School of Mathematical Sciences

APP MTH 3022 - Optimal Functions and Nanomechanics III

Tutorial 3 (Week 6)

1. Go through the steps of deriving the Euler-Poisson equation for a functional containing derivatives up to order three. That is,
$$F\{y\} = \int f(x, y, y', y'', y''')\, dx.$$

2. Calculate the form of geodesics in N-dimensional Euclidean space. In other words, assuming a vector space $\mathbb{R}^N$, so that $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)$ with a norm $\|\boldsymbol{q}\| = \left(\sum_{n=1}^{N} q_n^2\right)^{1/2}$, find the extremal of the functional
$$S\{\mathbf{q}(t)\} = \int ds.$$

3. Carbon nanotori are genus $g = 1$ surfaces. Assuming every atom of a carbon nanotorus is bonded to exactly three neighbours, how many pentagonal, hexagonal and heptagonal rings must occur when also assuming that:
   (a) The nanotorus comprises only hexagonal and heptagonal rings?
   (b) The nanotorus comprises only pentagonal and hexagonal rings?
   (c) The nanotorus comprises hexagonal and heptagonal rings and exactly p pentagonal rings?

4. $\star$ For some $n \in \mathbb{N}$, show that
$$\Gamma(n + 1/2) = \frac{\sqrt{\pi}(2n-1)!!}{2^n},$$
where $(2n-1)!! = (2n-1) \cdot (2n-3) \cdot (2n-5) \cdots 5 \cdot 3 \cdot 1$.

5. Show that
$$B(x, y)B(x + y, z) = B(y, z)B(y + z, x) = B(z, x)B(z + x, y).$$

6. From the series definition for the hypergeometric function given by
$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$
show that
   (a) $\dfrac{\mathrm{d}}{\mathrm{d}z} F(a, b; c; z) = \dfrac{ab}{c} F(a + 1, b + 1; c + 1; z)$,
   (b) $\left[\dfrac{\mathrm{d}}{\mathrm{d}z} F(a, b; c; z)\right]_{z=0} = \dfrac{ab}{c}$,
   (c) $\dfrac{1}{\sqrt{1-z}} = F(1/2, b; b; z)$.