Modelling with ODEs Assignment 3

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May 3, 2019

1. The 2D system

$$\dot{x} = 2x - 2y$$
$$\dot{y} = 2x - 3y$$

(a) In matrix-vector form, the system is

$$\dot{\mathbf{x}} = A\mathbf{x}$$

I.e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(b) First calculate the eigenvalues:

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= (2-\lambda)(-3-\lambda) - (-2)(2) \\ &= -6 + \lambda + \lambda^2 + 4 \\ &= -2 + \lambda + \lambda^2 \\ &= (\lambda - 1)(\lambda + 2) \\ \lambda &= 1, -2 \end{aligned}$$

Eigenvectors: $\lambda = 1$

$$(A - \lambda I)\mathbf{v} = 0 \implies \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \mathbf{v} = 0 \implies \begin{cases} v_1 - 2v_2 = 0 \\ 4v_1 - 2v_2 = 0 \end{cases} \implies \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$(A - \lambda I)\mathbf{v} = 0 \implies \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{v} = 0 \implies \begin{cases} 2v_1 - v_2 = 0 \\ 4v_1 - 2v_2 = 0 \end{cases} \implies \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(c) Since this is a linear, homogeneous, coupled system of ODEs, the solution for eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ has form:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

For initial condition $x(0) = x_0$ and $y(0) = y_0$, the coefficients are the solution to

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(2x_0 - y_0) \\ \frac{1}{3}(2y_0 - x_0) \end{pmatrix}$$

(d) The steady state for this problem is $(x, y) = \mathbf{0}$.
From theorem 3.1, for the system $\dot{x} = Ax$, the equilibrium $(0, 0)$ is asymptotically stable only if all eigenvalues of A have negative real part. In this case, $\lambda = 1, -2$, since $\lambda = 1$ is non-negative, the steady state is unstable. There is no other steady state.

(e) The steady and unsteady directions (stable and unstable directions) are obtained from the eigenvectors.
The stable direction corresponds to the negative eigenvalue. I.e. $\mathbf{v}$ for $\lambda = -2$: So the stable direction is:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

And the unstable direction for the positive eigenvalue, $\lambda = 1$:

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(f) The phase portrait is shown in figure 1f. It shows the vector field, stable and unstable directions ($\mathbf{v}$) and several solution curves. The solution curve on the stable direction goes to the steady state, and the solution curve on the unstable direction shoots away from the steady state.

