THE UNIVERSITY *of* ADELAIDE

Examination in the School of Mathematical Sciences

Semester 2, 2014

104831	**MATHS 2100**	**Real Analysis II**
104830	**MATHS 7100**	**Real Analysis**

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: 7 TOTAL MARKS: 70

Instructions

- Attempt all questions. Each is worth 10 marks.
- Begin each question on a new page.
- Examination materials may not be removed from the hall.

Materials

- One Blue Book is provided. You may request more if needed.
- Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Let $S \subseteq \mathbb{R}$ be a non-empty set.

   (a) What is a *lower bound* for S?

   (b) What is an *infimum* for S?

   (c) What does it mean for a sequence $(s_n)_{n=1}^{\infty}$ to *converge* to a number $L \in \mathbb{R}$?

   (d) Prove that $L \in \mathbb{R}$ is an infimum for S if and only if L is a lower bound for S and there is a sequence $(s_n)_{n=1}^{\infty}$ in S converging to L.

[2+2+2+4 = 10 marks.]

2. (a) Complete the statement of the following theorem that we proved in class: *A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if: for every $\epsilon > 0$ there is a* ______________ *such that* ____________________.

   [You do not need to define the various terms used in your answer.]

   (b) Write down the three key properties of the integral as proved in lectures.

   (c) Suppose that $f : [a,b] \to \mathbb{R}$ is a bounded integrable function. We often used the fact that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$ although it was never proved in class. Given that $|f|$ is also integrable, prove that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$, showing clearly what properties of the integral you are using.

[2+3+5 = 10 marks.]

3. (a) Find the average value of $f(x) = x^3 - 1$ on $[0,2]$.

   (b) Verify that $f(x) = x^3 - 1$ satisfies the conclusion of the Average Value theorem on $[0,2]$.

   (c) Part I of the Fundamental Theorem of Calculus (FTC) states that if $f : [a,b] \to \mathbb{R}$ is continuous then $F(x) := \int_a^x f(t)\,dt$ is differentiable on $[a,b]$ with $F'(x) = f(x)$ for each $x \in [a,b]$. State Part II of the FTC as presented in lectures.

   (d) Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable and $f' : [a,b] \to \mathbb{R}$ is continuous. By considering $\int_a^x f'(t)\,dt$, deduce Part II of the FTC from Part I in this case.

[2+2+2+4 = 10 marks.]

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is 3 times differentiable, with $f(1) = 2$, $f'(1) = -3$, $f''(1) = 6$ and with $|f'''(x)| \leq 6$ for any $x \in [1, 3]$.

   (a) Find the second Taylor polynomial p_2 for f at $x = 1$, and write down the expression for the remainder term $R(x)$ in $f(x) = p_2(x) + R(x)$ as given by the Lagrange Remainder theorem.

   (b) Use your polynomial to estimate $\int_1^3 f(x)\,dx$, and estimate the difference between this estimate and the true value of the integral, briefly explaining the steps in your estimation.

   (c) Consider the power series $\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{(3x-4)^n}{\sqrt[3]{n}}$. Explicitly determine the set of all real numbers x for which this series converges, briefly explaining the steps in your calculation.

[3+3+4 = 10 marks.]

5. (a) State the Comparison Test for $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ where $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

   (b) Suppose $r \in \mathbb{R}$. Under what condition on r does the series $\sum_{n=0}^{\infty} r^n$ converge, and when it does converge, to what number does it converge?

   (c) Suppose $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers with $a_n \neq 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ for some $L < 1$. Using (a) and (b), prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

[2+3+5 = 10 marks.]

6. Let (X, d) be a $\boxed{\text{metric space.}}$

Metric spaces are not part of the course this year (& were not last year also). Nevertheless, you should be able to do the modified question below where $S \subset \mathbb{R}$

   (a) What does it mean for a function $f : \cancel{X}\, S \to \mathbb{R}$ to be *continuous*?

   (b) If $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued functions on $\cancel{X}\, S$, what does it mean for the sequence to *converge uniformly* to some function $f : \cancel{X}\, S \to \mathbb{R}$?

   (c) If (f_n) is sequence of continuous real-valued functions on $\cancel{X}\, S$ that is converging uniformly to $f : \cancel{X}\, S \to \mathbb{R}$, prove that f is continuous.

[2+2+6 = 10 marks.]

7. Give an example of: [No justification required]

   (a) A bounded sequence $(a_n)_{n=1}^{\infty}$ that has two subsequences converging to different limits.

   (b) An uncountable open subset of $[1, 3]$.

   (c) A function $f : (0, 1) \to \mathbb{R}$ that is continuous but not uniformly continuous.

   (d) A series that converges conditionally.

   (e) ~~A metric space that is not complete~~.

[2 marks each.]

End of examination