APP MTH 3001 Applied Probability III

Class Exercise 3 Solutions

1. (a) In Figure 1 we see that $C_1 = \{1\}$, $C_2 = \{2\}$ and $C_3 = \{3\}$. State 1 is recurrent and states 2 and 3 are transient.

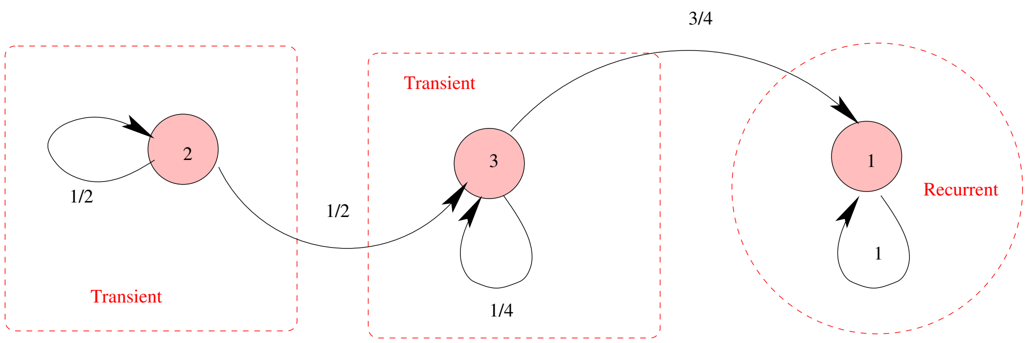

Figure 1: State diagram and communicating classes $\mathbb{P}_a$.

(b) In Figure 2 we see that $C_1 = \{1, 2, 3\}$ is a recurrent class and therefore the whole DTMC is irreducible and recurrent.

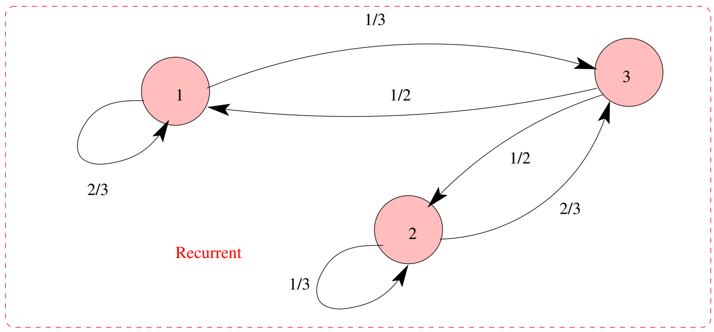

Figure 2: State diagram and communicating classes $\mathbb{P}_b$.

(c) In Figure 3 we see that $C_1 = \{5\}$ and $C_2 = \{2\}$ are distinct recurrent classes. $C_3 = \{1, 3\}$ is a transient class and state 4 is ephemeral since it does not communicate with itself or any other state.

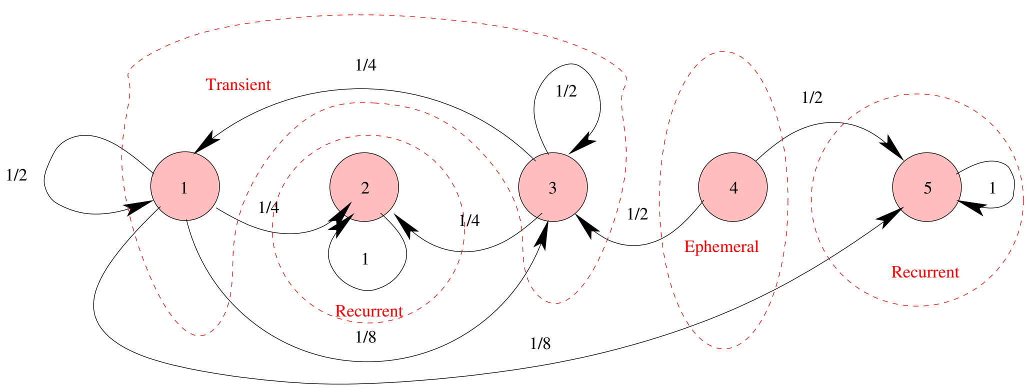

Figure 3: State diagram and communicating classes $\mathbb{P}_c$.

