Topic C Assignment 3

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1. (a)

$$\epsilon \frac{d^2y}{dx^2} + (\cosh x)\frac{dy}{dx} - y = 0$$

With $y(0) = y(1) = 1$ To leading order:

$$y = y_0 + \mathcal{O}(\epsilon)$$

$$\begin{aligned}
(\cosh x)\frac{dy_{0,out}}{dx} - y_0 &= 0 \\
\frac{1}{y_{0,out}}\frac{dy_{0,out}}{dx} &= \operatorname{sech} x \\
\log y_{0,out} &= 2\arctan(\tanh x/2) \\
\Aboxed{y_{0,out} &= a\exp\{2\arctan(\tanh x/2)\}}
\end{aligned}$$

For the boundary conditions:
Let $x = x_* + \delta_1 X$, and $y = \delta_2 Y$

$$\epsilon \frac{\delta_2^2}{\delta_1^2}\frac{d^2Y}{dX^2} + (\cosh(x_* + \delta_1 X))\frac{\delta_2}{\delta_1}\frac{dY}{dX} - \delta_2 Y = 0$$

With BCs $\delta_2 Y(0) = \delta_2 Y(1) = 1$ Hence $\delta_2 = 1$.

$$\epsilon \frac{1}{\delta_1}\frac{d^2Y}{dX^2} + (\cosh(x_* + \delta_1 X))\frac{1}{\delta_1}\frac{dY}{dX} - Y = 0$$

Expand the cosh term:

$$\begin{aligned}
\cosh(x_* + \delta_1 X) &= \cosh(x_*)\cosh(\delta_1 X) + \sinh(x_*)\sinh(\delta_1 X) \\
&= \sinh(x_*)\sum_{i=0}^{\infty}\frac{(\delta_1 X)^{2i+1}}{(2i+1)!} + \cosh(x_*)\sum_{i=0}^{\infty}\frac{(\delta_1 X)^{2i}}{(2i)!} \\
&= \sinh(x_*)(\delta_1 X) + \cosh(x_*) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Noting that $x^* = 0$ or $x^* = 1$.

$$\epsilon \frac{1}{\delta_1}\frac{d^2Y}{dX^2} + (\sinh(x_*)(\delta_1 X) + \cosh(x_*))\frac{1}{\delta_1}\frac{dY}{dX} - Y = 0$$

$$\epsilon \frac{d^2Y}{dX^2} + \delta_1 X \sinh(x_*)\frac{dY}{dX} + \cosh(x_*)\frac{dY}{dX} - \delta_1 Y = 0$$

Hence $\delta_1 = \epsilon$
If $x^* = 0$

$$\begin{aligned}\frac{d^2Y}{dX^2} + X\sinh(0)\frac{dY}{dX} - Y = 0\\ \frac{d^2Y}{dX^2} - Y = 0\\ Y = Ae^X + Be^{-X}\end{aligned}$$

$Y(0) = 1$:

$$\begin{aligned}Ae^0 + Be^0 = 1\\ A + B = 1\\ B = 1 - A\end{aligned}$$

Hence

$$\boxed{Y = Ae^X + (1-A)e^{-X}}$$

And the outer solution becomes

$$\begin{aligned}y_{0,out}(1) = 1 = a\exp\{2\arctan(\tanh 1/2)\}\\ a = \exp\{-2\arctan(\tanh 1/2)\}\end{aligned}$$

$$\boxed{y_{0,out} = \exp\{2\arctan(\tanh x/2) - 2\arctan(\tanh 1/2)\}}$$

Matching

$$\begin{aligned}\lim_{x\to 0} y_0(x) = \lim_{X\to\infty} Y_0(X)\\ a\exp\{\arctan(\tanh(0))\} = Ae^{\infty} + (1-A)e^{-\infty}\end{aligned}$$

