Random Processes III 2018: Assignment 2,

to be handed in by 1pm on Friday 24^{th} August.

[39 marks in total]

Question 0. [4 marks]

Make sure that in all your answers you:
(a) use full and complete sentences,
(b) include units where necessary,
(c) give ranges/constraints for parameters in formulae, and
(d) make sure your layout and mathematical arguments are clear and concise.

Question 1. [6 marks]

Let $(Y(t), t \geq 0)$ be a birth–death process on the state space $\mathcal{S} = \{0, 1, 2, \dots\}$ with the following nonzero transition rates:

$$
\begin{aligned}
q_{n,n+1} &= \lambda_n, \\
q_{n,n-1} &= \mu_n, \quad n \geq 1.
\end{aligned}
$$

(a) Write down the KFDEs for this CTMC for initial state $i \in \mathcal{S}$.

(b) With reasoning, what are the initial conditions for the differential equations in part (a)?

Question 2. [12 marks]

Let $(X(t), t \geq 0)$ be a CTMC on $\{1, 2\}$ with generator

$$
Q = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix},
$$

where $\lambda\mu > 0$.
(a) Write down the KFDEs.

(b) By noting that

$$P_{ij}(t) = 1 - P_{ii}(t) \text{ for } i \neq j,$$

(check this!) solve these forward equations for the transition probabilities $P_{ij}(t)$, for $i, j = 1,\ 2$, determining and using the initial conditions. Make sure you show all working.

Question 3. [17 marks]

Consider a single server queue where arrival instants occur according to a Poisson process with rate λ and the service time of each individual customer is exponentially distributed with rate μ. The interesting feature of this queue is that exactly two customers arrive at each instant.

(a) Define a suitable state space $\mathcal{S}$ for this continuous time Markov chain.

(b) Write down the transition rates and consequently the generator Q.

(c) Under what conditions will the equilibrium distribution exist for this system? In other words, under what conditions is this system stable?

(d) Under the stability conditions of part (c), write down the equilibrium equations for this system. *Do not attempt to solve this system of equations.*

(e) Use the probability generating function method to show for this system that

$$P(z) = \frac{\mu \pi_0}{\mu - \lambda z(z+1)} .$$

(f) Find π_0 in terms of λ and μ. *Do not attempt to find the full equilibrium distribution.*