(d) In Figure 4 we see that $C_1 = \{1,2,3,4,5\}$ is a recurrent class and therefore the whole DTMC is irreducible and recurrent.

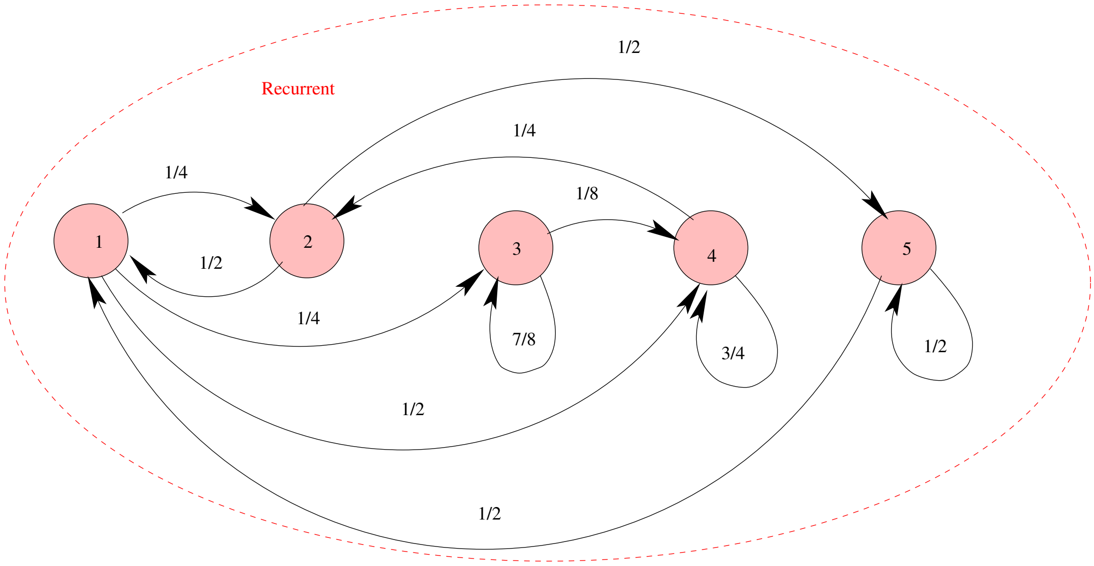

Figure 4: State diagram and communicating classes $\mathbb{P}_d$.

2. Case $p \neq q$. We saw in lectures that the general form of solution for the inhomogeneous system of difference equations is

$$k_i = A + B\left(\frac{q}{p}\right)^i + \frac{i}{q-p}, \qquad i \geq 0.$$

Make sure you are able to derive this general form of solution yourself ! The boundary condition $k_0 = 0$ gives $A = -B$, so that

$$k_i = B\left[\left(\frac{q}{p}\right)^i - 1\right] + \frac{i}{q-p}, \qquad i \geq 0. \tag{1}$$

(a) Now suppose $\underline{p < q}$. Then $q - p > 0$ and $\frac{q}{p} > 1$, and

$$\left(\frac{q}{p}\right)^i - 1 > 0 \qquad \text{for all } \ i \geq 1,$$

so the minimal non-negative solution (see Theorem 3.3) is obtained by setting $B = 0$. Thus

$$k_i = \frac{i}{q-p}, \qquad i \geq 1.$$

It is important to note that since $q - p > 0$, this solution is non-negative, as required.

For simplicity let's do part (c) now, then finish with part (b).

(c) Now suppose $\underline{p > q}$. Then $q - p < 0$ and $\frac{q}{p} < 1$, and

$$\left(\frac{q}{p}\right)^i - 1 < 0 \qquad \text{for all } \ i \geq 1.$$

Also, $\frac{i}{q-p} < 0$ for all $i \geq 1$. Note that for any finite B, we can find an i such that this is negative.
Therefore, there is no finite B that can make this non-negative, for all $i \geq 1$.
Thus, the only way in which (1) can be non-negative for all $i \geq 1$ is if we set $B = \infty$.
Therefore,

$$k_i = \infty, \qquad i \geq 1.$$

This makes sense, because we showed in lectures that the hitting probability on state 0, in this case, is $u_i^{\{0\}} = \left(\frac{q}{q}\right)^i < 1$. Hence, there is a positive probability that it never hits state 0, which means that with positive probability the (random) hitting time is ∞. Of course, we still have $k_0 = 0$.

