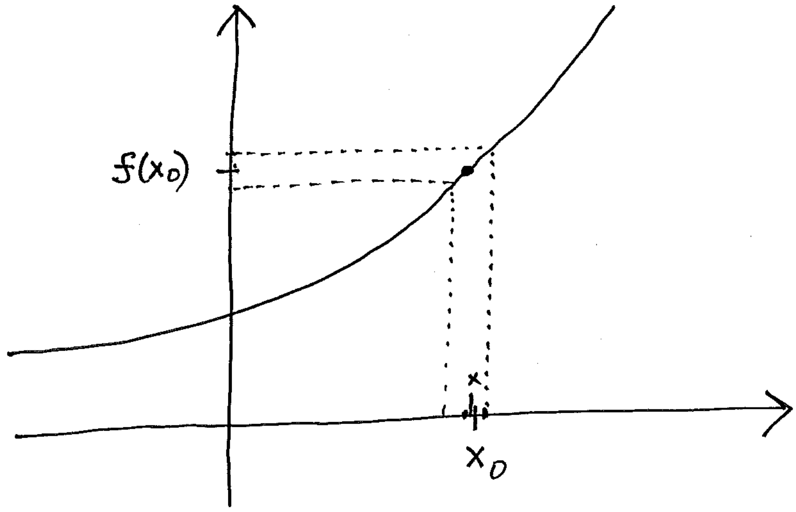
Last time

- Heine-Borel Th$^{m}$: $S$ seq. compact $\iff$ $S$ closed & bded
- $f: S \to \mathbb{R}$ <u>cts</u> at $x_0 \in S$ $\iff$ $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.th. if $x \in S$ & $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.




Ex. 1. any $f^n$ $f: \mathbb{N} \to \mathbb{R}$ is cts at every $x_0 \in \mathbb{N}$.

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$ is <u>not</u> cts at <u>any</u> $x_0 \in \mathbb{R}$.

3. $\exists\ f: \mathbb{R} \to \mathbb{R}$, s.th. $f$ is cts at every irrational $x$, but not cts at any $x \in \mathbb{Q}$.

Ex. $f: S \to \mathbb{R}$ is cts at $x_0 \in S$, suppose $T \subset S$, $x_0 \in T$. Then $f|_T : T \to \mathbb{R}$ is cts at $x_0$.
$\uparrow$ <u>restriction</u> of $f$ to $T$

<u>Def$^n$ 4.2</u>: We say $f: S \to \mathbb{R}$ is <u>cts on $S$</u> if $f$ is cts at every $x_0 \in S$.

<u>Ex</u>: if $f: S \to \mathbb{R}$ is cts on $S$ & $T \subset S$ then $f|_T : T \to \mathbb{R}$ is cts on $T$.

<u>Ex</u>: What does it mean for $f: S \to \mathbb{R}$ to <u>not be cts</u> at $x_0 \in S$?

$\exists\ \varepsilon > 0$ s.th. $\forall\ \delta > 0\ \exists\ x \in S$ s.th. $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$.

$\therefore$ ~~If~~ If $f$ is not cts at $x_0$ then $\exists\ \varepsilon > 0$ s.th. $\forall\ n \in \mathbb{N}\ \exists\ s_n \in S$ s.th. $\underbrace{|s_n - x_0| < \frac{1}{n}}_{\Rightarrow\ s_n \to x_0}$ and $|f(s_n) - f(x_0)| \geq \varepsilon$.

$\therefore$ $f$ not cts at $x_0 \Rightarrow \exists$ seq. $(s_n)$ in $S$ s.th. $s_n \to x_0$ but $f(s_n) \not\to f(x_0)$

<u>Th^m</u> 4.3: $f : S \to \mathbb{R}$ is cts at $x_0 \in S \iff$ for every seq. $(s_n)_{n=1}^{\infty}$ in $S$ s.th. $s_n \to x_0$, $f(s_n) \to f(x_0)$

<u>Pf</u>: ($\Leftarrow$) proved the contrapositive already.

($\Rightarrow$). Suppose $(s_n)$ is a seq in $S$ s.th. ~~$f(s_n)$~~ $s_n \to x_0$. (We've got to show that $f(s_n) \to f(x_0)$).

Let $\varepsilon > 0$.

- Since $f$ is cts at $x_0$, $\exists\ \delta > 0$ s.th. if $\underline{x \in S}$ & $\underline{|x - x_0| < \delta}$ then $|f(x) - f(x_0)| < \varepsilon$.
- Since $s_n \to x_0$, $\exists\ N \in \mathbb{N}$ s.th. $n \geq N \Rightarrow |s_n - x_0| < \delta$.
  (take $\varepsilon = \delta$). in the def$^n$ of cvgce for $s_n \to x_0$

$\therefore$ if $n \geq N$ then (since $s_n \in S\ \forall\ n$). $|s_n - x_0| < \delta$ and so $|f(s_n) - f(x_0)| < \varepsilon$.

