Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 8

Last lecture

- Looked at the special case of no explicit y dependence
- Considered the general case of geodesics of the unit sphere
- Refreshed material on coordinate transforms and Jacobian determinants
- Summarised the general case for geodesics

Invariance of the E-L equations

We side-track here to note that extremals found using the E-L equations don't depend on the coordinate system!

This can be very useful – a change of co-ordinates can often simplify a problem dramatically.

Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} \;=\; 0}$$

Invariance of the E-L equations

The extremals found using the E-L equations don't depend on the coordinate system!

For instance take co-ordinate transform

$$x = x(u, v)$$
$$y = y(u, v)$$

- **smooth:** if functions x and y have continuous partial derivatives.
- **non-singular:** if Jacobian is non-zero

For example, the path of a particle does not depend on the coordinate system used to describe the path!

Notation

Use the notation

$$x_u = \frac{\partial x}{\partial u}$$

For example, the Jacobian for transform $x = x(u, v)$ and $y = y(u, v)$ can be written

$$J = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

Note that if $J \neq 0$ the transform is invertible.

- treat u like the independent variable (like x)
- treat v like the dependent variable (like y)

Transforming dy/dx

Treat v like a function $v(u)$. The chain rule says for $x = x(u, v)$

$$\frac{dx}{du} = \frac{du}{du}\frac{\partial x}{\partial u} + \frac{dv}{du}\frac{\partial x}{\partial v}$$

so

$$\begin{aligned} \frac{dx}{du} &= x_u + x_v v' \\ \frac{dy}{du} &= y_u + y_v v' \end{aligned}$$

where $v' = dv/du$. So

$$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{y_u + y_v v'}{x_u + x_v v'}$$

Transforming functional

Transforming the functional, we get

$$
\begin{aligned}
F\{y\} &= \int_{x_0}^{x_1} f(x, y, y')\, dx \\
&= \int_{u_0}^{u_1} f\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)(x_u + x_v v')\, du \\
&= \int_{u_0}^{u_1} \tilde{f}(u, v, v')\, du
\end{aligned}
$$

Relabel the functional to get

$$
\tilde{F}\{v\} = \int_{u_0}^{u_1} \tilde{f}(u, v, v')\, du
$$

Fixed end-point problem

Find extremals of functional $F : C^2[x_0, x_1] \to \mathbb{R}$ given by

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

and the extremal is in the set S

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0 \text{ and } y(x_1) = y_1\}.$$

Becomes, find extremals of $\tilde{F} : C^2[u_0, u_1] \to \mathbb{R}$ given by

$$\tilde{F}\{v\} = \int_{u_0}^{u_1} \tilde{f}(u, v, v')\, du$$

and the extremal is in the set S

$$\tilde{S} = \{v \in C^2[u_0, u_1] \mid v(u_0) = v_0 \text{ and } v(u_1) = v_1\},$$

Relation between extremals

Theorem: Let $y \in S$ and $v \in \tilde{S}$ be two functions that satisfy the smooth, non-singular transformation $x = x(u, v)$, and $y = y(u, v)$, then y is an extremal for F if and only if v is an extremal for $\tilde{F}$.

Proof Sketch: The proof needs to show that the Euler-Lagrange equations for both problems produce the same extremals.
We can do so, by noting that

$$\frac{d}{du}\left(\frac{\partial \tilde{f}}{\partial v'}\right) - \frac{\partial \tilde{f}}{\partial v} = J\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right]$$

As the transform is non-singular $J \neq 0$, so if either side is zero, the Euler-Lagrange equation is satisfied for both problems.

Some of the details

$$\tilde{f}(u,v,v') = f\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)(x_u + x_v v')$$

$$\frac{\partial \tilde{f}}{\partial v} = \left(\frac{\partial f}{\partial x} x_v + \frac{\partial f}{\partial y} y_v + \frac{\partial f}{\partial y'} \frac{\partial}{\partial v}\left(\frac{y_u + y_v v'}{x_u + x_v v'}\right)\right)(x_u + x_v v') + f \frac{\partial}{\partial v}(x_u + x_v v')$$

$$\frac{\partial \tilde{f}}{\partial v'} = \frac{\partial f}{\partial y'}(x_u + x_v v') \frac{\partial}{\partial v'}\left(\frac{y_u + y_v v'}{x_u + x_v v'}\right) + x_v f$$

$$J = x_u y_v - x_v y_u$$

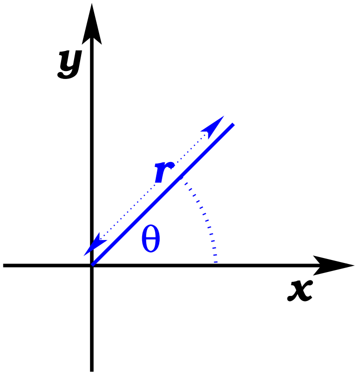
Example

Polar (circular) coordinates have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and inverse transform

