**Def$^n$ 4.9**: Let $f: S \longrightarrow \mathbb{R}$ be a function. Then we say $f$ is uniformly continuous on $S$ if

$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0$ s.th. $\forall\, x, y \in S$ if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

**Ex** $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2x+1$ is uniformly cts on $\mathbb{R}$ (given $\varepsilon > 0$, $\delta = \frac{\varepsilon}{2}$).

$f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ is not uniformly cts on $\mathbb{R}$ (see Tut. 4).

**Ex.** $f: S \longrightarrow \mathbb{R}$ is uniformly cts on $S$, $T \subset S$.
Then $f|_T : T \longrightarrow \mathbb{R}$ is uniformly cts on $T$.

**Note**: $f$ uniformly cts on $S \Rightarrow f$ cts on $S$

$f$ uniformly cts on $S \not\Leftarrow f$ cts on $S$

**Th$^m$ 4.10**: Let $f: S \longrightarrow \mathbb{R}$ be cts on $S$. If $S$ is seq. compact then $f$ is uniformly cts on $S$.

**Pf**: Suppose $f$ is not uniformly cts on $S$. $\therefore \exists\, \varepsilon > 0$ s.th. $\forall\, \delta > 0 \quad \exists\, x, y \in S$ s.th. $|x-y| < \delta$ & $|f(x) - f(y)| \geq \varepsilon$.

Taking $\delta = \frac{1}{n}$, $n = 1, 2, 3, \ldots$ we see that $\exists\, x_n, y_n \in S$ s.th. $|x_n - y_n| < \frac{1}{n}$ & $|f(x_n) - f(y_n)| \geq \varepsilon$.

Since $S$ is seq compact, $\exists$ subseq. $(x_{n_k})$ of $(x_n)$ s.th. $x_{n_k} \longrightarrow x_0$ for some $x_0 \in S$.

Since $S$ is seq. compact, $\exists$ subseq $(y_{n_{k_\ell}})$ of $(y_{n_k})$ s.th. $y_{n_{k_\ell}} \longrightarrow y_0$ for some $y_0 \in S$.

$$0 \leq |x_0 - y_0| \leq |x_0 - x_{n_{k_\ell}}| + |x_{n_{k_\ell}} - y_{n_{k_\ell}}| + |y_{n_{k_\ell}} - y_0|$$

$$< \underbrace{|x_0 - x_{n_{k_\ell}}| + \frac{1}{\ell} + |y_{n_{k_\ell}} - y_0|}_{\longrightarrow 0}$$

$$\therefore \quad |x_0 - y_0| = 0, \text{ i.e. } x_0 = y_0.$$

$f$ cb on $S$ $\Rightarrow$ $f(x_{n_{k_\ell}}) \longrightarrow f(x_0)$

$\qquad\qquad\qquad\qquad f(y_{n_{k_\ell}}) \longrightarrow f(y_0)$

$$0 < \varepsilon \leq |f(x_{n_{k_\ell}}) - f(y_{n_{k_\ell}})| \longrightarrow 0$$

$\therefore \quad \varepsilon \leq 0$ (preserv$^n$ of inequalities)

– contradiction $\quad \therefore f$ is uniformly cb.

Corr: $f: [a,b] \longrightarrow \mathbb{R}$ cb $\Rightarrow$ $f$ uniformly cb.

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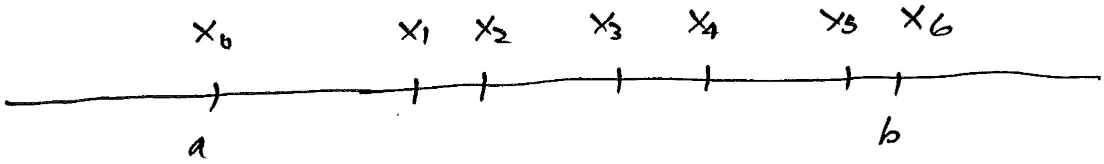
## § 5 Integration.

