APP MTH 3001 Applied Probability III
Class Exercise 5 Solutions

1. The size X_n of the n^{th} step $n \geq 1$ of a random walk on the integers starting at the origin has distribution $\mathrm{P}(X_n = j) = \dfrac{e^{-1}}{j!}, \quad j \geq 0$, which is the Poisson distribution with mean 1, and we define

$$\begin{aligned}
S_0 &= 0 \quad \text{and} \quad S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1, \\
Y_n &= S_n - n\,, \quad n \geq 0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[|Y_n|\right] &= \mathrm{E}\left[|S_n - n|\right] \\
&\leq n + \mathrm{E}\left[\left|\sum_{i=1}^{n} X_i\right|\right] \\
&= n + \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] \qquad \big(\text{since the } X_i \text{ are all non-negative random variables}\big) \\
&= n + \sum_{i=1}^{n} \mathrm{E}\left[X_i\right] \\
&= n + \sum_{i=1}^{n} 1 \qquad \big(\text{since the } X_i \text{ are Poisson distributed with mean } 1\big) \\
&= 2n \;<\; \infty.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{E}\left[Y_{n+1} \mid X_0, \ldots, X_n\right] &= \mathrm{E}\left[S_{n+1} - (n+1) \mid X_0, \ldots, X_n\right] \\
&= \mathrm{E}\left[X_{n+1} + S_n - (n+1) \mid X_0, \ldots, X_n\right] \\
&= S_n - (n+1) + \mathrm{E}\left[X_{n+1} \mid X_0, \ldots, X_n\right] \\
&= S_n - (n+1) + 1 \quad (X_i \text{ are IID Poisson, mean } 1) \\
&= S_n - n \;\equiv\; Y_n
\end{aligned}$$

and so $\{Y_n : n \in \mathbb{N}\}$ is a martingale wrt $\{X_n : n \in \mathbb{N}\}$.

2. For $k \leq \ell < m$ we have that

$$\begin{aligned}
\mathrm{E}\left[(X_m - X_\ell)X_k\right] &= \mathrm{E}\left[\mathrm{E}\left[(X_m - X_\ell)X_k \mid X_0, \ldots, X_\ell\right]\right] \\
&= \mathrm{E}\left[X_k \mathrm{E}\left[X_m \mid X_0, \ldots, X_\ell\right] - X_k X_\ell\right] \quad (X_k, X_\ell \text{ are conditionally known}) \\
&= \mathrm{E}\left[X_k X_\ell - X_k X_\ell\right] \quad (\text{by the martingale property}) \\
&= \mathrm{E}\left[0\right] \;=\; 0.
\end{aligned}$$

3. First,

$$
\begin{aligned}
E\left[|M_n|\right] &= E\left[\left|\sum_{m=1}^{n} f(X_m) - \sum_{m=0}^{n-1}\sum_{i\in\mathcal{S}} p_{X_m,i} f(i)\right|\right] \\
&\leq \sum_{m=1}^{n} E\left[|f(X_m)|\right] + \sum_{m=0}^{n-1}\sum_{i\in\mathcal{S}} E\left[|p_{X_m,i} f(i)|\right] \\
&< \infty,
\end{aligned}
$$

since $|f(j)|$ is bounded for all $j \in \mathcal{S}$.
Now,

$$
\begin{aligned}
E\left[M_{n+1}|X_0,\ldots,X_n\right] &= E\left[\sum_{m=1}^{n+1} f(X_m) - \sum_{m=0}^{n}\sum_{i\in\mathcal{S}} p_{X_m,i} f(i) \middle| X_0,\ldots,X_n\right] \\
&= E\left[M_n + f(X_{n+1}) - \sum_{i\in\mathcal{S}} p_{X_n,i} f(i) \middle| X_0,\ldots,X_n\right] \\
&= M_n + E\left[f(X_{n+1})|X_0,\ldots,X_n\right] - \sum_{i\in\mathcal{S}} p_{X_n,i} f(i), \\
&\qquad \text{since } X_0,\ldots,X_n \text{ are conditionally known,} \\
&= M_n + E\left[f(X_{n+1})|X_n\right], - \sum_{i\in\mathcal{S}} p_{X_n,i} f(i), \quad \text{since } X_n \text{ is Markov,} \\
&= M_n + \sum_{i\in\mathcal{S}} p_{X_n,i} f(i) - \sum_{i\in\mathcal{S}} p_{X_n,i} f(i) \\
&= M_n.
\end{aligned}
$$

4.

$$
\begin{aligned}
\mathrm{E}\left[X_{n+1}Z\right] &= \mathrm{E}\left[\mathrm{E}\left[X_{n+1}Z \mid Y_0,\ldots,Y_n\right]\right] \\
&= \mathrm{E}\left[Z\mathrm{E}\left[X_{n+1} \mid Y_0,\ldots,Y_n\right]\right] \\
&\geq \mathrm{E}\left[ZX_n\right] \quad \text{by the sub-martingale property,}
\end{aligned}
$$

noting that the inequality can only be preserved if $Z \geq 0$.