordinate system for a < r < 8a,
% 0 <= t <= pi and phi = 0, then evaluate stream function.

[r, t] = meshgrid(linspace(a, 8*a, 50), linspace(-pi, pi, 100));
x = r.*cos(t);
y = r.*sin(t);
psi = U*r.*sin(t).*(1 - a^2./r.^2) - 0.5*K/pi*log(r);

% Plot streamlines. On cylinder, r=a => psi = 0.5*K/pi*log(a)

psi0 = 0.5*K/pi*log(a);
contour(x, y, psi, [psi0+1e-6 psi0-1e-6 linspace(-8, 8, 30)])
hold on

% Plot stagnation points.

r = a*(Omega + sqrt(Omega^2 -1)*[-1 1])*(Omega > 1) + [a a]*(Omega <= 1);
t0 = asin(Omega);
t = 0.5*pi*[1 1]*(Omega > 1) + [t0 pi-t0]*(Omega <= 1);
x = r.*cos(t);
y = r.*sin(t);
plot(x, y, 'xk')

axis equal
axis([-4*a, 4*a, -4*a, 4*a])
xlabel('$z$')
ylabel('$x$')
c = colorbar;
ylabel(c, '$\psi$', 'Interpreter', 'Latex');
title(sprintf('$\\Omega$ = %3.1f', Omega))
set(gcf, 'units', 'inches', 'position', [4 4 4 4])
print(filename, '-depsc')
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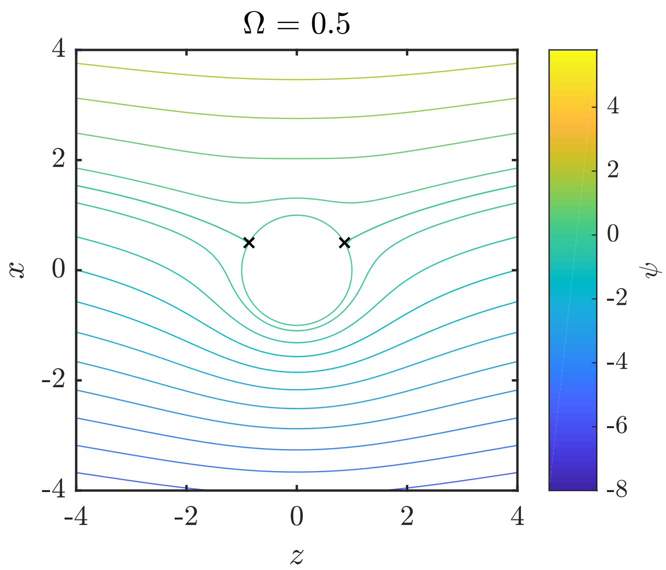
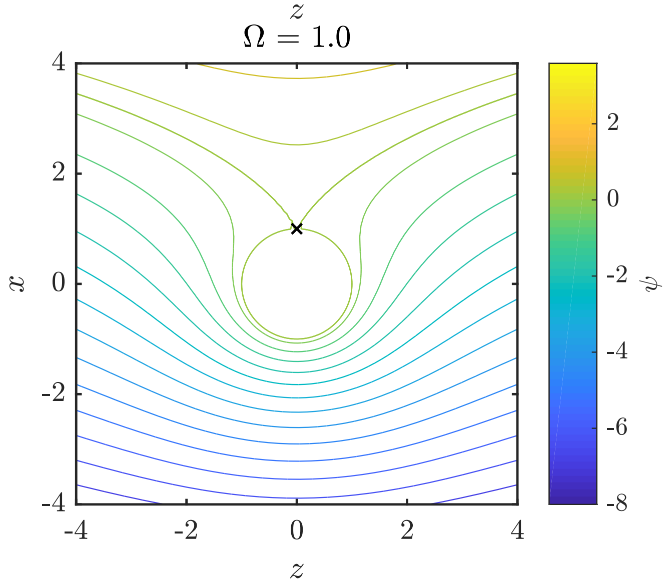
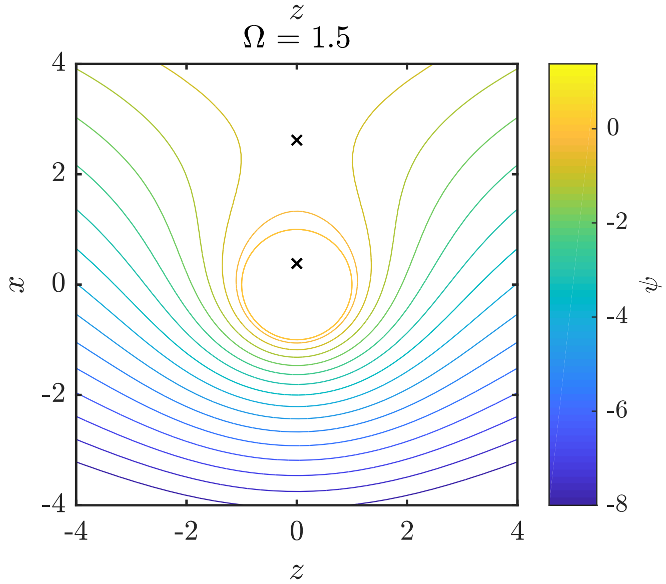






