Lecture 6: Kolmogorov differential equations – The key equations

Concepts checklist

At the end of this lecture, you should be able to:

- Derive the *Kolmogorov forward and backward differential equations*;
- Appreciate that the *Kolmogorov forward equations (KFDEs) don't always hold*, and that they *hold for finite-state processes and birth-death processes*; and,
- Understand a variety of approaches to *solve the KFDEs for certain CTMCs*, and effect these possibly with the assistance of a computer.

How do we gain information about the CTMC from the rates q_{ij}? That is, how does $P_{ij}(t)$ relate to Q? The Kolmogorov differential equations are key to answering this.

Kolmogorov Differential Equations

Theorem 4. *The Kolmogorov backward differential equations (KBDEs) of a continuous-time Markov chain are*

$$\frac{\mathrm{d}P_{ij}(t)}{\mathrm{d}t} = \sum_{k\in\mathcal{S}} q_{ik}P_{kj}(t) \qquad \textit{for } i,j\in\mathcal{S}.$$

Proof. Starting with the Chapman-Kolmogorov equation,

$$P_{ij}(t+h) = \sum_{k\in\mathcal{S}} P_{ik}(h)P_{kj}(t),$$

we have

$$\begin{aligned}
\lim_{h\to 0^+}\frac{P_{ij}(t+h)-P_{ij}(t)}{h} &= \lim_{h\to 0^+}\frac{\sum_{k\in\mathcal{S}} P_{ik}(h)P_{kj}(t) - P_{ij}(t)}{h} \\
&= \lim_{h\to 0^+}\left[\frac{\sum_{k\in\mathcal{S}} P_{ik}(h)P_{kj}(t)}{h} - \frac{P_{ij}(t)}{h}\right] \\
&= \lim_{h\to 0^+}\left[\left(\sum_{\substack{k\neq i\\ k\in\mathcal{S}}}\frac{P_{ik}(h)}{h}P_{kj}(t)\right) + \frac{P_{ii}(h)P_{ij}(t)}{h} - \frac{P_{ij}(t)}{h}\right] \\
&= \lim_{h\to 0^+}\left[\left(\sum_{\substack{k\neq i\\ k\in\mathcal{S}}}\frac{P_{ik}(h)}{h}P_{kj}(t)\right) - \left(\frac{1-P_{ii}(h)}{h}\right)P_{ij}(t)\right] \\
&\overset{\clubsuit}{=} \sum_{\substack{k\neq i\\ k\in\mathcal{S}}} q_{ik}P_{kj}(t) + q_{ii}P_{ij}(t) \\
&= \sum_{k\in\mathcal{S}} q_{ik}P_{kj}(t).
\end{aligned}$$

We need to justify the interchange of the *limit* and the *summation* in equality ♣, one such way to do so is by Fatou's Lemma. The details are omitted. □

Theorem 5. *The Kolmogorov forward differential equations (KFDEs) of a continuous-time Markov chain are*

$$\frac{\mathrm{d}P_{ij}(t)}{\mathrm{d}t} = \sum_{k\in\mathcal{S}} P_{ik}(t)q_{kj}.$$

Proof. Starting with the Chapman-Kolmogorov equations

$$P_{ij}(t+h) = \sum_{k\in\mathcal{S}} P_{ik}(t)P_{kj}(h),$$

we have

$$\begin{aligned}
\lim_{h\to 0^+} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} &= \lim_{h\to 0^+} \frac{\left(\sum_{k\in\mathcal{S}} P_{ik}(t)P_{kj}(h)\right) - P_{ij}(t)}{h} \\
&= \lim_{h\to 0^+} \left[\left(\sum_{\substack{k\neq j\\ k\in\mathcal{S}}} P_{ik}(t)\frac{P_{kj}(h)}{h}\right) - P_{ij}(t)\left(\frac{1-P_{jj}(h)}{h}\right)\right] \\
&\overset{\spadesuit}{=} \left(\sum_{\substack{k\neq j\\ k\in\mathcal{S}}} P_{ik}(t)q_{kj}\right) + P_{ij}(t)q_{jj} \\
&= \sum_{k\in\mathcal{S}} P_{ik}(t)q_{kj}.
\end{aligned}$$

In general, the interchange of the lim and $\sum$ at ♠ cannot be justified. The most that can be shown is

$$\frac{\mathrm{d}P_{ij}(t)}{\mathrm{d}t} \geq \sum_{k\in\mathcal{S}} P_{ik}(t)q_{kj}.$$

We can, however, prove equality for *finite-state* processes and *birth-and-death* processes. The details are omitted. □

In matrix form,

$$\begin{aligned}
\text{KFDEs:}\quad & \frac{\mathrm{d}}{\mathrm{d}t}P(t) = P(t)Q \\
\text{KBDEs:}\quad & \frac{\mathrm{d}}{\mathrm{d}t}P(t) = QP(t).
\end{aligned}$$

Remark 1. We can construct continuous-time Markov chains for which the KFDEs do not hold; these Markov chains have weird properties: they can traverse infinitely many states in a finite time. They are used, for example, in modelling nuclear explosions, but are not often encountered in the sort of situation we consider in this course.

