LECTURE 29

At the end of last lecture we observed the following fact: a series with non-negative terms converges if and only if its sequence of partial sums is bounded. We take advantage of this fact shortly to prove a very useful test for convergence.

Firstly, we make the following observation about the convergence of series: if $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers, then the associated series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=k}^{\infty} a_n$ converges for all $k \in \mathbb{N}$.

The sequence of partial sums for the series $\sum_{n=k}^{\infty} a_n$ is the sequence

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \ldots$$

Observe that this is equal to the sequence

$$s_k - s_{k-1}, s_{k+1} - s_{k-1}, s_{k+2} - s_{k-1}, \ldots$$

This sequence converges if and only if the sequence

$$s_k, s_{k+1}, s_{k+2}, \ldots$$

converges, i.e. if and only if the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ converges. Hence $\sum_{n=k}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 7.3 (Comparison Test): Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with non-negative terms, i.e. $a_n \geq 0$ and $b_n \geq 0$ for all n. If $a_n \leq b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Proof: Since $0 \leq a_n \leq b_n$ for all n, we have

$$a_1 + \cdots + a_N \leq b_1 + \cdots + b_N$$

for all N. Since $\sum_{n=1}^{\infty} b_n$ is a series of non-negative terms which is convergent, its sequence of partial sums is bounded above. By the inequality above, we see that the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is also bounded above. Therefore, since $\sum_{n=1}^{\infty} a_n$ is also a series of non-negative terms, the series $\sum_{n=1}^{\infty} a_n$ converges. ■

Note 1: The contrapositive of Theorem 7.3 can often be used to prove that a series does not converge: the contrapositive states that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series such that $0 \leq a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges.

Note 2: There is another very useful version of the Comparison Test. It states the following. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series such that there exists $N \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq N$ then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges.

Exercise: Prove the statement in Note 2 above.

Example: We show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. For any $n \geq 2$ we have

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)}$$

Therefore, since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ are series of non-negative terms, it follows by the Comparison Test that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ if $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges. But the series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges since its sequence of partial sums telescope:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{N(N-1)} = 1 - \frac{1}{N} \to 1 \text{ as } N \to \infty$$

using the fact that $\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$. More generally, we can show that $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges for any natural number k by comparison with the series

$$\sum_{n=k+1}^{\infty} \frac{1}{n(n-1)\cdots(n-k)}$$

which also can be shown to converge by another telescoping argument.

The Integral Test

Many series are of the form $\sum_{n=1}^{\infty} f(n)$, where $f\colon [1,\infty) \to \mathbb{R}$ is a continuous function taking non-negative values. If $f(x)$ is also decreasing, then there is a useful way to test for convergence of series of this form.

Let us first define

$$\int_1^{\infty} f(x)dx = \lim_{N\to\infty} \int_1^N f(x)dx$$

if this limit exists, in which case we say the *improper integral* $\int_1^{\infty} f(x)dx$ converges.

Theorem 7.4 (Integral Test): Let $f\colon [1,\infty) \to \mathbb{R}$ be continuous and decreasing, and suppose that $f(x) \geq 0$ for all $x \geq 1$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

Beware that $\int_1^{\infty} f(x)dx$ is not equal to $\sum_{n=1}^{\infty} f(n)$ in general.

Proof: Let $\mathscr{P} = \{1, 2, \ldots, N+1\}$ be a regular partition of $[1, N+1]$. Calculating the upper and lower sums of f with respect to the partition $\mathscr{P}$ gives the inequalities

$$\sum_{n=1}^{N} f(n+1) \leq \int_1^{N+1} f(x)dx \leq \sum_{n=1}^{N} f(n)$$

since $f(x)$ is decreasing.

Suppose first that $\sum_{n=1}^{\infty} f(n)$ converges. Since this is a series of non-negative terms, this happens if and only if its sequence of partial sums is bounded above. It follows that there is a $C > 0$ such that

$$\int_1^{N+1} f(x)dx \leq \sum_{n=1}^{N} f(n) \leq C$$

for all N. Therefore the sequence $(\int_1^{N+1} f(x)dx)_{N=1}^{\infty}$ is bounded above. This is an increasing sequence, since

$$\int_1^{N+2} f(x)dx = \int_1^{N+1} f(x)dx + \int_{N+1}^{N+2} f(x)dx \geq \int^{N+1} f(x)dx$$

on account of the fact that $f(x) \geq 0$ for all $x \geq 1$. Therefore the sequence $(\int_1^{N+1} f(x)dx)_{N=1}^{\infty}$, and hence the sequence $(\int_1^{N} f(x)dx)_{N=1}^{\infty}$, converges. Therefore the improper integral $\int_1^{\infty} f(x)dx$ converges.

