THE UNIVERSITY of ADELAIDE

Mathematics IA

Calculus Outline Notes

School of Mathematical Sciences
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Contents

Bibliography

Stewart, J. (2015), *Calculus*, 8th edn, Cengage Learning. Perhaps purchase e-chapters 2, 4, 6 and 7—albeit in 'CourseSmart' format—from http://www.cengagebrain.com.au/shop/isbn/9780538497817

1 Functions

Contents

1.1 Notation

iff	"if and only if"
$\Longrightarrow$	"implies"
$\Longleftrightarrow$	"is equivalent to"

1.2 Sets

A set is a collection of objects called *elements*. The notation $\{x \mid \ldots\}$ or $\{x : \ldots\}$ is read "the set of objects x such that …".

$x \in A$: the object x is an element of the set A

$\emptyset$: the *empty* set, that is, the set with no elements at all

$A \subset B$: the set A is contained in B, that is, every element of A is also an element of B. This does not exclude the possibility that $A = B$.

$A \cup B$: the *union* of the sets A and B: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B$: the *intersection* of the sets A and B: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A \backslash B$: the *difference* of the sets A and B: $A \backslash B = \{x \mid x \in A \text{ but } x \notin B\}$

$\mathbb{N}$: the set of natural numbers

$\mathbb{Q}$: the set of rational numbers

$\mathbb{R}$: the set of real numbers

$\mathbb{C}$: the set of complex numbers.

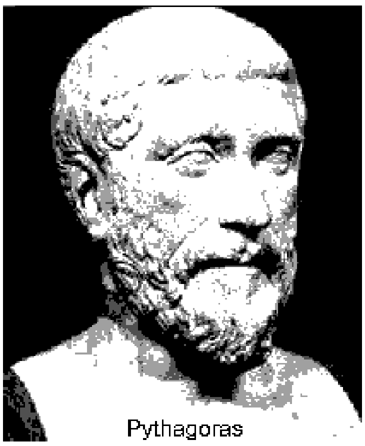
$\mathbb{R}^2$: $\{(x, y) \mid x, y \in \mathbb{R}\}$.

1.3 The Greek alphabet

A	α	alpha	B	β	beta	Γ	γ	gamma
Δ	δ	delta	E	ϵ	epsilon	Z	ζ	zeta
H	η	eta	Θ	θ	theta	I	ι	iota
K	κ	kappa	Λ	λ	lambda	M	μ	mu
N	ν	nu	Ξ	ξ	xi	O	o	omicron
Π	π	pi	P	ρ	rho	Σ	σ	sigma
T	τ	tau	Υ	υ	upsilon	Φ	ϕ	phi
X	χ	chi	Ψ	ψ	psi	Ω	ω	omega

1.4 The Real Numbers (Stewart, App. A)

$$\mathbb{N} = \text{the natural numbers} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \text{the integers} = \{0, \pm 1, \pm 2, \dots\}$$
$$\mathbb{Q} = \text{the rational numbers} = \left\{ \frac{m}{n} \,\middle|\, n, m \in \mathbb{Z},\ n \neq 0 \right\}$$



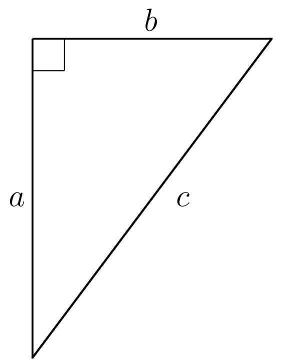
Any rational number can be represented in infinitely many ways: for example,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots = \frac{-1}{-2} \quad \text{etc.}$$

Historically the positive rational numbers were considered long before the negative numbers. Irrational numbers were considered long before negative numbers.

Around 500–600 B.C. the Pythagoreans were a sect in Ancient Greece whose philosophy was based on "number"; and number meant positive rational number (numbers that arose naturally from measurements, lengths). However, one of their members discovered irrational numbers (and was promptly expelled from the sect!)

Theorem 1.1 (Pythagoras).



$$a^2 + b^2 = c^2$$

If $a = b = 1$, then $c = \sqrt{2}$ ($c^2 = 2$) and they proved $\sqrt{2}$ is *irrational*.

Before doing this we give a proof of Pythagoras' Theorem; while the relationship $a^2 + b^2 = c^2$ was known 2,000–3,000 B.C. for many numbers for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$ etc. it is probable a proof of the general relationship first appeared around the time of the Pythagoreans.

◊ **1.0: Proof** ...

Theorem 1.2. *$\sqrt{2}$ is irrational.*

◊ **1.1: Proof** ...

The same argument works for $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt[3]{5}$, $\sqrt[6]{3}$ etc.

After the Greeks it was many years before a complete description of the real numbers was given; in fact it only occurred in the 19th century.

The real numbers $\mathbb{R}$ can be thought of as the numbers with decimal expansions such as $10.6123, -23.76\ldots, 0.3333\ldots$

The rational numbers are precisely the numbers with a decimal expansion which is eventually repeating.

For example,

- $0.333\ldots$ is $1/3$,
- 2.1532 is $\frac{21532}{10000}$ (repeating 0's), and
- $4.6198323333\ldots$ is also rational.

The irrational numbers have a decimal expansion which is not repeating—therefore we cannot write it down as a decimal. We can

only give an approximation by writing down some digits:

$$\begin{aligned}\sqrt{2} &\approx 1.414 \\ &\approx 1.4142137 \quad \text{(a better approximation)}\end{aligned}$$

Both these approximations are rational numbers.

One way we depict the real numbers is by a straight line which extends 'forever' in either direction (the x-axis or y-axis). Also we have no 'gaps' in the *real line*.

◊ 1.2: What would a line of rational numbers look like? . . .

1.5 Intervals (Stewart, App. A)

In mathematics we consider many subsets of the real numbers $\mathbb{R}$. The most common are the *intervals*.

We use square brackets to denote intervals which include endpoints, and round parentheses for intervals not including endpoints.

◊ 1.3: Example . . .

Note: "∞" is *not* a real number; it is a symbol to denote that the interval extends forever.

$$(-\infty, +\infty) = \mathbb{R}, \text{ the set of real numbers.}$$

We can never write $[a, \infty]$ or $[-\infty, b)$ etc.

The open interval $(0, 1)$ has no largest or smallest number (the same applies to any open interval). "There is no largest number in $(0, 1)$" means there is no real number closest to, but less than, 1.

◊ 1.4: Discussion . . .

1.6 Functions (Stewart, §1.1-1.2)

In this subject we deal with functions of one real variable.

Definition 1.1. (Stewart, pp.10–12) A real valued *function* f is a rule (or set of rules) which assigns to each element $a \in \mathcal{D}$ a unique real number denoted by $f(a)$. The set $\mathcal{D}$ is called the *domain*. Define the *range* of f to be the set of possible values of f:

$$\mathcal{R} = \mathcal{R}(f) = \{y \mid y = f(x) \text{ for some } x \in \mathcal{D}\}.$$

We write $f : \mathcal{D} \to \mathbb{R}$. The notation $f : \mathcal{D} \to \mathbb{R}$ means the range of f is a subset of the reals $\mathbb{R}$, $\mathcal{R}(f) \subset \mathbb{R}$.

In order to specify a function completely we must give the domain $\mathcal{D}$.

Example 1.1. If $f(x) = \sin x$, $x \in \mathbb{R}$ (that is, f has domain $\mathbb{R}$) and if $g(x) = \sin x$, $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ (that is, g has domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$), then f and g are *different* functions.

Usually we consider $\sin x$, $x \in \mathbb{R}$ (that is, function f), but sometimes we will want to consider $\sin x$ for some (proper) interval in $\mathbb{R}$; after a few lectures we return to this example. □

While the domain should always be specified, often we do not write the domain. In these cases we adopt the *convention* that the domain is the largest possible subset of $\mathbb{R}$ for which the function can be defined.

Example 1.2. If we write $f(x) = \sqrt{x+2}$, and do not specify the domain, then implicitly we mean

$$f(x) = \sqrt{x+2}, \quad x \geq -2;$$

that is, the domain is $[-2, \infty)$—the largest set of real numbers for which we can define $f(x) = \sqrt{x+2}$. □

Basic functions

Polynomials : (Stewart, p.27)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial function of degree n (if $a_n \neq 0$; each of the $a_0, a_1, \ldots, a_n$ is a (constant) real number).

◊ **1.5: Example** ...

By convention, each polynomial $p(x)$ has domain $\mathbb{R}$ (unless otherwise specified).

Absolute Value Function : The absolute value function $f(x) = |x|$ is defined by a two part rule:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$

Domain $\mathcal{D} = \mathbb{R}$; the range is $\{x \mid x \geq 0\} = [0, \infty)$.

