Lecture 11: Characterisation of States and Limiting Distributions

Concepts checklist

At the end of this lecture, you should be able to:

- Characterise states of a CTMC in terms of *communicating classes*, *irreducibility*, and *recurrence / transience*;
- Understand the relationship between these characteristics and equilibrium probabilities; and,
- State a theorem regarding the existence and uniqueness of a limiting distribution for irreducible, finite-state CTMCs.

Summary of three examples of equilibrium distributions

- In Example 6. Reliability (Pure Death), we have N equilibrium probabilities equal to 0, and one equilibrium probability equal to 1.
- In Example 4. Reliability (Birth and Death), we have $N+1$ equilibrium probabilities, all of which are greater than 0.

  — Both reliability models are finite state space continuous-time Markov chain.
- In Example 3. Single-server queue, we either have
  - all positive equilibrium probabilities if $\lambda < \mu$, or
  - no solution to the equilibrium equations which sums to 1, otherwise.

  — The single server queue is an infinite state space continuous-time Markov chain.

Question: What characteristics of a CTMC lead to these different types of behaviour?

Characterisation of States

Definition 8. *For $i, j \in \mathcal{S}$, state j is said to be* **accessible** *from state i if there is some path of transitions via which the Markov chain can move from state i to state j. In other words, there exists a sequence of states $\{i = i_0, i_1, i_2, \ldots, i_n = j\}$ such that*

$$q_{i_0,i_1} q_{i_1,i_2} \cdots q_{i_{n-1},i_n} > 0.$$

Definition 9. *States i and j are said to* **communicate** *(and we write $i \leftrightsquigarrow j$) if*

1. *$j = i$, or*
2. *j is accessible from i and i is accessible from j.*

Proposition 1. *The relation $\leftrightsquigarrow$, i.e., communication, is an equivalence relation.*

Proof. We need to show that $\leftrightsquigarrow$ is *Reflexive*, *Symmetric* and *Transitive*.

(*i*) *Reflexive* means that $i \leftrightsquigarrow i$. This follows directly from the definition.

(*ii*) *Symmetric* means that if $i \leftrightsquigarrow j$ then $j \leftrightsquigarrow i$. This also follows directly from the definition.

(*iii*) *Transitive* means that if $i \leftrightsquigarrow k$ and $k \leftrightsquigarrow j$ then we have $i \leftrightsquigarrow j$.
This follows because if k is accessible from i and j is accessible from k, then there exists a path from i to j (via k). This implies that j is accessible from i. Similarly, if k is accessible from j and i from k, then i is accessible from j. Hence, $i \leftrightsquigarrow j$.

□

Corollary 1. *The state space $\mathcal{S}$ of a continuous-time Markov chain can be partitioned into* **communicating classes** $\mathcal{S}_1, \mathcal{S}_2, \dots$ *such that $i, j \in \mathcal{S}_k$ if and only if $i \leftrightsquigarrow j$.*

Example 3. M/M/1 Queue

Here, all states are accessible from every other state. Thus, there is a single communicating class $\mathcal{S} = \{0, 1, 2, \dots\}$.

Example 6. Linear pure-death process

Recall, in this example we have N individuals, each subject dying after an exponentially distributed amount of time with rate μ. Here, state $n-1$ is accessible from state n, but state n is not accessible from $n-1$. Therefore, states n and $n-1$ do not communicate. Furthermore, each state is in its own communicating class,

$$\Rightarrow \mathcal{S} = \bigcup_{i=0}^{N} \mathcal{S}_i, \text{ where } \mathcal{S}_i = \{i\}.$$

Example 4. N-machine reliability (Birth and Death)

Each state is accessible from every other state – failure, and repair! Thus, there is a single communicating class, which is the whole state space $\mathcal{S} = \{0, 1, \dots, N\}$.

Let's introduce some further terminology, to label common communicating class structures, and also some properties possessed by states in communicating classes.