Since this is true for any $\varepsilon > 0$, it follows that $f(s_n) \to f(x_0)$.

$Th^{\underline{m}}$ 4.4: Suppose $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$, $s_0 \in S$

If $f, g$ are cts at $s_0$ then

(i) $cf$ is cts at $s_0$ $\forall\, c \in \mathbb{R}$ $((cf)(x) = c(f(x)))$.

(ii) $f+g$ is cts at $s_0$

(iii) $f \cdot g$ is cts at $s_0$

(iv) $f/g$ is cts at $s_0$, if $g(s) \neq 0$ for all $s \in S$.

exercise: Prove all of these statements using $Def^{\underline{n}}$ 4.1.

Pf: Use the Alg. Limit $Th^{\underline{m}}$ together with $Th^{\underline{m}}$ 4.3.

eg (iii).

Let $(s_n)$ be a seq. in $S$ s.th. $s_n \to s_0$.

(Check that $f \cdot g(s_n) \to f \cdot g(s_0)$).

$Th^{\underline{m}}$ 4.3 $\Rightarrow$ $f(s_n) \to f(s_0)$ ($f$ cts at $s_0$)

$g(s_n) \to g(s_0)$ ($g$ cts at $s_0$)

Alg. Limit $Th^{\underline{m}}$ $\Rightarrow$ $f(s_n)g(s_n) \to f(s_0)g(s_0)$.

i.e. $fg(s_n) \to fg(s_0)$

Since this is true for all seqs. $(s_n)$ in $S$ s.th. $s_n \to s_0$, $Th^{\underline{m}}$ 4.3 $\Rightarrow$ $f \cdot g$ is cts at $s_0$.

$Th^{\underline{m}}$ 4.5: If $f : S \to \mathbb{R}$ is cts at $s_0$, $f(S) \subset T$, $g : T \to \mathbb{R}$ is cts at $f(s_0)$ then $g \circ f : S \to \mathbb{R}$ is cts at $s_0$.

(see CEx 3).

Ex. 1. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = |x|$ is cts on $\mathbb{R}$.

use $||x| - |x_0|| \leq |x - x_0|$. to show $g$ is cts at $x_0$ for all $x_0 \in \mathbb{R}$.

2. if $f, g : S \to \mathbb{R}$ are cts at $s_0$ then $\max(f, g) : S \to \mathbb{R}$, $\min(f, g) : S \to \mathbb{R}$ are cts at $s_0$.

3. if $f : S \to \mathbb{R}$ is cts at $s_0$, then
$|f| : S \to \mathbb{R}$ is cts at $s_0$

4. $\exists\, f : \mathbb{R} \to \mathbb{R}$ s.th. $f$ is not cts at any $x_0 \in \mathbb{R}$
but $|f| : \mathbb{R} \to \mathbb{R}$ is cts on $\mathbb{R}$.

Thm 4.6: Suppose $f : S \to \mathbb{R}$ is cts where $S$ is non-empty and seq. compact. Then $f$ is bounded.
(i.e. $f(S) = \{ f(s) \mid s \in S \}$ is a bounded set,
i.e. $\exists\, K > 0$ s.th. $|f(s)| \leq K$ for all $s \in S$).

Corr: If $f : [a,b] \to \mathbb{R}$ is cts then $f$ is bounded.
closed & bded set $\therefore$ seq. compact.

Pf: Suppose $f$ is not bounded. Hence $\forall\, n \in \mathbb{N}$, $\exists\, s_n \in S$ s.th. $|f(s_n)| > n$. Observe: $(s_n)$ is a seq. in $S$. Since $S$ seq. compact $\therefore \exists$ subseq. $(s_{n_k})$ s.th. $s_{n_k} \to s_0$ for some $s_0 \in S$.
Since $f : S \to \mathbb{R}$ is cts, $f(s_{n_k}) \to f(s_0)$.
But $|f(s_{n_k})| > n_k \geq k \Rightarrow (f(s_{n_k}))$ is unbounded
– contradiction. $\therefore f$ is bounded.

Thm 4.7: Suppose $f : S \to \mathbb{R}$ is cts, where $S$ is non-empty & seq. compact. Then $f$ attains its max. & min on $S$, i.e. $f(S)$ has a max & a min, i.e. $\exists\, s_0, s_1 \in S$ s.th. $f(s_0) \leq f(s) \leq f(s_1)$ $\forall\, s \in S$.