Definition 1.2 (graphs)**.** The *graph* of a function $f : \mathcal{D} \to \mathbb{R}$ is the set of all points (x, y) in $\mathbb{R}^2$ with $x \in \mathcal{D}$ and $y = f(x)$. (Stewart, p.11)

♢ **1.6: example** . . .

The Greatest Integer Function : $h(x) = \lfloor x \rfloor$ = the greatest integer less than or equal to x. $h : \mathbb{R} \to \mathbb{R}$.

♢ **1.7: example** . . .

The domain is $\mathbb{R}$; the range is $\mathbb{Z}$ (the integers).

This is an example of a "*step function*"

Example 1.3. Graph of greatest integer function.

□

♢ **1.8: example** . . .

Rational Functions : (Stewart, p.30) If $p(x)$ and $q(x)$ are polynomial functions

$$r(x) = \frac{p(x)}{q(x)} \quad \text{is a } \textit{rational function}.$$

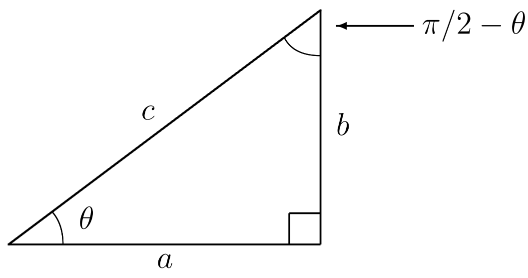
The domain of $r(x)$, is $\mathcal{D} = \{x \mid q(x) \neq 0\}$.

♢ **1.9: example** . . .

♢ 1.10: Graphs of rational functions . . .

1.7 Trigonometric Functions (Stewart, §1.2, App. D)

Recall the basic definitions of the trigonometric functions (Stewart, p.31–32):



$$\sin\theta = \frac{b}{c} \qquad \cos\theta = \frac{a}{c} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

In calculus, angles are always radians.

Formulae Think of the graphs and special cases in order to get these correct when using.

$$\begin{aligned}
&\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \\
&\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \\
&\sin^2\theta + \cos^2\theta = 1 \\
&\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
&\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

Put $\alpha = \beta = \theta$ to get the important double angle formulae:

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta\,, \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta\,.
\end{aligned}$$

Using $\sin^2\theta + \cos^2\theta = 1$,

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 \quad \text{or} \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}\,, \\
\cos 2\theta &= 1 - 2\sin^2\theta \quad \text{or} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}
\end{aligned}$$

Sign diagrams

$$\begin{array}{c|c} + & + \\ \hline - & - \end{array} \qquad \begin{array}{c|c} - & + \\ \hline - & + \end{array} \qquad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

$$\sin x \qquad\qquad \cos x \qquad\qquad \tan x$$

The other trigonometric functions are

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

◊ **1.11: Example** ...

Definition 1.3 (even and odd functions)**.**

- a function f is *even* if $f(-x) = f(x)$;
- a function f is *odd* if $f(-x) = -f(x)$;

for all x in its domain.

Emmy Noether proved symmetries underpin physical laws.

Even functions are symmetric about the y-axis (reflection) (Stewart, pp.17–18).

Odd functions are symmetric with respect to the origin (point/rotation).

◊ **1.12: Example** ...

1.8 Constructing new functions from old (Stewart, §1.3)

Recall we shift a functions by adding/subtracting a constant from the function or its argument (Stewart, p.36). Also, we stretch a function vertically or horizontally through multiplying or dividing, by some constant, the function or its argument (Stewart, p.37).

Combining two functions f, g we form new functions (Stewart, p.39):

the sum $f + g$	$(f + g)(x) = f(x) + g(x)$
difference $f - g$	$(f - g)(x) = f(x) - g(x)$
product $f \cdot g$	$(f \cdot g)(x) = f(x)g(x)$

quotient $\dfrac{f}{g}$ $\qquad\qquad$ $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

The domain of $f+g$, $f-g$, $f\cdot g$ is the intersection of the domains of f and g, $\mathcal{D}(f)\cap\mathcal{D}(g)$.

For $\frac{f}{g}$ we must also exclude the numbers x such that $g(x)=0$.

◊ **1.13: Example** . . .

Definition 1.4 (composition of functions)**.** Given any two functions f and g (Stewart, p.40),

$$(f\circ g)(x) = f\,(g(x)) \quad \text{and} \quad (g\circ f)(x) = g\,(f(x))$$

These are almost always *not* equal.

◊ **1.14: Example** . . .

Domains of $f\circ g$ and $g\circ f$ In general the domain of $f\circ g$ (or $g\circ f$) depends on both the domain of f and g. The composition $f\circ g$ has domain $\{x\in\mathcal{D}(g) \mid g(x)\in\mathcal{D}(f)\}$.

◊ **1.15: Examples** . . .

1.9 Inverse functions (Stewart, §6.1)

Definition 1.5 ((Stewart, §6.1, p.384)). A function $f : \mathcal{D} \to \mathcal{R}$ ($\mathcal{D}, \mathcal{R}$, the domain and range of f are subsets of $\mathbb{R}$) is *one-to-one*, *1–1*, if

$$\text{for any } x_1, x_2 \in \mathcal{D}, \text{ if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2).$$

Equivalently, f is 1–1 provided

$$\text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

◊ **1.16: Examples** . . .

Horizontal Line Test A function f is 1–1 if and only if any horizontal line meets the graph of f *at most once*.

Increasing, decreasing and monotonic functions

Definition 1.6 (monotonic functions (Stewart, p.19)). A function $f : \mathcal{D} \to \mathbb{R}$ is

- *increasing* if $f(x_1) < f(x_2)$ for $x_1 < x_2$;
- *decreasing* if $f(x_1) > f(x_2)$ for $x_1 < x_2$;

where x_1, x_2 all/any numbers in $\mathcal{D}$ with $x_1 < x_2$.

Functions which are increasing or decreasing are called *monotonic*.

◊ **1.17: Examples** . . .

Monotonic functions are 1–1 (use the horizontal line test).

Inverse Functions (Stewart, §6.1, pp.384–387) One-to-one functions are important because they are precisely the functions which have an inverse function.

Let $f : \mathcal{D} \to \mathcal{R}$ where $\mathcal{R} =$ range of f. If f is 1–1, then f has an inverse, $f^{-1} : \mathcal{R} \to \mathcal{D}$, with

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) = x \quad \text{for all } x \in \mathcal{D} \\ (f \circ f^{-1})(y) &= f\left(f^{-1}(y)\right) = y \quad \text{for all } y \in \mathcal{R} \end{aligned}$$

(That is, both $f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity functions—on $\mathcal{D}$, $\mathcal{R}$ respectively).

◊ **1.18: Example** . . .

Method for finding f^{-1}:

1. Write $y = f(x)$;
2. Solve this equation for x; $x = f^{-1}(y)$;
3. *Interchange* x and y;

and then $y = f^{-1}(x)$.

Note

1. If $y = f(x)$ then $x = f^{-1}(y)$ and if $x = f^{-1}(y)$ then $y = f(x)$. That is, $y = f(x)$ if and only if $x = f^{-1}(y)$.
2. f^{-1} does *not* denote $\frac{1}{f}$.
3. f must be 1–1 in order to define f^{-1}.
   Example 1.4. If $f(x) = x^2$, $x \in \mathbb{R}$, then $f(2) = f(-2) = 4$. If f^{-1} existed, what is $f^{-1}(4)$? Is it 2 or -2? Conclusion: f^{-1} cannot exist. □

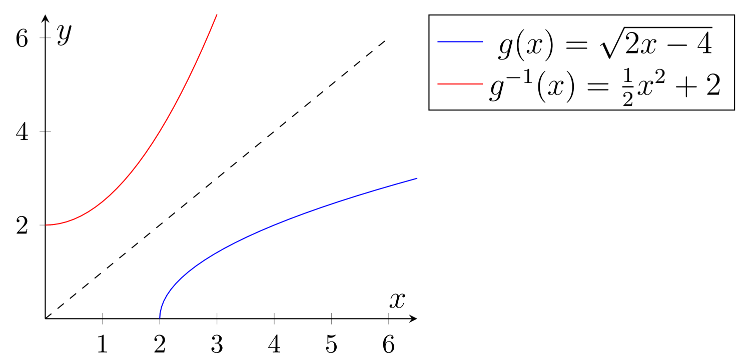
   In general, if f is not 1–1 then somewhere $f(x_1) = f(x_2) = y$, and then what is $f^{-1}(y)$? x_1? or x_2? For f^{-1} to be a function, $f^{-1}(y)$ must be a unique number.

◊ **1.19: example** . . .

The graph of the inverse If f has an inverse f^{-1}, then the graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line $y = x$.

As $y = f(x)$ if and only if $x = f^{-1}(y)$: if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}.