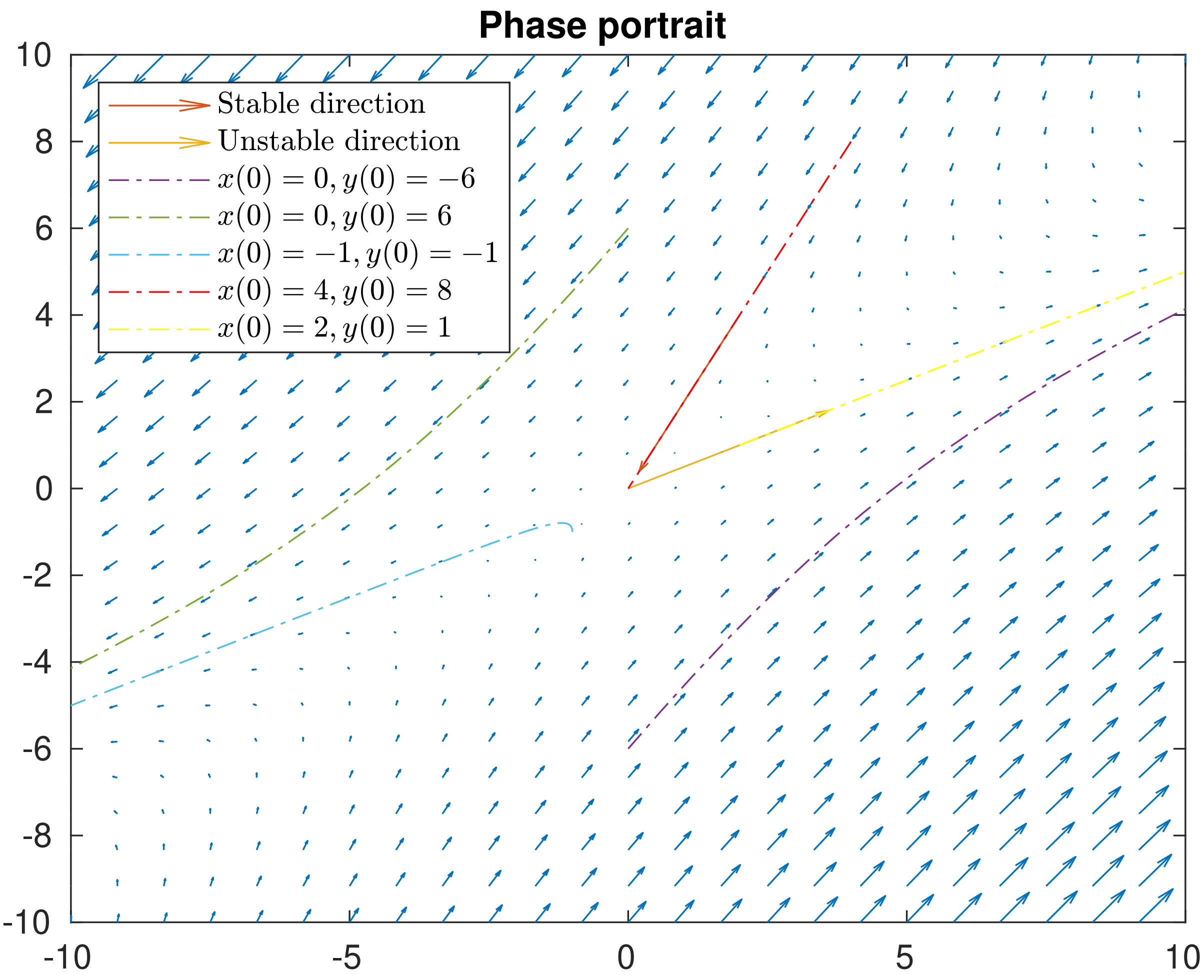

Figure 1: Phase Portrait for the 2D system

2. The 2 population interaction model

$$\frac{\partial}{\partial t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta xy \\ \gamma y + \delta xy \end{pmatrix}$$

With

$$\begin{cases} \alpha, \gamma > 0 \\ \beta, \delta < 0 \end{cases}$$

(a) From lectures the nullclines to this problem are

$$\eta_x = \begin{cases} x = 0, \\ y = -\alpha/\beta \end{cases}$$

and

$$\eta_y = \begin{cases} x = -\gamma/\delta, \\ y = 0 \end{cases}$$

Hence the steady states are

$$x = y = 0, \quad \text{and,} \quad x = -\gamma/\delta,\, y = -\alpha/\beta$$

(b) The steady states will behave like the linearised problem if they are *hyperbolic equilibria*, i.e. if all the eigenvalues of the jacobian have non-zero real part. Jacobian:

$$J(x) = \begin{pmatrix} \alpha + \beta y & \beta x \\ \delta y & \gamma + \delta x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$$

With eigenvalues $\lambda = \alpha, \gamma$ And we have assumed $\alpha, \gamma > 0$ so this will be equivalent to the linearised problem, and it will be unsteady.

$$J(-\gamma/\delta, -\alpha/\beta) = \begin{pmatrix} \alpha - \alpha & -\gamma\beta/\delta \\ -\alpha\delta/\beta & \gamma - \gamma \end{pmatrix} = \begin{pmatrix} 0 & -\gamma\beta/\delta \\ -\alpha\delta/\beta & 0 \end{pmatrix}$$

For the real part of both $\lambda's \neq 0$, i.e. neither of

- $tr\, J(x^*, y^*) = 0$ with $\det J(x^*, y^*) > 0$, or
- $\det J(x^*, y^*) = 0$

Since $tr\, J(x^*, y^*) = 0$ require $\det J(x^*, y^*) < 0$ for nonzero real parts

$$\begin{aligned} \det J(x^*, y^*) &= -(-\gamma\beta/\delta)(-\alpha\delta/\beta) \\ &= -\gamma\alpha \end{aligned}$$

Since $\gamma, \alpha > 1$ this means $\det J(x^*, y^*) < 0$ and hence the linearised model will work, and the steady state will be a saddle.

(c) The biologically relevant region is $x, y \geq 0$. $x = y = 0$ is clearly in this region. $(-\gamma/\delta, -\alpha/\beta)$ is also in the region since $\alpha, \gamma > 0, \beta, \delta < 0$, and hence the state is for positive x and y.

(d) Figure 2d shows the phase portrait for this non-linear system in the biologically relevant region. This region being $(x, y) > \mathbf{0}$.

(e) This particular competitive model with $\alpha = \gamma = -\beta = -\delta$ is an even competition. When either population exceeds the other, they will continue to increase in number, and the other population will stagnate in number. If the populations remain equal (and non-zero, without perturbation), they will both increase up to the steady state $(-\gamma/\delta, -\alpha/\beta)$, which in this case is $(1, 1)$. The phase portrait indicates that none of the steady states are stable.

