THE UNIVERSITY *of* ADELAIDE

Student ID: ______________________

Family name: ______________________

Other names: ______________________

Desk number: __________ Date: __________

Signature: ______________________

Examination in the School of Mathematical Sciences

Semester 1, 2018

108732	**APP MTH 4121**	**Modelling with Ordinary Differential Equations Hon**

Time for completing booklet: 180 mins (plus 10 mins reading time).

Question	Marks	
1	/15	
2	/20	
3	/16	
4	/20	
5	/14	
6	/15	
Total	/100	

Instructions to candidates

- Attempt all questions and write your answers in the space provided below that question.
- If there is insufficient space below a question, then use the space to the *right* of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

- Calculators are not permitted.

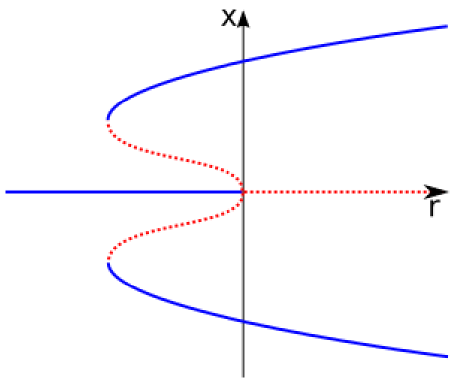
Do not commence writing until instructed to do so.

15 Total

Question 1.

Decide whether each of the following statements is true or false. Write 1–2 sentences of explanation to support each answer. You may also show an example or draw a figure if this assists your explanation.

1(a) The ODE $\dot{x} = f(x; r)$ with the bifurcation diagram below can demonstrate hysteresis. (Solid blue lines indicate stable branches and broken red lines denote unstable branches.)



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1(b) The solution $(x(t), y(t))$ of a two-dimensional autonomous system of the form

$$\dot{x}(t) = f(x, y) \quad \text{and} \quad \dot{y}(t) = g(x, y),$$

for smooth f and g, can only tend towards fixed points or become unbounded as $t \to \infty$.

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1(c) The backwards Euler method is unconditionally stable for the IVP

$$\frac{dx}{dt} = -(1+t)\,x \quad \text{with} \quad x(0) = 1. \tag{1}$$

/5 marks

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20 Total

Question 2.

Consider the ordinary differential equation

$$\frac{dX}{dT} = R\,X\left(1 - \frac{X}{K}\right) - C \quad \text{for} \quad X(T) \in \mathbb{R}, \quad T > 0, \qquad (2)$$

where R, K and C are positive parameters.

2(a) Show that ODE (2) can be transformed into the non-dimensional ODE

$$\frac{dx}{dt} = x\,(1 - x) - c \quad \text{for} \quad x(t) \in \mathbb{R}, \quad t > 0, \qquad (3)$$

using suitable scalings $X = X_c\,x$ and $T = T_c\,t$. Express the scales X_c and T_c, and non-dimensional parameter $c > 0$ in terms of R, K and C.

/5 marks

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2(b) Determine the steady states $x = x_*$ of ODE (3).

/3 marks

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2(c) Use your answer to 2(b) to determine the bifurcation value $c = \overline{c}$. Hence calculate the bifurcation point $x = \overline{x}$.

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2(d) Perform a phase-line analysis for $c < \overline{c}$, $c = \overline{c}$ and $c > \overline{c}$, including arrows to indicate the stability of the fixed points. Also state the stability of the fixed points, explaining your reasoning.

/4 marks

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2(e) State the type of bifurcation that occurs at $c = \overline{c}$, and give a reason.

/2 marks

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2(f) Sketch the bifurcation diagram, marking the stable and unstable branches and the bifurcation point and value.

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16 Total

Question 3.

Consider the dimensionless non-linear model of competition between biological populations $x(t)$ and $y(t)$,

$$\frac{dx}{dt} = x - x\,y, \qquad x(0) = x_0, \tag{4}$$

$$\frac{dy}{dt} = \mu\,y - x\,y, \qquad y(0) = y_0, \tag{5}$$

where t is time and μ is a real positive constant.

3(a) Determine the nullclines and the two steady states.

/4 marks

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3(b) Let $x(t) = x_* + w(t)$ and $y(t) = y_* + z(t)$ where (x_*, y_*) denotes a steady state and $w(t)$ and $z(t)$ are small perturbations to the steady state. Linearise the system given by Eqs. (4)–(5) to obtain the approximation

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix} = J(x_*, y_*)\begin{pmatrix} w \\ z \end{pmatrix}, \tag{6}$$

where $J(x, y)$ is the Jacobian matrix, which you are to derive.

