Lecture 17: Queueing Systems – Multiple Customer Classes

Concepts checklist

At the end of this lecture, you should be able to:

- *model* (some) multiple customer class queueing systems as k-variate birth-and-death processes;
- *specify* the equilibrium distribution of a stationary k-variate birth-and-death process; and,
- *specify* the equilibrium distribution of a truncated k-variate birth-and-death process (and more broadly reversible process).

Multiple Customer Classes

There are many situations in which different customer streams compete for the service resources available. Sometimes this involves priorities, different resource requirements, reservation, etc. In each of these cases, we have to keep track of the numbers of each type of customer in the system.

The simplest of these kinds of models is when they can be considered as *a set of k independent birth and death processes*, for some $k \geq 2$.

Definition 17. *A k-variate (sometimes called k-dimensional) birth-and-death process has*

- *state space* $\mathcal{S} = \{\boldsymbol{n} = (n_1, n_2, \ldots, n_k) : n_i \in \{0, 1, \ldots\}\}$,
- *arrival rate $\lambda_i(n_i)$ for Type i customers, when there are currently n_i of them, and*
- *service rate $\mu_i(n_i)$ of Type i customers, when there are n_i of them.*

Theorem 16. *A stationary k-variate birth-and-death process has equilibrium distribution*

$$\pi(\boldsymbol{n}) = C \prod_{r=1}^{k} \prod_{\ell=1}^{n_r} \frac{\lambda_r(\ell - 1)}{\mu_r(\ell)}, \quad \text{where } C \text{ is a normalising constant.} \tag{19}$$

Proof. We will use Theorem 15. First we guess that the process is reversible, that is

$$q^R(\boldsymbol{n}, \boldsymbol{m}) = q(\boldsymbol{n}, \boldsymbol{m}) \quad \text{for all } \boldsymbol{n}, \boldsymbol{m} \in \mathcal{S}.$$

By Corollary 2 on the properties of diagonal elements, we have that $q^R(\boldsymbol{n}, \boldsymbol{n}) = q(\boldsymbol{n}, \boldsymbol{n})$. Then, by Theorem 15 we only need to show that

$$\pi(\boldsymbol{n}) q(\boldsymbol{n}, \boldsymbol{m}) = \pi(\boldsymbol{m}) q^R(\boldsymbol{m}, \boldsymbol{n}) \overset{\text{by assumption}}{=} \pi(\boldsymbol{m}) q(\boldsymbol{m}, \boldsymbol{n}) \quad \text{for all } \boldsymbol{n}, \boldsymbol{m} \in \mathcal{S}.$$

In particular, we need to consider only the cases where

- $\boldsymbol{m} = \boldsymbol{n} + \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is a k-vector of zeros with a 1 in the ith position, and
- $\boldsymbol{m} = \boldsymbol{n} - \boldsymbol{e}_i$.

That is,

$$\begin{aligned}
\pi(\boldsymbol{n})q(\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_i) &= \left(C \prod_{r=1}^{k} \prod_{\ell=1}^{n_r} \frac{\lambda_r(\ell-1)}{\mu_r(\ell)} \right) \lambda_i(n_i) \\
&= \left(C \prod_{\substack{r=1 \\ r\neq i}}^{k} \prod_{\ell=1}^{n_r} \frac{\lambda_r(\ell-1)}{\mu_r(\ell)} \right) \left(\prod_{\ell=1}^{n_i+1} \frac{\lambda_i(\ell-1)}{\mu_i(\ell)} \right) \mu_i(n_i+1) \\
&= \pi(\boldsymbol{n}+\boldsymbol{e}_i)q(\boldsymbol{n}+\boldsymbol{e}_i, \boldsymbol{n}).
\end{aligned}$$

Therefore, the guess was correct, which implies that a k-variate birth-and-death process is reversible, and that its equilibrium distribution is given by (19). □

The following theorem tells what happens if we truncate a reversible process.

Theorem 17. *Consider a reversible continuous-time Markov chain with state space $\mathcal{S}$ and equilibrium distribution $\boldsymbol{\pi}$. For some set $\mathcal{A} \subseteq \mathcal{S}$, we change the transition rates as follows:*

$$\textit{alter } q_{jk} \textit{ to } cq_{jk} \textit{ for all } j \in \mathcal{A} \textit{ and } k \in \mathcal{S}\backslash\mathcal{A} \textit{ for some } c \geq 0.$$

Then, the resulting process is also a reversible continuous-time Markov chain with equilibrium distribution $\overline{\boldsymbol{\pi}}$ given by

$$\overline{\pi}_j = \begin{cases} B\pi_j & \textit{for } j \in \mathcal{A}, \\ Bc\pi_j & \textit{for } j \in \mathcal{S}\backslash\mathcal{A}, \end{cases} \tag{20}$$

where B is a normalising constant such that

$$B \sum_{j\in\mathcal{A}} \pi_j + Bc \sum_{i\in\mathcal{S}\backslash\mathcal{A}} \pi_i = 1.$$

In particular, if $c = 0$ then

$$\overline{\pi}_j = B\pi_j \quad \textit{for } j \in \mathcal{A},$$

and $B = \left(\sum_{j\in\mathcal{A}} \pi_j \right)^{-1}$.

Proof. First we show that with the probabilities $\overline{\boldsymbol{\pi}}$ given by (20), the necessary and sufficient conditions for reversibility are satisfied for the modified process. Then, we show that the probabilities given by (20) must be the equilibrium probabilities of the modified process.

