Practical Asymptotics

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Lecture 0: Outline/motivation (1)

Often real-world modelling problems give rise to equations which have no exact solution. One option is to solve them numerically; this can be great if (for example) interest lies in modelling of phenomena in geometrically complicated regions. Another option is to use asymptotic methods. This involves rigourously identifying the important mechanisms at play in a particular problem and writing down (approximate) analytic expressions to describe the solution behaviour.

Lecture 0: Outline/motivation (2)

The key advantage of such an approach is that these expressions can provide insight that is not possible with numerical simulations; it is much easier see how a varying a parameter value changes a solution if we have a formula, rather than having to wait for a simulation to be rerun. In practice when asymptotics approaches are used they are presented alongside computational results, but are attractive for this extra layer of insight they provide.

In this course we will:

- develop a tool-kit of useful asymptotic techniques; and
- apply these techniques to real-world modelling problems.

Lecture 0: Outline/motivation (3)

The plan for the course is as follows:

1. **Introduction to asymptotics**
   Key concepts involved in asymptotic methods; example application to the solution of differential equations.
   (Bender & Orszag, Chap. 3; Bowen & Witelski, Chap. 6)

2. **Perturbation methods** Introduce this broad class of techniques; examples from fluid mechanics and other case studies.
   (Bender & Orszag, Chap. 7; Bowen & Witelski, Chaps. 6, 8 & 12)

3. **Boundary layer theory and asymptotic matching** Discuss techniques for solving phenomena that vary over a thin region, and how to incorporate them into the bigger picture; examples from fluid mechanics and mathematical biology.
   (Bender & Orszag, Chap. 9; Bowen & Witelski, Chaps. 7 & 12)

Lecture 0: Outline/motivation (4)

4. **Multiscale methods and homogenisation theory**
   Discuss techniques for solving problems that involve multiple space/time scales. Techniques to average ('homogenise') over small scale variation. Examples from solid mechanics and mathematical biology.
   (Bender & Orszag, Chap. 11; Bowen & Witelski, Chaps. 9 & 10)

5. **Extension topics/more case studies** TBD: we can discuss more case studies or, subject to interest, look at extension topics. Possible topics include: asymptotic approximation of integrals; summation of divergent series; WKB theory; asymptotics beyond-all-orders.

Lecture 0: Outline/motivation (5)

In these notes, a grey box like this one indicates an example or further discussion of a topic.

References (none required, all helpful in their own way)

T. Witelski, M. Bowen, Methods of Mathematical Modelling: Continuous Systems and Differential Equations, Springer, 2015. (electronic version available from UoA library)

C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, Springer, 1999. (electronic version available from UoA library)

Lecture 0: Outline/motivation (6)

M. Van Dyke, Perturbation Methods in Fluid Mechanics, The Parabolic Press, 1978.

E.J. Hinch, Pertrubation Methods, Cambridge University Press, 1991.

S.W. McCue, Preface to fourth Special Issue on Practical Asymptotics, 63:153–154, Journal of Engineering Mathematics, 2009.

W.R. Smith, Preface to the sixth special issue on "Practical Asymptotics", 102:1–2, Journal of Engineering Mathematics, 2017.

Lecture 1: Introduction to asymptotics (1)

Let's find the real solution (say at $x = a$) to the following quintic (fifth-degree polynomial) equation

$$x^5 + x = 1. \tag{1}$$

We know a few things about this equation:

- the highest power is x^5 so there are 5 roots, some which are probably complex;
- there is no exact solution for these roots (that only works if the highest power is x^4).

Lecture 1: Introduction to asymptotics (2)

It'd be straightforward to solve this numerically, but another approach is to consider a related problem namely

$$x^5 + \epsilon x = 1, \tag{2}$$

where the second term on the right-hand side is now multiplied by a parameter ϵ. Let's denote the real root of this equation $x = a(\epsilon)$. This becomes the original equation when $\epsilon = 1$. We can view this as a 'perturbation' problem: it features a parameter (usually denoted ϵ) and when ϵ is set to zero the problem is easily solvable. It's really the behaviour when $\epsilon \neq 0$ that's of interest.

Working: series solution to the quintic (2)

Lecture 1: Introduction to asymptotics (3)

The process we went through was to convert an extremely difficult problem into a sequence of easy problems, then piece those together to get an approximate (and very informative) solution to the original problem. We'll extend these ideas to differential equations and more 'real-world' applications soon, but first need some new notation.

