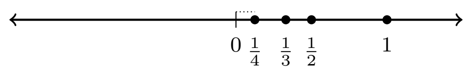
LECTURE 16

Last lecture we were discussing open and closed sets. Let's look at another example.

Example: Let S be the set

$$S = \{ \frac{1}{n} \mid \; n = 1, 2, 3, \ldots \} .$$

Then S is neither open nor closed. Here's an (impressionistic) picture of this set:



It should be easy to convince yourself that this set is not closed — if you draw a small ϵ neighbourhood around the point 1 for instance, then this ϵ-neighbourhood does not contain any points of S apart from 1. Therefore S is not open.

To show that S is not closed, we need to show that $\mathbb{R} \setminus S$ is not open. Here the problem is the point 0. Since $0 \notin S$ we have $0 \in \mathbb{R} \setminus S$. If $\mathbb{R} \setminus S$ were to be open, then there would have to exist an ϵ-neighbourhood of 0 which is disjoint from S. This can never happen: if $\epsilon > 0$ then by the Archimedean Property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$. But then $1/n \in I_\epsilon(0)$. Therefore every ϵ-neighbourhood of 0 contains points of S other than 0. Therefore $\mathbb{R} \setminus S$ is not open and so S is not closed.

This problem with the point 0 is the only obstruction to the set S being closed — the set $S \cup \{ 0 \}$ *is* closed. A good way to see that is to use the following very useful theorem, which gives a characterization of closed sets in terms of convergent sequences.

Theorem 3.3: A set $S \subset \mathbb{R}$ is closed if and only if for every sequence (x_n) such that $x_n \to x$, if $x_n \in S$ for all n then $x \in S$.

Proof: ($\Rightarrow$) Suppose S is closed. Let (x_n) be a sequence such that $x_n \in S$ for all n and suppose $x_n \to x$. We'll prove that $x \in S$. Suppose it's not, i.e. suppose $x \in \mathbb{R} \setminus S$. Since $\mathbb{R} \setminus S$ is open, there exists $\epsilon > 0$ such that $I_\epsilon(x) \subset \mathbb{R} \setminus S$. Therefore $y \in I_\epsilon(s) \implies y \notin S$. However, the sequence (x_n) converges to x. Therefore, since $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|x_n - x| < \epsilon$. If x_n satisfies the inequality $|x_n - x| < \epsilon$ then x_n belongs to the ϵ-neighbourhood $I_\epsilon(x)$. This is a contradiction since $x_n \in S$ for all n. Therefore $x \in S$.

($\Leftarrow$) Suppose that for all sequences (x_n) in S such that $x_n \to x$, we have $x \in S$. We will prove that S is closed. If it is not closed, then $\mathbb{R} \setminus S$ is not open. Therefore there is a point $x \in \mathbb{R} \setminus S$ such that for all $\epsilon > 0$, there exist points $y \in S$, $y \neq x$, such that $y \in I_\epsilon(x)$. In particular, for every $n \in \mathbb{N}$ there exists $x_n \in S$ such that $x_n \in I_\epsilon(x)$ (we see this by taking $\epsilon = 1/n$). Thus we have a sequence (x_n) in S which satisfies $|x_n - x| < 1/n$ for every $n \in \mathbb{N}$. The sequence (x_n) converges to x (since by the Squeeze Theorem we have $|x_n - x| \to 0$, which happens if and only if $x_n - x \to 0$, which happens if and only if $x_n \to x$). This is a contradiction, since $x \notin S$. Therefore S is closed. ∎

Definition 3.4: A set $N \subset \mathbb{R}$ is called a *neighbourhood* of a point $x \in \mathbb{R}$ if there exists $\epsilon > 0$ such that $I_\epsilon(x) \subset N$.

Remark: There is a similarity between the definition of nieghbourhood of a point and the definition of open set. There is a crucial difference however: in the definition of an open set U, for *every* $x \in U$ there exists an $\epsilon > 0$ such that $I_\epsilon(x) \subset U$; whereas in the definition of neighbourhood of a point x, we are only asserting that we can find such an ϵ for this particular x. It is not hard to show that a set is open if and only if it is a neighbourhood of each of its points.

Example: the set $[0, 1)$ is a neighbourhood of $1/2$, a neighbourhood of $1/3$ — in fact it is a neighbourhood of any point $x \in (0, 1)$. It is not a neighbourhood of 0 or of 1.

Definition 3.5: A set $S \subset \mathbb{R}$ is said to be *bounded* if there exists $K > 0$ such that $|x| \leq K$ for all $x \in S$.

Example: For instance the set $\mathbb{N}$ of natural numbers is not bounded; neither is the interval $(0, \infty)$. The interval $[a, b)$ is bounded.

Suppose that S is bounded and $(x_n)_{n=1}^{\infty}$ is a sequence in S. Since S is bounded the sequence $(x_n)_{n=1}^{\infty}$ is bounded and hence has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, by the Bolzano-Weierstrass Theorem. Suppose $x_{n_k} \to L$. If S is closed, in addition to being bounded, then by Theorem 3.3 we must have $L \in S$, since $x_{n_k} \in S$ for all k (remember that $(x_n)_{n=1}^{\infty}$ was a sequence in S).

Therefore, we've observed that if S is closed and bounded, then every sequence in S has a subsequence which converges in S (i.e. converges to a point of S). This property of such sets S turns out to be so useful it is given its own name.

Definition 3.6: A set $S \subset \mathbb{R}$ is said to be *sequentially compact* if for every sequence $(x_n)_{n=1}^{\infty}$ in S, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_k} \to x$ for some $x \in S$.

Notes: We observe the following:

1. From the discussion in the paragraphs above we see that if S is closed and bounded, then S is sequentially compact.

2. Suppose that S is sequentially compact. Then S is bounded. (Proof by contradiction: suppose S were not bounded, then for every $n \in \mathbb{N}$, there exists $x_n \in S$ such that $|x_n| > n$ (otherwise S would be bounded by some $n \in \mathbb{N}$). No subsequence of (x_n) can be bounded, and hence no subsequence of (x_n) can be convergent — contradiction.)

3. Suppose that S is sequentially compact. Then S is closed. (Proof using Theorem 3.3: let (x_n) be a sequence in S and suppose that $x_n \to x$. We will prove that $x \in S$. Since S is sequentially compact there exists a subsequence (x_{n_k}) of (x_n) which converges to a point $y \in S$. Since $x_n \to x$ we must have $x_{n_k} \to y$. Hence $x = y$. Hence $x \in S$.)

Putting these three observations together we see that we have proven the following important theorem (so important it gets a name):

Theorem 3.7 (Heine-Borel Theorem): A set $S \subset \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.