Topic C Assignment 1

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1. (a) There are 2 non-zero roots to the equation. This is shown in Figure 1a.

   (b) Use $x = \pm 1 + S$ where $S \ll 1$, and $tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$
   Note that $\tanh(x)$ is an odd function, so the solution around $x = -1$ will be the negative of the solution around $x = 1$.

   Take the taylor series of $\tanh(x/\epsilon)$ about $x = \pm 1$

$$\tanh(x/\epsilon) = \tanh(\pm\frac{1}{\epsilon}) + \overset{2}{\operatorname{sech}}(\frac{1}{\epsilon})$$

   So $x = \tanh(x/\epsilon)$

$$\begin{aligned}
x =& \tanh(x/\epsilon) \\
x =& \frac{2}{1+e^{-2x/\epsilon}} - 1 \\
x =& 2 - 2e^{-2x/\epsilon} + 2e^{-4x/\epsilon} - o(e^{-4/\epsilon}) - 1 \\
x =& 1 - 2e^{-2x/\epsilon} + 2e^{-4x/\epsilon} - o(e^{-4/\epsilon})
\end{aligned}$$

   Since $x = 1 + S + S_2$ Where $S \ll 1$ and $S_2 \ll S$, we get

$$\begin{aligned}
x =& 1 - 2e^{-2x/\epsilon} + 2e^{-4x/\epsilon} - o(e^{-4/\epsilon}) \\
1 + S + S_2 =& 1 - 2e^{-2(1+S+S_2)/\epsilon} + 2e^{-4(1+S+S_2)/\epsilon} - o(e^{-4/\epsilon}) \\
S + S_2 =& -2e^{-2(1+S+S_2)/\epsilon} + 2e^{-4(1+S+S_2)/\epsilon} - o(e^{-4/\epsilon})
\end{aligned}$$

   Since $S \ll 1$ and $S_2 \ll S$ $e^{-(1+S+S_2)/\epsilon} \approx e^{-1/\epsilon}$. Giving

$$\begin{aligned}
&S + S_2 = -2e^{-2(1+S+S_2)/\epsilon} + 2e^{-4(1+S+S_2)/\epsilon} - o(e^{-4/\epsilon}) \\
&S + S_2 = -2e^{-2/\epsilon} + 2e^{-4/\epsilon} - o(e^{-4/\epsilon}) \\
&\implies S = -2e^{2/\epsilon} \\
&\implies S_2 = 2e^{-4/\epsilon}
\end{aligned}$$

   Which then gives:

$$x_+ = 1 - 2e^{-2/\epsilon} + 2e^{-4/\epsilon} - \mathcal{O}(e^{-6/\epsilon})$$

   And

$$x_- = -(1 - 2e^{-2/\epsilon} + 2e^{-4/\epsilon} - \mathcal{O}(e^{-6/\epsilon}))$$

   (c) Figure 2c compares the 3 term asymptotic solution to the numerically obtained solution for $x = \tanh(x/\epsilon)$. As $\epsilon \to 0$ it is clear that the asymptotic solution quickly approaches the numerical solution