Lecture 1: Introduction to asymptotics (4)

Examples: ... but first some other notations

Lecture 2: Introduction to asymptotics (5)

Let's introduce some new notation. Say we have two functions $f(x)$ and $g(x)$, then

$$\underbrace{f(x) \sim g(x),}_{\text{“}f(x)\text{ is asymptotic to }g(x)\ldots} \qquad \underbrace{x \to x_0,}_{\text{as }x\text{ goes to }x_0\text{”}} \tag{3}$$

which means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{4}$$

There are two parts to this notation: the expression with the $\sim$ and the associated limit. Strictly, both parts are required for this to make sense.

Examples: use of $\sim$

Lecture 2: Introduction to asymptotics (6)

Let's introduce some more new notation. Say we have two functions $f(x)$ and $g(x)$, then

$$\underbrace{f(x) \ll g(x),}_{\text{“}f(x)\text{ is much smaller than }g(x)} \qquad \underbrace{x \to x_0,}_{\text{as }x\text{ goes to }x_0\text{”}} \tag{5}$$

which means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{6}$$

This notation works in a similar way to $\sim$ in that a $\ll$ statement should always be associated with a limit.

Examples: use of $\ll$

Lecture 2: Introduction to asymptotics (7)

Asymptotic relations can be manipulated in a similar way to equations. We can add, subtract, divide, cross-multiply, differentiate, integrate and so on. This should be done with care. For example, sometimes it is not necessarily valid to exponentiate both sides of an asymptotic relation (we'll see an example when we look at differential equations).

Switching between asymptotic and exact relations

Lecture 2: Introduction to asymptotics (8)

Let's look at the quintic problem (1) again. We artificially inserted an ϵ into that equation in front of the x term. What would have happened if we'd inserted it in front of the x^5? The problem is then

$$\epsilon x^5 + x = 1. \tag{7}$$

This is an example of a **singular perturbation** problem, since the behaviour of this equation is fundamentally different in the limit $\epsilon \to 0$.

Unpack this idea

Lecture 2: Introduction to asymptotics (9)

We are going to apply the **method of dominant balance** to (7); this is a systematic way of analysing the behaviour of the equation in the limit $\epsilon \to 0$ and converting the equation to a simpler asymptotic relation.

Introduce method of dominant balance

The three possibilities for $\epsilon x^5 + x = 1$ are:

1. $x \sim 1$ as $\epsilon \to 0$ (neglect ϵx^5).

Discuss this balance

Lecture 2: Introduction to asymptotics (10)

2. $\epsilon x^5 \sim 1$ as $\epsilon \to 0$ (neglect x).

Discuss this balance

3. $\epsilon x^5 \sim -x$ as $\epsilon \to 0$ (neglect 1).

Discuss this balance

What did all that tell us?

This illustrates the power of asymptotic techniques! By making a few relatively simple arguments we were able to convert this difficult singularly perturbed quintic problem (which we would otherwise need to treat numerically) into a some very simple problems. Similar techniques can be applied to differential equations, where we will use the method of dominant balance on more complicated expressions.

Lecture 3: Introduction to asymptotics (11)

Consider a linear second-order homogeneous differential equation in standard form:

$$y'' + a(x)y' + b(x)y = 0. \tag{8}$$

Let's say we're interested in the **local behaviour** of this equation near the point $x = x_0$ and want to construct an approximate solution. The form of solution depends on the behaviour of $a(x)$ and $b(x)$ near x_0:

1. $x = x_0$ is a **ordinary point** (that is $a(x)$ and $b(x)$ analytic in neighbourhood of x_0): use a Taylor series expansion,

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Lecture 3: Introduction to asymptotics (12)

2. $x = x_0$ is a **regular singular point** (that is $(x - x_0)a(x)$ and $(x - x_0)^2 b(x)$ are analytic in neighbourhood of x_0): use a Frobenius type expansion, for example

$$y = (x - x_0)^\alpha \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

3. $x = x_0$ is an **irregular singular point** (ie. not ordinary or regular singular): use asymptotics. We start by assuming the solution is of the form

$$y(x) = \mathrm{e}^{S(x)}, \tag{9}$$

and then construct a solution by repeatedly using the method of dominant balance.

