Optimal Functions and Nanomechanics III

APP MTH 3022/7106

Barry Cox

Lecture 20

Last lecture

- Revisited the catenary problem
- Looked at the Kantorovich method for numerical approximation

Constraints

We now include additional constraints into the problems:

- Integral constraints of the form

$$\int g(x, y, y')dx = \text{const}$$

  e.g., the Isoperimetric problem.
- Holonomic constraints, e.g., $g(x, y) = 0$
- Non-holonomic constraints, e.g., $g(x, y, y') = 0$
- We won't consider inequality constraints until later.

Integral Constraints

Integral constraints are of the form

$$\int g(x, y, y')dx = \text{const}$$

The standard example of such a problem is Dido's problem, leading to us referring to such constraints as **isoperimetric**. We solve these by introducing the functional analogy of a Lagrange multiplier.

Dido's problem

- Dido (Carthaginian queen) fled to North Africa, where a local chief offered her as much land as an oxhide could contain.
- Cut the oxhide into thin strips, and then use them to surround a patch of ground (in which to found Carthage).
- Obviously, she wanted to contain the largest possible land area
- Given a fixed length of oxhide, what shape would encompass the largest area?
- Hengist and Horsa had the same problem (semi-mythological rulers in southern England around Vortigen, preceding Arthur)

Isoperimetric problems

Dido's problem falls into the class of **isoperimetric** problems.

- iso- (from same) and perimetric (from perimeter), roughly meaning "same perimeter".
- in general, such problems involve a constraint
    - e.g. the length of the oxhide strip
    - But the constraint is not always to fix the perimeter length,
    - sometimes the constraint does not even involve a length,
    - but the term isoperimetric is still used.

Isoperimetric problems formulation

We can write the isoperimetric problems as the problem of finding extremals of the functional $F : C^2[x_0, x_1] \to \mathbb{R}$ given by

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

with all the usual conditions (e.g. on end points, and continuous derivatives) but in addition we must satisfy the extra functional constraint

$$G\{y\} = \int_{x_0}^{x_1} g(x, y, y')\, dx = L$$

A simplified form of Dido's problem

Imagine that the two end-points are fixed, along the coast (Carthage was a great sea power), and we wish to encompass the largest possible area inland with a fixed length L. We can write this problem as maximize the area

$$F\{y\} = \int_{x_0}^{x_1} y \, dx$$

encompassed by the curve y, such that the the curve y has fixed length L, e.g. as before the length of the curve is

$$G\{y\} = \int_{x_0}^{x_1} \sqrt{1 + y'^2} \, dx = L$$

subject to the end-point conditions $y(x_0) = 0$ and $y(x_1) = 0$.

A simplified form of Dido's problem

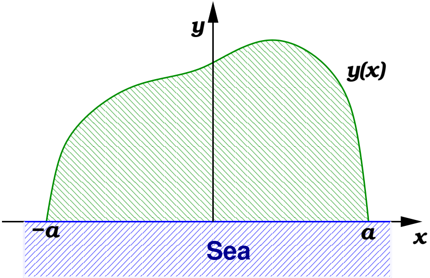
$$F\{y\} = \int_{x_0}^{x_1} y\,dx$$

subject to

$$G\{y\} = \int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx = L$$

$y(-a) = 0$ and $y(a) = 0$.

For simplicity take $2a < L \leqslant \pi a$



Approach

As before

- we perturb the curve, and consider the first variation,
- but we cannot perturb by an arbitrary function $\epsilon\eta$, because then the constraint $G\{y + \epsilon\eta\} = L$ might be violated.
- solution: use the same approach as we did earlier with constrained maximization, e.g. use Lagrange multipliers

Lagrange multiplier refresher

Problem: find the minimum (or maximum) of $f(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$ subject the constraints

$$g_i(\boldsymbol{x}) = 0, \quad i = 1, \ldots, m < n$$

Solution requires **Lagrange Multipliers**. Minimize (or maximize) a new function (of $m + n$ variables)

$$h(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x}),$$

where λ_i are the undetermined Lagrange multipliers.

Lagrange multipliers in functionals

To maximize

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

subject to

$$G\{y\} = \int_{x_0}^{x_1} g(x, y, y')\, dx = L$$

we instead consider the problem of finding extremals of

$$H\{y\} = \int_{x_0}^{x_1} h(x, y, y')\, dx = \int_{x_0}^{x_1} f(x, y, y') + \lambda g(x, y, y')\, dx$$

Euler-Lagrange equations

The Euler-Lagrange equations become

$$\boxed{\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) - \frac{\partial h}{\partial y} \;=\; 0}$$

where $h = f + \lambda g$, and λ is the unknown Lagrange multiplier.

