LECTURE 26

Suppose that $f\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$. Let $\mathscr{P} = \{\, a = x_0, x_1, \ldots, x_N = b \,\}$ be a partition of $[a,b]$. Then the restriction $f|_{[x_{i-1},x_i]}$ is differentiable on $[x_{i-1}, x_i]$ for $i = 1, \ldots, N$ and hence satisfies the hypotheses of the Mean Value Theorem. Therefore, for each $i = 1, \ldots, N$, there exists $c_i \in (x_{i-1}, x_i)$ such that

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$$

Therefore, summing from $i = 1$ to $i = N$, we find that

$$\sum_{i=1}^{N} (f(x_i) - f(x_{i-1})) = \sum_{i=1}^{N} f'(c_i)\Delta_i(x).$$

The sum on the left telescopes to $f(b) - f(a)$, in other words, all of the terms of the sum cancel, and we are left with $f(x_N) - f(x_0)$, i.e. $f(b) - f(a)$. Hence

$$\sum_{i=1}^{N} f'(c_i)\Delta_i(x) = f(b) - f(a).$$

Suppose that the derivative $f'\colon [a,b] \to \mathbb{R}$ is a bounded function on $[a,b]$. Then for each $i = 1, \ldots, N$ we have

$$m_i(f') \leq f'(c_i) \leq M_i(f').$$

Multliplying these inequalities by the positive number $\Delta_i(x)$ and summing from $i = 1$ to $i = N$, we find that

$$L(f', \mathscr{P}) \leq \sum_{i=1}^{N} f'(c_i)\Delta_i(x) \leq U(f', \mathscr{P}).$$

Hence

$$L(f', \mathscr{P}) \leq f(b) - f(a) \leq U(f', \mathscr{P}).$$

Suppose that f' is an integrable function on $[a,b]$. Since the partition $\mathscr{P}$ was arbitrary, we see that $f(b) - f(a)$ is an upper bound for the set $\{\, L(f', \mathscr{P}) \,\}$ and $f(b) - f(a)$ is a lower bound for the set of $\{\, U(f', \mathscr{P}) \,\}$. Therefore

$$L(f) \leq f(b) - f(a) \leq U(f).$$

Since f is integrable, $L(f) = U(f)$. Hence $\int_a^b f'(x)dx = f(b) - f(a)$.

Summarising this discussion, we have

Theorem 6.19 (The Fundamental Theorem of Calculus II): Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on $\mathbb{R}$. If the derivative $f'\colon [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

We usually rewrite this in the following form: suppose that $f\colon [a,b] \to \mathbb{R}$ is an integrable function, and that there exists a differentiable function $F\colon [a,b] \to \mathbb{R}$ with $F'(x) = f(x)$ for all $x \in [a,b]$. Then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

A function F satisfying $F' = f$ is called an *anti-derivative* for f. Not every function has an anti-derivative; for instance the function $f\colon [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \neq 1, \\ 1 & x = 1 \end{cases}$$

is integrable, but it does not have an anti-derivative. The reason that it does not have an anti-derivative is because of the following fact about derivatives: if $g\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, then the derivative $g'\colon [a,b] \to \mathbb{R}$ satisfies the Intermediate Value Property, in particular g' cannot have any jump discontinuities.

Remark: Two very good questions that you might ask are the following: are there functions $f\colon [a,b] \to \mathbb{R}$ which are differentiable on $[a,b]$ but f' is not bounded on $[a,b]$? and, are there functions $f\colon [a,b] \to \mathbb{R}$ which are differentiable on $[a,b]$, with a bounded derivative, but f' is not integrable on $[a,b]$? The answer to both questions turns out to be yes. Here is an example of a function $f\colon [-1,1] \to \mathbb{R}$ which is differentiable on $[-1,1]$, but whose derivative is unbounded:

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

It is easy to see that

$$f'(x) = \begin{cases} 2x \sin(1/x^2) - 2\cos(1/x^2)/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function f' takes arbitrarily large values near the origin (let $x = 1/\sqrt{2n\pi}$, where $n \in \mathbb{N}$). In particular, f' is not integrable on $[-1,1]$. It turns out that there are differentiable functions $f\colon [a,b] \to \mathbb{R}$ such that $f'\colon [a,b] \to \mathbb{R}$ is bounded but f' is not integrable on $[a,b]$. Such functions are much more difficult to describe; the most well-known one is a function called *Volterra's function*.

Theorem 6.20 (Fundamental Theorem of Calculus I): Let $f\colon [a,b] \to \mathbb{R}$ be integrable. Define a function $F\colon [a,b] \to \mathbb{R}$ by setting

$$F(x) = \int_a^x f(t)dt$$

for $x \in [a,b]$. Then

(i) F is continuous on $[a,b]$;

(ii) if f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0.

