Practical Asymptotics

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0 Outline/motivation

Often real-world modelling problems give rise to equations which have no exact solution. One option is to solve them numerically; this can be great if (for example) interest lies in modelling of phenomena in geometrically complicated regions. Another option is to use asymptotic methods. This involves rigourously identifying the important mechanisms at play in a particular problem and writing down (approximate) analytic expressions to describe the solution behaviour.

The key advantage of such an approach is that these expressions can provide insight that is not possible with numerical simulations; it is much easier see how a varying a parameter value changes a solution if we have a formula, rather than having to wait for a simulation to be rerun. In practice when asymptotics approaches are used they are presented alongside computational results, but are attractive for this extra layer of insight they provide.

In this course we will:

- develop a tool-kit of useful asymptotic techniques; and
- apply these techniques to real-world modelling problems.

The plan for the course is as follows:

1. **Introduction to asymptotics**
   Key concepts involved in asymptotic methods; example application to the solution of differential equations.
   (Bender & Orszag, Chap. 3; Bowen & Witelski, Chap. 6)

2. **Perturbation methods** Introduce this broad class of techniques; examples from fluid mechanics and other case studies.
   (Bender & Orszag, Chap. 7; Bowen & Witelski, Chaps. 6, 8 & 12)

3. **Boundary layer theory and asymptotic matching** Discuss techniques for solving phenomena that vary over a thin region, and how to incorporate them into the bigger picture; examples from fluid mechanics and mathematical biology.
   (Bender & Orszag, Chap. 9; Bowen & Witelski, Chaps. 7 & 12)

4. **Multiscale methods and homogenisation theory**
   Discuss techniques for solving problems that involve multiple space/time scales. Techniques to average ('homogenise') over small scale variation. Examples from solid mechanics and mathematical biology.
   (Bender & Orszag, Chap. 11; Bowen & Witelski, Chaps. 9 & 10)

5. **Extension topics/more case studies** TBD: we can discuss more case studies or, subject to interest, look at extension topics. Possible topics include: asymptotic approximation of integrals; summation of divergent series; WKB theory; asymptotics beyond-all-orders.

In these notes, a grey box like this one indicates an example or further discussion of a topic.

References (none required, all helpful in their own way)

T. Witelski, M. Bowen, Methods of Mathematical Modelling: Continuous Systems and Differential Equations, Springer, 2015. (electronic version available from UoA library)

C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, Springer, 1999. (electronic version available from UoA library) M. Van Dyke, Perturbation Methods in Fluid Mechanics, The Parabolic Press, 1978.

E.J. Hinch, Pertrubation Methods, Cambridge University Press, 1991.

S.W. McCue, Preface to fourth Special Issue on Practical Asymptotics, 63:153–154, Journal of Engineering Mathematics, 2009.

W.R. Smith, Preface to the sixth special issue on "Practical Asymptotics", 102:1–2, Journal of Engineering Mathematics, 2017.

1 Introduction to asymptotics

1.1 A difficult problem

Let's find the real solution (say at $x = a$) to the following quintic (fifth-degree polynomial) equation

$$x^5 + x = 1. \tag{1.1}$$

We know a few things about this equation:

- the highest power is x^5 so there are 5 roots, some which are probably complex;
- there is no exact solution for these roots (that only works if the highest power is x^4).

It'd be straightforward to solve this numerically, but another approach is to consider a related problem namely

$$x^5 + \epsilon x = 1, \tag{1.2}$$

where the second term on the right-hand side is now multiplied by a parameter ϵ.

If this were a real modelling problem, this parameter might have a physical meaning (and we might know if it was small or large).

Let's denote the real root of this equation $x = a(\epsilon)$. This becomes the original equation when $\epsilon = 1$. We can view this as a 'perturbation' problem: it features a parameter (usually denoted ϵ) and when ϵ is set to zero the problem is easily solvable. It's really the behaviour when $\epsilon \neq 0$ that's of interest.

Working: series solution to the quintic (1.2)

If we let $\epsilon = 0$ and $x = a_0$ we get the 'unperturbed' problem

$$a_0^5 = 1 \qquad \Longrightarrow \qquad a_0 = 1,$$

where we take just the real root.

NB: the other roots are $e^{i\frac{j2\pi}{5}}$, $j = 1, 2, 3, 4$.

This is interesting (kind of), it says that a first approximation to the real root is 1. To proceed let's write $a(\epsilon)$ in the form of a power series in the parameter ϵ

$$\begin{aligned} a &= \sum_{n=0}^{\infty} a_n \epsilon^n \\ &= a_0 + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 \ldots \\ &= 1 + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \ldots \end{aligned}$$

This is an example of an **ansatz** solution, which is an educated guess at an appropriate series representation of the solution. To obtain a better approximation to the real root of (1.2) we need to find a_1, a_2, a_3 and so on (assuming the series converges). This can be thought of as the correction to the unperturbed solution due to the perturbation. To do this we substitute the ansatz into (1.2) and compare powers of ϵ, this gives

$$\begin{aligned}(a_0 + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 \ldots)^5 + \varepsilon(a_0 + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 \ldots) &= 1 \\ \underbrace{(1 + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 \ldots)^5}_{\text{this expansion is fiddly}} + \varepsilon(1 + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 \ldots) &= 1\end{aligned}$$

where we've made use of unperturbed solution $x_0 = 1$. Now use binomial theorem to expand the first term, then collect powers of ϵ

Recall that $(1+B)^5 = 1 + 5B + 10B^2 + 10B^3 + 5B^4 + B^5$

$$\begin{aligned}&(1 + \epsilon a_1 + \epsilon^2 a_2 + \ldots)^5 \\ &= (1 + \epsilon(a_1 + \epsilon a_2 + \ldots))^5 \\ &= 1^5 + 5\epsilon(a_1 + \epsilon a_2 + \ldots)(1)^4 + 10\epsilon^2(a_1 + \epsilon a_2 + \ldots)^2(1)^3 \\ &\quad + 10\epsilon^3(a_1 + \epsilon a_2 + \ldots)^3(1)^2 + 5\epsilon^4(a_1 + \epsilon a_2 + \ldots)^4(1) \\ &\quad + \epsilon^5(a_1 + \epsilon a_2 + \ldots)^5 \\ &= 1 + 5\epsilon(a_1 + \epsilon a_2 + \ldots) + 10\epsilon^2(a_1^2 + 2\epsilon a_1 a_2 + \ldots) \\ &\quad + 10\epsilon^3(a_1 + \ldots)^3 + \ldots \\ &= 1 + \epsilon(5a_1) + \epsilon^2(5a_2 + 10a_1^2) + \epsilon^3(5a_3 + 20a_1 a_2 + 10a_1) + \ldots\end{aligned}$$

Substitute back into the equation and collect powers of ϵ to give

$$\begin{aligned}1 + \epsilon(5a_1 + 1) + \epsilon^2(5a_2 + 10a_1^2 + a_1) + \\ \epsilon^3(5a_3 + 20a_1 a_2 + 10a_1^3 + a_2) + \ldots = 1.\end{aligned}$$

Comparing powers of ϵ on the right- and left-hand sides:

$$\begin{aligned}(\epsilon^0):& \quad 1 = 1, \quad \text{(unperturbed solution)} \\ (\epsilon^1):& \quad 5a_1 + 1 = 0 \implies a_1 = -1/5, \\ (\epsilon^2):& \quad 5a_2 + 10a_1^2 + x_1 = 0 \implies a_2 = \tfrac{1}{5}(-\tfrac{10}{25} + \tfrac{1}{5}) = -\frac{1}{25}, \\ (\epsilon^3):& \quad 5a_3 + 20a_1 a_2 + 10a_1^3 + a_2 = 0 \implies a_3 = \ldots = -\frac{1}{125}\end{aligned}$$

This gives

$$a(\epsilon) = 1 - \tfrac{1}{5}\epsilon - \tfrac{1}{25}\epsilon^2 - \tfrac{1}{125}\epsilon^3 + \ldots$$

Guess the next term in this series?

Now we can find the solution to our original problem (1.1) by evaluating this expression at $\epsilon = 1$:

$$a(1) = 1 - \tfrac{1}{5} - \tfrac{1}{25} - \tfrac{1}{125} + \ldots \approx 0.752$$

The exact (numerical) answer is $a_{\text{exact}} = 0.7549\ldots$, so this four term series is a pretty impressive approximation.

The process we went through was to convert an extremely difficult problem into a sequence of easy problems, then piece those together to get an approximate (and very informative) solution to the original problem. We'll extend these ideas to differential equations and more 'real-world' applications soon, but first need some new notation.

1.2 Notation: $\sim$ and $\ll$

Examples: ... but first some other notations We are very used to dealing with equations (eg. $f(x) = g(x)$) and inequalities (eg. $f(x) > g(x)$). Another common notation is 'approximately equal' sign $\approx$, which allow us to make statements like

$$e^x \approx 1 + x + \frac{x^2}{2!}$$

without precisely defining what is meant by the two squiggly lines, or when this approximation is valid. Dots on the end of a truncated series indicate that an approximation could be 'better' if we'd included more terms. For instance

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

This a useful notation, but note that it doesn't give any information about the next term. We might even say something like

$$\sin(x) \approx x, \quad \text{for small values of } x.$$

This last expression is an asymptotic statement in disguise.

Let's introduce some new notation. Say we have two functions $f(x)$ and $g(x)$, then

$$\underbrace{f(x) \sim g(x),}_{\text{“}f(x)\text{ is asymptotic to }g(x)\ldots} \qquad \underbrace{x \to x_0,}_{\text{as }x\text{ goes to }x_0\text{”}} \tag{1.3}$$

which means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{1.4}$$

There are two parts to this notation: the expression with the $\sim$ and the associated limit. Strictly, both parts are required for this to make sense.

Use $f(x) = \sin x$, $g(x) = x$ and $x_0 = 0$ in (1.3).

Examples: use of $\sim$ We can rewrite the earlier statement as

$$\sin x \sim x, \quad \text{as} \quad x \to 0,$$

as a check recall that $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ (by l'Hôpital's rule). Some more examples:

$$\begin{aligned} e^x &\sim 1, \quad x \to 0, \\ \frac{1}{\epsilon} &\sim \frac{1}{\epsilon} + 1, \quad \epsilon \to 0, \\ \epsilon^2 &\sim \epsilon^2 + \epsilon^3, \quad \epsilon \to 0, \\ x^{1/3} &\sim 2, \quad x \to 8, \\ e^x &\sim e^x + x, \quad x \to \infty. \end{aligned}$$

Here are some things that are **NOT** asymptotic (denoted $\nsim$):

$$\begin{aligned} \epsilon^2 &\nsim \epsilon, \quad \epsilon \to 0, && \text{(these approach 0 at different rates)} \\ \sin x &\nsim x^2, \quad x \to 0, && \text{(again, approach 0 at different rates)} \end{aligned}$$

Also note that by the definition (1.4) no function can ever be asymptotic to zero. So statements like

$$x^2 \sim 0, \quad x \to 0. \quad \text{(NO! nothing is ever asymptotic to zero)}$$

are incorrect.

Let's introduce some more new notation. Say we have two functions $f(x)$ and $g(x)$, then

$$\underbrace{f(x) \ll g(x),}_{\text{“}f(x)\text{ is much smaller than }g(x)} \qquad \underbrace{x \to x_0,}_{\text{as }x\text{ goes to }x_0\text{”}} \tag{1.5}$$

which means that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0. \tag{1.6}$$

This notation works in a similar way to $\sim$ in that a $\ll$ statement should always be associated with a limit.

Examples: use of $\ll$ Here are some examples:

$$\begin{aligned}
\epsilon &\ll \frac{1}{\epsilon}, \quad \epsilon \to 0, \\
\frac{1}{x} &\ll x, \quad x \to \infty \\
x^{1/2} &\ll x^{1/3}, \quad x \to 0^+, \\
(\log x)^5 &\ll x^{1/4}, \quad x \to \infty, \\
x &\ll -10, \quad , x \to 0^+. && \text{(note the different signs)} \\
af(x) &\ll bg(x), \quad x \to x_0 \text{ if } f(x) \ll g(x), && a, b \text{ const.}
\end{aligned}$$

Asymptotic relations can be manipulated in a similar way to equations. We can add, subtract, divide, cross-multiply, differentiate, integrate and so on. This should be done with care. For example, sometimes it is not necessarily valid to exponentiate both sides of an asymptotic relation (we'll see an example when we look at differential equations).

Switching between asymptotic and exact relations Now we've established this notation, it is possible to convert (carefully) between exact and asymptotic statements. Say we have three function $f(x)$, $g(x)$ and $h(x)$. For instance,

$$\begin{aligned}
\text{if} \quad & f(x) = g(x) + h(x), \quad \text{with} \quad h(x) \ll g(x), \quad x \to x_0, \\
\text{then} \quad & f(x) \sim g(x), \quad x \to x_0.
\end{aligned}$$

It is also sometimes useful to go the other way, that is

$$\begin{aligned}
\text{if} \quad & f(x) \sim g(x), \quad x \to x_0, \\
\text{then} \quad & f(x) = g(x) + h(x), \quad \text{with} \quad h(x) \ll g(x), \quad x \to x_0.
\end{aligned}$$

Although this has to be done with care, for example

$$\begin{aligned}
& e^x \sim 1, \quad x \to 0, \\
\text{then} \quad & e^x = 1 + h(x), \quad \text{with} \quad h(x) \ll 1, \quad x \to 0.
\end{aligned}$$

There is obviously only one valid choice of $h(x)$ here, but the asymptotic statement $h(x) \ll 1$ doesn't give us enough information to determine what it actually is.

1.3 Method of dominant balance

Let's look at the quintic problem (1.1) again. We artificially inserted an ϵ into that equation in front of the x term. What would have happened if we'd inserted it in front of the x^5? The problem is then

$$\epsilon x^5 + x = 1. \tag{1.7}$$

This is an example of a **singular perturbation** problem, since the behaviour of this equation is fundamentally different in the limit $\epsilon \to 0$.

Unpack this idea Now when $\epsilon \to 0$ the unperturbed problem is no longer a quintic and has one rather than five solutions. In the original equation we were interested in the case $\epsilon = 1$, but this new problem we've constructed is actually pretty interesting itself. What happened to the four other roots of the equation when $\epsilon \to 0$?

We are going to apply the **method of dominant balance** to (1.7); this is a systematic way of analysing the behaviour of the equation in the limit $\epsilon \to 0$ and converting the equation to a simpler asymptotic relation.

Introduce method of dominant balance

- In the asymptotic limit $\epsilon \to 0$ it might be possible to neglect one of these term, and so convert the exact equation into a simpler asymptotic relation which tells us how the solution behaves in that limit.
- This must done systematically and in such a way that we don't introduce contradictions. Let's consider each of the three terms in (1.7) and see if they are negligible in the limit $\epsilon \to 0$.

The three possibilities for $\epsilon x^5 + x = 1$ are:

1. $x \sim 1$ as $\epsilon \to 0$ (neglect ϵx^5).

Discuss this balance Assume $\epsilon x^5 \ll x$ and $\epsilon x^5 \ll 1$ as $\epsilon \to 0$. We are looking to see if there are logical inconsistencies between this relation and either of the two assumptions. Let's do this in a lot of detail (this becomes quicker with practice). The first assumption is

$$\begin{aligned} \epsilon x^5 &\ll x, \quad \epsilon \to 0, \\ \implies \epsilon x^4 &\ll 1, \quad \epsilon \to 0, \end{aligned}$$

and since we said that $x \sim 1$ as $\epsilon \to 0$, it's easy to see that this is valid (self-consistent). The second assumption was

$$\epsilon x^5 \ll 1, \quad \epsilon \to 0,$$

and again, we said that $x \sim 1$ as $\epsilon \to 0$, so in the limit x^5 is around 1 and ϵ is getting small, so this is assumption is also valid. We conclude that $x \sim 1$ as $\epsilon \to 0$ is a consistent balance. This tells us that there's one real root near $x = 1$. Great!

2. $\epsilon x^5 \sim 1$ as $\epsilon \to 0$ (neglect x).

Discuss this balance Assume $x \ll 1$ and $x \ll \epsilon x^5$ as $\epsilon \to 0$. Let's look at this:

$$\begin{aligned} \epsilon x^5 &\sim 1, \quad \epsilon \to 0, \\ \implies x^5 &\sim \frac{1}{\epsilon} \quad \epsilon \to 0, \\ \implies x &\sim \frac{\omega}{\epsilon^{1/5}}, \quad \epsilon \to 0, \end{aligned}$$

where $\omega^5 = 1$ (ie. ω complex). As $\epsilon \to 0$ the right hand side blows up, that is $x \sim \omega/\epsilon^{1/5} \to \infty$ as $\epsilon \to 0$, **but** we assumed that $x \ll 1$. This is a contradiction (x can't tend towards ∞ and be $\ll 1$) and therefore this is not a valid balance.

3. $\epsilon x^5 \sim -x$ as $\epsilon \to 0$ (neglect 1).

Discuss this balance Assume $1 \ll \epsilon x^5$ and $1 \ll x$ as $\epsilon \to 0$. Note the x term has moved to the RHS since we can't have something asymptotic to zero. Looking at this:

$$\begin{aligned} \epsilon x^5 &\sim -x, \quad \epsilon \to 0, \\ \epsilon x^4 &\sim -1, \quad \epsilon \to 0, \\ x^4 &\sim -\frac{1}{\epsilon}, \quad \epsilon \to 0, \\ x &\sim \frac{\omega}{\epsilon^{1/4}}, \quad \epsilon \to 0, \end{aligned}$$

where $\omega^4 = -1$. So again we have x blowing up as $\epsilon \to 0$. The assumption that $1 \ll x$ is clearly fine. What about the other assumption? Manipulating the above relation gives

$$\begin{aligned} x^5 &\sim \frac{\omega^5}{\epsilon^{5/4}}, \quad \epsilon \to 0, \\ \implies \epsilon x^5 &\sim \frac{\omega^5}{\epsilon^{1/4}}, \quad \epsilon \to 0. \end{aligned}$$

The right hand side clearly goes to ∞ as $\epsilon \to 0$, and so the assumption that $1 \ll \epsilon x^5$ is valid. **This is also a valid balance**. In fact this is really informative. This balance has retrieves the four roots that went missing in the singular limit.

What did all that tell us? We found that the singular perturbation equation had two different valid balances. One corresponds to the real root (as suspected it was around 1). The other corresponds to the four complex roots that disappeared in the singular limit (we now know what's going on with these: they shoot off to ∞ as $\epsilon \to 0$). If we wanted to, we could proceed and obtain better approximations of each of these roots.

This balance also suggests that a rescaling of the equation might be useful. As it stands, an expansion of the four roots from the second balance start with an $\epsilon^{-1/4}$ term. Let's make a change of variable like this

$$x = \frac{X}{\epsilon^{1/4}}$$

the equation becomes

$$\begin{aligned} \epsilon \frac{X^5}{\epsilon^{5/4}} + \frac{X}{\epsilon^{1/4}} &= 1 \\ X^5 + X &= \epsilon^{1/4} \end{aligned}$$

which is now a regular perturbation problem. Great. But be careful - trying a for a series of the form $\sum a_n \epsilon^n$ won't work, can eyeball it and see that you need $\sum a_n \epsilon^{n/4}$.

