LECTURE 33

Last time we proved:

Theorem 8.5: Suppose $f_n\colon [a,b] \to \mathbb{R}$ is bounded for all $n \in \mathbb{N}$ and that $f_n \to f$ uniformly on $[a,b]$ for some function $f\colon [a,b] \to \mathbb{R}$. If f_n is integrable on $[a,b]$ for all n, then f is integrable on $[a,b]$.

We would like to have a way of calculating the integral $\int_a^b f(x)dx$ in terms of the integrals $\int_a^b f_n(x)dx$. The next proposition deals with this:

Proposition 8.6: Suppose that $f_n\colon [a,b] \to \mathbb{R}$ is integrable for all $n \in \mathbb{N}$ and that $f_n \to f$ uniformly on $[a,b]$. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Proof: Since $f_n \to f$ uniformly on $[a,b]$, we have $M_n \to 0$, where $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Therefore

$$\begin{aligned}\left|\int_a^b f(x)dx - \int_a^b f_n(x)dx\right| &= \left|\int_a^b (f(x) - f_n(x))dx\right| \\ &\leq \int_a^b |f(x) - f_n(x)|\ dx \\ &\leq \int_a^b M_n dx \\ &= M_n(b-a)\end{aligned}$$

since $|f(x) - f_n(x)| \leq M_n$ for all $x \in [a,b]$. Since $M_n \to 0$ the Squeeze Theorem shows that

$$\left|\int_a^b f(x)dx - \int_a^b f_n(x)dx\right| \to 0,$$

and hence

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

■

It is surprisingly more difficult to find conditions which guarantee that if a sequence of differentiable functions f_n converges to a function f, then f is differentiable. As we have seen, even if $f_n \to f$ uniformly this is not enough to guarantee that f is differentiable.

The following theorem, which we shall not prove, gives sufficient conditions to ensure that f is differentiable.

Theorem 8.7: Suppose $f_n\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ for all n, and that $f_n' \to g$ uniformly on $[a,b]$ for some function $g\colon [a,b] \to \mathbb{R}$. If $(f_n(x_0))$ converges for some $x_0 \in [a,b]$, then $f_n \to f$ uniformly on $[a,b]$ for some function $f\colon [a,b] \to \mathbb{R}$, differentiable on $[a,b]$, and $f' = g$.

Recall that a sequence of real numbers converges if and only if it is a Cauchy sequence. For uniform convergence of sequences of functions there is a similar phenomenon.

Definition 8.8: Let (f_n) be a sequence of functions $f_n \colon S \to \mathbb{R}$. We say (f_n) is *uniformly Cauchy* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n \geq N$ then $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in S$.

Note: if (f_n) is a uniformly Cauchy sequence of functions $f_n \colon S \to \mathbb{R}$, then for every $x \in S$ the sequence of real numbers $(f_n(x))$ is a Cauchy sequence.

Proposition 8.9: Let (f_n) be a sequence of functions $f_n \colon S \to \mathbb{R}$. Then (f_n) converges uniformly if and only if (f_n) is uniformly Cauchy.

Proof: Suppose first that (f_n) converges uniformly on S, say $f_n \to f$ uniformly on S. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in S$. Suppose $m, n \geq N$. Then, by the triangle inequality,

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $x \in S$. Hence (f_n) is uniformly Cauchy.

Now suppose that (f_n) is uniformly Cauchy. Let $x \in S$. Then $(f_n(x))$ is a Cauchy sequence of real numbers and hence is convergent. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Thus we have a function $f \colon S \to \mathbb{R}$. We will prove that $f_n \to f$ uniformly on S. Let $\epsilon > 0$. Since (f_n) is uniformly Cauchy we may choose $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n \geq N$ then $|f_m(x) - f_n(x)| < \epsilon/2$. Suppose $n \geq N$. Let $x \in S$. Then for any $k \in \mathbb{N}$, $n + k \geq N$ and hence

$$|f_{n+k}(x) - f_n(x)| < \epsilon/2.$$

Since the sequence of real numbers $(f_n(x))$ converges to $f(x)$, and the absolute value function is continuous, we have

$$\lim_{k\to\infty} |f_{n+k}(x) - f_n(x)| = |f(x) - f_n(x)| \leq \epsilon/2 < \epsilon.$$

Since $x \in S$ was arbitrary, it follows that we have show that $n \geq N$ implies $|f(x) - f_n(x)| < \epsilon$ for all $x \in S$. Hence $f_n \to f$ uniformly on S. ■