Topic C Assignment 4

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1. Use multiple scales to solve
$$y'' + y + \epsilon(y')^3 = 0$$
$\epsilon \ll 1$, $y(0) = 1$ and $y'(0) = 0$.

   Let $y(\tau) \sim y_0(t, T)$ where $T = \epsilon t$ is a slow timescale.

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T}$$
$$\frac{\partial^2}{\partial \tau^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2}$$

   Subbing this into the ODE gives:

$$\frac{d^2 y}{dt^2} + y + \epsilon \left(\frac{dy}{dt}\right)^3 = 0$$
$$\frac{\partial^2 y}{\partial t^2} + 2\epsilon \frac{\partial^2 y}{\partial t \partial T} + \epsilon^2 \frac{\partial^2 y}{\partial T^2} + y + \epsilon \left(\frac{\partial y}{\partial t} + \epsilon \frac{\partial y}{\partial T}\right)^3 = 0$$

   With initial conditions

$$y_0(0,0) = 1$$
$$\frac{\partial y_0(0,0)}{\partial t} = 0$$

   And

$$y_1(0,0) = 0$$
$$\frac{\partial y_1(0,0)}{\partial t} + y_0(0,0) = 0$$

   To leading order

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$$
$$y_0 = R(T)\cos(t + \theta(T))$$

   Boundary conditions:

$$y_0(0,0) = 1 \implies R(0) = 1$$
$$\frac{\partial y_0(0,0)}{\partial t} = 0 \implies R(0)(-\sin(\theta(0))) = 0 \implies \theta(0) = 0$$

To obtain the full forms of R and θ, find the second order:

$$\begin{aligned}
&\frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial^2 y_0}{\partial t \partial T} + y_1 + \left(\frac{\partial y_0}{\partial t}\right)^3 = 0 \\
&\frac{\partial^2 y_1}{\partial t^2} + 2\left(-R'(T)\sin(t+\theta(T)) - R(T)\cos(t+\theta(T))\theta'(T)\right) \\
&+ y_1 + \left(-R(T)\sin(t+\theta(T))\right)^3 = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= 2R'\sin(t+\theta) + 2R\theta'\cos(t+\theta) + R^3\sin^3(t+\theta) \\
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= 2R'\sin(t+\theta) + 2R\theta'\cos(t+\theta) + \frac{R^3}{4}\left(3\sin(t+\theta) - \sin(3(t+\theta))\right) \\
\frac{\partial^2 y_1}{\partial t^2} + y_1 &= (2R' + \frac{3}{4}R^3)\sin(t+\theta) + 2R\theta'\cos(t+\theta) - R^3\left(\sin(3(t+\theta))\right)
\end{aligned}$$

Hence we require

$$\begin{aligned}
(2R' + \frac{3}{4}R^3) &= 0 \\
2R\theta' &= 0
\end{aligned}$$

For non-trivial solutions this means

$$\begin{aligned}
\theta' = 0 \implies \theta &= c \\
2R' + \frac{3}{4}R^3 &= 0 \\
\frac{R'}{R^3} &= -\frac{3}{8} \\
-\frac{1}{2R^2} &= -\frac{3}{8}T + d_* \\
2R^2 &= \frac{1}{\frac{3}{8}T - d_*} \\
R &= \pm\frac{1}{\sqrt{\frac{3}{4}T + d}}
\end{aligned}$$

And using the condition from before, $R(0) = 1$

$$\begin{aligned}
R &= \frac{1}{\sqrt{d}} \\
\implies d &= 1
\end{aligned}$$

Hence

$$\boxed{y_0 = \frac{1}{\sqrt{3T+1}}\cos(t)}$$

Figure 1 shows the two solutions obtained. Clearly the two overlap very nicely even for $\epsilon = 0.1$.

