LECTURE 12

Recall from last time the definition of a subsequence:

Definition 2.11: Let $(a_n)_{n=1}^{\infty}$ be a sequence. A *subsequence* of $(a_n)_{n=1}^{\infty}$ is a sequence $(a_{\phi(n)})_{n=1}^{\infty}$ where $\phi\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

Notation: The strictly increasing function $\phi\colon \mathbb{N} \to \mathbb{N}$ is in particular a sequence, we denote its terms by $n_k := \phi(k)$. Thus we have the strictly increasing sequence

$$n_1 < n_2 < n_3 < \cdots$$

It is very common to omit mention of the function ϕ, and just describe the subsequence in terms of the sequence (n_k). Therefore the subsequence $(a_{\phi(n)})_{n=1}^{\infty}$ is typically denoted $(a_{n_k})_{k=1}^{\infty}$.

Example: (a_{2n}) and (a_{2n+1}) are subsequences of (a_n).

Example: If $(a_{n_k})_{k=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$ and $(a_{n_{k_l}})_{l=1}^{\infty}$ is a subsequence of $(a_{n_k})_{k=1}^{\infty}$, then $(a_{n_{k_l}})_{l=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. Ultimately, this statement boils down to the following fact: the composite of two strictly increasing functions is again strictly increasing. To see how the statement boils down to this fact, suppose that the subsequence $(a_{n_k})_{k=1}^{\infty}$ is determined by the strictly increasing function $\phi\colon \mathbb{N} \to \mathbb{N}$ and the subsequence $(a_{n_{k_l}})_{l=1}^{\infty}$ is determined by the strictly increasing function $\psi\colon \mathbb{N} \to \mathbb{N}$, so that $a_{n_{k_l}} = a_{\phi(\psi(l))}$. The point is that $\phi \circ \psi\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, and hence $(a_{n_{k_l}})_{l=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

Lemma 2.11: If $\phi\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function then $\phi(k) \geq k$ for all $k \in \mathbb{N}$. In other words $n_k \geq k$ for all $k \in \mathbb{N}$.

Proof: When $k = 1$ we have $\phi(k) = \phi(1) \geq 1 = k$. Assume that the statement is true for some natural number $k \geq 1$. Then $\phi(k+1) > \phi(k)$ since ϕ is strictly increasing. We have $\phi(k) \geq k$ by the inductive hypothesis. Therefore $\phi(k+1) > k$ and hence $\phi(k+1) \geq k+1$ since $\phi(k+1) \in \mathbb{N}$. ■

Theorem 2.12: If $a_n \to L$ then $a_{n_k} \to L$ for any subsequence (a_{n_k}) of (a_n).

Proof: Let (a_{n_k}) be a subsequence of (a_n). Let $\epsilon > 0$. Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L| < \epsilon$. If $k \geq N$ then $n_k \geq k$ by Lemma 2.11. Therefore, if $k \geq N$ then $|a_{n_k} - L| < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows that $a_{n_k} \to L$. ■

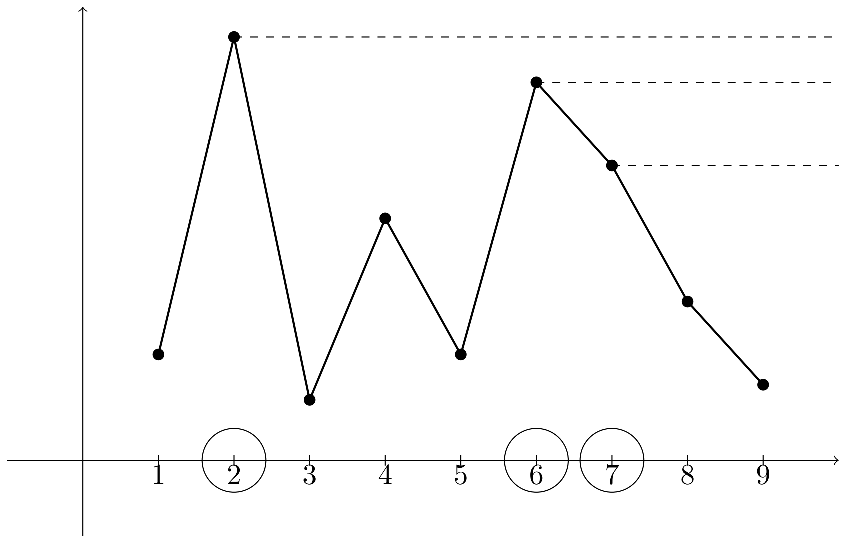
Example: Let (a_n) be the sequence defined by $a_n = (-1)^n$. Then (a_{2n}) is the constant sequence $1, 1, 1, \ldots$ while (a_{2n+1}) is the constant sequence $-1, -1, -1, \ldots$. Therefore the sequence (a_n) cannot converge. If $a_n \to L$ then we must have $a_{2n} \to L$ and $a_{2n+1} \to L$. But then $L = 1$ and $L = -1$ since limits are unique. This is a contradiction. Hence (a_n) does not converge.