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$



Find extremals of $F\{r\} = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + r'^2}\, d\theta$

Example

For the inverse transform

$$
\begin{aligned}
r_x &= x/\sqrt{x^2+y^2} \\
r_y &= y/\sqrt{x^2+y^2} \\
\theta_x &= (-y/x^2)/(1+(y/x)^2) = -y/(x^2+y^2) \\
\theta_y &= (1/x)/(1+(y/x)^2) = x/(x^2+y^2)
\end{aligned}
$$

using $\boxed{\dfrac{d}{dz}\arctan(z) = \dfrac{1}{1+z^2}}$

Example

The Jacobian

$$\begin{aligned} J &= \det \begin{pmatrix} r_x & \theta_x \\ r_y & \theta_y \end{pmatrix} \\ &= \det \begin{pmatrix} x/\sqrt{x^2+y^2} & -y/(x^2+y^2) \\ y/\sqrt{x^2+y^2} & x/(x^2+y^2) \end{pmatrix} \\ &= \frac{x^2+y^2}{(x^2+y^2)^{3/2}} \\ &= 1/\sqrt{x^2+y^2} \end{aligned}$$

$J \neq 0$ everywhere except $(x, y) = (0, 0)$, where it is undefined.

Example

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{r_x + r_y y'}{\theta_x + \theta_y y'} \\
&= \frac{x/\sqrt{x^2+y^2} + yy'/\sqrt{x^2+y^2}}{-y/(x^2+y^2) + xy'/(x^2+y^2)} \\
&= \sqrt{x^2+y^2}\,\frac{x+yy'}{-y+xy'} \\
r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (x^2+y^2) + (x^2+y^2)\left(\frac{x+yy'}{-y+xy'}\right)^2 \\
&= (x^2+y^2)\left[1 + \left(\frac{x+yy'}{-y+xy'}\right)^2\right]
\end{aligned}
$$

Example

$$
\begin{aligned}
r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (x^2+y^2)\left[1 + \left(\frac{x+yy'}{-y+xy'}\right)^2\right] \\
&= (x^2+y^2)\left[1 + \frac{x^2+2xyy'+y^2y'^2}{y^2-2xyy'+x^2y'^2}\right] \\
&= (x^2+y^2)\left[\frac{y^2-2xyy'+x^2y'^2+x^2+2xyy'+y^2y'^2}{y^2-2xyy'+x^2y'^2}\right] \\
&= (x^2+y^2)\left[\frac{x^2+y^2+(x^2+y^2)y'^2}{y^2-2xyy'+x^2y'^2}\right] \\
&= \frac{(x^2+y^2)^2(1+y'^2)}{(-y+xy')^2}
\end{aligned}
$$

Example

Now

$$
\begin{aligned}
\frac{d\theta}{dx} &= \frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y}\frac{dy}{dx} \\
&= -\frac{y}{(x^2+y^2)} + \frac{x}{(x^2+y^2)}y' \\
&= \frac{-y+xy'}{(x^2+y^2)} \\
\frac{dx}{d\theta} &= \frac{(x^2+y^2)}{-y+xy'} \\
r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1+y'^2)\left(\frac{dx}{d\theta}\right)^2
\end{aligned}
$$

Example

Given that

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + y'^2)\left(\frac{dx}{d\theta}\right)^2$$

The functional can be rewritten

$$\begin{aligned} F\{r\} &= \int_{\theta_0}^{\theta_1} \sqrt{r^2 + r'^2}\, d\theta \\ &= \int_{\theta_0}^{\theta_1} \sqrt{1 + y'^2}\, \frac{dx}{d\theta}\, d\theta \\ \tilde{F}\{y\} &= \int_{x_0(r_0,\theta_0)}^{x_1(r_1,\theta_1)} \sqrt{1 + y'^2}\, dx \end{aligned}$$

which is just the functional for finding shortest paths in the plane!

