MATHS 2103 Probability and Statistics
Assignment 5

Due: Thursday, 25 May 2017, 4 PM (week 11). $\boxed{\textbf{Total marks: 31}}$

Question 1 $\boxed{\text{8 marks}}$

Make sure that in all your answers you

$\boxed{2}$ (a) use full and complete sentences,

$\boxed{2}$ (b) include units where necessary,

$\boxed{2}$ (c) use logical arguments in your answers and proofs, and

$\boxed{2}$ (d) structure your answers and assignment clearly and precisely.

Question 2 $\boxed{\text{4 marks}}$

Let the sequence X_n be the value of a die on the nth roll.

$\boxed{3}$ (a) Show that X_n is a Markov chain.

$\boxed{1}$ (b) Determine the transition probabilities, p_{ij}.

Question 3 $\boxed{\text{6 marks}}$

Define a simple random walk Y_n on a finite state space $S = \{0, 1, 2, \dots, N\}$ to be a random process that

- increases by 1, when possible, with probability p,
- decreases by 1, when possible, with probability $1 - p$, and
- remains unchanged otherwise.

where $p \in (0,\ 1)$.

$\boxed{2}$ (a) Specify the transition matrix for Y_n.

$\boxed{4}$ (b) Assume that $N = 2$ and initially, the process is evenly distributed across S. Calculate the probability the process is in state 0 after 2 steps.

Question 4 $\boxed{\text{13 marks}}$

Let Z_n be a Markov chain, with a finite state space $S = \{0, 1, \dots, N\}$ with the following transition probabilities.

$$
\begin{aligned}
p_{i,i+1} &= \frac{p(N-i)}{p(N-i)+q} && \text{for } 1 \le i \le N-1, \\
p_{i,i-1} &= \frac{q}{p(N-i)+q} && \text{for } 1 \le i \le N, \\
p_{0,0} &= 1 \quad \text{and} \\
p_{N,N} &= 1.
\end{aligned}
$$

$\boxed{2}$ (a) With reasoning, determine which states, if any, in S are absorbing states.

5 (b) Assume that $N = 3$. Determine the probability that the process is absorbed after 3 steps in state i given that it starts in state j, denoted $A_{3,j}^{(i)}$ where $i = 0, N$ and $j = 0,\ 1,\ 2,\ 3. j = 0,\ 1,\ 2,\ 3$.

6 (c) Again for $N = 3$ with $p = 5$ and $q = 4$, determine the probability that the process is eventually absorbed in state i given that it starts in state j, denoted $X_j^{(i)}$, for $i = 0, N$ and $j = 0,\ 1,\ 2,\ 3$.

Extra questions for your edification (not to be handed up)

1. A vending machine can be in two states, (1=working, 0=out of order). If the machine is working on a particular day it will be out of order with probability δ on the next day. If the machine is out of order on a particular day then the probability that it will be working the next day is γ.

   (a) Write down the one step transition probability matrix for the Vending machine.

   (b) Assume the machine is working on Monday.

      i. What is the probability that the machine will remain working on all of Tuesday, Wednesday and Thursday?

      ii. What is the probability that the machine will be working on Thursday?

   (c) Calculate the equilibrium probabilities for the states of the vending machine.

2. Two players A and B are playing a game together. In total they have 5 dollars between them. Let the state $j \in \{0, 1, 2, 3, 4, 5\}$ be the amount of money that A has. The one-step probability transition matrix for the game is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   i. Rearranging the states in the order $\{1, 2, 3, 4, 5, 0\}$ write the matrix P in the form

$$P_A = \begin{bmatrix} R & S \\ 0 & I \end{bmatrix}.$$

   ii. Using Matlab, some other package or a graphics calculator find $(I - R)^{-1}S$.

   iii. Hence, using the fact that

$$\lim_{n\to\infty} P_A^n = \begin{bmatrix} 0 & (I - R)^{-1}S \\ 0 & I \end{bmatrix},$$

   find

$$X_1^{(5)} = P(\text{Player } A \text{ wins all the money}|\text{Player } A \text{ starts with \$1}).$$

3. **2012 exam** Consider a system comprising two jars A and B and 4 red marbles and 5 green marbles such that

- There are 3 marbles in jar A and 6 marbles in jar B.
- At each step, one marble is selected randomly from each jar. The two marbles are then swapped and returned to the opposite jar.
- The state of the system is the number of red marbles in jar A.

(a) Write down the state space S for this system.

(b) Complete the missing entries in the transition matrix.

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \spadesuit & \clubsuit & \diamondsuit & \heartsuit \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

(c) If the system starts with no red marbles in jar A, find the probability that there is exactly one red marble in jar A after 2 steps.

(d) Show that (you do not need to find this equilibrium probability distribution) the equilibrium probability distribution is $\boldsymbol{\pi} = \frac{1}{84}(10, 40, 30, 4)$.

(e) Suppose the initial allocation of marbles is performed randomly according to the probability vector $\boldsymbol{\pi}$, Find the probability that there are 2 red marbles in jar A after n draws.

4. Consider a person making a random walk on the non-negative integers. Assume that at a given time point the person is at an integer $i \geq 1$. Then the probability (at the next time point) that the person moves to $i+1$ is $1/5$, to $i-1$ is $3/5$ and stays in i is $1/5$. If the person is at 0, then the probability (at the next time point) of staying there is $2/5$ and the probability of moving to 1 is $3/5$. A Markov chain model of this system can be constructed where the state at time t is the person's position at time t.

(a) Write down the state space of the system.

(b) Write down the transition probabilities of this random walk.

(c) Write down the equilibrium equations.

(d) Use the difference (or recurrence) equation method to calculate the equilibrium distribution of this Markov chain.