Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 22

Last lecture

- Delved further into isoperimetric problems
- Solved the traditional version of Dido's problem (with a parametric formulation)
- Saw a hand-waving explanation as to why Lagrange multipliers work
- Very briefly discussed multiple integral constraints

Non-fixed end point problems

What happens when we don't fix the end-points of an extremal?

In this case **natural boundary conditions** are automatically introduced, and these can allow us to solve the Euler-Lagrange equations.

Non-fixed end point problems

What happens when we don't fix the boundary points?
There are lots of real problems like this, for instance

- a freely supported beam
  - end points fixed, but not derivatives
- a beam supported at only one end
  - one end point and derivative fixed, other free
- shortest path between two curves
  - end points lie of curves, but not fixed
- rocket changing between two orbits
  - end points lie on curves, and path is tangent to the two orbits.

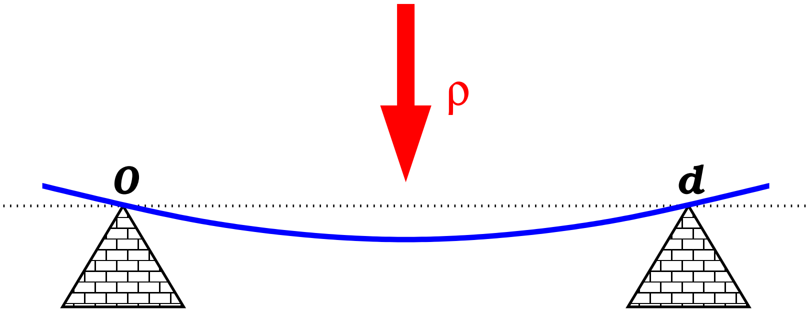
We then get **natural boundary conditions**

Free end points: Fixed x, Free y and/or y'

First we'll consider what happens when we allow y and/or y' to vary at the end-points, but we still keep the x values of the end-points fixed at x_0 and x_1.

Example: freely supported beam

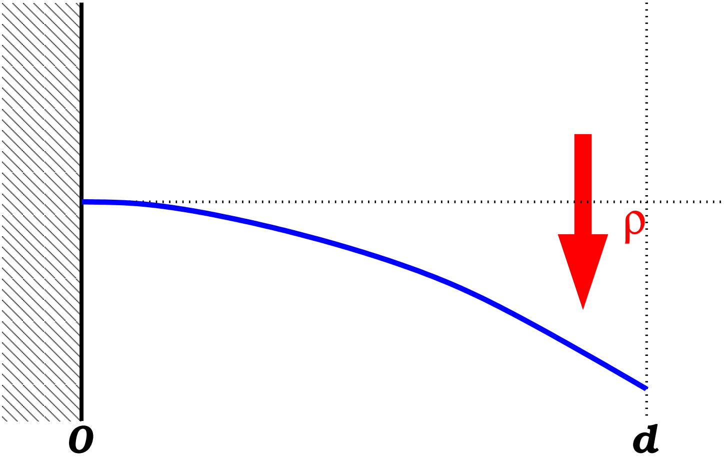
Freely supported beam



For the beam problems considered before, we had to specify the derivative at the boundary, but here it can vary.

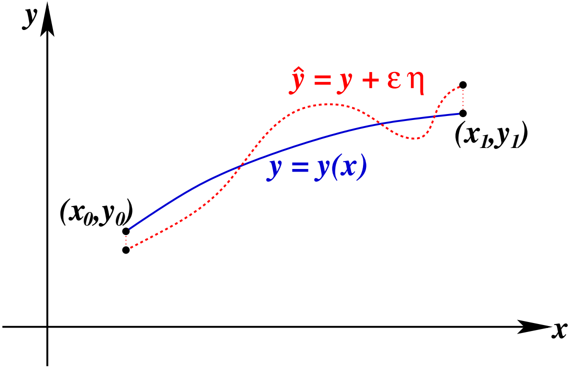
Example: beam fixed at one end point

Beam fixed at one end point



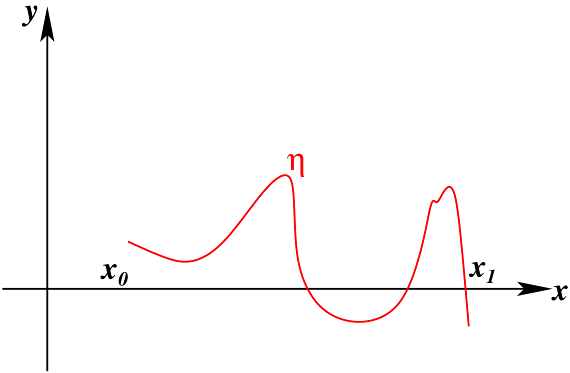
Perturbation again

We approach this the same way we did with all other variational problems, we perturb the curve and examine the First Variation, but this time, we allow $y(x_0)$ and $y(x_1)$ to vary as well.



