Class Exercise 2: Applied Probability

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1. Consider the "Two Gamblers" example
   It corresponds to a random walk on $\mathcal{S} = \{0, 1, \ldots, N\}$, with transition probabilities: $p_{0,0} = p_{N,N} = 1$ and
   $$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$
   In lectures we obtained $u_i = P(X_n \text{ reaches state } 0 \text{ before state } N | X_0 = i), i \in \mathcal{S}$.

   Consider the probabilities $v_i = P(X_n \text{ reaches state } N \text{ before state } 0 | X_0 = i)$, $i \in \mathcal{S}$, and use the same general method for solving second order homogeneous difference equations to find the $v_i, i \in \mathcal{S}$. Find $u_i + v_i$ and explain its meaning.

   Solution From lectures:
   $$u_i = \begin{cases} \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^i}{\left(\frac{q}{p}\right)^N - 1} & \text{for } p \neq q \\ \frac{N-i}{N} & \text{for } p = q \end{cases}$$
   $$v_i = P(X_n \to N \text{ before } X_n \to 0 | X_0 = i)$$
   Let W be the event $X_n \to N$.
   Boundaries: $v_0 = 0$ and $v_N = 1$.

   $$\begin{aligned} v_i &= P(W|X_0 = i) \\ &= \sum_{j \in S} P(W|X_1 = j, X_0 = i) \text{ probability of winning all the money given you moved to state j} \\ &= \sum_{j \in S} P(W|X_1 = j) P(X_1 = j | X_0 = i) \text{ Markov property} \\ &= \sum_{j \in S} P(W|X_0 = j) P(X_1 = j | X_0 = i) \text{ Time homogeneity} \\ &= \sum_{j \in S} u_j P_{ij} \\ &= p u_{i+1} + q u_{i-1} \end{aligned}$$

   With boundary conditions above.
   Try solutions $v_i = w^i$: This gives
   $$\begin{aligned} w^i &= p w^{i+1} + q w^{i-1} \\ \implies & p w^2 - 2 + q = 0 \end{aligned}$$

   If $p \neq q$: Solutions are of the form:
   $$v_i = A_1 + A_2 \frac{q^i}{p^i}$$

Using the boundary conditions:

$$v_0 = 0 \implies A_1 + A_2 = 0 \implies A_1 = -A_2$$

$$v_N = 1 \implies A_1 + A_2 \frac{q^N}{p^N} = 1 \implies A_2 \frac{q^N}{p^N} - A_2 = 1$$

$$A_2 = \frac{1}{\frac{q^N}{p^N} - 1}$$

$$A_1 = \frac{-1}{\frac{q^N}{p^N} - 1}$$

This gives:

$$v_i = \frac{-1}{\frac{q^N}{p^N} - 1} + \frac{1}{\frac{q^N}{p^N} - 1} \frac{q^i}{p^i}$$

I.e.

$$v_i = \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1}$$

For $p = q$, repeated root $w_1 = w_2 = 1$

$$v_1 = (A_1 + A_2 i) w^i = A_1 + A_2 i$$

Using the boundaries:

$$u_0 = 0 \implies A_1 = 0$$

$$u_N = 1 \implies A_2 N = 1 \implies A_2 = \frac{1}{N}$$

$$u_i = \frac{i}{N}$$

I.e. for $0 \le i \le N$.

$$v_i = \begin{cases} \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1} & \text{for } p \neq q \\ \frac{i}{N} & \text{for } p = q \end{cases}$$

$$\begin{aligned} u_1 + v_i &= \begin{cases} \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^i}{\left(\frac{q}{p}\right)^N - 1} + \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1} & \text{for } p \neq q \\ \frac{i}{N} + \frac{N-i}{N} & p = q \end{cases} \\ &= \begin{cases} 1, & p \neq q \\ 1, & p = q \end{cases} = 1 \forall\, p, q \end{aligned}$$

This result means that *eventually* the game will certainly end. Regardless of the probabilities. **As required.**

2. Show from first principles (i.e. without using covariance), that for independent X and W:

   (a) $E(XW) = E(X)E(W)$

   Solution Let $p_x = P(X = x)$ and $p_w = P(W = w)$. This gives:

$$E(X) = \sum_x x p_x \quad \textit{and} \quad E(W) = \sum_w w p_w$$

$$
\begin{aligned}
E(XW) &= \sum_{x,w} xp_x wp_w \\
&= \sum_x \sum_w xp_x wp_w \\
&= \sum_x xp_x \times \sum_w wp_w \text{ due to independence} \\
&= E(X)E(W)
\end{aligned}
$$

As required.

(b) $Var(X+W) = Var(X) + Var(W)$
Solution

$$
\begin{aligned}
var(X) &= E\left[(X-\mu)^2\right] \\
Var(X+W) &= E((X+W-E(X+W))^2) \\
&= E((X+W)^2 - 2(X+W)E(X+W) + E(X+W)^2) \\
&= E(X^2+W^2+2XW-2(X+W)(E(X)+E(W)) + E(X+W)^2) \text{ using tute result} \\
&= E(X^2+W^2+2XW-2(XE(X)+WE(X)+XE(W)+WE(W)) + (E(X)+E(W))^2) \\
&= E(X^2+W^2+2XW-2(XE(X)+WE(X)+XE(W)+WE(W)) \\
&\quad + E(X)^2 + E(W)^2 + E(X)E(W)) \\
&= E(X^2 - 2XE(X) + E(X)^2 + W^2 - 2WE(W) + E(W)^2 \\
&\quad - 2XE(W) - 2WE(X) + E(X)E(W)) \\
&= E((X-E(X))^2) + E((W-E(W))^2) + E(-2XE(W) - 2WE(X) + E(X)E(W))) \\
&= E((X-E(X))^2) + E((W-E(W))^2) + 2(E(X)E(W) - E(X)E(W)) \\
&= var(X) + var(W) + 0
\end{aligned}
$$

As required.

