Last time :

Th^{m} 5.6 : Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then f is integrable on $[a,b] \iff \forall\, c \in [a,b]$ f is integrable on $[a,c]$ & f is integrable on $[b,c]$. In this case

$$\int_a^b f(x)\,dx = \int_c^b f(x)\,dx + \int_a^c f(x)\,dx .$$

$$\int_a^a f := 0$$

$$\int_b^a f := -\int_a^b f$$

Pf: ($\Leftarrow$) (Use Th^{m} 5.3) Let $\varepsilon > 0$. Since f is int on $[a,c]$ $\exists$ partition $\mathcal{P}_1$ of $[a,c]$ s.th.

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2},$$

since f is int on $[c,b]$ $\exists$ part. $\mathcal{P}_2$ of $[c,b]$ s.th.

$$U(f,\mathcal{P}_2) - L(f,\mathcal{P}_2) < \frac{\varepsilon}{2}.$$

Let $\mathcal{P}_\varepsilon = \underbrace{\mathcal{P}_1}_{\{a,\dots,c\}} \cup \underbrace{\mathcal{P}_2}_{\{c,\dots,b\}}$. Then $\mathcal{P}_\varepsilon$ is a partition of $[a,b]$.

$$U(f,\mathcal{P}_\varepsilon) = U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2).$$

$$\hookrightarrow = \sum M_i(f)\Delta x_i$$

$$L(f,\mathcal{P}_\varepsilon) = L(f,\mathcal{P}_1) + L(f,\mathcal{P}_2).$$

$$\therefore\ U(f,\mathcal{P}_\varepsilon) - L(f,\mathcal{P}_\varepsilon) = U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) + U(f,\mathcal{P}_2) - L(f,\mathcal{P}_2)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ f is int. on $[a,b]$ by Th^{m} 5.3.

($\Rightarrow$). Suppose f int. on $[a,b]$. Without loss of generality $a < c < b$. Let $\varepsilon > 0$. Choose a partition $\mathcal{P}_\varepsilon$ of $[a,b]$ s.th. $U(f,\mathcal{P}_\varepsilon) - L(f,\mathcal{P}_\varepsilon) < \varepsilon$.

Let $P_\varepsilon' = P_\varepsilon \cup \{c\}$. Then P_ε' is a refinement of P_ε

$\therefore \quad U(f, P_\varepsilon') - L(f, P_\varepsilon') \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$

Let $P_1 = P_\varepsilon' \cap [a,c]$, $P_2 = P_\varepsilon' \cap [c,b]$. Then P_1 & P_2 are partitions of $[a,c]$ & $[c,b]$ resp.

$$U(f, P_\varepsilon') = U(f, P_1) + U(f, P_2)$$

$$L(f, P_\varepsilon') = L(f, P_1) + L(f, P_2)$$

$$\therefore \quad \underbrace{U(f, P_1) - L(f, P_1)}_{\geq 0} + \underbrace{U(f, P_2) - L(f, P_2)}_{\geq 0} = U(f, P_\varepsilon') - L(f, P_\varepsilon') < \varepsilon.$$

$$\Rightarrow \quad U(f, P_1) - L(f, P_1) < \varepsilon \quad \& \quad U(f, P_2) - L(f, P_2) < \varepsilon.$$

$\therefore$ f is int. on $[a,c]$ & f is int. on $[c,b]$

$$\int_a^c f + \int_c^b f \leq U(f, P_1) + U(f, P_2) = U(f, P_\varepsilon') < L(f, P_\varepsilon') + \varepsilon$$
$$\leq \int_a^b f + \varepsilon.$$

$$\therefore \quad \int_a^c f + \int_c^b f < \int_a^b f + \varepsilon. \quad \forall \varepsilon > 0.$$

$$\therefore \quad \int_a^c f + \int_c^b f \leq \int_a^b f. \quad —$$

$$\int_a^b f \leq \cancel{L} U(f, P_\varepsilon') < L(f, P_\varepsilon') + \varepsilon = L(f, P_1) + L(f, P_2) + \varepsilon$$
$$\leq \int_a^c f + \int_c^b f + \varepsilon.$$

$$\therefore \quad \int_a^b f < \int_a^c f + \int_c^b f + \varepsilon. \quad \forall \varepsilon > 0.$$

$$\therefore \quad \int_a^b f \leq \int_a^c f + \int_c^b f \quad —$$

$$\therefore \quad \int_a^b f = \int_a^c f + \int_c^b f.$$

$\underline{\text{Th}^{m}\ 5.7}$: If $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ then f is int. on $[a,b]$.

