Lecture 8: Evaluating moments, and generating functions

Concepts checklist

At the end of this lecture, you should be able to:

- *Perform calculations involving generating functions and KFDEs*, in particular with respect to simple CTMCs such as the Poisson process; and,
- *Evaluate the moments of simple CTMCs*, such as the Poisson process, via generating functions.

Let us consider evaluating moments of CTMCs, and return to the Poisson process.

Question 1: What is the expected number of events that happened by time t?

Let $N(t)$ be the random variable representing the number of events by time t. Then,

$$\begin{aligned}
\mathbb{E}[N(t)] &= \sum_{j=0}^{\infty} j \Pr(N(t) = j) \\
&= \sum_{j=0}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!} \\
&= e^{-\lambda t}\lambda t \sum_{j=1}^{\infty} \frac{(\lambda t)^{(j-1)}}{(j-1)!} \\
&= e^{-\lambda t}\lambda t \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \\
&= e^{-\lambda t}\lambda t e^{\lambda t} \\
&= \lambda t.
\end{aligned}$$

Question 2: What is the variance of the number of events that happened by time t?

$$\begin{aligned}
\mathrm{Var}(N(t)) &= \sum_{j=0}^{\infty} j^2 \Pr(N(t) = j) - (\mathbb{E}[N(t)])^2 \\
&= \sum_{j=0}^{\infty} j^2 \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \left(\sum_{j=0}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!}\right)^2 \\
&= \sum_{j=1}^{\infty} j^2 \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \left(\sum_{j=0}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!}\right)^2 \\
&= \sum_{j=1}^{\infty} j(j-1) \frac{e^{-\lambda t}(\lambda t)^j}{j!} + \sum_{j=1}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \left(\sum_{j=0}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!}\right)^2 \\
&= \sum_{j=2}^{\infty} j(j-1) \frac{e^{-\lambda t}(\lambda t)^j}{j!} + \sum_{j=1}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \left(\sum_{j=0}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!}\right)^2 \\
&= \sum_{j=2}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{(j-2)!} + \sum_{j=1}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \left(\sum_{j=0}^{\infty} j \frac{e^{-\lambda t}(\lambda t)^j}{j!}\right)^2 \\
&= e^{-\lambda t}(\lambda t)^2 \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} + \lambda t - (\lambda t)^2 \\
&= \lambda t,
\end{aligned}$$

so that the variance of the number of events by time t is identical to the mean λt.

Let us now consider evaluating the mean and variance of the Poisson process directly from the generating function.

First,

$$\begin{aligned}\mathbb{E}[N(t)] &= \sum_{j=0}^{\infty} j P_{0j}(t) \\ &= \sum_{j=0}^{\infty} \left(\left[\frac{\mathrm{d}}{\mathrm{d}z} z^j \right]_{z=1} P_{0j}(t) \right) \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{j=0}^{\infty} P_{0j}(t) z^j \right) \right]_{z=1} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}z} P(z,t) \right]_{z=1}.\end{aligned}$$

Hence, we can calculate the expected number of events from the generating function $P(z,t)$ by differentiating the function with respect to z and setting $z=1$.

Therefore, recalling $P(z,t) = e^{-(\lambda - \lambda z)t}$, the mean number of events is

$$\mathbb{E}[N(t)] = \left[\frac{\mathrm{d}}{\mathrm{d}z} e^{-(\lambda-\lambda z)t} \right]_{z=1} = \left[\lambda t e^{-(\lambda-\lambda z)t} \right]_{z=1} = \lambda t.$$

We can also calculate higher moments of $N(t)$ by taking higher order derivatives.

