Random Processes Assignment

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Question 1. A CTMC $(Z(t), t \geq 0)$ has the following transition rate matrix:

$$Q = \begin{pmatrix} -5 & 2 & 0 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 2 & 1 & 0 \\ 0 & 4 & 2 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

(a) Write down the state space S of $Z(t)$ and determine which pairs of states are said to communicate.
Solution

$$S = \{1, 2, 3, 4, 5, 6\}$$

The transitions are: $1 \to 2$,
$1 \to 6$,
$2 \to 3$,
$3 \to 2$,
$3 \to 4$,
$3 \to 5$,
$4 \to 2$,
$4 \to 3$,
$6 \to 1$. This means that 1 communicates with 6, and 2,3, and 4 each communicate with each other **As required.**

(b) Determine, with reasoning, whether each state is either transient or recurrent
Solution State 5 is recurrent, as it is an absorbing state - there is no way to leave 5, but it can be entered through state 3. States 1,2,3,4, and 6 are all transient as from all of them state 5 can be reached and then not left. So over an infinite period of time the system will all be in state 5. **As required.**

(c) Determine the communicating classes of S and, with reasoning, correctly label each class as either transient or recurrent.
Solution $C_1 = \{1, 6\}$ is a communicating class, $C_2 = \{2, 3, 4\}$ is a communicating class, and $C_3 = \{5\}$ is its own communicating class.
C_1 is transient as it can be left via the transition between states 1 and 2, and then never reentered.
C_2 is transient as it can be left via the transition between states 3 and 5, and then never reentered.
C_3 is recurrent as once entered, the class cannot be left. **As required.**

(d) Is $Z(t)$ reducible or irreducible? Why?
Solution $Z(t)$ is reducible, as it is impossible to get from some states to other states. **As required.**

Question 2. For Problem 1 from Tutorial 1
In short: A bird migrates between 3 islands. Tracking a particular bird. $p_{AB} = 1/2 = p_{AC} = 1/2$, $p_{BA} = 3/4$, $p_{BC} = 1/4$, $p_{CA} = 1$ The bird stays on average at A for 1/5 hours, B for 1/5 hours and C for 1/4 hours.

(a) Evaluate analytically the equilibrium distribution of the CTMC
Solution Firstly note the Q matrix:

$$Q = \begin{pmatrix} -5 & 5/2 & 5/2 \\ 15/4 & -5 & 5/4 \\ 4 & 0 & -4 \end{pmatrix}$$

$$\pi Q = 0$$

Subject to $\sum_i \pi_i = 1$ Which gives the system (we will ignore what would have been the second equation because 4 equations in 3 unknowns):

$$-5\pi_1 + 5/2\pi_2 + 5/2\pi_3 = 0 \tag{1}$$
$$4\pi_1 - 4\pi_3 = 0 \tag{2}$$
$$\pi_1 + \pi_2 + \pi_3 = 1 \tag{3}$$

Using (2) $\pi_1 = \pi_3$ Subbing this into (1) gives

$$-2\pi_1 + \pi_2 + \pi_3 = 0 \implies \pi_2 = \pi_1$$

And subbing into (3) finally gives

$$\pi_1 + \pi_2 + \pi_3 = 1 \implies \pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

As required.

(b) Explain what the equilibrium probabilities represent

Solution Over a long period of time of no observation, if we look at the system we expect to see it in each state with those corresponding probabilities. I.e. $\pi_1 = 1/3$ probability to be in state 1, etc.. **As required.**

(c) Provide code which numerically evaluates the equilibrium distribution by using a function which solves systems of linear equations

Solution

```
Q = [-5,5/2,5/2;
     15/4,-5,5/4;
     4,0,-4;
     1,1,1];

zer = [0;0;0;1];
x = zer\Q;
xnorm = x/sum(x)
```

Which gives solution

```
xnorm =

    0.3333    0.3333    0.3333
```

Where `xnorm` is the π such that $\sum_i \pi_i = 1$

As required.

Question 3. Consider a game of tennis when deuce is reached. If a player wins the next point, they are said to have advantage. On the following point, the player with advantage wins if they win the point, and the game returns to deuce if they lose the point. Each point tends to last for $1/\lambda_D$ minutes, on average, in deuce, and for $1/\lambda_A$ on average, if a player has advantage. Assume that the probability player A wins any game is p_A. Let the state space be $S = \{1, 2, 3, 4, 5\}$ where 1 : deuce, 2 : advantage A, 3 : advantage B, 4 : A wins, 5 : B wins

(a) Determine the generator matrix if this were modelled using a CTMC

Solution

$$Q = \begin{pmatrix} -\lambda_D & p_A\lambda_D & (1-p_A)\lambda_D & 0 & 0 \\ (1-p_A)\lambda_A & -\lambda_A & 0 & p_A\lambda_A & 0 \\ p_A\lambda_A & 0 & -\lambda_A & 0 & (1-p_A)\lambda_A \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As required.