Lecture 3: Introduction to asymptotics (13)

Examples of classifying points of DEs

Note: to look at the 'local' behaviour as $x \to \infty$, make a change of variables $x = 1/t$ and classify the point $t = 0$.

Lecture 3: Introduction to asymptotics (14)

Let's do an example. Consider the differential equation

$$x^3 y'' = y. \tag{10}$$

Let's examine the local behaviour of this equation as $x \to 0$.

Solve (10) as $x \to 0$ with method of dominant balance.

Putting this in standard form:

$$y'' = \frac{1}{x^3} y,$$

ie. easy to see that $x = 0$ is an irregular singular point. Going to try for a solution of the form $y = e^{S(x)}$. This gives:

$$\begin{aligned} y' &= S' \mathrm{e}^S \\ y'' &= S'' \mathrm{e}^S + (S')^2 \mathrm{e}^S = (S'' + (S')^2) \mathrm{e}^S \end{aligned}$$

Substitute all that into the DE to get

$$S'' + (S')^2 = \frac{1}{x^3}$$

We now apply the method of dominant balance to this equation.

There are 3 possible balances:

1. $S'' \sim 1/x^3$ as $x \to 0$, neglect $(S')^2$

   Integrating once gives $S' \sim -1/(2x^2)$, but then $(S')^2 \sim 1/(4x^4)$. This is a contradiction because we'd assumed that $(S')^2 \ll 1/x^3$ as $x \to 0$.

2. $(S')^2 \sim 1/x^3$ as $x \to 0$, neglect S''

   Rearranging gives $S' \sim \pm x^{-3/2} \implies S'' \sim \mp\frac{3}{2}x^{-5/2} \ll x^{-3}$ as $x \to 0$. There is no contradiction here! ✓

3. $S'' \sim -(S')^2$ as $x \to 0$, neglect $1/x^3$

   Let $T = S'$, then have $T' \sim -T^2$ or $-T^{-2}T' \sim 1$ as $x \to 0$. Integrating gives $T^{-1} \sim x + a \sim a$ as $x \to 0$, that is $(S')^2 \sim 1/a^2 \ll x^{-3}$, which give the contradiction.

This tells us that the appropriate dominant balance is $(S')^2 \sim 1/x^3$, as above this gives

$$S' \sim \pm x^{-3/2} \implies S \sim \mp 2x^{-1/2}, \quad \text{as } x \to 0.$$

Let's just focus on the positive solution (similar procedure for negative, there need to be two linearly independent solutions). Converting this to an (exact) equation gives

$$S(x) = \mp 2x^{-1/2} + C(x), \quad \text{where } C(x) \ll 2x^{-1/2} \quad \text{as } x \to 0.$$

Now we solve for $C(x)$. (We are iteratively finding corrections to $S(x)$) Substituting this into the DE for $S(x)$ above gives

$$\begin{aligned} S'' + (S')^2 &= x^{-3} \\ \mp\frac{3}{2}x^{-5/2} + C'' + (\pm x^{-3/2} + C')^2 &= x^{-3} \\ \mp\frac{3}{2}x^{-5/2} + C'' + x^{-3} \pm 2x^{-3/2}C' + (C')^2 &= x^{-3} \\ \mp\frac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 &= 0 \end{aligned}$$

(Note the cancellation of x^{-3} indicates we did the first bit correctly)

Now we apply the method of dominant balance to the equation:

$$\mp\frac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 = 0$$

Luckily we already know a few things about $C(x)$, namely that

$$\begin{aligned} C(x) &\ll x^{-1/2}, \quad \text{as } x \to 0, \\ C'(x) &\ll x^{-3/2}, \quad \text{as } x \to 0, \\ C''(x) &\ll x^{-5/2}, \quad \text{as } x \to 0. \end{aligned}$$

The last fact means that it will always valid to neglect C'' since there is a $x^{-5/2}$ term in the equation. Multiplying the both sides of the second by C' gives $(C')^2 \ll C'x^{-3/2}$ which similarly implies the $(C')^2$ term can be neglected.

This leaves the only possible balance (since we need at least two terms) as

$$2x^{-3/2}C' \sim \frac{3}{2}x^{-5/2}, \quad \text{as } x \to 0,$$
$$C' \sim \frac{3}{4}x^{-1}, \quad \text{as } x \to 0,$$
$$C \sim \frac{3}{4}\log x, \quad \text{as } x \to 0.$$

So we have

$$S \sim \mp 2x^{-1/2} + \frac{3}{4}\log x, \quad \text{as } x \to 0.$$

Let's keep going!

