SMI Assignment 1

Andrew Martin

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Question 1:
$(Y_i)_{i=1}^n$ are i.i.d $N(\mu, \sigma^2)$ r.vs with sample mean $\bar{Y}$.

a) Find $E\left[\bar{Y}^2\right]$

$$var(\bar{Y}) = E\left[\bar{Y}^2\right] - E\left[\bar{Y}\right]^2$$
$$\implies E\left[\bar{Y}^2\right] = var(\bar{Y}) + E\left[\bar{Y}\right]^2$$

$$= var(\frac{1}{n}\sum_{i=1}^n Y_i) + E\left[\frac{1}{n}\sum_{i=1}^n Y_i\right]^2$$
$$= \frac{1}{n^2}\sum_{i=1}^n var(Y_i) + \frac{1}{n^2}E\left[\sum_{i=1}^n Y_i\right]^2$$
$$= \frac{1}{n^2}\sum_{i=1}^n \sigma^2 + \frac{1}{n^2}(\sum_{i=1}^n E\left[Y_i\right])^2$$
$$= \frac{n}{n^2}\sigma^2 + \frac{1}{n^2}(\sum_{i=1}^n \mu)^2$$
$$= \frac{\sigma^2}{n} + \mu^2$$

b) For each i with $1 \leq i \leq n$, prove $\bar{Y}$ and $Y_i - \bar{Y}$ are uncorrelated

$$cov(\bar{Y}, Y_i - \bar{Y}) = E\left[(\bar{Y} - E[\bar{Y}])((Y_i - \bar{Y}) - E[Y_i - \bar{Y}])\right]$$
$$= E\left[(\bar{Y} - E[\bar{Y}])((Y_i - \bar{Y}) - E[Y_i] - E[\bar{Y}])\right]$$

From above $E[\bar{Y}] = \mu$, so:

$$= E\left[(\bar{Y} - \mu)((Y_i - \bar{Y}) - \mu - \mu)\right]$$
$$= E\left[(\bar{Y} - \mu)(Y_i - \bar{Y})\right]$$
$$= E[\bar{Y}Y_i - Y_i\mu + \mu\bar{Y} - \bar{Y}^2]$$
$$= E[\bar{Y}Y_i] - E[Y_i\mu] + E[\mu\bar{Y}] - E[\bar{Y}^2]$$
$$= E[\bar{Y}Y_i] - \mu E[Y_i] + \mu E[\bar{Y}] - \frac{\sigma^2}{n} - \mu^2$$

But From Tute 1: $E[\bar{Y}Y_i] = \frac{-\sigma^2}{n} + \mu^2$, so:

$$= \frac{-\sigma^2}{n} + \mu^2 - \mu E[Y_i] + \mu E[\bar{Y}] - \frac{\sigma^2}{n} - \mu^2$$

$$= -\mu\mu + \mu\mu = 0$$

Therefore they are uncorrelated.

Question 2:
$(Z_i)_{i=1}^p$ are i.i.d $N(0,1)$ random variables with

$$X = \sum_{i=1}^{p} Z_i^2$$

a) Find the moment generating function and distribution of X, assuming $M_{Z^2}(t) = (1-2t)^{-1/2}$, where $Z\sim N(0,1)$

$$M_X(t) = E\left[e^{tX}\right] = E\left[e^{t\sum_{i=1}^{p} Z_i^2}\right]$$

Note that
$$M_{Z^2}(t) = E[e^{tZ^2}] = E[e^t e^{Z^2}] = (1-2t)^{-1/2}$$

So:
$$M_X(t) = E\left[e^t e^{\sum_{i=1}^{p} Z_i^2}\right]$$
$$= E\left[e^t \prod_{i=1}^{p} e^{Z_i^2}\right]$$

Since Z_i are i.i.d
$$= \prod_{i=1}^{p} E\left[e^{tZ_i^2}\right]$$
$$= \prod_{i=1}^{p} M_{Z^2}(t)$$
$$= \prod_{i=1}^{p} (1-2t)^{-1/2}$$

b) $(Y_i)_{i=1}^p$ are independent normal random variables with different means and variances
i.e.
$Y_i\sim N(\mu_i, \sigma_i^2), i = 1, ..., p$
Show that $W = \sum_{i=1}^{p} \frac{(Y_i-\mu_i)^2}{\sigma_i^2} \sim \chi_p^2$
where χ_p^2 denotes the chi-squared dist with p degrees of freedom.