$$\begin{aligned}y_{comp} &= y_{in} + y_{out} - y_{overlap}\\ &=\end{aligned}$$

$$\begin{aligned}y_{0,out}(0) = 1 = a\exp\{2\arctan(\tanh 0)\}\\ a\exp^0 = 1\\ a = 1\end{aligned}$$

$$\lim_{x\to 1} y_0(x) = \lim_{X\to-\infty} Y_0(X)$$

(b) WKB ansatz solution for

$$\epsilon\frac{d^2y}{dx^2} + (\cosh x)\frac{dy}{dx} - y = 0$$

$$y(x) \sim \sum_{n=0}^{\infty} u_n(x)\epsilon^n + e^{-F(x)/\epsilon} \sum_{n=0}^{\infty} v_n(x)\epsilon^n$$

Leading order:

$$\begin{aligned} y &\sim u_0 + e^{-F/\epsilon} v_0 \\ y' &\sim u_0' + e^{-F/\epsilon} v_0' - \frac{F'}{\epsilon} e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots) \\ y'' &\sim u_0'' + e^{-F/\epsilon} v_0'' - 2\frac{F}{\epsilon} v_0' + \left(\frac{F'}{\epsilon}\right)^2 e^{-F/\epsilon}\left(v_0 + \epsilon v_1 + \ldots\right) - \frac{F''}{\epsilon} e^{-F/\epsilon} v_0 \end{aligned}$$

So the equation becomes:

$$\begin{aligned} \epsilon \frac{d^2y}{dx^2} + (\cosh x)\frac{dy}{dx} - y = 0 \\ \epsilon\left(u_0'' + e^{-F/\epsilon} v_0'' - 2\frac{F}{\epsilon} v_0' + \left(\frac{F'}{\epsilon}\right)^2 e^{-F/\epsilon}\left(v_0 + \epsilon v_1 + \ldots\right) - \frac{F''}{\epsilon} e^{-F/\epsilon} v_0\right) \\ + \cosh x \left(u_0' + e^{-F/\epsilon} v_0' - \frac{F'}{\epsilon} e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots)\right) - u_0 + e^{-F/\epsilon} v_0 = 0 \end{aligned}$$

$$\begin{aligned} \epsilon u_0'' + \epsilon e^{-F/\epsilon} v_0'' - 2F v_0' + \frac{F'^2}{\epsilon} e^{-F/\epsilon}\left(v_0 + \epsilon v_1 + \ldots\right) - F'' e^{-F/\epsilon} v_0 \\ + \cosh x \left(u_0' + e^{-F/\epsilon} v_0' - \frac{F'}{\epsilon} e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots)\right) - u_0 + e^{-F/\epsilon} v_0 = 0 \end{aligned}$$

$$\begin{aligned} \mathcal{O}(1) : 0 &= -2F v_0' + (\cosh x) u_0' - u_0 \\ \mathcal{O}(e^{-F(x)/\epsilon}\epsilon^{-1}) : 0 &= \left[F' - \cosh x\right] \\ \mathcal{O}(e^{-F(x)/\epsilon}) : 0 &= \end{aligned}$$

(c) First rewrite the BVP in a nicer format

$$\frac{d^2y}{dx^2} + \frac{1}{\epsilon}\left(\cosh x \frac{dy}{dx} - y\right) = 0$$

Figure 1

2.

$$\epsilon \frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$$

With $y(-2) = -4$ and $y(2) = 2$, $\epsilon \to 0$ over $-2 \le x \le 2$. There is an internal boundary layer shown via the numerical solution, in figure 2

The right outer solution y_R with $y_R(2) = 2$ to leading order:

$$\begin{aligned} x y_{R0}' + x y_{R0} &= 0 \\ y_{R0}' + y_{R0} &= 0 \\ y_{R0} &= Ae^{-x} \end{aligned}$$

