Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 23

Last lecture

- Considered the case of non-fixed end points
- For the time being we have just considered fix x and free y, y', etc.
- Solved the problem of the uniformly loaded beam with various end-point conditions
  - clamped
  - supported
  - cantilever

Line curvature

Be begin by thinking about what we mean by **curvature**.

For the curvature of a line, we mean how quickly the line changes direction.

So if we have some line given parametrically by

$$\boldsymbol{r}(s) = [x(s), y(s), z(s)],$$

where s is the arclength. Then the first derivative WRT s

$$\dot{\boldsymbol{r}}(s) = [\dot{x}(s), \dot{y}(s), \dot{z}(s)],$$

gives the tangent vector.

Line curvature

The "change of direction" we are thinking about is how quickly the tangent changes direction as a function of arclength. In other words, we want the second derivative of $\boldsymbol{r}$ WRT s. The magnitude of this is called the **line curvature**[1] $\kappa(s)$. That is

$$\begin{aligned}\kappa(s) &= |\ddot{\boldsymbol{r}}(s)| \\ &= |[\ddot{x}(s), \ddot{y}(s), \ddot{z}(s)]| \\ &= \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}.\end{aligned}$$

Note well that here dots denote differentiation with respect to the arclength s.

[1] Also sometimes called the *unsigned* line curvature.

How about $y(x)$ in $\mathbb{R}^2$?

Now if we have some curve given by $y(x)$ in $\mathbb{R}^2$ then we just need to work out the second derivatives with respect to arclength $\ddot{x}$ and $\ddot{y}$ and substitute this into the formula on the previous slide. (Of course in $\mathbb{R}^2$ then $\ddot{z} = 0$.) So by Pythagoras' theorem in differential form we have

$$ds^2 = dx^2 + dy^2.$$

Rearranging

$$\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + y'^2}} = \dot{x}$$

How about $y(x)$ in $\mathbb{R}^2$?

Differentiating WRT s using the chain rule we have

$$\begin{aligned}
\ddot{x} &= \frac{d}{ds}(\dot{x}) = \frac{d}{dx}(\dot{x}) \cdot \frac{dx}{ds} = \dot{x}(\dot{x})' \\
&= \frac{1}{\sqrt{1+y'^2}} \frac{d}{dx}\left(\frac{1}{\sqrt{1+y'^2}}\right) \\
&= \frac{1}{\sqrt{1+y'^2}} \left(-\frac{y'y''}{(1+y'^2)^{3/2}}\right) \\
&= -\frac{y'y''}{(1+y'^2)^2}.
\end{aligned}$$

Note that primes denote derivatives WRT x.

How about $y(x)$ in $\mathbb{R}^2$?

Again starting from $ds^2 = dx^2 + dy^2$ and rearranging

$$
\begin{aligned}
\frac{ds^2}{dy^2} &= \frac{dx^2}{dy^2} + 1 \\
\frac{ds}{dy} &= \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \\
\frac{dy}{ds} &= \frac{1}{\sqrt{1/y'^2 + 1}} \\
&= \frac{y'}{\sqrt{1 + y'^2}} = \dot{y}.
\end{aligned}
$$

Note too that $\dot{x}^2 + \dot{y}^2 = 1$ which we could have used to derive an expression for $\dot{y}$.

How about $y(x)$ in $\mathbb{R}^2$?

And again differentiating WRT s with the chain rule we have

$$
\begin{aligned}
\ddot{y} &= \frac{d}{ds}(\dot{y}) = \dot{x}(\dot{y})' \\
&= \frac{1}{\sqrt{1+y'^2}}\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) \\
&= \frac{1}{\sqrt{1+y'^2}}\left(\frac{y''\sqrt{1+y'^2} - y'^2y''/\sqrt{1+y'^2}}{1+y'^2}\right) \\
&= \frac{1}{\sqrt{1+y'^2}}\left(\frac{y''(1+y'^2) - y'^2y''}{(1+y'^2)^{3/2}}\right) \\
&= \frac{y''}{(1+y'^2)^2}
\end{aligned}
$$

How about $y(x)$ in $\mathbb{R}^2$?

So the line curvature squared is given by

$$\begin{aligned}\kappa^2 &= \ddot{x}^2 + \ddot{y}^2 \\ &= \frac{y'^2 y''^2}{(1+y'^2)^4} + \frac{y''^2}{(1+y'^2)^4} \\ &= \frac{y''^2(1+y'^2)}{(1+y'^2)^4} \\ &= \frac{y''^2}{(1+y'^2)^3}.\end{aligned}$$

and so the (signed) line curvature is

$$\boxed{\kappa = \frac{y''}{(1+y'^2)^{3/2}}.}$$

Tacit assumption in the bending beams examples

All the bending beam examples we have done so far, we have assumed the elastic energy in encapsulated in an integrand $\kappa y''$ where κ is some constant.

