STATS 3006 Mathematical Statistics III
Assignment 3
2018

Assignment 3 is due by 23:59 Tuesday 1^{st} May 2018.

Assignments are to be submitted as a single pdf file online on MyUni.

1. Suppose $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$ independently and let $Y = X_1 + X_2$. Use the method of moment generating functions to show that $Y \sim B(n_1 + n_2, p)$.

2. Suppose $X_1 \sim U(0, 1)$ and $X_2 \sim U(0, 1)$ independently and let $U = X_1/(X_1 + X_2)$. Use the method of regular transformations to show that U has PDF

$$f_U(u) = \begin{cases} \frac{1}{2(1-u)^2)} & \text{for } 0 < u \leq 1/2; \\ \frac{1}{2u^2} & \text{for } 1/2 < u < 1; \\ 0 & \text{otherwise.} \end{cases}$$

3. Suppose X_1, X_2 have joint PDF $f_X(x_1, x_2)$.

   (a) If $Y_1 = X_1$ and $Y_2 = -X_2$, show that the joint PDF of Y_1, Y_2 is

   $$f_Y(y_1, y_2) = f_X(y_1, -y_2).$$

   (b) Hence, show for $W = X_1 - X_2$ that

   $$f_W(w) = \int_{-\infty}^{\infty} f_X(x, x - w)dx.$$

   Hint: Express W in terms of Y_1 and Y_2.

   (c) Suppose $X_1 \sim \text{Exp}(\lambda)$ and $X_2 \sim \text{Exp}(\lambda)$ independently and let $W = X_1 - X_2$. Find the PDF of W.

4. Suppose $Z_1, Z_2 \sim N(0, 1)$ independently and let $X_1 = Z_1 + Z_2$ and $X_2 = Z_1 - Z_2$. Find the joint distribution of X_1, X_2.

5. (a) Suppose Σ is an $r \times r$ positive-definite symmetric matrix and let $\Sigma = E\Lambda E^T$ be the eigenvalue/eigenvector decomposition, where

   $$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r) \text{ with } \lambda_i > 0$$

   and E is an $r \times r$ orthogonal matrix, so that $E^T E = EE^T = I$. Define

   $$\Sigma^{-\frac{1}{2}} = E\Lambda^{-\frac{1}{2}}E^T$$

   where

   $$\Lambda^{-\frac{1}{2}} = \text{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \ldots, \frac{1}{\sqrt{\lambda_r}}\right).$$

   Show that $\Sigma^{-\frac{1}{2}}$ is symmetric and satisfies

   $$\Sigma^{-\frac{1}{2}}\Sigma\Sigma^{-\frac{1}{2}} = I.$$

(b) Suppose $\boldsymbol{Y} \sim N_r(\boldsymbol{\mu}, \Sigma)$ and let $\boldsymbol{Z} = \Sigma^{-\frac{1}{2}}(\boldsymbol{Y} - \boldsymbol{\mu})$. Find the distribution of $\boldsymbol{Z}$.

(c) Suppose $\boldsymbol{Y} \sim N_r(\boldsymbol{\mu}, \Sigma)$ and let

$$V = (\boldsymbol{Y} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{Y} - \boldsymbol{\mu}).$$

Show that $V \sim \chi^2_r$.
Hint: Express $V = \boldsymbol{Z}^T \boldsymbol{Z}$ for $\boldsymbol{Z}$ in (b).

Honours Questions

The following questions are compulsory for Honours and Masters students and may be attempted for bonus marks by all students.

7. Suppose X_1, X_2 are continuous random variables with joint PDF $f(x_1, x_2)$ and let $Y = X_1 X_2$. Obtain an expression for the PDF, $f_Y(y)$.
Hint: The construction is similar to that for the ratio of two random variables given in lectures.

8. Suppose $X_1 \sim U(0,1)$ and $X_2 \sim U(0,1)$ and let $Y = \sqrt{X_1} X_2$. Find the PDF, $f_Y(y)$ and perform a simulation in R to illustrate that your answer is correct.

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