LECTURE 35

At the end of last lecture we proved that if the series of real numbers $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, where $R > 0$, then the series of functions $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any $0 \le r < R$.

Example: One application of this fact is the following. Suppose that $-r < a < b < r$. Then

$$\int_a^b \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} \left(\int_a^b a_n x^n \, dx \right)$$

To see this, observe that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[a, b]$. Thus $s_N \to f$ uniformly on $[a, b]$, where $f\colon [a, b] \to \mathbb{R}$ is the function whose value at $x \in [a, b]$ is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and where $s_N = \sum_{n=0}^{N} a_n x^n$. Notice that each function s_N is integrable on $[a, b]$, since each such function is a polynomial. Therefore f is integrable on $[a, b]$ and

$$\int_a^b f(x) \, dx = \lim_{N \to \infty} \int_a^b s_N(x) \, dx = \lim_{N \to \infty} \sum_{n=0}^{N} \int_a^b a_n x^n \, dx = \sum_{n=0}^{\infty} \int_a^b a_x^n \, dx.$$

For example, consider the power series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$. This has radius of convergence $R = 1$ and we can identify

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|x| < 1$. Therefore, if $0 \le x < 1$ then

$$\arctan(x) = \int_0^x \frac{1}{1 + t^2} \, dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1}$$

The same formula holds for any $|x| < 1$. Thus we obtain a power series expansion for $\arctan(x)$.

Returning to the situation discussed at the end of last lecture, in which the series $\sum_{n=0}^{\infty} a_n x^n$ converges for any $|x| < R$, we observe that more is true: the power series $\sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n$ converges uniformly on $[-r, r]$ for any r with $0 \le r < R$.

To see this we again choose, given such an r, an x_0 with $r < x_0 < R$. Let $M > 0$ be such that $|a_n x_0^n| \le M$ for all n. Then, if $|x| \le r$, we have

$$\begin{aligned}
|(n + 1) a_{n+1} x^n| &= (n + 1) |a_{n+1}| \cdot |x|^n \\
&= \frac{(n + 1)}{|x_0|} |a_{n+1}| \cdot |x_0|^{n+1} \cdot \left(\frac{|x|}{|x_0|} \right)^n \\
&\le (n + 1) \frac{M}{|x_0|} s^n \quad (\text{where } s := |x|/|x_0|)
\end{aligned}$$

Therefore, if $|x| \le R$, then

$$|(n + 1) a_{n+1} x^n| \le C (n + 1) s^n$$

where $C = M/|x_0|$ is a constant. We would now like to apply the Weierstrass M-test to the series of functions $\sum_{n=0}^{\infty} f_n$, where $f_n(x) = (n+1)a_{n+1}x^n$ and where $M_n = C(n+1)s^n$. Therefore, we need to prove that the series $\sum_{n=0}^{\infty} C(n+1)s^n$ converges. Clearly, since C is a constant, it is enough to prove that the series $\sum_{n=0}^{\infty}(n+1)s^n$ converges. We apply the Ratio Test: we have

$$\frac{(n+1)s^{n+1}}{ns^n} = s\frac{n+1}{n} \to s$$

and $s = |x|/|x_0| < 1$. Therefore the series $\sum_{n=0}^{\infty} C(n+1)s^n$ converges and the series of functions $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ converges uniformly on $[-r, r]$ by the Weierstrass M-test.

To summarise, we have proved the following statements:

- $s_N = a_0 + a_1x + \cdots + a_Nx^N$ converges uniformly on $[-r, r]$ to $f(x) = \sum_{n=0}^{\infty} a_nx^n$;
- $s'_N = a_1 + 2a_2x + \cdots + (N+1)a_{N+1}x^N$ converges uniformly on $[-r, r]$ to $g(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$.