$$W = \sum_{i=1}^{p} \frac{(Y_i-\mu_i)^2}{\sigma_i^2}$$

Note that $\frac{(Y_i-\mu_i)}{\sigma_i} \sim Z \sim N(0,1)$ So from this:

$$W = \sum_{i=1}^{p} \frac{(Y_i-\mu_i)}{\sigma_i} = \sum_{i=1}^{p} Z_i^2$$

For which, the MGF is as above: $= \prod_{i=1}^{p}(1-2t)^{-1/2}$
Which is the form of a χ^2 MGF

Question 3

Let $X \sim Bin(n,p)$

And $\hat{p}^2 = \left(\frac{X}{n}\right)^2$

a) Find $E\left[\hat{p}^2\right]$ and state the bias

$$E[\hat{p}^2] = E[(\frac{X}{n})^2]$$

$$\implies E\left[(\frac{X}{n})^2\right] = var(\frac{X}{n^2}) + E\left[\frac{X}{n}\right]^2$$

$$= \frac{1}{n^2}var(X) + \frac{1}{n^2}E\left[X\right]^2$$

$$= \frac{1}{n^2}np(1-p) + \frac{1}{n^2}(np)^2$$

$$= \frac{p(1-p)}{n} + p^2$$

$$= \frac{p - p^2 + np^2}{n}$$

The bias is:

$$\frac{p-p^2}{n}$$

b) Show that

$$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$$

Start with the left hand side:

$$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{1}{n-1}E\left[\hat{p}(1-\hat{p})\right]$$

$$= \frac{1}{n-1}E\left[\hat{p} - \hat{p}^2\right]$$

$$= \frac{1}{n-1}(E[\hat{p}] - E[\hat{p}^2])$$

$$= \frac{1}{n-1}\left(E[\frac{X}{n}] - (\frac{p-p^2+np^2}{n})\right)$$

$$= \frac{1}{n^2-n}\left(E[X] - (p - p^2 + np^2)\right)$$

$$= \frac{1}{n^2-n}\left(np - p + p^2 - np^2\right)$$

$$= \frac{1}{n^2-n}\left(p(n-1+p-np)\right)$$

$$= \frac{n-1}{n^2-n}\left(p(1-p)\right)$$

$$= \frac{p(1-p)}{n}$$

c) Using a) and b) find an unbiased estimator for p^2.
If T is an estimator for θ then the bias is: $b_T(\theta) = E[T] - \theta$. For unbiased, set this to zero.
I.e. In this case, $b_T(p^2) = E[T] - p^2 = 0$
So aim to find T such that $E[T] = p^2$ By subtracting the expectations in a) and b)

$$\begin{aligned} &E[\hat{p}^2] - E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] \\ &= \frac{p - p^2 + np^2}{n} - \frac{p(1-p)}{n} \\ &= \frac{np^2}{n} \\ &= p^2 \end{aligned}$$

So an unbiased estimator, T, for p^2, would be:

