Practical Asymptotics (APP MTH 4048/7044)
Assignment 2 (5%)

Due 12 April 2019

1. (Bowen & Witelski) Consider the problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \epsilon v^2 + t = 0, \quad v(0) = 0, \quad \epsilon \to 0.$$

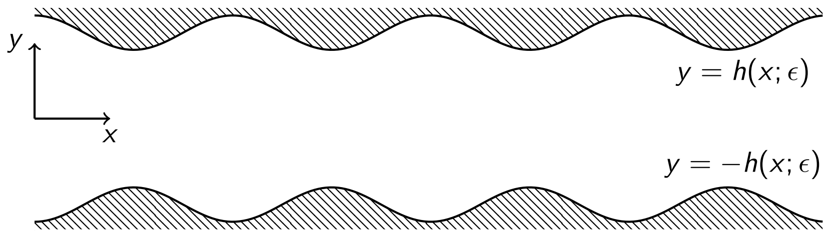
   (a) Find the first three terms in the expansion of the solution $v(t) \sim v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t)$, as $\epsilon \to 0$

   (b) Determine the range of times $0 \le t \le \mathcal{O}(\epsilon^\alpha)$, for which the terms in the expansion retain asymptotic ordering, i.e. $v_0 \gg \epsilon v_1 \gg \epsilon^2 v_2$.

2. (Hinch, adapted) The (shear) flow along a corrugated channel is described by a streamfunction $\psi(x, y)$ (the x and y components of velocity are given by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$). The streamfunction satisfies

$$\nabla^2\psi = -1, \quad \text{in } |y| < h(x, \epsilon) \equiv 1 + \epsilon \cos kx,$$

subject to the boundary condition $\psi = 0$ on the walls at $y = \pm h(x; \epsilon)$, and is periodic in x so that $\psi(0, y) = \psi(2\pi/k, y)$.

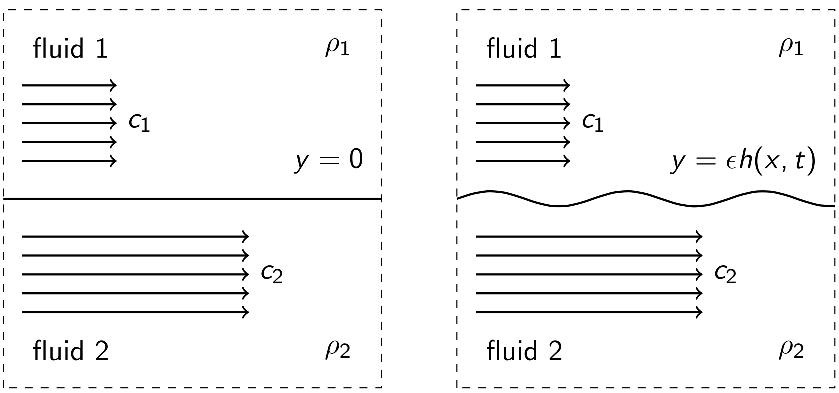


   (a) Obtain the first three terms in the perturbation expansion for ψ.

   (b) Plot a few streamlines in MATLAB for a different values of ϵ and k.

   (c) Comment on the validity of this solution.

3. (Kelvin-Helmholtz instability) Consider two fluid layers moving in parallel:



unperturbed perturbed

In the unperturbed state, the upper layer moves with speed c_1 and the lower layer moves with speed c_2. The upper layer is of density ρ_1 and the lower layer is of density ρ_2.

Assume the shape of the interface between the two layers is small in amplitude by writing $y = \epsilon h(x,t)$, where $\epsilon \ll 1$ and $h = \mathcal{O}(1)$.

The flow of the two layers is described by

$$\begin{aligned} \nabla^2\phi_1 &= 0, \qquad \text{for } y > \epsilon h(x,t), \\ \nabla^2\phi_2 &= 0, \qquad \text{for } y < \epsilon h(x,t). \end{aligned}$$

Far from the interface the layers are at their unperturbed speeds, namely

$$\begin{aligned} \phi_1 &= c_1 x, \quad y \to \infty, \\ \phi_2 &= c_2 x, \quad y \to -\infty. \end{aligned}$$

The kinematic conditions in each fluid are

$$\begin{aligned} \frac{\partial \phi_1}{\partial y} &= \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi_1}{\partial x}\frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x,t), \\ \frac{\partial \phi_2}{\partial y} &= \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi_2}{\partial x}\frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x,t). \end{aligned}$$

Finally, the Bernoulli condition is

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} - \frac{1}{2}c_1^2 + \frac{1}{2}|\nabla\phi_1|^2 + gy \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} - \frac{1}{2}c_2^2 + \frac{1}{2}|\nabla\phi_2|^2 + gy \right),$$

on $y = \epsilon h(x,t)$.

(a) Expand the various quantities about the unperturbed state to rewrite the kinematic and Bernoulli conditions on $y = 0$.

(b) Introduce perturbation series for the velocity potentials where

$$\begin{aligned} \phi_1(x,y,t) &= \phi_{10}(x,y,t) + \epsilon\phi_{11}(x,y,t) + \mathcal{O}(\epsilon^2), \\ \phi_2(x,y,t) &= \phi_{20}(x,y,t) + \epsilon\phi_{21}(x,y,t) + \mathcal{O}(\epsilon^2), \end{aligned}$$

as $\epsilon \to 0$. Write down a solution for ϕ_{10} and ϕ_{20} (the unperturbed problem), and then write down a problem for ϕ_{11} and ϕ_{21}.

(c) Assume the interface shape is a travelling wave, so that

$$h(x,t) = a e^{\mathrm{i}(kx - \omega t)}.$$

The stability of the system will be determined by ω. Use separation of variables to find ϕ_{11} and ϕ_{21} (each up to a multiplicative constant).

(d) Substitute $h(x,t)$, $\phi_{11}(x,y,t)$ and $\phi_{21}(x,y,t)$ into the boundary conditions to obtain an expression for ω.

(e) Use your expression for ω to determine the stability/instability of the interface in terms of the parameters c_1, c_2, ρ_1, ρ_2 and k. Briefly interpret the physical significance of this result.