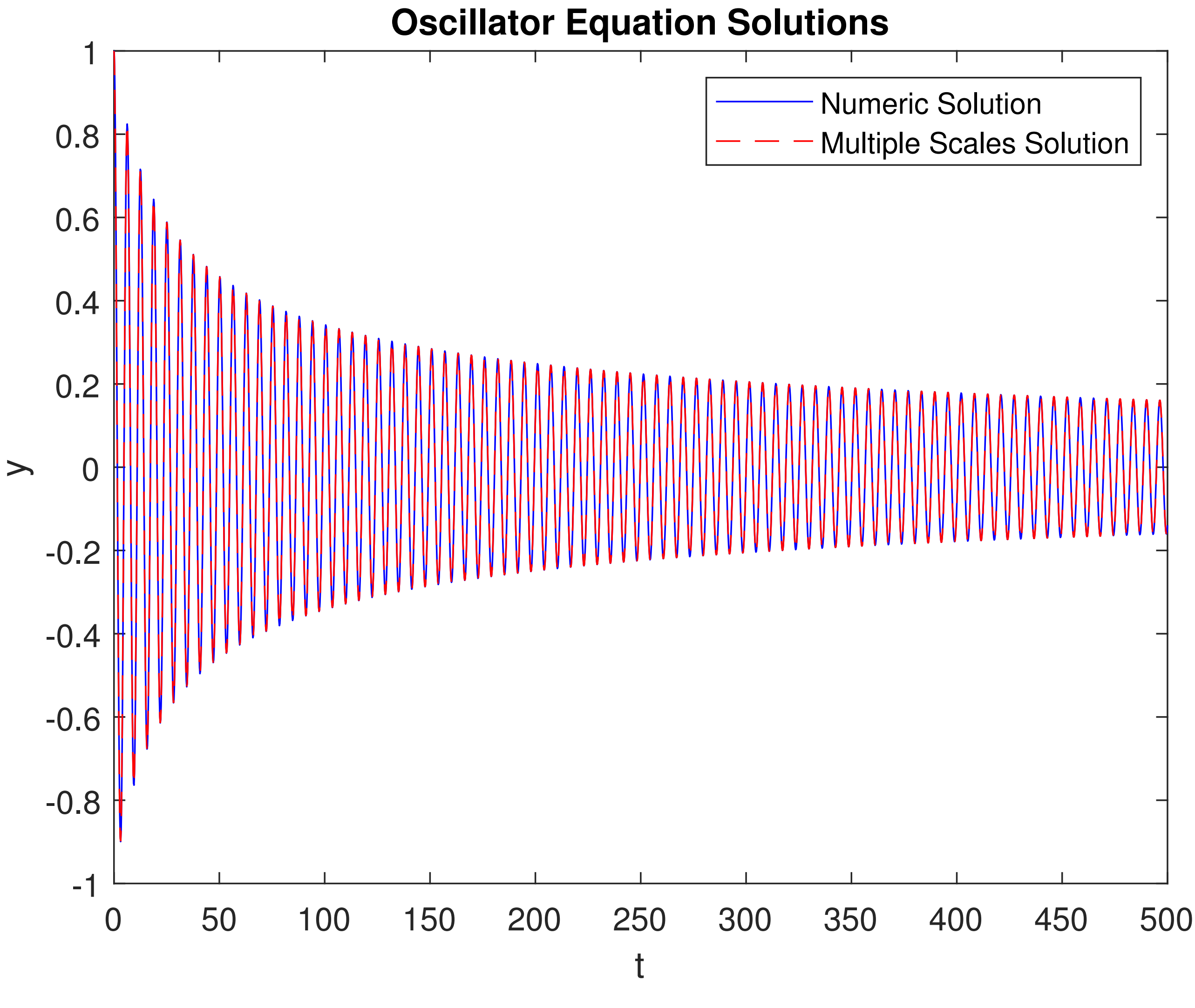

Figure 1: Comparison of numerical and multi-scale solutions for $\epsilon = 0.1$

2.
$$\frac{d^2y}{dt^2}+\epsilon(y^2-1)\frac{dy}{dt}+y=0,\quad y(0)=1,\ y'(0)=0,\quad \epsilon\ll 1$$