$$T = \hat{p}^2 - \frac{\hat{p}(1-\hat{p})}{n-1}$$

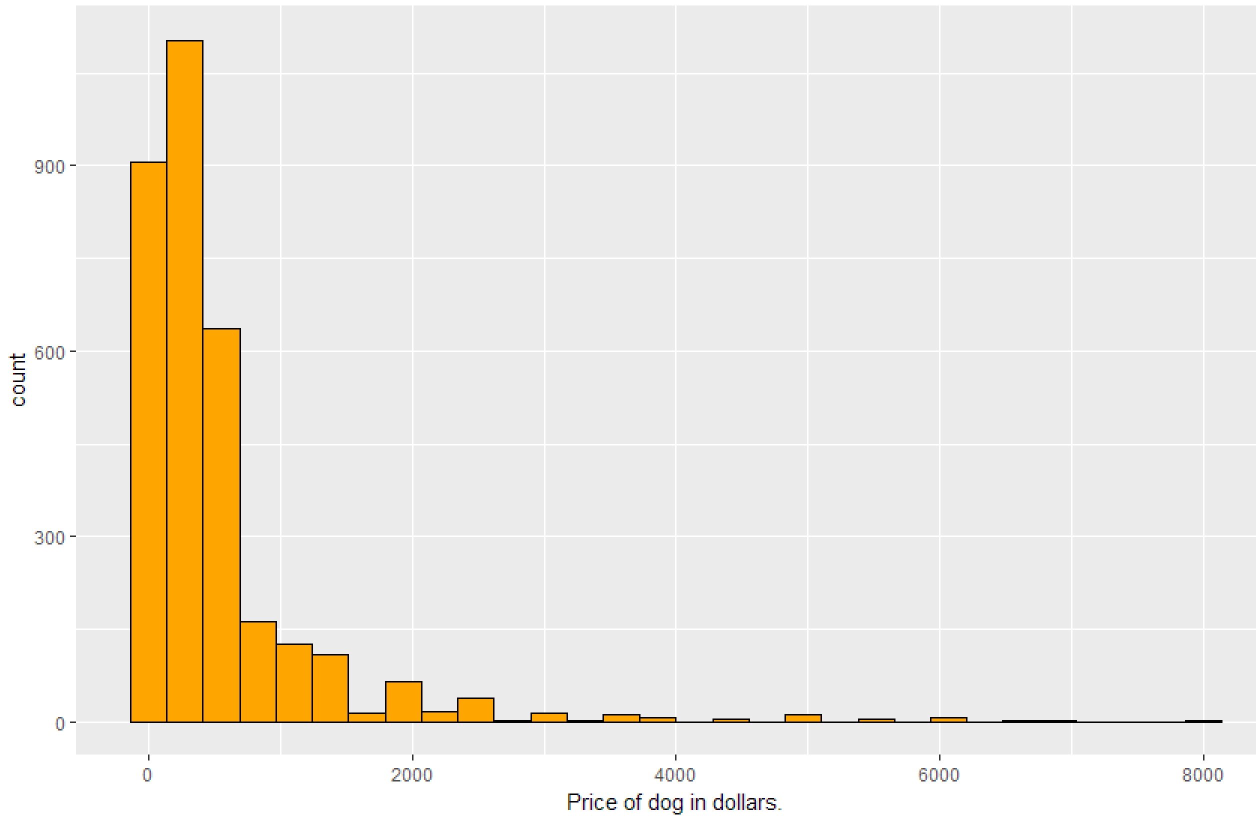
Question 4

1) Clean all the variables (include code)
2) Produce an appropriate plot for each variable
i.e for categorical - bar chart, for quantitative - histogram.
Label and caption each of these
3) For the quantitative variables, identify if they are: unimodal or bimodal, whether it is symmetric, left-skewed or right-skewed. For categorical, identify the most common level

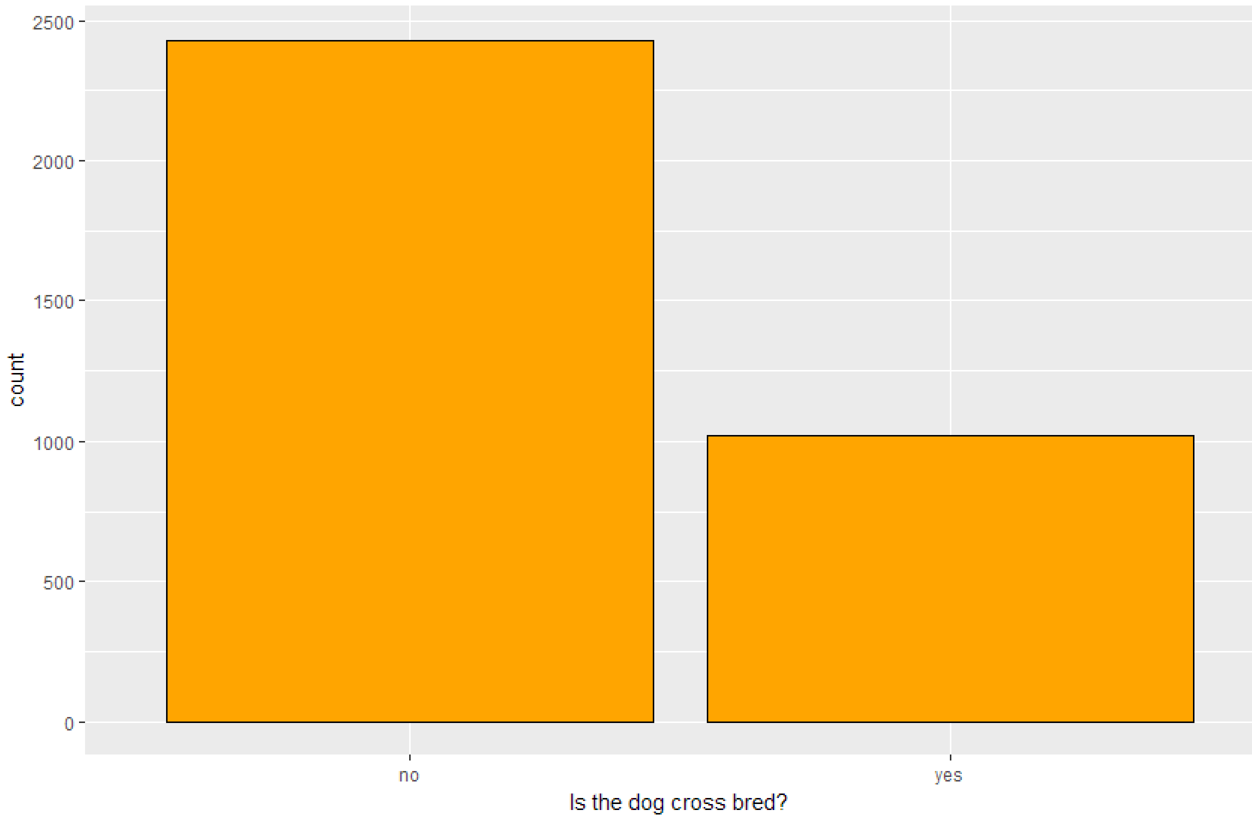
Price
Cross
Pet offered by
Microchip
Vaccination
Desexing status
Relinquished or not