(b) Case $p = q$. We know from lectures that the general form of solution for the **homogeneous** system of difference equations is

$$k_i = A + Bi, \qquad i \geq 0.$$

Make sure you are able to derive this general form of solution yourself!
For the particular solution to the **inhomogeneous** system, we can't use C or iC (as they are solutions to the homogeneous system!) so try i^2C.
Substituting into the **inhomogeneous** system gives,

$$\begin{aligned} i^2C &= 1 + p(i+1)^2C + q(i-1)^2C, \\ &= 1 + i^2C(p+q) + 2iC(p-q) + C(p+q). \end{aligned}$$

Since $p + q = 1$ and $p = q$, this simplifies to $0 = C + 1$ and so $C = -1$ and the general form of solution for the **inhomogeneous** system of difference equations is

$$k_i = A + Bi - i^2, \qquad i \geq 0.$$

The boundary condition $k_0 = 0$ gives $A = 0$, so that

$$k_i = Bi - i^2 = i(B - i), \qquad i \geq 0. \tag{2}$$

Note that for any finite B, we can find an i such that this is negative.
Therefore, there is no finite B that can make this non-negative, for all $i \geq 1$.
Thus, the only way in which (2) can be non-negative for all $i \geq 1$, is if we set $B = \infty$. Thus

$$k_i = \infty, \qquad i \geq 1.$$

We saw in lectures that for the case $p = q$, the probability that Player A eventually loses all of her money is equal to one. Here, we have just shown that the expected time taken for A to lose all of her money when $p = q$, is infinite.
In other words, the event {A eventually loses of all her money} occurs with probability one, but the expected time taken for this event to occur is infinite !
This phenomenon may appear somewhat counter-intuitive.
It occurs because Player B cannot ever be bankrupt, and can lose an arbitrarily large amount of money without losing the contest, and the excursions before A loses can be arbitrarily long!
This is a subtle point which we will explore further in the course.

3. (a) In Figure 5 we see that $\{0\}$ and $\{N\}$ are distinct recurrent communicating classes, as states 0 and N are absorbing states.
Each of the remaining states is a transient communicating class by itself, that is, $\{1\}, \{2\}, \ldots, \{N-1\}$ are all different transient communicating classes , since it is impossible to transition from state k to state $k-1$ for $k \geq 2$.
They are transient because they self-communicate and the process is eventually "trapped" in one of the absorbing states with probability one.

(i)

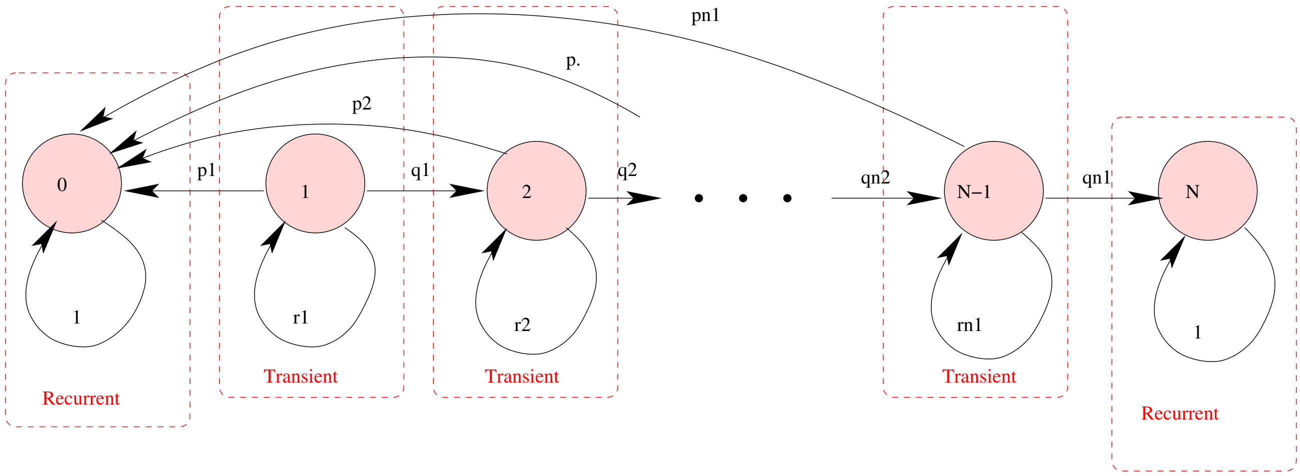

Figure 5: State diagram and communicating classes question 2.