Example 1.5. $g(x) = \sqrt{2x - 4}$

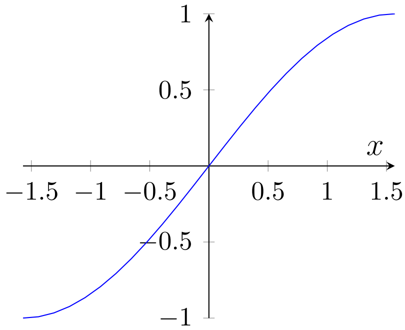


□

1.10 Inverse trigonometric functions (Stewart, §6.6)

We have already seen that $\sin x$, $x \in \mathbb{R}$, is not 1–1. In order to consider the inverse we "restrict the domain":

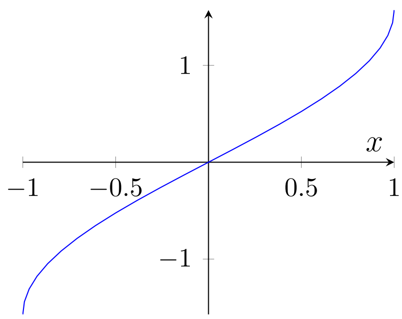
$$f(x) = \sin x, \quad \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$$



f is increasing on $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and so is 1–1. The inverse function $f^{-1}(x)$ is written as $\arcsin x$, or as $\sin^{-1} x$.

As the range of f, $\mathcal{R}(f) = [-1, 1]$, the domain $\mathcal{D}(\arcsin) = [-1, 1]$. Also the domain of f is $\mathcal{D}(f) = [\frac{-\pi}{2}, \frac{\pi}{2}]$, so the range $\mathcal{R}(\arcsin) = [\frac{-\pi}{2}, \frac{\pi}{2}]$.

$$f^{-1}(x) = \arcsin x = \sin^{-1} x, \quad -1 \leq x \leq 1$$

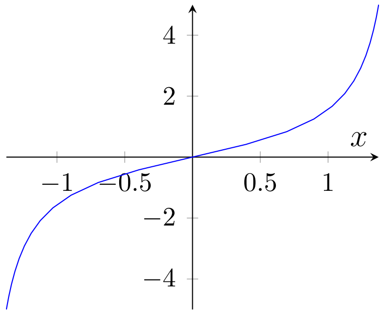


◊ 1.20: Example ...

In order to define the inverse cosine function, arccos, we restrict the domain of cos to $[0, \pi]$.

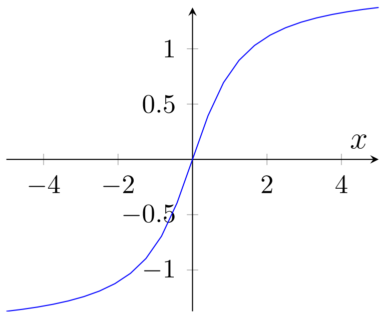
Inverse tangent Let $g(x) = \tan x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$. On the interval $(\frac{-\pi}{2}, \frac{\pi}{2})$ $\tan x$ is increasing and therefore 1–1.

$$g(x) = \tan x\,, \quad \tfrac{-\pi}{2} < x < \tfrac{\pi}{2}$$



The inverse tangent function is,

$$g^{-1}(x) = \arctan x \; = \tan^{-1}(x)$$



$$\text{Domain is } \mathbb{R} = (-\infty, \infty) \qquad \text{as } \mathcal{R}(\tan) = \mathbb{R}$$
$$\text{Range is } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \qquad \text{as } \mathcal{D}(\tan) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

◊ **1.21: Examples** . . .

1.11 Exponential and logarithmic functions (Stewart, §6.2–6.3)

Let a be a positive real number. Functions of the form $f(x) = a^x$ are called *exponential functions*.

Recall the basic laws of exponents:

$$a^0 = 1, \quad a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}$$
$$(ab)^x = a^x b^x, \quad a^{-x} = \frac{1}{a^x}$$

♢ 1.22: What does exponentiation mean when exponent x is irrational? ...

$$\begin{aligned} \text{For } a > 1, &\quad f(x) = a^x \text{ is increasing;} \\ a < 1, &\quad f(x) = a^x \text{ is decreasing;} \\ a = 1, &\quad f(x) = 1^x = 1 \text{ for all } x; \\ a \le 0, &\quad \text{avoid in this course.} \end{aligned}$$

For $a > 0$, $a \neq 1$, $f(x) = a^x$ has an inverse function as f is 1–1 on its domain $\mathbb{R}$.

Logarithms The inverse of $f(x) = a^x$, $f^{-1}(x) = \log_a x$, the logarithm to the base a, $a > 0$, $a \neq 1$.

$$\begin{aligned} &y = a^x \quad \text{iff} \quad x = \log_a y. \\ &\text{Also } \log_a(a^x) = x \quad \text{and} \quad a^{\log_a y} = y. \end{aligned}$$

♢ 1.23: Example ...

The logarithm has domain $(0, \infty)$ and range $\mathbb{R} = (-\infty, \infty)$.

The following laws for logarithms can be derived from the exponential laws:

$$\begin{aligned} &\log_a 1 = 0\,; && \log_a(bc) = \log_a b + \log_a c\,; \\ &\log_a b^d = d \log_a b\,; && \log_a c^{-1} = -\log_a c\,; \\ &\log_a b/c = \log_a b - \log_a c\,. \end{aligned}$$

♢ 1.24: Examples ...

2 Integration

Contents

Integration is used to solve the problem of finding the area under a curve $y = f(x)$, $a \leq x \leq b$. There are many applications in which the mathematical solution depends on an answer to this problem.

2.1 The area under $y = x^2$ (Stewart, §4.1)

We consider the problem of finding the area between $y = x^2$ and the x-axis over $x = 0$ to $x = 1$.

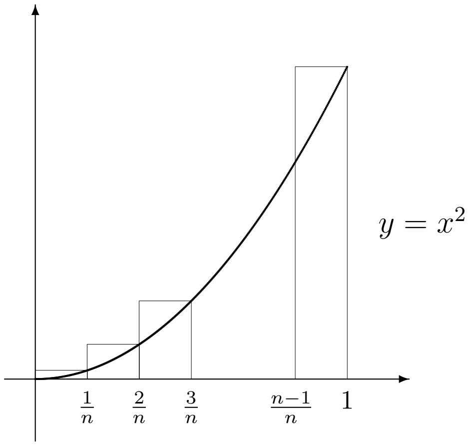
Begin by partitioning the interval into four equal subintervals (Stewart, p.285). Adding together four rectangles that just include the area, deduce area $A \leq 0.4688$. Adding together four rectangles just included by the area, deduce $0.2188 \leq A$.

Increasing the number of subintervals, improves the bounds on the area.

In general, partition the interval into n equal subintervals:

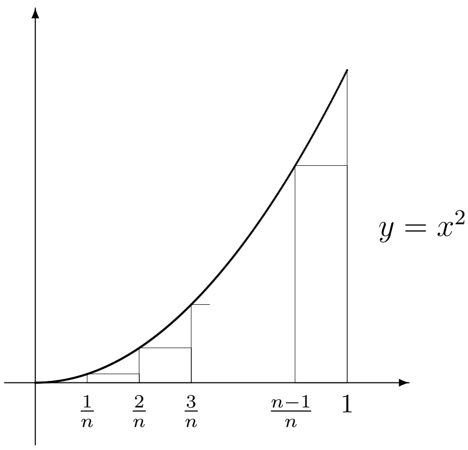
$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right].$$

On each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ let m_i and M_i respectively denote the minimum value and maximum value of $y = f(x)$ on that interval.



Adding the areas of all rectangles of height M_i and base $\frac{1}{n}$ gives the upper sum.

◊ 2.0: Calculating the upper sum ...



Similarly, adding the areas of all rectangles of height m_i and base $\frac{1}{n}$ gives the lower sum.

◊ 2.1: Calculating the lower sum. ...

L_n	$\leq$	A	$\leq$	U_n
area of rectangles below the curve		area under curve		area of rectangles above curve

As $n \to \infty$, that is, the width of the rectangles gets smaller and smaller, we expect the upper and lower sums to get closer together and better approximate the area.

◊ 2.2: What is the area A? ...

2.2 Summation notation (Stewart, App. E, §4.2)

In order to calculate areas using this method we need to evaluate sums. First we introduce some convenient notation to express the sums we encounter.

If $a_1, a_2, \ldots, a_n$ are real numbers, the sum of $a_1, \ldots, a_n$ is written

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

"i" is called the "index" of the sum (and is a "dummy index" as we can change it without altering the sum).

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

Often, a_i will be a function of i, $a_i = f(i)$ so the sum is $\sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n)$

◊ **2.3: Examples** ...

Properties of $\sum$:

1. $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$, for c a constant;
2. $\displaystyle\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$;
3. $\displaystyle\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$.
4. But $\displaystyle\sum_{i=m}^{n}(a_i \times b_i) \neq \left(\sum_{i=m}^{n} a_i\right) \times \left(\sum_{i=m}^{n} b_i\right)$;
5. and $\displaystyle\sum_{i=m}^{n} \frac{a_i}{b_i} \neq \frac{\sum_{i=m}^{n} a_i}{\sum_{i=m}^{n} b_i}$.