The R code is attached and will be referenced from here, onwards.

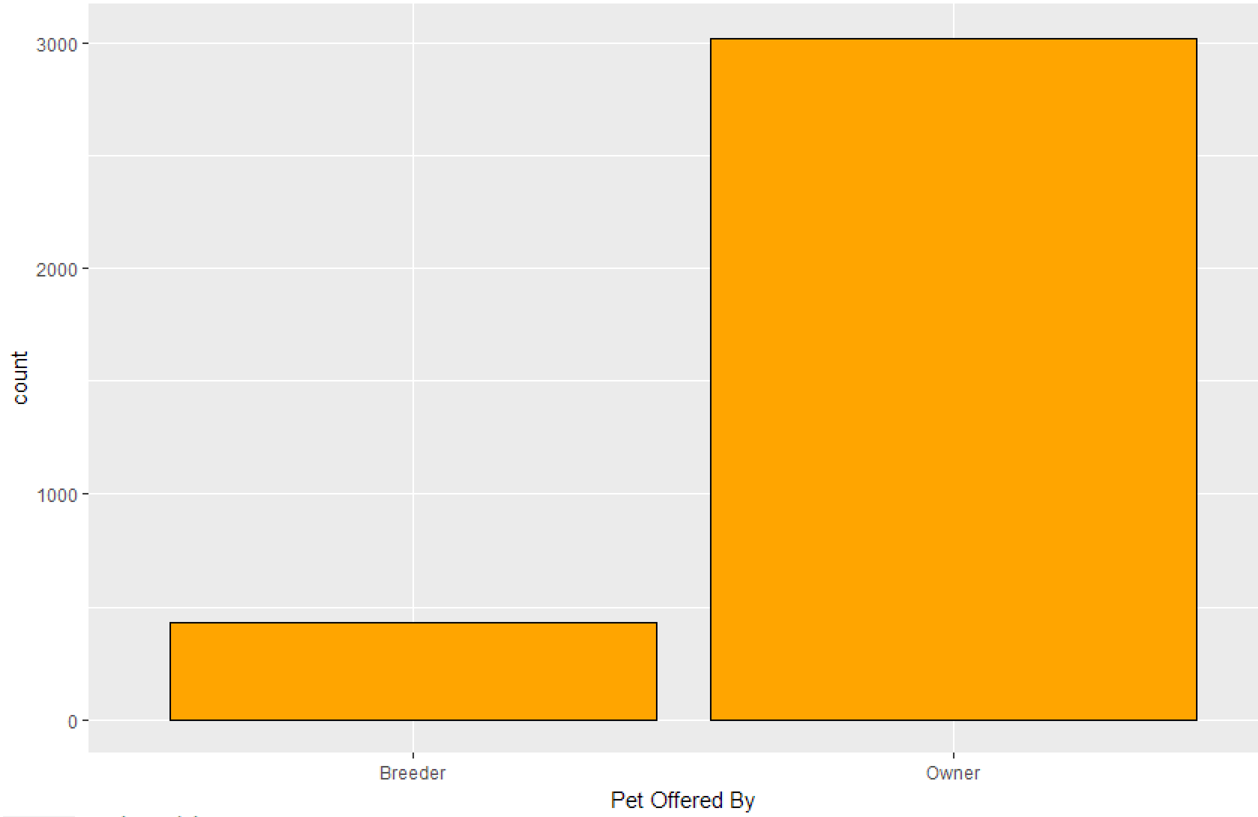
```
#R Code for SMI Assignment 1
#Andrew Martin
#10/08/2017
library('readxl')
library('tidyverse')
library('rmarkdown')
library('ggplot2')
setwd("F:/Documents/Uni/SMI")


gumtree = read_xlsx("Gumtree_dogs.xlsx")


####Price ----
##Cleaning -
#this should be a number so "NA" chars are removed
gumtree$price[gumtree$price=="NA"]=NA
class(gumtree$price)

#class is changed to numeric
gumtree$price=as.numeric(gumtree$price)

#since price is a quantitative, continuous variable, a histogram is appropriate

ggplot(gumtree,aes(x = price)) +
  geom_histogram(col = "black", fill = "orange") +
  labs(x = "Price of dog in dollars.")
#This graph is right - skewed, which suggests price is right skewed
#it has a single peak which suggests it is unimodal

