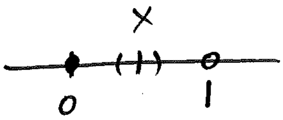
Correction: $[0,1)$ is a neighbourhood of $\frac{1}{2}, \frac{1}{3}$ - more generally a nbd of any $x \in (0,1)$.




It is not a nbd of $0$ & it is DEFINITELY NOT a nbd of $1$, $1$ isn't even IN the set!

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Last time: Thm 4.7: If $f: S \to \mathbb{R}$ is cts where $S$ is non-empty & seq. compact, then $f$ attains its max & min on $S$, i.e. $\exists\ s_0, s_1 \in S$ s.th. $f(s_0) \leq f(s) \leq f(s_1)$ $\forall\ s \in S$.

In particular any cts $f^n$ $f: \underbrace{[a,b]}_{\text{seq. compact.}} \to \mathbb{R}$ attains its max. & min.

Pf: We'll show first that $S$ seq. compact $\Rightarrow$ $f(S) = \{ f(s) \mid s \in S \}$ is seq. compact.

Let $(y_n)$ be a seq. in $f(S)$. $\therefore\ \exists\ s_n \in S$ s.th. $f(s_n) = y_n$ for $n = 1, 2, 3, \dots$ Since $S$ is seq. compact, $\exists$ subseq. $(s_{n_k})$ of $(s_n)$ s.th. $s_{n_k} \to s_0$ for some $s_0 \in S$. Since $f$ is cts on $S$, $\underbrace{f(s_{n_k})}_{= y_{n_k}} \to f(s_0)$.

$\therefore$ the subseq $(y_{n_k})$ converges to $f(s_0) \in f(S)$.

$\therefore$ $f(S)$ is seq. compact.

$\therefore$ $f(S)$ is closed & bounded

$\therefore$ $\inf f(S)$ & $\sup f(S)$ exist.

Since $f(S)$ is closed, $\inf f(S)$ & $\sup f(S)$ belong to $f(S)$. $\therefore$ The set $f(S)$ has a max & a min, i.e. $f$ attains it max & min.

if $C$ is closed (& bounded) then $\inf C$, $\sup C$ belong to $C$. ( Converse is not true: $[0,1) \cup (2,3]$. contains inf, sup but not closed ).

show $C$ closed & bounded $\Rightarrow \sup C \in C$.

$\exists$ seq $(c_n)$ in $C$ which converges to $\sup C$.

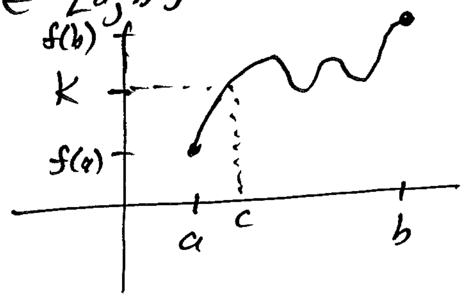
$\forall n \in \mathbb{N}\ \exists\, c_n \in C$ s.th. $\sup C - \frac{1}{n} < c_n \leq \sup C$

$$c_n \longrightarrow \sup C$$

$C$ closed, $\sup C \in C$.

<u>Th$^{m}$ 4.8</u> (Intermediate Value Th$^{m}$). Suppose $f : [a,b] \to \mathbb{R}$ is cts. Then $f$ attains every value between $f(a)$ & $f(b)$.

If $f(a) \leq K \leq f(b)$ or if $f(b) \leq K \leq f(a)$ then $\exists\ c \in [a,b]$ s.th. $f(c) = K$.




<u>Pf</u>: Let's suppose that $f(a) \leq f(b)$. – we can suppose this WLOG. (if $f(b) \leq f(a)$ consider the cts f$^{n}$ $-f : [a,b] \to \mathbb{R}$).

Wlog $f(a) < K < f(b)$. Let

$$S = \{x \in [a,b] \mid f(x) < K\}.$$

$S \neq \emptyset$ $(a \in S)$. & $S$ is bounded above (by $b$).

Let $s_0 = \sup S$. Then $a \leq s_0 < b$. For each $n \in \mathbb{N}$ $\exists$ $s_n \in S$ s.th. $s_0 - \frac{1}{n} < s_n \leq s_0$.

Then $(s_n)$ is a seq in $S$ (hence a seq. in $[a,b]$) which converges to $s_0$.

$f$ cts on $[a,b]$ $\Rightarrow f(s_n) \longrightarrow f(s_0)$.

$f(s_n) < K \quad \forall\, n \quad (s_n \in S)$.

$\boxed{\begin{array}{c} a_n \leq b_n \ \forall n \\ a_n \to L,\ b_n \to M \\ \Rightarrow L \leq M. \end{array}}$

$\therefore \lim_{n\to\infty} f(s_n) = f(s_0) \leq K$ (Preserv$^{n}$ of Inequalities).

Suppose $f(s_0) < K$. Then $s_0 \in S$.

