LECTURE 34

Series of functions

Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions $f_n\colon S \to \mathbb{R}$. Just as we can consider the series of real numbers $\sum_{n=1}^{\infty} a_n$ associated to a sequence of real numbers $(a_n)_{n=1}^{\infty}$, we can also consider the *series* of functions $\sum_{n=1}^{\infty} f_n$ associated to the sequence $(f_n)_{n=1}^{\infty}$.

Example: Let $(f_n)_{n=1}^{\infty}$ be the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x^n}{1 + x^{2n}}$$

for $x \in \mathbb{R}$. Then we can form the series of functions $\sum_{n=1}^{\infty} f_n$. We frequently denote such series by

$$\sum_{n=1}^{\infty} \frac{x^n}{1 + x^{2n}}$$

Note that there is a danger of confusion here: are we talking about a series of functions (in which case x is a variable), or are we talking about a series of real numbers (in which case x is a fixed real number)? The context of the problem will differentiate between the two situations.

Example: Power series are also examples of series of functions. Remember that a power series is an expression of the form $\sum_{n=0}^{\infty} a_n x^n$, where $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers. To any such sequence $(a_n)_{n=0}^{\infty}$ there is a corresponding sequence of functions $(f_n)_{n=0}^{\infty}$ with $f_n\colon \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = a_n x^n$.

Note: Just as for series of real numbers, sometimes it will be convenient to consider series of functions of the form $\sum_{n=0}^{\infty} f_n$ or $\sum_{n=k}^{\infty} f_n$.

In the case of series of real numbers, we declared that the series converges if and only if the sequence of partial sums (an ordinary sequence of real numbers) converges.

Given a series of functions $\sum_{n=1}^{\infty} f_n$ we can certainly form the sequence of partial sums $(s_N)_{N=1}^{\infty}$ where $s_N\colon S \to \mathbb{R}$ is defined by

$$s_N(x) = f_1(x) + f_2(x) + \cdots + f_N(x)$$

for $x \in S$. The sequence $(s_N)_{N=1}^{\infty}$ is a sequence of functions, and therefore there are two types of convergence associated to it, pointwise convergence and uniform convergence. We therefore have two types of convergence of the series of functions $\sum_{n=1}^{\infty} f_n$.

Definition: We say the series of functions $\sum_{n=1}^{\infty} f_n$ *converges pointwise* on S if the sequence of partial sums $(s_N)_{N=1}^{\infty}$ converges pointwise on S.

Definition: We say the series of functions $\sum_{n=1}^{\infty} f_n$ *converges uniformly* on S if the sequence of partial sums $(s_N)_{N=1}^{\infty}$ converges uniformly on S.

Note: If a series $\sum_{n=1}^{\infty} f_n$ converges uniformly on S, then it converges pointwise on S.

If a series of functions $\sum_{n=1}^{\infty} f_n$ converges pointwise (and hence if it converges uniformly) on a set $S \subset \mathbb{R}$, then we can define a function $f\colon S \to \mathbb{R}$ (the *sum* of the series) by

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Thus the value of the function f at x is the sum of the series of real numbers $\sum_{n=1}^{\infty} f_n(x)$ (this series converges because the sequence of partial sums $(s_N)_{N=1}^{\infty}$ converges pointwise on S, hence the sequence of real numbers $(s_N(x))_{N=1}^{\infty}$ converges).

We would like to have a method for determining when a given series of functions converges uniformly.

Weierstrass M-test

Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions $f_n\colon S \to \mathbb{R}$. If there exists a sequence of real numbers $(M_n)_{n=1}^{\infty}$ such that

1. for every $n \in \mathbb{N}$, $|f_n(x)| \leq M_n$ for all $x \in S$, and
2. the series (of real numbers) $\sum_{n=1}^{\infty} M_n$ converges,

then for every $x \in S$, the series (of real numbers)

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely, and the series (of functions)

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on S.

Before we give a proof of this statement, let's discuss an example.

Example: Discuss the convergence of the series of functions $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$. This is a series of functions of the form $\sum_{n=1}^{\infty} f_n$, where for each $n \in \mathbb{N}$, $f_n\colon \mathbb{R} \to \mathbb{R}$ is the function defined by $f_n(x) = 1/(x^2+n^2)$. Since $n^2 \leq x^2+n^2$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, we have

$$|f_n(x)| = \left|\frac{1}{x^2+n^2}\right| = \frac{1}{x^2+n^2} \leq \frac{1}{n^2}.$$

Let $(M_n)_{n=1}^{\infty}$ be the sequence of real numbers defined by $M_n = 1/n^2$. Then for every $n \in \mathbb{N}$, we have $|f_n(x)| \leq M_n$ for all $x \in \mathbb{R}$ and the series $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/n^2$ converges. Therefore, by the Weierstrass M-test, the series of functions $\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly on $\mathbb{R}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ whose value at $x \in \mathbb{R}$ is the sum of the series of real numbers

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$$

is the sum of the series of functions $\sum_{n=1}^{\infty} f_n$. Observe that this function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. This is because each function f_n is continuous on $\mathbb{R}$ and hence $s_N = f_1 + \cdots + f_N$ is continuous on $\mathbb{R}$ (since it is a sum of continuous functions); since $s_N \to f$ uniformly on $\mathbb{R}$ it follows that f is continuous on $\mathbb{R}$.

Note: When you are applying the M-test, it is absolutely critical that the sequence (M_n) does not depend on x in any way — (M_n) must be a sequence of real numbers, not a sequence of functions.

Let's look at the proof of the Weierstrass M-test.

