THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Practice

101488 APP MTH 3016 Random Processes III

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 7 TOTAL MARKS: 91

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators without remote communications facilities are permitted.
- Tables of Laplace Transforms are provided at the end of the Examination question book.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Answer *true* or *false* to each of the following assertions. You must also provide a *very brief* (1–3 lines) justification for each of your answers. (You might, for example, wish to refer to a theorem discussed in lectures.)

   (a) All elements of the generator matrix Q of a continuous-time Markov chain are in $[0, 1]$.

   (b) A continuous-time Markov chain is time-homogeneous if

   $$\mathbb{P}(X(t+s) = j \mid X(u) = i_u, X(s) = i_s, u \le s) = \mathbb{P}(X(t+s) = j \mid X(s) = i_s),$$

   for all $s, t \in [0, \infty)$ and all $i_u, i_s, j \in \mathcal{S}$.

   (c) Consider a queue where there is a Poisson arrival and five servers, each has exponential service time with rate μ. The transition rate of going from state 4 to state 3 is 5μ.

   (d) Let $P(z, t)$ be the generating function for a continuous-time Markov chain with transition probabilities $P_{0n}(t)$, for $n = 0, 1, \ldots, 2$, defined as

   $$P(z, t) = \sum_{n=0}^{\infty} P_{0n}(t) z^n.$$

   Then, $P(z, t)$ is well defined for $|z| \le 1$.

   (e) The Poisson process is an example of an irreducible continuous-time Markov chain.

   (f) If a continuous-time Markov chain is recurrent, then calculating expected hitting times is useful for identifying whether the model is positive-recurrent or null-recurrent.

   (g) Every Markov chain is reversible.

   (h) An assumption of the Erlang Fixed Point Method is that the links are independent.

   (i) Adding the possibility of feedback to an open Jackson network loses the product form structure for the equilibrium probability distribution of the network.

   (j) Let X and Y be two independent random variables with Laplace–Stieltjes transforms $\widehat{F}_X(s)$ and $\widehat{F}_Y(s)$ respectively. Then, the random variable $Z = X + Y$ has the Laplace–Stieltjes Transform $\widehat{F}_Z(s) = \widehat{F}_X(s) + \widehat{F}_Y(s)$.

[20 marks]

2. (a) Define a continuous-time Markov chain $\{X(t), t \geq 0\}$ on the finite state space $\mathcal{S}$.

   (b) For all $i, j \in \mathcal{S}$ and $s, t \geq 0$, let

$$P_{ij}(t) = \Pr(X(t+s) = j \mid X(s) = i).$$

   Define the *infinitesimal generator* Q of this Markov chain.

   (c) Give physical interpretations for the elements of the matrix Q.

   (d) Consider a continuous-time Markov chain $X(t)$ on finite state space $\mathcal{S}$ with generator Q. Show that, for all $i, j \in \mathcal{S}$,

$$\Pr\left(\text{moves to state } j \neq i \mid \text{leaves state } i \text{ at time } t\right) = \frac{q_{ij}}{-q_{ii}}.$$

[12 marks]

Please turn over for page 4

3. On quiet Wednesday nights, the Belgian Beer Cafe has two bartenders working. Customers arrive according to a Poisson process with rate γ, each bartender has exponential service time with rate μ, and the Cafe can cater up to 300 people.

   (a) Define a suitable state space $\mathcal{S}$ for this system, including a definition of each state.

   (b) Give the dimension of the generator matrix Q, and write down the transition rates for this system.

   (c) Write down the Kolmogorov forward differential equation for this Markov chain for $P_{3,6}(t)$ only, for $t \geq 0$, but do not solve.

   (d) Write down the Kolmogorov backward differential equation for this Markov chain for $P_{3,6}(t)$ only, for $t \geq 0$, but do not solve.

   (e) State the physical meaning of the quantity $P_{3,6}(t)$.

   (f) What initial condition should be satisfied by $P_{3,6}(0)$?

   (g) List all communicating classes of the Markov chain.

[14 marks]

4. To celebrate the beginning of spring, the School of Mathematical Sciences hosts a free BBQ for all math students. There is one queue, and for which there is one cook with exponential service time of rate α. Hungry students arrive according to a Poisson process with rate λ. With probability p the student leaves the queue; with probability $1-p$ the student decides to get more food and immediately rejoins the queue. We assume that there is an infinite population of students.

   (a) Write down an appropriate state space $\mathcal{S}$ to help keep track of the queue length.

   (b) Write down the transition rates for the system.

