THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2016

104831 MATHS 2100 Real Analysis II

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 62

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Let S be a non-empty subset of $\mathbb{R}$.
   (a) Let $f\colon S \to \mathbb{R}$ be a function and let $x_0 \in S$. Define what it means for f to be *continuous* at x_0.
   (b) Complete the statement of the following proposition from lectures:
   "*$f\colon S \to \mathbb{R}$ is continuous at $x_0 \in S \iff$ for all sequences (x_n) such that .... and $x_n \to x_0$, ...*"
   (c) Suppose that $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are both continuous at $x_0 \in S$. Use the proposition that you stated in part (b) and limit laws for sequences to prove that the function $f(x) + g(x)$ is continuous at x_0.
   (d) Suppose that $f\colon [a,b] \to \mathbb{R}$ is a continuous function. Using theorems from lectures about continuous functions defined on closed bounded intervals, prove that the range $f([a,b])$ of f is equal to $[c,d]$ for some real numbers c and d with $c \leq d$.

   [2+2+4+4=12 marks]

2. Let S be a non-empty subset of $\mathbb{R}$ and let x_0 be a limit point of S.
   (a) Suppose that $f\colon S \to \mathbb{R}$ is a function and $L \in \mathbb{R}$. Write down the $\epsilon - \delta$ definition of what it means for $\lim_{x\to x_0} f(x)$ to equal L.
   (b) Let $g\colon (-1,1) \to \mathbb{R}$ be the function defined by

   $$g(x) = \begin{cases} 2x+1 & \text{if } x \in (-1,1), x \neq 0 \\ 4 & \text{if } x = 0. \end{cases}$$

   Prove that $\lim_{x\to 0} g(x) = 1$ using the definition that you wrote down in part (a) above.
   (c) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{x\to 0} f(x) = 0$ and that $g\colon \mathbb{R} \to \mathbb{R}$ is a bounded function (i.e. there exists $M > 0$ such that $|g(x)| \leq M$ for all $x \in \mathbb{R}$). Prove that $\lim_{x\to 0} f(x)g(x) = 0$ using the definition that you wrote down in part (a) above.

   [2+3+4=9 marks]

3. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and let $\mathscr{P} = \{a = x_0, x_1, \ldots, x_N = b\}$ be a partition of $[a,b]$. For each $i = 1, \ldots, N$ let

   $$m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad M_i = \sup_{x\in[x_{i-1},x_i]} f(x).$$

   (a) Write down the definition of the upper and lower sums $L(f,\mathscr{P})$ and $U(f,\mathscr{P})$.
   (b) Write down the definition of the real numbers $L(f)$ and $U(f)$.
   (c) Complete the following statement of a proposition from lectures:
   "*f is integrable on $[a,b] \iff$ for all $\epsilon > 0$ there exists a .... such that ....*"

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(d) Suppose that for all $\epsilon > 0$, there is a partition $\mathscr{P}_\epsilon$ such that $U(f, \mathscr{P}_\epsilon) - L(f) < \epsilon/2$. Prove that f is integrable on $[a, b]$.

(e) Write down an example of a bounded function $f \colon [0, 1] \to \mathbb{R}$ which is not integrable (you do not need to prove that the function you define is not integrable).

[1+1+2+5+1=10 marks]

4. (a) State the Mean Value Theorem.

(b) Suppose that $f \colon (a, b) \to \mathbb{R}$ is differentiable on (a, b) and that $f'(x) = 0$ for all $x \in (a, b)$. Use the Mean Value Theorem to prove that f is a constant function.

(c) Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x)-f(y)| \leq |x-y|^2$ for all $x, y \in \mathbb{R}$. Use part (b) above to prove that f is a constant function.

(d) Suppose that $f \colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ with $f(x) > 0$ for all $x \in [a, b]$. Define a function $F \colon [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt.$$

Prove that if $x \in (a, b)$ then the inverse function F^{-1} is differentiable at $F(x)$ with derivative

$$(F^{-1})'(F(x)) = \frac{1}{f(x)}.$$

[2+4+3+3=12 marks]

5. (a) Define what it means for a series $\sum_{n=1}^{\infty} a_n$ to *converge*.

(b) Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms. Prove that the series converges if and only if the sequence of partial sums is bounded above.

(c) Complete the following statement of the Comparison Test for series:
"*Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be series such that* $0 \leq a_n \leq b_n$ *for all* $n \geq 1$. *If* .... *then* .... *If* .... *then* ...."

(d) Use the Comparison Test for series to prove that $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges.

[1+2+2+3=8 marks]

6. Let S be a subset of $\mathbb{R}$; let $f \colon S \to \mathbb{R}$ be a function and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n \colon S \to \mathbb{R}$.

(a) Define what it means for the sequence of functions $(f_n)_{n=1}^{\infty}$ to *converge uniformly* on S to f.

(b) Complete the statement of the following proposition from lectures:
"*If* $f_n \to f$ .... *on* S *and* .... *then* f *is continuous on* S."

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(c) Prove that the sequence of functions $(g_n)_{n=1}^{\infty}$ on $[0,1]$ defined by $g_n(x) = x^n$ for $x \in [0,1]$ does not converge uniformly to a function g on $[0,1]$.

(d) Suppose that f_n is continuous on S for all n and $(f_n)_{n=1}^{\infty}$ converges uniformly to f. Let $x_0 \in S$ and let $(x_n)_{n=1}^{\infty}$ be a sequence in S such that $x_n \to x_0$. Prove that $f_n(x_n) \to f(x_0)$.

[2+2+3+4=11 marks]