LECTURE 6

Example: Adapt the proof of Lemma 1.7 to prove that the set

$$\{ 2^n \mid n \in \mathbb{N} \}$$

is not bounded above.

Suppose that the set is bounded above; since it is non-empty it has a least upper bound by Axiom III. Suppose that $b = \sup \{ 2^n \mid \ n \in \mathbb{N} \}$. Let $n \in \mathbb{N}$. Then $n+1 \in \mathbb{N}$ and hence $2^{n+1} \leq b$ since b is an upper bound for $\{ 2^n \mid n \in \mathbb{N} \}$. But then $2(2^n) \leq b$ and hence $2^n \leq b/2$. Since this is true for all $n \in \mathbb{N}$ it follows that $b/2$ is an upper bound for $\{ 2^n \mid n \in \mathbb{N} \}$. But this is a contradiction since $b/2 < b$ and b is supposed to be the least upper bound. Therefore $\{ 2^n \mid n \in \mathbb{N} \}$ is not bounded above.

Example: does the subset $[0, \sqrt{2}] \cap \mathbb{Q}$ of $\mathbb{R}$ have a

(a) supremum?
(b) infimum?
(c) maximum?
(d) minimum?

The set $S = [0, \sqrt{2}] \cap \mathbb{Q}$ is non-empty and bounded, therefore it has a supremum. In fact $\sup(S) = \sqrt{2}$; at this point it is clear that $\sqrt{2}$ is an upper bound for S, but it is not so clear that $\sqrt{2}$ is the least upper bound. The set S is non-empty and bounded below, therefore it has an infimum. In fact $\inf(S) = 0$ but again it is not so easy to justify this at this point. Shortly in the lecture we'll clear this up. The set S does not have a maximum; this is because $\sqrt{2} = \sup(S)$ but $\sqrt{2} \notin S$ since $\sqrt{2}$ is irrational. However S does have a minimum; this is because $0 = \inf(S)$ and $0 \in S$ since 0 is rational.

Theorem 1.10: For every pair of real numbers a, b with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

(We say that $\mathbb{Q}$ is dense in $\mathbb{R}$.)

Proof: Suppose to begin with that $0 \leq a < b$. We want to find $m, n \in \mathbb{N}$ such that $a < m/n < b$. Since $a < b$ and hence $b - a > 0$ it follows by the Archimedean Property of $\mathbb{R}$ that there exists $n \in \mathbb{N}$ such that $1/n < b - a$. Hence $na + 1 < nb$. Let $m \in \mathbb{N}$ be the smallest natural number such that $na < m$ (clearly the set of such m is non-empty since $\mathbb{N}$ is bounded above, and so there is such a smallest number by the Well-Ordering Property of $\mathbb{N}$). Hence $a < m/n$. This is one of the two inequalities that we need. Since m is the smallest natural number with this property, we must have $m - 1 \leq na < m$. Hence $m < na + 1$ and so $m < nb$ since $na + 1 < nb$. Therefore $m/n < b$. Thus we have the two inequalities $a < m/n$ and $m/n < b$, in other words $a < m/n < b$.

Suppose now that $a < 0$. There are two possibilities: $a < b \leq 0$ or $a < 0 < b$. If $a < 0 < b$ then we are done since $0 \in \mathbb{Q}$. Suppose then that $a < b \leq 0$. Then $0 \leq -b < -a$ and hence there exists $r \in \mathbb{Q}$ such that $-b < r < -a$ by the case we have proved above. Therefore $a < -r < b$. Since $r \in \mathbb{Q}$, $-r \in \mathbb{Q}$ and so we have found a rational number between a and b in this case too.

■

Example: Let's revisit the second example above in light of this Theorem. We can now prove that $\sqrt{2} = \sup(S)$, where $S = [0, \sqrt{2}] \cap \mathbb{Q}$. We've seen that $\sqrt{2}$ is an upper bound, let's show that it is the least upper bound. Suppose that $0 < b < \sqrt{2}$ is an upper bound. By Theorem 1.10, there exists $r \in \mathbb{Q}$ such that $b < r < \sqrt{2}$. Hence b cannot be an upper bound. Therefore $b \geq \sqrt{2}$ since clearly b cannot be ≤ 0. The proof that $0 = \inf(S)$ is similar.