Converting the expression for C to an equation:

$$C = \frac{3}{4}\log x + D(x), \quad \text{where } D(x) \ll \frac{3}{4}\log x \quad \text{as } x \to 0.$$

Substituting into the DE for C gives

$0 = \mp\frac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2$

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Substituting into the DE for C gives

$$\begin{aligned} 0 &= \mp\tfrac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 \\ &= \mp\tfrac{3}{2}x^{-5/2} - \tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}\left(\tfrac{3}{4}x^{-1} + D'\right) + \left(\tfrac{3}{4}x^{-1} + D'\right)^2 \end{aligned}$$

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Substituting into the DE for C gives

$$\begin{aligned}
0 &= \mp\tfrac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 \\
&= \mp\tfrac{3}{2}x^{-5/2} - \tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}\left(\tfrac{3}{4}x^{-1} + D'\right) + \left(\tfrac{3}{4}x^{-1} + D'\right)^2 \\
&= -\tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}D' + \left(\tfrac{9}{16}x^{-2} + \tfrac{3}{2}x^{-1}D' + (D')^2\right)
\end{aligned}$$

Converting the expression for C to an equation:

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Substituting into the DE for C gives

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0 &= \mp\tfrac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 \\
&= \mp\tfrac{3}{2}x^{-5/2} - \tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}\left(\tfrac{3}{4}x^{-1} + D'\right) + \left(\tfrac{3}{4}x^{-1} + D'\right)^2 \\
&= -\tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}D' + \left(\tfrac{9}{16}x^{-2} + \tfrac{3}{2}x^{-1}D' + (D')^2\right) \\
&= -\tfrac{3}{16}x^{-2} + D'' + (\pm 2x^{-3/2} + \tfrac{3}{2}x^{-1})D' + (D')^2
\end{aligned}$$

Apply the method of dominant balance to

$$-\frac{3}{16}x^{-2} + D'' + (\pm 2x^{-3/2} + \frac{3}{2}x^{-1})D' + (D')^2 = 0$$

We can immediately neglect the $\frac{3}{2}x^{-1}D'$ term since $x^{-1} \ll x^{-3/2}$ as $x \to 0$. We know the following about D:

$$\begin{aligned} D(x) &\ll \log x, \quad \text{as } x \to 0, \\ D'(x) &\ll x^{-1}, \quad \text{as } x \to 0, \\ D''(x) &\ll -x^{-2}, \quad \text{as } x \to 0. \end{aligned}$$

The last of these means we can neglect the D'' term since there is and x^{-2} term present. Multiplying the second by D' gives $(D')^2 \ll x^{-1}D'$ and so the $(D')^2$ is negligible as $x \to 0$.

The remaining balance is then

$$2x^{-3/2}D' \sim \pm\tfrac{3}{16}x^{-2}, \quad \text{as } x \to 0$$

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\begin{aligned}
2x^{-3/2}D' &\sim \pm\tfrac{3}{16}x^{-2}, \quad \text{as } x \to 0 \\
D' &\sim \pm\tfrac{3}{32}x^{-1/2}, \quad \text{as } x \to 0
\end{aligned}
$$

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$$
\begin{aligned}
2x^{-3/2}D' &\sim \pm\tfrac{3}{16}x^{-2}, \quad \text{as } x \to 0\\
D' &\sim \pm\tfrac{3}{32}x^{-1/2}, \quad \text{as } x \to 0\\
\implies D(x) - d &\sim \pm\tfrac{3}{16}x^{1/2}, \quad \text{as } x \to 0
\end{aligned}
$$

where we have retained the constant of integration since the right-hand side vanishes $(x^{1/2} \ll 1)$ as $x \to 0$.

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\end{aligned}
$$

where we have retained the constant of integration since the right-hand side vanishes ($x^{1/2} \ll 1$) as $x \to 0$. We can write this as

$$D(x) = d + \delta(x)$$

where $\delta(x) \sim \pm\frac{3}{16}x^{1/2}$.