This illustrates the power of asymptotic techniques! By making a few relatively simple arguments we were able to convert this difficult singularly perturbed quintic problem (which we would otherwise need to treat numerically) into a some very simple problems. Similar techniques can be applied to differential equations, where we will use the method of dominant balance on more complicated expressions.

1.4 Local behaviour of differential equations

Consider a linear second-order homogeneous differential equation in standard form:

$$y'' + a(x)y' + b(x)y = 0. \tag{1.8}$$

Let's say we're interested in the **local behaviour** of this equation near the point $x = x_0$ and want to construct an approximate solution. The form of solution depends on the behaviour of $a(x)$ and $b(x)$ near x_0:

1. $x = x_0$ is a **ordinary point** (that is $a(x)$ and $b(x)$ analytic in neighbourhood of x_0): use a Taylor series expansion,

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

2. $x = x_0$ is a **regular singular point** (that is $(x - x_0)a(x)$ and $(x - x_0)^2 b(x)$ are analytic in neighbourhood of x_0): use a Frobenius type expansion, for example

$$y = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

3. $x = x_0$ is an **irregular singular point** (ie. not ordinary or regular singular): use asymptotics. We start by assuming the solution is of the form

$$y(x) = \mathrm{e}^{S(x)}, \tag{1.9}$$

and then construct a solution by repeatedly using the method of dominant balance.

Examples of classifying points of DEs Consider the following DEs, and the equivalent equation transformed via x=1/t.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2}y = 0, \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y}{2t^2} = 0.$$

All points ordinary except $x = \infty$ which is an irregular singular point.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2x}y = 0, \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y}{2t} = 0.$$

All points ordinary except $x = 0, \infty$ which are regular singular points.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2x^2}y = 0, \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y}{2} = 0.$$

All points ordinary except $x = 0$ which is an irregular singular points.

Note: to look at the 'local' behaviour as $x \to \infty$, make a change of variables $x = 1/t$ and classify the point $t = 0$.

Let's do an example. Consider the differential equation

$$x^3 y'' = y. \tag{1.10}$$

Let's examine the local behaviour of this equation as $x \to 0$.

Solve (1.10) as $x \to 0$ with method of dominant balance. Putting this in standard form:

$$y'' = \frac{1}{x^3}y,$$

ie. easy to see that $x = 0$ is an irregular singular point. Going to try for a solution of the form $y = e^{S(x)}$. This gives:

$$\begin{aligned} y' &= S'\mathrm{e}^S \\ y'' &= S''\mathrm{e}^S + (S')^2\mathrm{e}^S = (S'' + (S')^2)\mathrm{e}^S \end{aligned}$$

Substitute all that into the DE to get

$$S'' + (S')^2 = \frac{1}{x^3}$$

We now apply the method of dominant balance to this equation. There are 3 possible balances:

1. $S'' \sim 1/x^3$ as $x \to 0$, neglect $(S')^2$

   Integrating once gives $S' \sim -1/(2x^2)$, but then $(S')^2 \sim 1/(4x^4)$. This is a contradiction because we'd assumed that $(S')^2 \ll 1/x^3$ as $x \to 0$.

2. $(S')^2 \sim 1/x^3$ as $x \to 0$, neglect S''

   Rearranging gives $S' \sim \pm x^{-3/2} \implies S'' \sim \mp\frac{3}{2}x^{-5/2} \ll x^{-3}$ as $x \to 0$. There is no contradiction here! ✓

3. $S'' \sim -(S')^2$ as $x \to 0$, neglect $1/x^3$

   Let $T = S'$, then have $T' \sim -T^2$ or $-T^{-2}T' \sim 1$ as $x \to 0$. Integrating gives $T^{-1} \sim x + a \sim a$ as $x \to 0$, that is $(S')^2 \sim 1/a^2 \ll x^{-3}$, which give the contradiction.

This tells us that the appropriate dominant balance is $(S')^2 \sim 1/x^3$, as above this gives

$$S' \sim \pm x^{-3/2} \implies S \sim \mp 2x^{-1/2}, \quad \text{as } x \to 0.$$

Let's just focus on the positive solution (similar procedure for negative, there need to be two linearly independent solutions). Converting this to an (exact) equation gives

$$S(x) = \mp 2x^{-1/2} + C(x), \quad \text{where } C(x) \ll 2x^{-1/2} \quad \text{as } x \to 0.$$

Now we solve for $C(x)$. (We are iteratively finding corrections to $S(x)$) Substituting this into the DE for $S(x)$ above gives

$$\begin{aligned} S'' + (S')^2 &= x^{-3} \\ \mp\frac{3}{2}x^{-5/2} + C'' + (\pm x^{-3/2} + C')^2 &= x^{-3} \\ \mp\frac{3}{2}x^{-5/2} + C'' + x^{-3} \pm 2x^{-3/2}C' + (C')^2 &= x^{-3} \\ \mp\frac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 &= 0 \end{aligned}$$

(Note the cancellation of x^{-3} indicates we did the first bit correctly) Now we apply the method of dominant balance to the equation:

$$\mp\frac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 = 0$$

Luckily we already know a few things about $C(x)$, namely that

$$\begin{aligned} C(x) &\ll x^{-1/2}, \quad \text{as } x \to 0, \\ C'(x) &\ll x^{-3/2}, \quad \text{as } x \to 0, \\ C''(x) &\ll x^{-5/2}, \quad \text{as } x \to 0. \end{aligned}$$

The last fact means that it will always valid to neglect C'' since there is a $x^{-5/2}$ term in the equation. Multiplying the both sides of the second by C' gives $(C')^2 \ll C'x^{-3/2}$ which similarly implies the $(C')^2$ term can be neglected. This leaves the only possible balance (since we need at least two terms) as

$$\begin{aligned} 2x^{-3/2}C' &\sim \frac{3}{2}x^{-5/2}, \quad \text{as } x \to 0, \\ C' &\sim \frac{3}{4}x^{-1}, \quad \text{as } x \to 0, \\ C &\sim \frac{3}{4}\log x, \quad \text{as } x \to 0. \end{aligned}$$

So we have

$$S \sim \mp 2x^{-1/2} + \frac{3}{4}\log x, \quad \text{as } x \to 0.$$

Let's keep going!

Converting the expression for C to an equation:

$$C = \frac{3}{4}\log x + D(x), \quad \text{where } D(x) \ll \frac{3}{4}\log x \quad \text{as } x \to 0.$$

Substituting into the DE for C gives

$$\begin{aligned} 0 &= \mp\tfrac{3}{2}x^{-5/2} + C'' \pm 2x^{-3/2}C' + (C')^2 \\ &= \mp\tfrac{3}{2}x^{-5/2} - \tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}\left(\tfrac{3}{4}x^{-1} + D'\right) + \left(\tfrac{3}{4}x^{-1} + D'\right)^2 \\ &= -\tfrac{3}{4}x^{-2} + D'' \pm 2x^{-3/2}D' + \left(\tfrac{9}{16}x^{-2} + \tfrac{3}{2}x^{-1}D' + (D')^2\right) \\ &= -\tfrac{3}{16}x^{-2} + D'' + (\pm 2x^{-3/2} + \tfrac{3}{2}x^{-1})D' + (D')^2 \end{aligned}$$

Apply the method of dominant balance to

$$-\tfrac{3}{16}x^{-2} + D'' + (\pm 2x^{-3/2} + \tfrac{3}{2}x^{-1})D' + (D')^2 = 0$$

We can immediately neglect the $\frac{3}{2}x^{-1}D'$ term since $x^{-1} \ll x^{-3/2}$ as $x \to 0$. We know the following about D:

$$\begin{aligned} D(x) &\ll \log x, \quad \text{as } x \to 0, \\ D'(x) &\ll x^{-1}, \quad \text{as } x \to 0, \\ D''(x) &\ll -x^{-2}, \quad \text{as } x \to 0. \end{aligned}$$

The last of these means we can neglect the D'' term since there is and x^{-2} term present. Multiplying the second by D' gives $(D')^2 \ll x^{-1}D'$

and so the $(D')^2$ is negligible as $x \to 0$. The remaining balance is then

$$
\begin{aligned}
2x^{-3/2}D' &\sim \pm\tfrac{3}{16}x^{-2}, \quad \text{as } x \to 0\\
D' &\sim \pm\tfrac{3}{32}x^{-1/2}, \quad \text{as } x \to 0\\
\implies D(x) - d &\sim \pm\tfrac{3}{16}x^{1/2}, \quad \text{as } x \to 0
\end{aligned}
$$

3-¿where we have retained the constant of integration since the right-hand side vanishes ($x^{1/2} \ll 1$) as $x \to 0$.4-¿ We can write this as

$$D(x) = d + \delta(x)$$

where $\delta(x) \sim \pm\frac{3}{16}x^{1/2}$. Can continue further and write down a full series expression, but this is sufficient to give the 'leading behaviour' of y. That is, as y approaches the irregular singularity at $x = 0$ the bits of S that we have yet to work out will vanish and the behaviour is dominated by the stuff we found, namely

$$S(x) \sim \mp 2x^{-1/2} + \frac{3}{4}\log x + d, \quad \text{as } x \to 0.$$

Exponentiating both sides (why can we do this?) gives

$$
\begin{aligned}
y(x) &\sim \exp\left(\mp 2x^{-1/2} + \frac{3}{4}\log x + d\right), \quad \text{as } x \to 0,\\
&\sim c_1 x^{3/4} e^{\mp 2x^{-1/2}}, \quad \text{as } x \to 0,
\end{aligned}
$$

where $c_1 = e^d$.

To MATLAB ...

To recap: we assumed a solution of the form (1.9). We then iteratively approximated $S(x)$ until we obtained an approximate solution that was asymptotic to the solution to the original DE. The iterative steps in approximating $S(x)$ were

1. Drop all terms which are negligible (e.g. as $x \to 0$, $x \to \infty$) and replace the $=$ sign with a $\sim$ to give an asymptotic relation.
2. Solve this simpler asymptotic relation. Verify that the solution is consistent with the assumptions made about the negligible terms in step 1.
3. Replace the asymptotic relation with an equation by introducing an arbitrary function (that is $\ll$ the stuff we already found), repeat procedure to find that function.

We stop the procedure when a 'controlling factor' is found, that is enough terms in the approximation to $S(x)$ that when we put them back into $y = e^{S(x)}$ adding additional terms to the approximation for $S(x)$ would have a negligible effect on y.

Taking the exponent of an asymptotic relation. Consider the previous example. At some point we could have written

$$2x^{-1/2} \sim 2x^{-1/2} + \tfrac{3}{4}\log x, \quad \text{as } x \to 0.$$

If we exponentiated both sides we'd have

$$e^{2x^{-1/2}} \sim e^{2x^{-1/2}} x^{3/4}, \quad \text{as } x \to 0,$$

but is this actually true? Look at the following limit

$$\lim_{x\to 0} \frac{e^{2x^{-1/2}} x^{3/4}}{e^{2x^{-1/2}}} = 0 \neq 1$$

In general, if we have an expression $f(x) \sim g(x)$ then for $e^{f(x)} \sim e^{g(x)}$ this requires $f(x) - g(x) \ll 1$.

This technique also works for irregular singular points at ∞. Consider the following equation

$$y'' = xy. \tag{1.11}$$

This is the **Airy equation**, which is (among other things) used to describe rainbows. Let's examine its behaviour as $x \to \infty$.

Solve (1.11) as $x \to \infty$ with method of dominant balance. Check $x \to \infty$ is an irregular singular point, sub $x = 1/t$ to get

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} = \frac{1}{t^5}y$$

use $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2}$

The y term is divided by t^5 and $5 > 2$ so $t = 0$ is an irregular singular point. Make a substitution in (1.11) of $e^{S(x)}$, this gives:

$$S'' + (S')^2 = x.$$

We now apply the method of dominant balance to find the distinguished limit as $x \to \infty$.

1. $S'' \sim x$ as $x \to \infty$, neglect $(S')^2$

   Integrating gives $S' \sim \frac{1}{2}x^2 \implies (S')^2 \sim \frac{1}{4}x^4$ as $x \to \infty$. But we've assumed $(S')^2 \ll x$ as $x \to \infty$ so this is a contradiction.

2. $(S')^2 \sim x$ as $x \to \infty$, neglect S''

   Rearranging and differentiating gives $S'' \sim \pm\frac{1}{2}x^{-1/2}$ and $x^{-1/2} \ll x$ as $x \to \infty$ so this is a valid balance. ✓

3. $S'' \sim -(S')^2$ as $x \to \infty$, neglect x

   Integrating give $S' \sim 1/(x+a) \sim 1/x$ as $x \to \infty$, but we'd assumed $x \ll (S')^2$ so this is a contradiction.

The distinguished limit is

$$(S')^2 \sim x, \quad \text{as } x \to \infty,$$
$$\implies S \sim \pm\frac{2}{3}x^{3/2}, \quad \text{as } x \to \infty.$$

Proceeding, and choosing just the positive bit, we convert to an equation,

$$S(x) = \frac{2}{3}x^{3/2} + C(x), \quad \text{where } C(x) \ll x^{3/2}, \quad \text{as } x \to \infty.$$

Sub. back into the DE for S to get

$$\begin{aligned} S'' + (S')^2 &= x \\ \frac{1}{2}x^{-1/2} + C'' + (x^{1/2} + C')^2 &= x \\ \frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 &= 0 \\ \frac{1}{2}x^{-1/2} + C'' + C'(2x^{1/2} + C') &= 0 \end{aligned}$$

Apply dominant balance to

$$\frac{1}{2}x^{-1/2} + C'' + C'(2x^{1/2} + C') = 0$$

Here's what we know about C:

$$\begin{aligned} C(x) &\ll x^{3/2}, \quad \text{as } x \to \infty \\ C'(x) &\ll x^{1/2}, \quad \text{as } x \to \infty \\ C''(x) &\ll x^{-1/2}, \quad \text{as } x \to \infty \end{aligned}$$

Similarly to the previous example, as $x \to \infty$ we can therefore neglect the C'' and the $(C')^2$.2-¿ This leaves the dominant balance

$$\begin{aligned} 2C'x^{1/2} &\sim -\frac{1}{2}x^{-1/2}, \quad \text{as } x \to \infty \\ C' &\sim -\frac{1}{4}x^{-1}, \quad \text{as } x \to \infty \\ C &\sim -\frac{1}{4}\log x, \quad \text{as } x \to \infty. \end{aligned}$$

Converting that back to an equation we have

$$C(x) = -\frac{1}{4}\log x + D(x), \quad \text{where } D(x) \ll \log x, \quad \text{as } x \to \infty.$$

Sub. this into the equation for C:

$$
\begin{aligned}
\frac{1}{2}x^{-1/2} + C'' + 2C'x^{1/2} + (C')^2 &= 0\\
\frac{1}{2}x^{-1/2} + \frac{1}{4}x^{-2} + D'' + 2x^{1/2}(-\frac{1}{4}x^{-1} + D') + (-\frac{1}{4}x^{-1} + D')^2 &= 0\\
\frac{1}{4}x^{-2} + D'' + 2x^{1/2}D' + \frac{1}{16}x^{-2} - \frac{1}{2}x^{-1}D' + (D')^2 &= 0\\
\frac{5}{16}x^{-2} + D'' + D'(2x^{1/2} - \frac{1}{2}x^{-1}) + (D')^2 &= 0
\end{aligned}
$$

Apply dominant balance to

$$\frac{5}{16}x^{-2} + D'' + D'(2x^{1/2} - \frac{1}{2}x^{-1}) + (D')^2 = 0$$

We have $x^{-1} \ll x^{1/2}$, so that can be neglected. We know the following about D:

$$
\begin{aligned}
D(x) &\ll \log x, \quad \text{as } x \to \infty,\\
D'(x) &\ll x^{-1}, \quad \text{as } x \to \infty,\\
D''(x) &\ll x^{-2}, \quad \text{as } x \to \infty.
\end{aligned}
$$

Therefore neglect the D'' and the $(D')^2$ (in favour of the already neglected $x^{-1}D'$). This leaves

$$
\begin{aligned}
2D'x^{1/2} &\sim -\frac{5}{16}x^{-2}, \quad \text{as } x \to \infty,\\
D' &\sim -\frac{5}{32}x^{-5/2}, \quad \text{as } x \to \infty,\\
D &\sim d + \frac{5}{48}x^{-3/2} \sim d, \quad \text{as } x \to \infty,
\end{aligned}
$$

Then putting all that together, we have

$$
\begin{aligned}
S &\sim \tfrac{2}{3}x^{3/2} - \tfrac{1}{4}\log x + d, \quad \text{as } x \to \infty,\\
\implies y = \mathrm{e}^S &\sim c_1 \mathrm{e}^{\frac{2}{3}x^{3/2}} x^{-1/4}, \quad \text{as } x \to \infty,
\end{aligned}
$$

where $c_1 = \mathrm{e}^d$. If we go back and examine all that working, it's easy to see that the other solution is $c_2\mathrm{e}^{-\frac{2}{3}x^{3/2}}x^{-1/4}$.

These are approximations to **the Airy functions**

$$
\begin{aligned}
\mathrm{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-\frac{2}{3}x^{3/2}}x^{-1/4}\\
\mathrm{Bi}(x) &\sim \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{2}{3}x^{3/2}}x^{-1/4}
\end{aligned}
$$

as $x \to \infty$.

Lurking in our solutions to these differential equations are two important, very subtle ideas in asymptotics: **subdominance** and **behaviour**

of asymptotic relations in the complex plane. Let's discuss each of these in turn.

Example of subdominance/asymptotics in the complex plane Let's start with an example. Consider the sinh function:

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

As $x \to \infty$ have this

$$\sinh x \sim \tfrac{1}{2}e^x$$

However as $x \to -\infty$ have this

$$\sinh x \sim -\tfrac{1}{2}e^{-x}$$

This is interesting. The subdominant term in one limit becomes dominant in the other limit! Extending this to the complex plane ... (draw sketch). So region of validity can be thought of as dividing up the plane. Can't see it in this example, but these are 180 degree wedges, and can write (more properly)

$$\sinh z \sim \tfrac{1}{2}e^z, \quad \text{as } z \to \infty, -\pi/2 < \arg z < \pi/2,$$
$$\sinh z \sim \tfrac{1}{2}e^{-z}, \quad \text{as } z \to \infty, \pi/2 < \arg z < 3\pi/2.$$

The different asymptotic behaviours in different regions of the complex plane is **Stokes phenomenon**. The 'wedge' (in this case a half-plane) is called a **Stokes wedge**.

1.5 Some more notation: $\mathcal{O}$ and o

In the next section we will discuss some technical details on asymptotic series/expansions; but first we need one more piece of notation since to be more precise about ...

Why $\mathcal{O}$ why? Up to this point we've been truncating expression and writing dots. Things can be more precise. Similarly to using the asymptotic symbol instead of $\approx$, we can ditch the dots!

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \ldots$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \mathcal{O}(x^4)$$
$$\cos x = 1 - \frac{1}{2!}x^2 + o(x^2)$$

Ie. there's a few variations how this gets used depending on context.