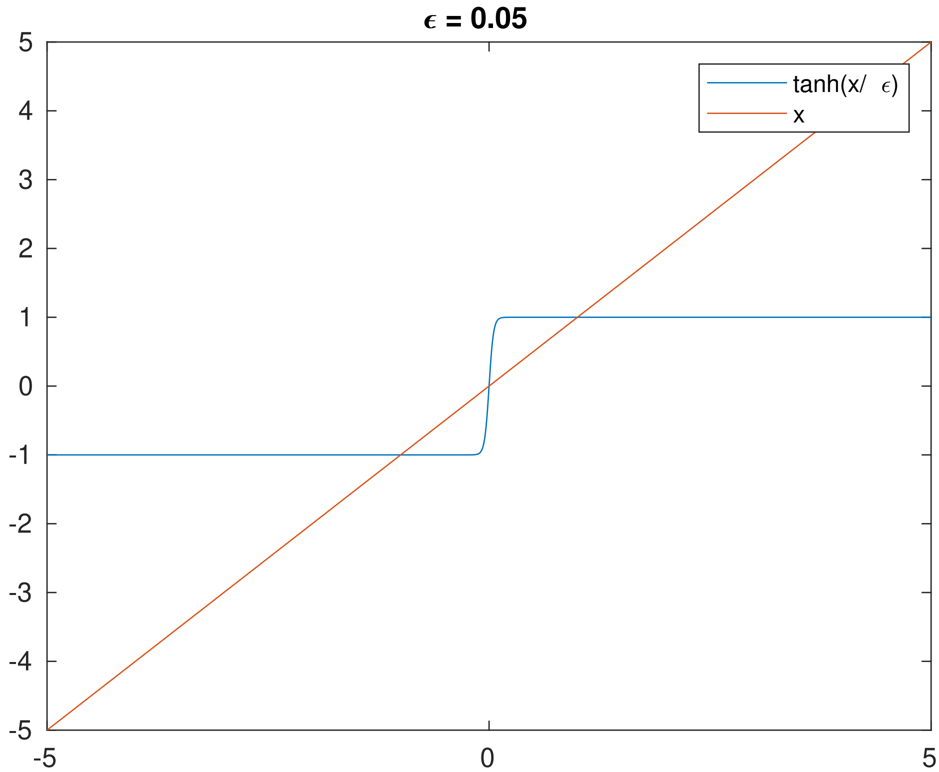


Figure 1: plot of $\tanh(x/\epsilon)$ and x for $\epsilon = 0.0001$

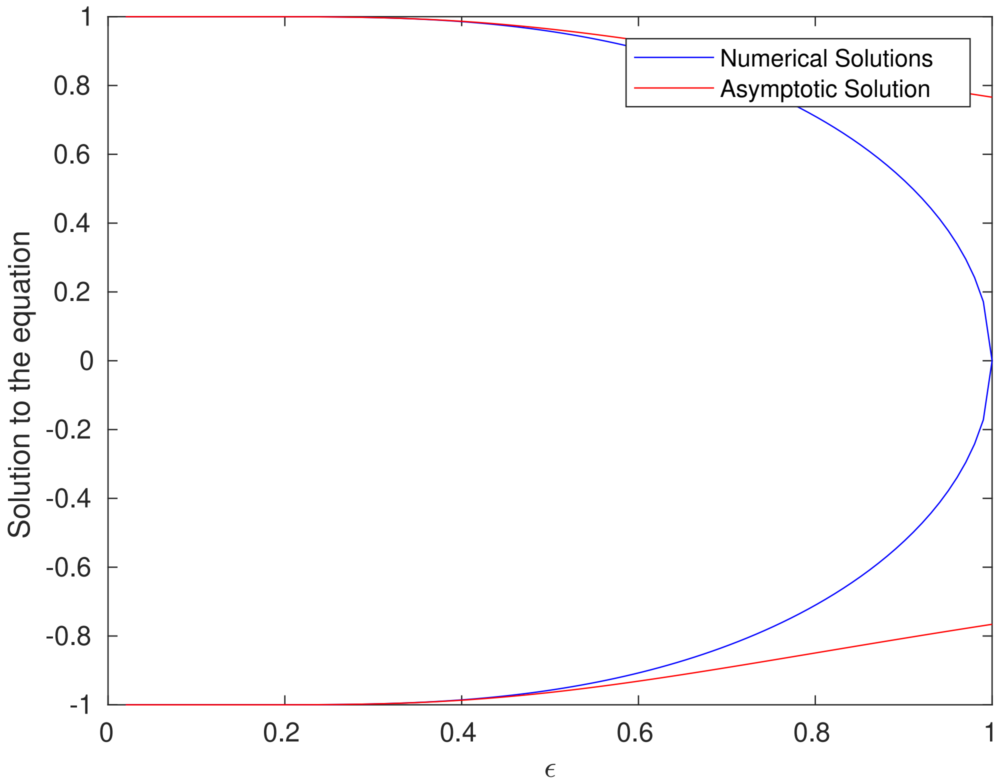


Figure 2: Comparison of asymptotic solution to numerical Solution of $x = \tanh(x/\epsilon)$

2. (a) If we rewrite the differential equation in standard form, we get:
$$y'' - \frac{1}{x^2}y' + \frac{1}{4x^4}y = 0$$
From this, $x = 0$ is an irregular singular point since $x\frac{-1}{x^2} = -\frac{1}{x} \to -\infty,\ as\ x \to 0.$ While all other values of x are ordinary points.

(b) Since $x = 0$ is an irregular singular point, use $y = e^{S(x)} := e^S$:
$$y' = S'e^S$$
$$y'' = S''e^S + (S')^2e^S$$
$$x^4(S''e^S + (S')^2e^S) - x^2(S'e^S) + \frac{1}{4}e^S = 0$$
$$S'' + (S')^2 - x^{-2}(S') + \frac{1}{4x^4} = 0$$

Now for asymptotic balance:

- $S'' \sim -(S')^2$ as $x \to 0$, assuming $x^{-2}(S'), \frac{1}{4x^4} \ll S'', -(S')^2$
$$\frac{S'}{S^2} \sim -1$$
$$S' \sim \frac{1}{x+a}$$
$$\implies -(S')^2 \sim -\frac{1}{(x+a)^2},\ x \to 0$$
Which is a **contradiction** as $\frac{1}{4x^4} \not\ll -(S')^2$ as $x \to 0$ and likewise $x^{-2}S' \not\ll -(S')^2$.
- $S'' \sim x^{-2}(S')$ as $x \to 0$, neglecting $-(S')^2$ and $\frac{1}{4x^4}$
$$\frac{S''}{S'} \sim x^{-2}$$
$$\log S' \sim \frac{-1}{x} + b$$
$$S' \sim ce^{-1/x}$$
$$\implies -(S')^2 \sim -ce^{-2/x}$$
Which is a **contradiction**, as $\frac{1}{4x^4} \gg e^{-2/x}$ as $x \to 0$. And $-(S')^2 \not\ll S''$
- $S'' \sim -\frac{1}{4x^4}$ as $x \to 0$, neglect $-(S')^2$ and $x^{-2}(S')$
$$S' \sim \frac{1}{12x^3}$$
$$-(S')^2 \sim \frac{-1}{144x^6}$$
Contradiction since we have neglected $-(S')^2$ but $S'' \ll -(S')^2$, as $x \to 0$. And similarly $x^{-2}S' \gg (\frac{1}{4x^4})$
- $(S')^2 \sim x^{-2}(S')$ as $x \to 0$, neglecting $\frac{1}{4x^4}$ and S''
$$S' \sim x^{-2}$$
$$\implies (S')^2 \sim x^{-2}(S') \sim \frac{1}{x^4}$$
$$S'' \sim \frac{-1}{2x}$$
But $x^{-4} \not\ll \frac{1}{4x^4}$ so this balance will be valid only if we include $\frac{1}{4x^4}$.

- $(S')^2 \sim -\frac{1}{4x^4}$ as $x \to 0$

$$\begin{aligned} S' &\sim \pm i \frac{1}{2x^2} \\ S'' &\sim \pm - i \frac{1}{x^3} \end{aligned}$$

Which follows the trend from the previous - this is only valid if we include $\frac{1}{4x^4}$.

- $x^{-2}(S') \sim \frac{1}{4x^4}$ as $x \to 0$, assuming S'' and $(S')^2 \ll x^{-2}(S')$ and $\frac{1}{x^4}$

$$\begin{aligned} S' &\sim \frac{1}{4x^2} \\ (S')^2 &\sim \frac{1}{16x^4} \\ S'' &\sim -\frac{1}{2x^3} \end{aligned}$$

Which also requires the inclusion of $\frac{1}{4x^4}$.

From this, conclude the correct balance is

$$(S')^2 \sim x^{-2}S' - \frac{1}{4x^4}, \quad x \to 0$$

Neglecting S''. Use the quadratic formula in S':

$$\begin{aligned} (S')^2 &\sim x^{-2}S' - \frac{1}{4x^4} \\ S' &\sim \frac{x^{-2} \pm \sqrt{x^{-4} - x^{-4}}}{2} \\ S' &\sim \frac{x^{-2}}{2} \\ S' &\sim \frac{1}{2x^2} \\ S &= \frac{-1}{2x} + C(x) \end{aligned}$$

$$\begin{gathered} S = \frac{-1}{2x} + C, \quad S' = \frac{1}{2x^2} + C', \quad S'' = \frac{-1}{x^3} + C'' \\ \implies C \ll \frac{-1}{2x}, \quad C' \ll \frac{1}{2x^2}, \quad C'' \ll \frac{-1}{x^3} \end{gathered}$$

Plug this back into the S equality:

$$\begin{aligned} S'' + (S')^2 - x^{-2}(S') + \frac{1}{4x^4} &= 0 \\ \frac{-1}{x^3} + C'' + (\frac{1}{2x^2} + C')^2 - \frac{1}{x^2}(\frac{1}{2x^2} + C') + \frac{1}{4x^4} &= 0 \\ \frac{-1}{x^3} + C'' + \frac{1}{4x^4} + (C')^2 - \frac{1}{x^2}C' - \frac{1}{2x^4} - \frac{1}{x^2}C' + \frac{1}{4x^4} &= 0 \\ -\frac{1}{x^3} + C'' + (C')^2 - \frac{2}{x^2}C' &= 0 \\ (C')^2 - \frac{2}{x^2}C' &\sim \frac{1}{x^3} \end{aligned}$$

- $(C')^2 \sim \frac{2}{x^2}C'$, negelct $\frac{1}{x^3}$

$$\begin{aligned}(C')^2 &\sim \frac{2}{x^2}C' \\ C' &\sim \frac{2}{x^2} \\ (C')^2 &\sim \frac{4}{x^4}\end{aligned}$$

  Which is a contradiction since we require $C' \ll \frac{1}{x^2}$

- $(C')^2 \sim \frac{1}{x^3}$ neglect $\frac{2}{x^2}C'$ as $x \to 0$.

$$\begin{aligned}(C')^2 &\sim \frac{1}{x^3} \\ C' &\sim \pm x^{-3/2} \\ \implies 2x^{-2}C' &\sim \pm 2x^{-7/2}\end{aligned}$$

  Which is a contradiction since we have neglected $\frac{2}{x^2}C'$

- $\frac{2}{x^2}C' \sim -x^{-3}$

$$\begin{aligned}\frac{2}{x^2}C' &\sim -\frac{1}{x^3} \\ C' &\sim \frac{-1}{2x} \\ C &\sim -\log(x)/2\end{aligned}$$

  Which is perfectly reasonable.