(e) By integrating over the surface of the cylinder, find the force per unit length on the cylinder.

Solution: The force per unit length of the cylinder is

$$\boldsymbol{F} = -\int_{\mathcal{C}} p\,\hat{\boldsymbol{n}}\,\mathrm{d}s,$$

where s is the arclength around the circumference of the cylinder $\mathcal{C}$. Since the flow is steady and irrotational, Bernoulli's equation is

$$\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 = C,$$

where C is a constant. The pressure is

$$p = \rho C - \frac{1}{2}\rho|\boldsymbol{u}|^2.$$

The velocity on the surface of the cylinder $r = a$ is

$$\boldsymbol{u} = \left(\frac{\kappa}{2\pi a} - 2U\sin\theta\right)\boldsymbol{e}_\theta.$$

The speed on the surface of the cylinder is

$$|\boldsymbol{u}|^2 = u_r^2 + u_\theta^2 = \left(\frac{\kappa}{2\pi a}\right)^2 - \frac{2\kappa U}{\pi a}\sin\theta + 4U^2\sin^2\theta.$$

On the surface of the cylinder, $\mathrm{d}s = a\,\mathrm{d}\theta$ and $\hat{\boldsymbol{n}} = \boldsymbol{e}_r = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$, hence the force per unit length is

$$\begin{aligned}\boldsymbol{F} &= -\int_{\mathcal{C}} p\,\hat{\boldsymbol{n}}\,\mathrm{d}s \\ &= -\int_0^{2\pi}\left(\rho C - \frac{1}{2}\rho|\boldsymbol{u}|^2\right)(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j})\,a\,\mathrm{d}\theta.\end{aligned}$$

But we know that

$$\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = 0,$$

hence the constant ρC term doesn't contribute anything. So we have

$$\begin{aligned}\boldsymbol{F} &= \frac{a\rho}{2}\int_0^{2\pi}|\boldsymbol{u}|^2(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j})\,\mathrm{d}\theta \\ &= \frac{a\rho}{2}\int_0^{2\pi}(A + B\sin\theta + C\sin^2\theta)(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j})\,\mathrm{d}\theta,\end{aligned}$$

where

$$A = \left(\frac{\kappa}{2\pi a}\right)^2, \quad B = -\frac{2\kappa U}{\pi a}, \quad C = 4U^2.$$

What a smorgasbord of integrals!

$$\int_0^{2\pi}\sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}\sin 2\theta\,\mathrm{d}\theta = \frac{1}{2}\left[-\cos 2\theta\right]_0^{2\pi} = 0$$

$$\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}(1 - \cos 2\theta)\,\mathrm{d}\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \pi$$

$$\int_0^{2\pi}\sin^2\theta\cos\theta\,\mathrm{d}\theta = \left[\frac{1}{2}\sin^3\theta\right]_0^{2\pi} = 0$$

$$\int_0^{2\pi}\sin^3\theta\,\mathrm{d}\theta = \int_0^{2\pi}\sin\theta(1 - \cos^2\theta)\,\mathrm{d}\theta = \frac{1}{2}\left[-\cos\theta + \frac{1}{3}\cos 3\theta\right]_0^{2\pi} = 0$$

