Real Analysis II **Semester 2, 2017**

Class Exercise 5, due 5 pm Monday 16 October 2017

1. (a) Let $f\colon (0,1) \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for all $x \in (0,1)$ and the third derivative $f'''(x)$ exists for every $x \in (0,1)$. If $f(c) = f(d) = 0$ for some $0 < c < d < 1$, prove that $f'''(x) = 0$ for some $x \in (0,1)$. (Hint: use Rolle's Theorem more than once.)

(b) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) \leq x^3$ for all $x \in [0,1]$ and such that $\int_0^1 f(x)\,dx = 1/4$. Prove that $f(x) = x^3$ for all $x \in \mathbb{R}$. (Hint: $\int_0^1 x^3\,dx = 1/4$.)

[6 points]

2. Suppose that $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0$ for all $x \in (a,b)$ prove that there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

[6 points]

3. Let $\ln\colon (0,\infty) \to \mathbb{R}$ be defined by

$$\ln(x) = \int_1^x \frac{1}{t}dt$$

for $x > 0$. Prove that

(i) ln is differentiable on $(0,\infty)$ with $\ln'(x) = 1/x$ for all $x > 0$.

(ii) for any $a > 0$, the function $\ln(x)$ is uniformly continuous on $[a,\infty)$.

[6 points]

4. Recall that $e = \exp(1)$.

(a) Prove that $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$. (Hint: what is $\ln'(1)$?)

(b) By considering the function $f(x) = \frac{\ln(1+x)}{x}$ and the sequence $x_n = 1/n$, use Proposition 6.5 to prove that $\lim_{n\to\infty} \ln(1+\frac{1}{n})^n = 1$.

(c) Use part (b) to prove that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

[6 points]

5. (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} a_n/b_n = c \neq 0$ prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges. (Hint: $c > 0$ — why? Therefore you can take $\epsilon = c$ in the definition of convergence for $\lim_{n\to\infty} a_n/b_n$. Use the Comparison Test.)

(b) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ converges and evaluate the sum of the series.

[6 points]