$\underline{\text{Pf}}$: f cts on $[a,b]$, f uniformly cts on $[a,b]$ ($[a,b]$ is closed ~~&~~ bded $\therefore$ seq. compact). Let $\varepsilon > 0$.
Since f uniformly cts on $[a,b]$ $\exists\ \delta > 0$ s.th.
$\forall\ x, y \in [a,b]$ if $|x-y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Let P_ε be any partition of $[a,b]$ s.th. the length of the largest subinterval in P_ε is $< \delta$.
Let $I = [x_{i-1}, x_i]$. Then

$$M_i(f) - m_i(f) = f(x_2) - f(x_1) \leq |f(x_2) - f(x_1)| < \frac{\varepsilon}{b-a}$$

for some $x_1, x_2 \in [x_{i-1}, x_i]$ (f cts on $[a,b] \Rightarrow$ $f|_{[x_{i-1}, x_i]}$ is cts on $[x_{i-1}, x_i]$ & attains its max. & min. on $[x_{i-1}, x_i]$.).

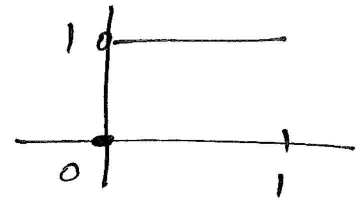
length of $[x_{i-1}, x_i]$ is less than δ.

$$\begin{aligned}
\therefore U(f, P_\varepsilon) - L(f, P_\varepsilon) &= \sum_{i=1}^{N} (M_i(f) - m_i(f)) \Delta(x_i) \\
&< \sum_{i=1}^{N} \frac{\varepsilon}{b-a} \Delta(x_i). \\
&= \frac{\varepsilon}{b-a} \sum_{i=1}^{N} \Delta(x_i) \\
&= \frac{\varepsilon}{b-a} \cdot b-a \\
&= \varepsilon.
\end{aligned}$$

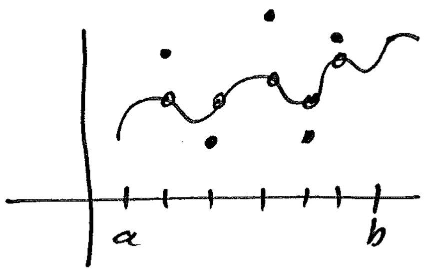
$\therefore f$ is int. on $[a,b]$ by Th^{m} 5.3.

Ex. $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



f is int.



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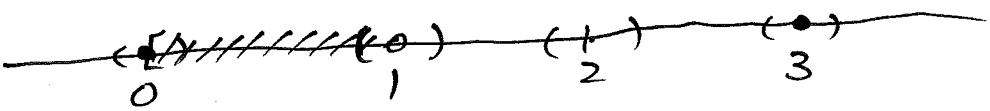
§ 6 Differentiability.

Def^n 6.1 : Let $S \subset \mathbb{R}$, $S \neq \emptyset$. A real number x_0 is said to be a <u>limit point</u> of S if $\forall \varepsilon > 0$ $\exists\, x \in S \setminus \{x_0\}$ s.th. $x \in I_\varepsilon(x_0)$.

(Note: x_0 need not belong to S).

Ex 1 : $S = [0,1) \cup \{3\}$.



- 0 is a limit point of S.
- 0.5 ———— " ————
- 1 ———— " ————
- 2 is not a limit point of S
- 3 ———— " ————

Ex 2. $S = \left\{ \frac{(-1)^n n}{n+1} \;\middle|\; n = 1, 2, 3, \ldots \right\}$

$= \left\{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots \right\}$

The only limit points of S are ± 1.

$\underline{\text{Prop}^{n}\ 6.2}$: Let $S \subset \mathbb{R}$, $S \neq \emptyset$, $x_0 \in \mathbb{R}$.

x_0 is a limit point of S

$\iff \exists$ seq. (x_n) in $S \setminus \{x_0\}$ s.th.

$x_n \longrightarrow x_0$.

$\underline{\text{Pf}}$: $(\Longrightarrow)$. Let $\varepsilon = \frac{1}{n}$, $n = 1, 2, 3, \ldots$ Since x_0 is a limit point of S, $\exists\, x_n \in S \setminus \{x_0\}$ s.th. $x_n \in I_{\frac{1}{n}}(x_0)$, i.e. $|x_0 - x_n| < \frac{1}{n}$ .... $\therefore x_n \rightarrow x_0$.

$(\Longleftarrow)$ Let $\varepsilon > 0$. Since $x_n \rightarrow x_0$ $\exists\, N \in \mathbb{N}$ s.th. $n \geqslant N \Rightarrow |x_n - x_0| < \varepsilon$. In particular $x_N \in I_\varepsilon(x_0)$ & $x_N \in S \setminus \{x_0\}$.