Can continue further and write down a full series expression, but this is sufficient to give the 'leading behaviour' of y. That is, as y approaches the irregular singularity at $x = 0$ the bits of S that we have yet to work out will vanish and the behaviour is dominated by the stuff we found, namely

$$S(x) \sim \mp 2x^{-1/2} + \frac{3}{4}\log x + d, \quad \text{as } x \to 0.$$

Exponentiating both sides (why can we do this?) gives

$$\begin{aligned} y(x) &\sim \exp\left(\mp 2x^{-1/2} + \frac{3}{4}\log x + d\right), \quad \text{as } x \to 0, \\ &\sim c_1 x^{3/4} \mathrm{e}^{\mp 2x^{-1/2}}, \quad \text{as } x \to 0, \end{aligned}$$

where $c_1 = \mathrm{e}^d$.

To MATLAB ...

Lecture 4: Introduction to asymptotics (15)

To recap: we assumed a solution of the form (9). We then iteratively approximated $S(x)$ until we obtained an approximate solution that was asymptotic to the solution to the original DE. The iterative steps in approximating $S(x)$ were

1. Drop all terms which are negligible (e.g. as $x \to 0$, $x \to \infty$) and replace the $=$ sign with a $\sim$ to give an asymptotic relation.
2. Solve this simpler asymptotic relation. Verify that the solution is consistent with the assumptions made about the negligible terms in step 1.
3. Replace the asymptotic relation with an equation by introducing an arbitrary function (that is $\ll$ the stuff we already found), repeat procedure to find that function.

Lecture 4: Introduction to asymptotics (16)

We stop the procedure when a ‘controlling factor’ is found, that is enough terms in the approximation to $S(x)$ that when we put them back into $y = \mathrm{e}^{S(x)}$ adding additional terms to the approximation for $S(x)$ would have a negligible effect on y.

Taking the exponent of an asymptotic relation.

Lecture 4: Introduction to asymptotics (17)

This technique also works for irregular singular points at ∞. Consider the following equation

$$y'' = xy. \tag{11}$$

This is the **Airy equation**, which is (among other things) used to describe rainbows. Let's examine its behaviour as $x \to \infty$.

Solve (11) as $x \to \infty$ with method of dominant balance.

Check $x \to \infty$ is an irregular singular point, sub $x = 1/t$ to get

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{2}{t}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t^5} y$$

The y term is divided by t^5 and $5 > 2$ so $t = 0$ is an irregular singular point. Make a substitution in (11) of $\mathrm{e}^{S(x)}$, this gives:

$$S'' + (S')^2 = x.$$

We now apply the method of dominant balance to find the distinguished limit as $x \to \infty$.

1. $S'' \sim x$ as $x \to \infty$, neglect $(S')^2$

   Integrating gives $S' \sim \frac{1}{2}x^2 \implies (S')^2 \sim \frac{1}{4}x^4$ as $x \to \infty$. But we've assumed $(S')^2 \ll x$ as $x \to \infty$ so this is a contradiction.

2. $(S')^2 \sim x$ as $x \to \infty$, neglect S''

   Rearranging and differentiating gives $S'' \sim \pm\frac{1}{2}x^{-1/2}$ and $x^{-1/2} \ll x$ as $x \to \infty$ so this is a valid balance. ✓

3. $S'' \sim -(S')^2$ as $x \to \infty$, neglect x

   Integrating give $S' \sim 1/(x+a) \sim 1/x$ as $x \to \infty$, but we'd assumed $x \ll (S')^2$ so this is a contradiction.

The distinguished limit is

$$
\begin{aligned}
(S')^2 &\sim x, \quad \text{as } x \to \infty, \\
\implies S &\sim \pm \frac{2}{3} x^{3/2}, \quad \text{as } x \to \infty.
\end{aligned}
$$

Proceeding, and choosing just the positive bit, we convert to an equation,

$$
S(x) = \frac{2}{3} x^{3/2} + C(x), \quad \text{where } C(x) \ll x^{3/2}, \quad \text{as } x \to \infty.
$$

Sub. back into the DE for S to get

$$
S'' + (S')^2 = x
$$

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$$(S')^2 \sim x, \quad \text{as } x \to \infty,$$
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Sub. back into the DE for S to get

$$S'' + (S')^2 = x$$
$$\frac{1}{2}x^{-1/2} + C'' + (x^{1/2} + C')^2 = x$$

The distinguished limit is

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Sub. back into the DE for S to get

$$\begin{aligned} S'' + (S')^2 &= x \\ \frac{1}{2}x^{-1/2} + C'' + (x^{1/2} + C')^2 &= x \\ \frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 &= 0 \end{aligned}$$