Dido's problem

$$H\{y\} = \int_{x_0}^{x_1} y + \lambda\sqrt{1+y'^2}\,dx$$

so

$$\begin{aligned}
\frac{\partial h}{\partial y} &= 1 \\
\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) &= \frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}}
\end{aligned}$$

and the Euler-Lagrange equations are

$$\frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}} = 1$$

Dido's problem

Integrating with respect to x we get

$$\frac{y'}{\sqrt{1+y'^2}} = (x+c_1)/\lambda$$

writing for the moment $\tilde{x} = (x+c_1)/\lambda$

$$\begin{aligned} y' &= \tilde{x}\sqrt{1+y'^2} \\ y'^2 &= \tilde{x}^2(1+y'^2) \\ y'^2 - \tilde{x}^2 y'^2 &= \tilde{x}^2 \\ y'^2 &= \frac{\tilde{x}^2}{1-\tilde{x}^2} \\ y' &= \tilde{x}\left(\frac{1}{1-\tilde{x}^2}\right)^{1/2} \end{aligned}$$

Dido's problem

Integrating with respect to x again

$$y = \int \tilde{x} \sqrt{\frac{1}{1-\tilde{x}^2}}\, dx$$

Change variables to $\tilde{x} = (x + c_1)/\lambda = \sin(\theta)$, then

$$\begin{aligned} y &= \int \sin(\theta) \frac{1}{\cos(\theta)} \frac{dx}{d\theta}\, d\theta \\ &= \lambda \int \sin(\theta)\, d\theta \\ &= -\lambda \cos(\theta) + c_2 \end{aligned}$$

where λ, c_1 and c_2 are determined by the two end-points, and the length of the curve L.

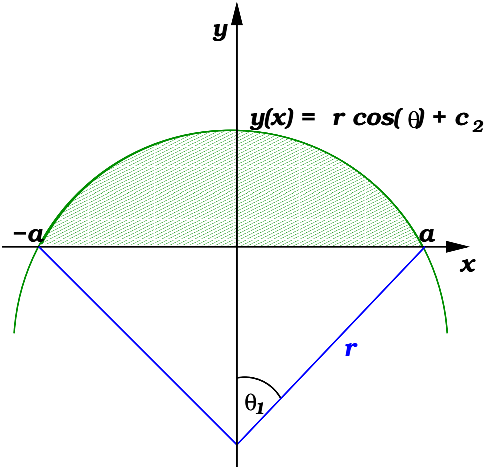
Dido's problem: constants

From the solution

$$
\begin{aligned}
x + c_1 &= \lambda \sin(\theta) \\
y &= -\lambda \cos(\theta) + c_2
\end{aligned}
$$

we may draw a sketch of the solution, and clearly we can identify $-\lambda = r$ the radius of a circle, of which our region is a segment.

Note we deliberately started with $2a < L \leqslant \pi a$



Dido's problem: constants

We can see that the arc length of the enclosing curve will be

$$L = 2\theta_1 r$$

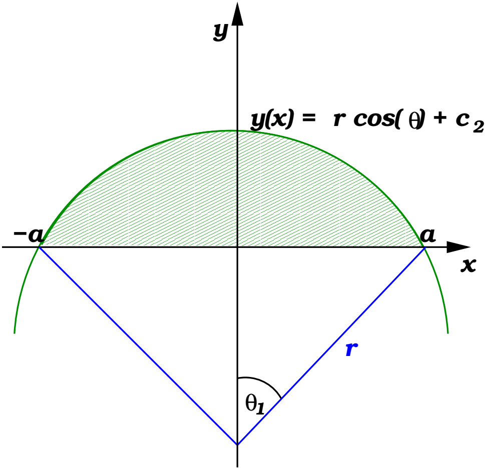
and that the value $x_1 = a$ determines that

$$r = a/\sin\theta_1$$

which combined give

$$L = 2a\theta_1/\sin\theta_1$$

from which we may determine θ_1.



Dido's problem: constants

We determine θ_1 from

$$\sin\theta_1 = \frac{2a}{L}\theta_1$$

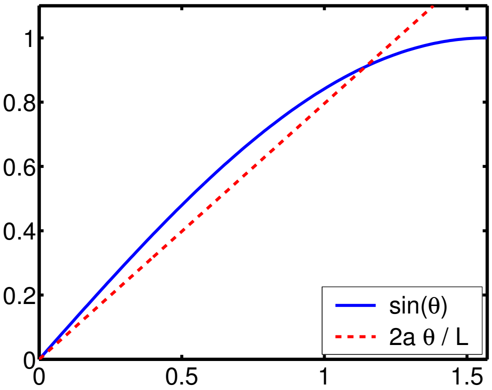
then we may compute

$$r = a/\sin(\theta_1)$$

and, we can easily see that

$$c_2 = -\cos(\theta_1)$$

from the condition that $y_1 = 0$.