These can each be verified by writing out the sums on each side

Some important sums are those of constants, linear and squares.

1. $\displaystyle\sum_{i=1}^{n} 1 = \underbrace{1 + \cdots + 1}_{n} = n.$
2. $\displaystyle\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = a\frac{1 - r^{n+1}}{1 - r}$ is the geometric sum.
3. $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

   ♢ **2.4: Derivation of formula** ...

4. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n}{6}(2n^2 + 3n + 1) = \frac{n}{6}(2n+1)(n+1)$

   ♢ **2.5: Derivation of formula** ...

♢ **2.6: Example** ...

Evaluate $\displaystyle\sum_{i=3}^{10} (i+2)^2$.

Example 2.1 (the distance problem)**.** (Stewart, p.291) Suppose a car is moving with increasing velocity, and suppose its velocity is recorded each second for five seconds:

Time t (sec)	0	1	2	3	4	5
Velocity v (m/sec)	7	10	13	15	17	19

How far has the car moved in these five seconds? □

2.3 The definite integral (Stewart, §4.2)

The idea used in §2.1 for computing the area—upper and lower sums—generalises to other functions, the most important class of such functions being "continuous" functions.

Definition 2.1 (Continuous functions—loosely)**.** Intuitively, a function f is continuous on an interval $[a, b]$ if the graph of $y = f(x)$ has no 'jumps' and if f does not become *unbounded.*

A function f is *unbounded* if $f(x) \to \pm\infty$ as $x \to c$ for some $c \in [a, b]$.

♢ 2.7: Some functions which are *not continuous* . . .

Continuous functions include all polynomials, rational functions (if the denominator is not 0 on the interval) $\sin x$, $\cos x$, a^x, $\log_c x$ (for intervals $[a, b]$ with $a > 0$).

A precise definition of continuity is given in later subjects. For the present this intuitive discussion suffices.

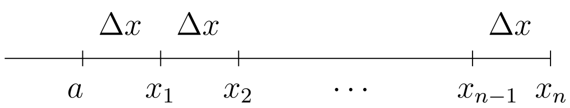
We now return to a discussion of areas under the graph of a function $y = f(x)$. Initially we take f as a *continuous* function on an interval $[a, b]$. Partition the interval $[a, b]$ into n equal parts:

$$a = x_0, x_1, x_2, \ldots, x_n = b$$

so for each i

$$\begin{aligned} x_i - x_{i-1} &= \frac{b-a}{n} \\ \text{and} \quad x_i &= a + \left(\frac{b-a}{n}\right) i. \end{aligned}$$

We write $\frac{b-a}{n} = \Delta x$. Such a partition is called a *regular partition.*



$$x_i = a + (\Delta x)i, \quad i = 1, 2, \ldots, n.$$

On each subinterval $[x_{i-1}, x_i]$ define (Stewart, p.289)

$$\begin{aligned} m_i &= \text{minimum value of } f \text{ on } [x_{i-1}, x_i], \\ M_i &= \text{maximum value of } f \text{ on } [x_{i-1}, x_i]. \end{aligned}$$

(That f is continuous guarantees the existence of a maximum and minimum value.)

As before we define the

$$\begin{aligned}
\text{Lower sum} \quad L_n &= m_1\Delta x + m_2\Delta x + \cdots + m_n\Delta x \\
&= \sum_{i=1}^{n} m_i\Delta x = \left(\sum_{i=1}^{n} m_i\right)\Delta x \\
\text{and Upper sum} \quad U_n &= M_1\Delta x + M_2\Delta x + \cdots + M_n\Delta x \\
&= \sum_{i=1}^{n} M_i\Delta x = \left(\sum_{i=1}^{n} M_i\right)\Delta x.
\end{aligned}$$

Definition 2.2 (The Definite Integral (Stewart, p.297))**.** If f is continuous on $[a, b]$, or if f has a finite number of jump discontinuities, then there is a unique number I which satisfies

$$L_n \le I \le U_n \quad \text{for all } n = 1, 2, 3, \ldots .$$

The quantity I is called the *definite integral* of f from a to b and symbolised as

$$I = \int_a^b f(x)\,dx.$$

◊ **2.8: Remarks** ...

◊ **2.9: Examples** ...

2.4 Properties of the definite integral (Stewart, §4.2)

First we extend the notation for the integral by defining (Stewart, p.303):

$$\begin{aligned}
\int_b^a f(x)\,dx &= -\int_a^b f(x)\,dx \\
\text{and} \quad \int_a^a f(x)\,dx &= 0.
\end{aligned}$$

Areas If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx$ gives the area under the graph of $y = f(x)$, $a \leq x \leq b$ (Stewart, p.297).

As we wish to use the integral to compute areas, we also must consider functions with negative values: if $f(x) \leq 0$, $a \leq x \leq b$, then the (positive) area *between* $y = f(x)$ and the x-axis is $-\int_a^b f(x)\,dx$.

The following theorem enables us to compute areas for general functions which are defined by more than one rule (for example, $|x|$ and $\lfloor x \rfloor$) by splitting the integrals:

Theorem 2.1. (Stewart, p.304) *If f is integrable (for example, continuous) on $[a, b]$ and $a < c < b$, then*

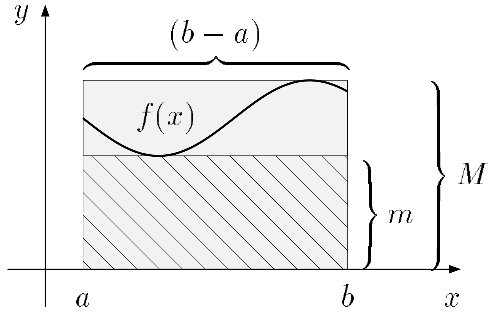
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

◊ 2.10: Examples ...

Properties Given two integrable functions f, g, the first few properties follow from the corresponding properties for sums (Stewart, p.303):

- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$,
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$,
- but $\int_a^b (f(x) \times g(x))\,dx \quad \neq \quad \left(\int_a^b f(x)\,dx\right) \times \left(\int_a^b g(x)\,dx\right)$,
- and $\int_a^b (f(x)/g(x))\,dx \quad \neq \quad \left(\int_a^b f(x)\,dx\right) \Big/ \left(\int_a^b g(x)\,dx\right)$,
- if $m \leq f(x) \leq M$, $x \in [a, b]$, then (Stewart, p.305)

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$



These bounds also hold for $f(x)$ that goes negative, provided one interprets area below the x-axis as negative area.

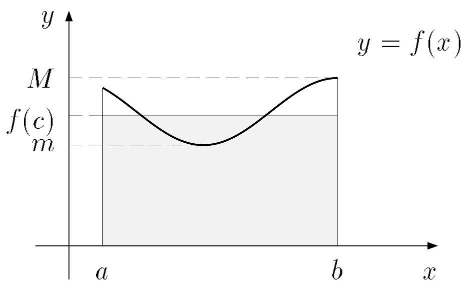
◊ **2.11: example** . . .

Theorem 2.2. *If f is continuous on $[a, b]$ then there exists a number $c \in [a, b]$ with*

$$f(c) \cdot (b - a) = \int_a^b f(x)\,dx$$

$$\textit{or} \qquad f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

The expression $\frac{1}{b-a}\int_a^b f(x)\,dx$ is called the *a*verage value of f on $[a, b]$.

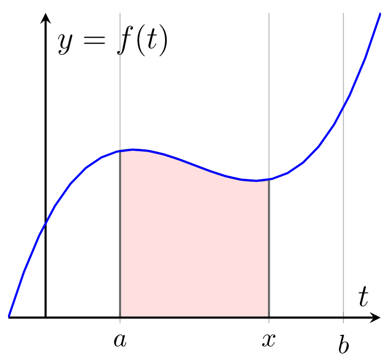


The area $f(c) \cdot (b - a) =$ the area under $y = f(x)$. Also $m \leq f(c) \leq M$

◊ **2.12: Examples** . . .

2.5 The Fundamental Theorem of Calculus (Stewart, §4.3)

As we have seen, even to compute the integral for $f(x) = x$ requires a good deal of algebra. Before Newton and Leibniz proved the Fundamental Theorem, the integral for each type of function was computed separately. This theorem gives a general method for finding the integral of a wide class of functions.