The detailed balance equations,

$$\bar{\pi}_j \bar{q}_{j,k} = \bar{\pi}_k \bar{q}_{k,j},$$

where $\bar{q}_{j,k}$ are the intensities for the new process.

Since the original process is reversible, we know that

$$\pi_j q_{j,k} = \pi_k q_{k,j}, \qquad \text{for all } j, k \in S.$$

Consider the following cases

1. $j, k \in A$:

$$\begin{aligned} \pi_j q_{j,k} &= \pi_k q_{k,j} \\ \Rightarrow (B\,\pi_j)\, q_{j,k} &= (B\,\pi_k)\, q_{k,j} \\ \Rightarrow\ \bar{\pi}_j \bar{q}_{j,k} &= \bar{\pi}_k \bar{q}_{k,j}. \end{aligned}$$

2. $j \in A,\ k \in (S \backslash A)$:

$$\begin{aligned} \pi_j q_{j,k} &= \pi_k q_{k,j} \\ \Rightarrow B\ c\ \pi_j q_{j,k} &= B\ c\ \pi_k q_{k,j} \\ \Rightarrow\ (B\,\pi_j)\ (c\ q_{j,k}) &= (B\ c\ \pi_k)\, q_{k,j} \\ \Rightarrow\ \bar{\pi}_j \bar{q}_{j,k} &= \bar{\pi}_k \bar{q}_{k,j}. \end{aligned}$$

3. $j, k \in (S \backslash A)$:

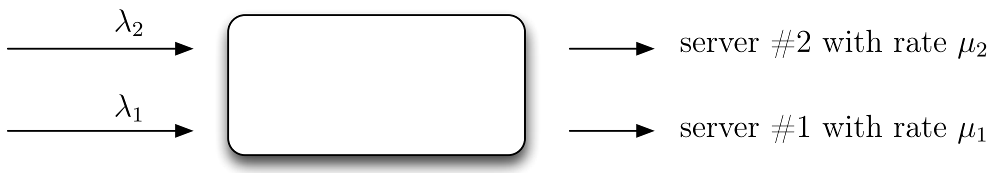
$$\begin{aligned} \pi_j q_{j,k} &= \pi_k q_{k,j} \\ \Rightarrow (B\ c\ \pi_j)\, q_{j,k} &= (B\ c\ \pi_k)\, q_{k,j} \\ \Rightarrow\ \bar{\pi}_j \bar{q}_{j,k} &= \bar{\pi}_k \bar{q}_{k,j}. \end{aligned}$$

Therefore, the new process is reversible with distribution given by the theorem and if $c = 0$, the process has been truncated to the set A and has equilibrium distribution given by $\bar{\pi}_j$.

□

Example 9. Shared Finite Buffer

Consider two independent single-server queues, with arrival rates λ_i and service rates μ_i for $i = 1, 2$. These queues share a common waiting room of size R and customers arriving to a full waiting room are lost. What is the equilibrium distribution for this queueing system?



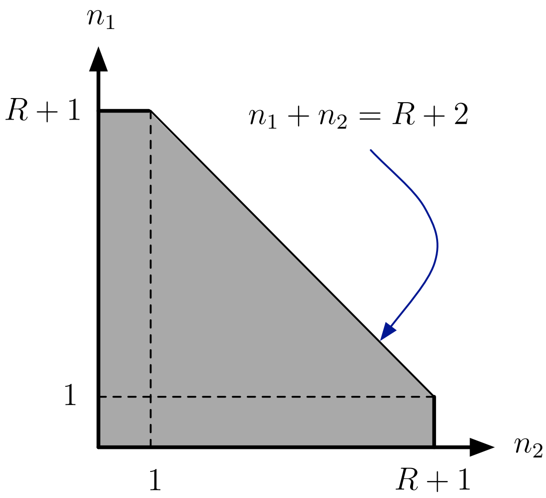
The state space for this system can be represented as follows

Step 1. Letting $[n]^+ = \max(0, n)$, we can write the state space $\mathcal{A}$ as follows

$$\mathcal{A} = \{(n_1, n_2) : [n_1 - 1]^+ + [n_2 - 1]^+ \leq R\}.$$

Step 2. Consider R to be ∞, which implies that the two queues are totally independent, reversible birth-and-death processes and have the joint equilibrium probability distribution given by

$$\pi(n_1, n_2) = \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(1 - \frac{\lambda_2}{\mu_2}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\ \left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$$



for $\lambda_1 < \mu_1$ and $\lambda_2 < \mu_2$, and with invariant measure $\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$.

Note: An invariant measure represents any positive multiple of the equilibrium distribution, and thus it is not normalised.

Step 3. We can use Theorem 17, which implies that since the original process was reversible, the truncated process must also be reversible with the same invariant measure.

With the state space $\mathcal{A} = \{(n_1, n_2) : [n_1 - 1]^+ + [n_2 - 1]^+ \le R\}$, the equilibrium distribution $\overline{\boldsymbol{\pi}}$ is given by

$$\overline{\pi}(n_1, n_2) = D\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2} \qquad \text{where } D = \left(\sum_{\mathcal{A}}\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}\right)^{-1}.$$

Out of interest, the normalising constant B of the theorem is given by

$$\frac{D}{\left(1-\frac{\lambda_1}{\mu_1}\right)\left(1-\frac{\lambda_2}{\mu_2}\right)} = \frac{1}{\left(1-\frac{\lambda_1}{\mu_1}\right)\left(1-\frac{\lambda_2}{\mu_2}\right)\sum_{\mathcal{A}}\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}}.$$