School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Tutorial 6 (Week 12)

1. Consider general functionals of the form

$$J\{y\} = \int_0^{x_1} K(x,y) e^{\tan^{-1} y'} \sqrt{1+y'^2}\, dx,$$

where $K(x,y) \neq 0$ and the desired extremals are subject to the endpoint conditions that (x_0, y_0) is prescribed and fixed but (x_1, y_1) is only constrained to lie on the curve $y = \phi(x)$. Find the traversality condition that applies at (x_1, y_1).

2. Using the Calculus of Variations, find the shortest distance from the point $(1,1,1)$ and the surface of the sphere $x^2 + y^2 + z^2 = 1$.

3. Consider the variational problem

$$M\{y\} = \int_0^1 y'^2 (1+y')^2\, dx, \quad y(0) = 0, \quad y(1) = m,$$

where $-1 < m < 0$. Find and sketch a broken extremal that yields the absolute minimum for $M\{y\}$.

4. $\star$ Consider the problem of determining the join region between two carbon nanotubes with different radii. By modelling the joining bonds as elastica with fixed length L, and choosing our baseline unit of distance as the difference in radius, i.e. $r_2 - r_1 = 1$, the problem can be posed as one of finding the extremal curve of the functional

$$F\{y\} = \int_0^1 \frac{y''^2}{(1+y'^2)^{5/2}}\, dx,$$

subject to the conditions

$$y(0) = 0, \quad y'(0) \to \infty, \quad y'(1) \to \infty,$$

as well as the isoperimetric constraint

$$\int_0^1 \sqrt{1+y'^2}\, dx = L.$$

Note that the problem does not specify the value of y at $x = 1$ and so a natural boundary condition is required here.

   (a) Find a parametric solution for the join region from $0 \leqslant x \leqslant 1$.

   (b) Assuming a value $L = 2$, find the value of y at $x = 1$ to four significant digits.

   (c) Produce a plot of your solution for $L = 2$.

Hint: this problem will require the curvature to change sign and you may assume from symmetry considerations that $\kappa = 0$ at $x = 1/2$.