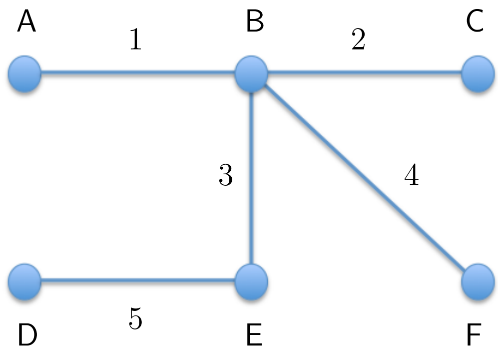
   (c) Let $f_i, i \in \mathcal{S}$ be the probability that the queue ever gets empty, given that it has i students at the beginning. Write down the appropriate set of equations satisfied by $f_i, i \in \mathcal{S}$.

   (d) What additional conditions do we need in order to determine which solution to this set of equations corresponds to $f_i, i \in \mathcal{S}$?

   (e) If $f_1 = 1$, what does this tell us about the transience or otherwise of this Markov chain? Why?

[11 marks]

5. Consider a simple circuit-switched loss network consisting of 6 nodes (labelled A, B, C, D, E, and F), and 5 links (labelled from 1 to 5), as shown below.



Links 1, 2, and 3 each has capacity of 20, links 4 and 5 each has capacity of 30. There are three routes in the network; we assume that calls arrive to these routes as independent Poisson processes, and that all calls have an exponentially distributed holding time with unit mean and use 1 circuit on each link it uses. Their arrival rates and links used are listed in the following table.

Route label	Route	Arrival Rate	Links Used
1	A–C	2	1, 2
2	A–F	1	1, 4
3	D–C	2	5, 3, 2

(a) By defining all necessary notation, write down an appropriate state space for a CTMC representation of this circuit-switched network.

(b) Write down an expression for the equilibrium distribution for this network.

(c) Write down an expression for the blocking probability of calls on Route 3 (that is between nodes D and C).

(d) Write down the expressions required to define the Erlang Fixed Point approximation for a circuit-switched network, including an expression for the blocking probability on a route.

[16 marks]

6. Consider a single-server queue with a Poisson arrival stream of customers of rate 4 where each customer requires an exponential amount of service with rate 6. Let $X(t)$ be the number of customers in the queue at time $t \geq 0$. Assume that all customers arriving when there is an even number of people already in the queue (2, 4, 6, etc) are considered *lucky* customers.

   Answer the following questions under *equilibrium conditions*.

   (a) Write down the detailed balance equations for this Markov chain and solve them.

   For parts (b)–(e), do not simplify your answer.

   (b) What is the average arrival rate of *lucky* customers?

   (c) What is the distribution of the number of customers in the queue as seen by arriving *lucky* customers?

   (d) What is the average queue length, as seen by arriving *lucky* customers?

   (e) What is the average waiting time for a *lucky* customer?

[14 marks]

7. Suppose that a renewal process $\{N(t) : t \geq 0\}$ has the lifetime density

$$f(t) = \lambda^2 t e^{-\lambda t} \quad \text{for } t \geq 0.$$

The Laplace-Stieltjes transform $\widehat{F}(s)$ of the distribution $F(t)$ of the inter-event-time is

$$\widehat{F}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2.$$

Show that the renewal function $M(t)$ is given by

$$M(t) = \frac{1}{2}\lambda t - \frac{1}{4}(1 - \mathrm{e}^{-2\lambda t}).$$

[4 marks]

Table of Laplace Transforms

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$1/s$	1
$1/s^2$	t
$1/s^n \quad (n = 1, 2, \ldots)$	$t^{n-1}/(n-1)!$
$1/\sqrt{s}$	$1/\sqrt{\pi t}$
$1/s^{3/2}$	$2\sqrt{t/\pi}$
$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^2}$	te^{at}
$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \ldots)$	$\dfrac{1}{(n-1)!}\, t^{n-1}e^{at}$
$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}\, t^{k-1}e^{at}$
$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(e^{at} - e^{bt})$
$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(ae^{at} - be^{bt})$

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\, \sin \omega t$
$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$
$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\, \sinh at$
$\dfrac{s}{s^2-a^2}$	$\cosh at$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}\, e^{at} \sin \omega t$
$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}\,(\omega t - \sin \omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}\,(\sin \omega t - \omega t \cos \omega t)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\, \sin \omega t$
$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}\,(\sin \omega t + \omega t \cos \omega t)$
$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}\,(\cos at - \cos bt)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$

Basic General Formulas for the Laplace Transformation

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\,\mathcal{L}(f)$	Integration of Function
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (1st Shifting Theorem)
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (2nd Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s}$	Differentiation of Transform Integration of Transform
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1-e^{-\ell s}} \int_0^\ell e^{-st} f(t)dt$	f Periodic with Period ℓ