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Sub. back into the DE for S to get

$$\begin{aligned}
S'' + (S')^2 &= x \\
\frac{1}{2}x^{-1/2} + C'' + (x^{1/2} + C')^2 &= x \\
\frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 &= 0 \\
\frac{1}{2}x^{-1/2} + C'' + C'(2x^{1/2} + C') &= 0
\end{aligned}$$

Apply dominant balance to

$$\frac{1}{2}x^{-1/2} + C'' + C'(2x^{1/2} + C') = 0$$

Here's what we know about C:

$$\begin{aligned} C(x) &\ll x^{3/2}, \quad \text{as } x \to \infty \\ C'(x) &\ll x^{1/2}, \quad \text{as } x \to \infty \\ C''(x) &\ll x^{-1/2}, \quad \text{as } x \to \infty \end{aligned}$$

Similarly to the previous example, as $x \to \infty$ we can therefore neglect the C'' and the $(C')^2$.

Apply dominant balance to

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Similarly to the previous example, as $x \to \infty$ we can therefore neglect the C'' and the $(C')^2$. This leaves the dominant balance

$$2C'x^{1/2} \sim -\frac{1}{2}x^{-1/2}, \quad \text{as } x \to \infty$$

Apply dominant balance to

$$\frac{1}{2}x^{-1/2} + C'' + C'(2x^{1/2} + C') = 0$$

Here's what we know about C:

$$\begin{aligned} C(x) &\ll x^{3/2}, \quad \text{as } x \to \infty \\ C'(x) &\ll x^{1/2}, \quad \text{as } x \to \infty \\ C''(x) &\ll x^{-1/2}, \quad \text{as } x \to \infty \end{aligned}$$

Similarly to the previous example, as $x \to \infty$ we can therefore neglect the C'' and the $(C')^2$. This leaves the dominant balance

$$2C'x^{1/2} \sim -\frac{1}{2}x^{-1/2}, \quad \text{as } x \to \infty$$

$$C' \sim -\frac{1}{4}x^{-1}, \quad \text{as } x \to \infty$$

Apply dominant balance to

$$\frac{1}{2}x^{-1/2} + C'' + C'(2x^{1/2} + C') = 0$$

Here's what we know about C:

$$\begin{aligned} C(x) &\ll x^{3/2}, \quad \text{as } x \to \infty \\ C'(x) &\ll x^{1/2}, \quad \text{as } x \to \infty \\ C''(x) &\ll x^{-1/2}, \quad \text{as } x \to \infty \end{aligned}$$

Similarly to the previous example, as $x \to \infty$ we can therefore neglect the C'' and the $(C')^2$. This leaves the dominant balance

$$\begin{aligned} 2C'x^{1/2} &\sim -\frac{1}{2}x^{-1/2}, \quad \text{as } x \to \infty \\ C' &\sim -\frac{1}{4}x^{-1}, \quad \text{as } x \to \infty \\ C &\sim -\frac{1}{4}\log x, \quad \text{as } x \to \infty. \end{aligned}$$

Converting that back to an equation we have

$$C(x) = -\frac{1}{4}\log x + D(x), \quad \text{where } D(x) \ll \log x, \quad \text{as } x \to \infty.$$

Sub. this into the equation for C:

$$\frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 = 0$$

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Sub. this into the equation for C:

$$\frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 = 0$$
$$\frac{1}{2}x^{-1/2} + \frac{1}{4}x^{-2} + D'' + 2x^{1/2}(-\frac{1}{4}x^{-1} + D') + (-\frac{1}{4}x^{-1} + D')^2 = 0$$

Converting that back to an equation we have

$$C(x) = -\frac{1}{4}\log x + D(x), \quad \text{where } D(x) \ll \log x, \quad \text{as } x \to \infty.$$

Sub. this into the equation for C:

$$\begin{aligned}
\frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 &= 0 \\
\frac{1}{2}x^{-1/2} + \frac{1}{4}x^{-2} + D'' + 2x^{1/2}(-\frac{1}{4}x^{-1} + D') + (-\frac{1}{4}x^{-1} + D')^2 &= 0 \\
\frac{1}{4}x^{-2} + D'' + 2x^{1/2}D' + \frac{1}{16}x^{-2} - \frac{1}{2}x^{-1}D' + (D')^2 &= 0
\end{aligned}$$

