Stochastic Assignment 1

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2. Consider the LP

$$\begin{aligned} \max \quad & z = -5x_1 + x_2 - 4x_3 \\ \text{such that} \quad & 2x_1 + 2x_2 - 4x_3 = 1 \\ & 2x_1 + 2x_2 + 2x_3 = 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) Construct its dual using the general relationship between primal and dual (instead of rewriting the above LP in standard form and then writing the dual of the LP in the standard form).

Solution Dual: Since $x_1, x_2, x_3 \geq 0$ we get $\sum_{i=1}^{m} y_i a_{ij} \geq c_j$

$$\begin{aligned} \min \quad & w = y_1 + 4y_2 \\ \text{such that} \quad & 2y_1 + 2y_2 \geq -5 \\ & 2y_1 + 2y_2 \geq 1 \\ & -4y_1 + 2y_2 \geq -4 \\ & y_1, y_2 \text{ free} \end{aligned}$$

Of course we have to rewrite this such that it is in standard form:

$$\begin{aligned} \min \quad & w = y_1 + 4y_2 \\ \text{such that} \quad & -2y_1 - 2y_2 \leq 5 \\ & -2y_1 - 2y_2 \leq -1 \\ & 4y_1 - 2y_2 \leq 4 \\ & y_1, y_2 \text{ free} \end{aligned}$$

As required.

(b) Provide the optimal solutions of both the primal and dual, using `linprog.m`. Include your `MATLAB` code and the output of the code.

Solution

The optimal solutions are $z = w = -0.5$.
Which are obtained when $x = [0, 1.5, 0.5]^T$ and $y = [\frac{5}{6}, -\frac{1}{3}]^T$ respectively

The Code:

```
%%%%Q2
%%Primal
%init the vectors
z =[5,-1,4];
A=[2,2,-4;2,2,2];
b = [1;4];
%bounded by 0
lb = [0,0,0];
[x,zval] = linprog(z,[],[],A,b,lb,[]);
%of course zval is actually giving the negative since it had to be swapped
zval = -zval;
%%Dual
w = [1,4];
B = [-2,-2;-2,-2;4,-2];
a = [5;-1;4];

[y,wval] = linprog(w,B,a);
%%%%Q3

%%%%Q4
```

As required.

3. Consider the LP

$$
\begin{aligned}
\max z &= 4x_1 + 8x_2 + 5x_3 \\
\text{such that } x_1 &\geq 10 (= b_1) \\
x_2 &\geq 9 (= b_2) \\
x_3 &\geq 3 (= b_3) \\
2x_1 + 3x_2 + x_3 &\leq 80 (= b_4) \\
x_1 + 2x_2 + 2x_3 &\leq 70 (= b_5) \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Suppose we place b_i with $b_i(\zeta)$ for $i = 1, 2, \ldots, 5$ where

$$
\zeta = \begin{cases} \epsilon_1 & \text{with probability } 0.3 \\ \epsilon_2 & \text{with probability } 0.5 \\ \epsilon_3 & \text{with probability } 0.2 \end{cases}
$$

Where

$$
\boldsymbol{b}(\zeta) = \begin{pmatrix} b_1(\zeta) \\ b_2(\zeta) \\ b_3(\zeta) \\ b_4(\zeta) \\ b_5(\zeta) \end{pmatrix}, \quad \text{with} \quad \boldsymbol{b}(\epsilon_1) = \begin{pmatrix} 8 \\ 6 \\ 1 \\ 80 \\ 70 \end{pmatrix}, \quad \boldsymbol{b}(\epsilon_2) = \begin{pmatrix} 10 \\ 10 \\ 3 \\ 80 \\ 70 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{b}(\epsilon_3) = \begin{pmatrix} 13 \\ 11 \\ 6 \\ 80 \\ 70 \end{pmatrix}
$$

Assume that we can meet any shortfall in demand through recourse at market, but we must pay q_1, q_2, q_3 units of currency per unit x_1, x_2, x_3 purchased at market respectively

(a) Formulate and write down a two-stage stochastic LP with fixed recourse

Solution Let y_1, y_2, y_3 be the constraint violation for x_1, x_2, x_3 respectively. This gives:

$$
\begin{aligned}
\max \quad & z = 4x_1 + 8x_2 + 5x_3 - \mathbb{E}[q_1 y_1(\zeta) + q_2 y_2(\zeta) + q_3 y_3(\zeta)] \\
\text{such that } & x_1 + y_1(\zeta) \geq b_1(\zeta) \\
& x_2 + y_2(\zeta) \geq b_2(\zeta) \\
& x_3 + y_3(\zeta) \geq b_3(\zeta) \\
& 2x_1 + 3x_2 + x_3 \leq 80 \\
& x_1 + 2x_2 + 2x_3 \leq 70 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

As required.

