APP MTH 3020 Stochastic Decision Theory

Tutorial 1

Week 3, Friday, August 10

1. Suppose we roll two (standard, six-sided) dice.

   (a) Specify the probability mass function for the sum of the numbers on their faces.

   Let X_i be the face value on the ith die, for $i \in \{1, 2\}$. Then we want to determine $P(Y = y)$ for the random variable $Y := X_1 + X_2 \in \mathcal{S} = \{2, 3, 4, \ldots, 12\}$. The probability mass function of Y is as follows:

$Y = y$	2	3	4	5	6	7	8	9	10	11	12
$P(Y = y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

   (b) Specify the probability mass function for the minimum of the numbers on their faces.

   Let X_i be the face value on the ith die, for $i \in \{1, 2\}$. Then we want to determine $P(T = t)$ for the random variable $T := \min(X_1, X_2) \in \mathcal{S} = \{1, 2, 3, \ldots, 6\}$. Note that

$$P(T > x) = P(X_1 > x)P(X_2 > x).$$

   Thus,

$$\begin{aligned}
P(T \leq 6) = 1 - P(T > 6) &= 1, \\
P(T \leq 5) = 1 - P(T > 5) &= \frac{35}{36} \Rightarrow P(T = 6) = \frac{11}{36}, \\
P(T \leq 4) = 1 - P(T > 4) &= \frac{32}{36} \Rightarrow P(T = 5) = \frac{9}{36}, \\
P(T \leq 3) = 1 - P(T > 3) &= \frac{27}{36} \Rightarrow P(T = 4) = \frac{7}{36}, \\
P(T \leq 2) = 1 - P(T > 2) &= \frac{20}{36} \Rightarrow P(T = 3) = \frac{5}{36}, \\
P(T \leq 1) = 1 - P(T > 1) &= \frac{11}{36} \Rightarrow P(T = 2) = \frac{3}{36}.
\end{aligned}$$

   Thus,

$$P(T = 1) = \frac{11}{36}.$$

   (c) Evaluate the expected value of the sum.

   The expected value of the sum is given by $\mathbb{E}[Y] = \sum_{i=2}^{12} iP(Y = i) = 7$.

   (d) Evaluate the expected value of the minimum.

   The expected value of the minimum is given by $\mathbb{E}[T] = \sum_{i=1}^{6} iP(T = i) = \frac{91}{36}$.

2. Prove **Jensen's Inequality** (Theorem 1.6 in the lecture notes) for the case of discrete random variables, which states that: If $h(x)$ be a convex function and X a discrete random variable, then

$$\mathbb{E}\left[h(X)\right] \geq h\left(\mathbb{E}[X]\right).$$

We say that the function h is **convex** on a set $\mathcal{S}$ if and only if (a) $\mathcal{S}$ is convex, and (b) for all $x_1, x_2 \in \mathcal{S}$, with $x_1 \neq x_2$, and for all $\alpha_1, \alpha_2 \in (0,1), \alpha_1 + \alpha_2 = 1$, we have

$$h(\alpha_1 x_1 + (1-\alpha_1)x_2) \leq \alpha_1 h(x_1) + (1-\alpha_1)h(x_2),$$

or, equivalently,

$$h(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 h(x_1) + \alpha_2 h(x_2).$$

Consider $x_i \in \mathcal{S}$ and $\alpha_i \in (0,1)$ for $i \in \{1,2,3\}$ with $x_i \neq x_j$ for $i \neq j$, such that

$$\sum_{i=1}^{3} \alpha_i = 1.$$

Then

$$\begin{aligned}
h\left(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\right) &= h\left(\alpha_1 x_1 + (1-\alpha_1)\frac{\alpha_2 x_2 + \alpha_3 x_3}{\alpha_2 + \alpha_3}\right) \qquad \text{since } (1-\alpha_1) = (\alpha_2 + \alpha_3) \\
&\leq \alpha_1 h(x_1) + (\alpha_2 + \alpha_3) h\left(\frac{\alpha_2}{\alpha_2 + \alpha_3} x_2 + \frac{\alpha_3}{\alpha_2 + \alpha_3} x_3\right) \\
&\leq \alpha_1 h(x_1) + (\alpha_2 + \alpha_3)\left(\frac{\alpha_2}{\alpha_2 + \alpha_3}\right) h(x_2) + (\alpha_2 + \alpha_3)\left(\frac{\alpha_3}{\alpha_2 + \alpha_3}\right) h(x_3) \\
&= \alpha_1 h(x_1) + \alpha_2 h(x_2) + \alpha_3 h(x_3).
\end{aligned}$$

Hence, by induction, we have

$$h\left(\sum_i \alpha_i x_i\right) \leq \sum_i \alpha_i h\left(x_i\right),$$

which is true for all $\alpha_i \in [0,1]$ such that $\sum_i \alpha_i = 1$ and for all $x_i \in \mathcal{S}$.