And applying the boundary condition:

$$\begin{aligned} y_{R0}(2) = Ae^{-2} &= 2 \\ A &= 2e^2 \end{aligned}$$

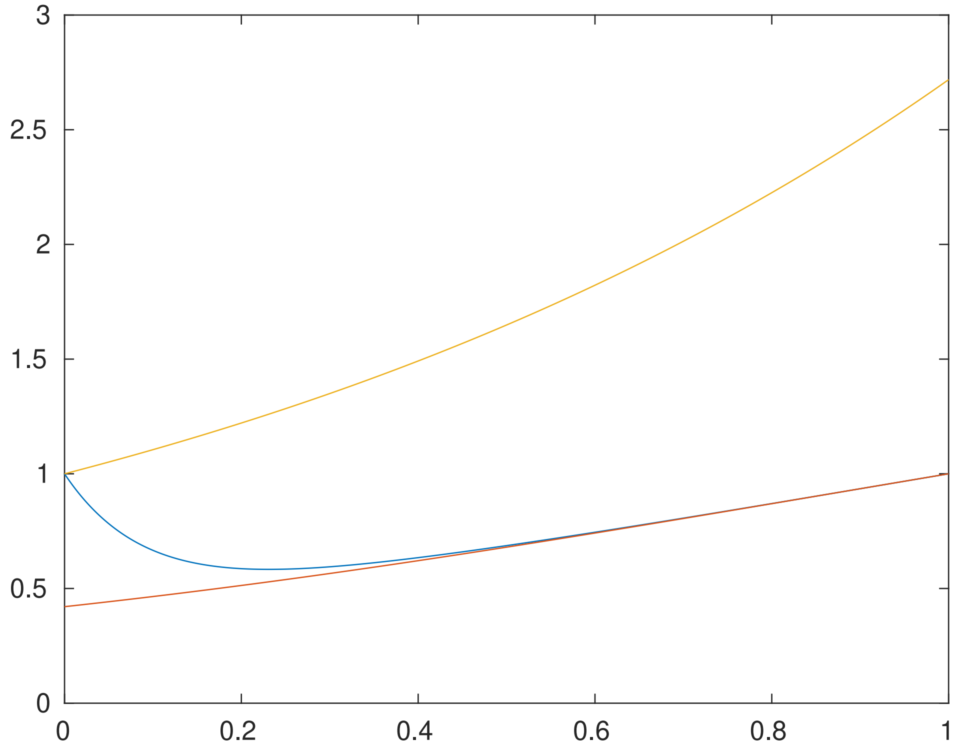

Figure 1: Comparison of Numerical, WKB and Composite solutions

$$\boxed{y_{R0} = 2e^2 e^{-x}}$$

The left outer solution y_L with $y_L(-2) = -4$

$$\begin{aligned} y_{L0} &= Be^{-x} \\ y_{L0}(-2) &= Be^{-2} = -4 \\ B &= -4e^{-2} \end{aligned}$$

Hence

$$\boxed{y_{L0} = -4e^{-2}e^{-x}}$$

For the inner solution $x = x_* + \delta_1 X$, and $y = \delta_2 Y$. Since the boundary conditions don't include ϵ, $\delta_2 = 1$.

$$\begin{aligned} \epsilon\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0 \\ \epsilon\frac{1}{\delta_1^2}\frac{d^2Y}{dX^2} + (x^* + \delta_1 X)\frac{1}{\delta_1}\frac{dY}{dX} + (x^* + \delta_1 X)Y = 0 \\ \epsilon\frac{d^2Y}{dX^2} + \delta_1(x^* + \delta_1 X)\frac{dY}{dX} + \delta_1^2(x^* + \delta_1 X)Y = 0 \\ \epsilon\frac{d^2Y}{dX^2} + \delta_1 x^*\frac{dY}{dX} + \delta_1^2 X\frac{dY}{dX} + \delta_1^2 x^* Y + \delta_1^3 XY = 0 \\ \epsilon\frac{d^2Y}{dX^2} + \delta_1 x^*\frac{dY}{dX} + \delta_1^2\left(X\frac{dY}{dX} + x^* Y\right) + \delta_1^3 XY = 0 \end{aligned}$$