Example

Given that $f(r, r') = \sqrt{r^2 + r'^2}$, does not depend explicitly on θ we can construct the constant function

$$H(r, r') = r' \frac{\partial f}{\partial r'} - f = \frac{r'^2}{\sqrt{r^2 + r'^2}} - \sqrt{r^2 + r'^2} = \text{const}$$

which we can rearrange to get $r' = r\sqrt{c_1^2 r^2 - 1}$, which we can rearrange to get

$$\theta = \int \frac{dr}{c_1 r^2 \sqrt{1 - 1/c_1^2 r^2}}$$

and integrate to get

$$\theta + c_2 = -\sin^{-1}\left(\frac{1}{c_1 r}\right) \quad \text{or} \quad Ar\cos(\theta) + Br\sin(\theta) = C$$

Special case 4

When $f = A(x,y)y' + B(x,y)$ we call this a degenerate case, because the E-L equations reduce to

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = 0$$

but we can't necessarily solve these, and when they are true, the functional's value only depends on the end-points, not the actual shape of the curve.

Degenerate case

Take $f = A(x,y)y' + B(x,y)$, so that the functional (for which we are looking for extrema) is

$$F\{y\} = \int_{x_0}^{x_1} \left(A(x,y)y' + B(x,y) \right) \, dx$$

Then the Euler-Lagrange equation can be written as

$$\begin{aligned}
\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} &= 0 \\
\frac{d}{dx}A(x,y) - \left[y' \frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} \right] &= 0 \\
\frac{\partial A}{\partial x} + y' \frac{\partial A}{\partial y} - \left[y' \frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} \right] &= 0
\end{aligned}$$

Degenerate case

So the extremals for

$$F\{y\} = \int_{x_0}^{x_1} A(x,y)y' + B(x,y)\,dx$$

satisfy

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = 0$$

This is not even a differential equation!

- may or may not have solutions depending on A and B
- no arbitrary constants, so can't impose conditions
- maybe true everywhere?

Degenerate case

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = 0$$

Where there is a solution, there exists a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial y} = A$$
$$\frac{\partial \phi}{\partial x} = B$$

Thus,

$$\frac{\partial A}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial B}{\partial y}$$

Degenerate case

In this case, the integrand $f(x, y)$ can be written

$$f = \frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial x} = \frac{d\phi}{dx}$$

So the functional can be written

$$\begin{aligned} F\{y\} &= \int_{x_0}^{x_1} f(x, y, y')\, dx \\ &= \int_{x_0}^{x_1} \frac{d\phi}{dx}\, dx \\ &= [\phi(x, y)]_{x_0}^{x_1} \\ &= \phi(x_1, y(x_1)) - \phi(x_0, y(x_0)) \end{aligned}$$

So the functional depends only on the end-points!

Example

Let $f(x, y, y') = (x^2 + 3y^2)y' + 2xy$ so the functional is

$$F\{y\} = \int_{x_0}^{x_1} \left[(x^2 + 3y^2)y' + 2xy\right] \, dx$$

Then $A(x, y) = (x^2 + 3y^2)$ and $B(x, y) = 2xy$, so the E-L equation reduces to

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = 2x - 2x = 0$$

which is always true, for any curve y!

this is what we mean by an identity

Hence the Euler-Lagrange equation is always satisfied.

Example

If we choose $\phi(x, y) = x^2 y + y^3 + k$ then

$$\frac{\partial \phi}{\partial y} = x^2 + 3y^2 = A$$
$$\frac{\partial \phi}{\partial x} = 2xy = B$$

So the functional is determined by the end-points, e.g.

$$F\{y\} = x_1^2 y_1 + y_1^3 - x_0^2 y_0 - y_0^3$$

and this does not depend on the curve between the two end points.

Theorem

Suppose that the functional F satisfies the conditions of such that its extremals satisfy the Euler-Lagrange equation, which in this case reduces to an identity. Then the integrand must be linear in y', and the value of the functional is independent of the curve y (except through the end-points).

Basically this says that the degenerate case above only occurs for $f(x, y, y') = A(x, y)y' + B(x, y)$.