Let's introduce this new notation. Say we have two function $f(x)$ and $g(x)$, then we write that

$$f(x) = \mathcal{O}(g(x)), \quad \text{as } x \to x_0 \tag{1.12}$$

if it is the case that

$$|f(x)| \le A|g(x)| \tag{1.13}$$

for some constant A.

Examples This is a bit like the asymptotic sign, but up to a multiplicative constant. In fact we could even write that if $f = \mathcal{O}(g)$ then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = A.$$

That gives us a bit of wriggle room compared to $\sim$. Here's some examples

$$\begin{aligned}
\sin x &= \mathcal{O}(x), \quad \text{as } x \to 0, \\
\sin x &= \mathcal{O}(1), \quad \text{as } x \to \infty \\
\sin x &= \mathcal{O}(1), \quad \text{as } x \to 0.
\end{aligned}$$

Note that this last statement is a bit useless - should use to lowest power that works. In general we'll use this to truncate series. Eg. in that particular solution to in the inhomogeneous Airy equation we could have written

$$y_p = \frac{1}{x} + \frac{2}{x^4} + \mathcal{O}(1/x^7).$$

Another, less common notation that we won't use but you should be aware of is 'little o'. Say we have two function $f(x)$ and $g(x)$, then we write

$$f(x) = o(g(x)), \quad \text{as } x \to x_0$$

if $f(x) \ll g(x)$.

Examples Here an example

$$\begin{aligned}
x &\sim \delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2, \\
x &= \delta_0 x_0 + \delta_1 x_1 + O(\delta_2) \\
x &= \delta_0 x_0 + \delta_1 x_1 + o(\delta_1)
\end{aligned}$$

So this is fine, but doesn't necessarily convey any extra info.

1.6 Asymptotic sequences and series

Recall the following definitions. A series $\sum_{n=0}^{\infty} f_n(z)$ **converges** at some fixed value of z if for an arbitrary $\epsilon > 0$ it is possible to find a number $N_0(z, \epsilon)$ such that

$$\left|\sum_{n=M}^{N} f_n(z)\right| < \epsilon \quad \text{for all } M, N > N_0.$$

A series $\sum_{n=0}^{\infty} f_n(z)$ **converges to a function** $f(z)$ at some fixed value of z if for an arbitrary $\epsilon > 0$ it is possible to find a number $N_0(z, \epsilon)$ such that

$$\left|f(z) - \sum_{n=0}^{N} f_n(z)\right| < \epsilon \quad \text{for all } N > N_0.$$

Equivalently, we can think of series converging if its terms decay sufficiently fast as $n \to \infty$.

Consider the error function, which is defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

A tale of two erfs It's easy to come up with a Taylor series representation of this function! Have

$$\mathrm{e}^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}$$

Analytic in the entire complex plane etc. Infinite radius of convergence.

$$\begin{aligned}
\mathrm{erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!} \\
&= \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} - \frac{z^{11}}{1320} + \cdots \right)
\end{aligned}$$

That was easy! This (should) be great, we now have a convergent series representation of that tricky function and we can, in principle, use this to evaluate it at **any** value of z. [To MATLAB, show $z = 1, 2, 5??$].

Let's try something else. Rewrite the integral as

$$\begin{aligned}
\mathrm{erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t, \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t - \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t, \\
&= 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t,
\end{aligned}$$

Now, instead of making a substitution from the exponential, let's integrate by parts:

$$\begin{aligned}
\int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t &= \int_z^\infty \frac{2t\mathrm{e}^{-t^2}}{2t}\mathrm{d}t \\
&= \frac{\mathrm{e}^{-z^2}}{2z} - \int_z^\infty \frac{\mathrm{e}^{-t^2}}{2t^2}\mathrm{d}t \\
&= \frac{\mathrm{e}^{-z^2}}{2z} - \int_z^\infty \frac{2t\mathrm{e}^{-t^2}}{4t^3}\mathrm{d}t \\
&= \frac{\mathrm{e}^{-z^2}}{2z} - \frac{\mathrm{e}^{-z^2}}{4z^3} + \int_z^\infty \frac{\mathrm{e}^{-t^2}}{12t^3}\mathrm{d}t
\end{aligned}$$

and so on until we get tired. This series gives

$$\mathrm{erf}(z) = 1 - \frac{\mathrm{e}^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{1.3}{(2z^2)^2} - \frac{1.3.5}{(2z^2)^3} + \cdots\right)$$

[To MATLAB] This is works great, because the first terms is pretty great!

This was an example of an asymptotic series. Let's say what we mean by that.

Say we have a sequence $\{f_n(\epsilon)\}$ this is an asymptotic sequence as $\epsilon \to 0$ if

$$f_{n+1}(\epsilon) \ll f_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

for all n.

Quick examples Some examples ...

- $\epsilon^{-1} + 1 + \epsilon + \epsilon^2 + ...$, as $\epsilon \to 0$
- $1 + 1/x^2 + 1/x^4 + 1/x^6 + ...$, as $x \to \infty$.

Say we have a function $f(\epsilon)$, a series $\sum_{n=0}^\infty f_n(\epsilon)$ is said to be an asymptotic expansion (or approximation) to this function if

$$f(\epsilon) - \sum_{n=0}^N f_n(\epsilon) \ll f_N(\epsilon), \quad \text{as } \epsilon \to 0.$$

That is the remainder between the approximation and the function (for $\epsilon \to 0$) is smaller than the last included term. This can be written as (we have already been doing this)

$$f(\epsilon) \sim \sum_{n=0}^\infty f_n(\epsilon), \quad \text{as } \epsilon \to 0,$$

The most common version of this is an asymptotic power series in ϵ, namely

$$f(\epsilon) \sim \sum_{n=0}^\infty a_n\epsilon^n, \quad \text{as } \epsilon \to 0.$$

As per previous examples, things like powers of $\epsilon^{1/2}$ might also show up in these series.

An interesting property of asymptotic approximations is that a function may have a variety of different asymptotic approximations. Here are a couple of approximations for $\tan(\epsilon)$ as $\epsilon \to 0$:

$$
\begin{aligned}
\tan(\epsilon) &\sim \epsilon + \frac{\epsilon^3}{3} + \frac{2\epsilon^5}{15} + ... \\
&\sim \sin\epsilon + \frac{1}{2}(\sin\epsilon)^3 + \frac{3}{8}(\sin\epsilon)^5 + ...
\end{aligned}
$$

Much of the rest of this course will consist of developing expansions in the form form $f(x;\epsilon)$, that is they involve an independent variable x as well as a small parameter ϵ. The most general form of an expansion of this type (say for a solution to a differential equation) is

$$
f(x;\epsilon) \sim \sum_{n=0}^{\infty} a_n(x)\delta_n(\epsilon), \quad \text{as } \epsilon \to 0.
$$

The most common version of this is where the $\delta_n(\epsilon) = \epsilon^n$, that is

$$
f(x;\epsilon) \sim \sum_{n=0}^{\infty} a_n(x)\epsilon^n, \quad \text{as } \epsilon \to 0.
$$

As we saw with the error function evaluating asymptotic series accurately is a bit of an art. Generally only a few terms are required, and care should be taken not to include too many terms if a series is divergent.

This in not usually a problem, since in practice (for the kind of problems we'll look at in this course) it is usual to find one, two or (at most!) three terms in an expansion.

The trick is to find an **optimal truncation** of a series (stop adding terms to an approximation); typically this involves looking a the magnitude of the terms and noticing when they start increasing in magnitude. Such an approximation is called (rather excitingly) a **superasymptotic** representation.

There are also numerical methods which can improve the convergence of divergent series to the 'right' answer (in certain circumstances). This might involve alternative ways of summing the terms in a series - it turns out just adding them up just about the most inefficient way to do this! Two such techniques are the Shanks tranformation and Padé summation (if interested see Bender and Orszag, Chapter 8).

2 Perturbation Methods

Much of the previous chapter we focussed on developing locally valid solutions to differential equations. In reality we are interested not just in how such problems behave near a point, but in their global behaviour. In the next chapter we will look at boundary-layer problems, for instance, where the solution behaviour in a thin region is different to the bulk of the domain. This chapter focusses on building techniques to consider such problems.

In the most general terms a perturbation problem is simply a problem that features a small parameter, often denoted ϵ. Such small parameters can be introduced to a modelling problem in a variety of ways. For instance,

- **initial conditions** a usually stable physical situation can be given a small 'kick', and we can look at its response.

  Example: Kelvin-Helmholtz instability

- **boundary conditions** similarly, a perturbation can be introduced at a boundary (inhomogeneity/boundary shape)

  Example: flow in a corrugated pipe

- **variable scaling** a modelling domain might, for instance, have different characteristic length scales in different directions (eg. long and thin) and therefore rescaling is appropriate.

  Example: fibre drawing

- **parameter scaling** assumptions around dominant physical processes simplify analysis; (eg. Reynolds number in fluids, Peclet number in heat tranfer) is large or small (physical scaling)

This is not an exhaustive list. It is also sometimes useful to arbitrarily introduce a small parameter into a problem to aid analysis.

In many examples in the previous chapter we were essentially already using perturbation methods! Let's revisit some definitions: regular and singular problems.

- **Regular perturbation problem:** The structure of the problem is unchanged in the limit $\epsilon \to 0$.

Some quick examples Something like

$$y'' + y' + y(1+\epsilon) = 0$$

- **Singular perturbation problem:** The solution behaves differently in the limit $\epsilon \to 0$. In a singularly perturbed DE, the highest order derivative might be multiplied by ϵ. Boundary-layer problems are a much studied class of these.

Examples, including a boundary-layer problem Consider this equation (OXF pert meth notes)

$$\epsilon y'' + y' + y = 0$$

Clearly when $\epsilon \to 0$ this equation goes from having two linearly indep solution to having one. Difficulties arise due to boundary conditions. ie. original problem has 2 BCs, say $y(0) = a$ and $y(1) = b$, unperturbed problem must still satisfy this, but only on constant of integration. Can sometimes

For many of our problems we seek solutions in the form

$$y(x) = \sum_{n=0}^{\infty} \epsilon^n y_n(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \mathcal{O}(\epsilon^2)$$

This is typically an iterative procedure. We find the leading-order solution $y_0(x)$, then solve for $y_1(x)$ which typically depends on $y_0(x)$ and can be thought of as a 'correction' term. In practice for regular perturbation problem we are interested in leading-order behaviour, sometimes interested in a first-order corrections and rarely in terms beyond that.

2.1 Ordinary differential equations

We have already used an expansion method to solve algebraic equations. Now let's use similar methods to solve some applied differential equation problems.

2.1.1 Flight of a projectile

Consider the flight of a projectile under the influence of gravity. Suitably non-dimensionalised (for details see Bowen & Witelski, Chapter 4) the height of the projectile $x(t)$ above the earth at time t is governed by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{1}{(1+\epsilon x)^2}, \quad x(0) = 1, \quad x'(0) = \alpha, \tag{2.1}$$

where $\epsilon \to 0$. The small parameter ϵ has a physical meaning: it is the ratio of the length scale of interest to that radius of the earth, so this approximation is valid as long as the projectile does not go too high.

We assume that $x(t)$ has the form of a perturbation series

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots$$

The solution procedure involves substituting this series into (2.1) and solving for $x_0(t)$, $x_1(t)$, ... in turn.

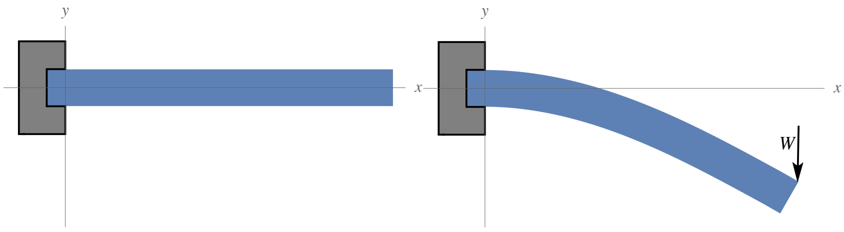
Derive perturbation series solution to the projectile problem.

A subtlety of this perturbation expansion solution (and all similar expansions) is that it is only valid when the asymptotic ordering of the terms is maintained.

Validity of the perturbation series approximation (discussion and MATLAB)

2.1.2 Shape of a diving board

Consider a diving board, as shown below.



If a weight is applied to the end of the board the deformation is described by

$$B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = W\cos\theta, \quad \theta(0) = \frac{\mathrm{d}\theta}{\mathrm{d}s}(L) = 0.$$

where θ is the angle of deflection along the board and s is an arc-length variable.

This is a dimensional equation where B is the bending stiffness of the board and W is the weight applied at the end of the board. The centre-line of the board $(x(s), y(s))$ is calculated via

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta.$$

We non-dimensionalise by scaling s with the length of the board L, so that $s = L\hat{s}$. Making this substitution (and dropping the hats) gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \delta\cos\theta, \quad \theta(0) = \frac{\mathrm{d}\theta}{\mathrm{d}s}(1) = 0.$$

Here the small parameter δ appears in the equation.

where $\delta = WL^2/B$ is now the single dimensionless parameter that in this problem. When this parameter is small we can seek an asymptotic solution to this problem.

Asymptotic solution to the diving board problem

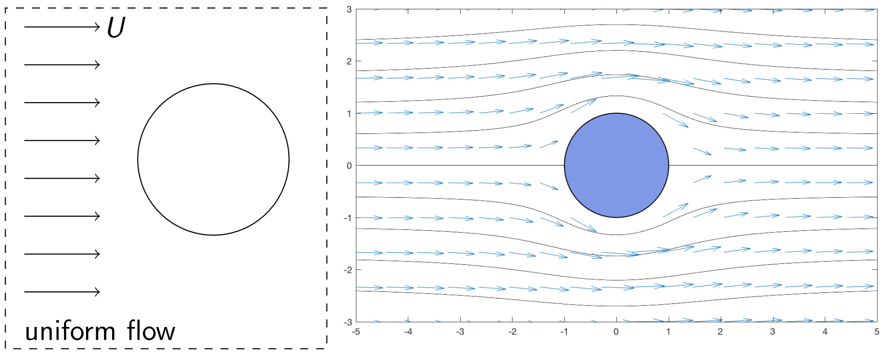
Comparison with numerical solution (MATLAB)

2.2 Partial differential equations

We now use perturbation methods to solve some partial differential equation models. The examples here are from fluid mechanics, a popular application area for asymptotic and perturbation methods.

2.2.1 Flow around an almost cylindrical obstacle

A very well-known problem from classical fluids mechanics is the inviscid flow around a cylinder. This problem is something of a rarity because it has an exact analytic solution!



First, a (very) brief review of some fluid mechanics ideas. We want to model the velocity (vector) field $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}$ for the flow of an inviscid, incompressible fluid. To do this we define a velocity potential ϕ such that $\mathbf{u} = \nabla\phi$. This velocity potential satisfies Laplace's equation:

$$\nabla^2\phi = 0.$$

A simple example is that uniform horizontal flow with velocity $\mathbf{u} = U\mathbf{i} + 0\mathbf{j}$ is given by $\phi = Ux$. In cartesian coordinates the velocity components in the x and y directions are just

$$u_x = \frac{\partial\phi}{\partial x}, \qquad u_y = \frac{\partial\phi}{\partial y}.$$

Recall that $x = r\cos\theta$ and $y = r\sin\theta$

In cylindrical polars (which we'll use for the next example) the radial and azimuthal velocity components are

$$u_r = \frac{\partial\phi}{\partial r}, \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}.$$

A related quantity is the streamfunction ψ, the Laplacian of which is vorticity ω (a measure of fluid rotation),

$$\nabla^2 \psi = -\omega,$$

and from which the velocity components are be calculated via:

$$\begin{aligned} &\text{(cartesian)} \qquad && u_x = \frac{\partial \psi}{\partial y}, \qquad && u_y = -\frac{\partial \psi}{\partial x}, \\ &\text{(cylindrical)} \qquad && u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad && u_\theta = -\frac{\partial \psi}{\partial r}. \end{aligned}$$

There's also a complex potential $F(z) = \phi + i\psi$.

The context (eg. geometry) of the problem determines choice of co-ordinate system, with velocity potential and streamfunction used interchangeably (more or less).

There's also a version of that use complex variables, defined as

$$F(z) = \phi + i\psi$$

where F is analytic. In this formulation get velocity by taking

$$\frac{\mathrm{d}F}{\mathrm{d}z} = u_x - \mathrm{i}u_y.$$

For examples, the uniform flow is just $F = Uz$.

This is a nice approach because it's possible to construct quite complicated velocity fields by adding together well-known analytic functions (z^n, $\log z$ and so on). We can also use conformal mapping techniques, to describe flows in geometrically interesting regions.

Flow around cylinder and exact solution: Working in cylindrical co-ordinates, the uniform flow of speed U around a cylinder of radius a is described by

$$\nabla^2 \phi = 0, \quad \text{in } r > a,$$

with boundary conditions

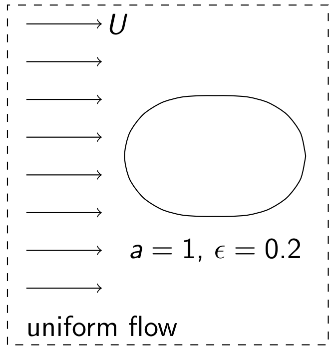
$$\begin{aligned} \mathbf{n} \cdot \nabla \phi = 0, &\quad \text{at } r = a, \\ \phi \sim Ur\cos\theta, &\quad \text{as } r \to \infty. \end{aligned}$$

Written in terms of streamfunction the first condition would be $\psi = 0$ at $r = a$.

The first says there is the component of velocity normal (perpendicular) to the cylinder surface is zero; that is the fluid goes around, rather than passing through the cylinder. The second is that far from the cylinder there is uniform flow with speed U. The exact solution is

$$\phi(r, \theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta.$$

Flow around an approximate ellipse: We now introduce a perturbation by perturb changing the shape of the obstacle. Instead of being circular, let's try an approximate ellipse given by $r = a(1 + \epsilon\cos 2\theta)$.



The problem is now

$$\nabla^2\phi = 0, \quad \text{in } r > a(1 + \epsilon \cos 2\theta),$$

with boundary conditions

$$\begin{aligned} \mathbf{n} \cdot \nabla\phi = 0, &\quad \text{at } r = a(1 + \epsilon \cos 2\theta), \\ \phi \sim Ur\cos\theta, &\quad \text{as } r \to \infty. \end{aligned}$$

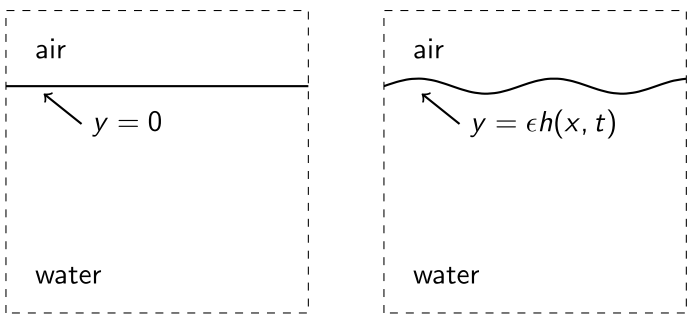
This is different in character than the problem we previously considered because the small parameter ϵ appears in the boundary condition. Our approach relies on perturbing about the circular solution, which in this case involves replacing the problem in $r > a(1 + \epsilon \cos 2\theta)$ with an equivalent problem in $r > a$.