This is in fact an approximation for a material where the elastic energy is proportional to the curvature for the beam and providing the deflection y' is small.

$$\frac{y''}{(1+y'^2)^{3/2}} = y''\left(1 - \frac{3}{2}y'^2 + \mathcal{O}(y'^4)\right).$$

So we could improve our models by including a y'^2 correction term but we shouldn't use those models for situations where y' is not small – that is, large deflections or non-horizontal beams. We should just use the proper curvature.

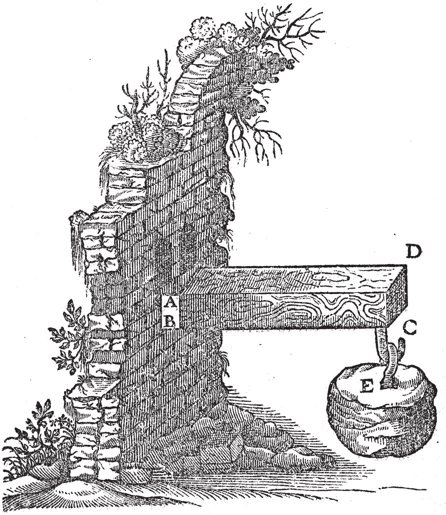
The elastica

The **elastica** are a family of curves derived from the Latin for a thin strip of elastic material. It is the curve that a thin strip of elastic material adopts when it is forced to bend.

The problem was first posed by **Jordanus de Nemore** in the 13th century in his book *De Ratione Ponderis*. His solution was fundamentally incorrect because he lacked the mathematical formalism required to solve the problem properly. However he did identify the circle as one possible solution (which it is *sometimes*).

In the 17th century the problem touched on by **Galileo**. He considered the problem of a prismatic beam set into a wall and loaded with a weight at the other end.

Galileo's problem



(From the *Discorsi*, Leiden 1638.)

How much weight is required to break the beam?

Hooke's law and Newton's calculus

Following Galileo, but still in the 17th century, we have **Hooke** who published a famous treatise on elasticity. In modern notation Hooke's law is given by

$$F \propto \Delta \ell.$$

At around the same time **Newton** used the calculus to derive an expression for the curvature. Specifically the **radius of curvature** $\rho = 1/\kappa$ was given by

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''}.$$

Note that ρ has dimensions of length and this is exactly the reciprocal of the line curvature we derived earlier.

James Bernoulli's "rectangular" elastica

Assuming a lamina AB of uniform thickness and width and negligible weight of its own, supported on its lower perimeter at A, and with a weight hung from its top at B, the force from the weight along the line BC sufficient to bend the lamina perpendicular, the curve of the lamina follows this nature:

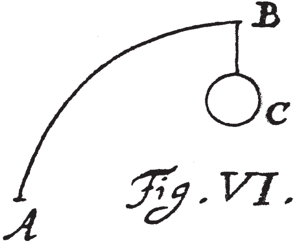


Fig. VI.

The rectangle formed by the tangent between the axis and its own tangent is a constant area.

Daniel Bernoulli suggests the problem to Euler

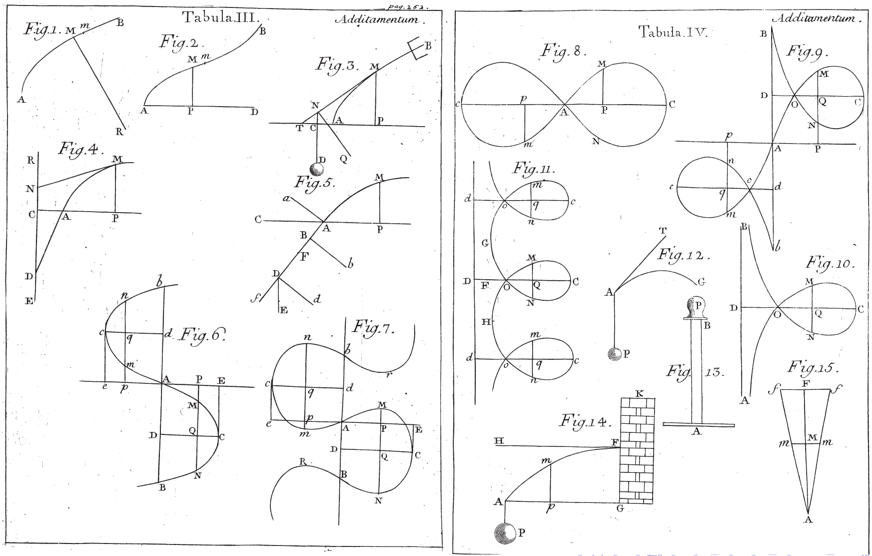
James Bernoulli's solution to the elastica (which didn't employ variational techniques) was criticised by Huygens because it only worked for the rectangular case, and around 50 years later **Daniel Bernoulli** proposed (to Euler) that using variational techniques a more general solution to the problem of the elastica might be found.

