THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Practice

101488 APP MTH 3016 Random Processes III

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 7 TOTAL MARKS: 91

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators without remote communications facilities are permitted.
- Tables of Laplace Transforms are provided at the end of the Examination question book.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Answer *true* or *false* to each of the following assertions. You must also provide a *very brief* (1–3 lines) justification for each of your answers. (You might, for example, wish to refer to a theorem discussed in lectures.)

   (a) All elements of the generator matrix Q of a continuous-time Markov chain are in $[0, 1]$.
   False. For example, the diagonal elements of Q are negative.

   [2 marks]

   (b) A continuous-time Markov chain is time-homogeneous if

   $$\mathbb{P}(X(t+s) = j \mid X(u) = i_u, X(s) = i_s, u \le s) = \mathbb{P}(X(t+s) = j \mid X(s) = i_s),$$

   for all $s, t \in [0, \infty)$ and all $i_u, i_s, j \in \mathcal{S}$.
   False. A continuous-time Markov chain is time-homogeneous if

   $$\mathbb{P}(X(t+s) = j \mid X(s) = i) = \mathbb{P}(X(t) = j \mid X(0) = i)$$

   for all $i, j \in \mathcal{S}$ and $s, t \in [0, \infty)$.

   [2 marks]

   (c) Consider a queue where there is a Poisson arrival and five servers, each has exponential service time with rate μ. The transition rate of going from state 4 to state 3 is 5μ.
   False. The rate of going from state 4 to state 3 is 4μ.

   [2 marks]

   (d) Let $P(z, t)$ be the generating function for a continuous-time Markov chain with transition probabilities $P_{0n}(t)$, for $n = 0, 1, \ldots, 2$, defined as

   $$P(z, t) = \sum_{n=0}^{\infty} P_{0n}(t) z^n.$$

   Then, $P(z, t)$ is well defined for $|z| \le 1$.
   True, because by the triangle inequality, for $|z| \le 1$, we have

   $$\left| \sum_{n=0}^{\infty} P_{0n}(t) z^n \right| \le \sum_{n=0}^{\infty} P_{0n}(t) \, |z^n| \le \sum_{n=0}^{\infty} P_{0n}(t) = 1.$$

   [2 marks]

   (e) The Poisson process is an example of an irreducible continuous-time Markov chain.
   False, because the Poisson process has infinitely many communicating classes, each communicating class has exactly one state $\{i\}$ for $i \in \mathcal{S}$.

   [2 marks]

(f) If a continuous-time Markov chain is recurrent, then calculating expected hitting times is useful for identifying whether the model is positive-recurrent or null-recurrent.

True. If the expected hitting time is finite, then the chain is positive-recurrent, otherwise it is null-recurrent.

[2 marks]

(g) Every Markov chain is reversible.

False. A Markov chain is reversible if and only if the detailed balance equations are satisfied, and this is not always the case.

[2 marks]

(h) An assumption of the Erlang Fixed Point Method is that the links are independent.

True. This is essential for calculating the route blocking probabilities in the EFPM, which is done by multiplying together the blocking probabilities on each link of the route.

[2 marks]

(i) Adding the possibility of feedback to an open Jackson network loses the product form structure for the equilibrium probability distribution of the network.

False. A theorem in the lecture notes says the product form structure is still maintained even with feedback.

[2 marks]

(j) Let X and Y be two independent random variables with Laplace–Stieltjes transforms $\widehat{F}_X(s)$ and $\widehat{F}_Y(s)$ respectively. Then, the random variable $Z = X+Y$ has the Laplace–Stieltjes Transform $\widehat{F}_Z(s) = \widehat{F}_X(s) + \widehat{F}_Y(s)$.

False. The convolution theorem states that $\widehat{F}_Z(s) = \widehat{F}_X(s)\widehat{F}_Y(s)$.

[2 marks]

[20 marks]

2. (a) Define a continuous-time Markov chain $\{X(t), t \geq 0\}$ on the finite state space $\mathcal{S}$.
A random process $\{X(t), t \geq 0\}$ is said to be a continuous time Markov chain if

$$\Pr(X(t+s) = k \mid X(u) = i, X(s) = j,\ u < s) = \Pr(X(t+s) = k \mid X(s) = j),$$

for all $s, t \in [0, \infty)$ and all $i, j, k \in \mathcal{S}$.