Converting that back to an equation we have

$$C(x) = -\frac{1}{4}\log x + D(x), \quad \text{where } D(x) \ll \log x, \quad \text{as } x \to \infty.$$

Sub. this into the equation for C:

$$\begin{aligned}
\frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 &= 0 \\
\frac{1}{2}x^{-1/2} + \frac{1}{4}x^{-2} + D'' + 2x^{1/2}(-\frac{1}{4}x^{-1} + D') + (-\frac{1}{4}x^{-1} + D')^2 &= 0 \\
\frac{1}{4}x^{-2} + D'' + 2x^{1/2}D' + \frac{1}{16}x^{-2} - \frac{1}{2}x^{-1}D' + (D')^2 &= 0 \\
\frac{5}{16}x^{-2} + D'' + D'(2x^{1/2} - \frac{1}{2}x^{-1}) + (D')^2 &= 0
\end{aligned}$$

Apply dominant balance to

$$\frac{5}{16}x^{-2} + D'' + D'(2x^{1/2} - \frac{1}{2}x^{-1}) + (D')^2 = 0$$

We have $x^{-1} \ll x^{1/2}$, so that can be neglected. We know the following about D:

$$\begin{aligned} D(x) &\ll \log x, \quad \text{as } x \to \infty, \\ D'(x) &\ll x^{-1}, \quad \text{as } x \to \infty, \\ D''(x) &\ll x^{-2}, \quad \text{as } x \to \infty. \end{aligned}$$

Therefore neglect the D'' and the $(D')^2$ (in favour of the already neglected $x^{-1}D'$). This leaves

$$\begin{aligned} 2D'x^{1/2} &\sim -\frac{5}{16}x^{-2}, \quad \text{as } x \to \infty, \\ D' &\sim -\frac{5}{32}x^{-5/2}, \quad \text{as } x \to \infty, \\ D &\sim d + \frac{5}{48}x^{-3/2} \sim d, \quad \text{as } x \to \infty, \end{aligned}$$

Then putting all that together, we have

$$
\begin{aligned}
S &\sim \tfrac{2}{3}x^{3/2} - \tfrac{1}{4}\log x + d, \quad \text{as } x \to \infty, \\
\implies \; y = \mathrm{e}^{S} &\sim c_1 \mathrm{e}^{\frac{2}{3}x^{3/2}} x^{-1/4}, \quad \text{as } x \to \infty,
\end{aligned}
$$

where $c_1 = \mathrm{e}^{d}$. If we go back and examine all that working, it's easy to see that the other solution is $c_2 \mathrm{e}^{-\frac{2}{3}x^{3/2}} x^{-1/4}$.

These are approximations to **the Airy functions**

$$
\begin{aligned}
\mathrm{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\frac{2}{3}x^{3/2}} x^{-1/4} \\
\mathrm{Bi}(x) &\sim \frac{1}{\sqrt{\pi}} \mathrm{e}^{\frac{2}{3}x^{3/2}} x^{-1/4}
\end{aligned}
$$

as $x \to \infty$.

Lecture 5: Introduction to asymptotics (18)

Lurking in our solutions to these differential equations are two important, very subtle ideas in asymptotics: **subdominance** and **behaviour of asymptotic relations in the complex plane**. Let's discuss each of these in turn.

Example of subdominance/asymptotics in the complex plane

Lecture 5: Introduction to asymptotics (19)

In the next section we will discuss some technical details on asymptotic series/expansions; but first we need one more piece of notation since to be more precise about ...

Why $\mathcal{O}$ why?

Let's introduce this new notation. Say we have two function $f(x)$ and $g(x)$, then we write that

$$f(x) = \mathcal{O}(g(x)), \quad \text{as } x \to x_0 \tag{12}$$

if it is the case that

$$|f(x)| \leq A|g(x)| \tag{13}$$

for some constant A.

Examples

Lecture 5: Introduction to asymptotics (20)

Another, less common notation that we won't use but you should be aware of is 'little o'. Say we have two function $f(x)$ and $g(x)$, then we write

$$f(x) = o(g(x)), \quad \text{as } x \to x_0$$

if $f(x) \ll g(x)$.

