# Fluid Mechanics Assignment 5

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1. Consider an accelerating sphere moving at speed $U(t)$ along the z-axis through an otherwise quiescent incompressible inviscid fluid. The velocity potential of the flow in a *stationary* reference frame is
$$\phi = -\frac{a^3U(t)}{2}\frac{(z-Z(t))}{[x^2+y^2+(z-Z(t))^2]^{3/2}}$$
Where $U(t)$ is the speed of the sphere at time $t$, $Z(t)$ is the position of the centre of the sphere along the $z$-axis and $a$ is the radius of the sphere.

   (a) Suppose that the centre of the sphere passes through the origin at time $t_0$. Show that
$$\phi|_{t=t_0} = -\frac{a^3U_0}{2}\frac{\cos\theta}{r^2}$$
$$\left.\frac{\partial\phi}{\partial t}\right|_{t=t_0} = -\frac{a^3U_0'}{2}\frac{\cos\theta}{r^2} + \frac{a^3U_0^2}{2}\frac{(1-3\cos^2\theta)}{r^3}$$
where $U_0 = U(t_0)$, $U_0' = U'(t_0)$ and $(r,\theta,\psi)$ are the coordinates of our usual spherical coordinate system.

**Solution** This implies that $Z(t_0) = 0$, and: $\sqrt{x^2+y^2+z^2} = r$
$$\begin{aligned}\phi|_{t=t_0} &= -\frac{a^3U(0)}{2}\frac{(z-Z(0))}{[x^2+y^2+(z-Z(0))^2]^{3/2}}\\ &= -\frac{a^3U_0}{2}\frac{z}{[x^2+y^2+z^2]^{3/2}}\\ &= -\frac{a^3U_0}{2}\frac{r\cos\theta}{r^3}\\ &= -\frac{a^3U_0}{2}\frac{\cos\theta}{r^2}\end{aligned}$$

Mix of product and quotient rules for the derivative. Note
$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{A(x)B(x)}{C(x)}\right) &= \frac{(A(x)B(x))'C(x) - A(x)B(x)C'(x)}{[C(x)]^2}\\ &= \frac{A'(x)B(x)C(x) + A(x)B'(x)C(x) - A(x)B(x)C'(x)}{[C(x)]^2}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial\phi}{\partial t} &= \frac{\partial}{\partial t}\left(-\frac{a^3U(t)}{2}\frac{(z-Z(t))}{[x^2+y^2+(z-Z(t))^2]^{3/2}}\right)\\
&= -\frac{a^3}{2}\frac{\partial}{\partial t}\left(U(t)\frac{(z-Z(t))}{[x^2+y^2+(z-Z(t))^2]^{3/2}}\right)\\
&= -\frac{a^3}{2}(\frac{\left(U'(t)(z-Z(t))[x^2+y^2+(z-Z(t))^2]^{3/2}\right)-U(t)Z'(t)[x^2+y^2+(z-Z(t))^2]^{3/2}}{[x^2+y^2+(z-Z(t))^2]^{6/2}}\\
&\quad -\frac{U(t)(z-Z(t))\left(-3(z-Z(t))Z'(t)[x^2+y^2+(z-Z(t))^2]^{1/2}\right)}{[x^2+y^2+(z-Z(t))^2]^{6/2}})\\
\frac{\partial\phi}{\partial t}\Big|_{t=t_0} &= -\frac{a^3}{2}(\frac{\left(U_0'z[x^2+y^2+z^2]^{3/2}\right)-U_0Z'(0)[x^2+y^2+z^2]^{3/2}}{[x^2+y^2+z^2]^{6/2}}-\frac{U_0z\left(-3zZ'(0)[x^2+y^2+z^2]^{1/2}\right)}{[x^2+y^2+z^2]^{6/2}})\\
&= -\frac{a^3}{2}(\frac{U_0'z-U_0Z'(0)}{[x^2+y^2+z^2]^{3/2}}-\frac{U_0z\left(-3zZ'(0)\right)}{[x^2+y^2+z^2]^{5/2}})\\
&= -\frac{a^3}{2}(\frac{U_0'r\cos\theta-U_0Z'(0)}{r^3}+\frac{3U_0r^2\cos^2\theta Z'(0)}{r^5})\\
&= -\frac{a^3}{2}(\frac{U_0'\cos\theta}{r^2}+\frac{3U_0Z'(0)\cos^2\theta-U_0Z'(0)}{r^3})
\end{aligned}
$$

