Lecture 26: Preliminary materials – Riemann-Stieltjes Integration

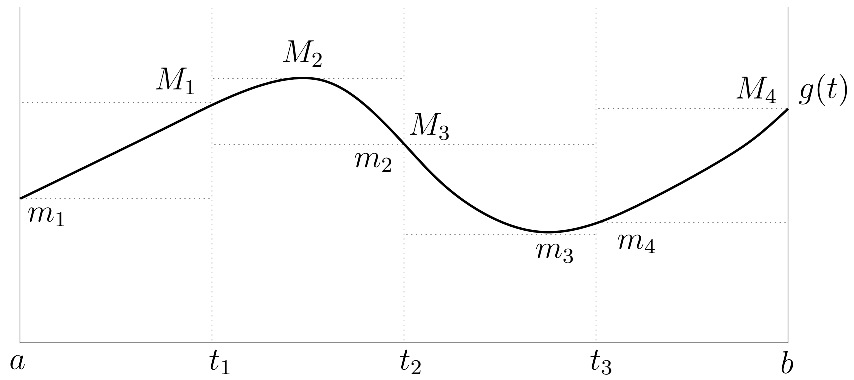
Concepts checklist

At the end of this lecture, you should be able to:

- *Understand* how the Riemann-Stieltjes integral generalises the Riemann integral; and,
- *evaluate* Riemann-Stieltjes integrals.

Riemann-Stieltjes Integration

In Maths I Calculus, you used the Riemann integral to calculate the area under some function $g(t)$ on some finite interval $[a, b]$. The formal definition of the Riemann integral is in terms of upper and lower sums of rectangular areas. That is, we partition $[a, b]$ into n sub-intervals with end points $a = t_0 < t_1 < \cdots < t_n = b$, and let M_i and m_i be the supremum and infimum of $g(t)$ on sub-interval i.



We know that

$$\sum_{i=1}^{n} m_i\,(t_i - t_{i-1}) \leq \text{ area under } g(t) \leq \sum_{i=1}^{n} M_i\,(t_i - t_{i-1})\,,$$

where

- the LHS is known as the lower (Riemann) sum with respect to the partition $\mathcal{P} = (t_0, t_1, \ldots, t_n)$, and
- the RHS is the upper (Riemann) sum with respect to $\mathcal{P}$.

Definition 22. *The Riemann integral exists if*

$$\sup_{\textit{all partitions } \mathcal{P}} \sum_{i=1}^{n} m_i\,(t_i - t_{i-1}) \quad = \quad \inf_{\textit{all partitions } \mathcal{P}} \sum_{i=1}^{n} M_i\,(t_i - t_{i-1}) \quad = \quad \ell,$$

in which case we write $\int_a^b g(t)\mathrm{d}t = \ell$.

Definition 23. *A function $f(t)$ is said to be right-continuous at a point τ, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(t) - f(\tau)| < \varepsilon \quad \text{for all} \quad \tau < t < \tau + \delta.$$

That is,

$$\lim_{t \to \tau^+} f(t) = f(\tau).$$

Definition 24. *A right-continuous function is a function that is right-continuous at all points.*

Definition 25. *A function $f(t)$ is monotonically non-decreasing if*

$$f(x) \leq f(y) \quad \text{for all} \quad x < y.$$

Distribution functions are examples of monotonically non-decreasing functions.

Note: Similar definitions exist for left-continuous and monotonically non-increasing functions.

If $\alpha(t)$ is a right-continuous, monotonically non-decreasing function on the interval $[a, b]$, define the lower and upper sums with respect to a partition $\mathcal{P} = \{t_0, t_1, \ldots, t_n\}$ (where $t_0 = a$ and $t_n = b$) and the function $\alpha(t)$ as

$$\sum_{i=1}^{n} m_i(\alpha(t_i) - \alpha(t_{i-1})) \quad \text{and} \quad \sum_{i=1}^{n} M_i(\alpha(t_i) - \alpha(t_{i-1})).$$

Definition 26. *The Riemann-Stieltjes integral of $g(t)$ with respect to $\alpha(t)$ over $[a, b]$ exists if*

$$\sup_{\text{all partitions } \mathcal{P}} \sum_{i=1}^{n} m_i\big(\alpha(t_i) - \alpha(t_{i-1})\big) = \inf_{\text{all partitions } \mathcal{P}} \sum_{i=1}^{n} M_i\big(\alpha(t_i) - \alpha(t_{i-1})\big) = \ell,$$

in which case we write

$$\int_a^b g(t) \mathrm{d}\alpha(t) = \ell.$$

If $\alpha(t)$ is *differentiable* on $[a, b]$ then

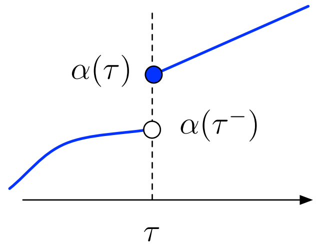
$$\int_a^b g(t) \mathrm{d}\alpha(t) = \int_a^b g(t) \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} \mathrm{d}t.$$

That is, the Riemann-Stieltjes integral is just the ordinary Riemann integral of the function

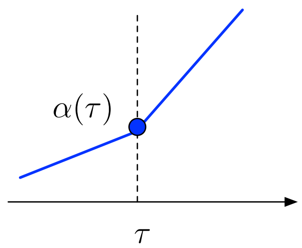
$$g(t) \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}.$$

However, the Riemann-Stieltjes integral is more general than the Riemann integral because it allows us to integrate with respect to non-differentiable $\alpha(t)$:

- The right-continuous function $\alpha(t)$ may have discontinuities. As it is monotonically non-decreasing, its left-hand limits must exist. That is, $\lim_{t\to\tau^-} \alpha(t) = \alpha(\tau^-)$.