Hence

$$C = -\log(x)/2 + D, \quad C' = \frac{-1}{2x} + D', \quad C'' = \frac{1}{2x^2} + D''$$

Where

$$D \ll \log(x)/2, \quad D' \ll \frac{1}{2x}, \quad D'' \ll \frac{1}{2x^2}$$

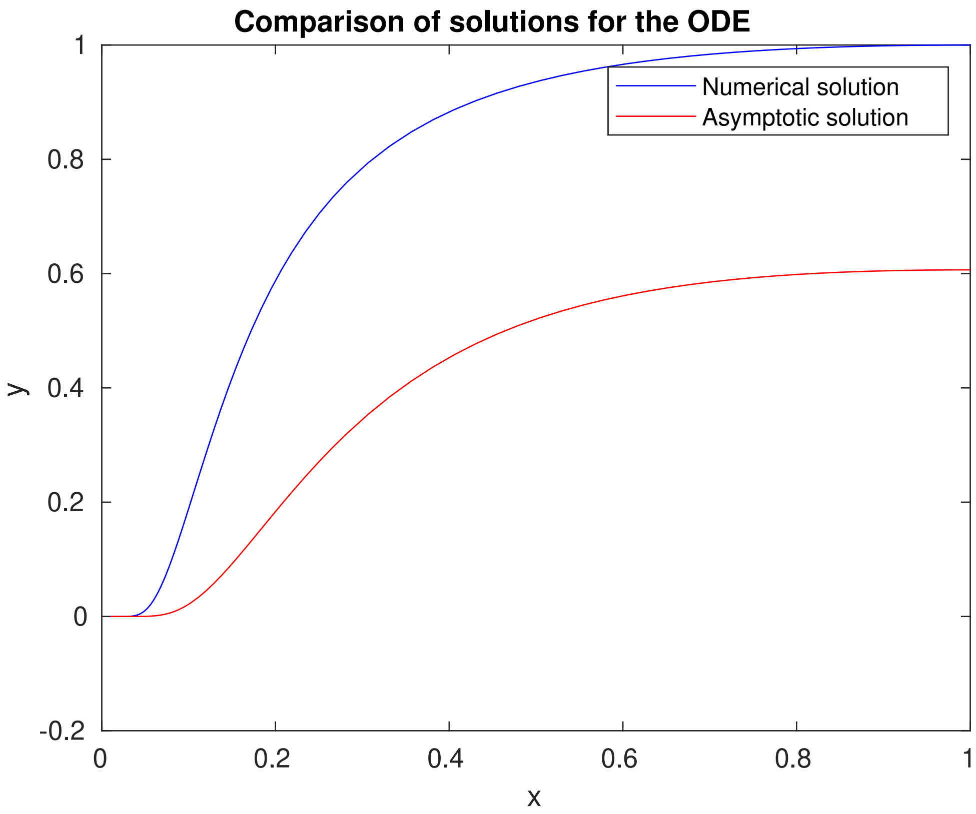
We could just say that the next term D will be constant given the previous term was log but lets continue anyway:

$$\begin{aligned}C'' + (C')^2 - \frac{2}{x^2}C' - \frac{1}{x^3} &= 0 \\ \frac{1}{2x^2} + D'' + (\frac{-1}{2x} + D')^2 - \frac{2}{x^2}\left(\frac{-1}{2x} + D'\right) - \frac{1}{x^3} &= 0 \\ \frac{1}{2x^2} + D'' + \frac{1}{4x^2} - \frac{D'}{x} + (D')^2 - \frac{2}{x^2}D' &= 0 \\ \frac{3}{4x^2} + D'' - \frac{D'}{x} + (D')^2 - \frac{2}{x^2}D' &= 0 \\ \frac{3}{4x^2} &\sim \frac{2}{x^2}D' \\ D' &\sim \frac{3}{2}\end{aligned}$$

Neglecting D'' since there is a $\frac{1}{x^2}$ term, and neglecting $\frac{1}{x}D' \ll \frac{1}{x^2}D'$. Lastly neglecting $(D')^2 \ll \frac{1}{x^2}$.