[2 marks]

(b) For all $i, j \in \mathcal{S}$ and $s, t \geq 0$, let

$$P_{ij}(t) = \Pr(X(t+s) = j \mid X(s) = i).$$

Define the *infinitesimal generator* Q of this Markov chain.
The infinitesimal generator Q of the Markov chain has entries

$$q_{ij} = \lim_{h \to 0^+} \frac{P_{ij}(h)}{h} \quad \text{for } j \in \mathcal{S}, j \neq i,$$
$$q_{ii} = \lim_{h \to 0^+} \frac{P_{ii}(h) - 1}{h}.$$

Or: In matrix notation,

$$Q = \lim_{h \to 0^+} \frac{P(h) - I}{h},$$

where $P(h) = [P_{ij}(h)]_{i,j \in \mathcal{S}}$ and I is an identity matrix.

[2 marks]

(c) Give physical interpretations for the elements of the matrix Q.
(i) For small $h > 0$ and for $i \neq j$, we have $P_{ij}(h) = q_{ij}h + o(h)$. Thus, the probability that the Markov chain moves out of state i and into state j in some small time h is approximately $q_{ij}h$. In other words, q_{ij} is the instantaneous rate (in probability terms) that the chain moves from i to j.
— *Just one of the three sentences will do.*

(ii) For small $h > 0$ and $i \in \mathcal{S}$, we have $1 - P_{ii}(h) = -hq_{ii} + o(h)$. Thus, the probability that the Markov chain moves out of state i in some small time h is approximately $-q_{ij}h$. In other words, $-q_{ii}$ is the instantaneous rate that the Markov chain moves out of state i.
— *Just one of the three sentences will do.*

[4 marks]

(d) Consider a continuous-time Markov chain $X(t)$ on finite state space $\mathcal{S}$ with generator Q. Show that, for all $i, j \in \mathcal{S}$,

$$\Pr\left(\text{moves to state } j \neq i \mid \text{leaves state } i \text{ at time } t\right) = \frac{q_{ij}}{-q_{ii}}.$$

Let T_{ij}, for $i, j \in \mathcal{S}$, denote the time to leave i for j, so $T_{ij} \sim \operatorname{Exp}(q_{ij})$. Define $M = \min_{k \neq i, k \in \mathcal{S}} T_{ik}$. Then,

$\Pr\left(\text{moves to state } j \neq i \mid \text{leaves state } i \text{ at time } t\right)$

$$\begin{aligned}
&= \frac{\Pr(T_{ij} \in [t, t+dt) \cap \min_{k \neq i, k \neq j, k \in \mathcal{S}} T_{ik} > t)}{\Pr(M \in [t, t+dt))} \\
&= \frac{\Pr(T_{ij} \in [t, t+dt)) \prod_{k \neq i, k \neq j} \Pr(T_{ik} > t)}{\Pr(M \in [t, t+dt))} \qquad \text{(independence)} \\
&= \frac{\Pr(T_{ij} \in [t, t+dt)) \Pr(Z > t)}{\Pr(M \in [t, t+dt))} \qquad \text{where } Z \sim \exp\left(\sum_{k \neq i, k \neq j, k \in \mathcal{S}} q_{ik}\right) \\
&= \frac{q_{ij}}{q_{ij} + \sum_{k \neq i, k \neq j, k \in \mathcal{S}} q_{ik}} \\
&= \frac{q_{ij}}{\sum_{k \neq i, k \in \mathcal{S}} q_{ik}} \\
&= \frac{q_{ij}}{-q_{ii}}.
\end{aligned}$$

[4 marks]

[12 marks]

3. On quiet Wednesday nights, the Belgian Beer Cafe has two bartenders working. Customers arrive according to a Poisson process with rate γ, each bartender has exponential service time with rate μ, and the Cafe can cater up to 300 people.

   (a) Define a suitable state space $\mathcal{S}$ for this system, including a definition of each state.

   $\mathcal{S} = \{0, 1, 2, \ldots, 300\}$, where state $i \in \mathcal{S}$ represents the state in which there are i customers in the Cafe.

   [2 marks]

   (b) Give the dimension of the generator matrix Q, and write down the transition rates for this system.

