THE UNIVERSITY *of* ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2011

104831	**Real Analysis - UG**
	MATHS 2100

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 8 TOTAL MARKS: 90

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- No calculators, books, notes, or other aids are permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. (a) Let $f : A \to \mathbb{R}$ be a function and $(A_i)_{i \in I}$ be a family of subsets of A. Show that

$$f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i).$$

(b) Find a function $f : \mathbb{R} \to \mathbb{R}$ and a subset B of $\mathbb{R}$ for which $f^{-1}(f(B)) \neq B$.

(c) Let $A_q = (-\infty, q)$ for $q \in \mathbb{Q}$, the rational numbers. Find (without proof) the following sets:

(i) $\bigcup_{q\in\mathbb{Q}} A_q$ (ii) $\bigcap_{q\in\mathbb{Q}} A_q$.

(d) Let $f : A \to B$ be a function. Show that if $f(C \cap D) = f(C) \cap f(D)$ for all $C, D \subset A$ then f is injective.

[3+2+4+3 = 12 marks]

2. (a) Let A be a subset of $\mathbb{R}$. Define what it means
   (i) for A to be bounded above
   (ii) for $b \in \mathbb{R}$ to be the least upper bound of A.

(b) State the axiom of completeness for real numbers.

(c) Find the following (proofs are not necessary).
   (i) $\sup\,[0, 20)$
   (ii) $\sup A$ if $A = (-1, 5) \cup (-5, 2] \cup \{3\}$
   (iii) $\inf A$ if $A = (-1, 5) \cup (-5, 2] \cup \{3\}$.

[3+2+3 = 8 marks]

3. (a) Define what it means for a sequence (a_n) in $\mathbb{R}$ to
   (i) converge to an element $a \in \mathbb{R}$
   (ii) be a Cauchy sequence.

(b) Use your definition to show that if $a_n = \dfrac{1}{3n+3}$ then $a_n \to 0$ as $n \to \infty$.

(c) Show directly from the definition of a Cauchy sequence that the sum of two Cauchy sequences is another Cauchy sequence.

(d) Let (a_n) be a sequence such that $a_n \geq 0$ for infinitely many $n \in \mathbb{N}$. Show that if $a_n \to a$ then $a \geq 0$.

[4+3+4+4 = 15 marks]

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4. (a) Define what it means for a subset A of $\mathbb{R}$ to be *open*.

   (b) Show that the intersection of finitely many open subsets of $\mathbb{R}$ is open.

   (c) (i) Complete the following (in your exam booklet). A subset $B \subset \mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if ______________________________

   (ii) Use the fact that the rational numbers are dense in $\mathbb{R}$ to prove that the *irrational* numbers are dense in $\mathbb{R}$.

   (d) Prove that any subset A of $\mathbb{R}$ can be written as the intersection of open sets.

[2+4+4+4 = 14 marks]

5. (a) (i) Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and let c be a limit point of A. Define what it means for the limit of f at c to exist and be equal to the real number L.

   (ii) Use your definition of limit to show that

$$\lim_{x \to 2}(7x - 12) = 2.$$

In the 2011 course the concept of 'compact set' was discussed. We discussed sequentially compact sets

   (b) (i) Define what it means for a subset C of $\mathbb{R}$ to be *compact.*

   (ii) Let $f : A \to \mathbb{R}$ be a continuous function and $C \subset A$ a compact subset of A. Show that $f(C)$ is compact.

replace 'compact' with 'sequentially compact'.

[5+7 = 12 marks]

6. (a) Complete the following (in your exam booklet). Suppose $f : I \to \mathbb{R}$ is a function defined on an interval $I \subset \mathbb{R}$ containing more than one point. f is said to be differentiable at $a \in I$ if ______________________________

   (b) Suppose that $f : (a, b) \to \mathbb{R}$ has a minimum at a point $c \in (a, b)$. Show that if f is differentiable at c then $f'(c) = 0$.

[2+4 = 6 marks]

7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let P be a partition of $[a, b]$ with elements $a = x_0 < x_1 < \cdots < x_n = b$. If $k \in \{1, 2, \ldots, n\}$ define

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

   (a) Define the upper sum $U(f, P)$ and lower sum $L(f, P)$ of f with respect to P.

   (b) Define what it means for f to be *integrable* on $[a, b]$.

   (c) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Show that if there is a point $c \in [a, b]$ with $f(c) > 0$ then $\int_a^b f(x)dx > 0$.

[2+2+4 = 8 marks]

8. (a) State what it means for a sequence (f_n) of functions on $A \subset \mathbb{R}$ to converge to a function $f : A \to \mathbb{R}$
   (i) pointwise
   (ii) uniformly.

   (b) Let $f_n(x) = nxe^{-n^2x}$ for $x \in [0, \infty)$.
   (i) Show that the sequence (f_n) converges uniformly to 0 (the zero function on the interval $[0, \infty)$).
   (ii) Does $f_n \to 0$ pointwise?

   (c) (i) Give the statement of the Weierstrass M-Test.
   (ii) Show that the formula

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

defines a continuous function f on the interval $[-1, 1]$.

[3+5+7 = 15 marks]