The variance of $N(t)$ is given by

$$\begin{aligned}\mathrm{Var}(N(t)) &= \sum_{j=0}^{\infty} j^2 P_{0j}(t) - \left(\sum_{j=0}^{\infty} j P_{0j}(t) \right)^2 \\ &= \sum_{j=0}^{\infty} \left(j(j-1) P_{0j}(t) + j P_{0j}(t) \right) - \left(\sum_{j=0}^{\infty} j P_{0j}(t) \right)^2.\end{aligned}$$

The second derivative of $P(z,t)$ with respect to z evaluated at $z=1$ is

$$\begin{aligned}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} \sum_{j=0}^{\infty} P_{0j}(t) z^j \right]_{z=1} &= \left[\sum_{j=0}^{\infty} P_{0j}(t) \frac{\mathrm{d}^2}{\mathrm{d}z^2} z^j \right]_{z=1} \\ &= \left[\sum_{j=0}^{\infty} P_{0j}(t) j(j-1) z^{j-2} \right]_{z=1} \\ &= \sum_{j=0}^{\infty} P_{0j}(t) j(j-1).\end{aligned}$$

Thus,

$$\sum_{j=0}^{\infty} \left(j(j-1) P_{0j}(t) + j P_{0j}(t) \right) = \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} P(z,t) \right]_{z=1} + \underbrace{\sum_{j=0}^{\infty} j P_{0j}(t)}_{\text{mean} = \lambda t}.$$

Therefore,

$$\mathrm{Var}(N(t)) = \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}P(z,t)\right]_{z=1} + \lambda t - (\lambda t)^2.$$

Substituting $P(z,t)$ for the Poisson process into the previous expression, we see that

$$\begin{aligned}
\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}P(z,t)\right]_{z=1} + \lambda t - (\lambda t)^2 &= \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}e^{-(\lambda-\lambda z)t}\right]_{z=1} + \lambda t - (\lambda t)^2 \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}z}\lambda t e^{-(\lambda-\lambda z)t}\right]_{z=1} + \lambda t - (\lambda t)^2 \\
&= \left[(\lambda t)^2 e^{-(\lambda-\lambda z)t}\right]_{z=1} + \lambda t - (\lambda t)^2,
\end{aligned}$$

and hence $\mathrm{Var}(N(t)) = \lambda t = \mathbb{E}[N(t)]$.

So far in this lecture, we avoided the need to determine the transition function from the generating function to evaluate the (first two) moments of $N(t)$ and instead worked directly with the explicit generating function. Can we take this back one step further, so that we do not need to evaluate the generating function explicitly, and still evaluate these moments?

Recall that

$$P(z,t) = \sum_{n=0}^{\infty} P_{0n}(t)z^n,$$

and

$$\mathbb{E}[N(t)] = \left[\frac{\mathrm{d}}{\mathrm{d}z}P(z,t)\right]_{z=1}.$$

Hence,

$$\begin{aligned}
\mathbb{E}[N(t)] &= \left[\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty} P_{0n}(t)z^n\right]_{z=1} \\
&= \sum_{n=0}^{\infty} nP_{0n}(t).
\end{aligned}$$

This is a very unsurprising equation! Now, let's differentiate with respect to time, t, to get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[N(t)] &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty} nP_{0n}(t) \\
&= \sum_{n=0}^{\infty} n\frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} \\
&= \sum_{n=0}^{\infty} \lambda P_{0n}(t) \qquad \text{(Substituting Poisson KFDEs)} \\
&= \lambda. \qquad \text{(Honest)}
\end{aligned}$$

Hence, $\mathbb{E}[N(t)] = \lambda t$.

Let us consider the variance. $\mathrm{Var}(N(t)) = \mathbb{E}(N(t)^2) - \mathbb{E}(N(t))^2$. We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[N(t)^2] &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty} n^2 P_{0n}(t) \\
&= \sum_{n=0}^{\infty} n^2 \frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} \\
&= \sum_{n=0}^{\infty}((n+1)^2 - n^2)\lambda P_{0n}(t) && \text{(Substituting Poisson KFDEs)} \\
&= \lambda \sum_{n=0}^{\infty}(2n+1)P_{0n}(t) \\
&= \lambda(2\mathbb{E}(N(t)) + 1) && \text{(Honest)} \\
&= 2\lambda^2 t + \lambda.
\end{aligned}
$$

Hence, $\mathbb{E}(N(t)^2) = \lambda^2 t^2 + \lambda t$. Therefore $\mathrm{Var}(N(t)) = \lambda^2 t^2 + \lambda t - (\lambda t)^2 = \lambda t$.