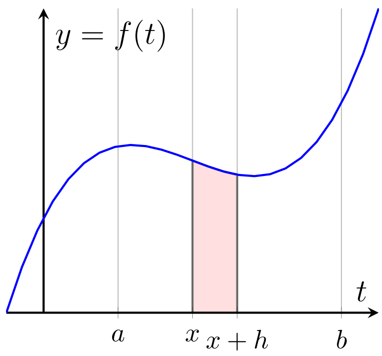
Let f be a continuous function on $[a,b]$ (for now consider $f(t) \geq 0$). For every value of x between a and b, $\int_a^x f(t)\,dt$ represents the area under the curve $y = f(t)$, $a \leq t \leq x$. This area is a function of x and so we define

$$F(x) = \int_a^x f(t)\,dt.$$

That is, $F(x)$ represents the area under $y = f(t)$ for $a \leq t \leq x$; F has domain $[a,b]$.

◊ 2.13: Example ...

The relationship between the two functions F and f



◊ 2.14: Derivation of formula ...

Theorem 2.3. (Stewart, p.312) *If f is continuous on $[a,b]$, the function $F(x) = \int_a^x f(t)\,dt$ satisfies $F'(x) = f(x)$ for all $x \in (a,b)$.*

◊ 2.15: Example ...

Definition 2.3. A function G is an *antiderivative* of f on (a,b) if $G'(x) = f(x)$, $a < x < b$.

Theorem 2.4 (The Fundamental Theorem of Calculus)**.** (Stewart, p.315) *Let f be continuous on $[a,b]$ and G any antiderivative of f on (a,b). Then*

$$\int_a^b f(t)\,dt = G(b) - G(a) = [G(x)]_a^b.$$

◊ 2.16: proof ...

Therefore many integrals may be found by finding an antiderivative of the function f.

It is not always possible to find an *algebraic* antiderivative: for example, integrands $f(t) = \sqrt{1 + t^3 + t^4}$ or $f(t) = \sin(t + \frac{1}{t})$.

3 Revision of differentiation

Contents

3.1 Rules for differentiation (Stewart, §2.3)

Recall $\frac{d}{dx}x^r = rx^{r-1}$ r rational (and irrational); that is, if $f(x) = x^r$, then $f'(x) = rx^{r-1}$ (Stewart, p.127).

Also $\frac{d}{dx}\sin x = \cos x$, and $\frac{d}{dx}\cos x = -\sin x$.

Using these derivatives and the general rules we find many more derivatives.

Theorem 3.1. *Let f and g be differentiable, and c and d be fixed real numbers, then* (Stewart, pp.128–132)

$$\frac{d}{dx}\left(cf(x) + dg(x)\right) = cf'(x) + dg'(x) \tag{3.1}$$

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x) \quad \textit{Product Rule} \tag{3.2}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad \textit{Quotient Rule} \tag{3.3}$$

$$\frac{d}{dx}\frac{1}{f(x)} = \frac{-f'(x)}{(f(x))^2} \quad \textit{Reciprocal Law.} \tag{3.4}$$

♢ **3.0: Examples** . . .

3.2 The chain rule (Stewart, §2.5)

Theorem 3.2 (The Chain Rule)**.** *Let f and g be differentiable functions, then*

$$\frac{d}{dx}f\circ g(x) = \frac{d}{dx}f\left(g(x)\right) = f'\left(g(x)\right)g'(x);$$

or using Leibniz' notation, writing $y = f\left(g(x)\right)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

◊ **3.1: Examples** ...

3.3 Evaluating the definite integral (Stewart, §4.3)

Let $f(x) = x^n$, then $F(x) = \frac{x^{n+1}}{n+1}$ is an antiderivative of $f(x)$; $F'(x) = f(x)$.

We may apply the facts about the derivative given above to do some calculations involving integrals which were discussed in the previous chapter.

◊ **3.2: Examples** ...

3.4 Derivatives of inverse functions (Stewart, §6.1)

Revision Recall that a function f which is 1–1 on its domain has an inverse f^{-1}. A function f is 1–1 if it satisfies the horizontal line test; that is, each horizontal line intersects the graph of f at most once.

Increasing or decreasing functions are 1–1.

◊ **3.3: example** ...

Derivative of the inverse

Suppose f is 1–1 and has an inverse f^{-1}; also suppose either $f'(x) > 0$ or $f'(x) < 0$ for all $x \in \mathcal{D}(f)$.

◊ 3.4: Derivation of a formula for the derivative of the inverse ...
(Stewart, p.388–9)

That is,

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))}, \quad \text{equivalently} \quad \frac{dy}{dx} = \frac{1}{dx/dy}.$$

The latter is more memorable. We generally substitute $y = f^{-1}(x)$ in the right-hand side to put the formula in terms of x.

◊ 3.5: example ...

3.5 Derivatives of inverse trigonometric functions (Stewart, §6.6, pp.457–9)

Example 3.1. Establish

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

Recall $y = \arcsin x$ iff $x = \sin y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d\sin y}{dy}} = \frac{1}{\cos y}.$$

We need to express $\cos y$ in terms of x. Recall $\cos^2 y = 1 - \sin^2 y = 1 - x^2$ as $x = \sin y$, so $\cos y = \pm\sqrt{1-x^2}$. Which *sign* $\pm$ do we use?

Recall for $y = \arcsin x$, $-1 \leq x \leq 1$ and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$. For $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos y \geq 0$. Thus $\cos y = +\sqrt{1-x^2}$, and hence $\frac{d}{dx}\arcsin x = 1/\sqrt{1-x^2}$. □

◊ 3.6: Examples ...

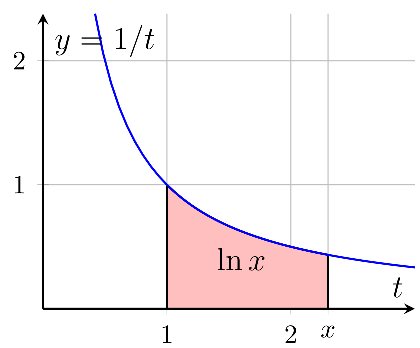
4 Logarithm and exponential functions

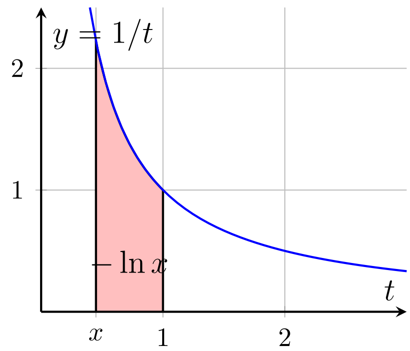
Contents

Previously we looked at logarithms and exponential functions, with logarithms defined as inverses of exponentials. The problem with this method is that we never properly defined what is meant by a^x when x is not rational. We now take a different approach, starting with a new definition of the natural logarithm using an integral, and from this we define exponentials and general logarithms. It is important to realise that in this section we are developing the theory and must carefully distinguish between definitions and results with are consequences of these definitions, at that these definitions replace the less rigorous ones which we used previously.

4.1 The natural logarithm ln (Stewart, §6.2*)

We know that $\frac{d}{dx}x^{n+1} = (n+1)x^n,\ n \neq -1$, and so we can evaluate integrals of the form $\int_a^b t^n dt$ unless $n = -1$. What about $\int_a^b \frac{1}{t}\,dt$?





Definition 4.1. The *natural logarithm* is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

the domain of $\ln x$ is $\{x \in \mathbb{R} \mid x > 0\}$. Thus $\ln x > 0$ if $x > 1$, $\ln 1 = 0$ and $\ln x < 0$ if $0 < x < 1$.

By the Fundamental Theorem,

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

As $x > 0$, $\ln x$ is increasing because $\frac{d}{dx}\ln x > 0$.

Theorem 4.1 (Properties of $\ln x$)**.** *If a, b are positive real numbers, then*

(a) $\ln 1 = 0$.

(b) $\ln ab = \ln a + \ln b$.

(c) $\ln \frac{1}{a} = -\ln a$

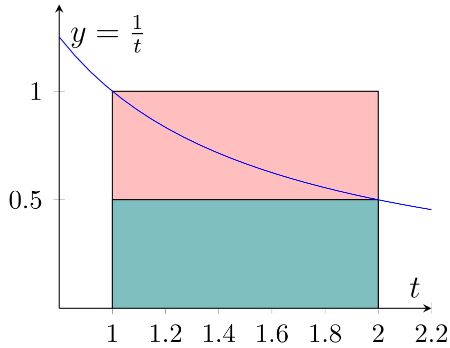
(d) $\ln \frac{a}{b} = \ln a - \ln b$

(e) $\ln a^r = r \ln a$ *for r rational (and ...).*

◊ **4.0: proof** ...

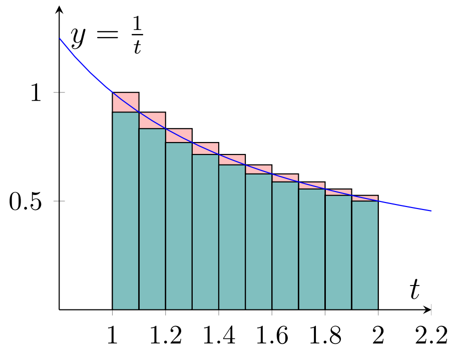
For any $x > 0$ we can approximate $\ln x$ by upper and lower sums.