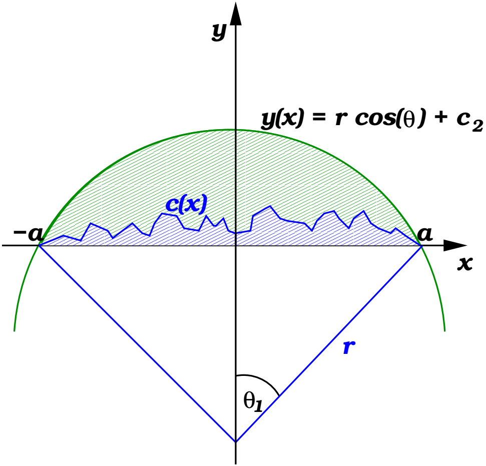
A realistic coast

What effect would a realistic coastline have?

Coast $c(x)$.

$$\text{Area} = \int_{x_0}^{x_1} y - c\,dx$$

But note that c doesn't depend on y or y', so the Euler-Lagrange equations are unchanged, provided $c(x) < y(x)$ for the extremal.



A realistic coast

Note the caveat: $c(x) < y(x)$. If this is not satisfied then the area integral includes negative components, so the problem we are maximizing is not really Dido's problem any more (she can't own negative areas).

We really want to maximize

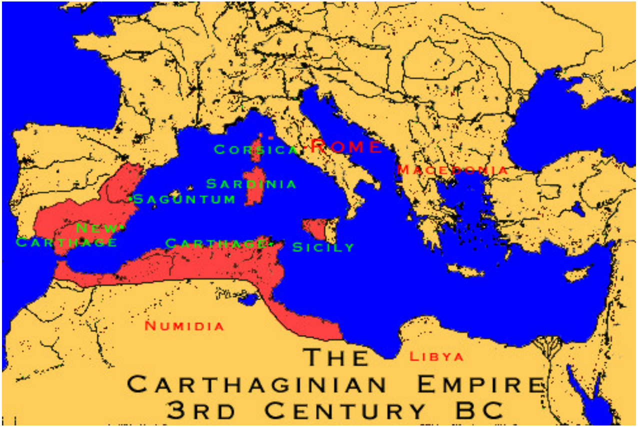
$$\text{Area} = \int_{x_0}^{x_1} [y - c]^+ \, dx$$

where

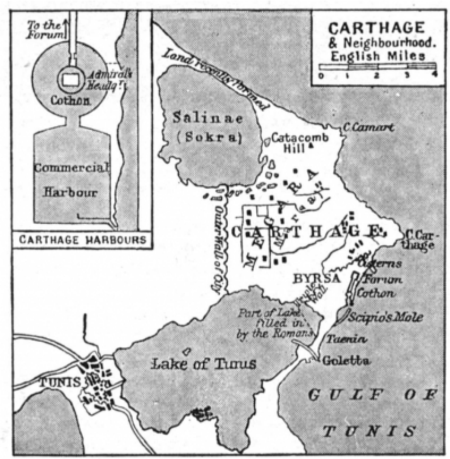
$$[x]^+ = \begin{cases} x, & \text{for } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

But this function does not have a derivative at $x = 0$.

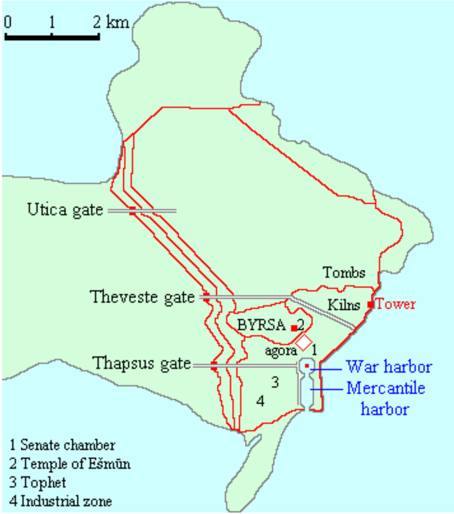
The real Carthage (modern Tunis)



The real Carthage (modern Tunis)



The real Carthage (modern Tunis)



Catenary redux

We previously computed the shape of a suspended wire, when we put no constraints on the length of the wire.

What happens to the shape of the suspended wire when we fix the length of the wire?