Proof: We prove statement (i). We prove that in fact F is uniformly continuous on $[a,b]$. Let $x, y \in [a,b]$. Then

$$F(x) - F(y) = \int_a^x f(t)dt - \int_a^y f(t)dt.$$

If $y \leq x$ then

$$\int_a^y f(t)dt + \int_y^x f(t)dt = \int_a^x f(t)dt$$

and hence

$$F(x) - F(y) = \int_y^x f(t)dt.$$

On the other hand, if $x < y$, then

$$\int_a^x f(t)dt + \int_x^y f(t)dt = \int_a^y f(t)dt$$

and hence

$$F(x) - F(y) = -\int_x^y f(t)dt = \int_y^x f(t)dt.$$

Therefore, for all $x, y \in [a, b]$, we have $F(x) - F(y) = \int_y^x f(t)dt$. Since f is integrable on $[a, b]$, f is bounded on $[a, b]$, hence there exists $C > 0$ such that $|f(t)| \leq C$ for all $t \in [a, b]$. Therefore $-C \leq f(t) \leq C$ for all $t \in [a, b]$. If $x \geq y$ then

$$-C|x-y| = -C(x-y) = \int_y^x (-C)dt \leq \int_y^x f(t)dt \leq \int_y^x Cdt = C(x-y) = C|x-y|$$

by the comparison property for integrals. If $x < y$ then

$$-C(y-x) = \int_x^y (-C)dt \leq \int_x^y f(t)dt \leq \int_x^y Cdt = C(y-x)$$

and hence

$$-C|x-y| = -C(y-x) \leq \int_y^x f(t)dt \leq C(y-x) = C|x-y|$$

Therefore, for all $x, y \in [a, b]$, we have

$$-C|x-y| \leq F(x) - F(y) \leq C|x-y|$$

and so

$$|F(x) - F(y)| \leq C|x-y|.$$

It is now easy to show that F is uniformly continuous on $[a, b]$: if $\epsilon > 0$ let $\delta = \epsilon/C$; then for all $x, y \in [a, b]$, if $|x-y| < \delta$ then $|F(x) - F(y)| \leq C|x-y| < \epsilon$. Since F is uniformly continuous on $[a, b]$, F is continuous on $[a, b]$.

We prove statement (ii). Let $x \in [a, b]$, $x \neq x_0$. Then

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x f(t)dt - f(x_0).$$

We have

$$f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x f(x_0)dt$$

since $\int_{x_0}^x f(x_0)dt = f(x_0)(x - x_0)$ if $x \geq x_0$ and $\int_{x_0}^x f(x_0)dt = -\int_x^{x_0} f(x_0)dt = -(x_0 - x)f(x_0) = f(x_0)(x - x_0)$ if $x < x_0$. Therefore

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x (f(t) - f(x_0))dt.$$

Let $\epsilon > 0$ and choose $\delta > 0$ so that $|t - x_0| < \delta \implies |f(t) - f(x_0)| < \epsilon/2$. Suppose that $|x - x_0| < \delta$. If t is between x and x_0 then we must have $|t - x_0| < \delta$ (if $x < t < x_0$ then $|x_0 - t| = x_0 - t < x_0 - x = |x - x_0|$ while if $x_0 < t < x$ then $|t - x_0| = t - x_0 < x - x_0 = |x - x_0|$) and so $-\epsilon/2 < f(t) - f(x_0) < \epsilon/2$. Therefore, if $|x - x_0| < \delta$ then

$$-\epsilon|x - x_0|/2 \leq \int_{x_0}^x (f(t) - f(x_0))dt \leq \epsilon|x - x_0|/2$$

by the same argument that was used in the proof of statement (i) above. Therefore,

$$\left| \int_{x_0}^{x} (f(t) - f(x_0))dt \right| \leq \epsilon |x - x_0|/2$$

and so

$$\left| \frac{F(x) - F(x_0)}{x - x_0} \right| = \frac{1}{|x - x_0|} \left| \int_{x_0}^{x} (f(t) - f(x_0))dt \right| \leq \epsilon/2 < \epsilon$$

if $|x - x_0| < \delta$. Therefore

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

and so F is differentiable at x_0 with $F'(x_0) = f(x_0)$. ■

Remark: In particular it follows that every continuous function $f\colon [a, b] \to \mathbb{R}$ has an anti-derivative.

Natural Logarithm and Exponential Functions

Definition 6.21: Define $\ln\colon (0, \infty) \to \mathbb{R}$ by the formula

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

for $x > 0$.

We make the following easy observations about the function $\ln(x)$:

- $\ln(1) = 0$ since $\int_1^1 \frac{1}{t} dt = 0$ by definition.
- if $x > 1$ then $\ln(x) > 0$ since $1/t \geq 1/x$ for $t \in [1, x]$ and hence $\int_1^x \frac{1}{t} dt \geq (x - 1)/x > 0$.
- if $0 < x < 1$ then $\ln(x) < 0$ since $1/t \geq 1/x$ for $t \in [x, 1]$ and hence $\int_x^1 \frac{1}{t} dt \geq (1 - x)/x$; therefore $\ln(x) = -\int_x^1 \frac{1}{t} dt \leq (x - 1)/x < 0$.
- by FTOC I, $\ln(x)$ is differentiable on $(0, \infty)$ with $\frac{d}{dx} \ln(x) = 1/x$.
- in particular, $\ln(x)$ is continuous on $(0, \infty)$. In fact, $\ln(x)$ is uniformly continuous on any interval of the form $[a, \infty)$ where $a > 0$.
- since $\frac{d}{dx} \ln(x) = \frac{1}{x} > 0$ for all x, it follows that $\ln(x)$ is strictly increasing on $(0, \infty)$.