Examples

Lecture 6: Introduction to asymptotics (21)

Recall the following definitions. A series $\sum_{n=0}^{\infty} f_n(z)$ **converges** at some fixed value of z if for an arbitrary $\epsilon > 0$ it is possible to find a number $N_0(z, \epsilon)$ such that

$$\left| \sum_{n=M}^{N} f_n(z) \right| < \epsilon \quad \text{for all } M, N > N_0.$$

A series $\sum_{n=0}^{\infty} f_n(z)$ **converges to a function** $f(z)$ at some fixed value of z if for an arbitrary $\epsilon > 0$ it is possible to find a number $N_0(z, \epsilon)$ such that

$$\left| f(z) - \sum_{n=0}^{N} f_n(z) \right| < \epsilon \quad \text{for all } N > N_0.$$

Equivalently, we can think of series converging if its terms decay sufficiently fast as $n \to \infty$.

Lecture 6: Introduction to asymptotics (22)

Consider the error function, which is defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

A tale of two erfs

Lecture 6: Introduction to asymptotics (23)

This was an example of an asymptotic series. Let's say what we mean by that.

Say we have a sequence $\{f_n(\epsilon)\}$ this is an asymptotic sequence as $\epsilon \to 0$ if

$$f_{n+1}(\epsilon) \ll f_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

for all n.

Quick examples

Lecture 6: Introduction to asymptotics (24)

Say we have a function $f(\epsilon)$, a series $\sum_{n=0}^{\infty} f_n(\epsilon)$ is said to be an asymptotic expansion (or approximation) to this function if

$$f(\epsilon) - \sum_{n=0}^{N} f_n(\epsilon) \ll f_N(\epsilon), \quad \text{as } \epsilon \to 0.$$

That is the remainder between the approximation and the function (for $\epsilon \to 0$) is smaller than the last included term. This can be written as (we have already been doing this)

$$f(\epsilon) \sim \sum_{n=0}^{\infty} f_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

Lecture 6: Introduction to asymptotics (25)

The most common version of this is an asymptotic power series in ϵ, namely

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n, \quad \text{as } \epsilon \to 0.$$

As per previous examples, things like powers of $\epsilon^{1/2}$ might also show up in these series.

Lecture 6: Introduction to asymptotics (26)

An interesting property of asymptotic approximations is that a function may have a variety of different asymptotic approximations. Here are a couple of approximations for $\tan(\epsilon)$ as $\epsilon \to 0$:

$$\begin{aligned}\tan(\epsilon) &\sim \epsilon + \frac{\epsilon^3}{3} + \frac{2\epsilon^5}{15} + \ldots \\ &\sim \sin\epsilon + \frac{1}{2}(\sin\epsilon)^3 + \frac{3}{8}(\sin\epsilon)^5 + \ldots\end{aligned}$$

Much of the rest of this course will consist of developing expansions in the form form $f(x;\epsilon)$, that is they involve an independent variable x as well as a small parameter ϵ.

Lecture 6: Introduction to asymptotics (27)

The most general form of an expansion of this type (say for a solution to a differential equation) is

$$f(x;\epsilon) \sim \sum_{n=0}^{\infty} a_n(x)\delta_n(\epsilon), \quad \text{as } \epsilon \to 0.$$

The most common version of this is where the $\delta_n(\epsilon) = \epsilon^n$, that is

$$f(x;\epsilon) \sim \sum_{n=0}^{\infty} a_n(x)\epsilon^n, \quad \text{as } \epsilon \to 0.$$

As we saw with the error function evaluating asymptotic series accurately is a bit of an art. Generally only a few terms are required, and care should be taken not to include too many terms if a series is divergent.

Lecture 6: Introduction to asymptotics (28)

The trick is to find an **optimal truncation** of a series (stop adding terms to an approximation); typically this involves looking a the magnitude of the terms and noticing when they start increasing in magnitude. Such an approximation is called (rather excitingly) a **superasymptotic** representation.

There are also numerical methods which can improve the convergence of divergent series to the 'right' answer (in certain circumstances). This might involve alternative ways of summing the terms in a series - it turns out just adding them up just about the most inefficient way to do this! Two such techniques are the Shanks tranformation and Padé summation (if interested see Bender and Orszag, Chapter 8).