Time Series A1

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Suppose $Z_0, Z_1, Z_2, Z_3, \dots$ are independent RVs with $Z_t \sim N(0,1)$. Let $Y_0 = Z_0$ and

$$Y_t = \sqrt{1 - \frac{1}{t^2}} \times Y_{t-1} + \frac{1}{t} \times Z_t$$

For $t = 1, 2, 3, \dots$

1. Show that $\{Y_t\}$ is marginally stationary

   Solution I.e. show that $F(Y_t) = F(Y_{t+k})$ Use induction:

$$Y_0 = Z_0$$

   Clearly $Y_0 \sim N(0,1)$

$$\begin{aligned} Y_1 &= \sqrt{1 - \frac{1}{1^2}} \times Y_0 + \frac{1}{1} \times Z_1 \\ &= 0 \times Y_0 + Z_1 \\ &= Z_1 \end{aligned}$$

   So $Y_1 \sim N(0,1)$ also. Assume $Y_{n-1} \sim N(0,1)$ Show that $Y_n \sim N(0,1)$

$$\begin{aligned} Y_n &= \sqrt{1 - \frac{1}{n^2}} \times Y_{n-1} + \frac{1}{n} \times Z_n \\ &= \sqrt{\frac{n^2}{n^2} - \frac{1}{n^2}} Y_{n-1} + \frac{Z_n}{n} \\ &= \frac{\sqrt{n^2 - 1}}{n} Y_{n-1} + \frac{Z_n}{n} \end{aligned}$$

   Which is the sum of two normal RVs so $Y_n \sim N(\mu, \sigma^2)$ Calculate μ and σ^2:

$$\begin{aligned} E[Y_n] &= E\left[\frac{\sqrt{n^2 - 1}}{n} Y_{n-1} + \frac{Z_n}{n}\right] \\ &= E\left[\frac{\sqrt{n^2 - 1}}{n} Y_{n-1}\right] + E\left[\frac{Z_n}{n}\right] \\ &= \frac{\sqrt{n^2 - 1}}{n} E\left[Y_{n-1}\right] + \frac{1}{n} E[Z_n] \\ &= 0 + 0 \text{ (by inductive assumption)} \\ &= 0 \end{aligned}$$

Calculate σ^2

$$
\begin{aligned}
\text{var}(Y_n) &= \text{var}\left[\frac{\sqrt{n^2-1}}{n}Y_{n-1} + \frac{Z_n}{n}\right] \\
&= \text{var}\left[\frac{\sqrt{n^2-1}}{n}Y_{n-1}\right] + \text{var}\left[\frac{Z_n}{n}\right] \text{ (the previous } (Y_{t-1}) \text{ is indep of the next } (Z_t)) \\
&= \frac{n^2-1}{n^2}\text{var}\left[Y_{n-1}\right] + \frac{1}{n^2}\text{var}\left[Z_n\right] \\
&= \frac{n^2-1}{n^2}\text{var}\left[Y_{n-1}\right] + \frac{1}{n^2}\text{var}\left[Z_n\right] \\
&= \frac{n^2-1}{n^2} + \frac{1}{n^2} \text{ (inductive assumption)} \\
&= \frac{n^2}{n^2} = 1
\end{aligned}
$$

Which means $Y_n \sim N(0,1)$. **As required.**

2. Calculate $\text{cov}(Y_t, Y_{t-1})$

   Solution Let $\mu_t = E(Y_t)$ (just to make it a little prettier):

$$
\begin{aligned}
\text{cov}(Y_t, Y_{t-1}) &= E\left((Y_t - \mu_t)(Y_{t-1} - \mu_{t-1})\right) \\
&= E(Y_t Y_{t-1}) \text{ expand } Y_t \\
&= E\left[\left(\sqrt{1-\frac{1}{t^2}} \times Y_{t-1} + \frac{1}{t} \times Z_t\right) Y_{t-1}\right] \\
&= E\left[\sqrt{1-\frac{1}{t^2}}Y_{t-1}^2 + \frac{1}{t}Z_t^2\right] \\
&= \sqrt{1-\frac{1}{t^2}}E\left[Y_{t-1}^2\right] + \frac{1}{t}E[Z_t Y_{t-1}]
\end{aligned}
$$

   Since $Y \sim N(0,1)$, then $Y^2 \sim \chi_1^2$. This gives:

$$E[Y^2] = 1$$

$$
\begin{aligned}
E[Z_t Y_{t-1}] &= cov(Z_t, Y_t{-}1) + E[Z_t]E[Y_t] \\
&= 0 + 0 \times 0 = 0
\end{aligned}
$$

   The covariance is 0 as Y_{t-1} has no relationship with Z_t

$$\implies \text{cov}(Y_t, Y_{t-1}) = \sqrt{1-\frac{1}{t^2}}$$

   As required.