```



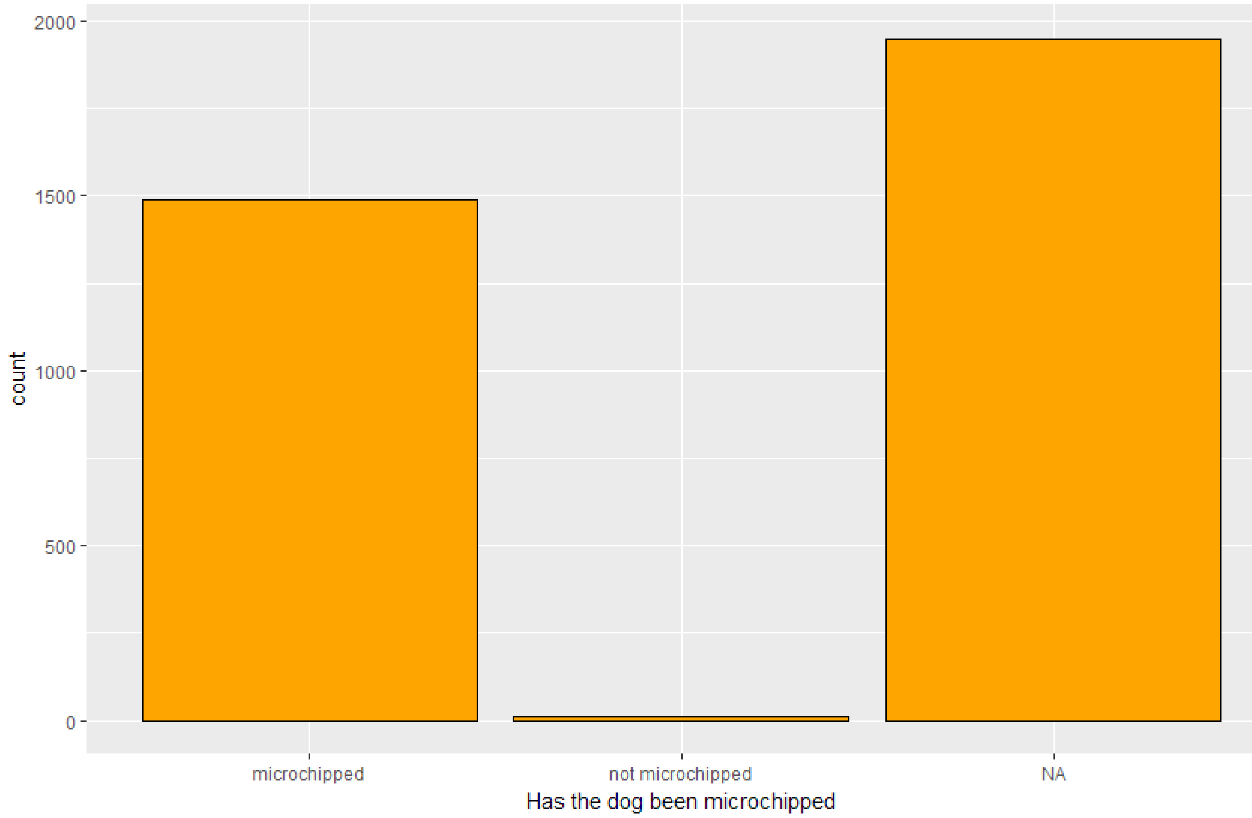

```
##cross ----
#told to ignore the "Cross" (uppercase) section and use the "cross" (lowercase)
#interesting note:
length(which(gumtree$Cross!=gumtree$cross)) #gives 635 -> that means 635 values are different!

#cross is chr which is good
class(gumtree$cross)
#nas are changed so R can read them
gumtree$cross[gumtree$cross=="NA"]=NA

#since cross has two states, yes and no, a bar chart is more appropriate
ggplot(gumtree,aes(x = cross)) +
  geom_bar(col = "black", fill = "orange") +
  labs(x = "Is the dog cross bred?")

#clearly a greater portion is not cross bred
```



