THE UNIVERSITY *of* ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2013

101488	**Random Processes III**
	APP MTH 3016

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 70

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators without remote communications facilities are permitted.
- Tables of Laplace Transforms are provided at the end of the Examination question book

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Consider a Continuous Time Markov chain (CTMC) $\{X(t), t \geq 0\}$ on the finite state space S. For all $i, j \in S$ and $s, t \geq 0$, let $P_{ij}(t) = P\left(X(t+s) = j | X(s) = i\right)$.

   (a) Define the *infinitesimal generator matrix*, or Q-matrix, of this CTMC.

   (b) The Kolmogorov Forward Differential Equations (KFDEs) for a CTMC can be written as
   $$\frac{dP_{ij}(t)}{dt} = \sum_{k \in S} P_{ik}(t) q_{kj}, \quad \text{for all } i, j \in S.$$
   Use the Chapman Kolmogorov equations to prove that the KFDEs must be obeyed for this CTMC (you may assume that the necessary swap of the order of limit and summation is justified in a finite state chain).

   (c) If the process is *stationary* and has equilibrium distribution given by $\pi_j, j \in S$, then show that the time-reversed CTMC has transition rates given by
   $$q_{ij}^R = \frac{q_{ji}\pi_j}{\pi_i}, \quad \text{for all } i, j \in S.$$

   [12 marks]

2. Consider a simple queue with exactly two servers and a Poisson arrival stream of customers of rate α. Each server takes an exponential period of time to serve an individual customer, with mean 1. When there is one customer in service, the remaining server remains idle.

   (a) Define a suitable state space, S, for this system, including a clear definition of each state.

   (b) Write down the transition rates for this system.

   (c) Write down the Kolmogorov Forward Differential Equations for this CTMC, only for $P_{0,i}(t)$, $i \in S$ and $t \geq 0$. **DO NOT SOLVE.**

   (d) State the physical meaning of the quantity $P_{0,i}(t)$.

   (e) What initial conditions should be satisfied by $P_{0,i}(0)$, $i \in S$.

   (f) Write down the global balance equations for this CTMC. **DO NOT SOLVE.**

   [16 marks]

3. Consider a single-server queue with a Poisson arrival stream of customers of rate β where each customer requires an exponential amount of service with mean $1/\gamma$. Let $X(t)$ be the number of customers in the queue at time $t,\ t \geq 0$.

   (a) Write down the detailed balance equations for this CTMC and solve them.

   For simplicity, let $\gamma = 5$ and $\beta = 4$. The company running this service has decided that it would like to advertise its performance more favourably and so designates some arrivals as "premium", with the remaining arrivals designated as "standard" customers. It therefore reports the mean waiting time *as seen by premium arrivals.* Interestingly, the company has not changed its processes at all and does nothing special for the premium customers. Instead, it designates all customers that arrive when there are at most 2 customers in the queue to be "premium".

   Answer the following questions *under equilibrium conditions.*

   (b) What is the average arrival rate of *premium* customers?

   (c) What is the distribution of the number of the customers in the queue, as seen by arriving *premium* customers?

   (d) What is the average queue length, as seen by arriving *premium* customers?

   (e) What is the average waiting time for a *premium* customer?

[15 marks]

4. Consider a simple open Jackson network consisting of 3 nodes (labelled $1, 2$, and 3). Nodes 1 and 2 are single-server queues, while node 3 is an infinite-server queue. Assume that all customers require an exponentially-distributed service time at each node, with mean $1/\mu_i$ at node i, $i = 1, 2, 3$.

   There is an exogenous Poisson arrival process of customers into node 1 of rate 1 and an exogenous Poisson arrival process into node 2 of rate 2. There are no exogenous arrivals into node 3. Customers that complete service at node i move instantaneously to node j with probability γ_{ij}, or exit the system with probability β_i. Here,

$$\Gamma = (\gamma_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{2}{5} & \frac{1}{5} & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{5} \end{pmatrix}.$$

   (a) Solve the traffic equations for this open Jackson network.

   (b) State the condition for existence of an equilibrium distribution.

   (c) Write down the equilibrium distribution.

[10 marks]

5. Consider a counting process, $N(t)$, with inter-event time distribution $F(t)$. Let $M(t)$ be the renewal function (also known as the mean-value function) for this renewal process, that is, $M(t) = E[N(t)]$. Prove Theorem 7.4.3:

   The renewal function $M(t)$ satisfies the renewal equation

$$M(t) = F(t) + \int_0^t M(t-x)dF(x).$$

[5 marks]

6. Consider the random variable X with distribution function given by

$$P(X \le t) = F(t) = 1 - ae^{-\lambda t}.$$

   (a) Determine the Laplace-Stieltjes Transform, $\hat{F}(s)$, of $F(t)$.

   (b) Show that the Laplace-Stieltjes Transform, $\hat{M}(s)$, of $M(t)$ is given by

$$\hat{M}(s) = \frac{1-a}{a} + \frac{\lambda}{as}.$$

   (c) Hence show that $M(t) = \dfrac{\lambda t + 1 - a}{a}$.

[12 marks]

Table of Laplace Transforms

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$1/s$	1
$1/s^2$	t
$1/s^n \quad (n = 1, 2, \ldots)$	$t^{n-1}/(n-1)!$
$1/\sqrt{s}$	$1/\sqrt{\pi t}$
$1/s^{3/2}$	$2\sqrt{t/\pi}$
$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^2}$	te^{at}
$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \ldots)$	$\dfrac{1}{(n-1)!}\, t^{n-1}e^{at}$
$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}\, t^{k-1}e^{at}$
$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(e^{at} - e^{bt})$
$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(ae^{at} - be^{bt})$

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\, \sin \omega t$
$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$
$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\, \sinh at$
$\dfrac{s}{s^2-a^2}$	$\cosh at$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}\, e^{at} \sin \omega t$
$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}\,(\omega t - \sin \omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}\,(\sin \omega t - \omega t \cos \omega t)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\, \sin \omega t$
$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}\,(\sin \omega t + \omega t \cos \omega t)$
$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}\,(\cos at - \cos bt)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$

Basic General Formulas for the Laplace Transformation

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (1st Shifting Theorem)
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (2nd Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s}$	Differentiation of Transform Integration of Transform
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ $= \int_0^t f(t-\tau)g(\tau)d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1-e^{-\ell s}} \int_0^\ell e^{-st} f(t)dt$	f Periodic with Period ℓ