As before we seek a minimum for the potential energy

$$W_p\{y\} = mg \int_{x_0}^{x_1} y\sqrt{1+y'^2}dx$$

but now we include the constraint that the length of the wire is L, e.g.

$$G\{y\} = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx = L$$

Catenary of fixed length

We instead seek extremals of

$$H\{y\} = \int_{x_0}^{x_1} (y + \lambda)\sqrt{1 + y'^2}\, dx$$

Notice that the above has no explicit dependence on x, and so we may compute

$$H(y, y') = y' \frac{\partial h}{\partial y'} - h = \text{const}$$

(apologies for reusing the notation H here).

Catenary of fixed length

$$H(y,y') = \frac{(y+\lambda)y'^2}{\sqrt{1+y'^2}} - (y+\lambda)\sqrt{1+y'^2} = \text{const}$$

Perform the change of variables $u = y + \lambda$, and note that $u' = y'$ so that the above can be rewritten as

$$\frac{uu'^2}{\sqrt{1+u'^2}} - u\sqrt{1+u'^2} = c_1$$

and as before (as with the earlier catenary example), this reduces to

$$\frac{u^2}{1+u'^2} = c_1^2$$

Equation of Catenary of Fixed Length

This is exactly the same equation (in u) as we had previously for the catenary in y. So the result is a catenary also, but shifted up or down by an amount such that the length of the wire is L.

$$\begin{aligned} y &= u - \lambda \\ &= c_1 \cosh\left(\frac{x - c_2}{c_1}\right) - \lambda \end{aligned}$$

so we have three constants to determine.

- we have two end points
- we have the length constraint

The length of the Catenary

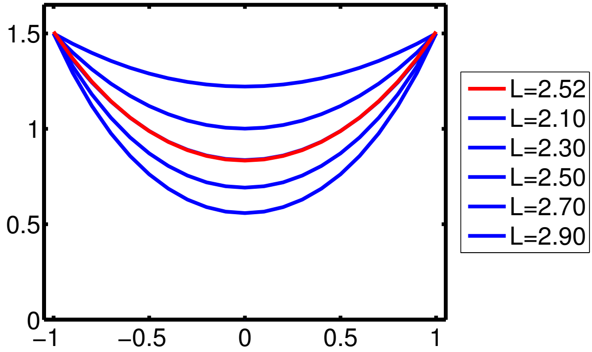
As before (taking the even solution with $y_0 = y_1$)

$$\begin{aligned} L\{y\} &= \int_{-1}^{1} \sqrt{1 + y'^2}\, dx \\ &= \int_{-1}^{1} \cosh(x/c_1)\, dx \\ &= c_1 \left[\sinh(x/c_1)\right]_{-1}^{1} \\ &= 2c_1 \sinh(1/c_1) \end{aligned}$$

But now we can use this as a constraint to calculate c_1 given L.
Once we know c_1 we can calculate λ to satisfy end heights $y_0 = y_1$.

Catenary of fixed length

All catenaries are valid, but one is **natural**



The red curve shows the natural catenary (without length constraints), and the blue curves show other catenaries with different lengths

Calculating the functional

As before its easy to calculate $F\{y\}$,

$$
\begin{aligned}
F\{y\} &= \int_{-1}^{1} (c_1 \cosh(x/c_1) - \lambda) \sqrt{1 + \sinh^2(x/c_1)}\, dx \\
&= \int_{-1}^{1} c_1 \cosh^2(x/c_1) - \lambda \cosh(x/c_1)\, dx \\
&= c_1 + \frac{c_1^2}{2} \sinh(2/c_1) - 2\lambda c_1 \sinh(1/c_1)
\end{aligned}
$$

Note however, that this assumes that $y < 0$ is possible. If not, then we have to truncate y and calculate the integral numerically.

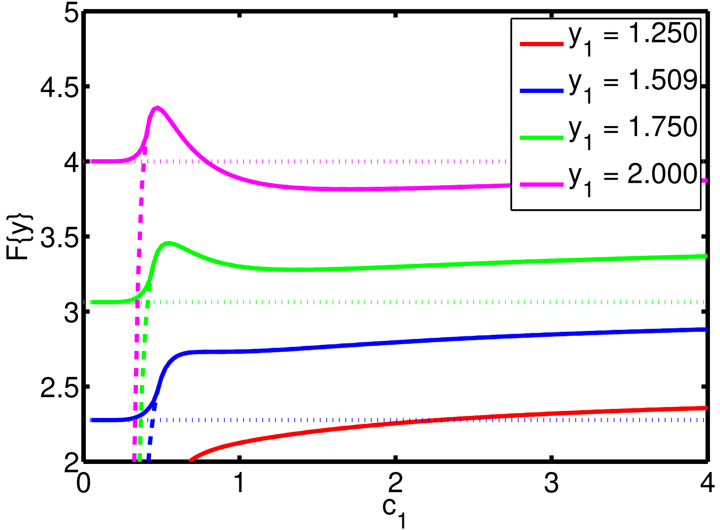
Degenerate solution

The degenerate solution has the wire lying on the ground, but we have to add in the energy of the wire leading from the pole to the ground at each end