(b) Write the above SLP in the extended form

Solution The objective function becomes

$$
\max \quad z = 4x_1 + 8x_2 + 5x_3 - \sum_{k=1}^{3} p_k[q_1 y_1(\epsilon) + q_2 y_2(\epsilon) + q_3 y_3(\zeta)]
$$

Inputting the values gives:

$$
\begin{aligned}
\max \quad z = 4x_1 + 8x_2 + 5x_3 - \Big(& 0.3[q_1 y_1(\epsilon_1) + q_2 y_2(\epsilon_1) + q_3 y_3(\epsilon_1)] \\
& + 0.5[q_1 y_1(\epsilon_2) + q_2 y_2(\epsilon_2) + q_3 y_3(\epsilon_2)] + 0.2[q_1 y_1(\epsilon_3) + q_2 y_2(\epsilon_3) + q_3 y_3(\epsilon_3)] \Big) \\
\text{such that } & x_1 + y_1(\epsilon_1) \geq b_1(\epsilon_1) \\
& x_1 + y_1(\epsilon_2) \geq b_1(\epsilon_2) \\
& x_1 + y_1(\epsilon_3) \geq b_1(\epsilon_3) \\
& x_2 + y_2(\epsilon_1) \geq b_2(\epsilon_1) \\
& x_2 + y_2(\epsilon_2) \geq b_2(\epsilon_2) \\
& x_2 + y_2(\epsilon_3) \geq b_2(\epsilon_3) \\
& x_3 + y_3(\epsilon_1) \geq b_3(\epsilon_1) \\
& x_3 + y_3(\epsilon_2) \geq b_3(\epsilon_2) \\
& x_3 + y_3(\epsilon_3) \geq b_3(\epsilon_3) \\
& 2x_1 + 3x_2 + x_3 \leq 80 \\
& x_1 + 2x_2 + 2x_3 \leq 70 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

As required.

(c) Solve the recourse DEP in (b) for $q_1 = 10$, $q_2 = 1$ and $q_3 = 20$. Provide matlab code and output of the code, with an interpretation

Solution

```
x =

   13.0000
   12.7500
   15.7500
         0
         0
         0
         0
         0
         0
         0
         0
         0


zval =

  232.7500
```

Are the outputs. Clearly all of the recourse variables y_i are zero. Meaning that no recourse is neccessary in this case. **As required.**

4. Suppose X is a continuous RV with density

$$f(x) = \lambda^2 x e^{-\lambda x}, \quad \text{for } x \geq 0$$

Suppose $\lambda = 3$

(a) Find the 99% CI for X

Solution Since X is bounded below, the 99% CI is defined as

$$\begin{aligned} 0.99 &= P(X < x) \\ &= P(X \leq x) \\ &= \int f(x)dx \\ &= \int \lambda^2 x e^{-\lambda x} dx \\ &= -e^{-\lambda x}(\lambda x + 1) + C \end{aligned}$$

We need this to be $= 0$ at $X = 0$ to be a valid CDF:

$$-e^0(0+1) + C = 0 \implies C = 1$$

Which gives:

$$\begin{aligned} 0.99 &= -e^{-\lambda x}(\lambda x + 1) + 1 \\ -0.01 &= -e^{-\lambda x}(\lambda x + 1) \end{aligned}$$

Plugging in $\lambda = 3$ and solving gives 2.213 as the upper bound for the 99% confidence interval So $x \in (0, 2.22)$ is a 99% CI for X. **As required.**

(b) Determine a discrete approximation for X with 10 realisations. Explain clearly how you obtain the realisations and their respective probabilities (include code if you have generated these)

Solution Obtain uniform RVs (u) and then invert the distribution, i.e. we want $X = F^{-1}(u)$ where $F(x) = \int f(x)dx$ (i.e. the CDF). Alternatively use the fact that this is a $Gamma(2, \lambda)$ distribution and use an inbuilt function to obtain gamma distributed values. I.e. in matlab

```
%reproducibility
rng(1704466)
%Matlab uses the inverse version so make sure
%b = 1/3 instead of 3
lambda = 3;
```

```
pd = makedist('Gamma','a',2,'b',1/lambda);
X= random(pd,[1,10]);
probx = ((1/length(X))* lambda^2 .*X .*exp(-lambda.* X));
%make probs sum to 1
probx = probx./sum(probx);
```

This gives:

```
X =

    1.2714    0.2829    0.8908    0.7810    0.5389    0.0870    0.4606    0.8522    0.6731    0.5698

probx =

    0.0336    0.1452    0.0738    0.0899    0.1283    0.0804    0.1387    0.0793    0.1071    0.1237
```

Where the probx are the densities for each X within this group. I.e. the probability associated with a particular X is the probability of being in that position.

As required.

(c) Propose a mathematical method/formula for assessing the error of your discretisation

Solution One method would be to run it for a large number of $X's$ and calculate residual sum of squares to the data fit. For this particular solution the probabilities turn the problem into a 10 state process with their own probabilities. These could be compared to the PDF values to obtain some form of error from this **As required.**

(d) Given the above, suggest how you would reduce this error

Solution A clear solution would be to increase the number of points simulated. **As required.**