Space of Perturbations

Now the space $\mathcal{H}$ of perturbations η contains functions whose value at x_0 and x_1 is no longer zero.



Same derivation of the first variation

Simple case where $F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$

$$
\begin{aligned}
f(x, \hat{y}, \hat{y}') &= f(x, y, y') + \epsilon \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right] + \mathcal{O}(\epsilon^2) \\
F\{\hat{y}\} - F\{y\} &= \int_{x_0}^{x_1} f(x, \hat{y}, \hat{y}')\, dx - \int_{x_0}^{x_1} f(x, y, y')\, dx \\
&= \epsilon \int_{x_0}^{x_1} \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right] dx + \mathcal{O}(\epsilon^2) \\
\delta F(\eta, y) &= \lim_{\epsilon \to 0} \frac{F\{y + \epsilon\eta\} - F\{y\}}{\epsilon} \\
&= \int_{x_0}^{x_1} \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right] dx
\end{aligned}
$$

The first variation

As before, we can vary the sign of ϵ, so for $F\{y\}$ to be a local minima it must be the case that

$$\delta F(\eta, y) = 0, \quad \forall \eta \in \mathcal{H}.$$

However, now $\mathcal{H}$ allows curves with arbitrary end-points, so that $\eta(x_0) \neq 0$, and $\eta(x_1) \neq 0$ are possible.

Hence when we integrate by parts we get

$$\delta F(\eta, y) \;\; = \;\; \left[\eta \frac{\partial f}{\partial y'}\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx,$$

and now the first term $\left[\eta \frac{\partial f}{\partial y'}\right]_{x_0}^{x_1}$ is not necessarily zero.

The first variation

However, $\delta F(\eta, y) = 0$ for all η, which includes cases where $\eta(x_0) = \eta(x_1) = 0$, and so the Euler-Lagrange equation must still be satisfied for such and extremal.

Given the E-L equation is satisfied by an extremal, the condition $\delta F(\eta, y) = 0$ next implies that

$$\left[\eta \frac{\partial f}{\partial y'} \right]_{x_0}^{x_1} = 0.$$

Likewise we can choose curves η such that $\eta(x_0) \neq 0$ and $\eta(x_1) \neq 0$, so that we must have

$$\left. \frac{\partial f}{\partial y'} \right|_{x_0} = 0, \quad \text{and} \quad \left. \frac{\partial f}{\partial y'} \right|_{x_1} = 0.$$

Euler-Lagrange again

Hence, as before, the extremal must satisfy the E-L equations

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.$$

But now that the boundary conditions were not specified as part of the problem, we get natural boundary conditions

$$\left.\frac{\partial f}{\partial y'}\right|_{x_0} = 0, \quad \text{and} \quad \left.\frac{\partial f}{\partial y'}\right|_{x_1} = 0,$$

which specify that these derivatives will be zero at the end-points.

Extensions (i)

What happens if we fix one end point, e.g. $y(x_0) = y_0$.

The result is we cannot vary this end-point when perturbing, so $\eta(x_0) = 0$, and therefore the condition

$$\left[\eta \frac{\partial f}{\partial y'}\right]_{x_0}^{x_1} = 0$$

collapses to give just one extra condition

$$\left.\frac{\partial f}{\partial y'}\right|_{x_1} = 0.$$

Hence the boundary conditions are **modular** in the sense that when we remove one, we replace it automatically with the natural boundary condition.

Extensions (ii)

The above results can be extended as before, in particular, consider a functional containing higher order derivatives:

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', y'')\, dx,$$

$$\begin{aligned}\delta F(\eta, y) = &\left[\eta\left(\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right) + \eta'\frac{\partial f}{\partial y''}\right]_{x_0}^{x_1} \\ &+ \int_{x_0}^{x_1} \eta\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''}\right] dx\end{aligned}$$

where we see integration by parts introduces terms including η and η'.

Extensions (ii)

The Euler-Lagrange equations are

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$

where the natural boundary conditions are

$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right]_{x_0} = 0 \qquad \text{and} \qquad \left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right]_{x_1} = 0$$

$$\left.\frac{\partial f}{\partial y''}\right|_{x_0} = 0 \qquad \text{and} \qquad \left.\frac{\partial f}{\partial y''}\right|_{x_1} = 0$$

where the first two replace absent conditions on the value of y at the end-points, and the second two replace absent conditions on y' at the end points.

Bent beam

Let $y : [0,d] \to \mathbb{R}$ describe the shape of the beam, and $\rho : [0,d] \to \mathbb{R}$ be the load on the beam.

