Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 7

Last lecture

- Analysed the brachistochrone problem (curve of quickest descent)
- Found that the solution to the brachistichrone problem were cycloids
- Considered Newton's aerodynamic problem
- Derived a parametric solution which was approximately the frustum of a cone and compared the solution with various other projectile shapes
- Saw that some bullets are manufactured with a similar blunted shapes (meplat)

Special Case 3

When f has no explicit dependence on y the E-L equations simplify to give

$$\frac{\partial f}{\partial y'} = \text{const}$$

An example where we might use this is in calculating geodesics on non-planar objects such as the sphere.

Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} \;=\; 0}$$

No explicit y dependence

Suppose the function is of the form

$$J\{y\} = \int_{x_0}^{x_1} f(x, y')\, dx,$$

where y does not appear explicitly.

The Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial y'} = c_1,$$

where c_1 is a constant.

Solving

$\dfrac{\partial f}{\partial y'}$ is a known function of x and y', so this is a first order ODE for y.

In principle for $\dfrac{\partial^2 f}{\partial y'^2} \neq 0$ can recast

$$\frac{\partial f}{\partial y'} = c_1 \quad \text{as} \quad y' = g(x, c_1)$$

for some g.

Geodesics on the unit sphere

Find the shortest path between two points on the unit sphere.

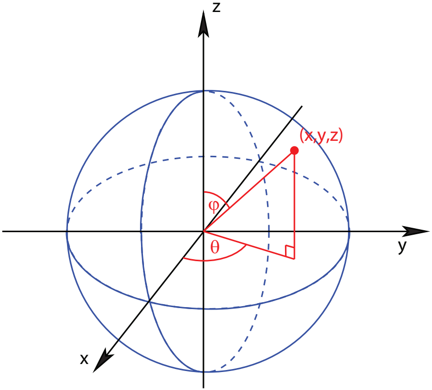
Spherical co-ordinates

Define

$$\begin{aligned} \phi &= \text{ latitude} \\ \theta &= \text{ longitude} \end{aligned}$$

Cartesian co-ordinates (x, y, z)

$$\begin{aligned} x &= \cos(\theta)\sin(\phi) \\ y &= \sin(\theta)\sin(\phi) \\ z &= \cos(\phi) \end{aligned}$$



Transformation to spherical co-ord.

$$
\begin{aligned}
x &= \cos(\theta)\sin(\phi) \\
y &= \sin(\theta)\sin(\phi) \\
z &= \cos(\phi)
\end{aligned}
$$

By the chain rule

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi = -\sin(\theta)\sin(\phi)d\theta + \cos(\theta)\cos(\phi)d\phi \\
dy &= \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi = \cos(\theta)\sin(\phi)d\theta + \sin(\theta)\cos(\phi)d\phi \\
dz &= \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}d\phi = -\sin(\phi)d\phi
\end{aligned}
$$

$$
ds^2 = dx^2 + dy^2 + dz^2 = \sin^2(\phi)d\theta^2 + d\phi^2
$$

Geodesics on the unit sphere

$$\int_{(x(s_0),y(s_0),z(s_0))}^{(x(s_1),y(s_1),z(s_1))} 1\,ds = \int_{\phi_0}^{\phi_1} \left[1 + \sin^2(\phi)\left(\frac{d\theta}{d\phi}\right)^2\right]^{\frac{1}{2}} d\phi$$

θ is like y, ϕ is like x, $\frac{d\theta}{d\phi} = \theta'$ is like y', hence EL eqn:

$$\frac{\partial}{\partial\theta'}\left[1 + \sin^2(\phi)\theta'^2\right]^{\frac{1}{2}} = c_1$$

$$\frac{\sin^2(\phi)\theta'}{\left[1 + \sin^2(\phi)\theta'^2\right]^{\frac{1}{2}}} = c_1$$

$$\frac{\sin^4(\phi)\theta'^2}{1 + \sin^2(\phi)\theta'^2} = c_1^2$$

The constant c_1

$$\frac{\sin^4(\phi)\theta'^2}{1+\sin^2(\phi)\theta'^2} = c_1^2$$

Now

$$\theta'^2 \sin^4(\phi) \leqslant \theta'^2 \sin^2(\phi) \leqslant 1 + \theta'^2 \sin^2(\phi)$$

So

$$c_1 \in [-1, 1]$$

So we can replace c_1 with

$$c_1 = \sin(\alpha)$$

Geodesics on the unit sphere

Re-arrange

$$\sin^4(\phi)\theta'^2 = \sin^2(\alpha)\left(1 + \sin^2(\phi)\theta'^2\right)$$

Re-arrange some more

$$\begin{aligned}
\theta'^2 &= \frac{\sin^2(\alpha)}{\sin^4(\phi) - \sin^2(\alpha)\sin^2(\phi)} \\
\theta' &= \left\{\frac{\sin^2(\alpha)}{\sin^2(\phi)(\sin^2(\phi) - \sin^2(\alpha))}\right\}^{\frac{1}{2}} \\
\theta' &= g(\phi, \alpha)
\end{aligned}$$

Analogous to $y' = g(x, c_1)$.