Of course, **Euler** needed no more hints. Armed with the calculus of variations a correct expression for curvature he realised that the elastica would be an extremal of the functional

$$F\{y\} = \int_0^L \kappa^2 \, ds,$$

usually subject to an arclength constraint, and he did what Euler did so often, completely solved the most general form of the problem.

Euler's elastica



Elastica: a "simple" example

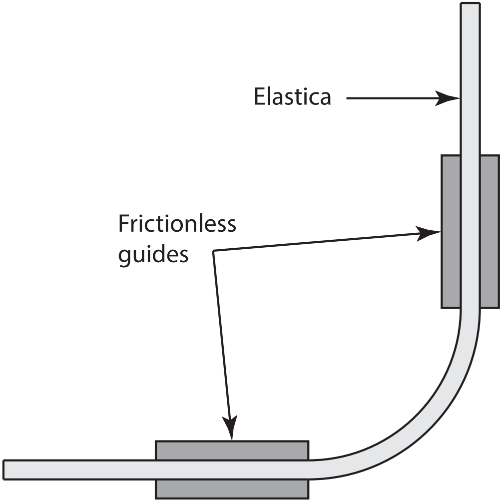
What is the equation of the elastica with end-point constraints

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = 1, \quad y'(1) \to \infty,$$

where we prescribe no length constraint?

This can be thought of as an elastica threaded through two frictionless guides that fix the position and derivative for the elastica at the end-points but do not constrain its length.

Elastica: a "simple" example



What is the shape of the curved elastica between the two guides?

Elastica: a "simple" example

Since we have fixed end-points and no length constraint there is no isoperimetric or natural boundary conditions with this problem. It is just a standard variational problem where the integrand of the functional is κ^2. That is

$$
\begin{aligned}
F\{y\} &= \int_0^L \frac{y''^2}{(1+y'^2)^3} ds \\
&= \int_{x_0}^{x_1} \frac{y''^2}{(1+y'^2)^3} \sqrt{1+y'^2}\, dx \\
&= \int_0^1 \frac{y''^2}{(1+y'^2)^{5/2}} dx.
\end{aligned}
$$

The Euler-Poisson equation

So the Euler-Poisson equation for this is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0.$$

Since f is independent of y we can immediately integrate once to derive

$$\frac{\partial f}{\partial y'} = \alpha + \frac{d}{dx}\frac{\partial f}{\partial y''}.$$

Now from the chain rule we have that

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + y'''\frac{\partial f}{\partial y''}.$$

The Euler-Poisson equation

Since this particular f is independent of x and y we may deduce that

$$\frac{df}{dx} = y''\frac{\partial f}{\partial y'} + y'''\frac{\partial f}{\partial y''},$$

and substituting for $\partial f/\partial y'$ from the previous slide we have

$$\begin{aligned}\frac{df}{dx} &= y''\left(\alpha + \frac{d}{dx}\frac{\partial f}{\partial y''}\right) + y'''\frac{\partial f}{\partial y''}\\ &= \frac{d}{dx}\left(\alpha y' + y''\frac{\partial f}{\partial y''}\right),\end{aligned}$$

and so integrating a second time we derive

$$f - y''\frac{\partial f}{\partial y''} = \alpha y' - \beta.$$

Curvature equation

We have succeeded in integrating twice and replaced a fourth-order equation (the original Euler-Possion equation) with a second order equation containing two arbitrary constants. The previous result is true for any functional which depends on y' and y'' alone. Now we need to substitute our particular f into this equation.

$$\begin{aligned}\frac{y''^2}{(1+y'^2)^{5/2}} - y''\frac{2y''}{(1+y'^2)^{5/2}} &= \alpha y' - \beta \\ \frac{y''^2}{(1+y'^2)^3} &= \frac{\beta - \alpha y'}{(1+y'^2)^{1/2}},\end{aligned}$$

or in other words

$$\kappa = \left(\frac{\beta - \alpha y'}{(1+y'^2)^{1/2}}\right)^{1/2}.$$

Substitution

Now we want to simplify things further and to this end we introduce the substitution

$$y' = \tan\theta.$$

Differentiating this substitution WRT x using the chain rule we have

$$y'' = \sec^2\theta \frac{d\theta}{dx}.$$

So the curvature is equal to

$$\kappa = \frac{y''}{(1+y'^2)^{3/2}} = \frac{\sec^2\theta}{(1+\tan^2\theta)^{3/2}}\frac{d\theta}{dx} = \cos\theta\frac{d\theta}{dx}$$

Note, here we use the signed curvature but in this problem we may assume that curvature is strictly positive.