Example 4.1. From the maximum and minimum values, $\frac{1}{2} < \ln 2 < 1$:



□

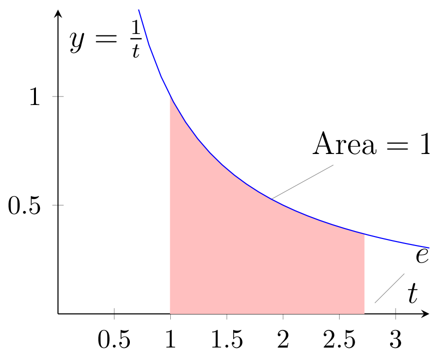
Example 4.2. To estimate $\ln 2$ better, partition $[1,2]$ into ten equal parts.



◊ **4.1: Calculation** ...

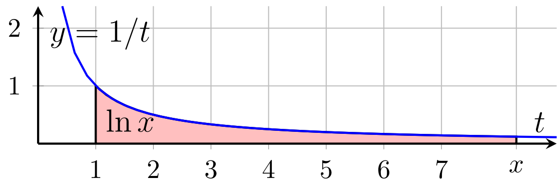
□

Definition 4.2. The number e is the unique real number with $\ln e = 1$ (Stewart, p.423).

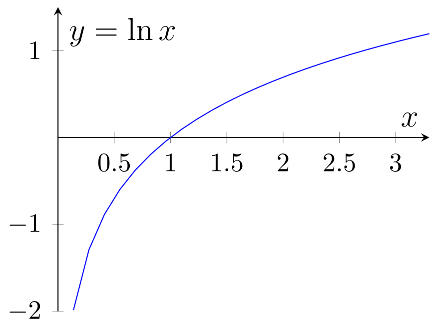


From above, as $\ln 2 < 1$ and $\ln x$ is an increasing function, then $2 < e$. Using the lower sum L_8 for $[1,3]$ we find $\int_1^3 \frac{1}{t}\,dt \geq L_8 = 1.0199 > 1$ which, since $\ln x$ is increasing, implies $e < 3$. Thus $2 < e < 3$.

◊ 4.2: What happens to $\ln x$ as $x \to \infty$ or $x \to 0$? ...



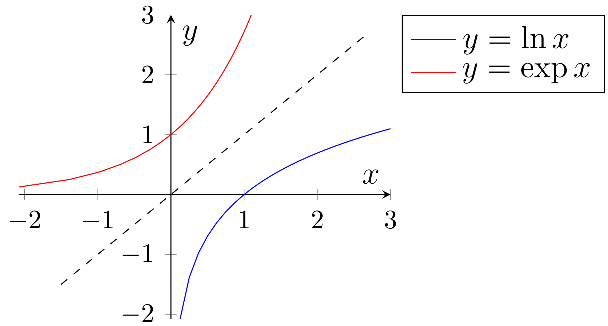
Putting together these facts we sketch $y = \ln x$:



4.2 The exponential function exp (Stewart, §6.3*)

As $\ln x$ is increasing for $x > 0$, it is 1–1 and therefore it has an inverse. The inverse function is called the exponential function, exp:

$$y = \exp x \quad \text{iff} \quad x = \ln y$$



Note

$$\text{Domain}(\exp) = \text{range}(\ln) = (-\infty, \infty) = \mathbb{R}$$
$$\text{Range}(\exp) = \text{domain}(\ln) = (0, \infty)$$

Thus $\exp(x)$ is defined for *all* real numbers (including irrational numbers).

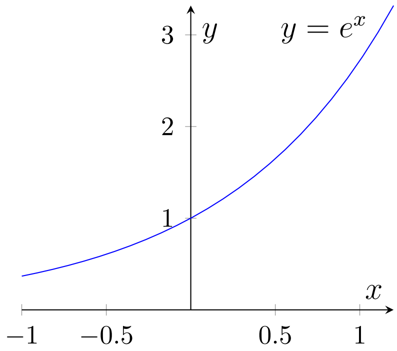
Special values

$$\begin{aligned}\exp(0) &= 1 \quad \text{as } \ln 1 = 0\\ \exp(1) &= e \quad \text{as } \ln e = 1.\end{aligned}$$

For r rational, $\ln(e^r) = r \ln e = r$. Thus $\exp(r) = e^r$ for r rational. This property leads us to *define*

$$e^x = \exp(x) \quad \text{for } \textit{all} \text{ real numbers } x.$$

Properties of e^x: $y = e^x$ iff $x = \ln y$; $e^{\ln x} = x$; and $\ln e^x = x$. The graph of $y = e^x$ is the same as that of $y = \exp x$.



The derivative of e^x: since $y = e^x$ iff $x = \ln y$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{y}} = y = e^x\\ \frac{d}{dx}e^x &= e^x\end{aligned}$$

◊ **4.3: Examples** ...

Theorem 4.2 (Laws of Exponents)**.** *Let a and b be real numbers, then*

(a) $e^a e^b = e^{a+b}$,

(b) $e^{-a} = \frac{1}{e^a}$,

(c) $(e^a)^r = e^{ar}$ *for r rational (and ...).*

◊ **4.4: Proof** ...

◊ **4.5: Example** ...

4.3 General exponential and logarithmic functions (Stewart, §6.4*)

We considered exponential functions such as $f(x) = 2^x$. While 2^r has a clear meaning for r rational, 2^x for x irrational was not properly defined. Using the exponential function we give a complete definition for exponential functions a^x, $a > 0$.

For r rational, $\ln a^r = r \ln a$. Thus $e^{\ln a^r} = e^{r \ln a}$, that is,

$$a^r = e^{r \ln a} \tag{4.1}$$

As e^x is defined for *all* real numbers, and $\ln x$ is defined for $x > 0$, we extend (4.1) to all real numbers and gives this as a definition of a^x.

Definition 4.3. For the positive number a, define the general exponential

$$a^x = e^{x \ln a} \quad \text{for all } x \in \mathbb{R}.$$

◊ 4.6: Remarks . . .

The derivative of a^x:

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}e^{x \ln a}\\ &= e^{x \ln a} \cdot \ln a\\ &= (\ln a)a^x.\end{aligned}$$

From e^x (Stewart, p.439): if $a > 1$, then a^x is increasing; whereas if $a < 1$, then a^x is decreasing. Consequently, an inverse function exists, called $\log_a$:

$$y = \log_a x \quad \text{iff } x = a^y.$$

By this definition, $a^{\log_a x} = x$ and $\log_a a^x = x$.

◊ 4.7: Example . . .

Logarithm to the base e: As there is an inverse to a^x for each $a \neq 1$, namely $\log_a x$, so there is an inverse function to the exponential function e^x; $\log_e x$. However, as the exponential e^x has a unique inverse, namely $\ln x$,

$$\ln x = \log_e x$$

What is the relationship between $\log_a x$ and $\ln x$?
$\log_a x = \frac{\ln x}{\ln a}$

◊ **4.8: Derivation of formula** ...

◊ **4.9: Example** ...
How fast is the increase in South Australia's population forecast in ten years time (in people/year)?

4.4 Hyperbolic functions (Stewart, §6.7)

Any function is the sum of an even and odd function:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}}$$

For a useful example,

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh x} .$$

These even and odd parts are often used functions and have amazingly similar properties to the trigonometric functions; they are given special names and called *hyperbolic functions*.

A chain, rope or suspension bridge cable hangs so its height $h = \cosh(kx)$. In ocean waves and tsunamis, for example, with $z = 0$ located at the sea-bed, the horizontal velocity profiles of the waves are $\propto \cosh(kz)$, whereas the vertical velocities are $\propto \sinh(kz)$.

Definition 4.4. The following functions all have domain $= \mathbb{R}$. The *hyperbolic sine* function *sinh* is defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

The *hyperbolic cosine* function *cosh* is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

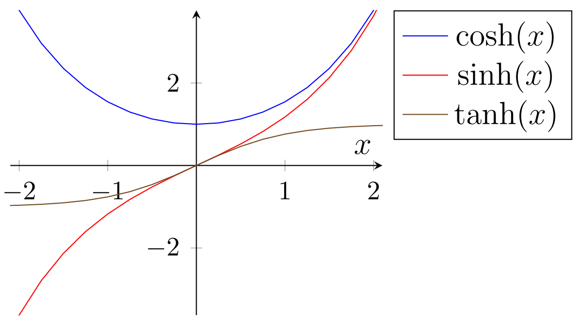
The *hyperbolic tangent* function *tanh* is defined by

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

And as for trigonometric functions:

$$\begin{aligned}
\operatorname{sech} x &= \frac{1}{\cosh x}\,; \\
\operatorname{cosech} x &= \frac{1}{\sinh x}\,; \\
\coth x &= \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}\,.
\end{aligned}$$

Graphs



Identities (think of their graphs)

1. $e^{\pm x} = \cosh x \pm \sinh x$
2. $\cosh^2 x - \sinh^2 x = 1$.

