Lecture 15: Stationary and Reversed-Time Processes

Concepts checklist

At the end of this lecture, you should be able to:

- *define* a stationary CTMC;
- *define* a reversed-time CTMC;
- *state and prove* a theorem regarding the transition rates and equilibrium distribution of the reversed-time process corresponding to a stationary CTMC; and,
- *define* a reversible CTMC.

We will show that some stochastic processes are such that when we consider them in reversed time, the behaviour of the process remains indistinguishable from the forward time behaviour.

Stationary Processes

Definition 13. *A continuous-time Markov chain $\{X(t)\}$ is stationary if*

$$\begin{aligned}&\Pr(X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n)\\ &= \Pr(X(t_1 + \delta) = i_1, X(t_2 + \delta) = i_2, \ldots, X(t_n + \delta) = i_n)\end{aligned}$$

for every positive integer n and for all $\delta, t_1, t_2, \ldots, t_n$.

In other words, the joint distribution of $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ is the same as that of $\{X(t_1 + \delta), X(t_2 + \delta), \ldots, X(t_n + \delta)\}$. In particular, when $n = 1$ the distribution of $X(t)$ is the distribution of $X(t + \delta)$.

Roughly speaking, a CTMC (that has an equilibrium distribution) operates in a stationary manner after it has been running for a long time. If we start a Markov chain with its equilibrium distribution, that is, let $\Pr(X(0) = j) = \pi_j$ for all $j \in \mathcal{S}$, then it will have the property

$$\Pr(X(t) = j) = \pi_j \text{ for all } j \in \mathcal{S} \text{ and } t \geq 0.$$

Reversed-time Processes

Definition 14. *For each stationary Markov chain $\{X(t)\}$, we define the reversed-time process $\{X^R(t)\}$ to be*

$$X^R(t) = X(\tau - t), \qquad \textit{for arbitrary } \tau.$$

Theorem 13. *If $X(t)$ is a stationary Markov chain with transition rates q_{jk} and equilibrium distribution $\boldsymbol{\pi}$, then the reversed-time process $X^R(t) = X(\tau - t)$, is a stationary Markov chain with*

$$\begin{aligned}\textit{transition rates} \quad & q^R_{kj} = \frac{q_{jk}\pi_j}{\pi_k}, \qquad \textit{for } k, j \in \mathcal{S}, k \neq j,\\ \textit{and equilibrium distribution} \quad & \pi^R_j = \pi_j.\end{aligned}$$

Proof. (i) Since $\{X(t)\}$ is stationary, the distribution of $X(\tau-t)(=_d X^R(t))$ and the distribution of $X(\tau-t-y)(=_d X^R(t-y))$ are the same for all t. Thus, the reversed process is also stationary.

For $\{X(t)\}$, we have that the future and the past are independent, conditional only on the present state. For $\{X^R(t)\}$,we interchange the concepts of future and past: the past becomes the future, the future becomes the past, and they are still independent from each other, conditional on the present $\Rightarrow X^R(t)$ too has the Markov property.

Consequently, if $X(t)$ is a stationary CTMC, then $X^R(t)$ is also a stationary CTMC.

(ii) For $h > 0$ and $j \neq k$,

$$\begin{aligned}\Pr(X(t) = j, X(t+h) = k) &= \Pr(X(t+h) = k | X(t) = j) \Pr(X(t) = j) \\ &\quad \text{or } \Pr(X(t) = j | X(t+h) = k) \Pr(X(t+h) = k).\end{aligned}$$

Then

$$\begin{aligned}&\lim_{h\to 0^+} \frac{\Pr(X(t+h) = k | X(t) = j)}{h} \Pr(X(t) = j) \\ &\qquad = \lim_{h\to 0^+} \frac{\Pr(X(t) = j | X(t+h) = k)}{h} \Pr(X(t+h) = k).\end{aligned}$$

Thus,

$$q_{jk}\pi_j = q^R_{kj}\pi_k \qquad \text{and} \qquad q^R_{kj} = \frac{q_{jk}\pi_j}{\pi_k}.$$

To verify that the π_j satisfy the equilibrium equations of the reversed-time Markov chain, we substitute the rates we have just established into the global balance equations and see if they are satisfied. That is,

$$\begin{aligned}\sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \pi_j q^R_{jk} &= \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \pi_k q^R_{kj} \\ \Rightarrow \quad \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \pi_j \frac{q_{kj}\pi_k}{\pi_j} &= \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \pi_k \frac{q_{jk}\pi_j}{\pi_k} \\ \Rightarrow \quad \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \pi_k q_{kj} &= \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \pi_j q_{jk}.\end{aligned}$$

□

Corollary 2. $q_{jj} = q^R_{jj}$.

Proof.

$$\begin{aligned}-q^R_{jj} &= \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} q^R_{jk} = \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} \frac{q_{kj}\pi_k}{\pi_j} \\ &= \frac{1}{\pi_j}\sum_{\substack{k\neq j\\ k\in\mathcal{S}}} q_{kj}\pi_k = \frac{1}{\pi_j}\sum_{\substack{k\neq j\\ k\in\mathcal{S}}} q_{jk}\pi_j \qquad \text{by equilibrium equations} \\ &= \sum_{\substack{k\neq j\\ k\in\mathcal{S}}} q_{jk} = -q_{jj}.\end{aligned}$$

□

Example 3. M/M/1 Queue

Recall that $\mathcal{S} = \{0, 1, 2, \dots\}$, and

$$q_{j,j+1} = \lambda \quad \text{and} \quad q_{j+1,j} = \mu \quad \text{for all } j \in \mathcal{S},$$

with equilibrium probabilities

$$\pi_j = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j \quad \text{for } \lambda < \mu.$$

Hence,

$$q^R_{j+1,j} = \frac{\pi_j q_{j,j+1}}{\pi_{j+1}} = \frac{\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j \lambda}{\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j+1}} = \mu$$

$$\text{and similarly} \qquad q^R_{j,j+1} = \frac{\pi_{j+1} q_{j+1,j}}{\pi_j} = \frac{\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j+1} \mu}{\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j}} = \lambda.$$

Definition 15. *A continuous-time Markov chain is reversible if the reversed-time process has the same transition rates as the forward-time process, that is,*

$$q^R_{jk} = q_{jk} \quad \textit{for all } j \textit{ and } k \in \mathcal{S}.$$

Note: There is a clear distinction here between a reversed-time process and a reversible process: All CTMCs may be looked at in reversed time, but *not all* CTMCs are *reversible*.

We will look at the special properties of reversible processes in the next lecture.