Solution of the flow around the approximate ellipse.

MATLAB: plots of this solution

2.2.2 Deep water waves

Another classic fluid mechanics problem is to look at the behaviour of small amplitude waves on the surface of water. This is a prototypical example of a ‘free-surface problem’, where the shape of the boundary is determined as part of the solution.



unperturbed perturbed

Assume surface of the waters is $y = \epsilon h(x,t)$, with $\epsilon \ll 1$ and $h = \mathcal{O}(1)$. The velocity is described by Laplaces equation in the water,

$$\nabla^2 \phi = 0, \qquad \text{for } y < \epsilon h(x,t).$$

The boundary conditions on the free-surface are a kinematic condition,

$$\frac{\partial \phi}{\partial y} = \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x,t),$$

which states that the normal velocity of the water is equation to the normal velocity of the surface, and the Bernoulli condition,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + gy = 0, \quad \text{on } y = \epsilon h(x,t),$$

which is essentially Newton's Second Law. This simply stated problem is an incredibly rich field of study.

Perturbation series solution of deep water waves

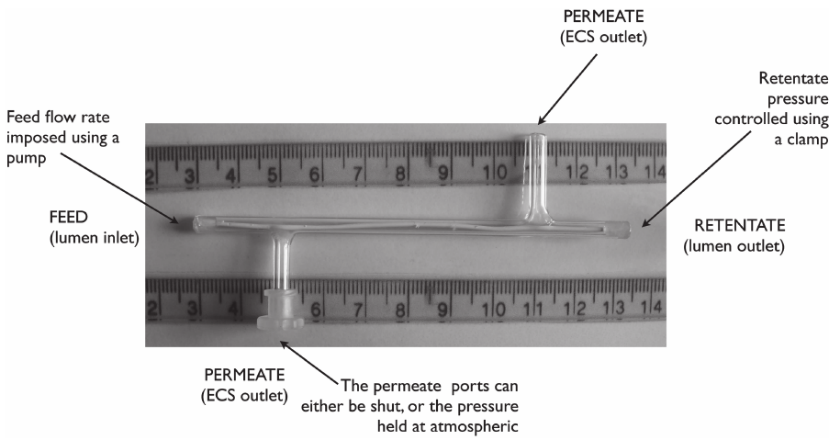
2.3 Case study: hollow fibre bioreactor

An important application area in mathematical biology is tissue engineering, the sub-field of bio-engineering which deals with growing artificial tissues. This typically involves seeding an engineered scaffold with a few cells and then culturing it in a bioreactor; after a culture period the goal is for the scaffold to have developed into a tissue (which might then be implanted).

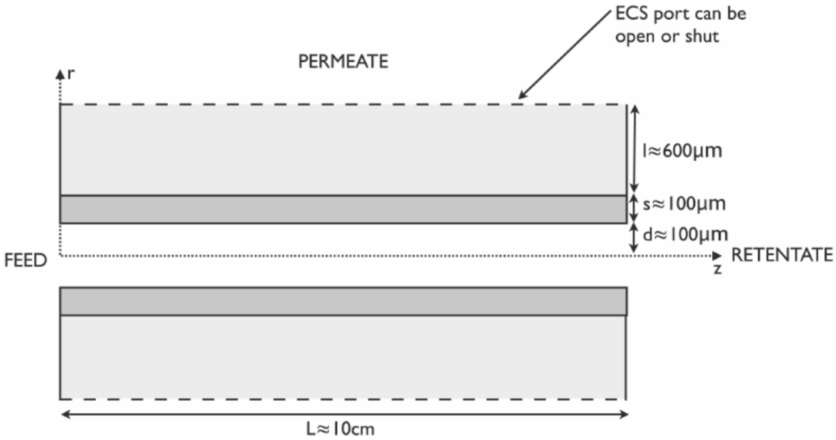
One common type of bioreactor is described in:

Shipley, R.J., Waters, S.L., *Fluid and mass transport modelling to drive the design of cell-packed hollow fibre bioreactors for tissue engineering applications*, **29**:329–359, Math. Med. Biol., 2012.

This treatment is of interest since it takes advantage of the slender geometry of this type of bioreactor (see below), which permits the introduction of a small parameter.



A hollow fibre bioreactor consists of a hollow outer tube and a coaxial lumen (permeable tube). Cells are seeded around the outside of the lumen (ECS, extra-capillary space). Fluid containing nutrients is pumped along the lumen; the idea being that nutrients pass through the permeable tube wall into the space containing to the cells.



Summary:

The approach taken in Shipley and Waters is:

- Develop a model for fluid flow in the lumen and couple this to a model for nutrient transport to the ECS (sections 3 and 4).
- Assumption of slenderness; aspect ratio of the lumen is small $\epsilon = d/L \approx 2 \times 10^{-3} \ll 1$, see Eq. (1).
- Derivation of reduced, leading-order models (sections 3.1 and 4.1).
- Application of model to investigate effect of opening ECS port on nutrient concentration and so effect on different cell types (section 5).

Use of asymptotics/perturbation methods:

The key step in the model reduction involves using lubrication theory, where in addition to the assumption of slender geometry it is further assumed that the radial flow is small compared to the axial flow. An interesting subtlety is in the choice of pressure scaling, which is different in the lumen (large), the membrane and the ECS (both small), see Eqs. (15) and (16).

An advantage to this approach is that the reduced model permits analytic solutions to the leading-order equations.

Repeat derivation of reduced model via lubrication theory Let's take these 'full' equations from the paper (equation 2):

r constant viscosity, conservation of fluid mass and momentum yield

$$\nabla \cdot \mathbf{u}_\mathrm{l} = 0, \quad \rho\left(\frac{\partial \mathbf{u}_\mathrm{l}}{\partial t} + (\mathbf{u}_\mathrm{l}\cdot\nabla)\,\mathbf{u}_\mathrm{l}\right) = -\nabla p_\mathrm{l} + \mu\nabla^2\mathbf{u}_\mathrm{l},$$

e the fluid density and dynamic viscosity, respectively. The membra

Apply the following scalings (eqn. 14, with $i = \mathrm{l, m, e}$ & $\epsilon = d/L$):

$u_z(r, z)$). Given the small aspect ratio of the lumen, we employ lubrica

$$r = d\tilde{r},\; z = L\tilde{z},\; u_{i,z} = U\tilde{u}_{i,z},\; u_{i,r} = \varepsilon U\tilde{u}_{i,r},\; p = P_i\tilde{p} + P_0,$$

ical lumen flow velocity and P_i for $i = \mathrm{l, m, e}$ is the pressure scale (in ea

... and end up with these 'reduced' equations (equation 17):

and ECS becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_{\mathrm{l},r}) + \frac{\partial u_{\mathrm{l},z}}{\partial z} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{l},z}}{\partial r}\right) = \frac{\mathrm{d}p_\mathrm{l}}{\mathrm{d}z},$$

Start with the conservation of mass equation:

$$\begin{aligned}
\nabla\cdot\mathbf{u}_\mathrm{l} &= 0,\\
\frac{1}{r}\frac{\partial}{\partial r}(r u_{\mathrm{l},r}) + \frac{\partial u_{\mathrm{l}.z}}{\partial z} &= 0, \quad \text{(this is dimensional)}\\
\frac{1}{d^2}d\epsilon U\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{u}_{\mathrm{l},r}) + \frac{U}{L}\frac{\partial\tilde{u}_{\mathrm{l}.z}}{\partial\tilde{z}} &= 0, \quad \text{(apply scalings)}\\
\frac{\epsilon}{d}\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{u}_{\mathrm{l},r}) + \frac{1}{L}\frac{\partial\tilde{u}_{\mathrm{l}.z}}{\partial\tilde{z}} &= 0, \quad \text{(div. by } U\text{)}\\
\frac{1}{L}\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{u}_{\mathrm{l},r}) + \frac{1}{L}\frac{\partial\tilde{u}_{\mathrm{l}.z}}{\partial\tilde{z}} &= 0, \quad \text{(since } \epsilon = d/L\text{)}\\
\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{u}_{\mathrm{l},r}) + \frac{\partial\tilde{u}_{\mathrm{l}.z}}{\partial\tilde{z}} &= 0.
\end{aligned}$$

Drop tildes to get:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_{\mathrm{l},r}) + \frac{\partial u_{\mathrm{l}.z}}{\partial z} = 0, \quad \text{(this is now dimensionless)}$$

Now let's do conservation of momentum:

$$\rho\left(\frac{\partial\mathbf{u}_\mathrm{l}}{\partial t} + (\mathbf{u}_\mathrm{l}\cdot\nabla)\,\mathbf{u}_\mathrm{l}\right) = -\nabla p_\mathrm{l} + \mu\nabla^2\mathbf{u}_\mathrm{l}.$$

To crack into this write out

$$\begin{aligned}
(\mathbf{u}_\mathrm{l}\cdot\nabla) &= \left(\mathbf{u}_\mathrm{l}\cdot\left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{\partial}{\partial z}\mathbf{e}_z\right)\right)\\
&\left(u_{\mathrm{l},r}\frac{\partial}{\partial r} + u_{\mathrm{l},z}\frac{\partial}{\partial z}\right).
\end{aligned}$$

We'll only need the z-component, this is (assume $\partial/\partial t = 0$):

$$\rho\left(u_{\mathrm{l},r}\frac{\partial}{\partial r} + u_{\mathrm{l},z}\frac{\partial}{\partial z}\right)u_{\mathrm{l},z} = -\frac{\partial p_\mathrm{l}}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{l},z}}{\partial r}\right) + \frac{\partial^2 u_{\mathrm{l},z}}{\partial z^2}\right)$$

Applying those scalings again:

$$\rho U^2\left(\tilde{u}_{\mathrm{l},r}\frac{\partial}{\partial \tilde{r}} + \frac{1}{L}\tilde{u}_{\mathrm{l},z}\frac{\partial}{\partial \tilde{z}}\right)\tilde{u}_{\mathrm{l},z} = -\frac{P_\mathrm{l}}{L}\frac{\partial \tilde{p}_\mathrm{l}}{\partial \tilde{z}} + \mu\left[\frac{U}{d^2}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{u}_{\mathrm{l},z}}{\partial \tilde{r}}\right) + \frac{U}{L^2}\frac{\partial^2 \tilde{u}_{\mathrm{l},z}}{\partial \tilde{z}^2}\right]$$

Multiply both sides by $d^2/(U\mu) = \epsilon^2 L^2/(U\mu)$:

$$\epsilon^2\left(\tilde{u}_{\mathrm{l},r}\frac{\partial}{\partial \tilde{r}} + \tilde{u}_{\mathrm{l},z}\frac{\partial}{\partial \tilde{z}}\right)\tilde{u}_{\mathrm{l},z} = -\frac{\epsilon^2 L}{U\mu}P_\mathrm{l}\frac{\partial \tilde{p}_\mathrm{l}}{\partial \tilde{z}} + \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{u}_{\mathrm{l},z}}{\partial \tilde{r}}\right) + \epsilon^2\frac{\partial^2 \tilde{u}_{\mathrm{l},z}}{\partial \tilde{z}^2}$$

where $\mathrm{Re} = \rho UL/\mu$. Drop tildes to get:

$$\epsilon^2\mathrm{Re}\left(u_{\mathrm{l},r}\frac{\partial}{\partial r} + u_{\mathrm{l},z}\frac{\partial}{\partial z}\right)u_{\mathrm{l},z} = -\frac{\epsilon^2 L}{U\mu}P_\mathrm{l}\frac{\partial p_\mathrm{l}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{l},z}}{\partial r}\right) + \epsilon^2\frac{\partial^2 u_{\mathrm{l},z}}{\partial z^2}$$

This is now ready to be 'reduced'.

- On the RHS the second term is $\mathcal{O}(1)$ but the third term is $\mathcal{O}(\epsilon^2)$.
- Assume small Reynolds number so $\epsilon^2\mathrm{Re} \ll 1$.
- Leaves a balance between first two terms on the RHS (no restriction on P_l as yet).

This gives (without saying anything yet about P_l) the following **distinguished limit**:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{l},z}}{\partial r}\right) \sim \frac{\epsilon^2 L}{U\mu}P_\mathrm{l}\frac{\partial p_\mathrm{l}}{\partial z}, \quad \epsilon \to 0.$$

and the equation in the paper is retrieved by setting $P_\mathrm{l} = \frac{\mu U}{\epsilon^2 L}$ (free to choose this to be anything we like) to give:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{l},z}}{\partial r}\right) = \frac{\partial p_\mathrm{l}}{\partial z}.$$

If we did something similar for the r component we'd find that $\partial p_l/\partial r = 0$, that is $p_l \equiv p_l(z)$, which gives the form in the paper:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{l,z}}{\partial r}\right) = \frac{\mathrm{d}p_l}{\mathrm{d}z}.$$

Case study project (5% of total mark)

This case study is a guide to the form of the short written project.

You should find a research paper that makes use of asymptotic and/or perturbation methods (ideally in an application area that you're interested in) and write a brief report.

A good way to structure this is as above, namely:

- **Background:** Provide some background to the application;
- **Summary:** Briefly summarise the mathematical approach used;
- **Use of asymptotics/perturbation methods:** Give some detail on the use of asymptotics/perturbation methods ...
- **Redo some working:** ... and repeat a few steps of working from the paper (eg. filling in the gaps between steps).

This should be a **short** report (2–3 pages).

3 Boundary layer theory and asymptotic matching

We've already seen that singular perturbation problems must be treated with care. Recall the dominant balance analysis of equation (1.7),

$$\epsilon x^5 + x = 1,$$

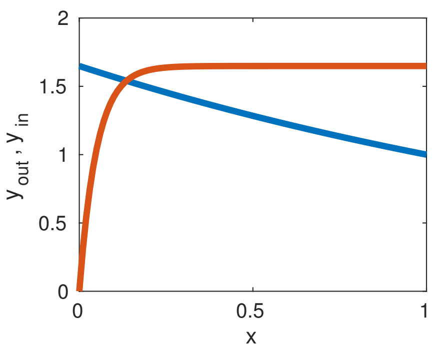
which revealed that as $\epsilon \to 0$ one of the roots was $x = 1$, while the four complex roots shot off to infinity; that is the solution exhibits behaviour on different scales (the real root is 'small'; the complex roots are 'big').

A similar kind of **multi-scale** structure is often seen in the solutions to singularly perturbed differential equations.

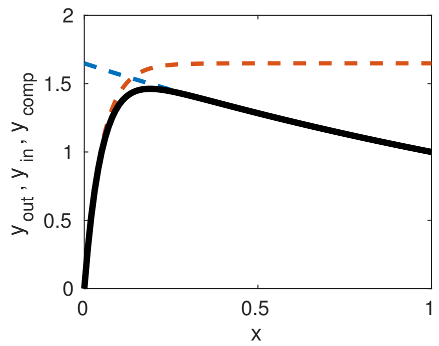
Such equations are used, for instance, to model phenomena which change rapidly in one region but more slowly in another region. A classic example (and origin of the term 'boundary layer') is from viscous fluid mechanics where fluid flows past objects/walls. In such cases there is usually a no-slip condition at the object/wall and therefore the flow behaves differently in a layer near this boundary than in the rest of the flow. Similar structure arise in a surprising array of applications.

Mathematically speaking in such cases is that a different distinguished limit of a governing equation is valid in different regions of the solution domain. To solve such problems we need to:

1. Construct solutions which are (asymptotically) valid in the various parts of the domain. Typically an **inner solution** near the edge of a domain, and an **outer solution** in the rest of the domain.



2. Join these solutions together by taking advantage of the fact that their regions of validity overlap: this is called **asymptotic matching**.



Shortly we'll go about constructing the above picture (which arise from a singularly perturbed second-order ODE). Before proceeding, note that

- Not all singularly perturbed DEs exhibit boundary layers (although it is very common).
- Asymptotic matching can be used in other contexts, not just for boundary layers (as we'll see in a later example).

3.1 Ordinary differential equations

3.1.1 An example: boundary layer in a second-order ODE

Consider the following second-order ODE:

$$\epsilon \frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0, \tag{3.1}$$

as $\epsilon \to 0$ with boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

Setting $\epsilon = 0$, the equation and boundary conditions become

$$2\frac{dy}{dx} + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

That's a first-order ODE with **two** boundary conditions. Fortunately, this equation has an exact, closed form solution. This is

$$y(x) = \frac{\exp\left(\frac{-1+\sqrt{1-\epsilon}}{\epsilon}x\right) - \exp\left(\frac{-1-\sqrt{1-\epsilon}}{\epsilon}x\right)}{\exp\left(\frac{-1+\sqrt{1-\epsilon}}{\epsilon}\right) - \exp\left(\frac{-1-\sqrt{1-\epsilon}}{\epsilon}\right)},$$

which features a $e^{-x/\epsilon}$-type dependence that is characteristic of boundary layers problems.

Construct inner and outer solutions to (3.1)

3.1.2 Brief guide to solving boundary layer problems

The solution to the above problem (and many problems like it involved the following steps:

- **Find outer solution:** Try for a regular perturbation series solution (as in the previous chapter), of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \cdots .$$

  If (as in the above example) the boundary conditions cannot be satisfied by a solution of this form then the solution likely features boundary layers and further analysis is required.

- **Determine the distinguished limits:** Use a dominant balance analysis to determine appropriate scalings; each balance corresponds to either the outer solution, or an inner solution. In the previous example we rescaled only the independent variable (in that case $x = \epsilon X$), but it is sometimes also necessary to scale the dependent variable too.

- **Find the inner solution(s):** Having determined the presence (and location) of a boundary layer, seek a regular perturbation series solution to the rescaled problem. In the above example, the inner solution was of the form

$$Y(x) = Y_0(X) + \epsilon Y_1(X) + \cdots .$$

- **Matching:** Apply an asymptotic matching procedure to connect the inner and outer solutions. The matching condition in the above problem was that: the outer limit of the inner solution equals the inner limit of the outer solution, which we wrote as

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x).$$

  This is a condition on the leading order solution; more sophisticated conditions are necessary at higher order.

- **Construct composite solution:** Add the inner and outer solutions together; subtract the overlaps to prevent doubling-up.

3.1.3 Another second-order ODE

The first example we considered had an exact solution (and so we knew about its boundary layer structure from the form of that). Let's know consider a problem with no exact solution:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x, \tag{3.2}$$

with $\epsilon \to 0$ over $0 \le x \le \pi$, and the boundary conditions

$$y(0) = 2, \qquad y(\pi) = -1.$$

Solve this problem. Outer problem:

Let $y(x) = y_0(x) + \epsilon y_1(x) + ...$ To leading order have

$$y_0' = \cos x,$$
$$\implies \quad y_0(x) = \sin x + A,$$

this can't satisfy both BCs, so A is determined the boundary condition the BC outside the boundary layer. Could have either:

$$y_0(x) = \sin x + 2, \quad \text{satisfying } y_0(0) = 2 \implies \text{BL at } x = \pi,$$
$$\text{or } y_0(x) = \sin x - 1, \quad \text{satisfying } y_0(\pi) = -1 \implies \text{BL at } x = 0.$$

It will be one of these - but first we need to figure out where the boundary layer actually is!