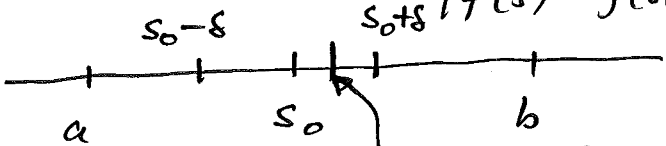
$f(s_0) < K \Rightarrow K - f(s_0) > 0$. Let $\varepsilon = K - f(s_0) > 0$.

Since $f$ is cts on $[a,b]$, $f$ is cts at $s_0$.

$\exists\ \delta > 0$ s.th.

(i) $(s_0 - \delta, s_0 + \delta) \subset [a,b]$ $(a < s_0 < b)$

(ii) if $s \in [a,b]$ & $|s - s_0| < \delta$ then $|f(s) - f(s_0)| < \varepsilon$. ($f$ cts at $s_0$).




$s_0 + \frac{\delta}{2} > s_0$ & $s_0 + \frac{\delta}{2} \in (s_0 - \delta, s_0 + \delta)$

$\therefore\ |f(s_0 + \frac{\delta}{2}) - f(s_0)| < \varepsilon$.

$\Rightarrow f(s_0 + \frac{\delta}{2}) - f(s_0) < \varepsilon = K - f(s_0)$

$\Rightarrow f(s_0 + \frac{\delta}{2}) < K$.

$\Rightarrow s_0 + \frac{\delta}{2} \in S$. $(s_0 < s_0 + \frac{\delta}{2})$.

– contradiction, since $s_0 = \sup S$.

$\therefore\ f(s_0) = K$.

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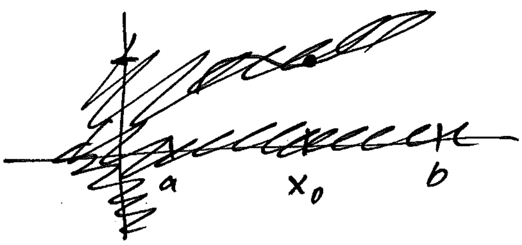
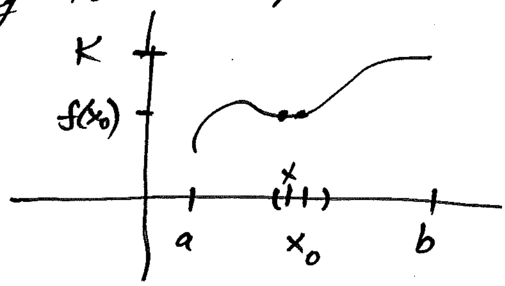
Note: we observed that if $f$ is cts on $[a,b]$, (say $f: [a,b] \to \mathbb{R}$), $x_0 \in [a,b]$), if $f(x_0) < K$ then there is an open interval containing $x_0$ s.th. $f(x) < K$ for all $x$ ~~in~~ belonging to the open interval



$\underline{\text{Ex}}$. $f : \mathbb{R} \to \mathbb{R}$ $\qquad$ $g : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+1$ $\qquad$ $g(x) = x^2$.

$f$ & $g$ are cts on $\mathbb{R}$.

eg: show $f$ is cts at $x_0 \in \mathbb{R}$.

Let $\varepsilon > 0$. $|f(x) - f(x_0)|$

$= |2x+1-2x_0-1|$

$= 2|x-x_0|$

$\therefore$ if $x \in \mathbb{R}$ & $|x-x_0| < \frac{\varepsilon}{2}$

then $|f(x) - f(x_0)| = 2|x-x_0|$

$< 2 \cdot \frac{\varepsilon}{2}$

$= \varepsilon$.

$\delta = \frac{\varepsilon}{2}$.

does not depend on $x_0$.

eg: show $g$ is cts at $x_0 \in \mathbb{R}$.

Let $\varepsilon > 0$.

$|g(x) - g(x_0)|$

$= |x^2 - x_0^2|$

$= |x-x_0| \cdot |x+x_0|$

Let $\delta = \min\left\{1, \frac{\varepsilon}{1+2|x_0|}\right\}$

if $|x-x_0| < \delta$ then

$\underline{|x-x_0| < 1}$ & $|x-x_0| < \frac{\varepsilon}{1+2|x_0|}$

$\Rightarrow |x| - |x_0| \le |x-x_0| < 1$

$\Rightarrow |x| \le 1 + |x_0|$.

$\therefore \underline{|x+x_0|} \le |x| + |x_0| < 1 + 2|x_0|$

$\therefore$ if $|x-x_0| < \delta$ then

$|g(x) - g(x_0)| = |x-x_0| \cdot |x+x_0|$

$< (1+2|x_0|) \cdot |x-x_0|$

$< \frac{\varepsilon}{1+2|x_0|} (1+2|x_0|)$

$= \varepsilon$.

$\delta = \min\left\{1, \frac{\varepsilon}{1+2|x_0|}\right\}$.

depends on $x_0$.