Revision Quiz: solutions

Q1. What are modelling, analysis and computation?

A1. Modelling: formulating a real world problem as mathematical equations, BCs and ICs.

Analysis: finding general patterns in and behaviour of a mathematical problem; is there a solution and is it unique; exact solution methods.

Computation: solving a mathematical problem by numerical methods to give a solution that is defined only at discrete points in space and/or time; solutions always contain error.

Q2. What is a well-posed problem? As a bonus, who is credited with this concept?

A2. A problem for which a unique solution exists that changes continuously with the initial conditions. Jacques Hadamard.

Q3. What are some important considerations when finding numerical solutions.

A3. Accuracy, efficiency and stability.

Q4. Given that most real problems must be solved numerically, what benefits are provided by exact solutions of idealised problems?

A4. Understanding likely behaviour. Testing and verifying numerical methods.

Q5. What is an autonomous ODE?

A5. An ODE that does not depend explicitly on the independent variable.

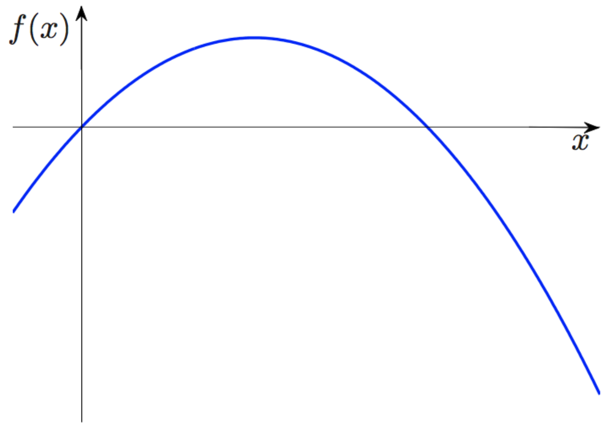
Q6. What is a fixed point and what other names can be used for it?

A6. A point at which the solution does not change with time; also called a steady state or an equilibrium.

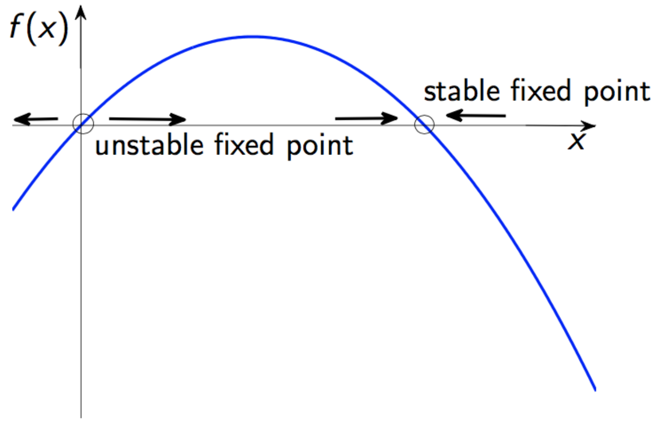
Q7. What is phase-line analysis of an ODE?

A7. Analysis of behaviour about fixed points to determine their stability.

Q8. The following plot shows the function $f(x)$ for the ODE $dx/dt = f(x)$. Perform a phase-line analysis.



A8.



Q9. Let x_* be a fixed point of the ODE $dx/dt = f(x)$.

(a) If $f'(x_*) > 0$ is the point stable or unstable?

(b) If $f'(x_*) < 0$ is the point stable or unstable?

(c) If $f'(x_*) = 0$ is the point stable or unstable?

A9. (a) unstable;

(b) stable;

(c) not known, need to look at higher derivatives.

Q10. What is a bifurcation?

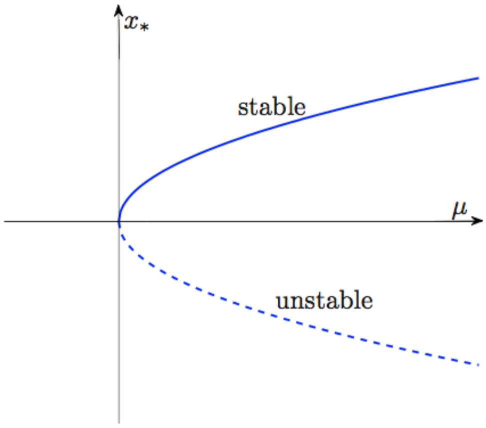
A10. A change in the number or stability of the steady states. In 1D we have $f(x_*, \mu_*) = 0$ and $f_x(x_*, \mu_*) = 0$.