3. Let Y be a RV with binomial distribution with parameters n and p. Recall the PMF is:

$$f_Y(k) = P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k},\ k = 0, 1, \ldots, n$$

Prove that as $n \to \infty$ and $p \to 0$, with $\lambda = np$, the binomial distribution converges to the Poisson distribution with parameter λ.s Use the identity

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Solution Rearrange $p = \frac{\lambda}{n}$ And recall poisson dist:

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$
\begin{aligned}
\lim_{n\to\infty} f_Y(k) &= \lim_{n\to\infty} \binom{n}{k} p^k(1-p)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} p^k(1-p)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k}(1-\frac{\lambda}{n})^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k}(1-\frac{\lambda}{n})^{n}(1-\frac{\lambda}{n})^{-k} \\
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} e^{-\lambda}(1) \\
&= e^{-\lambda} \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!}
\end{aligned}
$$

The fraction $\frac{n!}{n^k(n-k)!}$ can be written as $\frac{n(n-1)(n-2)...(n-k+1)}{n^k k!} = \frac{n^k + O(n^{k-1})}{n^k} = 1 + O(n^{-1}) \to 1$ as $n \to \infty$ This means:

$$
e^{-\lambda} \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} = e^{-\lambda} \lim_{n\to\infty} \lambda^k/k! = \frac{e^{-\lambda}\lambda^k}{k!}
$$

Which is the form of the poisson distribution. **As required.**

4. Create a visualisation of some aspect of piece of music given to your group for the group project. This can be as simple as a single figure, e.g., a time-series-like piano roll plot, a plot of the distribution of note values/lengths/intervals, an autocorrelation plot, a network, ... (you can get more creative if you wish!)
   Be sure to explain clearly what your visualisation shows, and use it to make a comment on your piece of music. All group members must submit different visualisations.

   Hint: try importing your MusicXML file into Matlab, converting into a MIDI-friendly Notematrix, and then exploring the various functions in the Matlab MIDI toolbox. The documentation for that package is very useful here.

   Solution Figure 1 shows the intervals generated in the lead part in Moose The Mooche. In this plot the x-labels have musical meanings: $P1$ represents a perfect unison, i.e. a repeated note, MI2 is a minor-second interval, MA2 is a major-second, and so on, with D5 being a Diminished fifth. These intervals represent the step from one note to another, only taking into account the distance between the two notes and the scale representation (major/minor/dim).

   From this plot we notice that the most common interval is the Major second. This is noticed when listening to the song, as there are a lot of trills over these 2 step intervals. **As required.**

```
load('D:\Documents\Uni\2018\App_Prob\Group_Project\musicxml_parser\output\musicxml.mat');
notes= all_songs.raw_merged_nmat;

lead = getmidich(notes,2);
figure
plotdist ( ivsizedist1 (lead ));
title ("Moose_the_Mooche_Lead_Intervals")
```

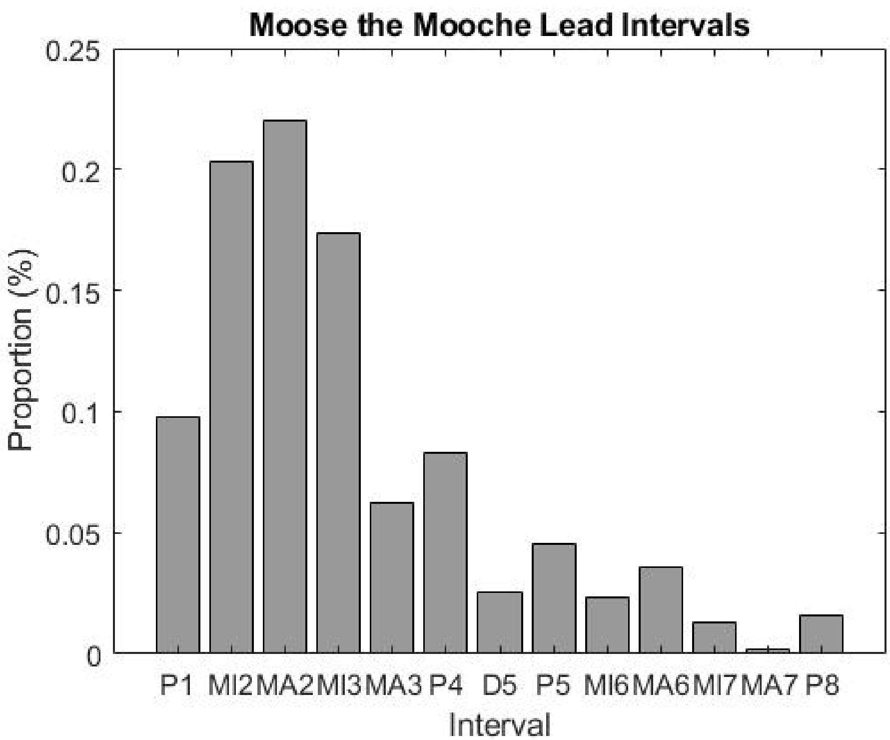

Figure 1: Intervals by the lead role in Moose the Mooche