(b) After the first time deuce has been reached in the game, determine the probability that player A eventually wins (before player B wins)

Solution Want to find the hitting probability for state 4, i.e. $f_1^{(4)}$.
$f_i^{(4)}$ is the minimal non-negative solution to

$$(-q_{ii})x_i = \sum_{\substack{k\in S\\ k\neq i}} q_{ik}x_k$$

Subject to $x_4 = 1$ and $x_5 = 0$.
Which gives the system:

$$\begin{aligned}
-\lambda_D x_1 &= p_A\lambda_D x_2 + (1-p_A)\lambda_D x_3\\
-\lambda_A x_2 &= (1-p_A)\lambda_A x_1 + p_A\lambda_A x_4\\
-\lambda_A x_3 &= p_A\lambda_A x_1 + (1-p_A)\lambda_A x_5
\end{aligned}$$

Plugging in the boundaries, and removing time dependence:

$$\begin{aligned}
x_1 &= p_A x_2 + (1-p_A)x_3\\
x_2 &= (1-p_A)x_1 + p_A\\
x_3 &= p_A x_1
\end{aligned}$$

$$\begin{aligned}
x_1 &= p_A\left((1-p_A)x_1 + p_A\right) + p_A(1-p_A)x_1\\
x_1 &= 2p_A(1-p_A)x_1 + p_A^2\\
x_1 - 2p_A(1-p_A)x_1 &= +p_A^2\\
x_1(1-2p_A(1-p_A)) &= +p_A^2\\
\therefore x_1 &= \frac{p_A^2}{1-2p_A(1-p_A)}
\end{aligned}$$

As required.

Question 4. Packets arrive to a switch as a Poisson process at rate λ and are then stored in a buffer. The switch processes packets at rate μ. The processing time for a packet in the switch is exponentially distributed.
After each packet has been processed, it is transmitted with probability p. However, with probability $1-p$, a processed packet returns to the buffer for re-processing by the switch, due to switch processing errors.

(a) Write down an appropriate state space for a CTMC model of the number of packets in the buffer

Solution

$$S = Z_+ = \{0, 1, 2, \dots\}$$

As required.

(b) Write down the transition rates q_{ij} for the CTMC

Solution

$$q_{ij} = \begin{cases} \lambda, & j = i+1\\ p\mu, & j = i-1\\ -\lambda - p\mu, & j = i \end{cases}$$

As required.

(c) Let f_j be the probability that the buffer *ever* empties, given that it starts with j packets. Write down a set of equations and boundary conditions satisfied by the f_j. Give an explanation for your boundary conditions and explain what other conditions we need in order to distinguish which solution to these equations gives us the f_j.

Solution f_j is the minimal non-negative (x) solution to

$$(-q_{jj})x_j = \sum_{\substack{k\in S\\ k\neq j}} q_{jk}x_k$$

Subject to $x_0 = 1$. Gives the system:

$$(\lambda + p\mu)x_j = \lambda x_{j+1} + p\mu x_{j-1}$$

The boundary condition $x_0 = 1$ corresponds to the conditional probability

$$P(\text{ever hit state 0}| \text{ in state 0}) = 1$$

As you trivially have hit the state if you are already in it. **As required.**

(d) Solve these equations, and give the criterion for the CTMC to be recurrent.

Solution Let $x_j = x^j$

$$\begin{aligned}
(\lambda + p\mu)x^j &= \lambda x^{j+1} + p\mu x^{j-1}\\
(\lambda + p\mu)x &= \lambda x^2 + p\mu\\
\lambda x^2 - (\lambda + p\mu)x + p\mu &= 0\\
(x-1)(\lambda x - p\mu) &= 0\\
\implies x &= 1, \frac{p\mu}{\lambda}
\end{aligned}$$

Which gives:

$$x_j = a1^j + b\left(\frac{p\mu}{\lambda}\right)^j = a + b\left(\frac{p\mu}{\lambda}\right)^j$$

if $\frac{p\mu}{\lambda} \neq 1$
And

$$x_j = a + bj$$

if $\frac{p\mu}{\lambda} = 1$

Using the boundary condition on the first:

$$x_0 = 1 \implies a + b = 1 \implies a = 1 - b$$

$$x_j = 1 - b + b\left(\frac{p\mu}{\lambda}\right)^j$$

Want the minimal non-negative solution. Consider $p\mu/\lambda$

- $\frac{p\mu}{\lambda} > 1$

  $$\lim_{j\to\infty} x_j \to 1 - b + b(\infty)$$

  So the minimal, non-negative solution for this is setting $b = 0$, i.e.

  $$f_j = 1$$

- $\frac{p\mu}{\lambda} < 1$

  $$\lim_{j\to\infty} x_j \to 1 - b + b(0)$$

  So the minimal non-negative solution for this is setting $b = 1$, i.e.

  $$f_j = \left(\frac{p\mu}{\lambda}\right)^j$$

- Looking back at $\frac{p\mu}{\lambda} = 1$: Boundary conditions give:

$$x_0 = a + b(0) = 1 \implies a = 1$$

$$x_j = 1 + bj$$

$$\lim_{j \to \infty} x_j \to 1 + b(\infty)$$

  So set $b = 0$, which gives

$$f_j = 1$$

Which gives the hitting probability solutions:

$$f_j = \begin{cases} \left(\frac{p\mu}{\lambda}\right)^j, & \frac{p\mu}{\lambda} < 1 \\ 1, & \frac{p\mu}{\lambda} \geq 1 \end{cases}$$

The CTMC will be recurrent if f_j is 1. Otherwise we have a finite probability to return to that state, and we may never hit it - making it a transient state. **As required.**