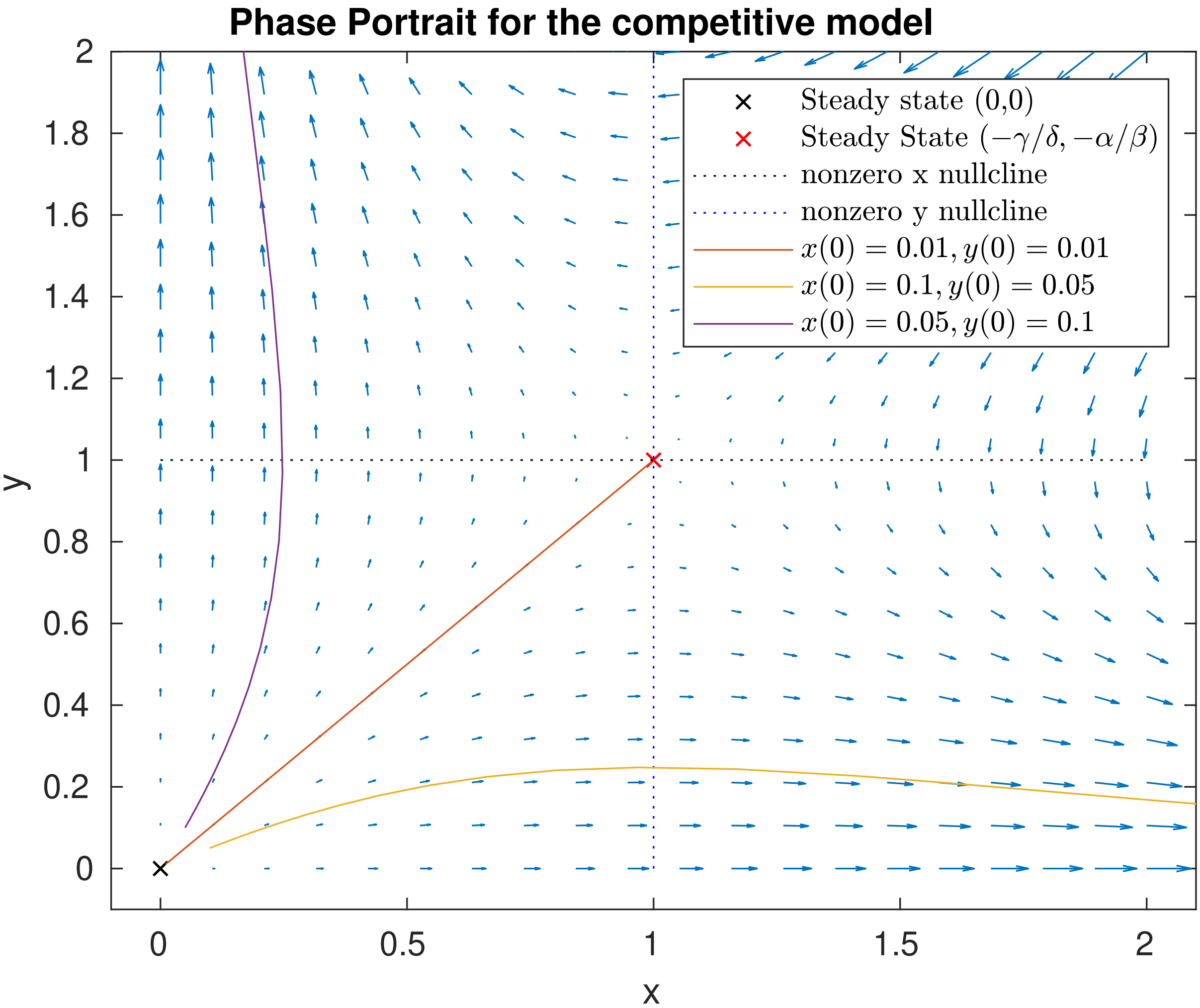

Figure 2: Phase Portrait for the Competitive model

Matlab code:

```
clear all
close all

%%
%%Q1f
A = [2,-2;2,-3];
[v,lambda] = eigs(A);
%since the eigenvector doesn't care about scaling,
%make it look nicer
v=2*v./v(1,1);
npts = 25;
[x,y] = meshgrid(linspace(-10,10,npts),linspace(-10,10,npts));
f = 2*x - 2*y;
g = 2*x - 3*y;
quiver(x,y,f,g,'HandleVisibility','off')
axis([-10,10,-10,10])
hold on
%steady and unsteady directions
quiver(v(1,1),v(2,1),-v(1,1),-v(2,1))
quiver(0,0,v(1,2),v(2,2))

%solve some example curves
t = linspace(0,5);

x = @(t,coeff) 2*coeff(1)* exp(t) + coeff(2)*exp(-2*t);
y = @(t,coeff) coeff(1)*exp(t) + coeff(2)*exp(-2*t)*2;

coeff = c1c2(0,-6);
plot(x(t,coeff),y(t,coeff),'-.')
coeff = c1c2(0,6);
plot(x(t,coeff),y(t,coeff),'-.')
coeff = c1c2(-1,-1);
plot(x(t,coeff),y(t,coeff),'-.')
coeff = c1c2(4,8);
plot(x(t,coeff),y(t,coeff),'r-.')
coeff = c1c2(2,1);
plot(x(t,coeff),y(t,coeff),'y-.')

legendlabels = ["Stable direction", ...
    "Unstable direction",...
    '$$x(0) = 0, y(0) = -6$$',...
    '$$x(0) = 0, y(0) = 6$$',...
    '$$x(0) = -1, y(0) = -1$$',...
    '$$x(0) = 4, y(0) = 8$$',...
    '$$x(0) = 2, y(0) = 1$$',...
    ];
legend(legendlabels,'location','northwest','interpreter','latex')
title("Phase portrait")
saveas(gcf,"ODEsA3Q1f.eps","epsc")
```

```
hold off

%%
%%Q2d
global alpha beta delta gamma
alpha =1;
beta  =-1;
delta =-1;
gamma =1;
npts=20;
[x,y] = meshgrid(linspace(0,2,npts),linspace (0,2, npts));
f = alpha*x + beta*x.*y;
g = gamma*y + delta*x.*y;

figure
quiver(x,y,f,g,'HandleVisibility','off');
hold on
plot (0,0, 'xk')
plot(-gamma/delta, -alpha/beta,'xr')
xlabel("x")
ylabel("y")

plot ([0,2],[- alpha/beta,-alpha/beta],':k')
plot([-gamma/delta,-gamma/delta],[0,2],':b')

%plot a couple of solution curves

[t,X] = ode45(@ode ,[0,10],[0.01,0.01]);
plot(X(:,1),X(:,2))
[t,X] = ode45(@ode ,[0,10],[0.1,0.05]);
plot(X(:,1),X(:,2))
[t,X] = ode45(@ode ,[0,10],[0.05,0.1]);
plot(X(:,1),X(:,2))
legendlabels = ["Steady state (0,0)",  ...
    "Steady State ($-\gamma/\delta,-\alpha/\beta$)",...
    "nonzero x nullcline ",...
    "nonzero y nullcline ",...
    '$$x(0) = 0.01, y(0) = 0.01$$',...
    '$$x(0) = 0.1, y(0) = 0.05$$',...
    '$$x(0) = 0.05, y(0) = 0.1$$'];