(a)
$$y(t)=y(t,T,\tau)$$

$T=\epsilon t$ and $\tau=\epsilon^2 t$. This gives the partial derivative expansions:

$$\begin{aligned}
\frac{d}{dt}&=\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T}+\epsilon^2\frac{\partial}{\partial \tau}\\
\frac{d^2}{dt^2}&=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T}+\epsilon^2\frac{\partial}{\partial \tau}\right)+\epsilon\frac{\partial}{\partial T}\left(\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T}+\epsilon^2\frac{\partial}{\partial \tau}\right)+\epsilon^2\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial T}+\epsilon^2\frac{\partial}{\partial \tau}\right)\\
&=\frac{\partial^2}{\partial t^2}+\epsilon^2\frac{\partial^2}{\partial T^2}+\epsilon^4\frac{\partial^2}{\partial \tau^2}+2\epsilon\frac{\partial}{\partial t\partial T}+2\epsilon^2\frac{\partial}{\partial t\partial \tau}+2\epsilon^3\frac{\partial}{\partial T\partial \tau}
\end{aligned}$$

Hence the ODE becomes

$$\frac{\partial^2 y}{\partial t^2}+\epsilon^2\frac{\partial^2 y}{\partial T^2}+\epsilon^4\frac{\partial^2 y}{\partial \tau^2}+2\epsilon\frac{\partial y}{\partial t\partial T}+2\epsilon^2\frac{\partial y}{\partial t\partial \tau}+2\epsilon^3\frac{\partial y}{\partial T\partial \tau}+\epsilon(y^2-1)\left(\frac{\partial y}{\partial t}+\epsilon\frac{\partial y}{\partial T}+\epsilon^2\frac{\partial y}{\partial \tau}\right)+y=0$$

With boundary conditions

$$y(0,0,0)=1$$
$$\frac{\partial y(0,0,0)}{\partial t}+\epsilon\frac{\partial y(0,0,0)}{\partial T}+\epsilon^2\frac{\partial y(0,0,0)}{\partial \tau}=0$$

(b) First expand the PDE

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2}+\epsilon^2\frac{\partial^2 y}{\partial T^2}+\epsilon^4\frac{\partial^2 y}{\partial \tau^2}+2\epsilon\frac{\partial y}{\partial t\partial T}+2\epsilon^2\frac{\partial y}{\partial t\partial \tau}+2\epsilon^3\frac{\partial y}{\partial T\partial \tau}\\
&+y^2\epsilon\frac{\partial y}{\partial t}+y^2\epsilon^2\frac{\partial y}{\partial T}+y^2\epsilon^3\frac{\partial y}{\partial \tau}-\left(\epsilon\frac{\partial y}{\partial t}+\epsilon^2\frac{\partial y}{\partial T}+\epsilon^3\frac{\partial y}{\partial \tau}\right)+y=0
\end{aligned}$$

We are only considering up to $\mathcal{O}(\epsilon^2)$, so dropping ϵ^3 and higher terms

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2}+\epsilon^2\frac{\partial^2 y}{\partial T^2}+2\epsilon\frac{\partial y}{\partial t\partial T}+2\epsilon^2\frac{\partial y}{\partial t\partial \tau}\\
&+y^2\epsilon\frac{\partial y}{\partial t}+y^2\epsilon^2\frac{\partial y}{\partial T}-\left(\epsilon\frac{\partial y}{\partial t}+\epsilon^2\frac{\partial y}{\partial T}\right)+y=0
\end{aligned}$$

Let
$$y(t,T,\tau)=y_0(t,T,\tau)+\epsilon y_1(t,T,\tau)+\epsilon^2 y_2(t,T,\tau)+\dots$$