Notice that $Z =$ position on $z$. This means $U =$ speed, over $z$ is $Z'(t)$.

$$
\begin{aligned}
\implies \frac{\partial\phi}{\partial t}\Big|_{t=t_0} &= -\frac{a^3}{2}(\frac{U_0'\cos\theta}{r^2}+\frac{3U_0U_0\cos^2\theta-U_0U_0}{r^3})\\
&= -\frac{a^3}{2}(\frac{U_0'\cos\theta}{r^2}+\frac{3U_0^2\cos^2\theta-U_0^2}{r^3})\\
&= -\frac{a^3}{2}(\frac{U_0'\cos\theta}{r^2}+\frac{U_0^2(3\cos^2\theta-1)}{r^3})\\
&= -\frac{a^3U_0'}{2}\frac{\cos\theta}{r^2}+\frac{a^3U_0^2}{2}\frac{(1-3\cos^2\theta)}{r^3}
\end{aligned}
$$

**As required.**

(b) Using spherical coordinates, calculate the velocity of the fluid at $t = t_0$

**Solution** Spherical coordinates:

$$x = r\sin\theta\cos\psi, \quad y = r\sin\theta\sin\psi, \quad z = r\cos\theta$$

Velocity $u = \nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\vec{e}_\psi$ We are already given that velocity in the $x, y$ directions are 0 by construction Meaning that $\frac{\partial\phi}{\partial\psi} = 0$.

$$
\begin{aligned}
u|_{t=t0} &= \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\vec{e}_\psi\\
&= \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\phi}{\partial r}\Big|_{t=t0} &= \frac{\partial}{\partial r}\left(-\frac{a^3U_0}{2}\frac{\cos\theta}{r^2}\right)\\
&= a^3U_0\frac{\cos\theta}{r^3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\phi}{\partial\theta}\Big|_{t=t0} &= \frac{\partial}{\partial\theta}\left(-\frac{a^3U_0}{2}\frac{\cos\theta}{r^2}\right)\\
&= \frac{a^3U_0}{2}\frac{\sin\theta}{r^2}
\end{aligned}
$$

$$
\begin{aligned}
\implies u|_{t=t0} = u_0 &= \frac{\partial \phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{e}_\theta \\
&= \left(a^3 U_0 \frac{\cos\theta}{r^2}\right)\vec{e}_r + \left(\frac{1}{r}\frac{a^3 U_0}{2}\frac{\sin\theta}{r^2}\right)\vec{e}_\theta \\
&= a^3 U_0 \frac{\cos\theta}{r^3}\vec{e}_r + \frac{a^3 U_0}{2}\frac{\sin\theta}{r^3}\vec{e}_\theta
\end{aligned}
$$

**As required.**

(c) Verify that the velocity found in part ($b$) satisfies the impermeability condition on the surface of the sphere $r = a$ at $t = t_0$.

**Solution** Impermeability implies zero velocity relative to the boundary at the boundary.

$$
(\mathbf{u} - \mathbf{U}) \cdot \hat{\mathbf{n}} = 0
$$

$$
\begin{aligned}
u_{0,r=a} &= a^3 U_0 \frac{\cos\theta}{a^3}\vec{e}_r + \frac{a^3 U_0}{2}\frac{\sin\theta}{a^3}\vec{e}_\theta \\
&= U_0 \cos\theta \vec{e}_r + \frac{U_0}{2}\sin\theta \vec{e}_\theta
\end{aligned}
$$

$\hat{\mathbf{n}} = \vec{e}_r$ and $\mathbf{U} = U(t)e_z = U(t)\cos\theta\vec{e}_r$

$$
\begin{aligned}
(\mathbf{u} - \mathbf{U}) \cdot \hat{\mathbf{n}} &= \left((U_0\cos\theta - U_0\cos\theta)\vec{e}_r + \frac{U_0}{2}\sin\theta\vec{e}_\theta\right) \cdot \vec{e}_r \\
&= 0 + 0 = 0
\end{aligned}
$$

**As required.**

(d) The drag on the sphere is $D = \mathbf{F} \cdot \mathbf{k}$, where $\mathbf{F}$ is the force

$$
\mathbf{F} = -\int_{\mathcal{S}} p\hat{n}dS
$$

$\mathcal{S}$ denotes the surface of the sphere, $p$ is the pressure and $\hat{n}$ is the normal to the sphere. Assuming that there are no external forces, use an appropriate form of Bernoullis equation, together with the results obtained in parts (a) and (b), to calculate the drag on the sphere at $t = t_0$.

