Random Processes Assignment 4

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Question 1. Tennis at deuce. Points take $1/\lambda_D$ on deuce, and $1/\lambda_A$ if at advantage. Probability player A wins is $0 < p_A < 1$. Let $S = \{1, 2, 3, 4, 5\}$ where 1 - deuce, 2 - advantage A, 3 - advantage B, 4 - A wins, 5 - B wins.

(a) Assume a game has reached deuce. Perform a first step analysis to get a system of equations which govern the expected time until the game has finished and either player has won.

Solution Denote the expected hitting time:
Let T be the time to reach state 4 or 5, and let $t_i = E(T|X(0) = i)$

$$\begin{aligned}
t_1 &= 1/\lambda_D + p_A t_2 + (1 - p_A)t_3 \\
t_2 &= 1/\lambda_A + (1 - p_A)t_1 + p_A t_4 \\
t_3 &= 1/\lambda_A + p_A t_1 + (1 - p_A)t_5 \\
t_4 &= 0 \\
t_5 &= 0
\end{aligned}$$

As required.

(b) Calculate the expected duration of a game starting from deuce.

Solution

$$\begin{aligned}
t_1 &= 1/\lambda_D + p_A t_2 + (1 - p_A)t_3 \\
t_1 &= 1/\lambda_D + p_A(1/\lambda_A + (1 - p_A)t_1) + (1 - p_A)(1/\lambda_A + p_A t_1) \\
t_1 &= \frac{1}{\lambda_D} + \frac{p_A}{\lambda_A} + p_A t_1 - p_A^2 t_1 + \frac{1}{\lambda_A} + p_A t_1 - \frac{p_A}{\lambda_A} - p_A^2 t_1 \\
t_1 &= \frac{1}{\lambda_D} + \frac{1}{\lambda_A} + 2p_A t_1 - 2p_A^2 t_1 \\
t_1(1 - 2p_A + 2p_A^2) &= \frac{1}{\lambda_D} + \frac{1}{\lambda_A} \\
t_1 &= \frac{\frac{1}{\lambda_D} + \frac{1}{\lambda_A}}{1 - 2p_A + 2p_A^2}
\end{aligned}$$

As required.

Question 2. The Dining Philosophers problem is often used in computer science as a model for systems which share resources. It works as follows.
N philosophers sit around a table. In between each pair of philosophers is a single chopstick. The philosophers alternately think and eat. When they wish to eat they pick up the chopsticks either side of them if both chopsticks are free, but if one of them is being used, they cannot eat and return immediately to thinking mode. Assume $N = 4$ and that each philosopher eats for a period which is exponentially distributed with parameter 1 and thinks for a period which is exponentially distributed with parameter γ.

(a) What is an appropriate state space for a continuous-time Markov chain model of this system, if I wish to know which philosophers are dining or thinking at any one time?

Solution Let

$$S = \{(w, x, y, z : w, x, y, z = 0, 1 \cap w + x + y + z \leq 2\}$$
$$S = \{(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}$$

Where the values of (w, x, y, z) equal to 1 if the relevant philosopher is eating, and 0 if they are thinking.
As required.

(b) What are the transition rates of this model?

Solution γ for any particular philosopher to try go from thinking to eating, and 1 to go back down. This is due to time homogeneity of the exponential distribution Write Q:

$$Q = \begin{pmatrix} -4\gamma & \gamma & \gamma & \gamma & \gamma & 0 & 0 \\ 1 & -1-\gamma & 0 & 0 & 0 & \gamma & 0 \\ 1 & 0 & -1-\gamma & 0 & 0 & 0 & \gamma \\ 1 & 0 & 0 & -1-\gamma & 0 & \gamma & 0 \\ 1 & 0 & 0 & 0 & -1-\gamma & 0 & \gamma \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}$$

As required.

(c) Write down the equilibrium equations for the model.

Solution

$$\pi Q = 0$$

$$\begin{aligned} -4\gamma\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 &= 0 \\ \gamma\pi_1 + (-1-\gamma)\pi_2 + \pi_6 &= 0 \\ \gamma\pi_1 + (-1-\gamma)\pi_3 + \pi_7 &= 0 \\ \gamma\pi_1 + (-1-\gamma)\pi_4 + \pi_6 &= 0 \\ \gamma\pi_1 + (-1-\gamma)\pi_5 + \pi_7 &= 0 \\ \gamma\pi_2 + \gamma\pi_4 - 2\pi_6 &= 0 \\ \gamma\pi_3 + \gamma\pi_5 - 2\pi_7 &= 0 \end{aligned}$$

Such that $\sum_{i=1}^{7} \pi_i = 1$

As required.

(d) Calculate the equilibrium (use theorem 14)

Solution Theorem 14 states that a stationary CTMC is reversible iff the equilibrium exists and satisfies the detailed balance:

$$\pi_j q_{jk} = \pi_k q_{kj}, \quad \forall j, k \in S$$

Such that

$$\sum_{i \in S} \pi_i = 1$$

Assume the equilibrium exists (and if we find it then we're good):

$$\begin{aligned} \gamma\pi_1 &= \pi_2 \\ \gamma\pi_1 &= \pi_3 \\ \gamma\pi_1 &= \pi_4 \\ \gamma\pi_1 &= \pi_5 \\ \gamma\pi_2 &= \pi_6 \implies \gamma^2\pi_1 = \pi_6 \\ \gamma\pi_3 &= \pi_7 \implies \gamma^2\pi_1 = \pi_7 \end{aligned}$$

Using the sum equation:

$$\sum_{i=1}^{7} \pi_i = 1 = \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7$$
$$1 = \pi_1(1 + \gamma + \gamma + \gamma + \gamma + \gamma^2 + \gamma^2)$$
$$1 = \pi_1(1 + 4\gamma + 2\gamma^2)$$
$$\pi_1 = \frac{1}{1 + 4\gamma + 2\gamma^2}$$
$$\therefore \pi_2 = \pi_3 = \pi_4 = \pi_5 = \frac{\gamma}{1 + 4\gamma + 2\gamma^2}$$
$$\therefore \pi_6 = \pi_7 = \frac{\gamma^2}{1 + 4\gamma + 2\gamma^2}$$

As required.

