LECTURE 14

Countable and uncountable sets, continued

Last time we proved the following lemma:

Lemma: Let S be a set. The following statements are equivalent.

(1) S is countable.

(2) S is empty or there exists an onto function $f\colon \mathbb{N} \to S$.

(3) There exists a 1-1 function $g\colon S \to \mathbb{N}$.

There is another standard fact about countable sets that we will have occasion to use several times; it is an easy consequence of the above lemma.

Lemma: If S is countable then every subset of S is countable.

Proof: By (3) of the lemma above, since S is countable, there exists a 1-1 function $g\colon S \to \mathbb{N}$. If $T \subset S$ then the restriction $g|_T\colon T \to \mathbb{N}$ of g to T is also 1-1. Therefore, by the lemma above, T is countable. ■

Note: Frequently, the contrapositive of this lemma is used — this is an efficient way to prove that a set is uncountable. The contrapositive is the statement that if S contains an uncountable subset then S itself must be uncountable.

Here is another lemma which is good to know about uncountable sets.

Lemma: if $A_1, A_2, \ldots$ are countable sets then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Proof: not difficult, but omitted for reasons of time. ■

Example: the following is a surprising example of a countable set: $\mathbb{Q}$ is countable! To see this, observe that $\mathbb{Q}$ is a finite union $\mathbb{Q} = \mathbb{Q}_{<0} \cup \{\, 0 \,\} \cup \mathbb{Q}_{>0}$ of the set of negative rational numbers $\mathbb{Q}_{<0}$, the set of positive rational numbers $\mathbb{Q}_{>0}$ and the set $\{\, 0 \,\}$; if we can show that each of these sets is countable then it will follow from the third lemma above that $\mathbb{Q}$ is countable.

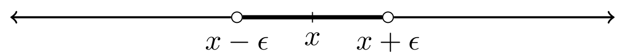
We will prove that $\mathbb{Q}_{>0}$ is countable; the proof that $\mathbb{Q}_{<0}$ is countable is analogous. We will construct an onto function $f\colon \mathbb{N} \to \mathbb{Q}_{>0}$, by the first lemma above it will then follow that $\mathbb{Q}_{>0}$ is countable. Constructing such an onto function is the same as listing out the elements of the set $\mathbb{Q}_{>0}$ as a sequence; the following picture shows how we can do this (note that the sequence will repeat some values but that's ok, what is important is that every element of $\mathbb{Q}_{>0}$ appears as a term of the sequence):

$$
\begin{array}{ccccccccc}
\frac{1}{1} & \text{—} & \frac{2}{1} & & \frac{3}{1} & \text{—} & \frac{4}{1} & & \cdots \\
 & / & & / & & / & & & \\
\frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \cdots & & \\
| & / & & / & & & & & \\
\frac{1}{3} & & \frac{2}{3} & & \cdots & & & & \\
 & / & & & & & & & \\
\vdots & & \vdots & & & & & &
\end{array}
$$

Example: What is perhaps equally surprising is that $\mathbb{R}$ is uncountable. In particular, it follows that the set of irrational numbers is uncountable; since $\mathbb{R}$ is the union of the rationals and the irrationals, if the irrationals formed a countable set then it would follow that $\mathbb{R}$ was countable, a contradiction. There are several ways of proving that $\mathbb{R}$ is uncountable. One such method uses the Nested Interval Theorem.

§3 The topology of $\mathbb{R}$

Let $x \in \mathbb{R}$ and let $\epsilon > 0$. The *ϵ-neighbourhood* of x is the set $I_\epsilon(x) = (x - \epsilon, x + \epsilon)$.



Exercise: If $0 < \epsilon_1 < \epsilon_2$ then $I_{\epsilon_1}(x) \subset I_{\epsilon_2}(x)$.

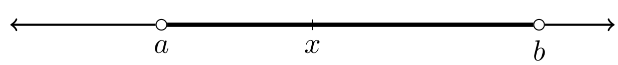
Example: A sequence (a_n) converges to L if and only if for all $\epsilon > 0$ all but finitely many terms of the sequence belong to $I_\epsilon(L)$. To see the relationship between this statement and the definition of convergence (Definition 2.1) suppose that $\epsilon > 0$ and all but finitely many terms of the sequence (a_n) belong to $I_\epsilon(L)$. Therefore, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $(a_n) \in I_\epsilon(L)$. Therefore, $n \geq N \implies a_n \in (L - \epsilon, L + \epsilon)$, i.e. $|a_n - L| < \epsilon$.

Definition 3.1: A set $S \subset \mathbb{R}$ is *open* if for all $s \in S$ there exists $\epsilon > 0$ such that $I_\epsilon(s) \subset S$.

Note: Therefore S is *not* open if and only if there exists $s \in S$ such that for all $\epsilon > 0$ there exists $x \in I_\epsilon(s)$ such that $x \notin S$.

Note: In general, ϵ in Definition 3.1 will depend on the point s. Thus choosing an ϵ for each s defines a function $\epsilon(s)$ of s.

Example 1: if $a < b$ then the open interval (a, b) is an open set (if it wasn't then 'open interval' would be a really unfortunate choice of name). Let $x \in (a, b)$. We need to show that there is an $\epsilon > 0$ such that $I_\epsilon(x) \subset (a, b)$. The following picture will help to motivate the choice of ϵ:



Let $\epsilon = \min\{\, x - a, b - x \,\}$. The claim is that $I_\epsilon(x) \subset (a, b)$. To prove this claim, let $y \in I_\epsilon(x)$; we have to show that $a < y < b$, i.e. $y \in (a, b)$. Since $y \in I_\epsilon(x)$, $x - \epsilon < y < x + \epsilon$. Since $\epsilon = \min\{\, x - a, b - x \,\}$, we have $\epsilon \leq x - a$ and $\epsilon \leq b - x$. Therefore

$$a = x - (x - a) \leq x - \epsilon < y < x + \epsilon \leq x + b - x = b.$$

Hence $y \in (a, b)$. Since this is true for all $y \in I_\epsilon(x)$ it follows that $I_\epsilon(x) \subset (a, b)$.