Suppose $a < b$. A partition of $[a,b]$ is a choice of points $x_0, x_1, \ldots, x_N$ in $[a,b]$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$




The $i^{th}$ sub-interval is $[x_{i-1}, x_i]$. It's length is $\Delta x_i = x_i - x_{i-1}$. We say $P$ is regular if

$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b-a}{N}.$$

Note: $\Delta x_1 + \Delta x_2 + \cdots + \Delta x_N = \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cdots + x_N - \cancel{x_{N-1}}$

$$= x_N - x_0 = b - a.$$

Now suppose $f: [a,b] \longrightarrow \mathbb{R}$ is a bounded function.

Let $\quad M = \sup_{x \in [a,b]} f(x) \quad (= \sup\{f(x) \mid x \in [a,b]\})$.

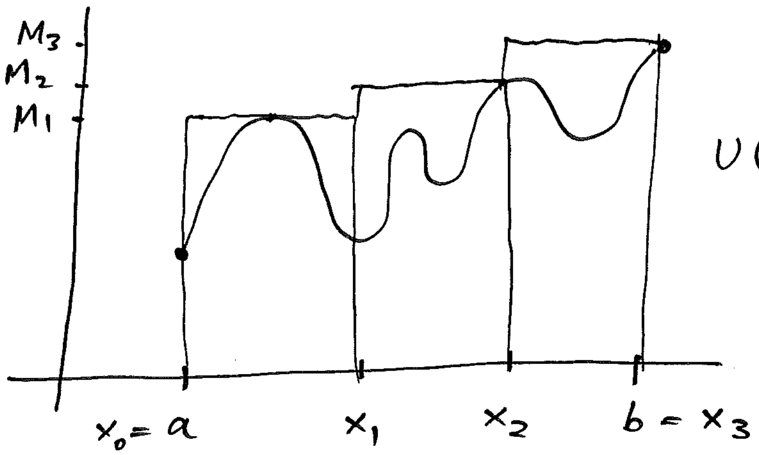
$\qquad m = \inf_{x \in [a,b]} f(x)$.

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$m_i = \cancel{\sup}\inf_{x \in [x_{i-1}, x_i]} f(x).$$

Let $L(f, \mathcal{P}) = \sum_{i=1}^{N} m_i \Delta x_i$ <u>lower sum</u>

$U(f, \mathcal{P}) = \sum_{i=1}^{N} M_i \Delta x_i$. <u>upper sum</u>.




$U(f, \mathcal{P}) \overset{\text{Area}}{=} \left( \square + \square + \square \right)$

<u>Fact</u>: $S \subset T$, $S, T$ bounded

$\sup S \le \sup T$

$\inf S \ge \inf T$

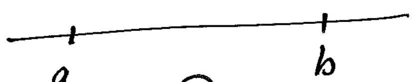
On $[x_{i-1}, x_i]$,

$$m \le m_i \le f(x) \le M_i \le M$$

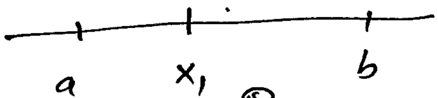
$$\sum_{i=1}^{N} m \Delta x_i \le \sum_{i=1}^{N} m_i \Delta x_i \le \sum_{i=1}^{N} M_i \Delta x_i \le \sum_{i=1}^{N} M \Delta x_i = M(b-a)$$

i.e. $m(b-a) \le L(f, \mathcal{P}) \le U(f, \mathcal{P}) \le M(b-a)$

We say a partition $\mathcal{P}_2$ of $[a,b]$ is a <u>refinement</u> of a partition $\mathcal{P}_1$ of $[a,b]$ if $\mathcal{P}_1 \subset \mathcal{P}_2$.




$\mathcal{P}_1$




$\mathcal{P}_2$

$m = \inf_{x \in [a,b]} f(x)$

$M = \sup_{x \in [a,b]} f(x)$.

$m_1 = \inf_{x \in [a, x_1]} f(x)$, $m_2 = \inf_{x \in [x_1, b]} f(x)$

$M_1 = \sup_{x \in [a, x_1]} f(x)$, $M_2 = \sup_{x \in [x_1, b]} f(x)$

$$L(f, \mathcal{P}_2) = m_1 \Delta x_1 + m_2 \Delta x_2 \geqslant m \Delta x_1 + m \Delta x_2 = L(f, \mathcal{P}_1)$$