Balances:

- $\epsilon\frac{d^2Y}{dX^2} \sim \delta_1 x^* \frac{dY}{dX}$ Hence $\delta_1 \sim \epsilon$, this is reasonable since the rejected terms will be $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ both of which are negligible.
- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1^2\left(X\frac{dY}{dX} + x^*Y\right)$ giving $\delta \sim \sqrt{\epsilon}$ neglecting terms of order $\epsilon^{1/2}$ and $\epsilon^{3/2}$. But $\epsilon^{1/2} \gg \epsilon$ so this is a contradiction.
- $\epsilon\frac{d^2Y}{dX^2} \sim -\delta_1^3 XY$ with $\delta_1 \sim \epsilon^{1/3}$, meaning we have neglected the $\epsilon^{1/2}$ and $\epsilon^{1/3}$ terms in favour of ϵ. This is a contradiction since $\epsilon^{1/3} \gg \epsilon$

Hence take $\delta_1 = \epsilon$

To leading order:

$$\begin{aligned} \frac{d^2Y_0}{dX^2} &= -x^*\frac{dY_0}{dX} \\ V' &= -x^*V \\ \implies V &= ae^{-x^*X} \\ \implies Y_0 &= a_0e^{-x^*X} + b \end{aligned}$$

We have to match this to the left and right solutions

Start with the right:

$$\begin{aligned} \lim_{x\to x^*} y_{R0} &= \lim_{X\to\infty} Y_0(X) \\ \lim_{x\to x^*} 2e^2e^{-x} &= \lim_{X\to\infty} a_0e^{-x^*X} + b \end{aligned}$$

The left:

$$\begin{aligned} \lim_{x\to x^*} y_{L0} &= \lim_{X\to-\infty} Y_0(X) \\ \lim_{x\to x^*} -4e^{-2}e^{-x} &= \lim_{X\to-\infty} a_0e^{-x^*X} + b \end{aligned}$$

Matlab Code

```
%%
%%1c
close all
clear all
epsilon = 0.1;
%obtain a numerical solution to the bvp
 solinit1 =bvpinit(linspace (0,1,11),[0  1]);
sol1=bvp4c(@(x,y)BVPODE1(x,y,epsilon),@boundaries1,solinit1);
xout1=linspace(0,1,1001);
yout1=deval(sol1,xout1);

plot(xout1,yout1(1,:))
hold on
%my solutions
x = linspace(0,1);
a = exp(-2*atan(tanh(1/2)));
youter = a*exp(2*atan(tanh(x/2)));
A = 1;
```