   The 301×301 transition matrix Q is given by

$$Q = \begin{bmatrix} -\gamma & \gamma & & & & \\ \mu & -(\mu+\gamma) & \gamma & & & \\ & 2\mu & -(2\mu+\gamma) & \gamma & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2\mu & -(2\mu+\gamma) & \gamma \\ & & & & 2\mu & -2\mu \end{bmatrix}.$$

   — *1 for arrivals, 1 for boundary , 1 for diagonals, 1 for dimension*

   [4 marks]

   (c) Write down the Kolmogorov forward differential equation for this Markov chain for $P_{3,6}(t)$ only, for $t \geq 0$, but do not solve.

   The Kolmogorov forward differential equations for $P_{3,6}(t)$, $t \geq 0$, are

$$\frac{dP_{3,6}(t)}{dt} = \sum_{k \in \mathcal{S}} P_{3k}(t) q_{k6} = \gamma P_{3,5}(t) - (2\mu + \gamma) P_{3,6}(t) + 2\mu P_{3,7}(t).$$

   [2 marks]

   (d) Write down the Kolmogorov backward differential equation for this Markov chain for $P_{3,6}(t)$ only, for $t \geq 0$, but do not solve.

   The Kolmogorov backward differential equations for $P_{3,6}(t)$, $t \geq 0$, are

$$\frac{dP_{3,6}(t)}{dt} = \sum_{k \in \mathcal{S}} q_{3k} P_{k6}(t) = 2\mu P_{2,6}(t) - (2\mu + \gamma) P_{3,6}(t) + \gamma P_{4,6}(t).$$

   [2 marks]

   (e) State the physical meaning of the quantity $P_{3,6}(t)$.

   $P_{3,6}(t)$ is the probability that the cafe has 6 customers at the end of a period of time t, given that the cafe initially had 3 customers.

   [2 marks]

(f) What initial condition should be satisfied by $P_{3,6}(0)$?
$P_{3,6}(0) = 0$.

[1 marks]

(g) List all communicating classes of the Markov chain.
There is only one: $\mathcal{S} = \{0, 1, \ldots, 300\}$.

[1 marks]

[14 marks]

4. To celebrate the beginning of spring, the School of Mathematical Sciences hosts a free BBQ for all math students. There is one queue, and for which there is one cook with exponential service time of rate α. Hungry students arrive according to a Poisson process with rate λ. With probability p the student leaves the queue; with probability $1-p$ the student decides to get more food and immediately rejoins the queue. We assume that there is an infinite population of students.

   (a) Write down an appropriate state space $\mathcal{S}$ to help keep track of the queue length.
   $\mathcal{S} = \{0, 1, 2, 3, \ldots, \infty\}$.

   [1 marks]

   (b) Write down the transition rates for the system.
   The transition rates are

$$
\begin{aligned}
q_{i,i+1} &= \lambda \quad \text{for } i = 0, 1, \ldots, \\
q_{i,i-1} &= p\alpha \quad \text{for } i = 1, 2, \ldots, \\
q_{i,i} &= -(p\alpha + \lambda) \quad \text{for } i = 1, 2, \ldots, \\
q_{0,0} &= -\lambda, \\
q_{i,j} &= 0 \quad \text{otherwise.}
\end{aligned}
$$

   — 1 *for arrivals,* 1 *for services,* 1 *for diagonals,* 1 *for boundary*

   [4 marks]

   (c) Let $f_i, i \in \mathcal{S}$ be the probability that the queue ever gets empty, given that it has i students at the beginning. Write down the appropriate set of equations satisfied by $f_i, i \in \mathcal{S}$.
   By first-step analysis, f_i has to satisfy the following equality

$$
f_i = \frac{\lambda}{p\alpha + \lambda} f_{i+1} + \frac{p\alpha}{p\alpha + \lambda} f_{i-1}. \tag{1}
$$

   [2 marks]

   (d) What additional conditions do we need in order to determine which solution to this set of equations corresponds to $f_i, i \in \mathcal{S}$?
   $f_0 = 1$, and $f_i \in [0, 1]$ is the minimum non-negative solution to the second-order difference equation (1).