Definition 10. *A continuous-time Markov chain is said to be* **irreducible** *if it has a single communicating class, and to be* **reducible** *otherwise.*

Definition 11. *A state is said to be* **recurrent** *if the probability that the continuous-time Markov chain returns to that state after it has left is 1. The state is* **transient** *otherwise.*

Definition 12. *A state that is recurrent is said to be* **positive recurrent** *if the mean return time is finite (or it is an absorbing state). Otherwise, it is called* **null recurrent**.

Within a communicating class, states are either all recurrent or all transient. Recurrence or transience is a property of communicating classes; hence in the irreducible case, recurrence or transience is a property of the CTMC itself.

The classification of states and hence communicating classes depends on the probability that a continuous-time Markov chain returns to a state after it has left it.

Theorem 6. *Consider a communicating class $\mathcal{C}$ and let $i \in \mathcal{C}$. If there exists $j \notin \mathcal{C}$ such that j is accessible from i, then $\mathcal{C}$ is transient.*

Proof. Note that state i cannot be accessible from j, because then j would be in $\mathcal{C}$. Therefore, the probability of returning to i having left it, must be less than 1. □

Theorem 7. *If $\mathcal{C}$ is finite and if for every $i \in \mathcal{C}$ there exists no $j \notin \mathcal{C}$ that is accessible from state i, then $\mathcal{C}$ is recurrent.*

Note, Theorem 7 does not extend to infinite communicating classes. For example, the single server queue with $\lambda > \mu$ is transient, and yet it has a single communicating class.

Now, returning to linking this characterisation of states and equilibrium distributions. We have

Theorem 8. *If j is in a transient communicating class $\mathcal{C}$, then there exists no solution $(\pi_i)_{i\in\mathcal{S}}$ with $\sum\limits_i \pi_i = 1$ and $\pi_j > 0$.*

This theorem shows that equilibrium probabilities for states in the transient communicating classes are equal to zero. This can arise in one of two ways:

1. The solution to the equilibrium equations for π_j is zero, as in Example 6 (Pure Death), for all states $j > 0$.
2. There exists a positive solution $(\pi_i)_{i\in\mathcal{S}}$ to the equilibrium equations, but it is impossible to normalise it such that $\sum_{i\in\mathcal{S}} \pi_i = 1$, as in the single-server queue (Example 3) with $\lambda > \mu$.

Theorem 9. *If j is in a recurrent communicating class $\mathcal{C}$, then there exists two possibilities:*

1. *$\mathcal{C}$ is positive-recurrent: There exists a solution $(\pi_i)_{i\in\mathcal{S}}$ with $\sum_{i\in\mathcal{S}} \pi_i = 1$ to the equilibrium equations, in which $\pi_j > 0$.*
2. *$\mathcal{C}$ is null-recurrent: There exists no solution $(\pi_i)_{i\in\mathcal{S}}$ with $\sum_{i\in\mathcal{S}} \pi_i = 1$ to the equilibrium equations, in which $\pi_j > 0$.*

Examples 6, 4, 3.

1. The communicating class $\mathcal{S} = \{0\}$ in Example 6 is positive-recurrent, since $\pi_0 = 1$ and $\pi_i = 0$ otherwise.
2. The communicating class $\mathcal{S} = \{0, 1, \ldots, N\}$ in Example 4 is positive-recurrent, since $\pi_i > 0$ for all $i \in \{0, 1, \ldots, N\}$.

3. The communicating class $\mathcal{S} = \{0, 1, 2, \dots\}$ in the single-server queue (Example 3)

   - with $\lambda < \mu$ is positive-recurrent,
   - with $\lambda = \mu$ is null-recurrent (will justify this later).

Finally, a theorem regarding the long-term behaviour of a certain class of CTMC.

Theorem 10. *For an irreducible finite-state CTMC $(X(t), t \geq 0)$ with state space S, there exists a unique limiting probability vector, $\pi = (\pi_i)_{i \in S}$, i.e., there exists a unique probability vector π such that*

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j, \ \ \forall i, j \in S.$$

Moreover, that limiting probability vector π is the unique stationary (and equilibrium) probability vector, i.e., if

$$\Pr(X(0) = j) = \pi_j, \ \ \forall j \in S,$$

then

$$\Pr(X(t) = j) = \pi_j, \ \ \forall j \in S \ \text{ and } t > 0.$$