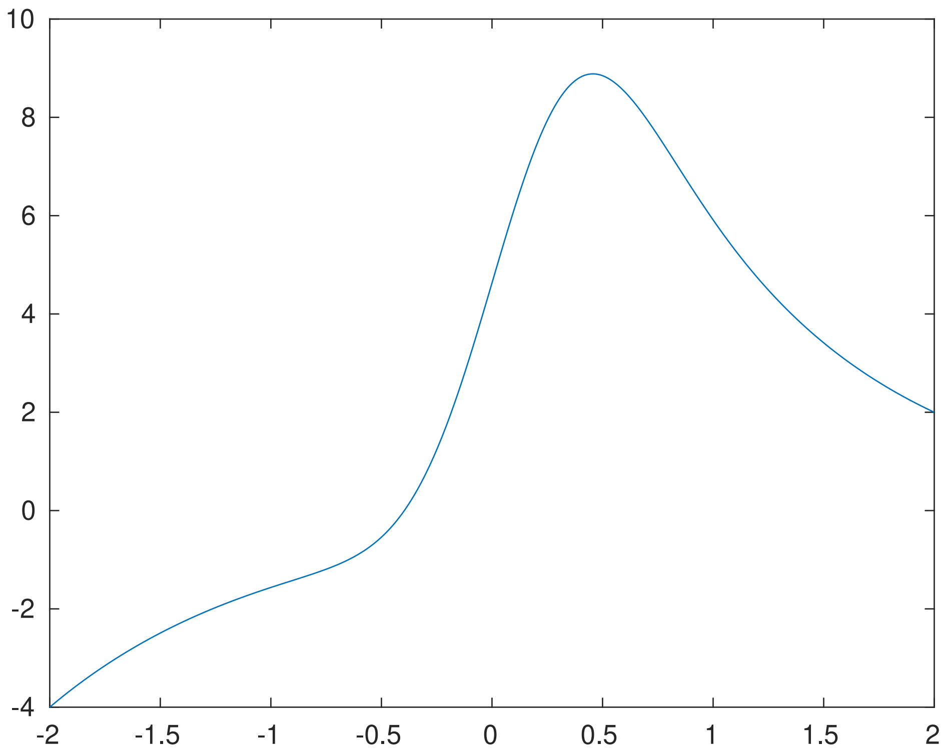

Figure 2: Caption here

```
yinner = A*(exp(x) -exp(-x)) + exp(-x);
plot(x,youter)
plot(x,yinner)
hold off
saveas(gcf,"TopicCA3Q1.eps",'epsc')
%%
%%2
epsilon = 0.1;
%numerical solution to the bvp
 solinit2 =bvpinit(linspace(-2,2,11),[0  1]);
sol2=bvp4c(@(x,y)BVPODE2(x,y,epsilon),@boundaries2,solinit2);
xout2=linspace(-2,2,1001);
yout2=deval(sol2,xout2);
 figure
plot(xout2,yout2(1,:))
hold on
x = linspace(-2,2);
xstar = -2;
yL = -4*exp(-2)*exp(-x);
yR = 2*exp(2)*exp(-x);
Y = 1*exp(-xstar*x) +0;
plot(x,yL)
plot(x,yR)
%plot(x,Y)
saveas(gcf,"TopicCA3Q2.eps",'epsc')
axis([-2,2,-4,10])
```

```
%%%FUNCTIONS
function res=boundaries1(ya,yb)
res=[ya(1)-1;yb(1)-1];
end
function dy=BVPODE1(x,y,epsilon)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=(1/epsilon)*(-(cosh(x)*y(2))+y(1));
end


function res=boundaries2(ya,yb)
res=[ya(1)+4;yb(1)-2];
end
function dy=BVPODE2(x,y,epsilon)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=(1/epsilon)*(-x*y(2)-x*y(1));
end
```

Practical Asymptotics (APP MTH 4051/7087)
Assignment 3 (5%)

Due 10 May 2019

1. Consider the following boundary value problem.

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\cosh x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to $y(0) = y(1) = 1$, for $\epsilon \to 0$ over the interval $0 \leq x \leq 1$.

   (a) Find a leading-order composite solution to this problem.

   (b) Apply a leading-order WKB ansatz to find a different approximate solution.

   (c) Compare these approximations with a numerical solution and comment briefly. How well do the approximate solutions satisfy the outer boundary condition?

2. This question involves an **internal boundary layer**, a region of rapid variation located away from the edges of the domain. Find leading-order outer and inner solutions to the following problem:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0,$$

subject to $y(-2) = -4$ and $y(2) = 2$, for $\epsilon \to 0$ over the interval $-2 \leq x \leq 2$. As part of your solution you should identify where the internal layer is located (discussing in detail why there is no boundary layer at $x = \pm 2$).

Compare the inner and outer solutions with a numerical solution.

(Note: A composite solution is not required here, although coming up with one might be fun!)

[Hints: it would be a **very** good idea to look at a numerical solution before starting your analysis. Different outer solutions are required each side of the internal layer (perhaps call these y_L and y_R) which require their own matching conditions.]