Gives

$$D' \sim \frac{3}{2} \implies D \sim \frac{3}{2}x + d \implies D \sim d$$



For a constant d, Since $ax \to 0$ as $x \to 0$

This means the asymptotic behaviour as $x \to 0$ for y is

$$y = e^{S(x)} = c\exp\left\{-\frac{1}{2}\left(\frac{1}{x} + \log(x)\right)\right\} = cx^{-1/2}e^{\frac{-1}{2x}}$$

Where $c = e^d$ Obtain c by dividing the true behaviour

(c) Write the ODE as

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \frac{1}{x^2}y_2 - \frac{1}{4x^4}y_1 \end{aligned}$$

We cannot include $x = 0$ due to the discontinuity, so consider the region just above $x = 0$, i.e $[0.02, 1]$. Figure 2c shows the comparison. It appears that they agree to a constant away from $x = 0$. This can't be shown since $y_{numeric}$ and $y_{asymptotic}$ both very quickly become equal to 0 near this point.

3. (a)

$$
\begin{aligned}
S(\epsilon) &= \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\,dt \\
&= \int_0^\infty e^{-t}\left(\frac{(-\epsilon t)^{N+1}}{1+\epsilon t} + \sum_{j=0}^{N}(-\epsilon t)^j\right)\,dt \\
&= \int_0^\infty e^{-t}\frac{(-\epsilon t)^{N+1}}{1+\epsilon t}\,dt + \int_0^\infty e^{-t}\sum_{j=0}^{N}(-\epsilon t)^j\,dt \\
&= (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt + \sum_{j=0}^{N}(-1)^j\epsilon^j\int_0^\infty e^{-t}t^j\,dt \\
&= (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt + \sum_{j=0}^{N}(-1)^j\epsilon^j j! \\
&= (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt + \sum_{j=0}^{N}(-1)^j\epsilon^j j! \\
&= \sum_{j=0}^{N}(-1)^j\epsilon^j j! + (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt \\
&= 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + 24\epsilon^4 - \dots
\end{aligned}
$$

Using $\int_0^\infty e^{-x}x^n dx = n!$
By truncating the series, at N, we get the error term as

$$err(N) = (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt$$

It is optimal to truncate the series at the smallest term, i.e. the value of j which gives the smallest value. So want the largest j such that (and terminate the series before the $j+1$ term)

$$
\begin{aligned}
\epsilon^j j! &\leq \epsilon^{j+1}(j+1)! \\
1 &\leq \epsilon(j+1) \\
(j+1) &\geq \frac{1}{\epsilon} \\
j &\geq \frac{1}{\epsilon} - 1
\end{aligned}
$$

(b) Using the error term

$$err(N) = (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt$$

And use

$$\frac{1}{1+\epsilon t} = \frac{1}{2[1+\frac{1}{2}(\epsilon t - 1)]}$$

$$
\begin{aligned}
err(N) &= (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+\epsilon t}\,dt \\
&= (-1)^{N+1}\epsilon^{N+1}\int_0^\infty \left(e^{-t}t^{N+1}\right)\frac{1}{2[1+\frac{1}{2}(\epsilon t-1)]}\,dt \\
&= (-1)^{N+1}\epsilon^{N+1}\frac{1}{2}\int_0^\infty \left(e^{-t}t^{N+1}\right)\left(\left(\sum_{j=0}^{M}(-\frac{1}{2}(\epsilon t-1))^j\right)+\frac{(-\frac{1}{2}(\epsilon t-1))^{M+1}}{1+\frac{1}{2}(\epsilon t-1)}\right)dt \\
&= (-1)^{N+1}\epsilon^{N+1}\frac{1}{2}\left(\int_0^\infty \left(e^{-t}t^{N+1}\right)\sum_{j=0}^{M}(-\frac{1}{2}(\epsilon t-1))^j\,dt+\int_0^\infty \left(e^{-t}t^{N+1}\right)\frac{(-\frac{1}{2}(\epsilon t-1))^{M+1}}{1+\frac{1}{2}(\epsilon t-1)}\,dt\right) \\
&= (-1)^{N+1}\epsilon^{N+1}\frac{1}{2}\left(\sum_{j=0}^{M}\int_0^\infty \left(e^{-t}t^{N+1}\right)(-\frac{1}{2})^j(\epsilon t-1)^j\,dt+\int_0^\infty \left(e^{-t}t^{N+1}\right)\frac{(-\frac{1}{2}(\epsilon t-1))^{M+1}}{1+\frac{1}{2}(\epsilon t-1)}\,dt\right)
\end{aligned}
$$