*Hint*: You may use technology to evaluate the integrals. If you do, please list all the results so that the reader can follow the calculation.

**Solution** Bernoulli equation for unsteady, incompressible and irrotational flow with no external forces (from tute 5)

$$
\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 = f(t)
$$

This gives that the pressure is:

$$
p = \rho\left(f(t) - \frac{\partial \phi}{\partial t} - \frac{1}{2}|\mathbf{u}|^2\right)
$$

From $b$ the velocity on the surface on the sphere at $r = a$ is:

$$
u = U_0\cos\theta\vec{e}_r + \frac{U_0}{2}\sin\theta\vec{e}_\theta
$$

Which gives that the speed is:

$$
\begin{aligned}
|\mathbf{u}|^2 = u_r^2 + u_\theta^2 &= U_0^2\cos^2\theta + \frac{U_0^2}{4}\sin^2\theta \\
&= \frac{U_0^2}{4}(4\cos^2\theta + \sin^2\theta) \\
&= \frac{U_0^2}{4}\left(3\cos^2\theta + 1\right)
\end{aligned}
$$

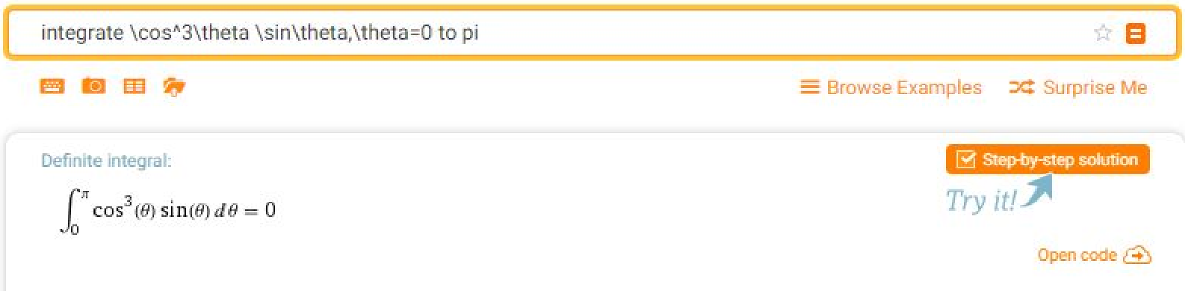


Figure 1: Evidence for $\cos^3\theta\sin\theta d\theta$

Next, $\frac{\partial\phi}{\partial t}$ at $r = a$

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= -\frac{a^3U_0'}{2}\frac{\cos\theta}{a^2} + \frac{a^3U_0^2}{2}\frac{(1-3\cos^2\theta)}{a^3}\\ &= -\frac{aU_0'\cos\theta}{2} + \frac{U_0^2(1-3\cos^2\theta)}{2}\end{aligned}$$

Sub into $p$:

$$\begin{aligned}p &= \rho\left(f(t_0) - \frac{\partial\phi}{\partial t} - \frac{1}{2}|\mathbf{u}|^2\right)\\ &= \rho\left(f(t_0) - \left(-\frac{aU_0'\cos\theta}{2} + \frac{U_0^2(1-3\cos^2\theta)}{2}\right) - \frac{1}{2}\left(\frac{U_0^2}{4}(3\cos^2\theta+1)\right)\right)\end{aligned}$$

We won't worry about simplifying this yet. The surface of the sphere $dS = 2\pi a^2\sin\theta d\theta$ (from lectures) and $\hat{n} = \vec{e}_r = \sin\theta\cos\psi i + \sin\theta\sin\psi j + \cos\theta k$ Now, using the formula for force:

$$\begin{aligned}\mathbf{F} &= -\int_S p\hat{n}dS\\ &= -\int_0^\pi \rho\left(f(t_0) - \left(-\frac{aU_0'\cos\theta}{2} + \frac{U_0^2(1-3\cos^2\theta)}{2}\right) - \frac{1}{2}\left(\frac{U_0^2}{4}(3\cos^2\theta+1)\right)\right)\\ &\quad(\sin\theta\cos\psi i + \sin\theta\sin\psi j + \cos\theta k)2\pi a^2\sin\theta d\theta\end{aligned}$$

