1 Tute 1

1. Revision

   (a) Degenerate cases when
   $$y = x^n$$

   (b) Geometrically it is the point between $\mathbf{x}$ and $\mathbf{y}$

   (c) Partial derivative
   $$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(\mathbf{x} + \mathbf{e}_i h) - f(\mathbf{x})}{h}$$
   $$\frac{\partial f(x,y)}{\partial x} = 2x - 12x^3$$
   $$\frac{\partial f(x,y)}{\partial x} = \sin(2x + yz) + 2x\cos(2x + yz)$$
   $$\frac{\partial f(x,y)}{\partial x} = \dots$$

   (d) Gradients:
   $$\Delta f(x,y) = \left(2x - 12x^3 y, 4y^3 - 3x^4\right)$$
   $$\dots$$

   (e) Taylor expansions

2. Chain rule for $\frac{dz}{dt}$
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

3. Taylor's Theorem for a polynomial approximation for
$$f(x,y) = \sin(x + y^2)$$
$$f(x + \delta x, y + \delta y) = f(x,y) + \delta\mathbf{x}\Delta f(x,y) + \frac{1}{2}\delta\mathbf{x}^T H(x,y)\delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x}^3)$$
$$\sin(x + \delta x + (y + \delta y)^2) = \sin(x + y^2) + \delta\mathbf{x}\begin{pmatrix}\cos(x + y^2)\\ 2y\sin(x + y^2)\end{pmatrix} + \frac{1}{2}\delta\mathbf{x}^T H(x,y)\delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x}^3)$$

4. Cylinder of largest volume inside a unit sphere Volume of a cylinder
$$V = \pi r^2 * h$$
Where r is the radius of the cylinder
$$r^2 + \left(\frac{h}{2}\right)^2 \leq 1$$
And $h, r > 0$. So we have
$$max\ V = \pi r^2 h$$
$$r^2 + \left(\frac{h}{2}\right)^2 - 1 = 0$$

So we want

$$H(r,h,\lambda) = \pi r^2 h + \lambda\left(r^2 + \left(\frac{h}{2}\right)^2 - 1\right)$$

Solve:

$$\frac{\partial H}{\partial r} = 0$$
$$\frac{\partial H}{\partial h} = 0$$
$$\frac{\partial H}{\partial \lambda} = 0$$

...

5.

2 Tute 2

1. (a)

$$F\{y\} = \int_0^{\pi/2} (y^2 + y'^2 - 2y\sin x)dx$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$
$$\frac{d}{dx}(2y') - 2y + 2\sin x = 0$$
$$2y'' - 2y + 2\sin x = 0$$
$$y'' - y + \sin x = 0$$

Homo: $y'' - y = 0$:

$$y_h = a\sin x + b\cos x$$

(b)

$$F\{y\} = \int_1^2 \frac{y'^2}{x^3}dx$$

No y dependence

$$\frac{\partial f}{\partial y'} = c_1^*$$
$$\frac{2y'}{x^3} = c_1^*$$
$$y' = c_1^+ x^3$$
$$y = x^4 c_1 + c_2$$

Use $y(1) = 0$ and $y(2) = 15$ but cbf

(c)

$$F\{y\} = \int_0^2 (xy' + y'^2)dx$$

No explicit y dependence vol 2

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y} = 0$$

So

$$\frac{\partial f}{\partial y'} = const$$

$$\begin{aligned} \frac{\partial f}{\partial y'} &= x + 2y' = const \\ 2y' &= c_1 - x \\ y' &= \frac{c_1}{2} - \frac{x}{2} \\ y &= \frac{c_1 x}{2} - \frac{x^2}{4} + c_2 \end{aligned}$$

$y(0) = 1 \implies c_2 = 1$ $y(2) = 0$

$$\begin{aligned} \frac{c_1 2}{2} - \frac{4}{4} + 1 = 0 \\ c_1 = 0 \end{aligned}$$

2. If you layer the shit out of glass then i guess so
3. Use conic coords

$$\begin{aligned} x &= r\cos\theta\sin\alpha \\ y &= r\sin\theta\sin\alpha \\ z &= r\cos\alpha \end{aligned}$$

Short path

$$\begin{aligned} dx &= \cos\theta\sin\alpha dr - r\sin\theta\sin\alpha d\theta \\ dy &= \sin\theta\sin\alpha dr + r\cos\theta\sin\alpha d\theta \\ dz &= \cos\alpha dr \end{aligned}$$