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3(c) For each steady state determine, if possible, its type using the eigenvalues of the Jacobian matrix. Explain why or why not you are able to determine the type of the steady states.

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/5 marks

3(d) Sketch a phase portrait in the biologically relevant part of the domain for $\mu = 1$, showing the nullclines, steady states and some trajectories. Show the direction of travel along the trajectories.

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20 Total

Question 4.

4(a) Explain what is meant by a well-posed problem.

/2 marks

4(b) State Lax's equivalence theorem and define each of the three concepts involved.

/4 marks

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4(c) Consider a function $y(t)$ and discrete times $t_n = n\,h$ for some step size $h > 0$ and $n \in \mathbb{N}$, and let $y_n = y(t_n)$.

Derive a finite difference formula for the second derivative $y_n'' = y''(t_n)$ by solving for the coefficients a_i in

$$y_n'' = \sum_{i=-1}^{1} a_i\, y_{n+i} + O(h^m).$$

Determine the truncation error associated with the formula, i.e. determine the value of m.

Perform a simple check of the coefficients that ensures the finite difference formula is correct for constant functions.

/6 marks

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4(d) Consider the finite difference formula

$$x_{n+1} = 4\,x_n - (3 - 4\,h)\,x_{n-1}, \tag{7}$$

where $h > 0$ is the step size. Perform a consistency analysis to show that the finite difference formula is consistent with the ODE

$$\frac{dx}{dt} = -2x \quad \text{for} \quad x(t). \tag{8}$$

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4(e) Show that the local discretisation error in using the finite difference formula (7) to approximate $x(t_n + h)$ is $O(h^3)$.

/2 marks

4(f) State the problem encountered when using the finite difference formula (7) to time step, starting with some known initial condition $x_0 = x(0)$. How could you overcome this problem using Euler's method? Does this compromise the global error in using the method?

/2 marks

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14 Total

Question 5.

Consider the IVP

$$\frac{dx}{dt} = x^3 \quad \text{with} \quad x(0) = 1. \tag{9}$$

5(a) Find the exact solution. Comment on existence and uniqueness of the solution.

/5 marks

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5(b) Define what it means for a function $f(x) : J \to \mathbb{R}$ to be Lipschitz continuous on J.

/1 mark

5(c) State the intervals on which $f(x) = x^3$ is Lipschitz continuous.

/1 mark

5(d) Does the Picard–Lindelof theorem guarantee a unique solution to IVP (9) for some interval of t following $t = 0$?

/2 marks

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5(e) Express the IVP (9) as an integral equation, and hence write down the associated Picard iteration scheme, including the value of the initial guess $x^{(0)}$. Calculate the first iterate $x^{(1)}$.

/5 marks

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15 Total

Question 6.

The Kermack–McKendrick model for the evolution of an epidemic is

$$\dot{x} = -kxy, \quad \dot{y} = kxy - ly, \quad \text{and} \quad \dot{z} = ly, \tag{10}$$

where t is time, $x(t)$ denotes the number of healthy people, $y(t) > 0$ denotes number of sick people, $z(t)$ denotes the number of dead people, and k and l are positive constants.

6(a) Determine the rates at which (i) healthy people get sick, and (ii) sick people die.

/1 mark

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6(b) Show that

$$x + y + z = N, \tag{11}$$

where N is constant. Interpret this equation.

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6(c) Use the first and third components of the model given in Eq. (10) to show that

$$x(t) = x_0 \exp(-kz/l), \tag{12}$$

where $x(0) = x_0$ and $z(0) = 0$. Hence show that

$$\dot{z} = l\,[N - z - x_0 \exp(-kz/l)]\,. \tag{13}$$

/4 marks

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6(d) You are given that ODE (13) can be nondimensionalised to

$$\frac{du}{d\tau} = a - bu - e^{-u}, \tag{14}$$

where $u(\tau) \geq 0$, and constants $a > 1$ and $b > 0$.

Use the definition of a bifurcation in terms of the right-hand side of the ODE and its derivative to show that for these biologically relevant ranges of a and b, ODE (14) has no bifurcations.

/4 marks

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6(e) By sketching the right-hand side of ODE (14) or otherwise, show that there is a single biologically relevant fixed point, u^*, and determine its stability.

/3 marks

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6(f) What modification might be made to this model to make it more appropriate for a flu epidemic?

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End of examination questions.

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