(ii) So in this problem, every communicating class has only one state! For all states $k \neq 0, N$, the process can "jump" directly to state 0 (with probability p_k), remain in state k (with probability r_k), or move to state $k+1$ (with probability q_k).

(b) In terms of the "Two Gamblers" example, this Markov chain could be used to model the modified contest having the following properties.
Recall that the contest consists of a sequence of games, and the contest ends when one of the players has lost of all their money.

- When Player A has k dollars, the result of the next game is a draw with probability r_k (originally, there was zero probability of a draw, since we had $p_{i,i} = 0$ for all $i \neq 0, N$).
- When Player A has k dollars, she wins one dollar with probability q_k.
- When Player A has k dollars, she loses her entire current capital k with probability p_k (originally, she could only lose one dollar at a time), and the contest ends.

(c) $u_k^{\{0\}}$ satisfies the equation

$$u_k^{\{0\}} = p_k + r_k u_k^{\{0\}} + q_k u_{k+1}^{\{0\}}, \qquad \text{for } k = 1, \ldots, N-1,$$

since $u_0^{\{0\}} = 1$.
However, it is easier to derive the required expression by first considering $u_k^{\{N\}}$, which satisifies

$$u_k^{\{N\}} = r_k u_k^{\{N\}} + q_k u_{k+1}^{\{N\}}, \qquad \text{for } k = 1, \ldots, N-1,$$

since $u_0^{\{N\}} = 0$. The other boundary condition is $u_N^{\{N\}} = 1$, so

$$\begin{aligned} u_{N-1}^{\{N\}} &= r_{N-1}u_{N-1}^{\{N\}} + q_{N-1} \\ u_{N-1}^{\{N\}}(1 - r_{N-1}) &= q_{N-1} \\ u_{N-1}^{\{N\}} &= \frac{q_{N-1}}{p_{N-1} + q_{N-1}}, \end{aligned}$$

where we have used the fact that $p_{N-1} + q_{N-1} + r_{N-1} = 1$.

Similarly,

$$\begin{aligned} u_{N-2}^{\{N\}} &= r_{N-2}u_{N-2}^{\{N\}} + q_{N-2}u_{N-1}^{\{N\}} \\ &= r_{N-2}u_{N-2}^{\{N\}} + q_{N-2}\left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right), \\ \Rightarrow u_{N-2}^{\{N\}} &= \left(\frac{q_{N-2}}{p_{N-2} + q_{N-2}}\right)\left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right), \end{aligned}$$

where we have used the fact that $p_{N-2} + q_{N-2} + r_{N-2} = 1$.

Proceeding recursively, we obtain

$$u_{N-k}^{\{N\}} = \left(\frac{q_{N-k}}{p_{N-k} + q_{N-k}}\right)\cdots\left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right),$$

and the transformation $(N - k) \to k$ gives

$$u_k^{\{N\}} = \left(\frac{q_k}{p_k + q_k}\right)\cdots\left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right),$$

which is valid for $k = 1, \ldots, N - 1$.

Finally, we note that for all $k = 1, \ldots, N - 1$, $u_k^{\{0\}} = 1 - u_k^{\{N\}}$, so that

$$u_k^{\{0\}} = 1 - \left(\frac{q_k}{p_k + q_k}\right)\cdots\left(\frac{q_{N-1}}{p_{N-1} + q_{N-1}}\right), \quad \text{for } k = 1, \ldots, N - 1.$$

4. Condition numbers for transition matrices will vary between pieces of music, and will depend on the particular method chosen to define the matrix (eg., what to do with the final note, etc). Condition numbers should typically be in the range 1-10000 , and the general rule of thumb is that for a condition number that is of order 10^k you might lose approximately k digits of precision due to numerical issues.