The force is

$$\boldsymbol{F} = \frac{a\rho}{2}(B\pi\,\boldsymbol{j}) = -\frac{a\rho}{2}\frac{2\pi\kappa U}{\pi a}\,\boldsymbol{j} = -\rho\kappa U\,\boldsymbol{j}.$$

Recall that far from the cylinder $\boldsymbol{u} \to U\,\boldsymbol{i}$, hence the force is perpendicular to the free-stream direction. The component of force that is perpendicular to the free-stream direction is called 'lift'. The component of force in the direction of the free-stream is called 'drag', which is zero in this case.

3. Two-dimensional, incompressible, irrotational flow produced by a source of strength Q located at $(0, d)$ and a plane impermeable boundary located at $y = 0$ can be modelled by introducing an 'image' source of equal strength at $(0, -d)$.

   (a) Write down the velocity potential and stream function for the above flow if $Q = 2\pi$ and $d = 1$.

   Solution: The velocity potential and stream function of a source located at the origin are

   $$\phi = \frac{Q}{2\pi}\ln r = \ln r, \quad \psi = \frac{Q}{2\pi}\theta = \theta.$$

   Since $x = r\cos\theta$ and $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. So in Cartesian coordinates,

   $$\phi = \frac{1}{2}\ln(x^2 + y^2), \quad \psi = \arctan\left(\frac{y}{x}\right).$$

   For a source located at $(0, d)$,

   $$\phi = \frac{1}{2}\ln\left[x^2 + (y - d)^2\right], \quad \psi = \arctan\left(\frac{y - d}{x}\right).$$

   Using the superposition principle, the velocity potential and stream function of a source located at $(0, 1)$ and its image at $(0, -1)$ is

   $$\phi = \frac{1}{2}\ln\left[x^2 + (y-1)^2\right] + \frac{1}{2}\ln\left[x^2 + (y+1)^2\right]$$
   $$\psi = \arctan\left(\frac{y-1}{x}\right) + \arctan\left(\frac{y+1}{x}\right)$$

   (b) Verify that $y = 0$ is an impermeable boundary and calculate the velocity along the boundary.

   Solution: The velocity components are

   $$u = \frac{\partial\phi}{\partial x} = \frac{x}{x^2 + (y-1)^2} + \frac{x}{x^2 + (y+1)^2},$$
   $$v = \frac{\partial\phi}{\partial y} = \frac{y-1}{x^2 + (y-1)^2} + \frac{y+1}{x^2 + (y+1)^2}.$$

   Along the x-axis $y = 0$, hence

   $$u(x, 0) = \frac{2x}{x^2 + 1}, \quad v(x, 0) = \frac{-1}{x^2 + 1} + \frac{1}{x^2 + 1} = 0.$$

   The x-axis is an impermable boundary because $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{j} = v = 0$ on $y = 0$.

(c) Locate the stagnation point.

Solution: For stagnation points, $u = v = 0$. We already know that $v = 0$ for $y = 0$. For $y = 0$,

$$u = \frac{2x}{x^2 + 1} = 0 \Rightarrow x = 0.$$

So the stagnation point is located at the origin $(0, 0)$.

(d) Show that the streamlines are given by

$$y = -Cx \pm \sqrt{(C^2 + 1)x^2 + 1},$$

where C is a constant.