Rescaling:

Let's rescale our equation to 'zoom in' on the boundary layer. We need to figure out both where the layer is and how it should be scaled. Let $x = x^* + \delta_1(\epsilon)X$ and $y = \delta_2(\epsilon)Y$. The problem becomes:

$$\epsilon\frac{\delta_2}{\delta_1^2}Y'' + \frac{\delta_2}{\delta_1}Y' = \cos(x^* + \delta_1 X),$$

with $\delta_2 Y(0) = 2$ (if $x^* = 0$) or $\delta_2 Y(0) = -1$ (if $x^* = \pi$).

The RHS of both BCs is $\mathcal{O}(1)$, so the LHS should be too $\implies \delta_2 = 1$. Equation is now:

$$\epsilon Y'' + \delta_1 Y' = \delta_1^2 \cos(x^* + \delta_1 X),$$

There are three possibilities here:

- $\delta_1 Y' \sim \delta_1^2 \cos(x^* + \delta_1 X) \implies \delta_1 = 1$, which is the outer solution.
- $\epsilon Y'' \sim \delta_1^2 \cos(x^* + \delta_1 X) \implies \delta_1$, which is a contradiction since $\delta_1 Y' \gg \epsilon Y''$.
- $\epsilon Y'' \sim -\delta_1 Y' \implies \delta_1 = \epsilon$, which is fine!

Rescaling with $\delta_1 = \epsilon$ gives ... **Inner problem:**

$$Y'' + Y' = \epsilon \cos(x^* + \epsilon X),$$

Let $Y = Y_0(X) + \epsilon Y_1(X) + ...$, then

$$Y_0'' + Y_0' = 0 \implies Y_0 = B + C\mathrm{e}^{-X}.$$

with B and C determined by matching and application of the final BC (still need to determine where the boundary layer is). **Matching condition:**

There are two possibilities here. If $x^* = 0$ then

$$\begin{aligned}
\lim_{x\to 0} y_0(x) &= \lim_{X\to\infty} Y_0(X),\\
\lim_{x\to 0} \sin x + A &= \lim_{X\to\infty} B + Ce^{-X}.
\end{aligned}$$

Or if $x^* = \pi$ then

$$\begin{aligned}
\lim_{x\to \pi} y_0(x) &= \lim_{X\to-\infty} Y_0(X),\\
\lim_{x\to \pi} \sin x + A &= \lim_{X\to-\infty} B + Ce^{-X}.
\end{aligned}$$

Here the LHS of the second option is divergent (bad news), so the boundary layer cannot be at $x^* = \pi$. It **is** then at $x^* = 0$. Now we can apply the boundary conditions:

$$\begin{aligned}
y(\pi) = -1 \quad &\implies \quad A = -1,\\
Y(0) = 2 \quad &\implies \quad B = 2 - C.
\end{aligned}$$

Applying the matching conditions then gives:

$$\begin{aligned}
\lim_{x\to 0} y_0(x) &= \lim_{X\to\infty} Y_0(X) \qquad (= y_{\text{overlap}})\\
\lim_{x\to 0} \sin x - 1 &= \lim_{X\to\infty} 2 + C(e^{-X} - 1),\\
-1 &= 2 - C, \quad \implies C = 3.
\end{aligned}$$

Putting all that together we have:

$$\begin{aligned}
y_0(x) &= \sin x - 1,\\
Y_0(X) &= 3e^{-X} - 1,\\
y_{\text{overlap}} &= -1.
\end{aligned}$$

The composite solution is then

$$\begin{aligned}
y_{\text{composite}} &= y_0 + Y_0 - y_{\text{overlap}},\\
&= \sin x - 1 + 3e^{-X} - 1 - (-1),\\
&= \sin x + 3e^{-x/\epsilon} - 1.
\end{aligned}$$

This all works because the inner and outer solutions have an overlapping region of validity (where they both behave like the overlap solution).

Let's figure out what those regions are.

When is $y_0 \sim y_{\text{overlap}}$? Consider:

$$\begin{aligned}
\sin x - 1 &\sim -1, \quad \epsilon \to 0,\\
\implies x - 1 &\sim -1 \quad \epsilon \to 0,\\
\implies x &\ll 1.
\end{aligned}$$

Now, when is $Y_0 \sim y_{\text{overlap}}$? Consider:

$$\begin{aligned} -1 + 3e^{-x/\epsilon} &\sim -1, \quad \epsilon \to 0, \\ \implies e^{-x/\epsilon} &\ll 1, \quad \epsilon \to 0, \\ \implies x/\epsilon &\gg 1, \quad \epsilon \to 0 \quad \text{with } x > 0, \\ \implies \epsilon &\ll x. \end{aligned}$$

Then we have that $y_0 \sim Y_0 \sim y_{\text{overlap}}$ in the combined region

$$\epsilon \ll x \ll 1.$$

3.1.4 Yet another second-order ODE

Consider the following second-order ODE:

$$\epsilon \frac{d^2 y}{dx^2} - \left(2 - x^2\right) y = -1, \quad \text{as } \epsilon \to 0,$$

subject to the boundary conditions $y'(0) = 0$ and $y(1) = 0$.

Solve this problem.

Outer solution:

Setting $\epsilon = 0$ we just get:

$$-(2 - x^2) y_0 = -1 \qquad \implies \qquad y_0(x) = \frac{1}{2 - x^2}.$$

Note that this satisfies the condition that $y_0'(0) = 0$. So there is no boundary layer at $y = 0$, and since we don't have any parameters to play with the must be a BL at $x^* = 1$.

Rescaling:

Set $x = 1 + \delta_1(\epsilon) X$ and $y = \delta_2(\epsilon) Y$. Previously the BC trivial gave us δ_2, but that doesn't work since the condition $y(1) = 0$ gives no information about the order of the function near the boundary. However the matching condition:

$$\lim_{x \to 1} y_0 = \lim_{X \to -\infty} \delta_2 Y$$

implies that $\delta_2 = 1$ since $\lim_{x \to 1} y_0 = 1 = \mathcal{O}(1)$.

Continuing, the equation becomes:

$$\begin{aligned} \frac{\epsilon}{\delta_1^2} Y'' - (2 - (1 + \delta_1 X)^2) Y &= -1, \\ \frac{\epsilon}{\delta_1^2} Y'' - (1 - 2\delta_1 X - \delta^2 X^2) Y &= -1. \end{aligned}$$

Assuming that $\delta_1 \ll 1$ we can write:

$$\frac{\epsilon}{\delta_1^2} Y'' - Y \sim -1 \quad \Longrightarrow \quad \delta_1 = \epsilon^{1/2}$$

Inner problem:

Rescaled problem is then:

$$Y'' - (1 - 2\epsilon^{1/2} X - \epsilon X^2) Y = -1, \quad Y(0) = 0.$$

Since there is an $\epsilon^{1/2}$ floating around, seek perturbation series solution of the form $Y = Y_0(X) + \epsilon^{1/2} Y_1(X) + \epsilon Y_2 + \ldots$, but we just want leading term. To get this, solve

$$Y_0'' - Y_0 = -1, \quad Y_0(0) = 0.$$

Which has a general solution of

$$Y_0(X) = A\mathrm{e}^{-X} + B\mathrm{e}^{X} + 1.$$

Hold off on satisfying BC, until after matching. **Matching:** The matching condition is

$$\begin{aligned} \lim_{x\to 1} y_0 &= \lim_{X\to -\infty} \delta_2 Y \\ 1 &= \lim_{X\to -\infty} A\mathrm{e}^{-X} + B\mathrm{e}^{X} + 1. \end{aligned}$$

In that limit require $A = 0$, otherwise the RHS will diverge. (Also $y_{\text{overlap}} = 1$ here.)

Applying that final BC gives $B = -1$. Now have

$$Y_0(X) = 1 - \mathrm{e}^{X}$$

Composite solution:

$$\begin{aligned} y_{\text{comp}} &= y_0 + Y_0 - y_{\text{overlap}}, \\ &= \frac{1}{2 - x^2} + 1 - \mathrm{e}^{X} - 1, \\ &= \frac{1}{2 - x^2} - \mathrm{e}^{(x-1)/\sqrt{\epsilon}} \end{aligned}$$

This illustrated two interesting points, namely that

- To obtain the outer solution we didn't even have to solve a differential equation!
- The form of the inner solution was in powers of $\epsilon^{1/2}$.

Taken as a whole, these ODE examples illustrate that there is no one-size-fits-all method for analysing boundary layers. Rather the issues around inner solutions, distinguished limits and matching must all be carefully considered to develop a consistent, coherent picture of a given problem.

Can these leading-order approximations be improved?

3.1.5 Higher-order matching

Our technique for matching the inner and outer solution of leading boundary layer problems was to use a matching condition of the form:

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x).$$

If we include more terms in these perturbation series we use a more general procedure/rule known as Van Dyke matching.

The same Van Dyke who wrote *Perturbation methods in fluid mechanics*

This rule is concisely (but rather confusingly) stated:

The m-term inner expansion of the n-term outer solution

matches with

the n-term outer expansion of the m-term inner solution.

This statement doesn't really make sense on its own, but describes a very general matching procedure. The first line means:

1. Find n terms in the outer solution.
2. Rewrite this expression in terms of the inner variable.
3. Expand/simplify the expression as appropriate.
4. Retain the first m terms.

Similarly, the third line tells us to

1. Find m terms in the inner solution.
2. Rewrite this expression in terms of the outer variable.
3. Expand/simplify the expression as appropriate.
4. Retain the first n terms.

Matching involves equating the two expression to determine any unknown coefficients.

Let's see how this works in practice by considering an example,

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x, \quad \epsilon\to 0, \tag{3.3}$$

subject to $y(0) = -2$ and $y(1) = 1$, for $0 \le x \le 1$.

Construct higher-order matched solution to this boundary layer

problem. Outer problem:

$$\begin{aligned}
\mathcal{O}(1): &\quad -y_0' + y_0 = 2x, \quad y_0(0) = -2, y_0(1) = 1 \\
\mathcal{O}(\epsilon): &\quad y_0'' - y_1' + y_1 = 0, \quad y_1(0) = 0, y_1(1) = 0
\end{aligned}$$

To leading order:

$$y_0 = A_0 e^x + 2x + 2,$$

which gives

$$-y_1' + y_1 = -A_0 e^x \implies y_1 = A_1 e^x + A_0 x e^x$$

Inner problem. Let $x = x^* + \delta_1 X$ and $y = \delta_2 Y$. BCs immediately give $\delta_2 = 1$. Get:

$$\begin{aligned}
&\epsilon \frac{1}{\delta_1^2} \frac{d^2 Y}{dX^2} - \frac{1}{\delta_1} \frac{dY}{dX} + Y = 2(x^* + \delta_1 X), \quad \epsilon \to 0, \\
&\epsilon \frac{d^2 Y}{dX^2} - \delta_1 \frac{dY}{dX} + \delta_1^2 Y = \delta_1^2 2(x^* + \delta_1 X)
\end{aligned}$$

Easy to spot that distinguished limit for $\delta_1 = \epsilon$. Ie.

$$\frac{d^2 Y}{dX^2} - \frac{dY}{dX} + \epsilon Y = \epsilon 2(x^* + \epsilon X)$$

Gives:

$$\begin{aligned}
\mathcal{O}(1): &\quad Y_0'' - Y_0' = 0, \quad Y_0(0) = -2 \text{ or } Y_0(0) = 1 \\
\mathcal{O}(\epsilon): &\quad Y_1'' - Y_1' + Y_0 = 2x^*, \quad Y_1(0) = 0.
\end{aligned}$$

Leading order:

$$Y_0' - Y_0 = B_0 \implies Y_0 = C_0 e^X - B_0$$

Diverges if the outer limit is $X \to +\infty$, so require $X \to -\infty$ ie. the boundary layer is on the right at $x^* = 1$. Applying the BC gives $1 = C_0 - B_0 \implies B_0 = C_0 - 1$, ie. $Y_0 = C_0(e^X - 1) + 1$, The $\mathcal{O}(\epsilon)$ inner equation becomes

$$\begin{aligned}
&Y_1'' - Y_1' = -C_0(e^X - 1) + 1, \\
&\implies Y_1 = C_1 e^X + B_1 - C_0 X(e^X + 1) - X, \\
&\qquad = C_1(e^X - 1) - C_0 X(e^X + 1) - X,
\end{aligned}$$

Applying BCs on outer solution gives $A_0 = -4$ and $A_1 = 0$. The outer and inner solutions are:

$$\begin{aligned}
y_{\text{out}(2)}(x) &= -4e^x + 2x + 2 - \epsilon 4 x e^x, \\
Y_{\text{in}(2)}(X) &= C_0(e^X - 1) + 1 + \epsilon \Big(C_1(e^X - 1) - C_0 X(e^X + 1) - X \Big).
\end{aligned}$$

for $\epsilon \to 0$. The coefficient C_0 and C_1 now need to be determined by matching. If only interested in leading order can still use:

$$\lim_{x\to 1} y_0 = \lim_{X\to -\infty} Y_0 \implies -4\mathrm{e}+4 = -C_0+1 \implies C_0 = 4\mathrm{e}-3$$

Equivalent to doing the Van Dyke matching with $m=1$ and $n=1$. Now let's do this using Van Dyke matching rule with $m=2$ and $n=2$. The outer expansion to order $n=2$ is rewritten in terms of the inner variable:

$$\begin{aligned}
-4\mathrm{e}^x + 2x + 2 - \epsilon 4x\mathrm{e}^x &= -4\mathrm{e}^{1+\epsilon X} + 2(1+\epsilon X) + 2 \\
&\quad - \epsilon 4(1+\epsilon X)\mathrm{e}^{1+\epsilon X} \\
\text{(expand etc.)} \quad &= -4\mathrm{e}(1+\epsilon X + ...) + 2(1+\epsilon X) + 2 \\
&\quad - \epsilon 4\mathrm{e}(1+\epsilon X)(1+\epsilon X + ...) \\
&= -4\mathrm{e} + 4 + \epsilon\left(-4\mathrm{e}X + 2X - 4\mathrm{e}\right) \\
&\quad + ... \\
\text{(retain only 2 terms)} \quad y_{\mathrm{out}(2,2)}(X) &= -4\mathrm{e} + 4 + \epsilon\left(-4\mathrm{e}X + 2X - 4\mathrm{e}\right)
\end{aligned}$$

The inner expansion to order $m=2$ is rewritten in terms of the outer variable is:

$$\begin{aligned}
&C_0(\mathrm{e}^X - 1) + 1 + \epsilon\left(C_1(\mathrm{e}^X - 1) - C_0X(\mathrm{e}^X + 1) - X\right) \\
&= C_0(\mathrm{e}^{(x-1)/\epsilon} - 1) + 1 \\
&\quad + \epsilon\left(C_1(\mathrm{e}^{(x-1)/\epsilon} - 1) - C_0\frac{x-1}{\epsilon}(\mathrm{e}^{(x-1)/\epsilon} + 1) - \frac{x-1}{\epsilon}\right) \\
&= -C_0 + 1 + \epsilon\left(-C_1 - (C_0+1)\frac{x-1}{\epsilon}\right) \\
&\quad + ... \text{(exponentially small terms)} \\
&= -C_0 + 1 - (C_0+1)(x-1) + \epsilon\left(-C_1\right) + ... \\
Y_{\mathrm{in}(2,2)}(x) &= -C_0 + 1 - (C_0+1)(x-1) + \epsilon\left(-C_1\right) \text{(retain only 2 terms)}
\end{aligned}$$

The matching rule says the two expression we've just developed must match, ie. be equal:

$$\begin{aligned}
y_{\mathrm{out}(2,2)}(X) &= Y_{\mathrm{in}(2,2)}(x) \\
-4\mathrm{e} + 4 + \epsilon\left(-4\mathrm{e}X + 2X - 4\mathrm{e}\right) &= -C_0 + 1 - (C_0+1)(x-1) + \epsilon\left(-C_1\right) \\
-4\mathrm{e} + 4 + \epsilon\left(-(4\mathrm{e}-2)X - 4\mathrm{e}\right) &= -C_0 + 1 + \epsilon\left(-(C_0+1)X - C_1\right)
\end{aligned}$$

Equating orders gives $C_0 = 4\mathrm{e} - 3$ as before. At $\mathcal{O}(\epsilon)$ have

$$\begin{aligned}
-(4\mathrm{e}-2)X - 4\mathrm{e} &= -(C_0+1)X - C_1 \\
-(4\mathrm{e}-2)X - 4\mathrm{e} &= -(4\mathrm{e}-2)X - C_1 \\
&\implies C_1 = 4\mathrm{e}
\end{aligned}$$

We've already calculated the overlap, it was $y_{\text{overlap}} = y_{\text{out}(2,2)}(X)$. The composite solution is then:

$$\begin{aligned}
y_{\text{comp}(2,2)} &= y_{\text{out}(2)}(x) + Y_{\text{in}(2)}(X) - y_{\text{overlap}} \\
&= -4e^{x} + 2x + 2 - \epsilon 4xe^{x} \\
&\quad + (4e-3)(e^{X}-1) + 1 \\
&\quad + \epsilon\left(-4e(e^{X}-1) - (4e-3)X(e^{X}+1) - X\right) \\
&\quad - [-4e + 4 + \epsilon(-4eX + 2X + 4e)] \\
&= -4e^{x} + 2x + 2 - \epsilon 4xe^{x} \\
&\quad + (4e-3)e^{X} - 4e + 4 + \epsilon\left(-4e^{X+1} + 4e - (4e-3)X(e^{X})\right. \\
&\quad \left. -4eX + 2X\right) - [-4e + 4 + \epsilon(-4eX + 2X + 4e)] \\
&= -4e^{x} + 2x + 2 + (4e-3)e^{X} \\
&\quad + \epsilon\left(4xe^{x} - 4e^{X+1} - (4e-3)Xe^{X}\right)
\end{aligned}$$

This example raised some key points which often come up in this matching procedure:

- neglect of exponentially small terms in the inner solution expressed in terms of the outer variable. In this example these were of the form $e^{(x-1)/\epsilon}$ if there was a boundary layer at $x = 0$ these might look like $e^{x/\epsilon}$;
- this is a self checking method: we found one matching coefficient from the leading order match, then found it again in the $\mathcal{O}(\epsilon)$ match;
- we determined the overlap behaviour along the way, as for leading order matching the overlap **function** is just one side of the matching criterion.

This type of matching works for a great many problems but not all problems; notably those involving $\log \epsilon$ are tricky. An approach to such problems is outlined in Hinch, Chapter 5.2.

Now let's look again at a previous example and do a match correct up to ϵ^2.

$$\epsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} = \cos x, \quad \epsilon \to 0, \tag{3.4}$$

subject to $y(0) = 2$ and $y(\pi) = -1$, over $0 \leq x \leq \pi$. Recall there was a boundary layer at $x = 0$ and we rescaled with $x = \epsilon X$, $y = Y$.