Expanding the first $\sum\int$ term:

$$
\begin{aligned}
&\sum_{j=0}^{M}\int_0^\infty \left(e^{-t}t^{N+1}\right)(-\frac{1}{2})^j(\epsilon t-1)^j\,dt \\
&= \sum_{j=0}^{M}(-\frac{1}{2})^j\int_0^\infty \left(e^{-t}t^{N+1}\right)\sum_{k=0}^{j}\binom{j}{k}(\epsilon t)^k(-1)^{j-k}\,dt \\
&= \sum_{j=0}^{M}(-\frac{1}{2})^j\sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}\int_0^\infty \left(e^{-t}t^{N+1}\right)\epsilon^k t^k\,dt \\
&= \sum_{j=0}^{M}(-\frac{1}{2})^j\sum_{k=0}^{j}\epsilon^k\binom{j}{k}(-1)^{j-k}\int_0^\infty \left(e^{-t}t^{N+k+1}\right)dt \\
&= \sum_{j=0}^{M}(-\frac{1}{2})^j\sum_{k=0}^{j}\epsilon^k\binom{j}{k}(-1)^{j-k}(N+k+1)! \\
&= \sum_{j=0}^{M}(\frac{1}{2})^j\sum_{k=0}^{j}\epsilon^k\binom{j}{k}(-1)^{-k}(N+k+1)!
\end{aligned}
$$

So it gives

$$
err(N) = (-1)^{N+1}\epsilon^{N+1}\frac{1}{2}\left(\sum_{j=0}^{M}(\frac{1}{2})^j\sum_{k=0}^{j}\epsilon^k\binom{j}{k}(-1)^{-k}(N+k+1)!\right)+errerr(N,M)
$$

Where

$$
\begin{aligned}
errerr(N,M) &= (-1)^{N+1}\epsilon^{N+1}\frac{1}{2}\left(\int_0^\infty \left(e^{-t}t^{N+1}\right)\frac{(-\frac{1}{2}(\epsilon t-1))^{M+1}}{1+\frac{1}{2}(\epsilon t-1)}\,dt\right) \\
errerr(N,M) &= (-1)^{N+M+2}\epsilon^{N+1}\left(\frac{1}{2}\right)^{M+2}\left(\int_0^\infty \left(e^{-t}t^{N+1}\right)\frac{(\epsilon t-1)^{M+1}}{1+\frac{1}{2}(\epsilon t-1)}\,dt\right)
\end{aligned}
$$

(c) Figure 3c shows the absolute error obtained in (a), while figure 3c shows the error, “errerr” obtained in (b), both for an epsilon value of $\epsilon = 0.1$. The optimal truncation

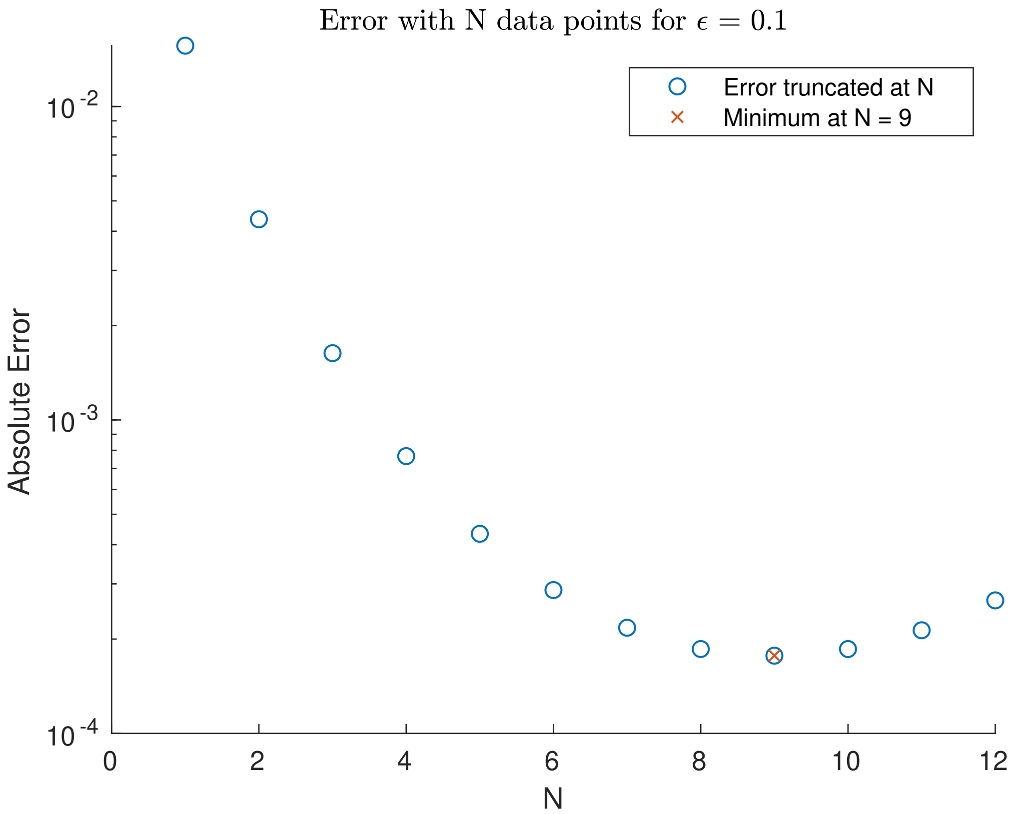


Figure 3: Plot of the absolute error term against N on a log y scale

for the first series (denoted by the red cross in figure 3c) occurs for $N = 9$. From figure 3c, the optimal truncation point shifts

