LECTURE 10

Example: Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers. For each $k, m \in \mathbb{N}$, let

$$A_{m,k} = (-\infty, a_k + \frac{1}{m}) \cap (a_k - \frac{1}{m}, \infty).$$

Suppose that

$$\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{m,n} \neq \emptyset.$$

What conclusions can you draw? Suppose that $x \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{m,k}$. Then $\forall\ m \in \mathbb{N}$, $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{m,k}$. Therefore, for all $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $x \in \bigcap_{k=N}^{\infty} A_{m,k}$. Therefore, for all $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $x \in A_{m,k}$. Now, $x \in A_{m,k}$ if and only if $a_k - 1/m < x < a_k + 1/m$. Hence $x \in A_{m,k}$ if and only if $|a_k - x| < 1/m$.

Therefore, the statement that the set is non-empty is equivalent to the statement that there exists a real number x belonging to the set such that for all $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - x| < 1/m$. In other words, the statement that the set is non-empty is equivalent to the statement that there is an x belonging to the set, to which the sequence (a_n) converges. Since limits are unique there can be only one such x. Therefore, if the set is non-empty, it contains only one element, which is the limit of the sequence.

The next theorem is an extremely useful tool.

Theorem 2.6: (**The Squeeze Theorem**) If $a_n \leq b_n \leq c_n$ for all n, and $a_n \to L$, $c_n \to L$, then $b_n \to L$.

Proof: Let $\epsilon > 0$. Since $a_n \to L$ there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies L - \epsilon < a_n < L + \epsilon$. Since $c_n \to L$ there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2 \implies L - \epsilon < c_n < L + \epsilon$. Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then $L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$. Therefore, $n \geq N \implies L - \epsilon < b_n < L + \epsilon$. Since $\epsilon > 0$ was arbitrary, it follows that $b_n \to L$. ∎

Note: There are some variants of this theorem which are sometimes useful. Here is one such variant. If $a_n \leq b_n \leq c_n$ for all but finitely many n and $a_n \to L, c_n \to L$, then $b_n \to L$. In other words, if there exists $N \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n$ for all $n \geq N$, and $a_n \to L$, $c_n \to L$ then $b_n \to L$.

Theorem 2.7: (**Preservation of Inequalities**) Suppose $a_n \to L$, $b_n \to M$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then $L \leq M$.

Proof: Suppose instead that $L > M$. Let $\epsilon = (L - M)/2$. Since $a_n \to L$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - L| < \epsilon$. Since $a_n \to M$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2 \implies |a_n - M|$. Let $N = \max\{N_1, N_2\}$. Choose $n \geq N$. Then, using the triangle inequality,

$$|L - M| \leq |L - a_n| + |a_n - M| < 2\epsilon = |L - M|.$$

This is a contradiction. Hence $L = M$. ∎

Note: if $a_n < b_n$ for all $n \in \mathbb{N}$ in the statement of Theorem 2.7, we cannot conclude that $L < M$. We can only conclude that $L \leq M$. The following example illustrates this: let $a_n = 0$ for all n and let $b_n = 1/n$. Then $a_n \to 0$, $b_n \to 0$ but $a_n < b_n$ for all $n \in \mathbb{N}$.

Note: there is a variant of Theorem 2.7 in which the inequality $a_n \leq b_n$ is only required to hold for all but finitely many n, i.e. there exists $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$.

Example: (Nested Interval Property revisited) Suppose that $a_n \le b_n$ are real numbers such that $a_n \le b_n$ for all $n \in \mathbb{N}$. Suppose that

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$$

We have seen earlier (Theorem 1.11) that $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$. We will show that if $b_n - a_n \to 0$ (i.e. if the lengths of the intervals $[a_n, b_n]$ become arbitrarily small) then $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{ x \}$ for a unique $x \in \mathbb{R}$. Suppose that $a_n \le x \le y \le b_n$ for all n, i.e. suppose that $x, y \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. Then $0 \le y - x \le b_n - a_n$ for all $n \in \mathbb{N}$. Therefore, by Preservation of Inequalities, we must have $0 \le y - x \le 0$, i.e. $y = x$.

Definition 2.8: We say (a_n) *diverges* to ∞, and we write $a_n \to \infty$, if for all $K > 0$ there exists $N \in \mathbb{N}$ such that $a_n > K$ for all $n \ge N$. Similarly we say that (a_n) *diverges* to $-\infty$, and we write $a_n \to -\infty$, if for all $K > 0$ there exists $N \in \mathbb{N}$ such that $-K < a_n$ for all $n \ge N$.

Note: If $a_n \to \infty$ or $a_n \to -\infty$ then the sequence (a_n) is not bounded and hence does not converge to any real number.

Note: Beware that it is not the case that if a sequence does not converge then it diverges to ∞ or diverges to $-\infty$. For instance the sequence $a_n = (-1)^n$ does not converge, but it is bounded, and hence it does not diverge to ∞ or diverge to $-\infty$.