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2}+\epsilon^2\frac{\partial^2 y}{\partial T^2}+2\epsilon\frac{\partial y}{\partial t\partial T}+2\epsilon^2\frac{\partial y}{\partial t\partial \tau}\\
&+y^2\epsilon\frac{\partial y}{\partial t}+y^2\epsilon^2\frac{\partial y}{\partial T}-\left(\epsilon\frac{\partial y}{\partial t}+\epsilon^2\frac{\partial y}{\partial T}\right)+y=0
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 y_0}{\partial t^2}+\epsilon\frac{\partial^2 y_1}{\partial t^2}+\epsilon^2\frac{\partial^2 y_2}{\partial t^2}+\epsilon^2\frac{\partial^2 y_0}{\partial T^2}+2\epsilon\frac{\partial y_0}{\partial t\partial T}+2\epsilon^2\frac{\partial y_1}{\partial t\partial T}+2\epsilon^2\frac{\partial y_0}{\partial t\partial \tau}\\
&+y_0^2\epsilon\frac{\partial y_0}{\partial t}+y_0^2\epsilon^2\frac{\partial y_1}{\partial t}+y_0^2\epsilon^2\frac{\partial y_0}{\partial T}-\left(\epsilon\frac{\partial y_0}{\partial t}+\epsilon^2\frac{\partial y_1}{\partial t}+\epsilon^2\frac{\partial y_0}{\partial T}\right)+y_0+\epsilon y_1+\epsilon^2 y_2=0
\end{aligned}$$

$$
\begin{aligned}
\mathcal{O}(1):& \frac{\partial^2 y_0}{\partial t^2} + y_0 = 0\\
\mathcal{O}(\epsilon):& \frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial y_0}{\partial t \partial T} + y_0^2 \frac{\partial y_0}{\partial t} - \frac{\partial y_0}{\partial t} + y_1 = 0\\
\mathcal{O}(\epsilon^2):& \frac{\partial^2 y_2}{\partial t^2} + \frac{\partial^2 y_0}{\partial T^2} + 2\frac{\partial y_1}{\partial t \partial T} + 2\frac{\partial y_0}{\partial t \partial \tau} + y_0^2 \frac{\partial y_1}{\partial t} + y_0 \frac{\partial y_0}{\partial T} - \frac{\partial y_1}{\partial t} - \frac{\partial y_0}{\partial T} + y_2 = 0
\end{aligned}
$$

With boundary conditions

$$
\begin{aligned}
\mathcal{O}(1):& y_0(0,0,0) = 1, \quad \frac{\partial y_0(0,0,0)}{\partial t} = 0\\
\mathcal{O}(\epsilon):& y_1(0,0,0) = 0, \quad \frac{\partial y_1(0,0,0)}{\partial t} + \frac{\partial y_0(0,0,0)}{\partial T} = 0\\
\mathcal{O}(\epsilon^2):& y_2(0,0,0) = 0, \quad \frac{\partial y_2(0,0,0)}{\partial t} + \frac{\partial y_1(0,0,0)}{\partial T} + \frac{\partial y_0(0,0,0)}{\partial \tau} = 0
\end{aligned}
$$

(c) Leading order equation:

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0$$

Gives

$$y_0 = R(T,\tau)\cos(t + \theta(T,\tau))$$

And using the boundary conditions, require $R(0,0) = 1$ and $\theta(0,0) = 0$

(d) To find y_1 first sub y_0 into the $\mathcal{O}(\epsilon)$ equation

$$
\begin{aligned}
\frac{\partial^2 y_1}{\partial t^2} + 2\frac{\partial y_0}{\partial t \partial T} + y_0^2 \frac{\partial y_0}{\partial t} - \frac{\partial y_0}{\partial t} + y_1 = 0\\
\frac{\partial^2 y_1}{\partial t^2} + 2\left(\right) + y_0^2 \frac{\partial y_0}{\partial t} - \frac{\partial y_0}{\partial t} + y_1 = 0
\end{aligned}
$$

(e)

$$\frac{\partial^2 y_2}{\partial t^2} + \frac{\partial^2 y_0}{\partial T^2} + 2\frac{\partial y_1}{\partial t \partial T} + 2\frac{\partial y_0}{\partial t \partial \tau} + y_0^2 \frac{\partial y_1}{\partial t} + y_0 \frac{\partial y_0}{\partial T} - \frac{\partial y_1}{\partial t} - \frac{\partial y_0}{\partial T} + y_2 = 0$$

(f)