   ♢ **4.10: Proof** ...

3. $\sinh 2x = 2 \sinh x \cosh x$.
4. $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Derivatives (think of their graphs)

$$\begin{aligned}
\frac{d}{dx}\cosh x &= \frac{d}{dx}\frac{e^x+e^{-x}}{2} = \frac{e^x-e^{-x}}{2} \\
&= \sinh x \quad (\text{No "}-\text{" sign!}) \\
\frac{d}{dx}\sinh x &= \frac{d}{dx}\frac{e^x-e^{-x}}{2} = \frac{e^x+e^{-x}}{2} \\
&= \cosh x \\
\frac{d}{dx}\tanh x &= \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh x\cdot\cosh x-\sinh x\cdot\sinh x}{\cosh^2 x} \\
&= \frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} \\
&= \operatorname{sech}^2 x
\end{aligned}$$

The sinh function is 1–1 on its domain $\mathbb{R}$ and so has an inverse denoted $\sinh^{-1} x$ and called $\operatorname{arcsinh} x$:

$$y = \operatorname{arcsinh} x \quad \text{iff } x = \sinh y.$$

Derivative of $\operatorname{arcsinh} x$ is $\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$.

◊ **4.11: Proof** ...

Exercise: find an explicit formula for $\operatorname{arcsinh} x$ in terms of logs and use it to verify this.

5 Techniques of integration

Contents

5.1 Antiderivatives (Stewart, §4.4)

Recall that if $F'(x) = f(x)$, then $F(x)$ is an *antiderivative* of $f(x)$.

◊ 5.0: Example . . .

If $F(x)$ is any antiderivative of $f(x)$, then by the Fundamental Theorem of Calculus, the definite integral

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

In order to *algebraically* evaluate definite integrals $\int_a^b f(x)\,dx$ we (usually) need to find an *algebraic* antiderivative $F(x)$ of $f(x)$. This is not always possible: functions such as e^{x^2} and $\sqrt{x^4+1}$ have no algebraic formula for the antiderivative.

To evaluate $\int_0^1 e^{x^2}\,dx$ we use numerical techniques—methods like the upper and lower sums considered earlier. However, for many functions we can find algebraic antiderivatives.

The *indefinite integral* $\int f(x)\,dx$ denotes the collection of *all* antiderivatives of $f(x)$ (whether known algebraically or not).

Thus if $F'(x) = f(x)$, that is, F is an antiderivative, then

$$\int f(x)\,dx = F(x) + C, \quad C \text{ a constant.}$$

Basic indefinite integrals (Stewart, Chap. 7)

$$
\begin{aligned}
&\int x^r \, dx = \frac{x^{r+1}}{r+1} + C, \quad r \neq -1 \\
&\int \frac{1}{x} \, dx = \ln|x| + C \\
&\int e^x \, dx = e^x + C \\
&\int a^x \, dx = \frac{1}{\ln a} a^x + C, \quad a > 0, a \neq 1 \\
&\int \sin x \, dx = -\cos x + C \\
&\int \cos x \, dx = \sin x + C \\
&\int \sec^2 x \, dx = \tan x + C \\
&\int \sinh x \, dx = \cosh x + C \\
&\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C, \quad |x| < 1 \\
&\int \frac{dx}{1+x^2} = \tan^{-1} x + C.
\end{aligned}
$$

◊ **5.1: The indefinite integral** $\int \frac{1}{x}\, dx = \ln|x| + C$ **requires explanation** . . .

5.2 Integration by substitution (Stewart, §4.5)

Example 5.1. We know $\int x^{23}\, dx = \frac{x^{24}}{24} + C$; but what about $\int (2x+7)^{23}\, dx$? We could expand to get a polynomial of degree 23, but there is a better way.

Substitution Let $u = 2x+7$; $\frac{du}{dx} = 2$ or $du = 2\, dx$ or $\frac{du}{2} = dx$.

$$
\begin{aligned}
\int (2x+7)^{23}\, dx &= \int u^{23} \frac{du}{2} \\
&= \frac{u^{24}}{24} \cdot \frac{1}{2} + C = \frac{(2x+7)^{24}}{48} + C.
\end{aligned}
$$

□

◊ **5.2: Examples** . . .

Common form of integration by substitution (Stewart, p.331)
For the integral $\int f(g(x))g'(x)\,dx$ put $u = g(x)$ so $du = g'(x)\,dx$ and then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

◊ 5.3: Examples ...

For definite integrals (Stewart, p.333)

Also change the limits of integration. For $\int_a^b f(g(x))g'(x)\,dx$ put $u = g(x)$ so $du = g'(x)\,dx$, $u_a = g(a)$, and $u_b = g(b)$. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{u_a}^{u_b} f(u)\,du.$$

◊ 5.4: Example ...

Sometimes substitution makes the integral easy to evaluate, even though it is not in the standard form described above.

◊ 5.5: Example ...

5.3 Integration by parts (Stewart, §7.1)

We have a product rule for derivatives, what, if anything, does this tell us about integrals?

◊ 5.6: Example of an integral of a product ...

Recall the Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.

Rearranging, $f(x)g'(x) = \frac{d}{dx}[f(x)g(x)] - f'(x)g(x)$.

Integrating both sides gives the integration by parts formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Note that you only add an integration constant when the very last indefinite integral disappears.

◊ 5.7: Examples ...

◊ 5.8: Does it matter which function we choose as $f(x)$ and which as $g'(x)$? ...

Most find the following form more memorable (Stewart, p.488): denoting $u = f(x)$ and $v = g(x)$ the integration by parts formula becomes $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$; that is,

$$\int u\,dv = uv - \int v\,du.$$

Integration by Parts is very useful for integrals of the form

$$\int x^n \cdot F(x)\,dx,$$

where $F(x)$ is a trigonometric, exponential function, etc. In this case take $u = f(x) = x^n$, $dv = g'(x)dx = F(x)dx$.

◊ 5.9: Example ...

Integration by Parts can often be used for integrals $\int F(x)\,dx$ where $F(x)$ is an inverse (trig, logarithmic) function and its derivative is known.

◊ **5.10: Examples** . . .

Sometimes by iterating the integration by parts procedure you end up with the original expression again, and sometimes this can still be used to evaluate the integral.

◊ **5.11: example** . . .

5.4 Trigonometric integrals and reduction formulae (Stewart, §7.2)

Reduction Formulae How do we evaluate integrals of (high) powers of trigonometric functions such as $\int \sin^n x\,dx$, $n = 1, 2, 3, \ldots$?

For $n = 1$, $\int \sin x\,dx = -\cos x + C$.

Example 5.2. Letting $I_n = \int \sin^n x\,dx$, show

$$I_n = -\frac{1}{n}\sin^{n-1} x\,\cos x + \left(\frac{n-1}{n}\right) I_{n-2}.$$

♢ 5.12: Calculation . . .

□

This is called a *reduction formula*. It does not give the integral but gives a method for calculating the integral by repeatedly using the formula until we reach either

$$I_1 = \int \sin x\,dx = -\cos x + C \quad \text{or} \quad I_0 = \int 1\,dx = x + C$$

♢ 5.13: Examples . . .

A reduction formula for $\int \cos^n x\,dx$ can be derived in a similar fashion.

♢ 5.14: Example . . .

Other trigonometric integrals

Example 5.3. Consider

$$\int \cos mx \cos nx \, dx$$

for $m \neq \pm n$.

We use the identity $\cos mx \cos nx = \frac{1}{2} [\cos(m+n)x + \cos(m-n)x]$

◊ 5.15: Solution . . .

□

A similar method works for other integrals involving $\cos mx$ and $\sin nx$.

Integrals of products of powers of $\sin, \cos$

(Stewart, §7.2) **◊ 5.16: Some examples of integrals of products of powers of $\sin$ and $\cos$.** . . .

In general for $\int \cos^n x \sin^m x \, dx$ (Stewart, p.497):

- if n is odd, put $u = \sin x$;
- if n even, m odd, put $u = \cos x$;
- for n, m both even, use double angle formulae until one power is odd

5.5 Trigonometric substitutions (Stewart, §7.3)

Integrands involving $\sqrt{a^2 \pm x^2}$, $\sqrt{x^2 \pm a^2}$ can often be evaluated by an appropriate trigonometric substitution. The strategy is to make a substitution of the type $x = a\sin\theta, x = a\sec\theta$ or something similar to simplify the expression using trigonometric identities. Often the result can then be expressed as an integral of trigonemtric functions of the type already dealt with above. We illustrate with examples.

◊ 5.17: Examples ...

5.6 Integration of rational functions by partial fractions

(Stewart, §7.4)

Recall that a rational function is a function of the form $\frac{p(x)}{q(x)}$ where p and q are polynomial functions. We consider how to integrate such functions, beginning with some basic examples established from previous integration.