We shall now consider how to solve the Kolmogorov differential equations for some of the examples we have considered. It is usually more convenient to work with the KFDEs (when possible), so this is what we will do.

Solving Kolmogorov Differential Equations

When $|\mathcal{S}| < \infty$, the Kolmogorov differential equations are a finite system of linear ordinary first-order differential equations. We could solve these equations using a variety of *differential equation solvers*, for example using `ode45` in `MATLAB`. In this finite case, the solution to the Kolmogorov differential equations is

$$P(t) = \mathrm{e}^{Qt}, \qquad \text{where } \mathrm{e}^{A} := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

This gives another potential method of solution, as a *matrix exponential.* This matrix exponential solution holds more broadly, but the details will be omitted here. This can assist the development of theory, however, typically we can still only evaluate the matrix exponential with the assistance of a computer. We might be able to evaluate the general form using an algebra package, such as `Wolfram Alpha`, `Mathematica` or `Maple`, or numerically for specific parameters / values of t, for example in `Matlab`.

Recalling Example 1, the problematic printers, I evaluated the transition function as a matrix exponential using `Wolfram Alpha`; I also checked using the `expm` function in `MATLAB` for a few values of t.

Another approach is to consider the *spectral representation* of Q. If $|\mathcal{S}| = d < \infty$, then assuming the eigenvalues of Q are all distinct[1] and labelled $\lambda_1, \lambda_2, \ldots, \lambda_d$, we have

$$Q = RDL = \lambda_1 M_1 + \lambda_2 M_2 + \ldots + \lambda_d M_d,$$

where, $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$, R has columns $r_1', r_2', \ldots, r_d'$, L has rows $l_1, l_2, \ldots, l_d$, and so $M_i = r_i' l_i$, with l_i $(1 \times d)$ the left eigenvector of Q corresponding to the eigenvalue λ_i and r_i' $(d \times 1)$ the right eigenvector of Q corresponding to the eigenvalue λ_i, such that $r_i l_j' = \delta_{ij}$. Hence, we have

$$P(t) = \exp(Qt) = e^{\lambda_1 t} M_1 + e^{\lambda_2 t} M_2 + \ldots + e^{\lambda_d t} M_d.$$

Use this approach to construct the transition function for the problematic printers example.

We will now return to the Poisson process, and consider the solution to the KFDEs.

Example 2. Poisson process as a CTMC (continued)

Recall that

$$\begin{aligned} q_{n,n+1} &= \lambda \quad \text{for } n \geq 0, \\ q_{nn} &= \sum_{m \neq n} q_{nm} = -\lambda \quad \text{for } n \geq 0. \end{aligned}$$

Recall also that the KFDEs are

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{ij}(t) = \sum_{k \in \mathcal{S}} P_{ik}(t) q_{kj}.$$

For $i = 0$ and $n > 0$, we have

$$\frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} = \sum_{k \in \mathcal{S}} P_{0k}(t) q_{kn} = P_{0,n-1}(t) q_{n-1,n} + P_{0n}(t) q_{nn}.$$

[1]The theory is more general than this.

Hence,

$$\frac{\mathrm{d}P_{0n}(t)}{\mathrm{d}t} = \lambda P_{0,n-1}(t) - \lambda P_{0n}(t) \quad \text{for} \quad n > 0,$$
$$\text{and} \qquad \frac{\mathrm{d}P_{00}(t)}{\mathrm{d}t} = -\lambda P_{00}(t).$$

We can recursively solve these differential-difference equations, starting with

$$P_{00}(t) = e^{-\lambda t} P_{00}(0) = e^{-\lambda t}.$$

Substituting we have

$$\frac{\mathrm{d}P_{01}(t)}{\mathrm{d}t} = \lambda e^{-\lambda t} - \lambda P_{01}(t),$$

giving

$$P_{01}(t) = (\lambda t)e^{-\lambda t}.$$

Substituting again, we have

$$\frac{\mathrm{d}P_{02}(t)}{\mathrm{d}t} = (\lambda^2 t)e^{-\lambda t} - \lambda P_{02}(t),$$

giving

$$P_{02}(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t}.$$

Continuing, we arrive at

$$P_{0n}(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$