Find composite solution correct to ϵ^2. Outer problem:

$$\begin{aligned}
\mathcal{O}(1): &\quad y_0' = \cos x, \quad y_0(\pi) = -1 \\
\mathcal{O}(\epsilon): &\quad y_0'' + y_1' = 0, \quad y_1(\pi) = 0, \\
\mathcal{O}(\epsilon^2): &\quad y_1'' + y_2' = 0, \quad y_2(\pi) = 0,
\end{aligned}$$

Gives:

$$y_{\text{out}} = \sin x - 1 + \epsilon(-\cos x - 1) - \epsilon^2 \sin x + \mathcal{O}(\epsilon^3)$$

Inner problem:

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \frac{\mathrm{d}Y}{\mathrm{d}X} = \epsilon \cos(\epsilon X) = \epsilon \left(1 - \epsilon^2 \frac{1}{2} X^2 + \ldots\right)$$

Gives:

$$\begin{aligned}
\mathcal{O}(1): &\quad Y_0'' + Y_0' = 0, \quad Y_0(0) = 2 \\
\mathcal{O}(\epsilon): &\quad Y_1'' + Y_1' = 1, \quad Y_1(0) = 0, \\
\mathcal{O}(\epsilon^2): &\quad Y_2'' + Y_2' = 0, \quad Y_2(0) = 0.
\end{aligned}$$

Solving that gives:

$$Y_{\text{in}}(X) = 2 + C_0(\mathrm{e}^{-X} - 1) + \epsilon(C_1(\mathrm{e}^{-X} - 1) + X) + \epsilon^2 C_2(\mathrm{e}^{-X} - 1) + \mathcal{O}(\epsilon^3)$$

Matching to determine C_0, C_1 and C_2. Look at:

$$\begin{aligned}
y_{\text{out}(3)}(X) &= \sin(\epsilon X) - 1 + \epsilon(-\cos(\epsilon X) - 1) - \epsilon^2 \sin(\epsilon X) \\
&= \left(\epsilon X - \frac{1}{6}\epsilon^3 X^3 + \ldots\right) - 1 \\
&\quad + \epsilon \left[-\left(1 - \frac{1}{2}\epsilon^2 X^2 + \ldots\right) - 1\right] \\
&\quad + \epsilon^2 (\epsilon X + \ldots), \\
y_{\text{out}(3,3)}(X) &= -1 + \epsilon(X - 2) \\
y_{\text{out}(3,3)}(x) &= -1 + x - 2\epsilon
\end{aligned}$$

This is also the overlap behaviour for the composite etc. Continuing sub $X = x/\epsilon$:

$$\begin{aligned}
Y_{\text{in}(3)} &= 2 + C_0(\mathrm{e}^{-x/\epsilon} - 1) + \epsilon(C_1(\mathrm{e}^{-x/\epsilon} - 1) + \frac{x}{\epsilon}) + \epsilon^2 C_2(\mathrm{e}^{-x/\epsilon} - 1) \\
&= 2 + C_0(-1) + \epsilon(C_1(-1) + \frac{x}{\epsilon}) + \epsilon^2 C_2(-1) + \ldots \text{ (e.s.t.)}
\end{aligned}$$

$$Y_{\text{in}(3,3)}(x) = 2 - C_0 + x - \epsilon C_1 - \epsilon^2 C_2$$

Matching then gives

$$
\begin{aligned}
y_{\text{out}(3,3)}(x) &= Y_{\text{in}(3,3)}(x) \\
-1 + x - 2\epsilon &= 2 - C_0 + x - \epsilon C_1 - \epsilon^2 C_2,
\end{aligned}
$$

So we have that: $2 - C_0 = -1 \implies C_0 = 3$ (as previously), $C_1 = 2$ and $C_2 = 0$. Getting $C_2 = 0$ is fine. Note that the x term is consistent in both expression (this is a good check). Composite is:

$$
\begin{aligned}
y_{\text{comp}(3,3)} &= y_{\text{out}(3)} + Y_{\text{in}(3)} - y_{\text{overlap}} \\
&= \sin x - 1 + \epsilon(-\cos x - 1) - \epsilon^2 \sin x \\
&\quad + 2 + 3(\mathrm{e}^{-X} - 1) + \epsilon(2(\mathrm{e}^{-X} - 1) + X) \\
&\quad - (-1 + x - 2\epsilon), \\
&= \sin x - 1 + 3\mathrm{e}^{-x/\epsilon} \\
&\quad + \epsilon(-1 - \cos x + 2\mathrm{e}^{-x/\epsilon}) \\
&\quad - \epsilon^2 \sin x.
\end{aligned}
$$

3.1.6 WKBJ approximation for boundary layers

An alternate way of solving boundary layer problems is to apply ansatz of the form:

$$
y(x) \sim \sum_{n=0}^{\infty} u_n(x)\epsilon^n + \mathrm{e}^{-F(x)/\epsilon} \sum_{n=0}^{\infty} v_n(x)\epsilon^n \tag{3.5}
$$

which is known as a **WKB ansatz** (sometime written in a variety of different forms). Those initials stand for Wentzel, Kramers and Brillouin. The naming of this is a bit contentious, sometimes Jeffreys is also credited and the acronym becomes either **WKBJ** or **JWKB**. It's also sometimes credited to some combination of Louiville, Green, Rayleigh or Carlini; in physics slightly less general versions are referred to as either geometric or physical optics. **These are all (essentially) the same thing.**

It's easy to see why such an approximation might be useful for looking at boundary layers: there's a exponential with a $1/\epsilon$ dependency built-in, which is what we saw in the inner solution to previous examples. Consider the following ODE:

$$
\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (2x + 1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0, \tag{3.6}
$$

with $\epsilon \to 0$ over $0 \le x \le \pi$, and the boundary conditions

$$
y(0) = 2, \qquad y(1) = 1.
$$

The leading-order solution by matched asymptotics (you might like to check this) is:

$$y_{\text{comp}}(x) = \frac{3}{1+2x} - e^{-x/\epsilon} \tag{3.7}$$

Lets see if we can 'improve' on that using the a WKB approximation.

Find leading order WKB solution, and compare with matched. For simplicity, let's reduce that ansatz to leading order:

$$\begin{aligned}
y &\sim u_0 + e^{-F/\epsilon} v_0, \\
y' &\sim u_0' + e^{-F/\epsilon} v_0' - \frac{F'}{\epsilon} e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots), \\
y'' &\sim u_0'' + e^{-F/\epsilon} v_0'' - 2\frac{F'}{\epsilon} e^{-F/\epsilon} v_0' + \left(\frac{F'}{\epsilon}\right)^2 e^{-F/\epsilon}(v_0 + \epsilon v_1 + \ldots) \\
&\quad - \frac{F''}{\epsilon} e^{-F/\epsilon} v_0.
\end{aligned}$$

Substituting in the WKB ansatz and equating to various orders gives:

$$\begin{aligned}
\mathcal{O}(1): &\quad 0 = (2x+1)u_0' + 2u_0, \\
\mathcal{O}(e^{-F(x)/\epsilon}\epsilon^{-1}): &\quad 0 = [F' - (2x+1)]F' v_0, \\
\mathcal{O}(e^{-F(x)/\epsilon}): &\quad 0 = (-2F' + 2x + 1)v_0' + (-F'' + 2)v_0 \\
&\qquad + [F' - (2x+1)]F' v_1
\end{aligned}$$

We 'know' the boundary layer is at $x = 0$, and we'd like u_0 and v_0 to balance there as per the composite solution, that is $e^{-F/\epsilon} = \mathcal{O}(1)$, so $F(0) = 0$. For F to be non-trivial the second equation then gives:

$$F' = 2x + 1,$$

which, applying $F(0)$, give $F(x) = x^2 + x$.

Solving the first equation and imposing the condition that $u_0(1) = 1$ gives:

$$u_0(x) = \frac{3}{2x+1}.$$

which is the outer solution.

The final equation is then:

$$-(2x+1)v_0' = 0.$$

So v_0 is just a constant here. Applying the final boundary condition $u_0(0) + v_0 = 2$ gives $v_0 = -1$. The solution is then:

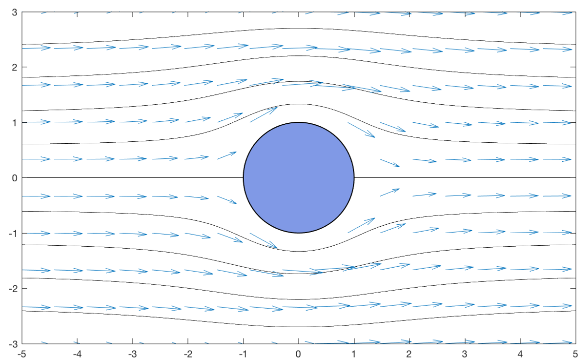
$$y_{\mathrm{WKB}}(x) = \left(\frac{3}{2x+1} - \mathrm{e}^{-(x^2+x)/\epsilon} \right) (1 + \mathcal{O}(\epsilon)).$$

This has slightly more detail in the second term that the matched version. [MATLAB: plot against each other to see that this is a slightly improvement]

3.2 Partial differential equations

3.2.1 Heat transfer from a cylinder to a fluid in uniform flow

Recall the flow of inviscid fluid past a cylinder looked like this: ...



Let's add an extra layer of detail on top of this: what if the cylinder is hot, and is heating the fluid as it goes past. Assume that the flow is quite fast, and so the heat from the cylinder doesn't penetrate very far into the fluid before the warmed fluid is advected downstream. This results in a warm **layer** around the cylinder.

To model this, let's assume the diffusion of temperature is small compared to advection of temperature (high Peclet number), that is

$$\mathbf{u} \cdot \nabla T = \epsilon \nabla^2 T \quad \text{in } r \geq 1.$$

We're only solving for T here since we already know that

$$\mathbf{u} = \nabla \phi, \quad \phi = \left(r + \frac{1}{r} \right) \cos \theta.$$

The boundary conditions are

$$T = 1, \quad \text{on } r = 1, \qquad T \to 0, \quad \text{as } r \to \infty.$$

That is the temperature is fixed at 1 on the surface of the cylinder, and far away from the cylinder the temperature is zero.

Boundary-layer solution for this problem. Outer solution:

Try for a perturbation series solution of the form:

$$T(r,\theta) = T_0(r,\theta) + \epsilon T_1(r,\theta) + \mathcal{O}(\epsilon^2),$$

with $\epsilon \to 0$. Gives:

$$\mathcal{O}(1): \quad \mathbf{u}\cdot\nabla T_0 = 0, \quad T_0(1,\theta) = 1,\, T_0(\infty,\theta) \to 0.$$

which trivially gives $T_0(r,\theta) = 0$ (since it must be constant on streamlines) and similarly for all T_n.

Thus, there is a boundary layer at $r = 1$.

Inner solution:

Written out in gory detail, the equation is

$$\left(1 - \frac{1}{r^2}\right)\cos\theta\frac{\partial T}{\partial r} - \left(1 + \frac{1}{r^2}\right)\frac{\sin\theta}{r}\frac{\partial T}{\partial\theta} = \epsilon\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right)$$

Let's make a change of variables to focus/zoom in on the boundary layer, put $r = 1 + \delta\rho$:

$$\begin{aligned}&\left(1 - \frac{1}{(1+\delta\rho)^2}\right)\frac{\cos\theta}{\delta}\frac{\partial T}{\partial\rho} - \left(1 + \frac{1}{(1+\delta\rho)^2}\right)\frac{\sin\theta}{(1+\delta\rho)}\frac{\partial T}{\partial\theta}\\ &= \epsilon\left(\frac{1}{\delta^2}\frac{\partial^2 T}{\partial\rho^2} + \frac{1}{\delta(1+\delta\rho)}\frac{\partial T}{\partial\rho} + \frac{1}{(1+\delta\rho)^2}\frac{\partial^2 T}{\partial\theta^2}\right)\end{aligned}$$

Continuing, expand denominators to give:

$$\begin{aligned}&(2\delta\rho + \ldots)\frac{\cos\theta}{\delta}\frac{\partial T}{\partial\rho} - (2 + \ldots)\sin\theta\frac{\partial T}{\partial\theta}\\ &= \epsilon\left(\frac{1}{\delta^2}\frac{\partial^2 T}{\partial\rho^2} + \frac{1}{\delta}(1+\ldots)\frac{\partial T}{\partial\rho} + (1+\ldots)\frac{\partial^2 T}{\partial\theta^2}\right)\end{aligned}$$

Rewrite slightly so we can see what order everything is at:

$$\begin{aligned}&(2\rho + \ldots)\cos\theta\frac{\partial T}{\partial\rho} - (2 + \ldots)\sin\theta\frac{\partial T}{\partial\theta}\\ &= \frac{\epsilon}{\delta^2}\frac{\partial^2 T}{\partial\rho^2} + \frac{\epsilon}{\delta}(1+\ldots)\frac{\partial T}{\partial\rho} + \epsilon(1+\ldots)\frac{\partial^2 T}{\partial\theta^2}\end{aligned}$$

Look at possible balances:

- $\delta = 1$ gives the outer solution (already seen what's happens there)
- $\delta = \epsilon^{1/2}$ brings in the first RHS term only without introducing contradictions ✓

Setting $\delta = \epsilon^{1/2}$ and introducing the inner perturbation series

$$T = \hat{T}_0(\rho, \theta) + \epsilon^{1/2}\hat{T}_1(\rho, \theta) + \mathcal{O}(\epsilon)$$

as $\epsilon \to 0$ gives to leading order:

$$2\rho\cos\theta\frac{\partial \hat{T}_0}{\partial \rho} - 2\sin\theta\frac{\partial \hat{T}_0}{\partial \theta} = \frac{\partial^2 \hat{T}_0}{\partial \rho^2}$$

subject to $\hat{T}_0 = 1$ on $\rho = 0$ (BC) and $\hat{T}_0 \to 0$ as $\rho \to \infty$ (matching condition).

This looks complicated to solve, but luckily there is a similarity solution of the form:

$$\hat{T}_0 = f(\eta), \quad \eta = \frac{\rho\sin\theta}{(1+\cos\theta)^{1/2}}.$$

Making this change of variable will let us write down a closed form solution to the PDE. Without going into any detail about where this comes from, but let's check that it works.Leading order-equation becomes:

$$\begin{aligned}
\frac{2\rho\cos\theta\sin\theta}{(1+\cos\theta)^{1/2}}f' - \frac{2\sin\theta}{(1+\cos\theta)^{1/2}}\left(\rho\cos\theta + \frac{\rho\sin^2\theta}{2(1+\cos\theta)}\right)f' &= \frac{\sin^2\theta}{1+\cos\theta}f'' \\
-\frac{2\sin\theta}{(1+\cos\theta)^{1/2}}\left(\frac{\rho\sin^2\theta}{2(1+\cos\theta)}\right)f' &= \frac{\sin^2\theta}{1+\cos\theta}f'' \\
-\frac{\rho\sin\theta}{(1+\cos\theta)^{1/2}}f' &= f'' \\
-\eta f' &= f''.
\end{aligned}$$

That's a separable ODE (!!) with solution:

$$f(\eta) = A\int_\eta^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u + B.$$

Applying the matching condition $f \to 0$ as $\eta \to \infty$ gives $B = 0$.

Applying the BC $f = 1$ on $\eta = 0$ (ie. $r = 1$) gives $A = \sqrt{2/\pi}$.

The leading order solution is then:

$$\hat{T}_0 = \sqrt{\frac{2}{\pi}}\int_\eta^\infty \mathrm{e}^{-u^2/2}\mathrm{d}u, \quad \eta = \epsilon^{-1/2}\frac{(r-1)\sin\theta}{(1+\cos\theta)^{1/2}}.$$

Plotting this suggests that we might need to be more careful.

- η diverges as $\theta \to \pi$.
- $\eta = 0$ for $\theta = 0$, regardless of r (that is $\hat{T}_0(r, 0) = 1$).

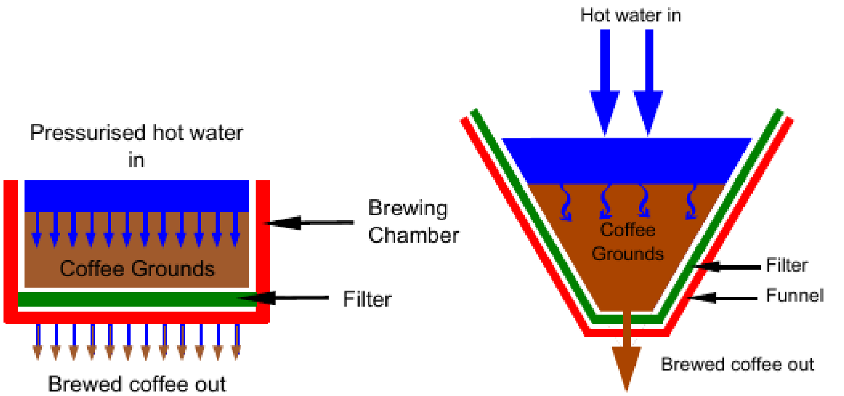
Teasing this out is part of Assignment 3.

3.2.2 Case study 2: boundary layers for coffee brewing

A very important area of applied mathematics is 'industrial mathematics'. This (often) involves working with a partner company to address a question of interest; perhaps to better understand or optimise an industrial process. Such processes often have a lot of moving parts, and quantifying the relationship between these is where modelling can help.

The following paper deals with one such application (some would say the most important application): brewing coffee.

Moroney, K.M., et al, *Asymptotic analysis of the dominant mechanisms in the coffee extraction process*, **76**(6):2196–2217, SIAM. J. Appl. Math., 2016.



This is a 'double porosity' model. Here, the coffee grinds as a porous material, then as an extra level of detail each individual grind is also thought of as porous.

The general idea is that the coffee concentration can be tracked as it come out of the very fine scale and then is transported through the porous bed (and then eventually drunk by a heroic mathematician).

Mathematically, the resulting model is a singularly perturbed PDE problem which is solved via matched asymptotic expansions.

Go through the analysis of the singular perturbation problem ('boundary layer' in time).

4 Multiple scales

Boundary layers problems typically feature behaviours on two different scales, acting in different regions of a solution domain (the outer and inner regions). Many problems of interest feature two different scales occuring simultaneously.

Some examples of real-world phenomena which such methods might be useful to model include:

- mechanical properties things like plywood, sponges or fibreglass, whose overall behaviour depends on their small scale structure;
- biological tissues which are comprised of much smaller scale cells that endow it with various functions and properties;
- propagation of waves through a region which contains small obstacles.

The class of asymptotic technique used to model such problems are variously described as **multiscale methods**, or equivalently **the method of multiple scales**.

The key idea is to assume that a solution, say $y(t)$, is of the form $y(t) \equiv y(\tau, T)$, where $\tau = t$ and $T = \epsilon T$ represent the fast and slow scales involved in the problem. Under this change of variables, it follow by the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\partial}{\partial\tau} + \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\partial}{\partial T} = \frac{\partial}{\partial\tau} + \epsilon\frac{\partial}{\partial T}$$

This then permits analysis of problems which are not otherwise amenable to the techniques of the perturbation problems already considered.

As a first example, let's look again at the Van der Pol oscillator.

4.1 Van der Pol oscillator

The Van der Pol equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \epsilon(y^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0, \tag{4.1}$$

with $\epsilon \ll 1$, $y(0) = 1$ and $y'(0) = 0$. We will see that for small ϵ the solution to this equation displays behaviour on two time scales simultaneously: rapid oscillations (fast scale) combined with a gradual change in amplitude (slow scale). Let's start our analysis by trying a regular perturbation expansion.