Q11. What is a bifurcation diagram?

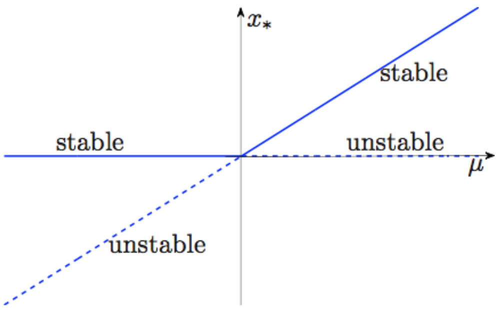
A11. A plot of steady states versus the bifurcation parameter(s).

Q12. What types of bifurcations are depicted in the bifurcation diagrams shown below?

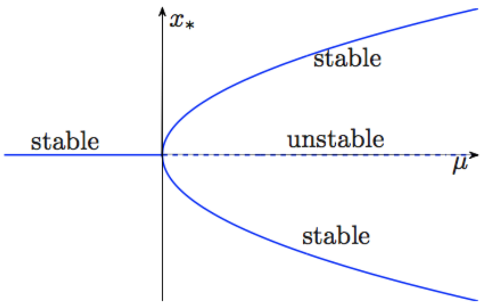
(a)



(b)



(c)

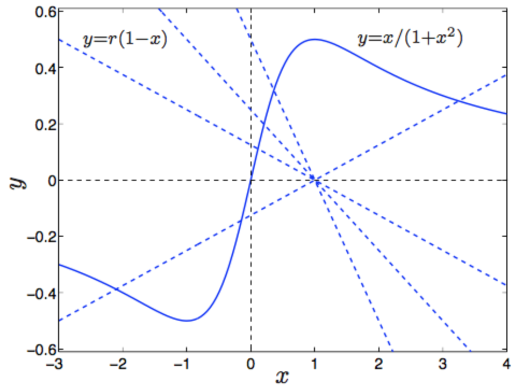


A12. (a) a saddle-node (fold, turning-point) bifurcation.
(b) a transcritical bifurcation.
(c) a supercritical pitchfork bifurcation.

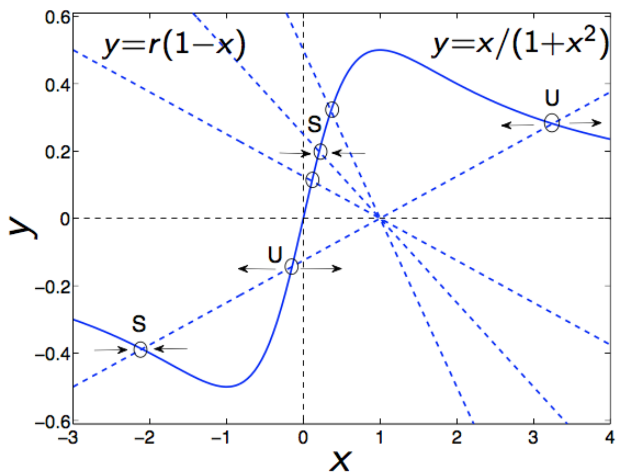
Q14. For the ODE

$$\frac{dx}{dt} = rx(1-x) - \frac{x^2}{1+x^2}$$

one steady state is $x = 0$. The other steady states can be determined from the diagram below that shows the curve $y = x/(1+x^2)$ (solid) and lines $y = r(1-x)$ (dashed). Mark them and determine their stability for both $r > 0$ and $r < 0$.



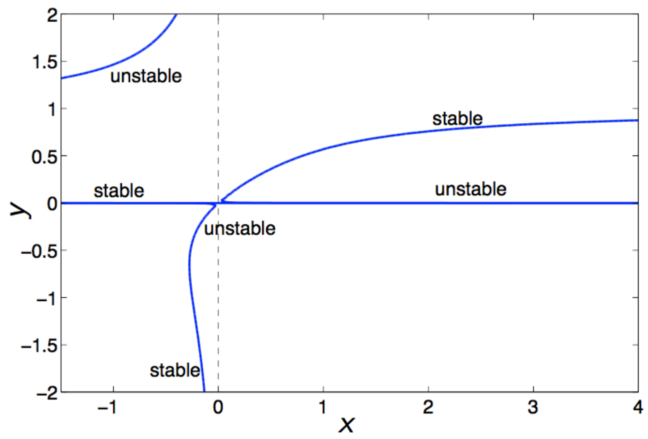
A14. Nontrivial steady states are given by $r(1-x) = x/(1+x^2)$. For $r > 0$ there is just one nontrivial steady state and $\dot{x} > 0$ for $r(1-x) > x/(1+x^2)$ and $\dot{x} < 0$ for $r(1-x) < x/(1+x^2)$, so it is stable.
For $r < 0$ there are three nontrivial steady states, one positive and two negative. The positive steady state is unstable. The most negative steady state is stable and the other is unstable, because, with $x < 0$ $\dot{x} < 0$ for $r(1-x) > x/(1+x^2)$.