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (\cos\theta\sin\alpha dr - r\sin\theta\sin\alpha d\theta)^2 + (\sin\theta\sin\alpha dr + r\cos\theta\sin\alpha d\theta)^2 + \cos^2\alpha dr^2 \\ &= \cos^2\theta\sin^2\alpha dr^2 - 2r\sin\theta\cos\theta\sin^2\alpha d\theta + r^2\sin^2\theta\sin^2\alpha d\theta^2 + \sin^2\theta\sin^2\alpha dr^2 \\ &+ 2r\sin\theta\cos\theta\sin^2\alpha d\theta + r^2\cos^2\theta\sin^2\alpha d\theta^2 + \cos^2\alpha dr^2 \\ &= \sin^2\alpha dr^2 + r^2\sin^2\alpha d\theta^2 + \cos^2\alpha dr^2 \\ &= dr^2 + r^2\sin^2\alpha d\theta^2 \end{aligned}$$

$$
\begin{aligned}
F\{y\} &= \int_{P_1}^{P_2} ds \\
&= \int_{P_1}^{P_2} \sqrt{dr^2 + r^2 \sin^2 \alpha d\theta^2} \\
&= \int_{P_1}^{P_2} \sqrt{1 + r^2 \sin^2 \alpha \left(\frac{d\theta}{dr}\right)^2} dr
\end{aligned}
$$

So

$$f(r, \theta, \theta') = 1 + r^2 \sin^2 \alpha \theta'^2$$

No θ dependence.

$$\frac{\partial f}{\partial \theta'} = c_1$$

4.

5.

3 Tutorial 3

1.

$$F\{y\} = \int f(x, y, y', y'', y''')dx$$

Taylor's theorem

$$
\begin{aligned}
&f(x, y + \epsilon\eta, y' + \epsilon'\eta', y'' + \epsilon\eta'', y''' + \epsilon\eta''') \\
&= f(x, y, y', y'', y''') + \epsilon(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''} + \eta'''\frac{\partial f}{\partial y'''}) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

$$F\{y + \epsilon\eta\} = \int_{x_0}^{x_1} f(x, y, y', y'', y''') + \epsilon(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''} + \eta'''\frac{\partial f}{\partial y'''}) + \mathcal{O}(\epsilon^2)dx$$

$$
\begin{aligned}
\delta F &= \lim_{\epsilon \to 0} \frac{F\{y + \epsilon\eta\} - F\{y\}}{\epsilon} \\
&= \lim_{\epsilon \to 0} \int_{x_0}^{x_1} f(x, y, y', y'', y''')/\epsilon + (\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''} + \eta'''\frac{\partial f}{\partial y'''}) + \mathcal{O}(\epsilon) - f(x, y, y', y'', y''')/\epsilon \\
&= \lim_{\epsilon \to 0} \int_{x_0}^{x_1} (\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''} + \eta'''\frac{\partial f}{\partial y'''}) + \mathcal{O}(\epsilon)
\end{aligned}
$$

Integrate by parts

2. Geodesics in N dim Euclidean space, assume $\mathbb{R}^N$ with $\mathbf{q} = (q_1, \ldots, q_n)$ with $||\mathbf{q}|| = \left(\sum_{n=1}^{N} q_n^2\right)^{1/2}$ find the extremal of

$$S\{\mathbf{q}(t)\} = \int ds$$

3. Assuming every atom of a carbon nanotorus with genus $g = 1$ is bonded to exactly 3 neighbours, how many pentagonal, hexagonal and heptagonal rings must occur when assuming that

   (a)

   (b)

   (c)

4. For some $n \in \mathbb{N}$ show

$$\Gamma(n + 1/2) = \frac{\sqrt{\pi}(2n-1)!!}{2^n}$$

   Where !! is the double factorial ($n!! = n(n-2)(n-4)\dots$)

5.

$$B(x, y)B(x + y, z) = B(y, z)B(y + z, x) = B(z, x)B(z + x, y)$$

$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$\begin{aligned}
B(x, y)B(x + y, z) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \frac{\Gamma(x+y)\Gamma(z)}{\Gamma(x+y+z)} \\
&= \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma(x+y+z)} \\
B(y, z)B(y + z, x) &= \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma(x+y+z)} \\
B(z, x)B(z + x, y) &= \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma(x+y+z)}
\end{aligned}$$

6.