Regular perturbation solution of the Van der Pol oscillator

Let $y(t) = y_0(t) + \epsilon y_1(t) + \dots$ to get:

$$\begin{aligned}
\mathcal{O}(1): &\quad y_{0tt} + y_0 = 0, && y_0(0) = 1, y_0'(0) = 0, \\
\mathcal{O}(\epsilon): &\quad y_{1tt} + y_1 = -(y_0^2 - 1)y_{0t}. && y_1(0) = 0, y_1'(0) = 0.
\end{aligned}$$

Solving the $\mathcal{O}(1)$ problem gives $y_0(t) = \cos t$. Then at $\mathcal{O}(\epsilon)$ have:

$$\begin{aligned}
y_{1tt} + y_1 &= (\cos^2 t - 1)\sin t, \\
&= -\sin^3 t, \\
&= \frac{1}{4}\sin(3t) - \frac{3}{4}\sin t
\end{aligned}$$

Solving that (by variation of parameters, for example) gives:

$$y_1 = \frac{3}{8} t\cos t - \frac{9}{32} - \frac{1}{32}\sin(3t)$$

This solution works for early times but at later times the first term in y_1 grows like t, and so we lose the ordering $\epsilon y_1 \ll y_0$.

The regular perturbation series solution has an $\mathcal{O}(\epsilon)$ term with a component proportional to t, as so grows unboundedly with time. **But** a numerical solution and/or integration of the original equation demonstrates that the solution is bounded as $t \to \infty$.

This behaviour is due to **secular** or **resonant** terms in the governing equations, namely terms that upon integration lead to this erroneous growth (typically featuring oscillations like those seen at leading-order). Multiple scale analysis can help us get around this difficulty.

Multiple scales analysis of the Van der Pol equation. Now let y depend on a fast time scale $\tau = t$ and a slow time scale $T = \epsilon t$, that is $y(t) \equiv y(\tau, T)$ and so

$$\begin{aligned}
y_t &= y_\tau + \epsilon y_T, \\
y_{tt} &= y_{\tau\tau} + 2\epsilon y_{\tau T} + \epsilon^2 y_{TT}.
\end{aligned}$$

The Van der Pol equation is then converted into a PDE, namely:

$$y_{\tau\tau} + 2\epsilon y_{\tau T} + \epsilon^2 y_{TT} + \epsilon(y^2 - 1)(y_\tau + \epsilon y_T) + y = 0,$$

subject to initial conditions $y(0,0) = 1$ and $y_\tau(0,0) + \epsilon y_T(0,0) = 0$.

Now let $y(\tau, T) = y_0(\tau, T) + \epsilon y_1(\tau, T) + \dots$, this gives

$$\begin{aligned}
\mathcal{O}(1): &\quad y_{0\tau\tau} + y_0 = 0, && y_0(0,0) = 1, y_{0\tau}(0,0) = 0, \\
\mathcal{O}(\epsilon): &\quad y_{1\tau\tau} + y_1 = -(y_0^2 - 1)y_{0\tau} - 2y_{0\tau T}, && y_1(0,0) = 0, y_{1\tau}(0) + y_{0T} = 0.
\end{aligned}$$

Our approach here is going to be to solve the leading order problem, and extract some useful information from the $\mathcal{O}(\epsilon)$ problem.

One form (is this the most convenient?) of the general solution to the leading-order equation is:

$$y_0(\tau, T) = R(T)\cos(\tau + \theta(T)),$$

where $R(T)$ and $\phi(T)$ need to be determined. The $\mathcal{O}(\epsilon)$ equation is then:

$$\begin{aligned}
y_{1\tau\tau} + y_1 &= -(y_0^2 - 1)y_{0\tau} - 2y_{0\tau T},\\
&= R\sin(\tau+\theta)(R^2\cos^2(\tau+\theta) - 1) + 2(R\sin(\tau+\theta))_T\\
&= R\sin(\tau+\theta)(R^2(1-\sin^2(\tau+\theta)) - 1)\\
&\quad + 2(R'\sin(\tau+\theta) + R\theta'\cos(\tau+\theta))\\
&= (R^3 - R + 2R')\sin(\tau+\theta) + R\theta'\cos(\tau+\theta) - R^3\sin^3(\tau+\theta)\\
&= (R^3 - R + 2R')\sin(\tau+\theta) + R\theta'\cos(\tau+\theta)\\
&\quad + (1/4)R^3(-4\sin^3(\tau+\theta) + 3\sin(\tau+\theta)) - (3/4)R^3\sin(\tau+\theta)\\
&= ((1/4)R^3 - R + 2R')\sin(\tau+\theta) + R\theta'\cos(\tau+\theta)\\
&\quad + (1/4)R^3\sin(3(\tau+\theta))
\end{aligned}$$

We've ended up with

$$\begin{aligned}
y_{1\tau\tau} + y_1 = &\boxed{((1/4)R^3 - R + 2R')}\sin(\tau+\theta) + \boxed{R\theta'}\cos(\tau+\theta)\\
&+ (1/4)R^3\sin(3(\tau+\theta)).
\end{aligned}$$

The presence of the $\sin(\tau+\theta)$ and $\cos(\tau+\theta)$ terms will lead to resonant behaviour (that is, growth in amplitude) in the solution. Terms like this force the equation at its natural frequency and so will lead to the behaviour we saw in the regular perturbation analysis.

However, R and θ are yet to be determined, so we can eliminate the resonant terms by setting:

$$(1/4)R^3 - R + 2R' = 0 \qquad \text{and} \qquad R\theta' = 0.$$

Just a couple of ODEs! These have (non-trivial) solution:

$$R(T) = \frac{2}{(1+\alpha e^{-T})^{1/2}}, \qquad \theta(T) = \beta, \qquad \alpha, \beta \text{ constant.}$$

Finally, the remaining constant are determined from the initial conditions. Let's do this carefully. The derivative condition was:

$$y_{0\tau}(0,0) = 0 \implies -\frac{2}{\sqrt{1+\alpha}}\sin(\beta) = 0 \implies \sin(\beta) = 0$$

so $\beta = 0$. The other IC is then

$$y_0(0,0) = 1 \implies \frac{2}{\sqrt{1+\alpha}}\cos(0) = 1 \implies \frac{2}{\sqrt{1+\alpha}} = 1,$$

so $\alpha = 3$. The leading order solution is then

$$\begin{aligned} y_0(\tau, T) &= \frac{2}{(1+\alpha e^{-T})^{1/2}}\cos(\tau), \\ \text{or} \quad y_0(t) &= \frac{2}{(1+\alpha e^{-\epsilon t})^{1/2}}\cos(t). \end{aligned}$$

This accurately captures the slow change in the amplitude of the oscillations for small ϵ. It doesn't capture the phase shift at later times - that requires an additional extra slow time scale.

The key steps in that solution procedure (which work for similar problems) were that:

- Make a change of variables to account for the fast and slow scales; apply to governing equation (in this case turning and ODE into a PDE);
- Removal of resonant terms from the next-to leading order governing equation (key in this sort of analysis).

That process featured some slightly unwieldy algebra, luckily we can use an equivalent analysis that is a bit more straightforward.

Alternate derivation of the same result. A different representation of the leading order solution is:

$$y_0(\tau, T) = A(T)e^{i\tau} + \overline{A(T)}e^{-i\tau},$$

where the overbar denotes a complex conjugate. This avoid mucking around with the trigonometric identities in the $\mathcal{O}(\epsilon)$ equation:

$$\begin{aligned} y_{1\tau\tau} + y_1 &= -(y_0^2 - 1)y_{0\tau} - 2y_{0\tau T}, \\ &= -(A^2e^{i2\tau} + 2A\overline{A} + \overline{A}^2e^{-i2\tau} - 1)i(Ae^{i\tau} - \overline{A}e^{-i\tau}) \\ &\quad - 2i(A'e^{i\tau} + \overline{A}'e^{-i\tau}), \\ &= -i(A^2\overline{A} - A + 2A')e^{i\tau} - i(-A\overline{A}^2 + A2\overline{A}')e^{-i\tau} \\ &\quad - iA^3e^{i3\tau} + i\overline{A}^3e^{-i3\tau} \end{aligned}$$

Here the $e^{i\tau}$ and $e^{-i\tau}$ terms are resonant. Note that their coefficients are complex conjugates so if one is zero then so is the other.

We now have:

$$y_{1\tau\tau} + y_1 = -\mathrm{i}\boxed{(A^2\overline{A} - A + 2A')}\,\mathrm{e}^{\mathrm{i}\tau} - \mathrm{i}\boxed{(-A\overline{A}^2 + A2\overline{A}')}\,\mathrm{e}^{-\mathrm{i}\tau}$$
$$- \mathrm{i}A^3\mathrm{e}^{\mathrm{i}3\tau} + \mathrm{i}\overline{A}^3\mathrm{e}^{-\mathrm{i}3\tau}$$

To eliminate those terms, choose:

$$A^2\overline{A} - A + 2A' = 0.$$

Setting $A = \rho(T)\mathrm{e}^{\mathrm{i}\theta(T)}$ gives:

$$\rho^2\mathrm{e}^{2\mathrm{i}\theta(T)}\rho\mathrm{e}^{-\mathrm{i}\theta(T)} - \rho\mathrm{e}^{\mathrm{i}\theta(T)} + 2(\rho' + \mathrm{i}\theta'\rho)\mathrm{e}^{\mathrm{i}\theta(T)} = 0,$$
$$\rho^3 - \rho + 2\rho' + \mathrm{i}\theta'\rho = 0.$$

Taking the real and imaginary parts gives:

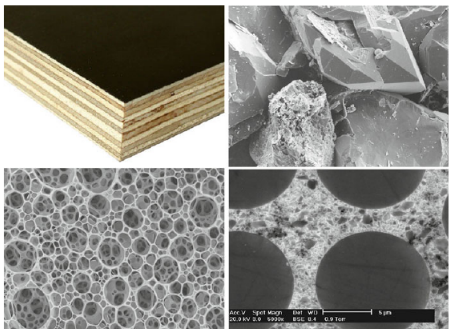
$$\rho^3 - \rho + 2\rho' = 0, \qquad \rho\theta' = 0.$$

and set $\rho = \frac{1}{2}R$ to get exactly what we had before:

$$(1/4)R^3 - R + 2R' = 0, \qquad R\theta' = 0.$$

4.2 Asymptotic homogenisation

Another popular use of the method of multiple scales is the derive effective governing equations for the behaviour of systems which feature complicated small-scale behaviour.



Asymptotic homogenisation involves converting our problem on that explicitly involves multiple scales, for instance by introducing

- the **macroscale** variable, x
- the **microscale** vairable, $X = x/\epsilon$.

We write our problem as a PDE in terms of these two variables (as in the previous oscillator problem). Homogenisation is then a formal procedure is to explicitly eliminate the microscale variation and derive an **effective** governing equation(s) in terms of the macroscale variable only. Information from the microscale is used to determine various coefficients of the effective equation(s).

Homogenisation techniques are especially useful for problems in mechanics - think perhaps of something like flow through a sponge. We'll focus on a toy problem from solid mechanics.

Let's introduce some basic solid mechanics first. Say we have a 1D object (think metal bar/beam) and we compress it along it's axis. The deformation (suitably non-dimensionalised) is described by:

$$E\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad u(0) = 0, u(1) = \alpha.$$

where u is **displacement** of the solid from a reference (undeformed) state, x is a Lagrangian co-ordinate along the beam and E is a Young's modulus. When E is a constant this equation is really easy to solve (linear elasticity).

What happens if E is instead a function of position, and varies across multiple scales?

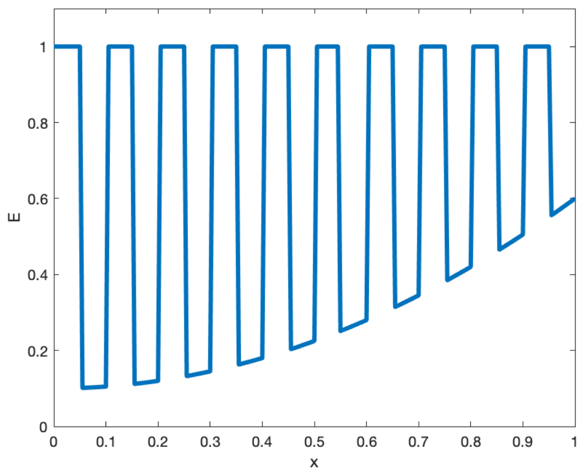
If we introduce a position dependent (and multiscale) $E(x, \xi)$, with $\xi = x/\epsilon$. our problem becomes:

$$E(x, \xi)\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad u(0) = 0, u(1) = \alpha.$$

Let's consider the specific example of

$$E(x, \xi) = \begin{cases} 1, & 0 < \xi < 1/2, \\ 0.1 + 0.5x^2, & 1/2 < \xi < 1. \end{cases}$$

Isn't this just averaging? Derivation of effective governing equation via homogenisation.



An overly simplistic approach to homogenisation might be to just average over this function ... but there are many ways to take such an average.

Multiple scales analysis will give us a rational way to do this!

We let $u(x) \equiv u(x, \xi)$ and so

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x} + \frac{1}{\epsilon}\frac{\partial u}{\partial \xi}.$$

Contrast this to the comparable expression in the VDP problem earlier - here we've added a small scale, rather than a slow scale.

The governing equation becomes:

$$E(x,\xi)\left(\frac{\partial u}{\partial x} + \frac{1}{\epsilon}\frac{\partial u}{\partial \xi}\right) = 1$$
$$\implies E(x,\xi)\left(\epsilon\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \xi}\right) = \epsilon.$$

The let $u(x, \xi) = u_0(x, \xi) + \epsilon u_1(x, \xi) + ...$, which at leading order gives:

$$\frac{\partial u_0}{\partial \xi} = 0 \qquad \implies \qquad \boxed{u_0(x,\xi) \equiv u_0(x)}$$

This looks simple but is a great result - the leading order component only varies on the macroscale!

Continuing to $\mathcal{O}(\epsilon)$, we have

$$E(x,\xi)\left(\frac{\mathrm{d}u_0}{\mathrm{d}x} + \frac{\partial u_1}{\partial \xi}\right) = 1,$$
$$\implies \quad \frac{\mathrm{d}u_0}{\mathrm{d}x} + \frac{\partial u_1}{\partial \xi} = \frac{1}{E(x,\xi)}$$

Integrating over one microscale period gives, which in terms of the microscale variable ξ goes from 0 to 1:

$$\int_0^1 \frac{\mathrm{d}u_0}{\mathrm{d}x}\mathrm{d}\xi + \int_0^1 \frac{\partial u_1}{\partial \xi}\mathrm{d}\xi = \int_0^1 \frac{1}{E(x,\xi)}\mathrm{d}\xi$$

Since u_0 is a function of x only we get:

$$\frac{\mathrm{d}u_0}{\mathrm{d}x}\int_0^1 \mathrm{d}\xi + [u_1]_{\xi=0}^{\xi=1} = \int_0^1 \frac{1}{E(x,\xi)}\mathrm{d}\xi.$$

We've assumed that u is periodic in $\xi \implies u_1(x,0) = u_1(x,1)$, so

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} + 0 = \int_0^1 \frac{1}{E(x,\xi)}\mathrm{d}\xi.$$

Rearranging this very slightly gives:

$$\frac{1}{\int_0^1 [1/E(x,\xi)]\mathrm{d}\xi} \frac{\mathrm{d}u_0}{\mathrm{d}x} = 1,$$

$$\text{or} \quad \boxed{E_{\text{eff}}(x)\frac{\mathrm{d}u_0}{\mathrm{d}x} = 1,} \qquad \text{where} \ \boxed{E_{\text{eff}}(x) = \frac{1}{\int_0^1 [1/E(x,\xi)]\mathrm{d}\xi}.}$$

Thus, the quantity $E_{\text{eff}}(x)$ is the **effective** modulus for this **homogenised** equation. Note that this is an ODE in the macroscale variable x with the microscale detail incorporated via the integral in the expression for E_{eff}. Note also that E_{eff} is the harmonic mean of $E(x,\xi)$.

Let's now calculate E_{eff} for our example.

Recall that we had

$$E(x,\xi) = \begin{cases} 1, & 0 < \xi < 1/2, \\ 0.1 + 0.5x^2, & 1/2 < \xi < 1. \end{cases}$$

Integrating the reciprocal of this over one period gives:

$$\begin{aligned} \int_0^1 [1/E(x,\xi)]\mathrm{d}\xi &= \int_0^{1/2} 1\mathrm{d}\xi + \int_{1/2}^1 \frac{1}{0.1+0.5x^2}\,\mathrm{d}\xi, \\ &= \frac{1}{2}\left(1 + \frac{1}{0.1+0.5x^2}\right) = \frac{1.1+0.5x^2}{2(0.1+0.5x^2)} \end{aligned}$$

Then the effective modulus is

$$E_{\text{eff}}(x) = \frac{1}{\int_0^1 [1/E(x,\xi)]\mathrm{d}\xi} = \frac{2(0.1+0.5x^2)}{1.1+0.5x^2}.$$

Great! See MATLAB code `???.mlx` for comparison between behaviour of the original and homogenised equation.

4.2.1 Homogenisation in 2D, with periodic microstructure

Many practical application for homogenisation involve periodic microstructure. Some practical examples might include composite materials (as we have already seen in 1D). We now extend the same ideas to more than one spatial dimension.

Another problem from solid mechanics describes antiplane strain $w(x,y)$. Say we have a 3D bar and deform it in such a way that two components of displacement are zero, the remaining component is w. This might happen if you apply a surface traction force to a bar, for instance. The strain is governed by

$$\nabla \cdot (\mu \nabla w) = 0,$$

here we have the antiplane strain $w \equiv w(x,y)$ and the shear modulus $\mu \equiv (x,y)$. We consider the case of small scale periodic structure in μ, corresponding to a composite material.

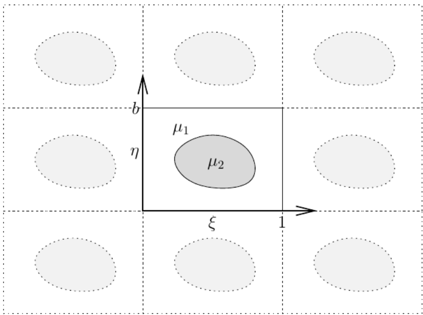
Derive effective equation for this 2D composite material. Here's an example of the type of composite mateiral we'll consider (pencils glued together).

The arrangement of the two materials is such that the shear modulus μ is periodic in cross-section and feature small-scale variation. More practically, similar arrangement in steel reinforced concrete pillars; multi-core optical fibres etc.