(e) What is the equilibrium probability that the number of eating philosophers is zero, one, two, three or four?

Solution Note that I have started with π_1 :=zero philosophers eating For zero:

$$\pi_1 = \frac{1}{1 + 4\gamma + 2\gamma^2}$$

One:

$$\pi_2 + \pi_3 + \pi_4 + \pi_5 = \frac{4\gamma}{1 + 4\gamma + 2\gamma^2}$$

Two:

$$\pi_6 + \pi_7 = \frac{2\gamma^2}{1 + 4\gamma + 2\gamma^2}$$

And for 3 and 4 it is zero, and no more than 2 can eat. **As required.**

Question 3. Consider customers arrive to the office of the SA Department of Planning, Transport and Infrastructure (DPTI) according to a Poisson process with rate λ. The office can only contain N customers (which includes the customers currently being served). Any customer who arrives to a full office immediately leaves. Each customer has one enquiry that will be classified into only one of five categories. The probability that a customer has an enquiry related to category i is p_i, such that $\sum_{i=1}^{5} p_i = 1$. There is a distinct queue associated with each enquiry and customers immediately join the appropriate queue on arrival.

(a) Assuming that N is infinite, we begin by proving that the arrivals to each queue are independent Poisson processes via the following steps.

i. What is the probability that in the time interval $[0, t]$, there were n arrivals to the office and x of those arrivals joined queue i?

Solution (Using the definition for $N_i(t)$ below)

The arrivals are Poisson process with rate λ. Probability of n arrivals in $[0, t]$ is

$$P(n\ arrivals) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(x\ join\ i \mid n\ arrivals) = \binom{n}{x} p_i^x (1 - p_i)^{n-x}$$

As required.

ii. Using the law of total probability, calculate the probability that in the time interval $[0, t]$, x arrivals joined queue i, $i \in \{1, 2, 3, 4, 5\}$. In other words, if $N_i(t)$ is the number of customers who arrive for queue i by time t, calculate the PMF, $p_i = P(N_i(t) = x)$ for all $x \in \{0, 1, ...\}$ and hence show that the arrivals to the ith queue follow a Poisson process with rate $p_i\lambda$.

Solution

$$
\begin{aligned}
p_i &= P(N_i(t) = x) \\
&= \sum_{n=0}^{\infty} P(N_i(t) = x | n \ arrivals) P(n \ arrivals) \quad (LOTP) \\
&= \sum_{n=0}^{\infty} \binom{n}{x} p_i^x (1-p_i)^{n-x} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{n!}{(n-x)!x!} p_i^x (1-p_i)^{n-x} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{p_i^x (1-p_i)^{n-x} (\lambda t)^n e^{-\lambda t}}{(n-x)!x!} \\
&= \frac{p_i^x e^{-\lambda t}}{x!} \sum_{n=0}^{\infty} \frac{(1-p_i)^{n-x} (\lambda t)^n}{(n-x)!} \\
&= \frac{p_i^x e^{-\lambda t}}{x!} \sum_{n=0}^{\infty} \frac{(1-p_i)^{n-x} (\lambda t)^{n-x} (\lambda t)^x}{(n-x)!} \\
&= \frac{p_i^x (\lambda t)^x e^{-\lambda t}}{x!} \sum_{n=0}^{\infty} \frac{((1-p_i)\lambda t)^{n-x}}{(n-x)!} \\
&= \frac{p_i^x (\lambda t)^x e^{-\lambda t} e^{(1-p_i)\lambda t}}{x!} \\
&= \frac{(p_i \lambda t)^x e^{-p_i \lambda t}}{x!}
\end{aligned}
$$

Which is the PDF of the Poisson distribution with rate $p_i\lambda$.
As required.

(b) Continue to assume that N is infinite, and also assume that queue i has i servers available, each with an exponential service rate of μ.

i. Specify an appropriate CTMC for modelling the number of customers in all of the queues.
Solution Inputs: $p_1\lambda$, $p_2\lambda$, $p_3\lambda \dots$
Outputs: μ, 2μ, $3\mu \dots$
Shared infinite buffer - 5 arrival rates and 5 service rates **As required.**

ii. Determine the stationary distribution for this CTMC.
Solution Assuming $p_i\lambda < i\mu$
Note that all of the queues are independent reversible birth-death processes (if R is infinite). Which gives the joint equilibrium distribution:

$$\pi(\mathbf{n}) = \prod_{i=1}^{5} \left(1 - \frac{p_i\lambda}{i\mu}\right) \left(\frac{p_i\lambda}{i\mu}\right)^{n_i}$$

As required.

(c) Now assuming that N is finite, determine the equilibrium distribution of the queues.
Solution Using theorem 17 - since this is a reversible process, the truncated process (finite) must be reversible with the same invariant measure

$$\pi(\mathbf{n}) = C \prod_{i=1}^{5} \left(\frac{p_i\lambda}{i\mu}\right)^{n_i}$$

Where

$$C = \left(\sum_{\mathcal{A}} \prod_{i=1}^{5} \left(\frac{p_i\lambda}{i\mu}\right)^{n_i}\right)^{-1}$$

As required.