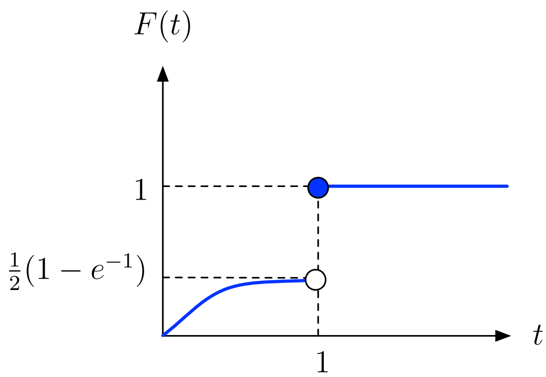


- The function $\alpha(t)$ may also be continuous but not differentiable at τ.



Example 24.

$$F(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ \dfrac{1}{2}(1 - e^{-t}) & \text{if} \quad 0 \le t < 1, \\ 1 & \text{if} \quad t \ge 1. \end{cases}$$



Now consider (a little roughly!)

$$\begin{aligned} \int_0^\infty g(t)\mathrm{d}F(t) &= \lim_{N\to\infty} \sum_{n=1}^{N} M_n \left[F(t_n) - F(t_{n-1})\right] \quad \text{for an appropriate partition } t_0, t_1, \ldots, \\ &= \lim_{\varepsilon\to 0^+} \int_0^{1-\varepsilon} g(t)\frac{\mathrm{d}F(t)}{\mathrm{d}t}\mathrm{d}t + g(1)\left[F(1) - F(1-\varepsilon)\right] + \int_{1+\varepsilon}^{\infty} g(t)\frac{dF(t)}{\mathrm{d}t}\mathrm{d}t, \\ &= \int_0^\infty g(t)\frac{\mathrm{d}F(t)}{\mathrm{d}t}\mathrm{d}t + g(1)\left[F(1) - F(1-)\right]. \end{aligned}$$

Extensions to improper integrals:

$$\lim_{b\to\infty} \int_0^b \quad \text{or} \quad \int_0^\infty$$

follow the same line as for standard Riemann integrals.

Two minor problems with the definition:

1. If $\alpha(t)$ has a discontinuity at a, we want to include it in any integral that starts (or finishes) at a. That is, we want to include the mass $(\alpha(a) - \alpha(a^-))$ at a. Thus, we interpret the lower limit of the integration to be to the left of a (and the upper limit to be to the right of b), and so we could write $\displaystyle\int_{a^-}^{b^+} g(t)\mathrm{d}\alpha(t)$.

2. There is a problem if $g(t)$ and $\alpha(t)$ have a discontinuity at the same point τ, because the upper and lower sums always differ by at least

$$\big[g(\tau) - g(\tau^-)\big]\big[\alpha(\tau) - \alpha(\tau^-)\big].$$

This can really only be resolved by resorting to more advanced measure theory. However, we can get over this by regarding the correct contribution of this point to the integral as

$$g(\tau)\big(\alpha(\tau) - \alpha(\tau^-)\big).$$

Examples (Exercises in class)

Calculate the Laplace-Stieltjes transform

$$\int_0^\infty \mathrm{e}^{-st}\mathrm{d}F(t)$$

with respect to the following distribution functions $F(t)$, where $\lambda > 0$ and $\rho \in [0, 1)$.

(a)

$$\begin{aligned}
F(t) &= \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - e^{-\lambda t} & \text{for} \quad t \geq 0 \end{cases} \\
\Rightarrow \widehat{F}(s) &= \int_0^\infty e^{-st} dF(t) \\
&= \int_0^\infty e^{-st}\lambda e^{-\lambda t}\mathrm{d}t \\
&= \lambda \int_0^\infty e^{-(s+\lambda)t}\mathrm{d}t \\
&= \frac{\lambda}{(s+\lambda)}.
\end{aligned}$$

(b)

$$\begin{aligned}
F(t) &= \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - \rho\, e^{-\lambda t} & \text{for} \quad t \in [0, \infty) \end{cases} \\
\Rightarrow \quad \widehat{F}(s) &= \int_0^\infty e^{-st}\mathrm{d}F(t) \\
&= (1-\rho)e^{-s.0} + \lambda\rho \int_0^\infty e^{-(s+\lambda)t}\mathrm{d}t \\
&= (1-\rho) + \frac{\lambda\rho}{(s+\lambda)}.
\end{aligned}$$

(c)

$$
\begin{aligned}
F(t) &= \begin{cases} 0 & \text{for} \quad t < 0 \\ \rho & \text{for} \quad t \in [0, \lambda) \\ 1 & \text{for} \quad t \geq \lambda \end{cases} \\
\Rightarrow \qquad \widehat{F}(s) &= \int_0^\infty e^{-st} \mathrm{d}F(t) \\
&= \rho e^{-s.0} + (1-\rho) e^{-s\lambda} \\
&= \rho + (1-\rho) e^{-s\lambda}.
\end{aligned}
$$