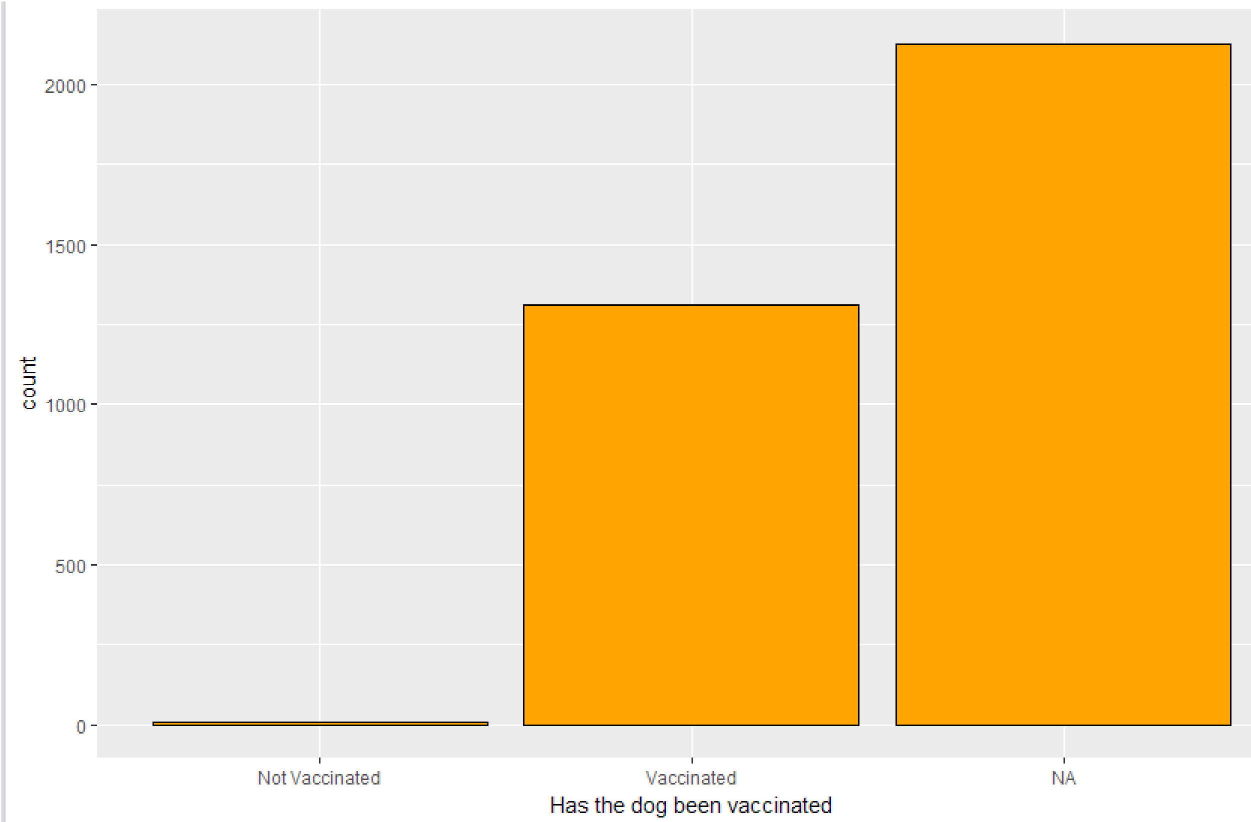

```
##Pet offered by ----
gumtree$"Pet Offered By:"[gumtree$"Pet Offered By:"=="NA"]=NA


ggplot(gumtree,aes(x = gumtree$'Pet Offered By:')) +
  geom_bar(col = "black", fill = "orange") +
  labs(x = "Pet Offered By")

#a majority of the pets were offered by an owner.
```