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

   (a)

$$\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a + 1, b + 1; c + 1; z)$$

$$\begin{aligned}
\frac{d}{dz} F(a, b; c; z) &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} n z^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1} (n+1)!} (n+1) z^n
\end{aligned}$$

   Note

$$(a)_{n+1} = \frac{\Gamma(a + n + 1)}{\Gamma(a)} = \frac{a\Gamma(a + n)}{\Gamma(a)} = a(a)_n$$

$$\frac{d}{dz}F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(n+1)!}(n+1)z^n$$
$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(n+1)!}(n+1)z^n$$
$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}z^n$$
$$= \frac{ab}{c}F(a+1,b+1;c+1;z)$$

(b)

$$\left[\frac{d}{dz}F(a,b;c;z)\right]_{z=0} = \frac{ab}{c}$$

The only non-zero term is $z^0 = 0^0 = 0$ and hence you get $\frac{ab}{c}$

(c)

$$\frac{1}{\sqrt{1-z}} = F(1/2,b;b;z)$$

$$F(1/2,b;b;z) = \sum_{n=0}^{\infty} \frac{(1/2)_n(b)_n}{(b)_n n!}z^n$$
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!}z^n$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(1/2+n)}{\Gamma(1/2)n!}z^n$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(1/2+n)}{\Gamma(1/2)\Gamma(n+1)}z^n$$
$$= \sum_{n=0}^{\infty} \binom{1/2}{n}z^n$$
$$= \frac{1}{\sqrt{1-z}}$$

$$\frac{1}{\sqrt{1-z}} = \sum_{n=0}^{\infty} z^n\binom{1/2}{n}$$

Note

$$\binom{x}{y} = \frac{x!}{y!(x-y)!} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Assuming int at first then for all numbers.

4 Tute 4

1. Assume $\Phi(\rho) = -A\rho^{-6} + B\rho^{-12}$ calculate the interaction of

   (a) A point $P = (0,0,\delta)$ and a line ℓ_1 collinear with the $x-axis$, $-\infty < x < \infty$, with uniform line density η_1.

$\ell_1 = (x, 0, 0)$ The distance from the point to a point on the line is

$$d = \sqrt{x^2 + \delta^2}$$

$$\begin{aligned} E &= \eta_1 \eta_1 \int_{S_2} \int_{S_1} \Phi(\rho) dA_1 dA_2 \\ &= \end{aligned}$$

(b) ℓ_1 from A and ℓ_2 parameterised by $(t\cos\theta, t\sin\theta, \delta)$

(c) Whats the interaction between ℓ_1 and ℓ_2 for $\theta = \pi/2$

(d) Same as before but $\theta = \pi$.

2. The surface $\mathcal{T}$:

$$\mathbf{r}(\theta, \phi) = ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$

With radius r and R distance from the centre of the torus to the centre of the tube, $-\pi < \theta \leq \pi$, $-\pi < \phi \leq \pi$.

(a) tangent vector in θ direction:

$$\frac{\partial r}{\partial \theta} = \begin{pmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

(b) tangent vector in ϕ direction

$$\frac{\partial r}{\partial \phi} = \begin{pmatrix} -(R + r\cos\theta)\sin\phi \\ (R + r\cos\theta)\cos\phi \\ 0 \end{pmatrix}$$

(c) Determine dA for $\mathcal{T}$.

$$dA = \frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial \phi} d\theta d\phi$$

$$dA = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta \\ -(R + r\cos\theta)\sin\phi & (R + r\cos\theta)\cos\phi & 0 \end{vmatrix} d\theta d\phi$$

$$dA = \begin{pmatrix} 0 - r\cos\theta(R + r\cos\theta)\cos\phi \\ -r\cos\theta(R + r\cos\theta)\sin\phi \\ -r\sin\theta\cos\phi(R + r\cos\theta)\cos\phi - r(R + r\cos\theta)\sin\phi\sin\theta\sin\phi \end{pmatrix} d\theta d\phi$$

$$dA = -r(R + r\cos\theta) \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} d\theta d\phi$$

(d) Derive an expression for the SA of $\mathcal{T}$.

