LECTURE 31

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and let

$$S = \{ x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \}.$$

Our first aim is to try and understand what this set looks like. Recall that we have already observed that S is non-empty since $0 \in S$.

At the end of last lecture we proved (Proposition 7.9) that if $\sum_{n=0}^{\infty} a_n x_0^n$ converges for some $x_0 \in \mathbb{R}$, then the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for any $x \in \mathbb{R}$ with $|x| < |x_0|$. We can use this fact to get a better understanding of S.

To begin with, if S is not bounded above, then $S = \mathbb{R}$. To see this, we observe that under this assumption, for any $n \in \mathbb{N}$, there exists $x_n \in S$ such that $x_n > n$. Suppose that $|x| < x_n$. Since $x_n \in S$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely (Proposition 7.9), and hence converges. Therefore $x \in S$. Therefore $(-x_n, x_n) \subset S$ and so $(-n, n) \subset S$. Since this is true for all $n \in \mathbb{N}$, we must have $S = \mathbb{R}$.

Now suppose that S is bounded above. Let $R = \sup(S)$. Note that $R \geq 0$. The series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$. To see this, suppose that $|x| < R$. There exists $y \in S$ such that $|x| < y < R$, since otherwise $|x|$ would be an upper bound for S smaller than R. Therefore $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely by Proposition 7.9 again. Suppose that $|x| > R$. If $\sum_{n=0}^{\infty} a_n x^n$ converges, then $\sum_{n=0}^{\infty} a_n y^n$ converges absolutely, and hence converges, if $R < y < |x|$. But this is a contradiction, since R is an upper bound for S.

Suppose that $R' \geq 0$ is another real number such that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R'$ and diverges if $|x| > R'$. Therefore R' is an upper bound for S (since $x \in S$ implies $|x| \leq R'$). Therefore $R \leq R'$. If $R < R'$ then there exists $x \in \mathbb{R}$ with $R < x < R'$ and hence $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. But then $x \in S$, contradiction. Therefore $R = R'$. Therefore R is uniquely determined by the requirement that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$.

It follows that

$$(-R, R) \subset S \subset [-R, R]$$

and so S must be an interval. We call S the *interval of convergence* of the power series $\sum_{n=0}^{\infty} a_n x^n$.

Example: Here are some power series along with their corresponding intervals of convergence:

1. $\sum_{n=0}^{\infty} x^n$, $S = (-1, 1)$
2. $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$, $S = [-1, 1)$
3. $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$, $S = [-1, 1]$

We look at the second example in some detail. To find the radius of convergence, we want to find the unique real number $R \geq 0$ such that $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ converges absolutely if $|x| < R$ and diverges if $|x| > R$. Suppose $x \neq 0$. By the Ratio Test, $\sum_{n=0}^{\infty} \left| \frac{x^n}{n+1} \right|$ converges if $|x| < 1$ and diverges if $|x| > 1$. Therefore the series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ converges absolutely if $|x| < 1$ and diverges[1] if $|x| > 1$. To determine the interval of convergence we need to examine the cases $x = \pm 1$. If $x = 1$ then the series becomes the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which diverges. If $x = -1$ then the series becomes the alternating harmonic series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, which converges. Hence the interval of convergence is $[-1, 1)$. The other two cases are similar.

In the previous example we introduced the terminology 'radius of convergence'. We make this precise in the following definition.

Definition 7.10: Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the unique 'number' $R \in [0, \infty]$, such that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$ is called the *radius of convergence*.

Proposition 7.11: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and suppose $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, with $L \in [0, \infty]$. Then $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 1/L$.

Implicit in the statement of Proposition 7.11 is the understanding that $L = \infty$ means that the sequence $\left|\frac{a_{n+1}}{a_n}\right|$ diverges to ∞.

Proof: Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < \infty$. If $x \neq 0$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}|x|}{a_n}\right| = L|x|$. Therefore, by the Ratio Test, $\sum_{n=0}^{\infty} |a_n x^n|$ converges if $L|x| < 1$ and diverges if $L|x| > 1$. The series $\sum_{n=0}^{\infty} a_n x^n$ must diverge if $L|x| > 1$, for if it converged, then the series $\sum_{n=0}^{\infty} a_n y^n$ would have to converge absolutely for some y with $1 < Ly < L|x|$, contradicting the Ratio Test. Hence the radius of convergence of the series is $1/L$. The case where $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ is left as an exercise. ■

Example: We look at how the previous proposition plays out in two examples.

First, consider the geometric series $\sum_{n=0}^{\infty} x^n$. This is of the form $\sum_{n=0}^{\infty} a_n x^n$ with $a_n = 1$ for all n. Therefore $a_{n+1}/a_n = 1$ for all n. Proposition 7.11 implies that the radius of convergence is $R = 1$, which is consistent with our previous experience (note that the interval of convergence is $(-1, 1)$ in this example).