We can drop all terms not in the $k$ direction (because we only care about the drag)

$$\begin{aligned}\mathbf{F} = &-\int_0^\pi \rho\left(f(t_0) - \left(-\frac{aU_0'\cos\theta}{2} + \frac{U_0^2(1-3\cos^2\theta)}{2}\right) - \frac{1}{2}\left(\frac{U_0^2}{4}(3\cos^2\theta+1)\right)\right)\cos\theta k 2\pi a^2\sin\theta d\theta\\ &= -\rho 2\pi a^2\int_0^\pi \left(f(t_0) - \left(-\frac{aU_0'\cos\theta}{2} + \frac{U_0^2(1-3\cos^2\theta)}{2}\right) - \frac{1}{2}\left(\frac{U_0^2}{4}(3\cos^2\theta+1)\right)\right)\cos\theta k\sin\theta d\theta\\ &= -\rho 2\pi a^2\int_0^\pi \left(f(t_0) + \frac{aU_0'\cos\theta}{2} - \frac{U_0^2(1-3\cos^2\theta)}{2} - \frac{U_0^2}{8}(3\cos^2\theta+1)\right)\cos\theta k\sin\theta d\theta\\ &= -\rho 2\pi a^2(\int_0^\pi f(t_0)\cos\theta k\sin\theta d\theta + \int_0^\pi \frac{aU_0'\cos\theta}{2}\cos\theta k\sin\theta d\theta\\ &- \int_0^\pi \frac{U_0^2}{2}\cos\theta k\sin\theta d\theta + \int_0^\pi \frac{U_0^2 3\cos^2\theta}{2}\cos\theta k\sin\theta d\theta\\ &- \int_0^\pi \frac{U_0^2}{8}3\cos^2\theta\cos\theta k\sin\theta d\theta - \int_0^\pi \frac{U_0^2}{8}\cos\theta k\sin\theta d\theta)\end{aligned}$$

Note $\int_0^\pi \cos\theta\sin\theta d\theta = 0$, Using wolframalpha.com, we also get $\int_0^\pi \cos^3\theta\sin\theta d\theta = 0$ shown in figure 1. This simplifies the integral down to:

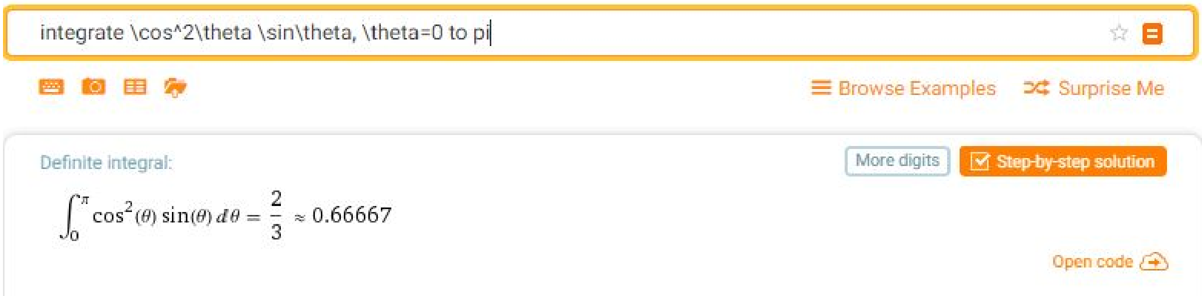


Figure 2: Evidence for $\cos^2\theta\sin\theta d\theta$

$$\begin{aligned}
\mathbf{F} &= -\rho\pi a^2 \int_0^\pi aU_0' \cos\theta\cos\theta k \sin\theta d\theta \\
&= -\rho\pi a^3 U_0' \int_0^\pi \cos^2\theta k \sin\theta d\theta \\
&= \frac{-2\rho\pi a^3 U_0'}{3}\mathbf{k}
\end{aligned}$$

Figure 2 evidences the step that the integral is equal to 2/3.
Finally, calculate the drag:

$$\begin{aligned}
D &= \mathbf{F}\cdot k \\
&= \frac{-2\rho\pi a^3 U_0'}{3}
\end{aligned}$$

**As required.**

2. Consider a pair of parallel line vortices propagating at constant speed in an incompressible irrotational fluid. Suppose that we observe this flow from a reference frame that moves with the two vortices. In this frame, the flow is steady and consists of one vortex circulating in a clockwise sense at $z = id$ with circulation $-\Gamma$, another vortex circulating in an anticlockwise sense at $z = -id$ with circulation $\Gamma$, and a uniform flow of speed $\Gamma/(4\pi d)$ in the positive $x$-direction. Here, $z = x + iy$ is a point on the complex plane, $d$ is half the distance between the vortices and $\Gamma > 0$.

