STATS 3006 Mathematical Statistics III
Assignment 2
2018

Assignment 2 is due by 23:59 Tuesday 17^{th} April 2018.

Assignments are to be submitted online on Myuni.

1. Suppose X_1 and X_2 are discrete random variables such that $X_1 \sim B(n, \pi)$ and $X_2 | X_1 = x_1 \sim B(x_1, \rho)$.
   (a) Write down the joint probability function of (X_1, X_2).
   (b) Derive the marginal distribution of X_2

2. (a) Consider pairs of random variables

$$(Y_{11}, Y_{12}), (Y_{21}, Y_{22}), \ldots, (Y_{n1}, Y_{n2})$$

   such that $E(Y_{i1}) = \mu_1$, $E(Y_{i2}) = \mu_2$, $\mathrm{cov}(Y_{i1}, Y_{i2}) = \sigma_{12}$ for $i = 1, 2, \ldots, n$ and Y_{ij}, Y_{kl} are independent for $i \neq k$.

   If $X_1 = \sum_{i=1}^{n} Y_{i1}$ and $X_2 = \sum_{j=1}^{n} Y_{i2}$, show that $\mathrm{cov}(X_1, X_2) = n\sigma_{12}$.

   (b) Consider an experiment that results in exactly one of
   - Outcome 1, with probability π_1;
   - Outcome 2, with probability π_2;
   - Outcome 3, with probability $1 - \pi_1 - \pi_2$

   and let Y_1 and Y_2 be indicator variables defined by

$$Y_1 = \begin{cases} 1 & \text{for Outcome 1} \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } Y_2 = \begin{cases} 1 & \text{for Outcome 2} \\ 0 & \text{otherwise.} \end{cases}$$

   Show that $\mathrm{cov}(Y_1, Y_2) = -\pi_1 \pi_2$.

   (c) Hence show that $\mathrm{cov}(X_1, X_2) = -n\pi_1\pi_2$ if (X_1, X_2) have the trinomial distribution with parameters n and π_1, π_2.

3. Suppose X_1, X_2 have joint PDF

$$f(x_1, x_2) = k(x_1 + x_2^2), \text{ for } 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1.$$

   (a) Find the value of k for which $f(x_1, x_2)$ is a valid PDF.
   (b) Find $P(X_1 > X_2)$.
   (c) Find $P(X_1 + X_2 \leq \frac{1}{2})$.
   (d) Find $P(X_1 \leq \frac{1}{4})$.

4. Suppose (X_1, X_2) have the Dirichlet distribution,

$$f(x_1, x_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} (1 - x_1 - x_2)^{\alpha_3 - 1}.$$

(a) Prove that the marginal distribution of X_1 is a Beta distribution.

(b) Find the conditional density function $f_{X_2|X_1}(x_2|x_1)$.

5. Suppose X_1, X_2 have the uniform distribution on the region $|x_1| + |x_2| \leq 1$.

   (a) Give an expression for the joint PDF.
   Hint: Sketch the region $|x_1| + |x_2| \leq 1$.

   (b) Find $E(X_1)$ and $E(X_2)$.

   (c) Find $\text{cov}(X_1, X_2)$.

   (d) Find the marginal distribution of X_1 and also of X_2.

   (e) Are X_1 and X_2 independent? Comment on this example.

6. Suppose $U \sim U(0, 1)$ and $V|u \sim U(0, u)$.

   (a) Write down the joint probability density function of (U, V) including its domain.

   (b) Find the marginal PDF of V.

Honours Questions

The following questions are compulsory for Honours and Masters students and may be attempted for bonus marks by all students.

7. Suppose $U \sim U(0, 1)$ and $V|u \sim U(0, u)$.

   (a) State $E(U)$ and $\text{var}(U)$.

   (b) Find $E(V)$.

   (c) Find $\text{var}(V)$.

   (d) Find $\text{cor}(U, V)$.

   (e) Use `R` to simulate 1,000,000 pairs of observations of (U, V) and use your simulations to demonstrate the marginal distribution of V from question 6 and the moment calculations in this question.

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