1. $\int \frac{1}{x+3}\,dx = \ln|x+3| + C$
2. $\int \frac{1}{(x+3)^2}\,dx = \frac{-1}{x+3} + C$
3. $\int \frac{1}{(x+3)^{10}}\,dx = \frac{-1}{9}\frac{1}{(x+3)^9} + C$
4. $\int \frac{1}{x^2+1}\,dx = \arctan x + C$
5. $\int \frac{x-3}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx - \int \frac{3}{x^2+1}\,dx$
   $= \frac{1}{2}\ln(x^2+1) - 3\arctan x + C$

- Whenever the degree of the numerator is greater than or equal to the denominator, we first simplify the integrand by performing a long division

  ♢ **5.18: example** ...

- When the denominator is an irreducible quadratic, completing the square is helpful.

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$$
$$= \arctan(x+1) + C$$

  Here x^2+2x+2 is irreducible because it cannot be factored into real linear factors.

  ♢ **5.19: example** ...

- When the denominator is a product of linear factors, then write the integrand as a sum of partial fractions; for example,

$$\int \frac{1}{(x+1)(x+2)}\,dx = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx\,.$$

◊ **5.20: example** ...

- In the next example, the denominator x^2+2x+2 is irreducible: for such irreducible quadratics we must *seek* the numerator as a *linear function*.

◊ **5.21: Example** ...

- When a factor occurs repeated several times, include terms of all powers up to the multiplicity of the factor.

◊ **5.22: Example** ...

Method for integrating a rational function $p(x)/q(x)$

1. If the degree of $p(x) \geq$ degree of $q(x)$, use (long) division to write the function as the sum of a polynomial and a rational function with the degree of the numerator less than the degree of the denominator $q(x)$.

   Thus we only need to further consider rational functions $\frac{p(x)}{q(x)}$ with the degree of $p(x) <$ the degree of $q(x)$.

2. Factorize $q(x)$ into a product of *linear* and *irreducible quadratic* terms:

   $$q(x) = k(x-a)^r \cdots (x^2 + 2bx + c)^s \cdots ,$$

   where k is a constant, $(x-a)^r$ is a typical linear factor repeated r times, and $x^2 + 2bx + c$ is an irreducible quadratic factor ($b^2 < c$) repeated s times.

3. Write the rational function as a sum of partial fractions: for each linear term $(x-a)^r$ there is a sum of terms

   $$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r};$$

   and for each quadratic term $(x^2 + 2bx + c)^s$ there is a sum of terms

   $$\frac{B_1 x + C_1}{x^2 + 2bx + c} + \frac{B_2 x + C_2}{(x^2 + 2bx + c)^2} + \cdots + \frac{B_s x + C_s}{(x^2 + 2bx + c)^s}.$$

   Determine the constants $A_1, \ldots, A_r$, $B_1, \ldots, C_s$ using the techniques of substitution or comparison of coefficients.

4. Finally, integrate the partial fractions.

◊ **5.23: example** ...

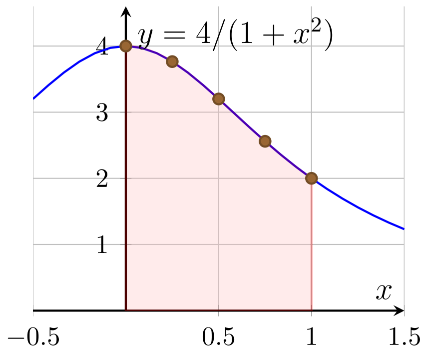
6 Numerical integration

There are two reasons for needing numerical integration.

- Most integrals cannot be evaluated analytically, just one example is $\int_a^b \sqrt{1+x^3+x^4}\,dx$. Instead we use numerical integration.
- In order to integrate a quantity obtained from experiment or observation, then we need numerical integration.

There are various ways we numerically estimate integrals. Recall we defined integrals as the area under a curve: let's approximate this area by adding together rectangles the fit the curve. Easy choices for the heights of the rectangles are the left and right ends of the subinterval, or their average, or the midpoint of the subinterval.

Example 6.1. $I = \int_0^1 \frac{4}{1+x^2}\,dx$



◊ 6.0: Calculation ...

□

In general

Left-end : Use the height of the function at the left-end of each subinterval to obtain the left-end point approximation, with $y_i = f(x_i)$,

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = (y_0 + y_1 + \cdots + y_{n-1})\Delta x\,.$$

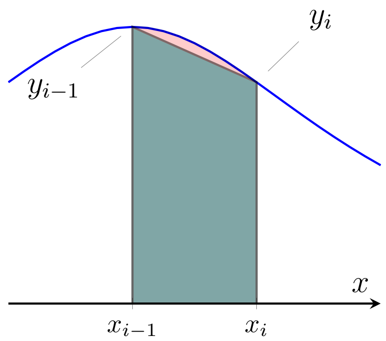
Right-end : Use the height of the function at the left-end of each subinterval to obtain the right-end point approximation

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = (y_1 + y_2 + \cdots + y_n)\Delta x\,.$$

Trapezoidal : (Stewart, p.532)
Definition 6.1. The trapezoidal approximation to $\int_a^b f(x)\,dx$ with $\Delta x = \frac{b-a}{n}$ is

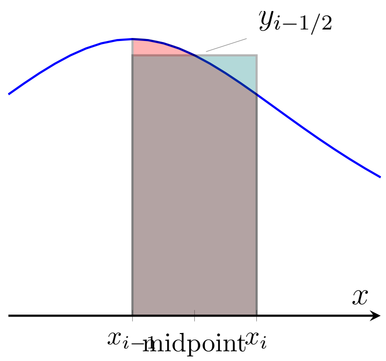
$$\begin{aligned} T_n &= \frac{L_n + R_n}{2} \\ &= \frac{1}{2} \cdot \Delta x \cdot [y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n]. \end{aligned}$$

Geometrically The area of the ith trapezium is $\frac{1}{2}\Delta x \cdot [y_{i-1} + y_i]$ which gives a better approximation to the area than either rectangle.



Midpoint : (Stewart, p.531) Use the height of the function at the midpoint of $[x_{i-1}, x_i]$; that is, denote $x_{i-1/2} = \frac{x_{i-1}+x_i}{2}$, and define $y_{i-1/2} = f(x_{i-1/2})$, then the midpoint approximation is

$$M_n = \Delta x[y_{1/2} + y_{3/2} + \cdots + y_{n-1/2}].$$



◊ **6.1: Example** ...

Simpson's Rule

To get a better approximation, we use parabolas rather than straight lines to approximate a curve.

◊ **6.2: Derivation** ...

Thus we get the following formula for Simpson's Rule:

$$S_n = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

This usually gives a better approximation to $\int_a^b f(x)\,dx$, furthermore it can also be thought of as a weighted average of the Trapezoidal and Midpoint approximations.

$$S_{2n} = \frac{1}{3}[2M_n + T_n]$$

◊ **6.3: Examples** ...

Error estimates

In terms of *accurate estimation* of $\int_a^b f(x)\,dx$:

poor : upper, lower, left-end, right-end estimates;

fair : trapezoidal and midpoint choices;

good : Simpson's rule

Example 6.2. Apply Simpson's rule with $n = 2$ to approximate $\int_0^1 \frac{4}{1+x^2}\,dx$.

◊ **6.4: Calculation** ...

□

Science and engineering require errors for every estimate. Hence it is highly desirable to get an estimate of errors involved in numerical integration. Since exact = approx + error, define the following.

Definition 6.2. Suppose $\int_a^b f(x)\,dx$ is calculated using the Trapezoidal or Simpson s Rule . Let the error be denoted by

$$E(T_n) = \int_a^b f(x)\,dx - T_n\,,$$
$$E(S_n) = \int_a^b f(x)\,dx - S_n\,.$$

The following theorem gives us *upper bounds* on such errors.

Theorem 6.1. *(a)* (Stewart, p.534) *Suppose $f''(x)$ is continuous on $[a,b]$ and that $|f''(x)| \le K_2$, for all $x \in [a,b]$, then*

$$|E(T_n)| \le \frac{K_2(b-a)^3}{12n^2}\,.$$

(b) (Stewart, p.538) *Suppose the fourth derivative $f^{(iv)}(x)$ is continuous on $[a,b]$ and that $|f^{(iv)}(x)| \le K_4$ for all $x \in [a,b]$, then the error in Simpson's rule is*

$$|E(S_n)| \le \frac{K_4(b-a)^5}{180n^4}\,.$$

Important Theorem 6.1 says:

$$|E(T_n)| \propto \frac{1}{n^2} \quad \text{and} \quad |E(S_n)| \propto \frac{1}{n^4}\,.$$

For example, a ten-fold increase in n gives

- two more decimal places of accuracy for T_n, and
- four more decimal places of accuracy for S_n.

$\Diamond$ **6.5: Examples** ...