3. Hence determine whether Y_t is second order stationary

   Solution Second order stationary if $\gamma(s+\tau, t+\tau) = \gamma(s,t)$ I.e. if γ is independent of τ. In this case, we can see that $\text{cov}(Y_t, Y_{t-1})$ depends on t, which corresponds to γ having dependence on τ. So Y_t is not second order stationary **As required.**

4. Simulate a realisation of the sequence $Y_0, Y_1, \ldots, Y_{999}$ and plot it in `R`

   Solution **As required.**

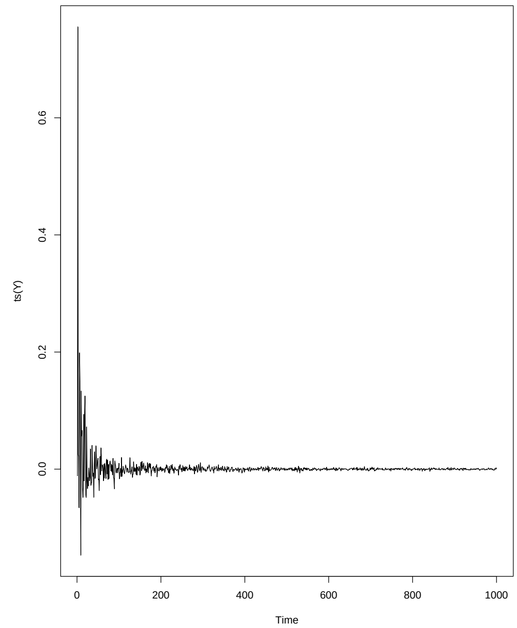

Figure 1: Plot of Y_t against Time

5. Compare the behaviour of the series in Question 4 to a white noise series

   Solution As shown in figure 2, which is a plot of white noise over 1000 points, the series behave differently as time increases. The two series are both 'sharp' as at any point they can jump up or down. However the sequence of Y_t clearly approaches its expectation as time increases, whereas the white noise series continues to jump up and down. **As required.**

6. Explain how marginal stationarity occurs in the process defined here [**Hint:** Repeat the simulation in Question 4 a few times.]

   Solution Figure 3 shows the effect of repeating the simulation and superimposing it on the same graph. Colours are randomised to be different for each separate simulation. Marginal stationarity appears as we see that regardless of time, the mean of the separate simulations sits approximately at 0. This becomes more apparent as time increases, but over multiple simulations it is clear that the average is 0 throughout. **As required.**

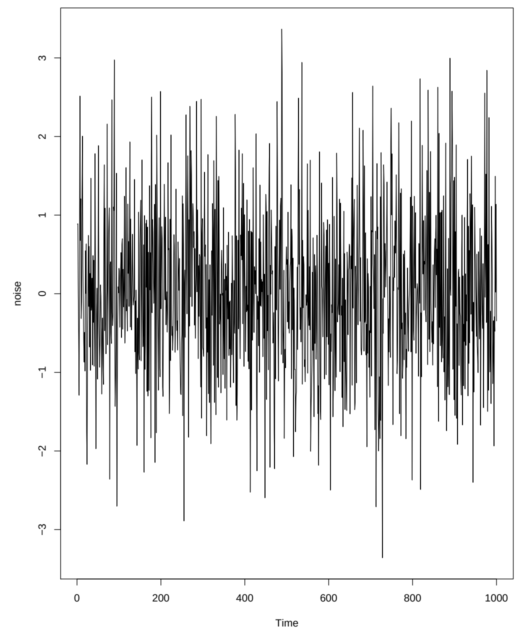

Figure 2: Plot of 1000 white noise points

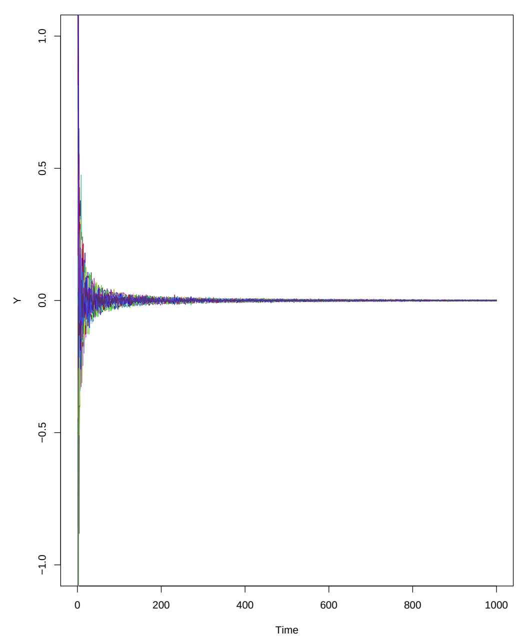

Figure 3: Repeat samples for Y_t

1 Code

```
library(MASS)
setwd("~/Uni/2018/Sem2/Time Series")


#R indexing starts at 1 so we will have to shift everything forward one element
#So t= 2,3,4,...,1000
t = seq(2,1000,1)
Z = rnorm(1000,0,1)
Y = vector(mode ="double", length = 1000)
Y[1] = Z[1]
Y[t] = sqrt(1-1/(t)^2) * Y[t-1] + 1/(t) * Z[t]
plot(Y)

#White noise series
noise=rnorm(1000)

noise=ts(noise)
plot(noise)
lines(Y,col="blue")


#Q 6
#To compare

plot(NULL, xlim=c(0,1000), ylim=c(-1,1), ylab="Y", xlab="t")

for (i in 1:100){
  Z = rnorm(1000,0,1)
Y = vector(mode ="double", length = 1000)
Y[1] = Z[1]
Y[t] = sqrt(1-1/(t)^2) * Y[t-1] + 1/(t) * Z[t]
#plotting each y set with a different colour
lines(Y, col=rgb(runif(1,min=0,max=1),runif(1,min=0,max=1),runif(1,min=0,max=1),1))

}
```