Practical Asymptotics

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0 Outline/motivation

Often real-world modelling problems give rise to equations which have no exact solution. One option is to solve them numerically; this can be great if (for example) interest lies in modelling of phenomena in geometrically complicated regions. Another option is to use asymptotic methods. This involves rigourously identifying the important mechanisms at play in a particular problem and writing down (approximate) analytic expressions to describe the solution behaviour.

The key advantage of such an approach is that these expressions can provide insight that is not possible with numerical simulations; it is much easier see how a varying a parameter value changes a solution if we have a formula, rather than having to wait for a simulation to be rerun. In practice when asymptotics approaches are used they are presented alongside computational results, but are attractive for this extra layer of insight they provide.

In this course we will:

- develop a tool-kit of useful asymptotic techniques; and
- apply these techniques to real-world modelling problems.

The plan for the course is as follows:

1. **Introduction to asymptotics**
   Key concepts involved in asymptotic methods; example application to the solution of differential equations.
   (Bender & Orszag, Chap. 3; Bowen & Witelski, Chap. 6)
2. **Perturbation methods** Introduce this broad class of techniques; examples from fluid mechanics and other case studies.
   (Bender & Orszag, Chap. 7; Bowen & Witelski, Chaps. 6, 8 & 12)
3. **Boundary layer theory and asymptotic matching** Discuss techniques for solving phenomena that vary over a thin region, and how to incorporate them into the bigger picture; examples from fluid mechanics and mathematical biology.
   (Bender & Orszag, Chap. 9; Bowen & Witelski, Chaps. 7 & 12)
4. **Multiscale methods and homogenisation theory**
   Discuss techniques for solving problems that involve multiple space/time scales. Techniques to average ('homogenise') over small scale variation. Examples from solid mechanics and mathematical biology.
   (Bender & Orszag, Chap. 11; Bowen & Witelski, Chapters 9 & 10)

5. **Extension topics/more case studies** TBD: we can discuss more case studies or, subject to interest, look at extension topics. Possible topics include: asymptotic approximation of integrals; summation of divergent series; WKB theory; asymptotics beyond-all-orders.

In these notes, a grey box like this one indicates an example or further discussion of a topic.

References (none required, all helpful in their own way)

T. Witelski, M. Bowen, Methods of Mathematical Modelling: Continuous Systems and Differential Equations, Springer, 2015. (electronic version available from UoA library)

C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, Springer, 1999. (electronic version available from UoA library) M. Van Dyke, Perturbation Methods in Fluid Mechanics, The Parabolic Press, 1978.

E.J. Hinch, Pertrubation Methods, Cambridge University Press, 1991.

S.W. McCue, Preface to fourth Special Issue on Practical Asymptotics, 63:153–154, Journal of Engineering Mathematics, 2009.

W.R. Smith, Preface to the sixth special issue on "Practical Asymptotics", 102:1–2, Journal of Engineering Mathematics, 2017.

1 Introduction to asymptotics

1.1 A difficult problem

Let's find the real solution (say at $x = a$) to the following quintic (fifth-degree polynomial) equation

$$x^5 + x = 1. \tag{1.1}$$

We know a few things about this equation:

- the highest power is x^5 so there are 5 roots, some which are probably complex;
- there is no exact solution for these roots (that only works if the highest power is x^4).

It'd be straightforward to solve this numerically, but another approach is to consider a related problem namely

$$x^5 + \epsilon x = 1, \tag{1.2}$$

where the second term on the right-hand side is now multiplied by a parameter ϵ.

If this were a real modelling problem, this parameter might have a physical meaning (and we might know if it was small or large).

Let's denote the real root of this equation $x = a(\epsilon)$. This becomes the original equation when $\epsilon = 1$. We can view this as a 'perturbation' problem it features a parameter (usually denoted ϵ) and when ϵ is set to zero the problem is easily solvable. It's really the behaviour when $\epsilon \neq 0$ that's of interest.

Working: series solution to the quintic (1.2)

The process we went through was to convert an extremely difficult problem into a sequence of easy problems, then piece those together to get an approximate (and very informative) solution to the original problem. We'll extend these ideas to differential equations and more 'real-world' applications soon, but first need some new notation.

1.2 Notation: $\sim$ and $\ll$

Examples: ... but first some other notations

Let's introduce some new notation. Say we have two functions $f(x)$ and $g(x)$, then

$$\underbrace{f(x) \sim g(x),}_{\text{“}f(x)\text{ is asymptotic to }g(x)\ldots} \qquad \underbrace{x \to x_0,}_{\text{as }x\text{ goes to }x_0\text{”}} \tag{1.3}$$

which means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{1.4}$$

There are two parts to this notation: the expression with the $\sim$ and the associated limit. Strictly, both parts are required for this to make sense.