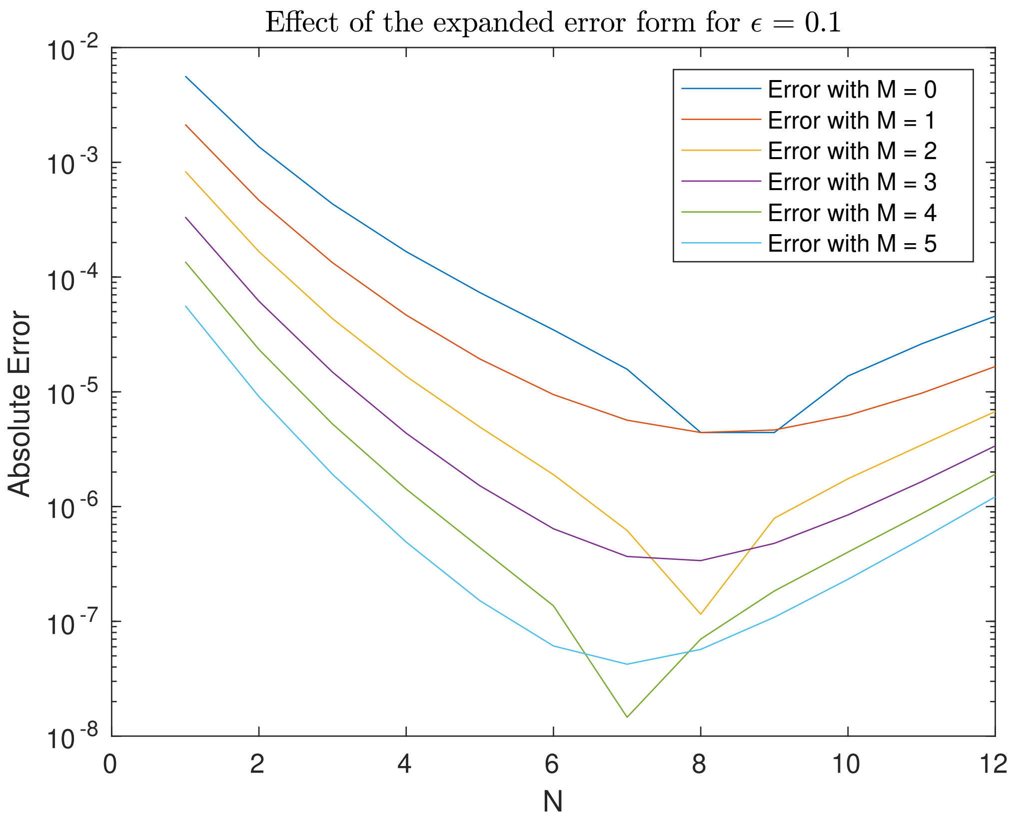


Figure 4: Plot of the second absolute error term ("errerr") for various M on a log y scale