Letting the x_i be the values the random variable X may take and the α_i be their associated probability mass, we have

$$h\left(\mathbb{E}[X]\right) \leq \mathbb{E}[h(x)].$$

3. A furniture maker makes two products, P_1 and P_2, where there is a total production limit of a total of 1500 units of furniture products P_1 and P_2. Both carpentry and finishing are required resources for the manufacturing process. The requirements measured in hours per unit are known and shown in the below table, along with the profit per unit of product.

Product parameters	P_1	P_2
Carpentry hours	4	8
Finishing hours	3	2
profit per unit	15	25

Our problem is to select the product mix to maximise total profit, but the availability of the resources are unknown. Rather, we have two equally likely estimates of the hours available for each resource:

$$\text{Available carpentry hours} = \begin{cases} 4950 & \text{with probability } p_{c_1} = 0.5 \\ 5850 & \text{with probability } p_{c_2} = 0.5, \end{cases}$$

$$\text{Available finishing hours} = \begin{cases} 3636 & \text{with probability } p_{f_1} = 0.5 \\ 4064 & \text{with probability } p_{f_2} = 0.5. \end{cases}$$

(a) Write down the expected time available for carpentry, and the expected time available for finishing.

The expected time available for carpentry is 5400 hours; the expected time available for finishing is 3850 hours.

(b) Write down the LP using the **expected time availabilities** for the RHS of your problem constraints.

$$\begin{aligned} \max \quad & z = 15x_1 + 25x_2 \\ \text{s.t.} \quad & 4x_1 + 8x_2 \leq 5400 \\ & 3x_1 + 2x_2 \leq 3850 \\ & x_1 + x_2 \leq 1500 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(c) Write down the dual to this LP.

$$\begin{aligned} \min \quad & w = 5400y_1 + 3850y_2 + 1500y_3 \\ \text{s.t.} \quad & 4y_1 + 3y_2 + y_3 \geq 15 \\ & 8y_1 + 2y_2 + y_3 \geq 25 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

(d) Verify that 1250 units of P_1 and 50 units of P_2 is a solution to the original LP, with a profit of \$20,000.

Show all constraints are satisfied and that the objective function is $15 \times 1250 + 25 \times 50 = 20,000$.

(e) Given that a solution to the dual may also be found such that its objective function is \$20,000, is this solution an optimal solution?

Yes, this solution is an optimal solution according to the Strong Duality Theorem.

(f) A solution to the averaged value LP is not very acceptable because it does not allow for the stochastic variation of available carpentry and finishing hours. Assume that additional carpentry hours may be purchased at \$5/hr and that extra finishing hours may be purchased at \$12/hr. Also assume that any unused base carpentry hours are wasted and must be costed at \$4/hr and similarly any unused base hours of finishing must be costed at \$9/hr.

(i) Write down the expanded version of a recourse model considering all realisations.

$$
\begin{aligned}
\max \quad & z = 15_1 + 25x_2 - 4\frac{1}{2}y_1 - 4\frac{1}{2}y_2 - 9\frac{1}{2}y_3 - 9\frac{1}{2}y_4 - 5\frac{1}{2}y_5 - 5\frac{1}{2}y_6 - 12\frac{1}{2}y_7 - 12\frac{1}{2}y_8 \\
\text{s.t.} \quad & 4x_1 + 8x_2 + y_1 - y_5 = 4950 \\
& 4x_1 + 8x_2 + y_2 - y_6 = 5850 \\
& 3x_1 + 2x_2 + y_3 - y_7 = 3636 \\
& 3x_1 + 2x_2 + y_4 - y_8 = 4064 \\
& x_1 + x_2 \leq 1500 \\
& x_1, x_2, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \geq 0.
\end{aligned}
$$

(ii) Solve this problem in expanded form and give an interpretation of the solution.
Using Matlab, the optimal solution is given by $x_1 = 1086.7$ units, $x_2 = 187.9$ units, with profit $z = \$16,822$.

Note that this solution always uses 5850 hours of carpentry and 3636 hours of finishing, which are greater than and less than the average available hours of each, respectively. They are in fact the maximum and minimum variable amount of hours available for carpentry and finishing respectively, meaning sometimes we purchase additional carpentry hours and sometimes we waste base finishing hours.