   (a) Write down the complex potential for this flow
   **Solution** For a line vortex, with circulation $\Gamma$ centered at $z = -id$

$$w = -i\frac{\Gamma}{2\pi}\ln(z + id)$$

   For the line vortex with circulation $-\Gamma$ centered at $z = id$

$$w = i\frac{\Gamma}{2\pi}\ln(z - id)$$

   Uniform flow at speed $\Gamma/(4\pi d)$

$$w = z\Gamma/(4\pi d)$$

   Complex potential will be the sum:

$$\begin{aligned} w &= -i\frac{-\Gamma}{2\pi}\ln(z - id) - i\frac{\Gamma}{2\pi}\ln(z + id) + \frac{z\Gamma}{4\pi d} \\ &= \frac{\Gamma}{2\pi}\left(\frac{z}{2d} + i\left(-\ln(z + id) + \ln(z - id)\right)\right) \end{aligned}$$

   **As required.**

   (b) Find the complex velocity
   **Solution** The complex velocity is

$$\frac{dw}{dz} = \frac{\partial\psi}{\partial y} - i\frac{\partial\phi}{\partial y} = u - iv$$

$$\begin{aligned} \frac{dw}{dz} &= \frac{\partial w}{\partial z}\left(\frac{\Gamma}{2\pi}\left(\frac{z}{2d} + i\left(-\ln(z + id) + \ln(z - id)\right)\right)\right) \\ &= \frac{\Gamma}{2\pi}\frac{\partial w}{\partial z}\left(\frac{z}{2d} + i\left(-\ln(z + id) + \ln(z - id)\right)\right) \\ &= \frac{\Gamma}{2\pi}\left(\frac{1}{2d} + i\left(-\frac{1}{z + id} + \frac{1}{z - id}\right)\right) \\ &= \frac{\Gamma}{2\pi}\left(\frac{1}{2d} + i\left(\frac{-(z - id) + (z + id)}{(z + id)(z - id)}\right)\right) \\ &= \frac{\Gamma}{2\pi}\left(\frac{1}{2d} + i\left(\frac{2id}{(z + id)(z - id)}\right)\right) \\ &= \frac{\Gamma}{2\pi}\left(\frac{1}{2d} - \left(\frac{2d}{z^2 + d^2}\right)\right) \end{aligned}$$

   **As required.**

   (c) Find the stagnation points
   **Solution** Stagnant when velocity $= 0$ i.e.

$$\frac{dw}{dz} = 0$$

$$
\begin{aligned}
0 &= \frac{\Gamma}{2\pi}\left(\frac{1}{2d} - \left(\frac{2d}{z^2+d^2}\right)\right) \\
-\frac{1}{2d} &= \frac{-2d}{z^2+d^2} \\
z^2 + d^2 &= 4d^2 \\
z^2 &= -d^2 + 4d^2 \\
z &= \pm\sqrt{3d^2} \\
&= \pm d\sqrt{3} \\
\implies x &= \pm d\sqrt{3} \text{ and } y = 0
\end{aligned}
$$

**As required.**

(d) Find and plot the stream function

**Solution** And we want to find $\psi$ i.e. the imaginary part of $w$.

$$
w = \frac{\Gamma}{2\pi}\left(\frac{z}{2d} + i\left(-\ln(z+id) + \ln(z-id)\right)\right)
$$

Use $z - id = r_1 e^{i\theta_1}$ and $z + id = r_2 e^{i\theta_2}$

$$
\begin{aligned}
\implies w &= \frac{\Gamma}{2\pi}\left(\frac{z}{2d} + i\left(-\ln(r_2 e^{i\theta_2}) + \ln(r_1 e^{i\theta_1})\right)\right) \\
&= \frac{\Gamma}{2\pi}\left(\frac{x+iy}{2d} + i\left(-\ln r_2 - i\theta_2 + \ln r_1 + i\theta_1\right)\right) \\
&= \frac{\Gamma}{2\pi}\left(\frac{x+iy}{2d} - i\ln r_2 + \theta_2 + i\ln r_1 - \theta_1\right) \\
&= \frac{\Gamma}{2\pi}\left(\frac{x}{2d} - \theta_2 - \theta_1\right) + \frac{i\Gamma}{2\pi}\left(\frac{y}{2d} - \ln r_2 + \ln r_1\right)
\end{aligned}
$$