Matlab Code

```
clear all
close all
%%Q1

epsilon = 0.1;
[t,yNumeric] = ode45(@Q1OscillatorEqn,[0,500],[1,0],[],epsilon);
T = t*epsilon;
R = 1./sqrt(0.75*T+1);
theta = 0;
```

```
yAsymp = R .* cos(t + theta);
plot(t,yNumeric(:,1),'b')
hold on
plot(t,yAsymp,'--r')
hold off
xlabel('t')
ylabel('y')
legend("Numeric Solution","Multiple Scales Solution")
title('Oscillator Equation Solutions')
saveas(gcf,'TopicCA4Q1.eps','epsc')

%%
%%Q2
epsilon = 0.01;
[t,yNumeric] = ode45(@Q2VanderPol,[0,100],[1,0],[],epsilon);
plot(t,yNumeric(:,1))
xlabel('t')
ylabel('y')
legend("Numeric Solution","Multiple Scales Solution")
title('Van der Pol Oscillator Solutions')

%%
%obtain symbolic solutions

%syms y0(t,T,tau) y1(t,T,tau) y2(t,T,tau) theta(T,tau) R(T,tau) t T tau
%y0 = R*cos(t+theta)
%y1eqn = diff(y1,t,2) + 2*diff(diff(y0,t,1),T,1)+y0^2*diff(y0,t,1) - diff(y0

syms t tau y0(t) y1(t,T,tau) y2(t,T,tau) R(T,tau) theta(T,tau)
dy0 = diff(y0,t);
ddy0 = diff(y0,t,2);
eqn0 = ddy0 + y0 ==0;
cond0= [y0(0)==1, dy0(0) ==0];
y0(t) = dsolve(eqn0,cond0)
y0(t,T,tau) = R*y0(t+theta)
eqn1 =diff(y1,t,2) + 2*diff(y0,t,T) + y0^2* diff(y0,t) - diff(y0,t) + y1 ==0



function dy = Q1OscillatorEqn(t,y,epsilon)

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```
dy = [y(2);-y(1) - epsilon*(y(2)^3)];

end
function dy = Q2VanderPol(t,y,epsilon)

dy = [y(2);-epsilon*(y(1)^2 - 1)*y(2) - y(1)];
end
```

Practical Asymptotics (APP MTH 4051/7087)
Assignment 4 (5%)

Due 27 May 2019

1. Apply the method of multiple scales to find a leading-order solution to the following oscillator equation:

$$y'' + y + \epsilon \left(y'\right)^3 = 0,$$

with $\epsilon \ll 1$, subject to $y(0) = 1$ and $y'(0) = 0$. Seek a solution of the form $y(t) \sim y_0(t, T)$, where $T = \epsilon t$ is a slow timescale. Compare this leading-order solution with a numerical solution and comment.

2. Recall from lectures that the numerical solution to the Van der Pol oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \left(y^2 - 1\right) \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0, \quad y(0) = 1, y'(0) = 0, \quad \epsilon \ll 1,$$

exhibited a phase shift, but the leading-order solution did not. To capture this phase shift we require an additional, extra slow timescale.

   (a) Introduce an extra slow timescale by letting $y(t) \equiv y(t, T, \tau)$, where $T = \epsilon t$ and $\tau = \epsilon^2 t$, then use the chain rule to transform the above ODE into a PDE in terms of these three variables.

   (b) Let $y(t, T, \tau) = y_0(t, T, \tau) + \epsilon y_1(t, T, \tau) + \epsilon^2 y_2(t, T, \tau) + ...$ and write down the leading-order, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ problems, including boundary conditions.

   (c) Find y_0 by solving the leading-order problem and eliminating resonant terms from the $\mathcal{O}(\epsilon)$ equation.
   [Hint: This should include arbitrary functions of τ, but otherwise be identical to that found in lectures (you may reuse working).]

   (d) Having eliminated these resonant terms, find y_1 by solving the $\mathcal{O}(\epsilon)$ problem (in terms of aribtrary functions of T and τ). [Hint: strongly recommend using computer algebra for this and the next part.]

   (e) Identify the resonant terms from the $\mathcal{O}(\epsilon^2)$ equation that contain derivatives of the unknown function of τ in y_0, and set these terms to zero by finding these unknown function. [Hint: One of these is easy to solve, the other needs to be considered in the 'long time' limit as $T \to \infty$.]

   (f) Compare your solution for y_0 with a numerical solution and comment.