For a bent elastic beam the potential energy from elastic forces is

$$V_1 = \frac{\kappa}{2} \int_0^d y''^2 \, dx, \qquad \kappa = \text{flexural rigidity.}$$

The potential energy is

$$V_2 = -\int_0^d \rho(x) y(x) \, dx.$$

Thus the total potential energy is

$$V = \int_0^d \left[\frac{\kappa y''^2}{2} - \rho(x) y(x) \right] dx.$$

Bent Beam: see earlier

The Euler-Lagrange equation is

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} &= 0 \\ y^{(4)} &= \frac{\rho(x)}{\kappa} \end{aligned}$$

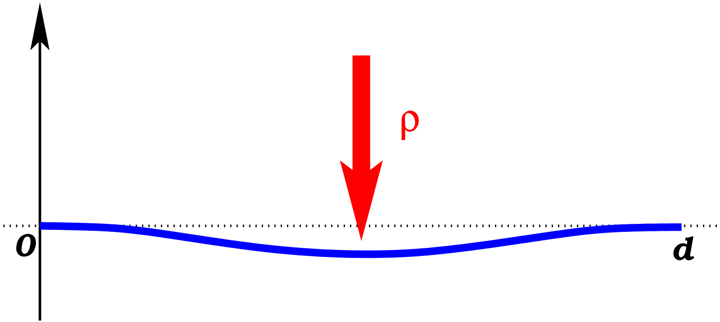
This DE has the solution

$$y(x) = P(x) + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

where the c_k's are the constants of integration, and $P(x)$ is a particular solution to $P^{(4)}(x) = \rho(x)/\kappa$.

Bent Beam: Example 1

Doubly clamped: see earlier lectures.



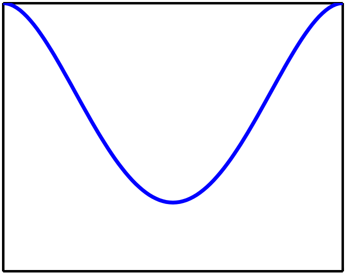
Two end-points are fixed, and clamped so that they are level, e.g. $y(0) = 0$, $y'(0) = 0$, and $y(d) = 0$ and $y'(d) = 0$.

Bent Beam: Example 1

Doubly clamped, uniform load: see earlier lectures.

Choose a solution of the form

$$y(x) = \frac{\rho(d-x)^2 x^2}{24\kappa}$$



Then the derivative

$$y'(x) = \frac{2\rho(d-x)x^2}{12\kappa} + \frac{\rho(d-x)^2 x}{12\kappa}$$

We can see that the constraints are satisfied

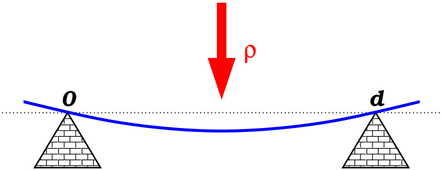
$$y(0) = 0 \qquad \text{and} \qquad y(d) = 0$$
$$y'(0) = 0 \qquad \text{and} \qquad y'(d) = 0$$

Bent Beam Example 2

Freely supported, uniform load

The natural constraints are

$$\left.\frac{\partial f}{\partial y''}\right|_{x_0} = \kappa y''(x_0) = 0$$
$$\left.\frac{\partial f}{\partial y''}\right|_{x_1} = \kappa y''(x_1) = 0$$



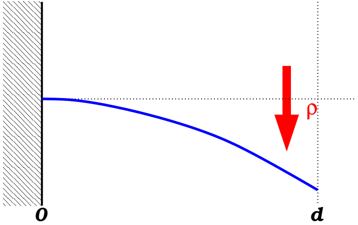
The fixed end-points are $y(0) = y(d) = 0$, so uniform load solution looks like

$$y(x) = \frac{\rho x \left(d^3 - 2dx^2 + x^3\right)}{24\kappa}$$

Bent Beam Example 3

One end-point fixed, and clamped.
Called a **Cantilever**
The natural constraints are

$$
\begin{aligned}
\left.\frac{\partial f}{\partial y''}\right|_{x_1} &= \kappa y''(x_1) = 0 \\
\frac{\partial f}{\partial y'} - \frac{d}{dx}\left.\frac{\partial f}{\partial y''}\right|_{x_1} &= -\frac{d}{dx}\left.\kappa y''\right|_{x_1} \\
&= \kappa y'''(x_1) = 0
\end{aligned}
$$



The clamped end-point introduces constraints $y(0) = 0$ and $y'(0) = 0$ so the solution for uniform load is

$$y(x) = \frac{\rho x^2(6d^2 - 4dx + x^2)}{24\kappa} \quad \text{and} \quad y(d) = \frac{\rho d^4}{8\kappa}$$

Bent beam, end-points conditions

End-point conditions are modular: i.e., we can use different end-point conditions at each end of the beam.

- **clamped:** specifies the position, and the derivative.
- **freely supported:** specifies the position. Natural boundary condition is that the second derivative is zero at the end point.
- **no condition:** neither position, nor end-point are specified, so the natural boundary conditions fix the second and third derivatives at the end point to be zero.

Bent Beam Example 4

One end-point fixed, but not clamped.

In this case the beam just collapses, and lies vertical.

The approach doesn't work, but this is a failure of the **model**, not the **method**.

In this case, the cantilever approximation (that x_1 is fixed) no longer works, and we need to consider a more general model that allows x_1 to vary as well.