Similarly one may prove that (a, ∞) and $(-\infty, a)$ are open sets. The set $[0, 1)$ is *not* open: if $\epsilon > 0$ then $I_\epsilon(0) = (-\epsilon, \epsilon)$ contains both positive and negative numbers and hence cannot be a subset of $[0, 1)$ (the set $[0, 1)$ does not contain any negative numbers). Therefore $[0, 1)$ is not open.

Example 2: the set $\mathbb{R}$ is open. This is very easy to prove: if $x \in \mathbb{R}$ then $I_\epsilon(x) \subset \mathbb{R}$ for any $\epsilon > 0$.

Example 3: the empty set $\emptyset$ is also open. This is slightly trickier to understand. The point is that the statement $\forall\ s \in \emptyset$ there exists $\epsilon > 0$ such that $I_\epsilon(s) \subset \emptyset$ is always true, since the statement $s \in \emptyset$ is always false.

Example 4: If S_1 and S_2 are open then $S_1 \cup S_2$ is open. To see this, suppose that $s \in S_1 \cup S_2$. Then $s \in S_1$ or $s \in S_2$. Without loss of generality $s \in S_1$. Since S_1 is open there exists $\epsilon > 0$ such that $I_\epsilon(s) \subset S_1$. Since $S_1 \subset S_1 \cup S_2$, we have $I_\epsilon(s) \subset S_1 \subset S_1 \cup S_2$. It follows that $S_1 \cup S_2$ is open.

In fact a more general statement is true. We first need some notation. Suppose that I is a set and that for every $i \in I$ we are given a set S_i of real numbers. Thus we have a collection of sets labelled or 'indexed' by the set I; such a collection is often called a 'family' of sets (indexed by I). In this case we can form a new subset of $\mathbb{R}$, called the *union* of the 'family' $(S_i)_{i \in I}$, it is defined by

$$\bigcup_{i \in I} S_i = \{\, x \in \mathbb{R} \mid \ \exists\, i \in I \text{ such that } x \in S_i \,\}\,.$$

Thus $\bigcup_{i \in I} S_i$ is the set of real numbers which belong to at least one of the sets S_i in the family. For example, let $I = [0, 1]$ and for each $i \in [0, 1]$ let $S_i = (3 + i, 6 - i)$. Then $\bigcup_{i \in I} S_i = (3, 6)$.

Example 5: Generalizing the previous example, if S_i is open for all $i \in I$ then the union $\bigcup_{i \in I} S_i$ is open. The proof of this is an excellent exercise.

Example 6: Another excellent exercise is to prove the following: a subset $S \subset \mathbb{R}$ is open if and only if $S = \bigcup_{i \in A} I_i$ where I_i is an open interval for each $i \in A$. In one direction, if S is open then for each $s \in S$ we can choose $\epsilon(s) > 0$ such that $I_{\epsilon(s)}(s) \subset S$. Then $S = \bigcup_{s \in S} I_{\epsilon(s)}(s)$.

Example 7: If S_1, S_2 are open then $S_1 \cap S_2$ is open. More generally, if $S_1, S_2, \ldots, S_n$ are open then $S_1 \cap S_2 \cap \cdots \cap S_n$ is open. This is *not true* for infinite intersections. For example the sets $(-1/n, 1/n)$ are open for all $n \in \mathbb{N}$ but $\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{\, 0 \,\}$ is not open.

Definition 3.2: A subset $S \subset \mathbb{R}$ is *closed* if and only if $\mathbb{R} \setminus S$ is open. (Here $\mathbb{R} \setminus S = \{\, x \in \mathbb{R} \mid \ x \notin S \,\}$.)

Warning: Sets are not like doors!

$$\begin{array}{l} S \text{ not open } \not\Longrightarrow \ S \text{ closed} \\ S \text{ not closed } \not\Longrightarrow \ S \text{ open} \end{array}$$

For example, the set $[0, 1)$ is not open (as we have already seen), but it is not closed either since $\mathbb{R} \setminus [0, 1) = (-\infty, 0) \cup [1, \infty)$ is not open.

However,

$$S \text{ not closed } \iff \mathbb{R} \setminus S \text{ not open}$$

Example: The following statements are true:

1. $[a, b]$ is closed.

2. $\mathbb{R}$ and $\emptyset$ are closed ($\emptyset$ is closed because $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open; $\mathbb{R}$ is closed because $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open). Thus both $\mathbb{R}$ and $\emptyset$ are closed and open; they are the *only* subsets of $\mathbb{R}$ with this property (proving this is a slightly challenging exercise; you will need to make use of the Least Upper Bound property of $\mathbb{R}$).

3. If S_i is closed for all $i \in I$ then $\bigcap_{i \in I} S_i$ is closed. Here $\bigcap_{\in I} S_i$ is the set $\bigcap_{i \in I} S_i = \{\, x \in \mathbb{R} \mid \ x \in S_i \text{ for all } i \in I \,\}$.

4. If $S_1, S_2, \ldots, S_n$ are closed then $S_1 \cup S_2 \cup \cdots \cup S_n$ is closed.

The proofs of 3 and 4 use *de Morgan's Laws*: suppose $S_i \subset T$ for all $i \in I$, then

$$T \setminus \bigcup_{i \in I} S_i = \bigcap_{i \in I} T \setminus S_i$$
$$T \setminus \bigcap_{i \in I} S_i = \bigcup_{i \in I} T \setminus S_i.$$