Matlab Code

```
%reproducibility
clear all
close all

%%Q1a
x=linspace(-5,5,10000);
epsilon = 0.05;
plot(x,tanh(x/epsilon))
hold on
plot(x,x)

hold off
legend('tanh(x/\epsilon)','x')
title('\epsilon = 0.05')
saveas(gcf,"TopicCA1Q1a.eps","epsc")

%%Q1c
%symbolically solve the equation
%and plot against asymptotic solution

syms x
epsilon = linspace(1,0.02);
S = zeros(2,length(epsilon));
mysol = zeros(2,length(epsilon));
for i=1:length(epsilon)
    eqn = x == tanh(x/epsilon(i));
    S(1,i) = vpasolve(eqn,x,-1);
    S(2,i) = vpasolve(eqn,x,1);
    mysol(1,i) = 1 -2*exp(-2/epsilon(i)) +2*exp(-4/epsilon(i));
    mysol(2,i) = -mysol(1,i);

end


figure
plot(epsilon,S(1,:), 'b')
hold on
%HandleVisibility off makes the legend work nicely
plot(epsilon,S(2,:), 'b','HandleVisibility','off')
plot(epsilon,mysol(1,:), 'r')
plot(epsilon,mysol(2,:), 'r','HandleVisibility','off')
%plot(epsilon,mysol)
legend("Numerical Solutions","Asymptotic Solution")
xlabel("$$\epsilon$$",'interpreter','latex')
ylabel("Solution to the equation")
saveas(gcf,"TopicCA1Q1c.eps","epsc")

%%Q2c
%obtain numeric solution
```

```
%plot it against asmptotic
[x,ynum] = ode45(@odefun,[1,0.01],[1,0]);
%we only want to plot y which is the first column of ynum
figure
plot(x,ynum(:,1),'b')
hold on
S = @(x) -0.5*(1./x +log(x));
yasymp = exp(S(x));
plot(x,yasymp,'r')
%axis([0,0.2,-1,1])
legend("Numerical solution", "Asymptotic solution")
xlabel("x")
ylabel("y")
title("Comparison of solutions for the ODE")
saveas(gcf,"TopicCA1Q2c.eps","epsc")


%%Q3c
%%
%with epsilon = 0.1 we expect the optimal trunctation at j= 1/eps -1 = 9
epsilon = 0.1;
syms tsym
Nmax = 12;
%keep Mmax*Nmax relatively small to lower computation time
Mmax = 5;
xvals = 1:Nmax;
solvalsa =zeros(size(xvals));
solvalsb =zeros(Nmax,Mmax);
for N = 1:Nmax
    % %i've written this slightly differently and omitted the (-1) terms
    % since we are only concerned with absolute error
    erraint = int((exp(-tsym)*tsym^(N+1)/(1+(epsilon*tsym))),tsym,[0,inf]);
    erra = (epsilon).^(N+1) * erraint;
        solvalsa(N) = erra;
    for M = 0:Mmax
        errbint = int(exp(-tsym)*tsym^(N+1)...
            * ((epsilon*tsym - 1)^(M+1))/(1+0.5*(epsilon*tsym-1)) ,tsym,[0,inf]);
        errb = (epsilon).^(N+1) * (1/2)^(M+2) * errbint;
        solvalsb(N,M+1) = errb;
    end

 end
[minimum,index] = min(abs(solvalsa));
figure
scatter(xvals,abs(solvalsa))
hold on
textflaga = "Minimum at N = "+ num2str(index);
scatter(index,minimum,'x')
xlabel("N")
ylabel("Absolute Error")
legend("Error truncated at N", textflaga)
```

```
title ("Error with N data points for $\epsilon$ = "+num2str(epsilon), 'interpreter','latex')
hold off
set(gca,'yscale','log')
saveas(gcf,"TopicCA1Q3c1.eps","epsc")


figure
semilogy(xvals,abs(solvalsb ))
xlabel("N")
ylabel("Absolute Error")
textflagb = "Error with M = "+[0:Mmax];
legend(textflagb)
title ("Effect of the expanded error form for $\epsilon$ = "+num2str(epsilon), 'interpreter','latex')
saveas(gcf,"TopicCA1Q3c2.eps","epsc")
%%



%%Function for 2
function dy = odefun (x,y)
dy = [y(2);y(2)./x.^2 - y(1)./(4*x.^4)];
%y'' - y' x^(-2) + y/(4x^4) = 0
%y'' = y'/(x^2) - y/(4x^4)
end
```

Practical Asymptotics (APP MTH 4048/7044)
Assignment 1 (5%)

Due 22 March 2019

1. Consider the transcendental equation
$$x = \tanh\left(\frac{x}{\epsilon}\right).$$
   (a) How many (non-zero) real solutions are there to the above equation for $\epsilon \to 0$? [Hint: Sketch x and $\tanh(x/\epsilon)$.]
   (b) For each of the non-zero solutions find three terms in an asymptotic expansion as $\epsilon \to 0$.
   (c) Compare your expansion with a numerical solution.

2. Consider the differential equation
$$x^4 y'' - x^2 y' + \tfrac{1}{4} y = 0, \quad \text{as } x \to 0.$$
   (a) Classify the ordinary, regular singular and irregular singular points of this equation.
   (b) Use the method of dominant balance to find the leading behaviours as $x \to 0$.
   [Hint: it is possible to have a balance between three terms]
   (c) Solve the differential equation numerically over a suitable range, subject to initial conditions of your choice. Discuss how this numerical solution relates to the behaviours you found in part (b).

3. Consider the integral representation of the Stieljes function:
$$S(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\,dt.$$
   (a) Develop a series representation of this function using the definition of a geometric series,
$$\frac{1}{1+z} = \sum_{j=0}^{N} (-z)^j + \frac{(-z)^{N+1}}{1+z}.$$
   What is the error if the resulting series is truncated after N terms? This is a divergent series, how many terms are required for an optimal truncation for a given value of ϵ?
   (b) Having optimally truncated the series, let's now try and improve this by examining the error term. To do this, develop a series representation of the error from part (a) using the fact that
$$\frac{1}{1+y} = \frac{1}{2[1+\frac{1}{2}(y-1)]}.$$
   Find a series expression for the error term of this new representation (it will depend on the number of terms included in both series).
   (c) Use MATLAB to plot the approximate errors found in parts (a) and (b) for a fixed value of ϵ.