Proof: There are two things that we must do. First, we must show that for every $x \in S$, the series of real numbers $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Therefore, we must show that for every $x \in S$, the series of real numbers $\sum_{n=1}^{\infty} |f_n(x)|$ converges. This is a series of non-negative terms; since the series $\sum_{n=1}^{\infty} M_n$ converges and we have $|f_n(x)| \leq M_n$ for every $n \in \mathbb{N}$, we see that the series $\sum_{n=1}^{\infty} |f_n(x)|$ converges by the Comparison Test.

Next, we need to show that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on S. For this, we will use Theorem 8.9 from last lecture: we will show that the sequence of partial sums $(s_N)_{N=1}^{\infty}$ is a uniformly Cauchy sequence of functions $s_N \colon S \to \mathbb{R}$.

Let $\epsilon > 0$. Suppose that $M_1 > M_2$. For any $x \in S$ we have

$$\begin{aligned} |s_{M_1}(x) - s_{M_2}(x)| &= \left| \sum_{n=1}^{M_1} f_n(x) - \sum_{n=1}^{M_2} f_n(x) \right| \\ &= \left| \sum_{n=M_2+1}^{M_1} f_n(x) \right| \\ &\leq \sum_{n=M_2+1}^{M_1} |f_n(x)| \\ &\leq \sum_{n=M_2+1}^{M_1} M_n \end{aligned}$$

by the Triangle Inequality, and using the fact that for each $n \in \mathbb{N}$, $|f_n(x)| \leq M_n$ for all $x \in S$. Since the series $\sum_{n=1}^{\infty} M_n$ converges, its sequence of partial sums is a Cauchy sequence. Hence, there exists $N \in \mathbb{N}$ such that if $M_1 > M_2 \geq N$ then

$$\sum_{n=M_2+1}^{M_1} M_n < \epsilon.$$

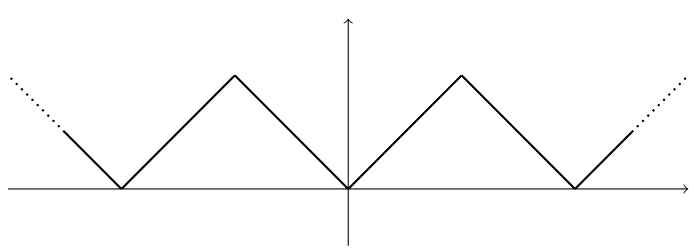
Suppose then that $M_1 > M_2 \geq N$. Then for every $x \in S$ we have

$$|s_{M_1}(x) - s_{M_2}(x)| \leq \sum_{n=M_2+1}^{M_1} M_n < \epsilon.$$

Therefore $(s_M)_{M=1}^{\infty}$ is a uniformly Cauchy sequence of functions on S. Hence $\sum_{n=1}^{\infty} f_n$ converges uniformly on S. ■

An every-where continuous function that is no-where differentiable

Here is an example of a function $T \colon \mathbb{R} \to \mathbb{R}$ which is continuous every-where, but is differentiable no-where (this function is called *Takagi's function*). To begin with, let $f \colon [-1, 1] \to \mathbb{R}$ be defined by $f(x) = |x|$. This is a simple example of a function which is not differentiable at $x = 0$. Extend f to a function $f \colon \mathbb{R} \to \mathbb{R}$, by requiring that f is periodic with period 2, i.e. $f(x + 2) = f(x)$ for all $x \in \mathbb{R}$. Thus the function f has graph



Notice that the function f is now non-differentiable at every integer. Notice also that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. Define a sequence $(f_n)_{n=0}^{\infty}$ of functions $f_n \colon \mathbb{R}$ by setting

$$f_n(x) = \frac{1}{2^n} f(2^n x).$$

Thus each function f_n is periodic with period 2^{1-n} and $|f_n(x)| \leq M_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$. Notice that each function f_n is continuous, but is getting progressively less differentiable as n increases. Let

$$T(x) = \sum_{n=0}^{\infty} f_n(x).$$

The Weierstrass M-test shows that $T \colon \mathbb{R} \to \mathbb{R}$ is well-defined and that the series $\sum_{n=0}^{\infty} f_n$ converges uniformly to T. Since each function f_n is continuous, it follows that T must be continuous (since the sequence of partial sums $(s_N)_{N=0}^{\infty}$ is a sequence of continuous functions such that $s_N \to T$ uniformly, T must be continuous).

But you can show that T is not differentiable at x for every $x \in \mathbb{R}$ (see separate hand-out for the proof of this).

Power series again

Suppose $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, where $R > 0$.

Theorem: the series of functions $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any $0 \leq r < R$.

Proof: Suppose that $0 \leq r < R$. Choose x_0 with $r < x_0 < R$. The series $\sum_{n=0}^{\infty} a_n x_0^n$ converges, and hence $a_n x_0^n \to 0$. Therefore the sequence $(a_n x_0^n)$ is bounded since it is convergent. Therefore there exists $M > 0$ such that $|a_n x_0^n| \leq M$ for all n.

Suppose that $|x| \leq r$. Then

$$\begin{aligned}
|a_n x^n| &= |a_n| \cdot |x|^n \\
&= |a_n| \cdot |x_0|^n \cdot \frac{|x|^n}{|x_0|^n} \\
&= |a_n x_0^n| \cdot s^n \quad \text{(where } s = |x|/|x_0|\text{)} \\
&\leq M \cdot s^n
\end{aligned}$$

Let $M_n = M \cdot s^n$ for every n. Then, for any $|x| \leq r$, we have $|a_n x^n| \leq M_n$. Moreover, since $|x| \leq r < x_0$, we see that $0 \leq s < 1$. Hence the series $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} M \cdot s^n$ converges (it is a geometric series with $0 \leq s < 1$). We can apply the Weierstrass M-test to the series $\sum_{n=0}^{\infty} f_n$ with $f_n(x) = a_n x^n$ and deduce that the series $\sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$. $\blacksquare$