STATS 3001 Statistical Modelling III
Tutorial 1
Week 2, Semester 1, 2018

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(1) Let $\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ be a random vector with

$$\mathrm{E}\left[\boldsymbol{Y}\right] = \boldsymbol{\eta} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \text{ and } \mathrm{Var}\left[\boldsymbol{Y}\right] = \Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$

Also define,

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Find $A\boldsymbol{Y} + \boldsymbol{b}$ and call this random vector $\boldsymbol{W}$.

(b) Find $\mathrm{E}\left[W_1\right]$ and $\mathrm{Var}\left[W_1\right]$, that is, the expectation and variance of the first element of $\boldsymbol{W}$.

(c) Find $\mathrm{E}\left[A\boldsymbol{Y} + \boldsymbol{b}\right]$ and $\mathrm{Var}\left[A\boldsymbol{Y} + \boldsymbol{b}\right]$ using Lemma 1.1 from lectures.

(d) Are the results in (b) and (c) consistent?

(e) Using part (c) above, what is $\mathrm{Cov}\left[W_1, W_2\right]$ and thus the correlation between W_1 and W_2.

(2) If X is a matrix of dimension $n \times p$ with linearly independent columns (so that $X^T X$ is invertible) then prove that the matrix $P = X(X^T X)^{-1} X^T$ satisfies

$$P^T = P = P^2$$

and hence show that

$$(I - P)^T = (I - P) = (I - P)^2.$$

(3) Suppose $\boldsymbol{y} \in \mathbb{R}^n$ and X is a $n \times p$ matrix with linearly independent columns.

(a) What are the dimensions of $\boldsymbol{\beta}$ and $\hat{\boldsymbol{\beta}}$?

(b) Give interpretations of what vectors $\boldsymbol{\beta}$ and $\hat{\boldsymbol{\beta}}$ are, and how they differ.

(c) What is contained the vector $\boldsymbol{y} - X\hat{\boldsymbol{\beta}}$?

(d) Prove that

$$\boldsymbol{y} - X\hat{\boldsymbol{\beta}}$$

is orthogonal to

$$X\hat{\boldsymbol{\beta}} - X\boldsymbol{\beta}.$$

(4) Suppose A and B are matrices of dimension $n \times m$ and $m \times n$ respectively. Prove that the trace satisfies

$$\mathrm{tr}\left(AB\right) = \mathrm{tr}\left(BA\right).$$

(5) If A is a constant $n \times n$ matrix and $\mathcal{Y}$ is an $n \times n$ matrix whose elements are random variables, prove that

$$\mathrm{E}\left[\mathrm{tr}\left(A\mathcal{Y}\right)\right] = \mathrm{tr}\left(A\mathrm{E}\left[\mathcal{Y}\right]\right).$$
