Random Processes Assignment 1

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1. $M/M/1$ queue from lecture 3. $\mathcal{X} = (X(t), t \geq 0)$ is the queue length at time t, inclusive of the person being served.
   Assume there are arrivals to the queue according to a Poisson process with rate λ, and service occurs as a Poisson process with rate μ when there is at least one customer. State space $\mathcal{S} = \{0, 1, 2, \ldots\}$ where each $i \in S$ means there are i people in the system.
   Transition rates are
$$q_{ij} = \begin{cases} \lambda & j = i+1 \\ \mu & j = i-1 \\ 0 & j \neq i, i+1, i-1 \end{cases}$$

   (a) Show that in state $i \in S \backslash \{0\}$ the sojourn (waiting) time is exponentially-distributed with rate $\lambda + \mu$
   Solution Denote a new arrival as A and a customer leaving as L.
   $A \sim Exp(\lambda)$ and $L \sim Exp(\mu)$ and let $M = \min\{A, L\}$
$$\begin{aligned} P(M > t) &= P(\min\{A, L\} > t) \\ &= P(A > t \cap L > t) \\ &= P(A > t)P(L > t) \\ &= (1 - P(A \leq t))(1 - P(L \leq t)) \\ &= (1 - (1 - e^{-\lambda t}))(1 - (1 - e^{-\mu t})) \\ &= e^{-\lambda t} e^{-\mu t} \\ &= e^{-t(\lambda + \mu)} \end{aligned}$$
   Which is exponential with rate $\lambda + \mu$ **As required.**

   (b) Show that in state $i \in S \backslash \{0\}$ the probability of arrival at the time of an event is independent of the timing of the event and is equal to $\frac{\lambda}{\lambda + \mu}$
   Solution First time independence:
$$\begin{aligned} P(A > t + s | A > s) &= \frac{P(A > t + s \cap A > t}{P(A > s)} \\ &= \frac{P(A > t + s)}{P(A > s)} \\ &= \frac{1 - P(A \leq t + s)}{1 - P(A \leq s)} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\ &= e^{-\lambda t - \lambda s + \lambda s} \\ &= e^{-\lambda t} \end{aligned}$$

Hence time independence holds. Now The probability of an arrival at any particular time:

$$\begin{aligned}
P(\text{Arrival when an event occurs}) &= P(A < L | M \in [t, t+dt]) \\
&= \frac{P(A \in [t, t+dt] \cap L > t)}{P(M \in [t, t+dt])} \\
&= \frac{P(A \in [t, t+dt])P(L > t)}{P(M \in [t, t+dt])} \\
&= \frac{\lambda e^{-\lambda t} e^{-\mu t}}{(\lambda + \mu)e^{-t(\lambda+\mu)}} \\
&= \frac{\lambda e^{-\lambda t - \mu t}}{(\lambda + \mu)e^{-t(\lambda+\mu)}} \\
&= \frac{\lambda}{\lambda + \mu}
\end{aligned}$$

As required.

2. Two machines, machine $i \in \{1, 2\}$ functions for an exponentially distributed time with mean $1/\gamma_i$ before it breaks down. Repair time for machine i is exponentially distributed with mean $1/\beta_i$. Only one can be repaired at a time
Assume that the machines are repaired in the order in which they fail.
Construct a CTMC to model the system where $X(t)$ represents the status of the machines at time $t \geq 0$:

   (a) Write down the state space for the CTMC
   Solution Since it is necessary to keep track of the machines, the state space will be the pair $\{M_1, M_2\}$ where M_i denotes the status of Machine i. The state space will be

   $$X(t) = \{W_{1,2}, W_2B_1, W_1B_2, B_{1,2}, B_{2,1}\}$$

   Where $W_{i,j}$ means that M_i and M_j are working. and B_{ij} indicates that i and j are broken and that i is being repaired first. **As required.**

   (b) Draw a state transition diagram corresponding to the CTMC
   Solution This is shown in figure 1. I have used $1/B_i$ in place of $1/\beta_i$ and $1/g_i$ in place of $1/\gamma_i$, as I do not have the Greek Locale. **As required.**

   (c) Specify the generator of the CTMC
   Solution The generator is:

   $$Q = \begin{pmatrix}
-\frac{1}{\gamma_1} - \frac{1}{\gamma_2}, & \frac{1}{\gamma_1}, & \frac{1}{\gamma_2}, & 0, & 0 \\
\frac{1}{\beta_1}, & -\frac{1}{\beta_1} - \frac{1}{\gamma_2}, & 0, & \frac{1}{\gamma_2}, & 0 \\
\frac{1}{\beta_2}, & 0, & -\frac{1}{\beta_1} - \frac{1}{\gamma_1}, & 0, & \frac{1}{\gamma_1} \\
0, & 0, & \frac{1}{\beta_2}, & -\frac{1}{\beta_2} & 0 \\
0, & \frac{1}{\beta_1}, & 0, & 0, & -\frac{1}{\beta_1}
\end{pmatrix}$$

   As required.

