TUTORIAL #2

1/

(a) $$F\{y\} = \int_0^{\pi/2} \left(y^2 + y'^2 - 2y\sin x\right) dx$$

$$y(0) = 0 \qquad y(\pi/2) = 3/2$$

The Euler-Lagrange equations give

$$(2y - 2\sin x) - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow \quad y'' - y = -\sin x$$

Homogeneous sol'n $\quad y_h = A\sinh x + B\cosh x$

Undetermined coeff. $\quad y_p = C_1 \sin x + C_2 \cos x$

$$y_p'' = -C_1 \sin x - C_2 \cos x$$

Sub into ODE $\Rightarrow$ $-C_1 \sin x - C_2 \cos x - C_1 \sin x - C_2 \cos x = -\sin x$.

So $C_2 = 0 \qquad C_1 = \frac{1}{2}$

Thus sol'n is $\quad y = A\sinh x + B\cosh x + \frac{1}{2}\sin x$

$y(0) = 0 \quad \Rightarrow \quad B = 0$

$y(\pi/2) = 3/2 \quad \Rightarrow \quad \frac{3}{2} = A\sinh\frac{\pi}{2} + \frac{1}{2} \quad \Rightarrow \quad A = \frac{1}{\sinh \pi/2}$

So solution to whole problem is

$$y(x) = \frac{\sinh x}{\sinh \pi/2} + \frac{1}{2}\sin x$$

1/ (b)

$$F\{y\} = \int_1^2 \frac{y'^2}{x^3}\,dx, \quad y(1)=0 \quad y(2)=15$$

This functional is y-absent so.

$$\frac{\partial f}{\partial y'} = C$$

$$\frac{2y'}{x^3} = C$$

$$y' = \frac{C}{2}x^3$$

Integrating $\quad y = \frac{C}{8}x^4 + D$

$y(1)=0 \quad \Rightarrow \quad 0 = \frac{C}{8} + D \quad \Rightarrow \quad y = D - Dx^4$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = D(1-x^4)$.

$y(2) = 15 \quad \Rightarrow \quad 15 = D(1-16) \quad \Rightarrow \quad D = -1$

So solution is $\quad y = x^4 - 1$

1/ (c)

$$F\{y\} = \int_0^2 (xy' + y'^2)\, dx, \quad y(0) = 1, \quad y(2) = 0.$$

Again the functional is y-absent.

$$\frac{\partial f}{\partial y'} = c$$

$$x + 2y' = c$$

$$y' = \frac{c - x}{2}$$

Integrating.

$$y = D + \frac{c}{2}x - \frac{x^2}{4}$$

$$y(0) = 1 \Rightarrow 1 = D$$

$$y(2) = 0 \Rightarrow 0 = 1 + C - 1 \Rightarrow C = 0$$

so $\underline{y = 1 - \frac{x^2}{4}}$

2/

Using Fermat's priciple of least time, we want to design a material that gives circular arcs as solutions. Following the notes we will use CoV on a time functional

$$T\{y\} = \int \frac{\sqrt{1+y'^2}}{c}\,dx.$$

But let's first convert to polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$$

So $\quad y' = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

and so $\quad 1 + y'^2 = \dfrac{r'^2 + r^2}{(r'\cos\theta - r\sin\theta)^2}$

So our functional becomes

$$T\{y\} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{c}\,dx$$

$$\Rightarrow \quad T\{r\} = \int_{\theta_0}^{\theta_1} \frac{\sqrt{r'^2+r^2}}{c(r'\cos\theta - r\sin\theta)} \cdot (r'\cos\theta - r\sin\theta)\,d\theta.$$

$$= \int_{\theta_0}^{\theta_1} \frac{\sqrt{r'^2+r^2}}{c}\,d\theta.$$

This is a θ-absent functional so

$$H = r'\frac{\partial f}{\partial r'} - f = \alpha \qquad \text{(a constant)}$$

<u>provided</u> c doesn't explicitly depend on θ.