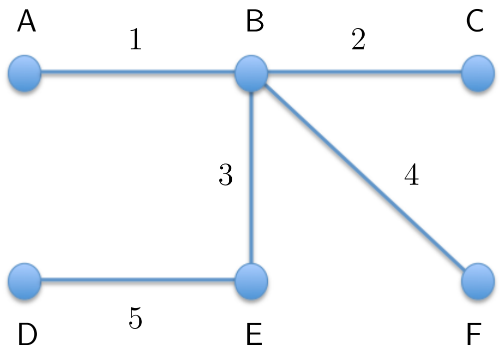
   [2 marks]

   (e) If $f_1 = 1$, what does this tell us about the transience or otherwise of this Markov chain? Why?
   This tells us that the Markov chain is recurrent, because (a) the process is irreducible and (b) the probability of returning to 0, once the process starts at zero, is 1.

   [2 marks]

[11 marks]

5. Consider a simple circuit-switched loss network consisting of 6 nodes (labelled A, B, C, D, E, and F), and 5 links (labelled from 1 to 5), as shown below.



Links 1, 2, and 3 each has capacity of 20, links 4 and 5 each has capacity of 30. There are three routes in the network; we assume that calls arrive to these routes as independent Poisson processes, and that all calls have an exponentially distributed holding time with unit mean and use 1 circuit on each link it uses. Their arrival rates and links used are listed in the following table.

Route label	Route	Arrival Rate	Links Used
1	A–C	2	1, 2
2	A–F	1	1, 4
3	D–C	2	5, 3, 2

(a) By defining all necessary notation, write down an appropriate state space for a CTMC representation of this circuit-switched network.

The routing matrix A and the capacity vector C are

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 30 \\ 30 \end{bmatrix}.$$

Let $\boldsymbol{n} = (n_1, n_2, n_3)$ be the number of calls in progress on each route.

The state space $\mathcal{S}$ is

$$\mathcal{S} = \left\{ \boldsymbol{n} \in \mathbb{Z}_+^3 : A\boldsymbol{n} \le C \right\}.$$

[5 marks]

(b) Write down an expression for the equilibrium distribution for this network.

$$\pi(\boldsymbol{n}) = G(C,3)^{-1} \frac{2^{n_1}}{n_1!} \frac{1^{n_2}}{n_2!} \frac{2^{n_3}}{n_3!}, \qquad \text{where } G(C,3) = \sum_{\boldsymbol{n}:A\boldsymbol{n} \le C} \frac{2^{n_1}}{n_1!} \frac{1^{n_2}}{n_2!} \frac{2^{n_3}}{n_3!}.$$

[4 marks]

(c) Write down an expression for the blocking probability of calls on Route 3 (that is between nodes D and C).

$$B_3 = 1 - \frac{G(C - A\boldsymbol{e}_3, 3)}{G(C, 3)}, \qquad \text{where } \boldsymbol{e}_3 = (0, 0, 1).$$

[2 marks]

(d) Write down the expressions required to define the Erlang Fixed Point approximation for a circuit-switched network, including an expression for the blocking probability on a route.

The reduced load y_i on link i is given by

$$y_i = \sum_{r:i\in r} \lambda_r \prod_{j\in r, j\neq i} (1 - \alpha_j).$$

The blocking probability of link i is

$$\alpha_i = B(C_i, y_i).$$

The route blocking probability B_r is

$$B_r = 1 - \prod_{j\in r} (1 - \alpha_j).$$

[5 marks]

[16 marks]

Please turn over for page 11

6. Consider a single-server queue with a Poisson arrival stream of customers of rate 4 where each customer requires an exponential amount of service with rate 6. Let $X(t)$ be the number of customers in the queue at time $t \geq 0$. Assume that all customers arriving when there is an even number of people already in the queue (2, 4, 6, etc) are considered *lucky* customers.

   Answer the following questions under *equilibrium conditions*.

   (a) Write down the detailed balance equations for this Markov chain and solve them.

   The detailed balance equation is $\pi_k q_{ki} = \pi_i q_{ik}$ for $i, k \in \mathcal{S}$. Thus,

$$4\pi_i = 6\pi_{i+1} \quad \text{for } i \geq 0. \tag{2}$$

   By 2 and the fact that $\sum_{i \in \mathcal{S}} \pi_i = 1$, we have

$$\pi_i = \frac{1}{3}\left(\frac{2}{3}\right)^i \quad \text{for } i \geq 0.$$

[4 marks]

   For parts (b)–(e), do not simplify your answer.