Example: If $a_n \to L$ then $a_{n+k} \to L$ for any natural number k. In other words, if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+k} = L$.

Example: Suppose $0 < x < 1$ and let (a_n) be the sequence defined by $a_n = x^n$. Previously we saw (since (a_n) is decreasing and bounded below) that $a_n \to L$ for some real number L. It follows that $a_{2n} \to L$. But $a_{2n} = x^{2n} = (x^n)^2 = (a_n)^2$. Therefore $a_{2n} = (a_n)^2 \to L^2$ by the Algebraic Limit Theorem. Since limits are unique, we must have $L^2 = L$. Therefore $L = 0$ or $L = 1$. Recall that $L = \inf\{\, x^n \mid n \in \mathbb{N} \,\}$. Clearly $L < 1$ since 1 is not a lower bound for $\{\, x^n \mid n \in \mathbb{N} \,\}$. Therefore the only possibility is $L = 0$.

We introduce the concept of 'peak point' for a sequence (a_n). We say $m \in \mathbb{N}$ is a *peak point* of (a_n) if $n > m \implies a_n < a_m$.

The following picture illustrates the concept – the peak points are circled:



Either a sequence (a_n) has infinitely many peak points or it doesn't.

(i) Suppose (a_n) does have infinitely many peak points — suppose that $n_1 < n_2 < n_3 < \cdots$ are all the peak points. Then

$$a_{n_1} > a_{n_2} > a_{n_3} > \cdots$$

and hence (a_{n_k}) is a strictly decreasing subsequence of (a_n).

(ii) Suppose that (a_n) does not have infinitely many peak points, i.e. suppose that (a_n) has only finitely many peak points. Choose $n_1 \in \mathbb{N}$ which is greater than the largest peak point. By assumption, n_1 is not a peak point and hence there exists $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. By assumption, n_2 is not a peak point and hence there exists $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continue in this way to obtain a strictly increasing sequence $n_1 < n_2 < n_3 < \cdots$. The corresponding subsequence (a_{n_k}) satisfies $a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \cdots$ and hence is an increasing subsequence.

Therefore we have proved the following result.

Proposition 2.13: Any sequence (a_n) has a monotonic subsequence.

The main application of this proposition is to prove the following key result:

Theorem 2.14 (The Bolzano-Weierstrass Theorem): Every bounded sequence of real numbers has a convergent subsequence.

Proof: Let (a_n) be a bounded sequence of real numbers. By Proposition 2.13 there exists a monotonic subsequence (a_{n_k}) of (a_n). Since (a_n) is bounded, the subsequence (a_{n_k}) is also bounded. Therefore (a_{n_k}) is a bounded, monotonic sequence. Therefore it converges. Therefore (a_n) has a convergent subsequence. ■

Example: Consider the following sequence (a_n) whose terms are as follows:

$$1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, \ldots$$

The sequence (a_n) is bounded and hence it has convergent subsequences. Find all real numbers L such that there is a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to L$.

Some convergent subsequences are easy to spot. For example there is the subsequence $1/2, 1/3, 1/4, 1/5, \ldots$ converging to 0. Another is $1/2, 2/3, 3/4, 4/5, 5/6, \ldots$ converging to 1. Clearly there are no subsequences converging to L if $L < 0$ or if $L > 0$.

Observe also that *every rational number between 0 and 1 occurs infinitely often* in this sequence. It follows that there is a subsequence converging to every rational number between 0 and 1. But more is true. If $0 < x < 1$ is irrational, then we can find a sequence (x_n) of rational numbers in $(0, 1)$ such that $x_n \to x$. To see this, choose for each $n \in \mathbb{N}$ a rational number x_n such that $x_n \in (x - 1/n, x + 1/n) \cap (0, 1)$ (the density of the rationals inside the reals ensures that we can always do this). But any such sequence (x_n) is a subsequence of (a_n). Hence there is a subsequence of (a_n) which converges to x. Therefore the set of such numbers L is exactly the closed interval $[0, 1]$.