So let's assume $c = c(r)$ a function of r alone. Then

$$H = \frac{r'^2}{c\sqrt{r'^2+r^2}} - \frac{\sqrt{r'^2+r^2}}{c} = \alpha$$

$$-r^2 = \alpha c \sqrt{r'^2+r^2} \qquad (*)$$

Solving for c we have. $c(r) = -\dfrac{r^2}{\alpha\sqrt{r'^2+r^2}}$

For a circular path we would like $r' = 0$. Thus we would have.

$$c(r) = -\frac{r}{\alpha}$$

So some material with this property then we would have (from $(*)$)

$$-r^2 = \alpha\left(-\frac{r}{\alpha}\right)\sqrt{r'^2+r^2}$$

$$r = \sqrt{r'^2+r^2}$$

$$r^2 = r'^2 + r^2 \quad \Rightarrow r' = 0 \quad r = R.$$

So we derive a path which is a circular path.

Note: to make our material we need to know α since

$$c = -r/\alpha$$

but how to find α?

3/ The cone surface can be thought of as a surface in polar spherical coordinates

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi$$

where $\phi = \text{constant} = \alpha$. So

$$x = r\cos\theta\sin\alpha, \quad y = r\sin\theta\sin\alpha, \quad z = r\cos\alpha$$

and we consider x, y, z as functions of r and θ.

Following the formulae from lectures.

$$P = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \cos^2\theta\sin^2\alpha + \sin^2\theta\sin^2\alpha + \cos^2\alpha = 1$$

$$Q = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}$$

$$= -r\sin\theta\cos\theta\sin^2\alpha + r\sin\theta\cos\theta\sin^2\alpha + 0 = 0$$

$$R = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\sin^2\theta\sin^2\alpha + r^2\cos^2\theta\sin^2\alpha + 0 = r^2\sin^2\alpha$$

So

$$L = \int_{r_0}^{r_1} \sqrt{1 + \theta'^2 r^2 \sin^2\alpha}\, dr$$

or

$$L = \int_{\theta_0}^{\theta_1} \sqrt{r'^2 + r^2\sin^2\alpha}\, d\theta.$$

Considering the second formulation, we note it is autonomous. So

$$H = r' \frac{\partial f}{\partial r'} - f = -c \qquad \text{(say)}$$

$$\frac{r'^2}{\sqrt{r'^2 + r^2\sin^2\alpha}} - \sqrt{r'^2 + r^2\sin^2\alpha} = -c$$

$$\Rightarrow \qquad r^2\sin^2\alpha = c\sqrt{r'^2 + r^2\sin^2\alpha}$$

$$r^4\sin^4\alpha = c^2\left(r'^2 + r^2\sin^2\alpha\right)$$

$$\frac{r^4}{c^2}\sin^4\alpha - r^2\sin^2\alpha = r'^2$$

$$r^2\sin^2\alpha\left(\frac{r^2\sin^2\alpha}{c^2} - 1\right) = r'^2$$

$$\frac{dr}{d\theta} = r\sin\alpha\sqrt{\frac{r^2\sin^2\alpha}{c^2} - 1}$$

$$\int d\theta = \int \frac{dr}{r\sin\alpha\sqrt{\frac{r^2\sin^2\alpha}{c^2} - 1}}$$

$$\theta - \theta_0 = -\frac{1}{\sin\alpha}\tan^{-1}\left(\frac{1}{\sqrt{\frac{r^2\sin^2\alpha}{c^2} - 1}}\right).$$

$$\sin\alpha\left(\theta_0 - \theta\right) = \tan^{-1}\left(\frac{1}{\sqrt{\frac{r^2\sin^2\alpha}{c^2} - 1}}\right)$$

$$\tan\left(\sin\alpha\left(\theta_0 - \theta\right)\right) = \frac{1}{\sqrt{\frac{r^2\sin^2\alpha}{c^2} - 1}}$$

$$\cot\left(\sin\alpha\left(\theta_0 - \theta\right)\right) = \sqrt{\frac{r^2\sin^2\alpha}{c^2} - 1}$$

$$\frac{r^2 \sin^2\alpha}{c^2} = 1 + \cot^2(\sin\alpha(\theta_0 - \theta)).$$

now $1 + \cot^2\phi = \csc^2\phi$.

so taking the square root we have.

$$\frac{r\sin\alpha}{c} = \csc(\sin\alpha(\theta_0 - \theta)).$$

$$r = \frac{c}{\sin\alpha}\csc(\sin\alpha(\theta_0 - \theta)).$$

so if we call $\mu = \frac{c}{\sin\alpha}$ and $\nu = \sin\alpha\,\theta_0$

then $r = \mu\csc(\nu - \sin\alpha\,\theta)$.

