Lecture 19: Queueing Systems - Loss Networks and Reduced Load Approximations

Concepts checklist

At the end of this lecture, you should be able to:

- *evaluate* the Erlang Blocking/Loss Formula for Erlang Loss Systems; and,
- *evaluate* approximate blocking probabilities using the Erlang Fixed Point Method (EFPM).

Example 12. The $M/M/N/N$ queue

This queue is known as the Erlang Loss System, which can model a group of N circuits handling a Poisson stream of calls where arrivals which occur when all circuits are occupied are lost. The notation refers to *arrival process/service time distribution/number of servers/capacity.* Hence, calls arrive in a Poisson stream, of some rate λ; if a circuit is available, the call grabs the circuit and holds it for the connection time, which is exponentially distributed with some parameter μ; and the state space is $S = \{0, 1, 2, \dots, N\}$.

The rates are as follows

$$\lambda_\ell = \begin{cases} \lambda & 0 \le \ell < N \\ 0 & \ell = N \end{cases}$$
$$\mu_\ell = \ell\mu \qquad \ell \le N.$$

We have already derived the equilibrium probability distribution $\boldsymbol{\pi}$, where

$$\pi_i = \left[\sum_{\ell=0}^{N} \frac{1}{\ell!} a^\ell\right]^{-1} \frac{1}{i!} a^i \quad \text{for } i \in \{0, 1, 2, \dots, N\} \text{ and } a = \frac{\lambda}{\mu}.$$

The Erlang-B Formula (also known as *Erlang Blocking Formula* or *Erlang Loss Formula*) $B(N, a) = \pi_N$ is the probability that an arriving call is lost (as given by PASTA – *Poisson Arrivals See Time Averages*), and is also the blocking probability of an arriving call under equilibrium conditions. The Erlang Loss Formula is valid when connection times follow any distribution with mean $1/\mu$. That is, the Erlang Loss formula is insensitive to service time distributions.

Evaluating the (Erlang-B) blocking formula

$$B(N, a) = \pi_N = \frac{1}{N!} a^N \left[\sum_{\ell=0}^{N} \frac{1}{\ell!} a^\ell\right]^{-1}$$

becomes a formidable task when the value of N becomes large, because such things as

$$\frac{100^{200}}{200!}$$

are very difficult to evaluate and use. Therefore, it is best to use an iterative method for calculation, as follows (which is not hard to derive):

$$B(N, a) = \frac{aB(N-1, a)}{N + aB(N-1, a)} \quad \text{with} \quad B(0, a) = 1.$$

Reduced Load Approximations

Reduced load approximations are commonly used to find approximate performance measures for stochastic systems which either

1. do not have *closed form* equilibrium distributions, or
2. the equilibrium distribution has a closed form but cannot be numerically calculated because of the size of the system's state space.

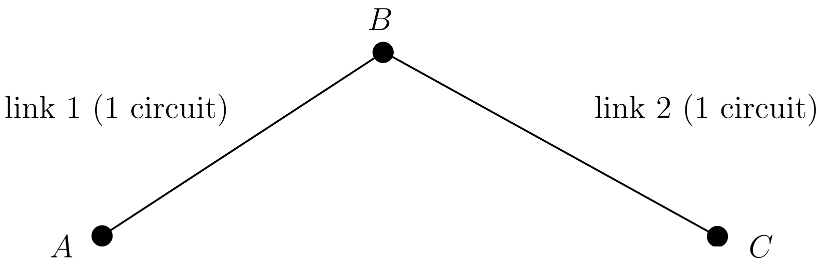
The general idea of a reduced load approximation is

1. consider sections of the network in isolation, and
2. reduce the traffic load offered to that section in accord with calls rejected by the remainder of the network.

The reduced load technique we shall demonstrate is the Erlang fixed point method (EFPM), a network decomposition technique, which can be used to approximate many performance measures of interest for a variety of loss networks.

