APP MTH 3001 Applied Probability III
Class Exercise 5

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1. In a random walk on the non-negative integers starting at the origin, the size X_n of the n^{th} step, $n \geq 1$ has distribution

$$\mathrm{P}(X_n = j) = \frac{e^{-1}}{j!}, \quad j \geq 0.$$

Define

$$\begin{aligned} S_0 &= 0, \\ S_n &= \sum_{i=1}^{n} X_i, \quad n \geq 1, \\ Y_n &= S_n - n, \quad n \geq 0. \end{aligned}$$

Show that $\{Y_n : n \in \mathbb{N}\}$ is a martingale wrt $\{X_n : n \in \mathbb{N}\}$.

Solution Clearly the X_ns are independent. Martingale if $E[|Y_n|] < \infty$, and

$$E[Y_{n+1}|X_0, \dots, X_n] = Y_n$$

Noting that $E(X) = \sum_{i=0}^{\infty} x_i p_i = \sum_{j=0}^{\infty} \frac{je^{-1}}{j!}$

Checking convergence of $E(X)$ (using ratio test):

$$\begin{aligned} \lim_{j\to\infty} \left(\frac{\frac{(j+1)e^{-1}}{(j+1)!}}{\frac{je-1}{j!}} \right) &= \lim_{j\to\infty} \left(\frac{(j+1)j!}{j(j+1)!} \right) \\ &= \lim_{j\to\infty} \left(\frac{1}{j} \right) \\ &= 0 \end{aligned}$$

So the series converges.

$$\begin{aligned} E[|Y_n|] &= E\left[\Big| \ S_n - n \ \Big| \right] \\ &= E\left[\left| \sum_{i=1}^{n} X_i - n \right| \right] \\ &= \begin{cases} E\left[\sum_{i=1}^{n} X_i - n\right] & \sum_{i=1}^{n} X_i - n >= 0 \\ E\left[-\sum_{i=1}^{n} X_i + n\right] & \sum_{i=1}^{n} X_i - n < 0 \end{cases} \\ &= \begin{cases} \sum_{i=1}^{n} E[X_i] - n & \sum_{i=1}^{n} X_i - n >= 0 \\ -\sum_{i=1}^{n} E[X_i] + n & \sum_{i=1}^{n} X_i - n < 0 \end{cases} \end{aligned}$$

Since $E(X_i)$ is convergent, a finite sum of it must be finite. Therefore, $E(|Y_n|) < \infty$. Now the martingale property:
Aside: Recall, $\sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x$ In this case, let $x_j = 1$, i.e.:

$$\sum_{j=0}^{\infty} \frac{e^{-1}}{j!} = e^{-1} \sum_{j=0}^{\infty} \frac{1}{j!} = e^{-1} e^1 = 1$$

For the expectation, we have:

$$E(X) = \sum_{j=1}^{\infty} j \frac{e^{-1}}{j!} = \sum_{j=1}^{\infty} \frac{e^{-1}}{(j-1)!} = 1$$

End aside

$$\begin{aligned}
E[Y_{n+1}|X_0, \ldots, X_n] &= E\left[\sum_{i=1}^{n+1} X_i - (n+1) | X_0 \ldots, X_n\right] \\
&= E\left[\sum_{i=1}^{n+1} X_i | X_0 \ldots, X_n\right] - (n+1) \\
&= \sum_{i=1}^{n+1} E\left[X_i | X_0 \ldots, X_n\right] - (n+1) \\
&= \sum_{i=1}^{n} X_i - n + E[X_{n+1}] - 1 \\
&= Y_n + 1 - 1 \\
&= Y_n
\end{aligned}$$

Therefore Y_n is a martingale.
As required.

2. If $\{X_n : n \in \mathbb{N}\}$ is a martingale wrt to itself, show that for any non-negative integers, $k \leq \ell < m$, the difference $X_m - X_\ell$ is uncorrelated with X_k. That is, show that

$$E\left[\left(X_m - X_\ell\right) X_k\right] = 0.$$

Solution Note that we are given, $k \leq \ell < m$.
If X_n is a martingale w.r.t itself, then we know

$$E(X_{n+1}|X_0, \ldots, X_n) = X_n$$

Tower property:

$$E(X) = E(E(X|Y))$$

$$\begin{aligned}
E\left[\left(X_m - X_\ell\right) X_k\right] &= E\left[E\left[\left(X_m - X_\ell\right) X_k \;\middle|\; X_0, \ldots, X_\ell\right]\right] \quad \text{tower property} \\
&= E\left[E\left[X_m X_k \;\middle|\; X_0, \ldots, X_\ell\right] - E\left[X_\ell X_k \;\middle|\; X_0, \ldots, X_\ell\right]\right] \\
&= E\left[X_k E\left[X_m \;\middle|\; X_0, \ldots, X_\ell\right] - X_\ell X_k\right] \quad \text{since } k \leq \ell,\ X_k \text{ is given} \\
&= E\left[X_k X_\ell - X_\ell X_k\right] \\
&= E(0) = 0
\end{aligned}$$

Used the fact that:

$$E[X_{n+k}|Y_0, \ldots, Y_n] = X_n \quad a.s.$$

As required.