3. Assume you are modelling an arrival process using a Poisson process, where the rate of arrivals is $\lambda = 24$ per day

   (a) Calculate the probability of seeing 2 arrivals in the first hour, and 5 arrivals in total during the second and third hours.
   Solution Convert into hours $\lambda = 1$ arrival(s) per hour. Let $X(t)$ be the process $X(t) \sim Po(1)$

   $$f_X(t) = e^{-1}\frac{1^t}{t!}$$

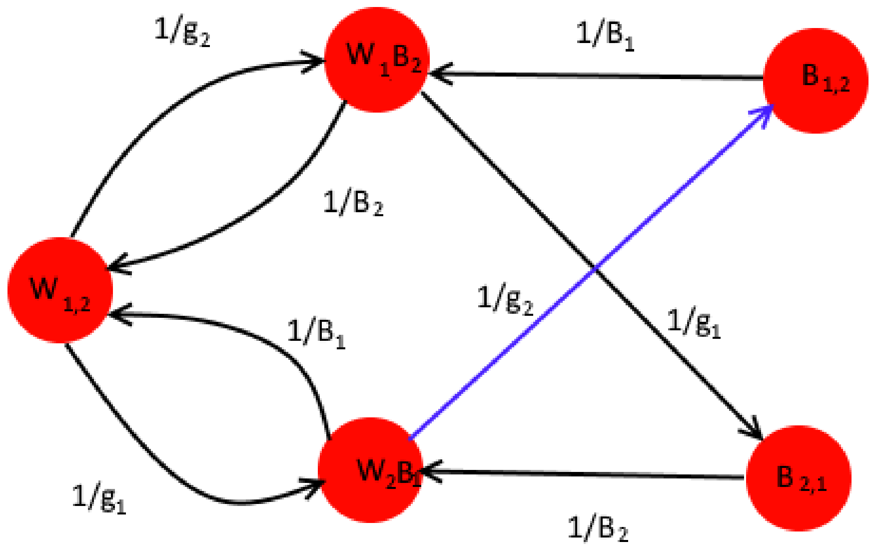

Figure 1: Transition State Diagram

$$
\begin{aligned}
P(\text{ 2 arrivals in 1 hour and 5 in the next two hours}) &= P(\text{ 2 arrivals in 1 hour})P(\text{5 in the next two hours}) \\
&= P(x=2)P(x_2=5) \\
&= e^{-1}\frac{1^2}{2!} \times e^{-2}\frac{2^5}{5!} \\
&= \frac{2e^{-3}}{15} \approx 0.0066
\end{aligned}
$$

As required.

4. Consider Example 1 of Lecture 2 - revisited in lecture 4

   (a) Evaluate $P_{01}(200)$, the probability of having only 1 working printer 200 days after having both printers working, given the transition function in the lecture notes

   Solution Plugging it into `Matlab` Gives

   ```
   P = 1/25 * (6 - 4*exp(-t/6) - 2*exp(-t/12))
   P = 0.2400
   ```

   As required.

   (b) What happens to the transition function as $t \to \infty$? Compare, appropriately, to your answer in (a).

   Solution As $t \to \infty$ We get

   $$
   \begin{aligned}
   P &= \frac{1}{25}(6 - 4e^{-\infty} - 2e^{-\infty}) \\
   &= 6/25 = 0.24
   \end{aligned}
   $$

   Which again is the same as (a). **As required.**

   (c) Evaluate $\exp(Q * 200)$ where exp is the matrix exponential (in `Matlab` use `expm`). Compare your answer to (b)

   Solution Running the code:

```
Q = [-2/60, 2/60,0;1/10,-7/60,1/60;0,1/10,-1/10];
expQ= expm(Q*200)
```

Gives

```
expQ =

    0.7200    0.2400    0.0400
    0.7200    0.2400    0.0400
    0.7200    0.2400    0.0400
```

So $expQ_{01} = 0.24$, implying that starting in state 0, after 200 steps we have a 0.24 probability to be in state 1. This is exactly the same as the answer found in (a). **As required.**

(d) Modify `Repairman2.m` provided to estimate $P_{01}(200)$. Provide details of your inputs. Compare to your answer in (a)

Solution Calling `RepairmanN(0,1,200,10000,2)` gives

$$EstProb = 0.2378$$

Modified code (i got carried away and made it work for N machines):

```
function EstProb = RepairmanN(istate,endstate, T, numsims,N)
%RepairmanN
%Simulates the repairman problem for N machines
%%Inputs:
%istate - initial state of the markov chain takes values (0,1,2)
%endstate - state to consider on termination, takes values (0,1,2)
%T - duration for markov chain simulation T must be positive or the program
%will terminate
%numsims - Number of repetitions for the simulation (restarting at time 0)
%must be at least 1, must be an integer
%N - number of machines
%%Output:
%EstProb - the probability of reaching endstate based on the number of
%simulations

figure
hold
counter = 0;

%sim number
for si = 1:numsims
    %initialise time and timestep to 0
    t = 0;
    ts = 0;
    %Initial state is state
    state = istate;
    ss = istate;
    %Repeat until we exit the allowed time
    while t < T
        %if all machines are working
        if state == 0
            t = t + exprnd(30);
            ts = [ts; t];
            state = 1;
            ss = [ss; state];
```

```
        elseif state== N
            t = t + exprnd(10);
            state = state-1;
            ts = [ts; t];
            ss = [ss; state];
        else
            t = t + exprnd(1/((1/10) + (1/60)));
            ts = [ts; t];
            if rand < (1/10)/((1/10) + (1/60))
                state = state-1;
            else
                state = state +1;
            end
            ss = [ss; state];

        end
    end
    if ss(end-1) == endstate
        counter = counter + 1;
    end
    plot(si, counter/si, 'o')
end
EstProb = counter/si;
```

As required.