Example 13. Two-link network

Consider two links with a single circuit on each (as illustrated below), one route only from A to C, which requires both circuits and an offered load of 1 (Erlang).



Obtaining the exact blocking probability:

The system behaves exactly the same as a single link with one circuit and offered load $a = 1$. Therefore, using the Erlang blocking formula we have

$$\Pr(\text{a call is blocked}) = B(1,1) = \left.\frac{a}{1+a}\right|_{a=1} = \frac{1}{1+1} = 0.5 \quad \textit{– the correct probability!}$$

Obtaining an approximation to the blocking probability, using EFPM:

EFPM is a reduced load technique which considers each link in turn and reduces the load offered to that link by the proportion of that load which is blocked at the other links in the network. The method assumes that

- links are **independent**, and
- a call is accepted only if accepted on all links.

Thus, in the above example, if for $i = 1, 2$ we let

$$\alpha_i = \Pr(\text{call is blocked on link } i \text{ under EFPM assumptions}),$$

then $\alpha_1 = \alpha_2$ by symmetry.

The traffic offered to link 1 is $a(1-\alpha_2) = a(1-\alpha_1)$. Thus,

$$\alpha_1 = \Pr(\text{call is blocked on link 1}) = B(1, 1-\alpha_1) = \frac{1-\alpha_1}{1+1-\alpha_1}.$$

That is,

$$\begin{aligned} & 2\alpha_1 - \alpha_1^2 = 1 - \alpha_1, \\ \Longrightarrow \quad & \alpha_1^2 - 3\alpha_1 + 1 = 0, \end{aligned}$$

which has solution $\alpha_1 \approx 0.38$.

Then we calculate

$$\begin{aligned} \Pr(\text{call is accepted by network}) &= \Pr(\text{accepted by link 1}) \Pr(\text{accepted by link 2}) \\ & \quad \textit{(by our } \textbf{\textit{independence}} \textit{ assumption)}, \\ &\approx (1-0.38)(1-0.38) = 0.62^2. \\ \Rightarrow \Pr(\text{a call is blocked}) &\approx 1 - (0.62)^2 = 0.62. \end{aligned}$$

Even in this case, where **it is blatantly wrong to assume independence** between the links, the approximate blocking probability of 0.62 is *not too far* from the true value of 0.5.

Example 14 - Two link, slightly more complex routing

Consider the same two link network, but with a slightly more complex routing pattern.

Route Number	Route	Offered load (a_i)	Links used
1	AB	0.5E	1
2	BC	0.6E	2
3	AC	0.3E	1, 2

If the vector $\boldsymbol{n} = (n_1, n_2, n_3)$ records the number of calls on routes $1, 2$ and 3 respectively, then we can then write down all the possible states for the network as

- $(0,0,0)$ – no calls present
- $(0,0,1)$ – one call using route 3, which takes both links
- $(1,0,0)$ – one call on route 1, which only uses link 1
- $(0,1,0)$ – one call on route 2, which only uses link 2
- $(1,1,0)$ – two calls, one using route 1 and the other route 2.

We can then work out the exact equilibrium distribution by:

$\boldsymbol{n}$	Invariant Measure $= \prod_{i=1}^{3} \frac{a_i^{n_i}}{n_i!}$	$\boldsymbol{\pi}(\boldsymbol{n})$
$(0,0,0)$	1	$1/2.7 = 0.3704$
$(0,0,1)$	0.3	$0.3/2.7 = 0.1111$
$(1,0,0)$	0.5	$0.5/2.7 = 0.1852$
$(0,1,0)$	0.6	$0.6/2.7 = 0.2222$
$(1,1,0)$	$0.5 \times 0.6 = 0.3$	$0.3/2.7 = 0.1111$
	Total $= 2.7$	Total $= 1.0$

We can calculate the exact blocking probabilities as follows:

Route	
AB	$0.1111 + 0.1852 + 0.1111 = 0.4074$
BC	$0.1111 + 0.2222 + 0.1111 = 0.4444$
AC	$1.0 - 0.3704 = 0.6296$