School of Mathematical Sciences

Mathematical Biology (Honours)

Assignment 1 question sheet

Due: Friday, 30 August, by 11am
(bring to lecture, or leave in box on office door)

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1. **Dimensional analysis of rowing** A boat carries N similar rowers, each of whom we assume puts the same power, P, into propelling the boat.

   (a) Assuming that they each require the same volume of boat, V, to accommodate them, show that the wetted area of the boat is $A \propto (NV)^{\frac{2}{3}}$.

   (b) Assume that the drag force on the boat as it moves through the water depends on the wetted area of the boat, A, its speed U, and the density of the water, ρ. Hence show the drag force must be proportional to $\rho U^2 A$, and the rate of energy dissipation due to drag must be proportional to $\rho U^3 A$

   (c) Hence show
   $$U \propto N^{\frac{1}{9}} P^{\frac{1}{3}} \rho^{-\frac{1}{3}} V^{-\frac{2}{9}}.$$

   (d) If we make the further crude assumption that both P and V are proportional to body mass, would we expect size to be an advantage to a rower?

   [4 marks]

2. Is it true that water flows out of a bathtub with a clockwise swirl in the southern hemisphere, and an anticlockwise swirl in the northern hemisphere?

   Consider a fluid flow subject to the force of gravity. In a coordinate system rotating with angular velocity, $\boldsymbol{\Omega}$, the dimensional Navier-Stokes equations become
   $$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{2\boldsymbol{\Omega} \times \boldsymbol{u}}_{\text{Coriolis acceleration}} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \underbrace{\rho(\boldsymbol{g} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}))}_{\text{apparent gravity}},$$
   where here we will take the origin of the coordinate system to be the centre of the Earth.

   The angular velocity, $\boldsymbol{\Omega}$, of the Earth is 2π per 24 hours, or 7.3×10^{-5} s^{-1} (where $\boldsymbol{\Omega}$ points in the direction of the Earth's axis of rotation), and the radius of the earth is around 6,400 km. Now assume that you are dealing with a bathtub with a typical lengthscale of around 1m, and the water flows at a typical speed of around 1 m s^{-1}. (You can take the density of water to be 1000 kg m^{-3} and its viscosity to be 8.9×10^{-4} Pa s.) By nondimensionalising, determine if the Coriolis acceleration will be significant (and hence, if the swirl direction is likely to change depending on which hemisphere you are in). [4 marks]

3. Consider the following reaction where m molecules of A and n molecules of B react to produce the product, C:

$$mA + nB \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C.$$

(a) In lectures, we discussed that the equation for the concentration of the product, c would take the form:

$$\frac{dc}{dt} = k_1 a^m b^n - k_{-1} c.$$

Write down the corresponding equations for a and b, the concentrations of the two reactants.

(b) Find the two conserved quantities, and hence eliminate a and b from the equation for c above.

[4 marks]

4. **(An enzymatic reaction with reversible product formation)** Consider the Michaelis-Menten reaction from lectures, but relax the assumption that the reaction in which the product and enzyme are formed from the complex is irreversible. Hence the reaction scheme is now

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} E + P.$$

(a) Write down a modified set of equations for the concentrations s, e, c and p, based on the scheme above. The initial conditions are:

$$s(0) = s_i > 0, \qquad e(0) = e_i = \epsilon s_i > 0, \qquad c(0) = p(0) = 0.$$

(b) Obtain a conservation law, and hence eliminate e from the system.

(c) Nondimensionalise the system of three equations as in lecture, with $\epsilon = e_i/s_i \ll 1$. Show the scaled system involves 3 dimensionless parameters, which you should give in terms of the original parameters.

(d) By neglecting terms of $O(\epsilon)$ or smaller, find a leading-order expression for the dimensionless complex concentration, and hence show that the approximate dimensionless reaction velocity takes the form

$$\frac{d\tilde{p}}{d\tilde{t}} = \frac{A_3 \tilde{s} - A_1 A_2 \tilde{p}}{\tilde{s} + A_2 \tilde{p} + A_1 + A_3},$$

where the A_i are constants.

(e) Show that, at leading order, the steady state ratio of product and substrate concentrations is

$$\frac{\bar{p}}{\bar{s}} = \frac{k_1 k_2}{k_{-1} k_{-2}}.$$

(Note: This is known as the Haldane relationship.)

[12 marks]

5. **(A substrate that can be broken down by two different enzymes)** Consider the following system of reactions:

$$E_1 + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1 \overset{k_3}{\rightarrow} E_1 + P, \qquad E_2 + S \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C_2 \overset{k_4}{\rightarrow} E_2 + P$$

(a) Write down the system of equations for the concentrations of S, E_1, E_2, C_1, C_2 and P. Show that there are two conserved quantities, and hence reduce your system to three equations involving only the concentrations of S, C_1 and C_2.

(b) Assume that the initial conditions for your system of equations are:

$$s(0) = s_i > 0, \qquad e_1(0) = e_2(0) = e_i = \epsilon s_i > 0.$$

Nondimensionalise your system of three equations, to obtain a system of the form:

$$\begin{aligned}
\frac{ds}{dt} &= -s(1+\alpha) + c_1(\mu_1 + s) + \alpha c_2(\mu_2 + s) \\
\epsilon\frac{dc_1}{dt} &= s(1 - c_1) - \lambda_1 c_1 \\
\epsilon\frac{dc_2}{dt} &= \alpha[s(1 - c_2) - \lambda_2 c_2].
\end{aligned}$$

You define the dimensionless parameters in terms of the original dimensional paramaters.

(c) Find the leading-order solution for c_1 and c_2, and hence s (Using technology to help is fine).

(d) By making a suitable rescaling of time, find the leading-order inner (short-time) solution for s, c_1 and c_2.

[16 marks]

Total: 40 marks