Solution: For streamlines,

$$\psi = \arctan\left(\frac{y - 1}{x}\right) + \arctan\left(\frac{y + 1}{x}\right) = \theta_1 + \theta_2 = K,$$

where K is a constant and

$$\tan\theta_1 = \frac{y - 1}{x}, \quad \tan\theta_2 = \frac{y + 1}{x}.$$

Then

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} \\
&= \frac{\frac{y-1}{x} + \frac{y+1}{x}}{1 - \frac{y-1}{x}\frac{y+1}{x}} \\
&= \frac{x(y - 1 + y + 1)}{x^2 - (y - 1)(y + 1)} \\
&= \frac{2xy}{x^2 - y^2 + 1} \\
&= \tan K = \frac{1}{C}
\end{aligned}$$

Rearranging,

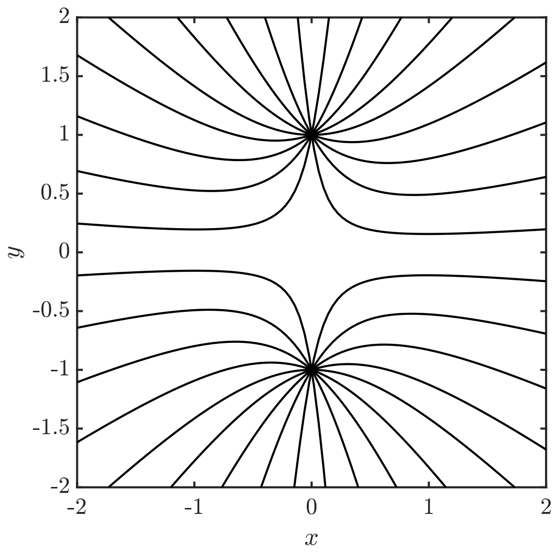
$$2Cxy = x^2 - y^2 + 1 \Rightarrow y^2 + 2Cxy - x^2 - 1 = 0.$$

Streamlines are given by the solution

$$y = \frac{-2Cx \pm \sqrt{4C^2x^2 - 4(1)(-x^2 - 1)}}{2} = -Cx \pm \sqrt{(C^2 + 1)x^2 + 1}.$$

(e) Sketch or plot the streamlines.

Solution:



4. A flow field in the xy-plane has the velocity components

$$u = 3x + y, \quad v = 2x - 3y.$$

Show that the circulation around the circle $(x-1)^2 + (y-6)^2 = 4$ is 4π.

Solution: The circulation is

$$\Gamma = \oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \oint_{\mathcal{C}} u \,\mathrm{d}x + v \,\mathrm{d}y.$$

A parametric expression of the curve $\mathcal{C}$ is

$$\begin{aligned} x &= 1 + 2\cos\theta \Rightarrow \mathrm{d}x = -2\sin\theta, \\ y &= 6 + 2\sin\theta \Rightarrow \mathrm{d}y = 2\cos\theta. \end{aligned}$$

Evaluating the velocity along the curve $\mathcal{C}$,

$$\begin{aligned} u &= 3(1 + 2\cos\theta) + 6 + 2\sin\theta \\ &= 9 + 6\cos\theta + 2\sin\theta, \\ v &= 2(1 + 2\cos\theta) - 3(6 + 2\sin\theta) \\ &= -16 + 4\cos\theta - 6\sin\theta. \end{aligned}$$

Substituting this into the integral,

$$\begin{aligned} \Gamma &= \int_0^{2\pi} (9 + 6\cos\theta + 2\sin\theta)(-2\sin\theta) + (-16 + 4\cos\theta - 6\sin\theta)(2\cos\theta)\,\mathrm{d}\theta \\ &= \int_0^{2\pi} (-18\sin\theta - 24\sin\theta\cos\theta - 4\sin^2\theta - 32\cos\theta + 8\cos^2\theta)\,\mathrm{d}\theta. \end{aligned}$$

Using the results

$$\int_0^{2\pi} \sin\theta \,\mathrm{d}\theta = \int_0^{2\pi} \cos\theta \,\mathrm{d}\theta = \int_0^{2\pi} \sin\theta\cos\theta \,\mathrm{d}\theta = 0,$$

this simplifies to

$$
\begin{aligned}
\Gamma &= \int_0^{2\pi} (-4\sin^2\theta + 8\cos^2\theta)\,\mathrm{d}\theta \\
&= \int_0^{2\pi} -2(1-\cos 2\theta) + 4(1+\cos 2\theta)\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 2 + 6\cos 2\theta\,\mathrm{d}\theta \\
&= [2\theta + 3\sin 2\theta]_0^{2\pi} = 4\pi
\end{aligned}
$$