U = unstable, S = stable.

Q15. If you are told that the trivial steady state for the previous question is stable for $r < 0$ and unstable for $r > 0$, plot the bifurcation diagram. Note: what is the non-trivial steady state as $r \to \infty$?

A15.



Q16. Does the existence and uniqueness theorem tell you when a unique solution does not exist?

A16. No. It tells you when a unique solution is guaranteed to exist.

Q17. What does it mean for a function $f(x)$ to be Lipschitz continuous on the interval $J = [x_0, x_1]$?

A17. There exists a value $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in J$. L is the Lipschitz constant.

Q18. What is an ill-conditioned problem?

A18. A small change in the input data causes a big change in the solution.

Q19. What is an unstable algorithm?

A19. An algorithm that gives a solution to a problem very far from the problem we are attempting to solve. Also the global error is magnified from one time step to the next.

Q20. Would you expect to get a good solution to an ill-conditioned problem if you were using a stable algorithm?

A20. No.

Q21. What is the difference between an explicit and implicit finite difference method?

A21. An explicit method uses only information from previous time steps; an implicit method uses information from the current time step as well.

Q22. A finite difference formula must be consistent with the ODE being solved by it. What does this mean?

A22. The local discretisation error must go to zero as the step size goes to zero.

Q23. What is convergence?

A23. The global discretisation error goes to zero as the step size goes to zero.

Q24. What is Lax's equivalence theorem?

A24. Consistency + stability $\rightarrow$ convergence.

Q25. What test problem is used to generate stability diagrams?

A25.

$$\frac{dx}{dt} = \lambda x \quad \text{with} \quad \lambda \in \mathbb{C}.$$

Q26. What are unconditional and conditional stability?

A26. An algorithm/numerical method is unconditionally stable if it remains stable for any choice of stepwise. Else it is conditionally stable.

Q27. What is a stiff problem?

A27. A problem with a region of rapid change compared to nearby regions. The problem involves (at least) two different time scales.

Q28. What should be thought about when solving a stiff problem numerically?

A28. Using a method that allows a large time step in regions of slow change and a small time step in regions of rapid change, without becoming unstable. You also need a means for determining when the time step needs to change which does not slow down solution too much.

Q29. What is a strictly diagonally dominant matrix?

A29. For a matrix A with components a_{ij}

$$|a_{ii}| > \sum_{j=1, j\neq i}^{N} |a_{ij}|.$$

Q30. For an $n \times n$ matrix system of equations, what does strict diagonal dominance of the matrix guarantee?

A30. Stable numerical solution.

Q31. What type of model is this

$$\dot{x} = \alpha x + \beta xy, \quad \dot{y} = \gamma y - \delta xy$$

if $\alpha, \beta, \gamma, \delta$ are all positive constants.

A31. A predator–prey model.

Q32. What type of model is this:

$$\dot{x} = \alpha x - \beta xy, \quad \dot{y} = \gamma y - \delta xy$$

if $\alpha, \beta, \gamma, \delta$ are all positive constants.

A32. A competition model.

Q33. For the model

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

what are the x and y nullclines?

A33. x-nullcline: solutions of $f(x, y) = 0$;
y-nullcline: solutions of $g(x, y) = 0$.

Q34. For the model

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y),$$

suppose that the x and y nullclines in part coincide (i.e. lie on top of one another). What does this signal?

A34. Along the portion of the curve/line where the nullclines coincide there is a continuous ray of steady states.

Q35. How do we examine the stability of steady states of a nonlinear 2D model?

A35. Linearise about the steady states. If the Hartman–Grobman theorem is satisfied (eigenvalues of the Jacobian matrix are all non zero) then the linear model tells us the local stability of the steady state.

Q36. What is required for asymptotic stability of a steady state?

A36. All eigenvalues of the Jacobian matrix have negative real part.

Q37. What does stability of a steady state mean physically?

A37. An unstable steady state would not be observable, except under very controlled conditions. Stable steady states are observable steady states.

Q38. What is a limit cycle?

A38. An isolated closed trajectory.