Examples: use of $\sim$

Let's introduce some more new notation. Say we have two function $f(x)$ and $g(x)$, then

$$\underbrace{f(x) \ll g(x),}_{\text{“}f(x)\text{ is much smaller than }g(x)} \qquad \underbrace{x \to x_0,}_{\text{as }x\text{ goes to }x_0\text{”}} \tag{1.5}$$

which means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{1.6}$$

This notation works in a similar way to $\sim$ in that a $\ll$ statement should always be associated with a limit.

Examples: use of $\ll$

Comment: switching between asymptotic and exact relations

Finally, note that asymptotic relations can be manipulated in a similar way to equations. We can add, subtract, divide, cross-multiply, differentiate, integrate and so on. This should be done with care. For example, sometimes it is not necessarily valid to exponentiate both sides of an asymptotic relation (we'll see an example when we look at differential equations).

1.3 Method of dominant balance

Let's look at the quintic problem (1.1) again. We artificially inserted an ϵ into that equation in front of the x term. What would have happened if we'd inserted it in front of the x^5? The problem is then

$$\epsilon x^5 + x = 1. \tag{1.7}$$

This is an example of a **singular perturbation** problem, since the behaviour of this equation is fundamentally different in the limit $\epsilon \to 0$.

Unpack this idea

We are going to apply the **method of dominant balance** to (1.7); this is a systematic way of analysing the behaviour of the equation in the limit $\epsilon \to 0$ and converting the equation to a simpler asymptotic relation.

Introduce method of dominant balance

The three possibilities are:

1. $x \sim 1$ as $\epsilon \to 0$ (neglect ϵx^5).

   Discuss this balance

2. $\epsilon x^5 \sim 1$ as $\epsilon \to 0$ (neglect x).

   Discuss this balance

3. $\epsilon x^5 \sim -x$ as $\epsilon \to 0$ (neglect 1).

   Discuss this balance

What did all that tell us?

This illustrates the power of asymptotic techniques! By making a few relatively simple arguments we were able to convert this difficult singularly perturbed quintic problem (which we would otherwise need to treat numerically) into a some very simple problems. Similar techniques can be applied to differential equations, where we will use the method of dominant balance on more complicated expressions.

1.4 Local behaviour of differential equations

Consider a linear second-order homogeneous differential equation in standard form:

$$y'' + a(x)y' + b(x)y = 0. \tag{1.8}$$

Let's say we're interested in the **local behaviour** of this equation near the point $x = x_0$ and want to construct an approximate solution. The form of solution depends on the behaviour of $a(x)$ and $b(x)$ near x_0:

ordinary point: ie. $a(x)$ and $b(x)$ analytic in neighbourhood of x_0

1. $x = x_0$ is a **ordinary point**: use a Taylor series expansion,

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

reg. singular point: $(x - x_0)a(x)$ and $(x - x_0)^2 b(x)$ are analytic in neighbourhood of x_0

2. $x = x_0$ is a **regular singular point**: use a Frobenius type expansion, for example

$$y = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

irreg. singular point: not one of the other two

3. $x = x_0$ is an **irregular singular point**: use asymptotics. We start by assuming the solution is of the form

$$y(x) = \mathrm{e}^{S(x)}, \tag{1.9}$$

   and then construct a solution by repeatedly using the method of dominant balance.

Examples of classifying points of DEs

Note: to look at the 'local' behaviour as $x \to \infty$, make a change of variables $x = 1/t$ and classify the point $t = 0$.

Let's do an example. Consider the differential equation

$$x^3 y'' = y. \tag{1.10}$$

Let's examine the local behaviour of this equation as $x \to 0$.

Solve (1.10) as $x \to 0$ with method of dominant balance.

To recap: we assumed a solution of the form (1.9). We then iteratively approximated $S(x)$ until we obtained an approximate solution that was asymptotic to the solution to the original DE. The iterative steps in approximating $S(x)$ were

1. Drop all terms which are negligible (e.g. as $x \to 0$, $x \to \infty$) and replace the $=$ sign with a $\sim$ to give an asymptotic relation.
2. Solve this simpler asymptotic relation. Verify that the solution is consistent with the assumptions made about the negligible terms in step 1.
3. Replace the asymptotic relation with an equation by introducing an arbitrary function (that is $\ll$ the stuff we already found), repeat procedure to find that function.

We stop the procedure when a 'controlling factor' is found, that is enough terms in the approximation to $S(x)$ that when we put them back into $y = e^{S(x)}$ adding additional terms to the approximation for $S(x)$ would have a negligible effect on y.

Taking the exponent of an asymptotic relation.

This technique also works for irregular singular points at ∞. Consider the following equation

$$y'' = xy. \tag{1.11}$$

This is the **Airy equation**, which is (among other things) used to describe rainbows. Let's examine its behaviour as $x \to \infty$.

Solve (1.11) as $x \to \infty$ with method of dominant balance.

Lurking in our solutions to these differential equations are two important, very subtle ideas in asymptotics: **subdominance** and **behaviour of asymptotic relations in the complex plane**. Let's discuss each of these in turn.

