APP MTH 3002 Fluid Mechanics III
Assignment 4
Due: 12 noon, Friday 25 May.

When presenting your solutions to the assignment, please include some explanation in words to accompany your calculations. It is not necessary to write a lengthy description, just a few sentences to link the steps in your calculation. Messy, illegible or inadequately explained solutions may be penalised. The marks awarded for each part are indicated in boxes.

This assignment has 2 questions, for a total of 30 marks.

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1. Consider the 2π-periodic function

$$u(x) = \sin(2x)\sin(3x)$$

on the interval $0 \leq x < 2\pi$.

$\boxed{4}$ (a) Use the discrete Fourier transform to estimate the derivative $u'(x_j)$, where $x_j = jh$, $h = 2\pi/N$ and $N = 8$. This calculation must be done by hand, but you are welcome to use Matlab to check your answers.

$\boxed{2}$ (b) Is the spectral estimate of the derivative obtained in part (a) accurate? Why or why not? If not, what is the minimum value of N that would be needed to obtain an accurate estimate of the derivative?

2. Consider the Kuramoto–Sivashinsky equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^4 u}{\partial \xi^4} = 0. \tag{1}$$

The equation arises in a number of applications, one of which is liquid-film flow down a vertical wall. You can think of $u(\xi, t)$ as the perturbation of the surface of the film from its mean at a streamwise position ξ along the wall (relative to a reference frame moving at a certain speed) and time t.

We will consider L-periodic solutions of (1), that is, solutions for which $u(\xi, t) = u(\xi + L, t)$, subject to the initial condition

$$u(\xi, 0) = f(\xi). \tag{2}$$

$\boxed{1}$ (a) In order to apply the spectral methods that we have developed for 2π-periodic functions, we will transform this into an equivalent 2π-periodic problem.[1]

The interval $0 \leq \xi < L$ is mapped to $0 \leq x < 2\pi$ by $x = \alpha\xi$, where $\alpha = 2\pi/L$. Show that $x = \alpha\xi$ transforms the Kuramoto–Sivashinsky equation (1) into

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha^4\frac{\partial^4 u}{\partial x^4} = 0, \tag{3}$$

where u is now 2π-periodic.

$\boxed{4}$ (b) Suppose that u is small. Neglecting products of u and its derivatives, the linearised equation is

$$\frac{\partial u}{\partial t} + \alpha^2\frac{\partial^2 u}{\partial x^2} + \alpha^4\frac{\partial^4 u}{\partial x^4} = 0. \tag{4}$$

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[1] It is fairly easy to modify our formulation of spectral methods to L-periodic functions.

Let

$$u(x_j, t) \approx u_j(t) = \sum_{-N/2+1}^{N/2} \hat{u}_k(t) e^{\mathrm{i}kx_j},$$

where $x_j = jh$, $h = 2\pi/N$ and N is even. Use the pseudospectral method to derive a system of uncoupled ODEs for the Fourier coefficients $\hat{u}_k(t)$, $k = -N/2+1, \dots, N/2$.

2 (c) Solve the system of ODEs obtained in part (b) subject to the initial condition (2).

2 (d) In the linearised model (4), the amplitudes of some modes may grow without limit, thereby violating the assumption that u is small.

i. Using the solution you obtained in part (c), determine the range of wave numbers for which the amplitude $|\hat{u}_k(t)|$ increases with time.

ii. What is the minimum value of $L = 2\pi/\alpha$ necessary for the amplitude of any mode to increase with time?

4 (e) If u is not small, we must solve the nonlinear equation (3). Describe a pseudospectral method to solve (3) in physical space.

Hint: Your answer must include the system of ODEs to solve, all expressions needed to compute the spatial derivatives, and any initial conditions.

5 (f) Write a MATLAB script or function that uses the scheme described in (e) to solve (3) subject to the random initial condition

```
f = @(x) 0.01*(2*rand(size(x)) - 1);
```

Please upload your script or function when you submit your assignment.

Hints:

i. The system of ODEs is very stiff owing to the fourth-order *hyperdiffusion* term. You may find it better to use `ode15s` rather than `ode45`.

ii. You may find that `waterfall` does not produce very informative plots when there are more than about 30–40 times. Instead, you might like to use:

```
contourf(x, t, u, 'EdgeColor', 'none')
```

Read the documentation to find out more about `contourf`. Make sure you include a `colorbar` so that the reader knows what the colours mean.

iii. Make sure you document your code. The minimum documentation includes a description of the script and any functions, author name, date, and a description of any input and output arguments.

6 (g) Run your code and plot the solution $u(x, t)$ for $L = 3, 12, 14, 15, 20, 60$ and $0 \le t \le 400$.

For each plot, write a sentence or two that describes the nature of the solution (for example, reaches a steady state, travelling wave, pulsation between two states, chaos, etc.)

Hints:

i. Experiment a little with N. Setting $N \sim 6L$ seems to work reasonably well. Keep in mind that the random initial conditions will mean that your solution will be a little different each time you run.

ii. Make sure your plots are well resolved in time, otherwise you may not be able to see the details of the temporal development.