PDE's Assignment 1

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Question 1: Consider the ODE:

$$y'' - y' + xy = 0$$

Seek power series solutions.
Solutions have form:
$y = \sum_{n=0}^{\infty} a_n x^n$
From this:

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$-y' = -\sum_{n=1}^{\infty} n a_n x^{n-1} = -\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

So

$$y'' - y' + xy = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

Extracting the $n = 0$ terms:

$$= 2a_2 - a_1 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$= 2a_2 - a_1 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} x^n - (n+1) a_{n+1} x^n + a_{n-1} x^n\right]$$

$$= 2a_2 - a_1 + \sum_{n=1}^{\infty} \left[\left((n+2)(n+1) a_{n+2} - (n+1) a_{n+1} + a_{n-1}\right) x^n\right]$$

For equality all coefficients must equate to zero.

$$2a_2 - a_1 = 0 \implies a_2 = \frac{a_1}{2}$$

$$(n+1)(n+1)a_{n+2} - (n+1)a_{n+1} + a_{n-1} = 0$$

So for $n = 1$

$$4a_3 - 2a_2 + a_0 = 0$$

Question 2:
Approximate the function $f(x) = x$ with sine
Using the interval $(0, \pi)$. Assume the function is odd over the interval $(-\pi, \pi)$
Where $g(x)$ is the function approximating $f(x)$, the function will have form:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(\left(\frac{n\pi x}{\pi}\right)) = \sum_{n=1}^{\infty} b_n \sin((nx))$$

Where:

$$b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right)$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin(nx)$$

Integration by parts gives

$$b_n = \frac{\sin(n\pi) - \pi n \cos(\pi n)}{\pi n^2}$$

$$b_n = -\frac{\cos(\pi n)}{n} = \frac{(-1)^{n+1}}{n}$$

So:

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin((nx))$$

Using this, the second coefficient, i.e. b_2 Will be when $n = 2$ and

$$b_2 = \frac{(-1)^3}{2} = \frac{-1}{2}$$

So b_2 is negative.

Question 3:
The given MATLAB code written algebraically gives a 1x2 matrix for x and y $x = [1.1 \quad 2.6]$ and $y = [0.4 \quad 3]$
The line z=linspace(0,3) creates a set of 100 points, linearly spaced between 0 and 3 (inclusive)
When written more neatly the function gives

$$p = y(1)\frac{z - x(2)}{x(1) - x(2)} + y(2)\frac{z - x(1)}{x(2) - x(1)}$$

Which is a simple linear regression function.
It finds the line which passes through the points (x_1, y_1) and (x_2, y_2), with endpoints 0 and 3.
The last line, plot$(x, y,' o', z, p)$, generates a plot with circle markings for the points (x_1, y_1) and (x_2, y_2), and plots the line (z, p)

Question 4:
(a)
(b)

Question 5:
Solve $\frac{\delta^2 u}{\delta t^2} = 4\frac{\delta^2 u}{\delta x^2}$
With the boundary conditions

$$u_x(0,t) = u(\pi,t) = 0$$

And initial condition $u(x,0) = 0$ Separation of variables for u gives

$$u(x,t) = X(x)T(x)$$

Where X is a function of x, and T is a function of t.
Finding the second derivatives gives:

$$\frac{\delta^2 u}{\delta t^2} = X\ddot{T}$$

$$4\frac{\delta^2 u}{\delta x^2} = 4X''T$$

$$X\ddot{T} = 4X''T$$
$$\frac{X''}{X} = \frac{\ddot{T}}{4T} = -\lambda$$

Where λ is a constant.
These are solved separately:

$$X'' + \lambda X = 0$$

There are a few cases for the solutions of this:

$$\lambda < 0$$

Gives solutions
$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

But $X'(0) = 0$

$$X'(x) = \sqrt{\lambda}c_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}x} \implies c_1 = c_2$$

So
$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_1 e^{-\sqrt{\lambda}x}$$

Second initial condition: $X(\pi) = 0$

$$X(\pi) = c_1 e^{\sqrt{\lambda}\pi} + c_1 e^{-\sqrt{\lambda}\pi} = 0 \implies c_1 = 0$$

Which is a trivial solution.

$$\lambda = 0$$

$$X'' = 0 \implies X(x) = c_1 x + c_2$$

$$X'(0) = 0 \implies c_1 = 0$$

So $X(\pi) = c_2 = 0 \implies c_2 = 0$
Which once again is trivial

$$\lambda > 0$$

Gives solutions:

$$X(x) = c_1 \sin(x\sqrt{\lambda}) + c_2 \cos(x\sqrt{\lambda})$$
$$X'(x) = \sqrt{\lambda}c_1 \cos(x\sqrt{\lambda}) - \sqrt{\lambda}c_2 \sin(x\sqrt{\lambda})$$
$$X'(0) = 0 = \sqrt{\lambda}c_1 \cos(0) \implies c_1 = 0$$
$$X(\pi) = c_2 \cos(\pi\sqrt{\lambda})$$

Which will have non trivial solutions when $\pi\sqrt{\lambda} = n\pi + \frac{\pi}{2}$, $n \in \boldsymbol{N}$

$$\lambda = n^2 + \frac{1}{4}$$

Solutions for T:

$$\ddot{T} + 4\lambda T = 0$$

But $\lambda = n^2 + \frac{1}{4}$.

$$\ddot{T} + (4n^2 + 1)T = 0$$