Conversely, suppose the improper integral converges. Therefore the sequence $(\int_1^{N+1} f(x)dx)_{N=1}^{\infty}$ converges and hence is bounded. Therefore there exists $C > 0$ such that

$$\sum_{n=1}^{N} f(n+1) \leq \int_1^{N+1} f(x)dx \leq C$$

for all N. It follows that the sequence of partial sums of the series $\sum_{n=1}^{\infty} f(n)$ is bounded above. Since this is a series of non-negative terms, the series $\sum_{n=1}^{\infty} f(n)$ converges. ■

Note: If you examine the proof above carefully, you will see that the hypothesis that f is continuous is never used (remember that f decreasing implies f integrable). In practice though this theorem is most useful when the function f is continuous, and in fact differentiable — since it is then much easier to check that f is decreasing.

Example: Let $p > 0$ be a real number. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. To see this we apply the integral test. Let $f\colon [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^p}$. It is easy to see that f is decreasing, and clearly f takes non-negative values. If $p \neq 1$ then an anti-derivative for x^{-p} is $x^{1-p}/(1-p)$ and so

$$\int_1^N \frac{1}{x^p} dx = \frac{1}{1-p}(N^{1-p} - 1)$$

Clearly N^{1-p} converges if and only if $p > 1$. If $p = 1$ it is easy to see that $\int_1^{\infty} \frac{1}{x} dx$ does not converge. Hence the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Example: Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\alpha}}$. Let $f\colon [2, \infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{1}{x \ln x)^{\alpha}}.$$

Then $f(x) \geq 0$ for all $x \geq 2$. We have $f'(x) < 0$ for all $x \geq 2$ and hence f is decreasing. Therefore the Integral Test applies. We have

$$\int_1^N \frac{1}{x(\ln x)^{\alpha}} dx = \int_{\ln 2}^{\ln N} x^{-\alpha} dx$$

which converges as $N \to \infty$ if and only if $\alpha > 1$. Therefore the series converges if and only if $\alpha > 1$.

Theorem 7.5 (The Ratio Test): Suppose $a_n > 0$ for all n. Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ (possibly $L = \infty$). The series $\sum_{n=1}^{\infty} a_n$

- converges if $L < 1$,
- diverges if $L > 1$,
- may or may not converge if $L = 1$.

Notice that since $a_n > 0$ and hence $a_{n+1}/a_n > 0$, we must have $L \geq 0$.

Proof: First of all notice that for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ we have $a_{n+1}/a_n = n/(n+1) \to 1$, and the harmonic series diverges. For the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we have $a_{n+1}/a_n = n^2/(n+1)^2 \to 1$. Therefore there exist convergent and divergent series of the above form with $L = 1$, so the test is inconclusive in this case.

Suppose now that $L < 1$. Choose r so that $L < r < 1$. Since $a_{n+1}/a_n \to L$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies a_{n+1}/a_n < r$. Therefore

$$a_{N+1}/a_N < r \implies a_{N+1} < a_N r.$$

and so

$$a_{N+2}/a_{N+1} < r \implies a_{N+2} < a_{N+1} r < a_N r^2.$$

Continuing in this way we see that

$$a_{N+k} < a_N r^k = a_N r^{-N} r^{N+k}$$

for any natural number k. We use the Comparison Test on the series $\sum_{n=1}^{\infty} a_n$ and $a_N r^{-N} \sum_{n=1}^{\infty} r^n$. The inequality above shows that we have $a_n \leq a_N r^{-N} r^n$ for all $n \geq N+1$. Therefore, since these are series of non-negative terms, we see that $\sum_{n=1}^{\infty} a_n$ converges by comparison with $a_N r^{-N} \sum_{n=1}^{\infty} r^n$, since $0 < r < 1$.

In the case $L > 1$ we choose r so that $L > r > 1$; then $a_{N+1} > r a_N > a_N$, and so we see that $a_{N+2} > a_{N+1} > a_N$, ...etc, so that $a_n > a_N > 0$ for all $n > N$. Therefore we cannot have $\lim_{n\to\infty} a_n = 0$ and hence the Vanishing Criterion fails. It follows that $\sum_{n=1}^{\infty} a_n$ diverges. $\blacksquare$