Substitution

Also, since $d\theta/dx = (dy/dx)(d\theta/dy) = \tan\theta(d\theta/dy)$ we have

$$\kappa = \cos\theta\frac{d\theta}{dx} = \sin\theta\frac{d\theta}{dy}.$$

Now making the same substitution into the curvature equation from two slides ago we may also derive

$$\begin{aligned}\kappa &= \left(\frac{\beta - \alpha y'}{(1+y'^2)^{1/2}}\right)^{1/2}\\ &= \left(\frac{\beta - \alpha\tan\theta}{(1+\tan^2\theta)^{1/2}}\right)^{1/2}\\ &= (\beta\cos\theta - \alpha\sin\theta)^{1/2}.\end{aligned}$$

Parametric equations for x and y in terms of θ

Now considering x and y to be functions of the parameter θ we can state a system of first order differential equations

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{\cos\theta}{(\beta\cos\theta - \alpha\sin\theta)^{1/2}},\\ \frac{dy}{d\theta} &= \frac{\sin\theta}{(\beta\cos\theta - \alpha\sin\theta)^{1/2}}.\end{aligned}$$

The domain of the parameter can be deduced from the boundary conditions. At $x = 0$ we have $y'(0) = 0$ and since $y' = \tan\theta$, this means $\theta = 0$. Likewise as $x \to 1$ then $y' \to \infty$, so formally we take this to correspond to $\theta = \pi/2$.

Now α and β are still arbitrary so let's choose some more convenient constants.

New constants (and parameter)

To begin with we let

$$A = \frac{1}{(\alpha^2+\beta^2)^{1/4}}, \quad B = \cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right).$$

Substitution into our system of DEs yields

$$\frac{dx}{d\theta} = \frac{A\cos\theta}{\sqrt{\cos(\theta+B)}}, \quad \frac{dy}{d\theta} = \frac{A\sin\theta}{\sqrt{\cos(\theta+B)}}.$$

Now let $\phi = (\theta+B)/2$ then

$$\frac{dx}{d\phi} = \frac{2A\cos(2\phi-B)}{\sqrt{\cos 2\phi}}, \quad \frac{dy}{d\phi} = \frac{2A\sin(2\phi-B)}{\sqrt{\cos 2\phi}}.$$

More new constants

Now we expand the numerators using the multiple angle formula. The x equation is

$$\frac{dx}{d\phi} = 2A\cos B\sqrt{\cos 2\phi} + 2A\sin B\frac{\sin 2\phi}{\sqrt{\cos 2\phi}}.$$

It makes sense to let $c_1 = 2A\cos B$, and $c_2 = 2A\sin B$ and therefore

$$\frac{dx}{d\phi} = c_1\sqrt{\cos 2\phi} + c_2\frac{\sin 2\phi}{\sqrt{\cos 2\phi}},$$
$$\frac{dy}{d\phi} = c_1\frac{\sin 2\phi}{\sqrt{\cos 2\phi}} - c_2\sqrt{\cos 2\phi}.$$

Parametric solution

Now we have the following results

$$\int \frac{\sin 2\phi}{\sqrt{\cos 2\phi}} d\phi = -\sqrt{\cos 2\phi} + \text{const.}$$

$$\begin{aligned}\int \sqrt{\cos 2\phi}\, d\phi &= \int \sqrt{1 - 2\sin^2\phi}\, d\phi \\ &= E(\phi, \sqrt{2}) + \text{const.}\end{aligned}$$

So we can now integrate our system to give

$$\begin{aligned} x(\phi) &= c_3 + c_1 E(\phi, \sqrt{2}) - c_2\sqrt{\cos 2\phi}, \\ y(\phi) &= c_4 - c_1\sqrt{\cos 2\phi} - c_2 E(\phi, \sqrt{2}), \end{aligned}$$

and we will need to solve for the arbitrary constants using the end-point conditions.

Finding the constants

Finding the constants for this problem is not entirely straightforward. Part of the problem is our new parametric variable ϕ is tied up with the constant B and so even if you know the starting and ending values you are still going to have a transcendental equations to solve numerically.