$$SA = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dA$$

3. Explicitly told to avoid this one
4. Use Ritz's method to minimise

$$J\{y\} = \int_0^{2\pi} (y'^2 + \lambda^2 y^2) dx$$

With $y(0) = 1$ and $y(2\pi) = 1$ and λ is positive integer. Use

$$\phi_0 = 1, \quad \phi_n(x) = \sin[(n - 1/2)x]$$

And

$$y_N = \phi_0 + \sum_{n=1}^{N} c_n \phi_n(x)$$

$$y_n' = c_1(n - 1/2)\cos[(n - 1/2)x]$$

$$\begin{aligned}
J_1(c_1) &= \int_0^{2\pi} (y'^2 + \lambda^2 y^2) dx \\
&= \int_0^{2\pi} (c_1(n-1/2)\cos[(n-1/2)x])^2 + \lambda^2 (c_1 \sin[(n-1/2)x])^2 dx \\
&= c_1^2 \int_0^{2\pi} ((n-1/2)\cos[(n-1/2)x])^2 + \lambda^2 (\sin[(n-1/2)x])^2 dx \\
&= c_1^2 \int_0^{2\pi} ((n-1/2)^2 - \lambda^2)\cos^2[(n-1/2)x] + \lambda^2 dx \\
&= c_1^2 \left(((n-1/2)^2 - \lambda^2) \int_0^{2\pi} \cos^2[(n-1/2)x] dx + \int_0^{2\pi} \lambda^2 dx \right) \\
&= c_1^2 \left(\frac{((n-1/2)^2 - \lambda^2)(\sin(2(n-1/2)x) + 2(n-1/2)x)}{4(n-1/2)} \right)_0^{2\pi} + 2\pi\lambda^2 \\
&= c_1^2 \pi + 2\pi\lambda^2
\end{aligned}$$

5 Tute 4

1. Consider the cat problem again but with

$$W_p\{y\} = g \int_0^1 c_1 + c_2 y dx$$

Include the isoperimetric constraint to get

$$W_p\{y\} = \int_0^1 my + \lambda\sqrt{1 + y'^2} dx$$

Solve for the shape of the cable

$$\frac{\partial f}{\partial y} = my', \quad \frac{\partial f}{\partial y'} = \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

$$\begin{aligned}
\frac{\lambda y'^2}{\sqrt{1+y'^2}} - (my + \lambda\sqrt{1+y'^2}) &= c \\
\lambda y'^2 - (my\sqrt{1+y'^2} + \lambda(1+y'^2)^2) &= c\sqrt{1+y'^2} \\
\lambda y'^2 - (my\sqrt{1+y'^2} + \lambda(1+2y'^2+y'^4)) &= c\sqrt{1+y'^2} \\
\lambda &= \sqrt{1+y'^2}(c+my)
\end{aligned}$$

2. Two boats one towing the other. Only the first boat has a motor, and the second boat only uses its rudder to maintain a constant horizontal distance = 1.

   The force on a section of the tow rope is proportional to the length of that section.

3.

4.

$$F\{y\} = \int_0^R \frac{x}{1+y'^2}dx$$

   With $y(0) = L$ and $y(R) = 0$ ($y' \leq 0$ and $y'' \geq 0$) Approximate $1 + y'^2 \approx y'^2$

   (a) Solve it given the total surface area

   (b) Find the optimal profile of the nose-cone.

6 Tute 6

1. Note the derivative of arctan(y) is $\frac{1}{1+y'^2}$

$$f = K(x,y)e^{\arctan y'}\sqrt{1+y'^2}$$

$$\begin{aligned}
p &= \frac{\partial f}{\partial y'} \\
&= K\left(e^{\arctan y'}\frac{y'}{\sqrt{1+y'^2}} + \frac{1}{1+y'^2}e^{\arctan y'}\sqrt{1+y'^2}\right) \\
&= Ke^{\arctan y'}\left(\frac{y'+1}{\sqrt{1+y'^2}}\right)
\end{aligned}$$

$$\begin{aligned}
H &= y'\frac{\partial f}{\partial y'} - f \\
&= Ke^{\arctan y'}\left(\frac{y'^2+y'}{\sqrt{1+y'^2}}\right) - K(x,y)e^{\arctan y'}\sqrt{1+y'^2} \\
&= \frac{Ke^{\arctan y'}(y'-1)}{\sqrt{1+y'^2}}
\end{aligned}$$

Hence the transversality condition is

$$\begin{aligned}\left[p\frac{\partial y_\Gamma}{\partial \xi} - H\frac{\partial x_\Gamma}{\partial \xi}\right]_{x_1} &= 0\frac{Ke^{\arctan y'}}{\sqrt{1+y'^2}}(y'+1, y'-1)\cdot(1,\phi') &= 0\\ (y'+1,y'-1)\cdot(1,\phi') &= 0\\ (\cos\pi/4 - \sin\pi/4y', \cos\pi/4+\sin\pi/4y')\cdot(1,\phi') &= 0\end{aligned}$$

The last step is cheating.

I.e. the extremal will approach at angle $\pi/4$.

2. Distance from the point $(1,1,1)$ to

$$x^2+y^2+z^2=1$$

Using CoV

$$F(y) =$$

3. Broken extremal

4.