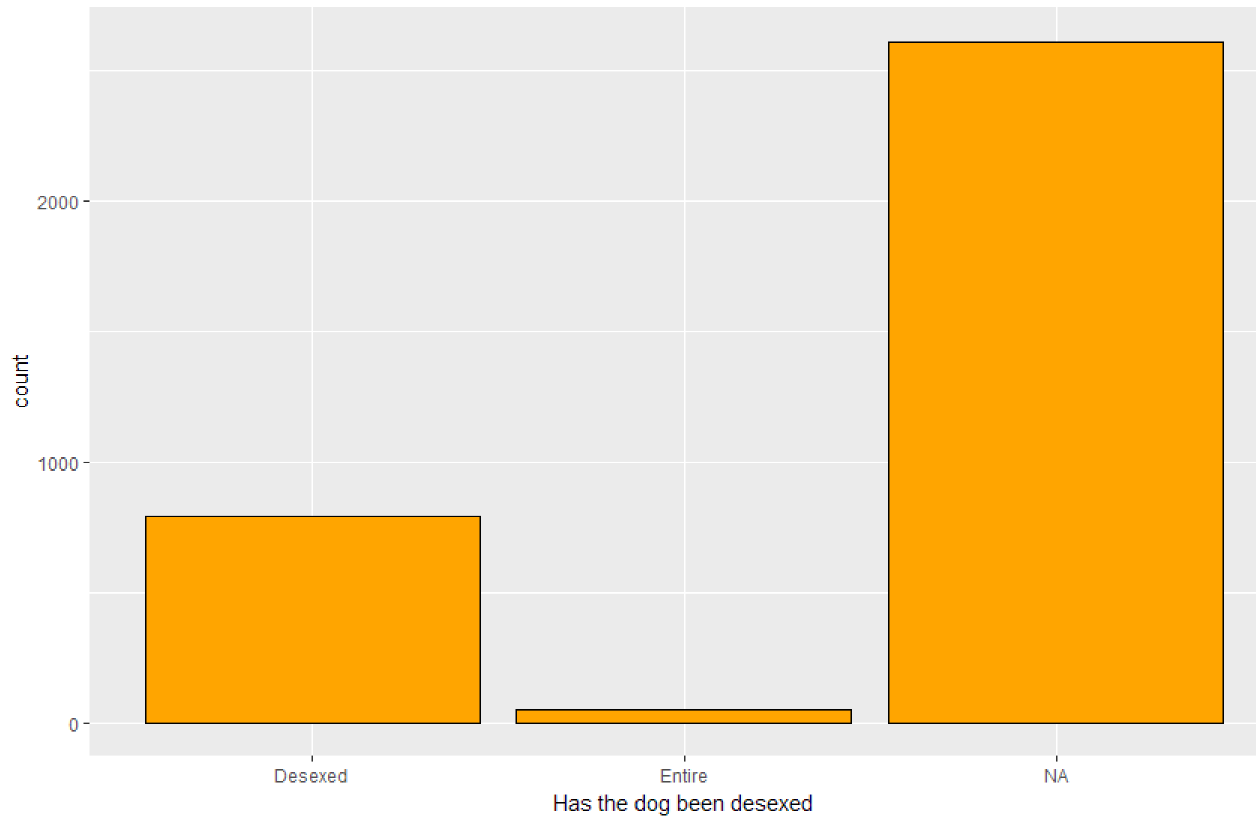

```
##Microchip ----
gumtree$micro[gumtree$micro=="NA"]=NA

#na.rm is not working here for some reason
ggplot(gumtree,aes(x = micro)) +
  geom_bar(col = "black", fill = "orange", na.rm = TRUE) +
  labs(x = "Has the dog been microchipped")


#most fields are NA, but few to no dogs are not microchipped

