APP MTH 3001 Applied Probability III
Class Exercise 3, 2018
Due: 3pm, 20 April 2018, via Canvas (PDF only).

1. Consider the following stochastic matrices

$$\mathbb{P}_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix}, \quad \mathbb{P}_b = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

$$\mathbb{P}_c = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{P}_d = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{7}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

(i) Draw state transition diagrams for the DTMCs having $\mathbb{P}_a$, $\mathbb{P}_b$, $\mathbb{P}_c$, and $\mathbb{P}_d$ as their one-step transition matrices.

(ii) Determine the communicating classes for each of the DTMCs having $\mathbb{P}_a$, $\mathbb{P}_b$, $\mathbb{P}_c$, and $\mathbb{P}_d$ as their one-step transition matrices.

(iii) Classify the states in the classes as either recurrent or transient.

2. Consider Example 3.13 from lectures. This is the contest between Players A and B, in which Player B is infinitely rich, and corresponds to a random walk on the state space $\mathcal{S} = \{0, 1, 2, \ldots\}$, with state 0 an absorbing state. As in lectures, let X_n, represent the fortune of Player A at time n. The transition probabilities governing the Markov process $X_n, n \in \mathbb{N}$, are as follows. For $i \geq 1$,

$$p_{i,j} = \begin{cases} p & \text{if } \ j = i+1 \\ q & \text{if } \ j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

and at the (single) boundary of the state space we have $p_{0,0} = 1$.

Find the mean hitting times $k_i = k_i^{\{0\}}, i = 0, 1, 2, \ldots$, for the cases

(a) $p < q$

(b) $p = q$ and

(c) $p > q$.

3. Consider the DTMC on $N+1$ states (labelled $0, 1, 2, \ldots, N$), whose transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ p_1 & r_1 & q_1 & 0 & \cdots & 0 \\ p_2 & 0 & r_2 & q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ p_{N-1} & 0 & 0 & 0 & \cdots & q_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $p_i + r_i + q_i = 1$ for all $i = 1, \dots, N-1$, where $p_i, q_i, r_i > 0, \quad \forall i$.

(a) Identify the communicating classes, and state whether they are recurrent or transient.

  (i) Draw a state transition diagram for this Markov chain.

  (ii) Give a brief qualitative description (in words) of the dynamics associated with this Markov chain.

(b) This Markov chain can be viewed as a modified version of the "Two Gamblers" example seen in lectures.

  (i) Describe the (modified) contest that is modelled by this Markov chain.

(c) Show that the absorption probabilities $u_k^{\{0\}}$, for $k = 1, \dots, N-1$ are given by

$$u_k^{\{0\}} = 1 - \left(\frac{q_k}{p_k + q_k}\right) \times \dots \times \left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right).$$

Hint: first evaluate $u_k^{\{N\}}$, by starting with $u_{N-1}^{\{N\}}$, and proceed by recursion.

4. Consider the transition matrix $\mathbb{P}$ from Approach 1 in the Group Project, for your group's piece of music. With reference to the condition number of $\mathbb{P}$, determine on the number of digits of numerical accuracy you expect to lose when solving systems of equations using this $\mathbb{P}$. Hence, comment on the appropriateness of computing expected hitting times for this problem (i.e., in part 1(h) of the project).

Note: even if you haven't encountered condition numbers before through e.g., Numerical Methods II, a quick Wikipedia search will give you all the theory you need for this question.