Modelling With ODEs Tutorials

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0.1 Tute 1

1. Fishery Model where $N(t)$ is the number of fish

$$\frac{dN}{dt} = f(N) = BN - DN^2 - Y$$

Y fishing yield, B birth rate, D death rate. In lectures we showed that a non-dimensional version of the model is

$$\frac{d\hat{N}}{d\hat{t}} = \hat{N}(1-\hat{N}) - y$$

(a) Show that the steady state is:

$$\hat{N}_*^{\pm} = \frac{1 \pm \sqrt{1-4y}}{2}$$

and which values of y does it exist: Steady state when

$$\begin{aligned} \hat{N}(1-\hat{N}) - y &= 0 \\ \hat{N} - \hat{N}^2 - y &= 0 \\ \hat{N}^2 - \hat{N} + y &= 0 \\ \hat{N} &= \frac{1 \pm \sqrt{1-4y}}{2} \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Exists if $\Delta \geq 0$, i.e. $1 - 4y \geq 0$.

$$\implies 4y \leq 1 \implies y \leq \frac{1}{4}$$

(b) Stability of the steady states?

$$\begin{aligned} f'(\hat{N}) &= 1 - 2\hat{N} \\ \implies f'(\hat{N}_+^*) &< 0 \\ \implies f'(\hat{N}_-^*) &> 0 \end{aligned}$$

So $\hat{N}_+^*$ is stable, and $\hat{N}_-^*$ is unstable. When $\hat{N}(0) < \hat{N}_-^*$ the population will go to 0.

(c) When $y = .25$ we get a repeated root $\hat{N}^* = \frac{1}{2}$.

$$f'(N^*) = 0, \quad f''(N^*) = -2$$

So since the slope is 0 and it is a turning point, it is semi-stable (if we shift to the right it will come back, to the left it will continue to the left)

(d) What happens when $y > 0.25$? There are no real roots so there are no steady states...

2. (a) Fixed points are where $C(x)$, $S(x : \mu)$ intersect, treat $S(x)$ as x, so rotate the coordinate system.

(b) i. Bifurcation points when

$$f(\bar{x}:\bar{\mu}) = 0, \frac{\partial f}{\partial x} = 0$$

$$\begin{aligned} \mu x + x^3 - x^5 &= 0 \\ x(\mu + x^2 - x^4) &= 0 \\ x = 0 \ or \ x^4 - x^2 - \mu &= 0 \\ \implies x^2 &= \frac{1 \pm \sqrt{1+4\mu}}{2} \\ \implies x &= \pm\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}} \end{aligned}$$

The four will exist/cease to exist for values of μ need both square roots to exist

$$x_{\pm+} \implies \mu > -1/4$$

$$x_{\pm-} \implies \mu \in \{-1/4, 1/4\}$$

I.e. bifurcation at $(x,\mu) = (1/2, -1/4), (-1/2, -1/4)$ And $\bar{x} = 0$ works for all μ.

ii. This was found at the start:

$$x = \pm\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}}, 0$$

iii.

iv.

(c) i.

ii.

0.2 Tute 3

1. Trajectories of form

$$\mathbf{x} = e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}$$

$\alpha = 0$ gives centres and $\alpha \neq 0$ gives spirals.

(a) Effect of β on the direction of trajectory: For $0 \le t \le \beta 2\pi$ $-\sin\beta t > 0$ when $\beta t \in (\pi, 2\pi)$ $\cos\beta t > 0$ when $\beta t \in (-\pi/2, \pi/2)$

(b)

2. Done in matlab

3.

$$\frac{dx}{dt} = 3x - x^2 - xy, \quad \frac{dy}{dt} = 2y - y^2 - xy$$

(a) Competition model

(b) $n_x = x = 0, x = 3 - y,$
$n_y = y = 0, y = 2 - x$
Biologically relevant where $x, y \ge 0$, steady states are:
$x, y = 0$
$x = 3, y = 0,\ x = 0, y = 2$

(c) Linearisation

$$J(x) = \begin{pmatrix} 3-2x-y & -x \\ -y & 2-x-2y \end{pmatrix}$$

At the steady states:

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \implies \lambda_{1,2} > 0 \implies \mathit{unstable}$$

$$J(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \implies \lambda = -1, -3 \implies \mathit{asymptotically\ stable}$$

$\det(J(3,0)) = 3$ $trace(J(3,0)) = -4$ $\frac{1}{4}tr(J)^2 = 4$ stable node.

$$J(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$$

(d) n ty

(e)