Example of subdominance/asymptotics in the complex plane

1.5 Some more notation: $\mathcal{O}$ and o

In the next section we will discuss some technical details on asymptotic series/expansions; but first we need one more piece of notation since to be more precise about ...

Why $\mathcal{O}$ why?

Let's introduce this new notation. Say we have two function $f(x)$ and $g(x)$, then we write that

$$f(x) = \mathcal{O}(g(x)), \quad \text{as } x \to x_0 \tag{1.12}$$

if it is the case that

$$|f(x)| \leq A|g(x)| \tag{1.13}$$

for some constant A.

Examples

Another, less common notation that we won't use but you should be aware of is 'little o'. Say we have two function $f(x)$ and $g(x)$, then we write

$$f(x) = o(g(x)), \quad \text{as } x \to x_0$$

if $f(x) \ll g(x)$.

Examples

1.6 Asymptotic sequences and series

Recall the following definitions. A series $\sum_{n=0}^{\infty} f_n(z)$ **converges** at some fixed value of z if for an arbitrary $\epsilon > 0$ it is possible to find a number $N_0(z, \epsilon)$ such that

$$\left| \sum_{n=M}^{N} f_n(z) \right| < \epsilon \quad \text{for all } M, N > N_0.$$

A series $\sum_{n=0}^{\infty} f_n(z)$ **converges to a function** $f(z)$ at some fixed value of z if for an arbitrary $\epsilon > 0$ it is possible to find a number $N_0(z, \epsilon)$ such that

$$\left| f(z) - \sum_{n=0}^{N} f_n(z) \right| < \epsilon \quad \text{for all } N > N_0.$$

Equivalently, we can think of series converging if its terms decay sufficiently fast as $n \to \infty$.

Consider the error function, which is defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt.$$

A tale of two erfs

This was an example of an asymptotic series. Let's say what we mean by that.

Say we have a sequence $\{f_n(\epsilon)\}$ this is an asymptotic sequence as $\epsilon \to 0$ if

$$f_{n+1}(\epsilon) \ll f_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

for all n.

Quick examples

Say we have a function $f(\epsilon)$, a series $\sum_{n=0}^{\infty} f_n(\epsilon)$ is said to be an asymptotic expansion (or approximation) to this function if

$$f(\epsilon) - \sum_{n=0}^{N} f_n(\epsilon) \ll f_N(\epsilon), \quad \text{as } \epsilon \to 0.$$

That is the remainder between the approximation and the function (for $\epsilon \to 0$) is smaller than the last included term. This can be written as (we have already been doing this)

$$f(\epsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

The most common version of this is an asymptotic power series in ϵ, namely

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n, \quad \text{as } \epsilon \to 0.$$

As per previous examples, things like powers of $\epsilon^{1/2}$ might also show up in these series.

An interesting property of asymptotic approximations is that a function may have a variety of different asymptotic approximations. Here are a couple of approximations for $\tan(\epsilon)$ as $\epsilon \to 0$:

$$\begin{aligned} \tan(\epsilon) &\sim \epsilon + \frac{\epsilon^3}{3} + \frac{2\epsilon^5}{15} + ... \\ &\sim \sin\epsilon + \frac{1}{2}(\sin\epsilon)^3 + \frac{3}{8}(\sin\epsilon)^5 + ... \end{aligned}$$

Much of the rest of this course will consist of developing expansions in the form form $f(x;\epsilon)$, that is they involve an independent variable x as well as a small parameter ϵ. The most general form of an expansion of this type (say for a solution to a differential equation) is

$$f(x;\epsilon) \sim \sum_{n=0}^{\infty} a_n(x)\delta_n(\epsilon), \quad \text{as } \epsilon \to 0.$$

The most common version of this is where the $\delta_n(\epsilon) = \epsilon^n$, that is

$$f(x;\epsilon) \sim \sum_{n=0}^{\infty} a_n(x)\epsilon^n, \quad \text{as } \epsilon \to 0.$$

This in not usually a problem, since in practice (for the kind of problems we'll look at in this course) it is usual to find one, two or (at most!) three terms in an expansion.

As we saw with the error function evaluating asymptotic series accurately is a bit of an art. Generally only a few terms are required, and care should be taken not to include too many terms if a series is divergent.

The trick is to find an **optimal truncation** of a series (stop adding terms to an approximation); typically this involves looking a the magnitude of the terms and noticing when they start increasing in magnitude. Such an approximation is called (rather excitingly) a **superasymptotic** representation.

There are also numerical methods which can improve the convergence of divergent series to the 'right' answer (in certain circumstances). Roughly, these involves alternative ways of summing the terms in a series - it turns out just adding them up just about the most inefficient way to do this! Two techniques are the Shanks tranformation and Padé summation (if interested see Bender and Orszag, Chapter 8).