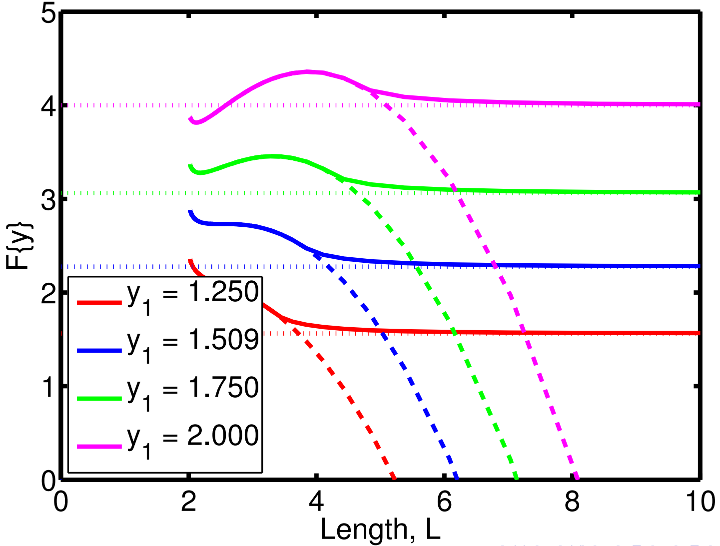
$$
\begin{aligned}
F\{y\} &= \int_0^{y_0} s\,ds + \int_0^{y_1} s\,ds \\
&= \frac{y_0^2}{2} + \frac{y_1^2}{2}
\end{aligned}
$$

- This is the energy of the degenerate solution
- It isn't necessarily the minimum energy configuration

Calculating the functional (energy)



Energy as a function of length



Pathologies

Notice in both cases above

- the approach only works for certain ranges of L.
- if L is too small, there is no physically possible solution.
    - e.g. if wire length $L < x_1 - x_0$
    - e.g. if oxhide length $L < x_1 - x_0$
- if L is too large in comparison to y_1, the solution may have our wire dragging on the ground.

Rigid Extremals

A particular problem to watch for are **rigid extremals**

- Extremals that cannot be perturbed, and still satisfy the constraint.
- For example

$$G\{y\} = \int_0^1 \sqrt{1 + y'^2}\, dx = \sqrt{2}$$

with the boundary constraints $y(0) = 0$ and $y(1) = 1$.

The only possible y to satisfy this constraint is $y(x) = x$, so we cannot perturb around this curve to find conditions for viable extremals.

Rigid Extremals

Rigid extremal cases have some similarities to maximization of a function, where the constraints specify a single point:

- e.g. maximize $f(x, y) = x + y$, under the constraint that $x^2 + y^2 = 0$.

In the extremal case above, the constraint, and the end-points leave only one choice of function, $y(x) = x$

Interpretation of λ

- Consider finding extremals for

$$H\{y\} = F\{y\} + \lambda G\{y\},$$

  where we include G to meet an isoperimetric constraint $G\{y\} = L$
- One way to think about λ is that we are trying to minimize $F\{y\}$ and $G\{y\} - L$
  - λ is a tradeoff between F and G
  - if λ is big, we give a lot of weight to G
  - if λ is small, then we give most weight to F
- So λ might be thought of as how hard we have to "pull" towards the constraint in order to make it

Interpretation of λ

- For example:
  - in the catenary problem, the size of λ is the amount we have to shift the $\cosh$ function up or down to get the right length.
  - when $\lambda = 0$ we get the natural catenary
    - i.e., in this case, we didn't need to change anything to get the right shape, so the constraint had no affect

Interpretation of λ

Write the problem (including the constant) as minimise

$$H\{y\} = \int f + \lambda(g - k)\,dx,$$

for constant $k = L/\int 1\,dx$, then

$$\frac{\partial H}{\partial k} = \lambda,$$

- we can also think of λ as the rate of change of the value of the optimum with respect to k
- when $\lambda = 0$, the functional H has a stationary point
  - e.g., in the catenary problem this is a local minimum corresponding to the natural catenary