legend(legendlabels,'location','northeast','interpreter','latex')
 title ("Phase Portrait for the competitive model")
axis ([-.1,2.1,-.1,2])
saveas(gcf,"ODEsA3Q2d.eps","epsc")

%%%%
%%FUNCTIONS
%%%%
%get the coefficients for Q1
function coeff = c1c2(x0,y0)
A = [2,1;1,2];
```

```
b = [x0;y0];
coeff = A\b;
end

%Q2 ODE for numerical solving
function dx =ode(t,x)
global alpha beta delta gamma
dx = [alpha*x(1) + beta*x(1).*x(2) ; gamma*x(2) + delta*x(1).*x(2)];
end
```

School of Mathematical Sciences

Modelling with ODEs

Semester 1, 2019

Assignment 2

Due 5pm Monday, Week 8: Submit via MyUni

You will be marked on the presentation of your answers (including clarity of explanations)!

1. Consider the linear 2D system

$$\dot{x} = 2x - 2y \tag{1a}$$
$$\dot{y} = 2x - 3y. \tag{1b}$$

   (a) Write the system in matrix–vector form.
   (b) Calculate the eigenvalues and eigenvectors of the system.
   (c) Find the solution for initial values $x(0) = x_0$ and $y(0) = y_0$.
   (d) Classify the steady state with reason.
   (e) State the steady and unsteady directions with reason.
   (f) Produce the phase portrait for the system, i.e. typical solutions in the phase plane, and include the stable and unstable directions. You can use MatLab/other technology or sketch by hand.

2. Consider the two population interaction model

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha\,x + \beta\,x\,y \\ \gamma\,y + \delta\,x\,y \end{pmatrix} \quad \text{for} \quad \begin{cases} \alpha, \gamma > 0 \\ \beta, \delta < 0. \end{cases} \tag{2}$$

   (a) Write down the two steady states (no need to calculate). For each of the steady states, determine the nature of the associated linearised problems.*
   (b) For each of the steady states of the nonlinear system (2), state with reason whether the behaviour is equivalent to the linearised problem.
   (c) State, with reason, if each of the states is biologically relevant.
   (d) Produce a phase portrait for system (2) with $\alpha = \gamma = 1$ and $\beta = \delta = -1$, in the biologically relevant region of the phase plane (either using MatLab/other technology or by hand).
   (e) Give a biological interpretation of the phase portrait.

*You can use results from lectures for part (a).