Solving the DE (i)

So integrating

$$
\begin{aligned}
\theta' &= \frac{\sin(\alpha)}{\sin(\phi)\left[\sin^2(\phi) - \sin^2(\alpha)\right]^{\frac{1}{2}}} \\
\theta &= \int \frac{\sin(\alpha)}{\sin(\phi)\left[\sin^2(\phi) - \sin^2(\alpha)\right]^{\frac{1}{2}}}\, d\phi \\
&= \int \frac{\csc^2(\phi)}{\left[\csc^2(\alpha) - \csc^2(\phi)\right]^{\frac{1}{2}}}\, d\phi \\
&= \int \frac{\csc^2(\phi)}{\left[\cot^2(\alpha) - \cot^2(\phi)\right]^{\frac{1}{2}}}\, d\phi \qquad \text{as } \csc^2 x = 1 + \cot^2 x
\end{aligned}
$$

Solving the DE (ii)

$$\theta = \int \frac{\csc^2(\phi)}{\left[\cot^2(\alpha) - \cot^2(\phi)\right]^{\frac{1}{2}}}\, d\phi = \frac{1}{\cot(\alpha)} \int \frac{\csc^2(\phi)}{\left[1 - \frac{\cot^2(\phi)}{\cot^2(\alpha)}\right]^{\frac{1}{2}}}\, d\phi.$$

Substitute $u = \cot(\phi)/\cot(\alpha)$
Then $d\phi = \frac{\cot(\alpha)}{\csc^2(\phi)} du$

$$\theta = \int \frac{1}{\left[1 - u^2\right]^{\frac{1}{2}}}\, du = \sin^{-1}\left(\frac{\cot(\phi)}{\cot(\alpha)}\right) - \beta,$$

since $\dfrac{d}{du}\sin^{-1}(u) = \dfrac{1}{\sqrt{1-u^2}}$

The solution

$$\sin(\theta + \beta) = \frac{\cot(\phi)}{\cot(\alpha)}$$

Note we can write this

$$\begin{aligned} \sin(\beta + \theta) &= \frac{1}{\cot(\alpha)} \frac{\cos(\phi)}{\sin(\phi)} \\ \cot(\alpha)\sin(\phi)\sin(\beta + \theta) &= \cos(\phi) \\ \cot(\alpha)\sin(\phi)\left[\sin(\beta)\cos(\theta) + \cos(\beta)\sin(\theta)\right] &= \cos(\phi) \end{aligned}$$

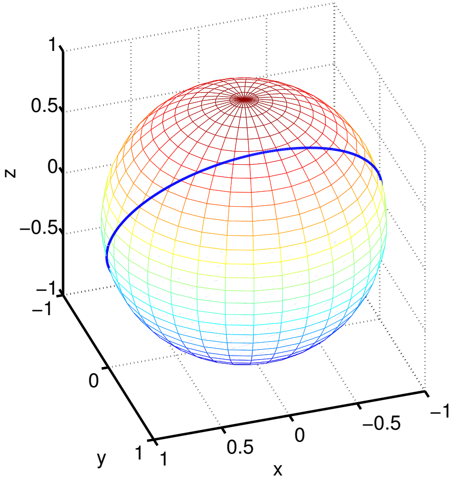
Convert back to Cartesian co-ordinates,

$$\cot(\alpha)\sin(\beta)x + \cot(\alpha)\cos(\beta)y = z \quad \Rightarrow \quad Ax + By = z.$$

which is the equation of a plane, through the origin.
Hence, solution is a **great circle**, the intersection of plane (through the origin) and the sphere.

Example

We can find the solution because three points (the origin plus the start and end point of the curve) define a plane, and therefore the solution is the intersection of this plane with the sphere.



Co-ordinate transformation

More generally, spherical co-ordinates

$$
\begin{aligned}
x &= r\cos(\theta)\sin(\phi)\\
y &= r\sin(\theta)\sin(\phi)\\
z &= r\cos(\phi)
\end{aligned}
$$

And

$$
\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = J \begin{pmatrix} dr \\ d\phi \\ d\theta \end{pmatrix}, \quad J = \begin{pmatrix} x_r & x_\phi & x_\theta \\ y_r & y_\phi & y_\theta \\ z_r & z_\phi & z_\theta \end{pmatrix}
$$

Where J is the Jacobian matrix and subscripts denote partial differentiation.

Jacobians

If

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1(\boldsymbol{x}) \\ y_2(\boldsymbol{x}) \\ \vdots \\ y_n(\boldsymbol{x}) \end{pmatrix}$$

Then the Jacobian matrix is

$$J(\boldsymbol{x}) = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

The Jacobian determinant

Then the determinant of the Jacobian matrix is also sometimes called the Jacobian

$$|J(\boldsymbol{x})| = \left|\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right|$$

This gives the ratio of n-dimensional volume between the two co-ordinate systems, i.e.

$$d\boldsymbol{y} = |J(\boldsymbol{x})|\, d\boldsymbol{x}$$

Transforms and integrals

Substitution in 1D:

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{u(x_0)}^{u(x_1)} f(x(u))\frac{dx}{du}\,du$$

In 2D

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Geodesics

Can we find a geodesic on other surfaces in $\mathbb{R}^3$?
Consider a surface parameterised by $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$, and minimize the arc length

$$L = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dx^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\, du\, dv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2$$

and likewise for dy^2 and dz^2.

Geodesics

So we can write the path length as

$$\begin{aligned} L &= \int \sqrt{P + 2Qv' + Rv'^2}\, du \\ &= \int \sqrt{Pu'^2 + 2Qu' + R}\, dv \end{aligned}$$

where $u' = du/dv$ and $v' = dv/du$ and

$$\begin{aligned} P &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ Q &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ R &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned}$$

Geodesics

Then the Euler-Lagrange equations become

$$\frac{\dfrac{\partial P}{\partial v} + 2v'\dfrac{\partial Q}{\partial v} + v'^2\dfrac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + Rv'^2}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}}\right) = 0$$

References:

http://mathworld.wolfram.com/GreatCircle.html

http://mathworld.wolfram.com/Geodesic.html