0.3 Tute 4

1. (a) Saddle node bifurcation if you can create/destroy 2 fixed points by changing the parameter. Nullclines:

$$\dot{x} = -ax + y, \quad and \quad \dot{y} = \frac{x^2}{1+x^2} - y$$

$$\begin{aligned} \eta_x &= x = \frac{y}{a} \implies y = ax \\ \eta_y &= y = \frac{x^2}{1+x^2} \end{aligned}$$

Bifurcation: Hence fixed points if $x, y = 0$ for all a, or

$$\begin{aligned} \frac{x^2}{1+x^2} - ax &= 0 \\ x^2 - ax(1+x^2) &= 0 \\ x - a + ax^2 &= 0 \\ a &= \end{aligned}$$

(b) Show pitchfork for:

$$\dot{x} = -bx + y + \sin x \quad \dot{y} = x - y$$

$$\begin{aligned} \eta_x &= y = bx - \sin x \\ \eta_y &= x = y \end{aligned}$$

$$\begin{aligned} x &= bx - \sin x \\ (1-b)x - \sin x &= 0 \end{aligned}$$

$x = 0, y = 0$ for all b is a solution

2. The ODE

$$2tx^3 + 3t^2x^2\frac{dx}{dt} = 0$$

$$\begin{aligned} 2tx^3 + 3t^2x^2\frac{dx}{dt} &= 0 \\ 2tx^3dt + 3t^2x^2dx &= 0 \end{aligned}$$

Exact if it can be written as $f(x,t)dt + g(x,t)dx = 0$ its exact.

$$\begin{aligned} 2tx^3 + 3t^2x^2\frac{dx}{dt} &= 0 \\ \frac{2}{3tx} + \frac{dx}{dt} &= 0 \end{aligned}$$

Hence linear

$$\frac{2}{3t} + x\frac{dx}{dt} = 0$$

Hence separable All of these show it is homogeneous.

3.

$$f(x) = \log|x|$$

This can't be globally Lipschitz continuous since it is not continuous about $x = 0$. Lipschitz continuous if

$$|f(x) - f(y)| \leq L|x - y|$$

hence

$$\frac{df}{dx} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ -\frac{1}{x} & \text{if } x < 0 \end{cases}$$

For $x, y > 0$

$$\begin{aligned} |\log|x| - \log|y|| &= |\log|\frac{x}{y}|| \\ &= ||x| - |y|\frac{1}{c}| \quad (mvt) \\ &= \frac{1}{c}|x - y| \end{aligned}$$

However $\frac{1}{c}$ is not bounded above. However for the intervals $(x, y) \in [-\infty, 0)$ or $(x, y) \in (0, \infty]$

0.4 Tute 5

1.

$$\frac{du}{dt} = u^2, \quad u(0) = 1$$

PL - if the interval $t_- \leq t_0 < t_+$ and $u_- < u_0 < u_+$ and u^2 is continuous in there, and Lipschitz continuous on the u interval then it has a unique solution It is continuous. Lipschitz:

$$|x^2 - y^2| = 2C|x - y|$$

Hence by PL there exists a unique solution. Pick the interval with $t_0 = 0$ and $u_0 = 1$

Pick α as the radius of temporal interval and δ as radius of spatial interval.

Define

$$M = ||f|| = \sup_{I \times J} |f| = \sup_{J} |u^2| = (1 + \delta)^2$$

$$\frac{\delta}{M} = \delta/(1 + \delta)^2$$

$$\epsilon = \min\{\alpha, \delta/M\} = \min\{\alpha, \delta/(1 + \delta)^2\} < 1$$

And

$$\max\{\frac{\delta}{(1 + \delta)^2}\} = \frac{1}{4}, \quad \delta = 1$$

(apparently)

2.

$$\frac{du}{dt} = 1 + u^2 \ u(0) = 0$$

$$\begin{aligned} u_1 = 0 + \int_0^t 1 ds \\ u_1 = t \\ u_2 = 0 + \int_0^t 1 + s^2 ds \\ u_2 = t + \frac{t^3}{3} \\ u_n = \int_0^t 1 + u_{n-1}^2(s) ds \end{aligned}$$

```
syms t
u = 0
for k=1:10
   u = int(1+u^2,t,0,t);
end
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3. Finite diff formula

$$x_{n+1} = (2 + h^2)x_n - x_{n-1}, \quad h = step$$

(a) Which ODE does this pair with

$$\begin{aligned} x(t+h) = (2+h^2)x(t) - x(t-h) \\ x(t+h) = 2x(t) + h^2 x(t) - x(t-h) \\ x(t) = \frac{x(t+h) - 2x(t) + x(t-h)}{h^2} \\ x(t) = x''(t-h) \\ x(t) = x''(t), \quad h \to 0 \end{aligned}$$

(b) Order of local discretisation error

(c) Is this explicit or implicit