So if we have starting and ending points

$r(\theta_0) = r_0$ and $r(\theta_1) = r_1$ then we would use them to determine μ and ν and find the shortest distance path for our cone.

4/

$$F\{y\} = \int_0^R \frac{x}{y'^2}\,dx.$$

y-absent functional so

$$\frac{\partial f}{\partial y'} = \text{const.}$$

$$-\frac{2x}{y'^3} = \text{const}$$

$$y'^3 = -\frac{2}{\text{const}}\,x$$

$$y' = \alpha x^{1/3}$$

$$y = \frac{3\alpha}{4}x^{4/3} + \beta$$

at $x=0,\ y=L \Rightarrow \beta = L.$

at $x=R,\ y=0 \Rightarrow 0 = \frac{3\alpha}{4}R^{4/3} + L$

$$\Rightarrow \alpha = -\frac{4L}{3R^{4/3}}$$

So $y = L\left(1 - \left(\frac{x}{R}\right)^{4/3}\right)$

Lets assume $L=1, R=1$ $\quad y = 1 - x^{4/3}$

$$y'^2 = \frac{16}{9}x^{2/3}$$

So $F = \int_0^1 \frac{x}{1+\frac{16}{9}x^{2/3}}\,dx \simeq 0.2200$

But remember this is not in the $L \gg R$ regime.

5/ (a) if $f(y, y')$ then by the Chain rule

$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$= y' f_y + y'' f_{y'}$$

$$\Rightarrow \quad y' f_y = \frac{df}{dx} - y'' f_{y'} \qquad (1)$$

From the Euler-Lagrange eqn.

$$f_y - \frac{d}{dx}(f_{y'}) = 0.$$

$$y' f_y - y' \frac{d}{dx}(f_{y'}) = 0.$$

substituting from (1).

$$\frac{df}{dx} - y'' f_{y'} - y' \frac{d}{dx}(f_{y'}) = 0$$

but second and third terms are integrable.

$$\frac{df}{dx} - \frac{d}{dx}(y' f_{y'}) = 0.$$

so integrating.

$$f - y' f_{y'} = \text{Const}.$$

as required.

5/ (b) if $f(y, y', y'')$ then by Chain rule.

$$\frac{df}{dx} = y' f_y + y'' f_{y'} + y''' f_{y''}$$

$$\Rightarrow y' f_y = \frac{df}{dx} - y'' f_{y'} - y''' f_{y''} \qquad (2)$$

Euler-Lagrange equations give.

$$f_y - \frac{d}{dx} f_{y'} + \frac{d^2}{dx^2} f_{y''} = 0.$$

$$y' f_y - y' \frac{d}{dx} f_{y'} + y' \frac{d^2}{dx^2} f_{y''} = 0$$

Substituting from (2)

$$\frac{df}{dx} - y'' f_{y'} - y''' f_{y''} - y' \frac{d}{dx} f_{y'} + y' \frac{d^2}{dx^2} f_{y''} = 0.$$

integrable.

$$\frac{df}{dx} - \frac{d}{dx}(y' f_{y'}) - y''' f_{y''} + y' \frac{d^2}{dx^2} f_{y''} = 0.$$

now add $y'' \frac{d}{dx} f_{y''} - y'' \frac{d}{dx} f_{y''} = 0.$

This now makes the last two terms integrable and so.

$$\frac{d}{dx}\left\{ f - y' f_{y'} - y'' f_{y''} + y' \frac{d}{dx} f_{y''} \right\} = 0.$$

So now integrating.

$$f - y' f_{y'} - y'' f_{y''} + y' \frac{d}{dx} f_{y''} = \text{const}.$$