Therefore, by Theorem 8.7, f is differentiable on $(-R, R)$ and

$$f'(x) = g(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n.$$

Therefore f' has a representation as a power series on $(-R, R)$. We can then apply the previous discussion to conclude that f' is differentiable on $(-R, R)$ with

$$f''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

This is another power series, and so we can apply the previous discussion *again* to conclude that f'' is differentiable on $(-R, R)$ with

$$f'''(x) = \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}x^n.$$

Clearly we can continue this indefinitely to deduce that for any $k \geq 1$, f is k-times differentiable with

$$f^{(k)}(x) = \sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}x^n.$$

Notice in particular (setting $x = 0$ in the expression for $f^{(k)}(x)$) that $k!a_k = f^{(k)}(0)$. Therefore we have reached the following conclusion:

Theorem: if $f\colon (-R, R) \to \mathbb{R}$ has a power series expansion around $x_0 = 0$, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_nx^n$$

for $x \in (-R, R)$, then f is infinitely differentiable on $(-R, R)$, with

$$f^{(k)}(x) = \sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}x^n$$

for $x \in (-R, R)$. Moreover $a_n = f^{(n)}(0)/n!$ for every $n \geq 0$.

We can ask the converse question: if f is infinitely differentiable in an open interval $(-R, R)$, does f have a power series expansion around $x_0 = 0$?

The answer is, not always! A good example is the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp(-x^{-2}) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It can be shown, using L'Hôpital's Rule, that f is infinitely differentiable, and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Therefore f is not identically equal to the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

in any open interval of the form $(-R, R)$.

If f is a fucntion which is infinitely differentiable on an interval of the form $(-R, R)$, then the *Taylor series* of f at $x_0 = 0$ is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

More generally, if f is a function which is infinitely differentiable on an interval of the form $(x_0 - R, x_0 + R)$ for some $x_0 \in \mathbb{R}$, then the *Taylor series* of f at x_0 is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Question 1: If f is a function which is infinitely differentiable on an interval of the from $(-R, R)$, when is f equal to the Taylor series of f at $x_0 = 0$ on an open interval containing $x_0 = 0$?

More generally,

Question 2: If f is a function which is infinitely differentiable on an open interval of the form $(x_0 - R, x_0 + R)$, when is f equal to the Taylor series of f at x_0 on an open interval containing x_0?

The Lagrange Remainder Theorem can be used to give partial answers to both questions. Let's consider the first question.

The partial sums for the Taylor series $\sum_{n=0}^{\infty} \frac{^{(n)}(0)}{n!} x^n$ are the Taylor polynomials $p_N(x)$. Therefore, Question 1 above is the question of whether or not $\lim_{N \to \infty} p_N(x) = f(x)$ for all $x \in (-R, R)$.

By the Lagrange Remainder Theorem we have, for any $x \in (-R, R)$,

$$f(x) - p_N(x) = \frac{f^{(N+1)}(c_x)}{(N+1)!} x^{N+1}$$

where c_x is between x and 0. Therefore

$$|f(x) - p_N(x)| = \frac{|f^{(N+1)}(c_x)}{(N+1)!} |x|^{N+1} \leq \frac{|f^{(N+1)}(c_x)}{(N+1)!} R^{N+1}$$

since $|x| < R$. Suppose that the derivatives $f^{(n)}(x)$ are all bounded on $(-R, R)$ independently of n, i.e. there exists $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in (-R, R)$ and for all $n \geq 1$. Then

$$|f(x) - p_N(x)| \leq \frac{M}{(N+1)!} R^{N+1}$$

for all $x \in (-R, R)$. Therefore, if

$$\lim_{N \to \infty} \frac{MR^{N+1}}{(N+1)!} = 0$$

then f will equal its Taylor series on $(-R, R)$. We will use the Ratio Test to prove that

$$\frac{R^{N+1}}{(N+1)!} \to 0$$

Consider the series

$$\sum_{n=0}^{\infty} \frac{R^n}{n!}.$$

We have

$$\frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}} = \frac{R}{n+1} \to 0$$

and hence the series $\sum_{n=0}^{\infty} \frac{R^n}{n!}$ converges. Therefore $\frac{R^n}{n!} \to 0$. Therefore f is equal to its Taylor series on $(-R, R)$.

A function $f \colon (-R, R) \to \mathbb{R}$ which is equal to its Taylor series at x_0 on an open interval containing x_0, for every $x_0 \in (-R, R)$, is called a *real analytic* function. The functions $\sin(x)$, $\cos(x)$, $\exp(x)$ are all examples of real analytic functions.