```



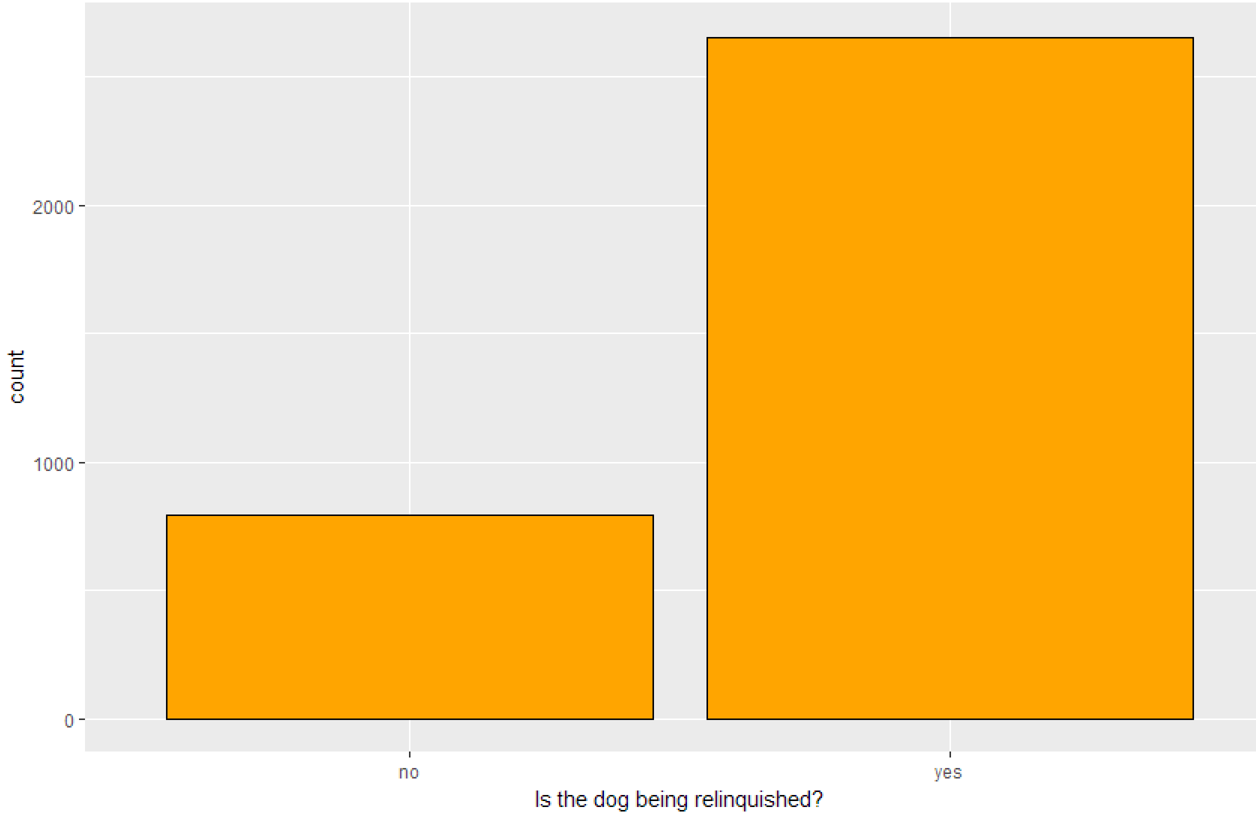

```
##Vaccination ----


gumtree$vacc[gumtree$vacc=="NA"]=NA

#na.rm is not working here for some reason
ggplot(gumtree,aes(x=vacc))+
geom_bar(col = "black", fill = "orange", na.rm=TRUE) +
  labs(x = "Has the dog been vaccinated")

#A very small percentage are labelled as not vaccinated, and most are vaccinated.
```




```
##Desexing ----

gumtree$desex[gumtree$desex=="NA"]=NA

#na.rm is not working here for some reason
ggplot(gumtree,aes(x=desex))+
  geom_bar(col = "black", fill = "orange", na.rm=TRUE) +
  labs(x = "Has the dog been desexed")

```




```r
##Relinquished ----
gumtree$relinquished[gumtree$relinquished=="NA"]=NA

ggplot(gumtree,aes(x = relinquished)) +
  geom_bar(col = "black", fill = "orange", na.rm=TRUE) +
  labs(x = "Is the dog being relinquished?")
#Most dogs have been relinquished

```


count
2000
1000
0
no
yes
Is the dog being relinquished?