APP MTH 3002 Fluid Mechanics III
Assignment 1
Due: 12 noon, Friday 16 March.

When presenting your solutions to the assignment, please include some explanation in words to accompany your calculations. It is not necessary to write a lengthy description, just a few sentences to link the steps in your calculation. Messy, illegible or inadequately explained solutions may be penalised. The marks awarded for each part are indicated in brackets.

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1. Suppose that the position of a fluid particle is

$$\boldsymbol{X}(\boldsymbol{x}_0, t) = R_0 \cos(\omega t + \phi_0)\, \boldsymbol{i} + R_0 \sin(\omega t + \phi_0)\, \boldsymbol{j},$$

where $\boldsymbol{x}_0 = \boldsymbol{X}(\boldsymbol{x}_0, 0)$ is the initial position and t is the time.

   (a) Write down $\boldsymbol{x}_0$ in terms of R_0 and ϕ_0. What do R_0 and ϕ_0 represent? [1 mark]

   (b) Find expressions for the velocity and acceleration using Lagrangian variables, that is, $\boldsymbol{U}(\boldsymbol{x}_0, t)$ and $\boldsymbol{A}(\boldsymbol{x}_0, t)$. [2 marks]

   (c) Find expressions for the velocity and acceleration using Eulerian variables, that is, $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{a}(\boldsymbol{x}, t)$. [2 marks]

   (d) Is this flow steady? Justify your answer. [1 mark]

2. Consider the Eulerian velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = (x + \sin t)\, \boldsymbol{i} - y\, \boldsymbol{j},$$

where $\boldsymbol{u} = (u, v)$, $\boldsymbol{x} = (x, y)$, and t is the time.

   (a) Find a parametric expression for the pathline of the particle that passes through (X_0, Y_0) at $t = 0$. [4 marks]

   (b) Find a parametric expression for the streakline emanating from the point $\boldsymbol{x}^* = (x^*, y^*)$. [3 marks]

   (c) Find a non-parametric expression for the streamlines. [3 marks]

   (d) On a single figure, plot the following:

      i. The pathline of the particle that passes through $(-1/2, 2)$ at $t = 0$ for $0 \leq t \leq \pi$. [1 mark]

      ii. The streakline emanating from $(-1/2, 2)$ at $t = \pi$. [1 mark]

      iii. The streamline passing through the the point $(-1/2, 2)$ at $t = \pi$. [1 mark]

(e) Sketch or plot the streamlines at $t = 0$ and $t = \pi/2$. What happens to the streamline pattern as time passes? [2 marks]

3. Consider the set of particles initially on the unit sphere

$$\boldsymbol{x}_0(\theta, \phi) = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi,$$

at $t = 0$. Suppose that these particles move according to the Eulerian velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \alpha x\,\boldsymbol{i} + \beta y\,\boldsymbol{j} + \gamma z\,\boldsymbol{k} \tag{1}$$

where $\boldsymbol{x} = (x, y, z)$ and α, β and γ are constants.

(a) Show that this set of particles is deformed into an ellipsoid as time passes. [4 marks]

You are given that an ellipsoid satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b and c are the lengths of the semi-axes.

(b) As we shall see, *incompressible* flows satisfy the constraint

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

For the flow given by (1), this means that

$$\alpha + \beta + \gamma = 0. \tag{2}$$

What happens to the volume enclosed by the ellipsoid when constraint (2) is satisfied? Justify your answer. [2 marks]

You are given that the volume of an ellipsoid is $V = 4\pi abc/3$.

(c) In order to satisfy the constraint (2), one of the parameters (let's choose α) must be positive, another (let's choose γ) must be negative, while the remaining one ($\beta = -\alpha - \gamma$) can be of either sign, depending on the magnitudes of the other two.

Describe how the sphere deforms when:

i. $\alpha = 0.75$, $\beta = 0.25$ and $\gamma = -1$,

ii. $\alpha = 1.25$, $\beta = -0.25$ and $\gamma = -1$.

In each case, explain what happens as $t \to \infty$? Illustrate your answer with sketches or plots. [3 marks]

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