Example: the set $S = \{ r \in \mathbb{Q} \mid r \geq 1 \text{ and } r^2 < 2 \}$ does not have a least upper bound in $\mathbb{Q}$. In other words, there does not exist $b \in \mathbb{Q}$ such that b is a least upper bound for S. Suppose instead that there did exist such a b. Then either $1 < b < \sqrt{2}$ or $b > \sqrt{2}$ since $\sqrt{2} \notin \mathbb{Q}$. Suppose that $1 < b < \sqrt{2}$. By Theorem 1.10 there exists $r \in \mathbb{Q}$ such that $b < r < \sqrt{2}$. Hence b cannot be an upper bound for S. On the other hand $b > \sqrt{2}$ then by Theorem 1.10 there exists $q \in \mathbb{Q}$ such that $\sqrt{2} < q < b$. But then q is an upper bound for S which is smaller than the least upper bound — contradiction. Therefore our assumption that S has a least upper bound in $\mathbb{Q}$ is false.

Example: Suppose $x \notin \mathbb{Q}$. Then $x = \sup \{ q \in \mathbb{Q} \mid q < x \}$. Clearly x is an upper bound for $S = \{ q \in \mathbb{Q} \mid q < x \}$. Let $b = \sup(S)$. Then $b \leq x$ since x is an upper bound for S. Suppose that $b < x$. Then by Theorem 1.10 there exists $q \in \mathbb{Q}$ such that $b < q < x$. But then b cannot be an upper bound for S — contradiction. Therefore $b = x$.

Corollary: If $a, b \in \mathbb{R}$ with $a < b$ then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.

Proof: Since $a < b$ we have $a - \sqrt{2} < b - \sqrt{2}$. Therefore, by Theorem 1.10 there exists $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. Therefore $a < r + \sqrt{2} < b$. We must have $r + \sqrt{2} \notin \mathbb{Q}$ since otherwise $(r + \sqrt{2}) - r = \sqrt{2} \in \mathbb{Q}$, which is a contradiction. ■

Thus the irrational numbers are also densely packed in $\mathbb{R}$!

Before we state the last result of this lecture, let's recall some set theoretic notation.

Notation: if $A_1, A_2, A_3, \ldots$ are subsets of $\mathbb{R}$, then we define

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{ x \in \mathbb{R} \mid x \in A_n \text{ for all } n \in \mathbb{N} \}$$

and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{ x \in \mathbb{R} \mid \text{there exists } n \in \mathbb{N} \text{ such that } x \in A_n \}.$$

Theorem 1.11: (the Nested Interval Property of $\mathbb{R}$) For each $n \in \mathbb{N}$, suppose that $I_n = [a_n, b_n]$ is a closed interval for some $a_n, b_n \in \mathbb{R}$ with $a_n \leq b_n$. Suppose that $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof: Let $A = \{ a_n \mid n \in \mathbb{N} \}$. Let b_m be the end point of I_m. We will show that b_m is an upper bound for A. If $n \geq m$ then $I_n \subset I_m$ and hence $a_n \leq b_m$. If $n < m$ then $a_n \leq a_m$ since $I_m \subset I_n$. Hence $a_n \leq a_m \leq b_m$ and so $a_n \leq b_m$. Therefore b_m is an upper bound for A. Let $x = \sup(A)$. Then $a_n \leq x$ for all $n \in \mathbb{N}$ since x is an upper bound for A. But we have seen above that each b_n is an upper bound for A. Hence $x \leq b_n$ for all $n \in \mathbb{N}$ since x is the least upper bound for A. Therefore $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$. Hence $x \in I_n$ for all $n \in \mathbb{N}$; hence $x \in \bigcap_{n=1}^{\infty} I_n$. Hence $\bigcap_{n=1}^{\infty} I_n$ is non-empty. ■