School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Assignment 2 question sheet

Due: Thursday, 29 August, at 12 noon (in the hand-in box on level 6)

When presenting your solutions to the assignment, please include some explanation in words to accompany your calculations. It is not necessary to write a lengthy description, just a few sentences to link the steps in your calculation. Messy, illegible or inadequately explained solutions may be penalised. The marks awarded for each part are indicated in brackets.

All students are to attempt Questions 1–3.

1. Find the extremals of the following functionals:

   (a) $F\{y\} = \int_0^1 (y^2 + y'^2 + 2ye^x)\, dx, \qquad y(0) = 0, \quad y(1) = 1.$

   (b) $F\{y\} = \int_0^1 (y^2 - y'^2 - 2y \sin x)\, dx, \quad y(0) = 0, \quad y(1) = 1.$

   [8 marks]

2. Consider the functional

$$F\{y\} = \int_0^1 \left(\frac{1}{2} y'^2 + yy' + y' + y \right) dx, \quad y(0) = 0, \quad y(1) = \frac{3}{2}.$$

   (a) Determine a differential expression H that takes a constant value for extremals of F.

   (b) Derive the function $y(x)$ which is an extremal of F.

   [6 marks]

3. In lectures we consider the Brachistochrone functional

$$T\{y\} = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \sqrt{\frac{1 + y'^2}{y_0 - y}}\, dx, \tag{1}$$

   together with the end-points $y(x_0) = y_0$ and $y(x_1) = y_1$. The parametric solution we derive is

$$x = x_0 + \kappa(\theta - \sin\theta), \quad y = y_0 - \kappa(1 - \cos\theta), \quad 0 \leqslant \theta \leqslant \theta_1. \tag{2}$$

   To fully specify a solution curve we must determine θ_1, the value of the parameter that corresponds to the $x = x_1$ end-point, and the constant κ.

   (a) By substituting the solution (2) into the the functional (1) and evaluating the integral, give an explicit formula for T in terms of θ_1, κ and g.

   (b) Assuming $(x_0, y_0) = (0, 2)$ and $(x_1, y_1) = (5, 1)$, determine the values of θ_1 and κ for three different solution curves that satisfy the end-points. Give your answers to four significant digits and provide any computer code you wrote to find these solutions.

   (c) Taking meters as the units of length and $g = 9.807$ m/s^2, determine the value of T for the three different solution curves found in part (b). Give your answers to four significant digits.

   (d) Using a computer package, plot the three curves from part (b) clearly labelling each path with the value of T calculated in part (c).

   [10 marks]