So the stream-function $\psi$:

$$
\psi = \frac{\Gamma}{2\pi}\left(\frac{y}{2d} - \ln r_2 + \ln r_1\right)
$$

back to cartesian:

$$
\begin{aligned}
r_1^2 &= x^2 + (y-d)^2 \\
\implies r_1 &= \sqrt{x^2 + (y-d)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
r_2^2 &= x^2 + (y+d)^2 \\
\implies r_2 &= \sqrt{x^2 + (y+d)^2}
\end{aligned}
$$

Which gives:

$$
\begin{aligned}
\psi &= \frac{\Gamma}{2\pi}\left(\frac{y}{2d} - \ln\sqrt{x^2+(y+d)^2} + \ln\sqrt{x^2+(y-d)^2}\right) \\
&= \frac{\Gamma}{2\pi}\left(\frac{y}{2d} - \frac{1}{2}\ln\left(x^2+(y+d)^2\right) + \frac{1}{2}\ln\left(x^2+(y-d)^2\right)\right)
\end{aligned}
$$

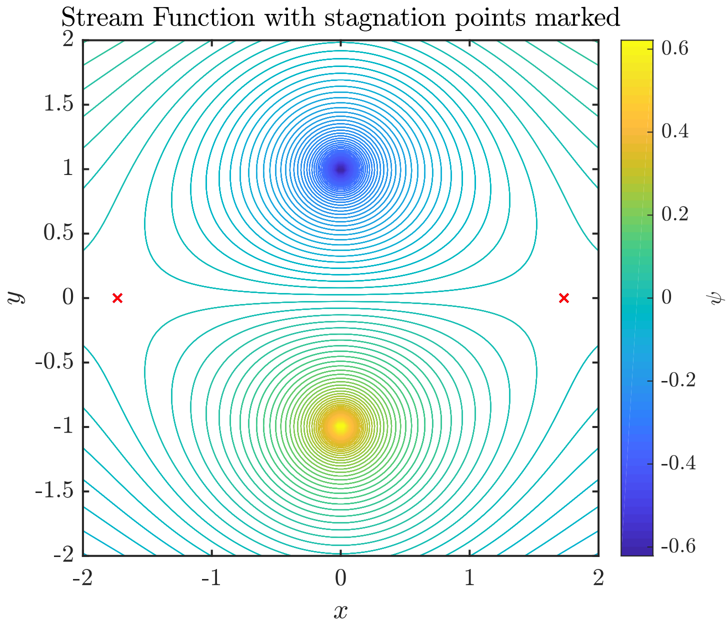


Figure 3: Plot of Streamfunction found in $2d$ for $\Gamma = d = 1$ with stagnation points marked for $x = \pm d\sqrt{3}$

`Matlab` code:

```
%%Q2 part d
%Plot stream function for two line vortices in a uniform flow
%7/06/2018
%Andrew Martin
set(groot, 'DefaultLineLineWidth', 1, ...
    'DefaultAxesLineWidth', 1, ...
    'DefaultAxesFontSize', 12, ...
    'DefaultTextFontSize', 12, ...
    'DefaultTextInterpreter', 'latex', ...
    'DefaultLegendInterpreter', 'latex', ...
    'DefaultColorbarTickLabelInterpreter', 'latex', ...
    'DefaultAxesTickLabelInterpreter','latex');
% Parameters
gamma = 1;
d = 1;

[x,y] = meshgrid(linspace(-2*d,2*d),linspace(-2*d,2*d));
psi = gamma/(2*pi) *(y/(2*d) - 0.5* log(x.^2 + (y+d).^2) + 0.5 * log(x.^2 + (y-d).^2));


% Create contour plot
contour(x,y,psi,100)
axis equal
axis([-2*d, 2*d, -2*d, 2*d])
xlabel('$x$')
ylabel('$y$')
c = colorbar;
ylabel(c, '$\psi$', 'Interpreter', 'Latex');
hold on
plot([-1,1]*d*sqrt(3),0,'rx')
title('Stream Function with stagnation points marked')
```

**As required.**