3. Let $\{X_n : n \in \mathbb{N}\}$ be a DTMC on the state space $\mathcal{S}$ with one-step transition probability matrix $\mathbb{P} = (p_{ij})$ and let $f : \mathcal{S} \to \mathbb{R}$ be a bounded function. Then, define

$$M_n = \sum_{m=1}^{n} f(X_m) - \sum_{m=0}^{n-1} \sum_{i \in \mathcal{S}} p_{X_m,i} f(i).$$

Show that $\{M_n : n \in \mathbb{N}\}$ is a martingale wrt $\{X_n : n \in \mathbb{N}\}$.

Solution Note that $p_{i,j} \geq 0$. Since f is a bounded function, $|f(x)| < \infty$, $\forall x \in \mathcal{S}$.
Show the expectation is bounded:

$$\begin{aligned} E\left(|M_n|\right) &= E\left(\left| \sum_{m=1}^{n} f(X_m) - \sum_{m=0}^{n-1} \sum_{i \in \mathcal{S}} p_{X_m,i} f(i) \right|\right) \\ &\leq E\left(\sum_{m=1}^{n} |f(X_m)| - \sum_{m=0}^{n-1} \sum_{i \in \mathcal{S}} |p_{X_m,i} f(i)|\right) \\ &\leq \sum_{m=1}^{n} E(|f(X_m)|) - \sum_{m=0}^{n-1} 1 \sum_{i \in \mathcal{S}} E(|f(i)|) \\ &= \sum_{m=1}^{n} E(|f(X_m)|) - n \sum_{i \in \mathcal{S}} E(|f(i)|) \\ &< \infty \end{aligned}$$

As a finite sum of a bounded function is bounded.

$$\begin{aligned} E(M_{n+1}|X_0, \dots, X_n) &= E\left(M_n + f(X_{N+1}) - \sum_{i \in \mathcal{S}} p_{X_n,i} f(i) \;\middle|\; X_0, \dots, X_n\right) \\ &= E\left(M_n \;\middle|\; X_0, \dots, X_n\right) + E\left(f(X_{N+1}) - \sum_{i \in \mathcal{S}} p_{X_n,i} f(i) \;\middle|\; X_0, \dots, X_n\right) \\ &= M_n + E(f(X_{N+1})|X_n) - E\left(\sum_{i \in \mathcal{S}} p_{X_n,i} f(i) \;\middle|\; X_0, \dots, X_n\right) \quad \text{memoryless} \\ &= M_n + \sum_{i \in \mathcal{S}} p_{X_n,i} f(i) - \sum_{i \in \mathcal{S}} p_{X_n,i} f(i) \text{ markov} \\ &= M_n \end{aligned}$$

Therefore $\{M_n : n \in \mathbb{N}\}$ is a martingale wrt $\{X_n : n \in \mathbb{N}\}$.

As required.

4. If $\{X_n : n \in \mathbb{N}\}$ is a sub-martingale wrt $\{Y_n : n \in \mathbb{N}\}$ and $Z \geq 0$ is a (measurable) function of $Y_0, \dots, Y_n$, show that

$$E[X_n Z] \leq E[X_{n+1} Z]$$

Solution Pretty sure it should start from Y_1 given $0 \notin \mathbb{N}$...
Sub-martingale means:

$$E(|X_n|) < \infty$$

$$E[X_{n+1}|Y_0, \dots, Y_n] \geq X_n$$

$$\begin{aligned}
E[X_{n+1}Z] &= E[E[X_{n+1}Z|Y_0, \dots Y_n]] \text{ tower property} \\
&= E[E[X_{n+1}|Y_0, \dots, Y_n][E[Z|Y_0, \dots Y_n]] \text{ indep} \\
&= E[ZE[X_{n+1}|Y_0, \dots, Y_n]] \\
&= E[ZE[X_{n+1}|Y_0, \dots, Y_n]] \\
&\geq E[ZX_n]
\end{aligned}$$

I.e.

$$E[X_n Z] \leq E[X_{n+1}Z]$$

As required.