   (b) What is the average arrival rate of *lucky* customers?

$$\overline{\lambda} = \sum_{i=2,4,\ldots} \frac{4}{3}\left(\frac{2}{3}\right)^i.$$

[2 marks]

   (c) What is the distribution of the number of customers in the queue as seen by arriving *lucky* customers?

$$\pi_i^{(L)} = \begin{cases} \dfrac{4\pi_i}{\sum_{i=2,4,\ldots} 4\pi_i} = \dfrac{\pi_i}{\sum_{i=2,4,\ldots} \pi_i} & \text{for } i = 2, 4, \ldots. \\ 0 & \text{otherwise.} \end{cases}$$

[3 marks]

   (d) What is the average queue length, as seen by arriving *lucky* customers?

$$\overline{L} = \sum_{i=2,4,\ldots} i\pi_i^{(L)} = \frac{1/3}{\sum\limits_{i=2,4,\ldots} \pi_i} \sum_{i=2,4,\ldots} i\left(\frac{2}{3}\right)^i.$$

[2 marks]

(e) What is the average waiting time for a *lucky* customer?

$$\overline{W} = \frac{\overline{L}}{\overline{\lambda}} = \frac{\sum_{i=2,4,\ldots} i\pi_i^{(L)}}{\frac{4}{3}\sum_{i=2,4,\ldots}\left(\frac{2}{3}\right)^i}.$$

[3 marks]

[14 marks]

7. Suppose that a renewal process $\{N(t) : t \geq 0\}$ has the lifetime density

$$f(t) = \lambda^2 t e^{-\lambda t} \quad \text{for } t \geq 0.$$

The Laplace-Stieltjes transform $\widehat{F}(s)$ of the distribution $F(t)$ of the inter-event-time is

$$\widehat{F}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2.$$

Show that the renewal function $M(t)$ is given by

$$M(t) = \frac{1}{2}\lambda t - \frac{1}{4}(1 - \mathrm{e}^{-2\lambda t}).$$

$$\widehat{M}(s) = \frac{\widehat{F}(s)}{1 - \widehat{F}(s)} = \frac{\lambda^2}{s(s+2\lambda)} = \frac{\lambda}{2s} - \frac{2\lambda}{4(s+2\lambda)}$$
$$\Rightarrow M'(t) = \frac{\lambda}{2} - \frac{2\lambda}{4}\mathrm{e}^{-2\lambda t}.$$

Finally,

$$M(t) = \frac{1}{2}\lambda t - \frac{1}{4}(1 - \mathrm{e}^{-2\lambda t}), \text{ since } M(0) = 0.$$

[4 marks]

[4 marks]

Table of Laplace Transforms

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$1/s$	1
$1/s^2$	t
$1/s^n \quad (n = 1, 2, \ldots)$	$t^{n-1}/(n-1)!$
$1/\sqrt{s}$	$1/\sqrt{\pi t}$
$1/s^{3/2}$	$2\sqrt{t/\pi}$
$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{(s-a)^2}$	te^{at}
$\frac{1}{(s-a)^n} \quad (n = 1, 2, \ldots)$	$\frac{1}{(n-1)!}\ t^{n-1}e^{at}$
$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)}\ t^{k-1}e^{at}$
$\frac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)}\ (e^{at} - e^{bt})$
$\frac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)}\ (ae^{at} - be^{bt})$

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$\frac{1}{s^2+\omega^2}$	$\frac{1}{\omega}\ \sin\omega t$
$\frac{s}{s^2+\omega^2}$	$\cos\omega t$
$\frac{1}{s^2-a^2}$	$\frac{1}{a}\ \sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{1}{(s-a)^2+\omega^2}$	$\frac{1}{\omega}\ e^{at}\sin\omega t$
$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
$\frac{1}{s(s^2+\omega^2)}$	$\frac{1}{\omega^2}(1-\cos\omega t)$
$\frac{1}{s^2(s^2+\omega^2)}$	$\frac{1}{\omega^3}\ (\omega t - \sin\omega t)$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}\ (\sin\omega t - \omega t\cos\omega t)$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t}{2\omega}\ \sin\omega t$
$\frac{s^2}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega}\ (\sin\omega t + \omega t\cos\omega t)$
$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2-a^2}\ (\cos at - \cos bt)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$

Basic General Formulas for the Laplace Transformation

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\,\mathcal{L}(f)$	Integration of Function
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (1st Shifting Theorem)
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (2nd Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$	Differentiation of Transform
$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s}$	Integration of Transform
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1-e^{-\ell s}} \int_0^\ell e^{-st}f(t)dt$	f Periodic with Period ℓ