Alternatively, we can try to work with the simple final form of the solution. What do we know?

- The solution will have reflective symmetry about $y = 1 - x$.
- $E(\phi, \sqrt{2})$ and $\sqrt{\cos 2\phi}$ are odd or even
- and both are real for $-\pi/4 \leqslant \phi \leqslant \pi/4$.

Considering these factors we would expect the solution to have some features like

- Domain of ϕ to be centred on 0. That is $-\phi_0 \leqslant \phi \leqslant \phi_0$.
- Since $0 \leqslant \theta \leqslant \pi/2$ then we expect $\phi_0 = \pi/8$.

Finding the constants

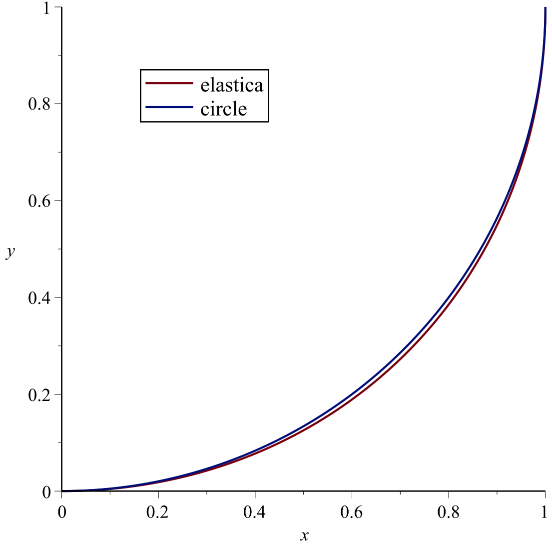
Bearing these considerations in mind we can find the values of c_1, c_2, c_3, and c_4 that satisfy the equations

$$x(-\pi/8) = 0, \quad x(\pi/8) = 1, \quad y(-\pi/8) = 0, \quad y(\pi/8) = 1.$$

After some algebra we find

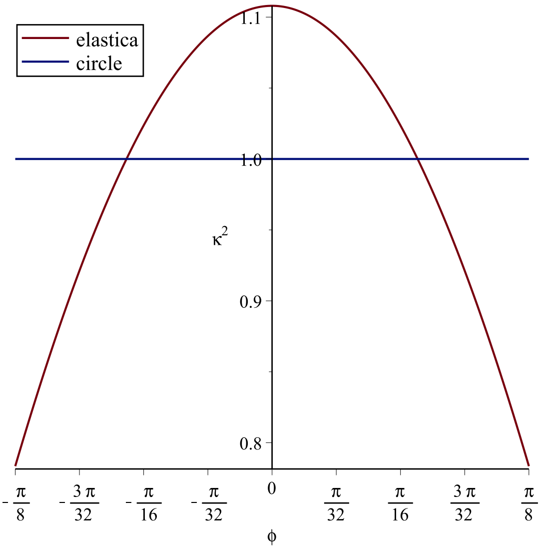
$$\begin{aligned}
c_1 &= \frac{1}{2E(\pi/8, \sqrt{2})}, \\
c_2 &= -\frac{1}{2E(\pi/8, \sqrt{2})} = -c_1, \\
c_3 &= \frac{1}{2} - \frac{1}{2^{5/4}E(\pi/8, \sqrt{2})} = \frac{1}{2} - \frac{c_1}{2^{1/4}}, \\
c_4 &= \frac{1}{2} + \frac{1}{2^{5/4}E(\pi/8, \sqrt{2})} = \frac{1}{2} + \frac{c_1}{2^{1/4}}.
\end{aligned}$$

Solution: plot of $(x(\phi), y(\phi))$



So we find the elastica for our problem without a length constraint is not quite a circle.

Solution: plot of κ^2



Or in other words, in minimising κ^2 with our particular end-point constraints, then a curve with constant curvature (a circle) is not an extremal.

Isn't a circle close enough?

It seems that the difference between the elastica and a quarter circle is not great and couldn't we just ignore it?
We could have done out problem with slightly different boundary conditions to generate a "semicircle" instead.
If we choose $y(-1) = 1$ and $y'(-1) \to -\infty$ as the lower end-point all the equations above are still valid, and all we need to do is solve

$$x(-\pi/4) = -1, \quad y(-\pi/4) = 1, \quad x(\pi/4) = 1, \quad y(\pi/4) = 1,$$

which leads to

$$c_1 = \frac{1}{E(\pi/4, \sqrt{2})}, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 1.$$

"Semicircular" elastica