THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2016

107352 APP MTH 3022 Optimal Functions and Nanomechanics III

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 60

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Formulae sheets are provided.
- Only scientific calculators with basic capabilities are permitted. Graphics calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Consider the functional

$$F\{y\} = \int_0^1 xy^2y'^3\,dx.$$

   (a) If we constrain our investigation of possible functions for y to $y(x) = x^\epsilon$, what is the value of $\epsilon > 1/5$ which leads to an extremum of F.

   (b) What value does the functional take for this value of ϵ.

   (c) Is this a maximum or a minimum? Justify your answer.

[8 marks]

2. Find the extremals for the following functionals:

   (a) $F\{y\} = \displaystyle\int_0^1 (y^2 - y'^2 - 2y\sin x)\,dx, \qquad y(0) = 0, \quad y(1) = 1.$

   (b) $F\{y\} = \displaystyle\int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)\,dx, \quad y(0) = 0, \quad y(1) = \frac{3}{2}.$

   (c) $F\{y\} = \displaystyle\int_0^1 (y''^2 - 360x^2y)\,dx, \quad y(0) = 0, \quad y'(0) = 1, \quad y(1) = 1, \quad y'(1) = 5/2.$

[18 marks]

3. (a) From the integral definitions given on the formula sheet show that

$$K(k) = \frac{\pi}{2}F(1/2, 1/2; 1; k^2).$$

   (b) A spheroidal surface $\mathcal{P}$ is given parametrically by the position vector $\boldsymbol{x}(\theta, \phi)$ as

$$\boldsymbol{x}(\theta, \phi) = (b\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi),$$

   where $-\pi < \theta \leqslant \pi$, and $0 \leqslant \phi \leqslant \pi$ and the constant b is the minor semi-axis length and c is the major semi-axis length.

   (i) Derive an expression for the scalar surface element dA for $\mathcal{P}$.

   (ii) Integrate your answer from part (i) to find the surface area of $\mathcal{P}$ as a function of b and c.
   Hint: it is easiest to express this in terms of a hypergeometric function.

[13 marks]

4. Consider the functional
$$F\{y\} = \int_0^1 \left(\frac{1}{2}y' + \frac{1}{2}y^2 - y\right)\,dx,$$
subject to the end-point constraints $y(0) = 0$, $y(1) = 0$. Moreover consider a Ritz trial function of the form
$$y_n = \phi_0 + \sum_{i=1}^{n} c_i\phi_i.$$

   (a) Write down the approximate solution (with one undetermined coefficient) y_1, by assuming $\phi_0 = 0$ and $\phi_i = x^i(1-x)^i$.

   (b) Determine a function $F(c_1)$ which approximates the functional for the y_1 from part (a).

   (c) Determine the value of c_1 that leads to an extremal value for $F(c_1)$.

   (d) Is this extremum a maximum or minimum? Justify your answer.

[6 marks]

5. Consider the problem of finding extremals curves for
$$F\{y\} = \int_0^1 (y'^2 + x^2)\,dx,$$
subject to $y(0) = y(1) = 0$, and the integral constraint
$$G\{y\} = \int_0^1 y^2\,dx = 2.$$

   (a) Using the method of Lagrange multipliers, form a new functional $H\{y\}$ which incorporates the original functional $F\{y\}$ and the integral constraint.

   (b) Derive the Euler-Lagrange equation for $H\{y\}$.

   (c) Show that the only non-trivial solutions to the Euler-Lagrange equation and boundary conditions involve trigonometric functions and a discrete spectrum of Lagrange multipliers.

   (d) Thus find the curve or curves) y that is both an extremal of $F\{y\}$ and satisfies the constraint $G\{y\} = 2$. Calculate the value of $F\{y\}$ for this extremal.

Hint: You may find the following results useful
$$\int_0^1 \cos^2(n\pi x)\,dx = \int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}, \quad \text{for } n = 1, 2, 3, \ldots$$

[15 marks]

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Formula Sheet, Special Functions

Gamma function, definition	$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \quad \text{for } \Re(z) > 0.$
Gamma function, duplication	$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2).$
Beta function, definition	$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad \text{for } \Re(x), \Re(y) > 0.$
Beta function, gamma relation	$B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$
Pochhammer symbol, definition	$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}.$
Hypergeometric function, series	$F(a,b;c;z) = \sum_{n=0}^\infty \dfrac{(a)_n(b)_n}{(c)_n n!} z^n.$
Hypergeometric function, integral	$F(a,b;c;z) = \dfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$
Hypergeometric function, derivative	$\dfrac{d^n}{dz^n}F(a,b;c;z) = \dfrac{(a)_n(b)_n}{(c)_n}F(a+n,b+n;c+n;z).$
Elliptic integral, first kind	$F(\varphi,k) = \int_0^\varphi \dfrac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}.$
Elliptic integral, second kind	$E(\varphi,k) = \int_0^\varphi \sqrt{1-k^2\sin^2\vartheta}\,d\vartheta.$
Complete elliptic integrals	$K(k) = F\left(\dfrac{\pi}{2},k\right), \quad E(k) = E\left(\dfrac{\pi}{2},k\right).$
Lennard-Jones potential	$\Phi(\rho) = -\dfrac{A}{\rho^6} + \dfrac{B}{\rho^{12}}.$

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Formula Sheet, Variational

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\,dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \{y \in C^2[x_0, x_1] \mid y(x_0) = y_0, y(x_1) = y_1\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0} \qquad \text{The Euler-Lagrange equation}$$

Theorem 2.3.1: Let J be a functional of the form

$$J\{y\} = \int_{x_1}^{x_2} f(y, y')dx$$

and define the function H by

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f(y, y')$$

Then H is constant along any extremal of y.

Generalisation: Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\,dx,$$

where f has continuous partial derivatives of second order with respect to $x, y, y', \ldots, y^{(n)}$, and $x_0 < x_1$, and the values of $y, y', \ldots, y^{(n-1)}$ are fixed at the end-points, then the extremals satisfy the condition

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} = 0$$

Natural boundary condition: When we extend the theory to allow a free x and y, we find the additional constraint

$$\Big[p\,\delta y - H\,\delta x\Big]_{x_0}^{x_1} = 0,$$

where $p = f_{y'}$ and $H = y'f_{y'} - f$.

Weierstrass-Erdman corner conditions: For a broken extremal

$$p\Big|_{x^{\star -}} = p\Big|_{x^{\star +}}, \quad H\Big|_{x^{\star -}} = H\Big|_{x^{\star +}},$$

must hold at any "corner".