Secondly, consider the power series $\sum_{n=0}^{\infty} n! x^n$. This is of the form $\sum_{n=0}^{\infty} a_n x^n$ with $a_n = n!$ for all n. Therefore $a_{n+1}/a_n = n + 1 \to \infty$. Proposition 7.11 implies that the radius of convergence is $R = 0$.

Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then we can define a function $f\colon (-R, R) \to \mathbb{R}$ whose value at x with $|x| < R$ is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

There are several natural questions that we may ask about this function:

1. Is f continuous?

2. Is f differentiable? Moreover, if f is differentiable, is the derivative f' given by differentiating the power series 'term-by-term', i.e. is $f'(x) = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n$?

3. Is f integrable?

The value of the function f at x with $|x| < R$ is given by $f(x) = \lim_{N\to\infty} s_N(x)$, where

$$s_N(x) = a_0 + a_1 x + \cdots + a_N x^N.$$

[1] The reason that the series diverges if $|x| > 1$ is because, if it converged for some x with $|x| > 1$ then the series $\sum_{n=0}^{\infty} \frac{y^n}{n+1}$ would have to converge absolutely for y with $1 < y < |x|$, contradicting the Ratio Test.

Thus f is the limit of a sequence of functions. Therefore we embark on a study of sequences of functions.

Sequences of functions

Consider the following general situation: suppose $S \subset \mathbb{R}$ and for every $n \in \mathbb{N}$ we have a function $f_n \colon S \to \mathbb{R}$, so that we have a sequence of functions

$$f_1, f_2, f_3, \ldots$$

Suppose that $f \colon S \to \mathbb{R}$ is a function. We want to think about what it means for the sequence of functions $(f_n)_{n=1}^{\infty}$ to converge to the function f. There are two possibilities.

Definition 8.1: We say that the sequence $(f_n)_{n=1}^{\infty}$ *converges pointwise* to f on S if for every $x \in S$ the sequence of real numbers $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$, i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$.

Definition 8.2: We say that the sequence $(f_n)_{n=1}^{\infty}$ *converges uniformnly* to f on S if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$.

Notes: we make the following observations.

1. A necessary condition for f_n to converge uniformly to f on S is that the sequence of functions $|f_n - f|$ is eventually bounded on S, i.e. there is a an $N \in \mathbb{N}$ such that for all $n \geq N$, the function $|f_n - f|$ is bounded (for example, taking $\epsilon = 1$ in Definition 8.2 above shows that there is an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| \leq 1$ for all $x \in S$ if $n \geq N$. Therefore, the sequence (M_n), defined by $M_n := \sup_{x\in S} |f_n(x) - f(x)|$ is well-defined. Note that $f_n \to f$ uniformly on S if and only if $M_n \to 0$.

To see the equivalence of these statements, suppose firstly that $f_n \to f$ uniformly on S. We prove that $M_n \to 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in S$. Hence if $n \geq N$ then $M_n = \sup_{x\in S} |f_n(x) - f(x)| \leq \epsilon/2 < \epsilon$. Therefore $M_n \to 0$. The proof of the converse statement is left as an exercise.

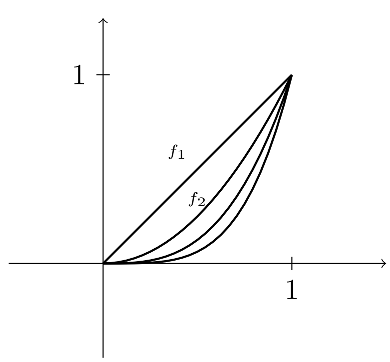
2. If $f_n \to f$ uniformly on S then $f_n \to f$ pointwise on S.

3. If $T \subset S$ and $f_n \to f$ uniformly on S, then $f_n \to f$ uniformly on T.

Example: Consider the sequence of functions $(f_n)_{n=1}^{\infty}$ on $[0,1]$ defined by $f_n(x) = x^n$. If $0 \leq x < 1$ then $\lim_{n\to\infty} x^n = 0$, while if $x = 1$ then $\lim_{n\to\infty} x^n = 1$. Therefore $f_n \to f$ pointwise on $[0,1]$ where $f \colon [0,1] \to \mathbb{R}$ is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

The graphs of the first few functions in this sequence are pictured below



However, $f_n \nrightarrow f$ uniformly on $[0,1]$ (i.e. the convergence is not uniform on $[0,1]$). To see this, note that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n - 0| = \sup_{x\in[0,1)} x^n = 1$$

Therefore $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$ is the constant sequence (1) and hence does not converge to 0. Therefore (f_n) does not converge uniformly to f on $[0,1]$.