That is, in the governing equation

$$\nabla \cdot (\mu \nabla w) = 0,$$

we have $\mu \equiv \mu(x, y, \xi, \eta)$ where $(\xi, \eta) = (x, y)/\epsilon$. In cross-section, the microstructure might look like:



with (assumed) periodicity in ξ (period 1) and η (period b) so,

$$\mu(x, y, \xi + 1, \eta) = \mu(x, y, \xi, \eta), \qquad \mu(x, y, \xi, \eta + b) = \mu(x, y, \xi, \eta).$$

Similalry, let the displacement $w \equiv w(x, y, \xi, \eta)$ and has the same periodicity as μ. As in the previous problem we apply the chain rule to give

$$\nabla \to \nabla_x + \frac{1}{\epsilon}\nabla_\xi$$

where

$$\nabla_x = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j},$$
$$\nabla_\xi = \frac{\partial}{\partial \xi}\mathbf{i} + \frac{\partial}{\partial \eta}\mathbf{j}.$$

This transforms the governing equation to

$$\nabla_\xi \cdot (\mu \nabla_\xi w) + \epsilon\left[\nabla_\xi \cdot (\mu \nabla_x w) + \nabla_x \cdot (\mu \nabla_\xi w)\right] + \epsilon^2 \nabla_x \cdot (\mu(\nabla_x w)) = 0.$$

For the purpose of our homogenisation procedure, let's consider on microscale cell (ultimately to average over it, like we did in the 1D example). Say that our representative cell has the domain $0 < \xi < 1$ and $0 < \eta < b$.

Periodicity of w means that its ξ and η derivatives are also periodic, that is:

$$w(x,y,1,\eta) = w(x,y,0,\eta), \qquad w(x,y,\xi,b) = w(x,y,\xi,0),$$
$$\frac{\partial w}{\partial \xi}(x,y,1,\eta) = \frac{\partial w}{\partial \xi}(x,y,0,\eta), \quad \frac{\partial w}{\partial \eta}(x,y,\xi,b) = \frac{\partial w}{\partial \eta}(x,y,\xi,0).$$

This periodicity is used extensively in the following derivation (hence the emphasis).

As usual, seek a solution of the form:

$$w \sim w_0(x,y,\xi,\eta) + \epsilon w_1(x,y,\xi,\eta) + \epsilon^2 w_2(x,y,\xi,\eta) + ...$$

Substituting this into the governing equation:

$$\nabla_\xi \cdot (\mu \nabla_\xi w) + \epsilon\left[\nabla_\xi \cdot (\mu \nabla_x w) + \nabla_x \cdot (\mu \nabla_\xi w)\right] + \epsilon^2 \nabla_x \cdot (\mu(\nabla_x w)) = 0.$$

We get (remembering that $\mu \equiv \mu(x,y,\xi,\eta)$):

$$\begin{aligned}
\mathcal{O}(1): &\quad 0 = \nabla_\xi \cdot (\mu \nabla_\xi w_0),\\
\mathcal{O}(\epsilon): &\quad 0 = \nabla_\xi \cdot (\mu \nabla_\xi w_1) + \left[\nabla_\xi \cdot (\mu \nabla_x w_0) + \nabla_x \cdot (\mu \nabla_\xi w_0)\right].\\
\mathcal{O}(\epsilon^2): &\quad 0 = \nabla_\xi \cdot (\mu \nabla_\xi w_2) + \left[\nabla_\xi \cdot (\mu \nabla_x w_1) + \nabla_x \cdot (\mu \nabla_\xi w_1)\right]\\
&\qquad + \nabla_x \cdot (\mu(\nabla_x w_0)).
\end{aligned}$$

Given what we're doing here, can you spot why we've written this out to $\mathcal{O}(\epsilon^2)$?

Consider the leading-order equation:

$$0 = \nabla_\xi \cdot (\mu \nabla_\xi w_0),$$

subject to the boundary conditions

$$\begin{aligned} w_0(x,y,1,\eta) &= w_0(x,y,0,\eta), & w_0(x,y,\xi,b) &= w_0(x,y,\xi,0), \\ \frac{\partial w_0}{\partial \xi}(x,y,1,\eta) &= \frac{\partial w_0}{\partial \xi}(x,y,0,\eta), & \frac{\partial w_0}{\partial \eta}(x,y,\xi,b) &= \frac{\partial w_0}{\partial \eta}(x,y,\xi,0). \end{aligned}$$

Easy to see that to these are satisfied when w is independent of the microscale variables, that is

$$w_0 \equiv w_0(x,y).$$

Which is great! And also what we want from a homogenisation procedure.

Note that this is also the unique solution to the equation, which is important to know (but we won't show it here).

Continuing to $\mathcal{O}(\epsilon)$, we have:

$$\begin{aligned} &0 = \nabla_\xi \cdot (\mu \nabla_\xi w_1) + \nabla_\xi \cdot (\mu \nabla_x w_0) + \nabla_x \cdot (\mu \nabla_\xi w_0) \\ &0 = \nabla_\xi \cdot (\mu \nabla_\xi w_1) + \nabla_\xi \cdot (\mu \nabla_x w_0) \qquad \text{(since } w_0 = w_0(x,y)\text{)} \end{aligned}$$

Expanding and rearranging slightly gives:

$$\frac{\partial}{\partial \xi}\left(\mu \frac{\partial w_1}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\mu \frac{\partial w_1}{\partial \eta}\right) = -\frac{\partial \mu}{\partial \xi}\frac{\partial w_0}{\partial x} - \frac{\partial \mu}{\partial \eta}\frac{\partial w_0}{\partial y}$$

Two things here:

1. The form of this equation is a PDE (linear, elliptic) for w_1 driven $\partial w_0/\partial x$ and $\partial w_0/\partial y$.
2. Equation has periodic solution if RHS has zero mean, that is

$$\int_0^1 \int_0^b \left(\frac{\partial \mu}{\partial \xi}\frac{\partial w_0}{\partial x} - \frac{\partial \mu}{\partial \eta}\frac{\partial w_0}{\partial y}\right) \mathrm{d}\xi \mathrm{d}\eta = 0.$$

The solution for w_1 thus has the form:

$$w_1(x,y,\xi,\eta) = W^{(1)}(x,y,\xi,\eta)\frac{\partial w_0}{\partial x} + W^{(2)}(x,y,\xi,\eta)\frac{\partial w_0}{\partial y}.$$

Substitute this into the $\mathcal{O}(\epsilon)$ equation (and equate 'coefficients' of $\partial w_0/\partial x$ and $\partial w_0/\partial y$) to get

$$\frac{\partial}{\partial \xi}\left(\mu \frac{\partial W^{(1)}}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\mu \frac{\partial W^{(1)}}{\partial \eta}\right) = -\frac{\partial \mu}{\partial \xi},$$
$$\frac{\partial}{\partial \xi}\left(\mu \frac{\partial W^{(2)}}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\mu \frac{\partial W^{(2)}}{\partial \eta}\right) = -\frac{\partial \mu}{\partial \eta}$$

We have turned the problem into on for the functions $W^{(1)}$ and $W^{(2)}$, along with corresponding periodic boundary conditions.

This problem only depends on the microscale! It has unique periodic solutions (up to arbitrary function of x and y) and for a given $\mu(x, y, \xi, \eta)$ can be solved numerically.

Continuing to $\mathcal{O}(\epsilon^2)$ (and rearranging the original equation)

$$\begin{aligned}\nabla_\xi \cdot (\mu \nabla_\xi w_2) &= -\nabla_\xi \cdot (\mu \nabla_x w_1) - \nabla_x \cdot (\mu \nabla_\xi w_1) - \nabla_x \cdot (\mu(\nabla_x w_0)) \\ &= -\left[\frac{\partial}{\partial \xi}\left(\mu \frac{\partial w_1}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\mu \frac{\partial w_1}{\partial y}\right)\right] \\ &\quad -\left[\frac{\partial}{\partial x}\left(\mu \frac{\partial w_1}{\partial \xi}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w_1}{\partial \eta}\right)\right] \\ &\quad -\left[\frac{\partial}{\partial x}\left(\mu \frac{\partial w_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w_0}{\partial y}\right)\right]\end{aligned}$$

As for w_1, if w_2 has a periodic solution the RHS of this equation must have zero mean. We aren't going to solve for w_2, but we are going to use this fact.

Taking $\int_0^1 \int_0^b \ldots \mathrm{d}\xi \mathrm{d}\eta$ can see that the first RHS term is identically zero (similar argument to the one we skipped earlier).

We are requiring that (and slightly rearranging):

$$0 = -\int_0^1 \int_0^b \left(\frac{\partial}{\partial x}\left[\mu \frac{\partial w_1}{\partial \xi} + \mu \frac{\partial w_0}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu \frac{\partial w_1}{\partial \eta} + \mu \frac{\partial w_0}{\partial y}\right]\right) \mathrm{d}\xi \mathrm{d}\eta$$

Substituting in the solution for w_1 (recall that this was in terms of w_0), gives:

$$\begin{aligned}0 = -\int_0^1 \int_0^b &\left(\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial W^{(1)}}{\partial \xi} + 1\right)\frac{\partial w_0}{\partial x} + \mu \frac{\partial W^{(2)}}{\partial \xi}\frac{\partial w_0}{\partial y}\right]\right. \\ &\left. + \frac{\partial}{\partial y}\left[\mu \frac{\partial W^{(1)}}{\partial \eta}\frac{\partial w_0}{\partial x} + \mu\left(\frac{\partial W^{(2)}}{\partial \eta} + 1\right)\frac{\partial w_0}{\partial y}\right]\right) \mathrm{d}\xi \mathrm{d}\eta\end{aligned}$$

which looks very complicated, but can be rearranged to give ...

A homogenised eqaution for $w_0(x.y)$, namely

$$\frac{\partial}{\partial x}\left(\overline{\mu}_{11}\frac{\partial w_0}{\partial x}\right)+\frac{\partial}{\partial x}\left(\overline{\mu}_{12}\frac{\partial w_0}{\partial y}\right)+$$
$$\frac{\partial}{\partial y}\left(\overline{\mu}_{21}\frac{\partial w_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(\overline{\mu}_{22}\frac{\partial w_0}{\partial u}\right)=0,$$

where the effective moduli $\overline{\mu}_{11}$, $\overline{\mu}_{21}$, $\overline{\mu}_{12}$ and $\overline{\mu}_{22}$ come from the solution to the microscale problem:

$$\overline{\mu}_{11}=\int_0^1\int_0^b \mu\left(\frac{\partial W^{(1)}}{\partial\xi}+1\right)\mathrm{d}\eta\mathrm{d}\xi,\quad \overline{\mu}_{21}=\int_0^1\int_0^b \mu\frac{\partial W^{(1)}}{\partial\eta}\mathrm{d}\eta\mathrm{d}\xi,$$
$$\overline{\mu}_{22}=\int_0^1\int_0^b \mu\left(\frac{\partial W^{(2)}}{\partial\eta}+1\right)\mathrm{d}\eta\mathrm{d}\xi,\quad \overline{\mu}_{12}=\int_0^1\int_0^b \mu\frac{\partial W^{(2)}}{\partial\xi}\mathrm{d}\eta\mathrm{d}\xi.$$

To summarise, to actually apply this procedure for a given $\mu(x,y,\xi,\eta)$ the steps would be:

1. Solve the microscale problem to get the coefficient functions $W^{(1)}$ and $W^{(2)}$.
2. Use these functions to obtain the effective modui $\overline{\mu}_{11}$, $\overline{\mu}_{21}$, $\overline{\mu}_{12}$ and $\overline{\mu}_{22}$.
3. Solve the homogenised equation to obtain the effective macroscale behaviour.

Let's continue this example by going through the full details for a particular choice of μ.

One choice might be:

$$\mu(x,y,\xi,\eta)=(1+\sin(\pi x)\sin(\pi y))(1+0.1\sin(\pi\xi)\sin(\pi\eta))$$

Solve over a macroscale domain of $x\in[0,1]$ and $y\in[0,1]$, subject to the boundary conditions:

$$\begin{aligned} &w(0,y,\xi,\eta)=0.1\sin(\pi y), && w(1,y,\xi,\eta)=0.2\sin(\pi y),\\ &w(x,0,\xi,\eta)=0, && w(x,1,\xi,\eta)=0. \end{aligned}$$

For completeness let's start by solving the full problem (this will be computationally intensive), then homogenise via the procedure we previously outlined.

Go through homogenisation procedure for this choice of μ.

Some further points:

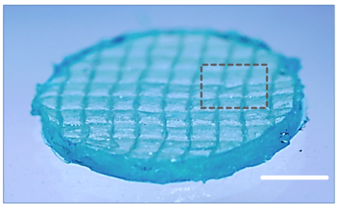
- Although it can sometimes looks like solving the microscale problem can computationally challenging - this only needs to be done once per material.

- ... plus often there are microstructure symmetries which make life easier.
- Extension to three-dimensional problems follows in much the same way, and results in a similar homogenisation procedure (ie. use a microscale problem to find effective parameters).
- A significant complication in doing this for elasticity (solid mechanics) problems is that it requires tensors (or rank 4).

Let's look at a case study from tissue engineering/soft materials to see what this looks like.

4.2.2 Case study 1: tissue engineering constructs

Our first study involves a homogenisation procedure applied to a soft composite material used in tissue engineering.



This composite consists of two materials:

1. hydrogel - good cell culture (but mechanical weak)
2. polymer fibres - 3D printed in a lattice structure

Discussion of Chen et al., EJAM 2019

4.2.3 Case study 2: Faraday cages

Next we'll look at an application that is important to shielding electromagnetic radiation, namely Faraday cages (as seen in science museums and in microwave oven doors).

Discussion of Chapman et al., SIAM Review 2015

4.2.4 Case study 3: solar panels

A common technique to manufacture solar panels results in complex microstructure in the panel. Interest lies in tuning this microstructure, and understanding how this complicated system turns light into electricity.

Discussion of Richardson et al., EJAM 2018

5 Integrals

This chapter is all about the asymptotic evaluation of integrals. In particular, those of the following forms:

$$I_1(x) = \int_a^b f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t,$$
$$I_2(x) = \int_a^b f(t)\mathrm{e}^{\mathrm{i}x\psi(t)}\mathrm{d}t,$$
$$I_3(x) = \int_C f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t.$$

Why do we care? These come up, for example, in

- Solve DEs with transforms.
- Approximating PDF from the moment generating function (not covered here).

Let's focus, as we usual do, on solving DEs. Recall that we can hit a DE and it's initial/boundary condition with a transform to get an integral representation of the solution. For instance a heat equation:

$$\psi_t = \psi_{xx}, \qquad -\infty < x < \infty, \qquad t > 0,$$

with BCs $\psi \to 0$ as $|x| \to \infty$ and initial data $\psi(x, 0) = \psi_0(x)$. Take a Fourier transform, muck around for a bit, and invert to get:

$$\psi(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b_0(k)\mathrm{e}^{ikx-k^2t}\mathrm{d}k.$$

where $b_0(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-ikx}\psi_0(x)\mathrm{d}x$.

It's lovely that we've written down a closed form solution, but what's actually going on with that integral. For instance, how does it behave for large t?

First off, use integration by parts on 'Laplace-type' integrals like

$$I(x) = \int_a^b f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t,$$

to determine the behaviour for $x \to \infty$.

Let's look at an example:

$$I(x) = \int_0^{\infty} \frac{\mathrm{e}^{-xt}}{(1+t^2)^2}\mathrm{d}t.$$

Evaluate this integral with integration by parts. Do the usual integration by part thing:

$$\begin{aligned}
I(x) &= \int_0^\infty \frac{e^{-xt}}{(1+t^2)^2} dt, \\
&= \left[\frac{e^{-xt}}{-x} \frac{1}{(1+t^2)^2} \right]_0^\infty - \int_0^\infty \frac{e^{-xt}}{-x} \frac{(-2)2t}{(1+t^2)^3} dt, \\
&= \frac{1}{x} + \mathcal{O}(1/x^3).
\end{aligned}$$

Or, converting that to an asymptotic relation:

$$\implies I(x) \sim \frac{1}{x}, \quad x \to \infty.$$

NB. find the order of the 'remainder' term by directly evaluating the integral, or making some other argument.

That was easy - keep going to generate more terms in the asymptotic series. This only work for some integrals:

- need $\phi(t)$ to be monotonic;
- can't pick up on fractional powers.

So it's easy to use, let's us approximate the error ... but doesn't always work.

Look at

$$I(x) = \int_0^\infty e^{-xt^2} dt.$$

Attempt to evaluate this integral with integration by parts. Try the usual thing:

$$\begin{aligned}
I(x) &= \int_0^\infty e^{-xt^2} dt \\
&= \int_0^\infty \left(-\frac{1}{2xt} \right) \left(-2xte^{-xt^2} \right) dt \\
&= \left[\frac{e^{-xt^2}}{-2xt} \right]_0^\infty - \int_0^\infty \frac{1}{2xt^2} e^{-xt^2} dt.
\end{aligned}$$

The first bit doesn't exist!

In fact there's an exact solution:

$$\int_0^\infty e^{-xt^2} dt = \frac{\sqrt{\pi}}{2\sqrt{x}}$$

So this is an example where integration by parts misses a fractional power. Also fails when the dominant bit of the integrand is in the interior.

One technique to evaluate integrals like

$$I(x) = \int_a^b f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t,$$

for $x \to \infty$ is by **Laplace's Method**. This method makes use of the fact that the most important contributions to the integral come from the regions where the integrand is largest. Let's do an example first by way of illustration, and then see how this works in general.

Consider the integral

$$I(x) = \int_0^{10} \frac{\mathrm{e}^{-xt}}{(1+t)}\mathrm{d}t,$$

let's find an asymptotic approximation as $x \to \infty$.

Evaluate this integral with Laplace's method First off, let's look at the integrand $\mathrm{e}^{-xt}/(1+t)$ and see what happens as $x \to \infty$. (MATLAB)

Looks like the integrand is mostly zero, except near $t = 0$.

Let's take advantage of this and the split the range of integration:

$$\begin{aligned} I(x) &= \int_0^{10} \frac{\mathrm{e}^{-xt}}{(1+t)}\mathrm{d}t, \\ &= \int_0^{\epsilon} \frac{\mathrm{e}^{-xt}}{(1+t)}\mathrm{d}t + \int_{\epsilon}^{10} \frac{\mathrm{e}^{-xt}}{(1+t)}\mathrm{d}t, \end{aligned}$$

We assume here that $x^{-1} \ll \epsilon \ll 1$ (ie. small, but not too small).

The second term is $\mathcal{O}(\mathrm{e}^{-x\epsilon})$ and so negligible compared to the first as $x \to \infty$.

Converting this to an asymptotic relation gives:

$$I(x) \sim \int_0^{\epsilon} \frac{\mathrm{e}^{-xt}}{(1+t)}\mathrm{d}t, \quad x \to \infty.$$

To proceed we make a change of variables and let $s = xt$:

$$I(x) \sim \int_0^{x\epsilon} \frac{\mathrm{e}^{-s}}{1+s/x}\frac{1}{x}\mathrm{d}s.$$

Then we Taylor expand the bit that is not the integral as:

$$\frac{1}{1+s/x} = \sum_{n=0}^{\infty} \frac{(-s)^n}{x^n}.$$

Putt that in the integral gives:

$$I(x) \sim \frac{1}{x}\int_0^{x\epsilon} \mathrm{e}^{-s}\sum_{n=0}^{\infty}\frac{(-s)^n}{x^n}\mathrm{d}s,$$
$$= \frac{1}{x^{n+1}}\int_0^{x\epsilon} \mathrm{e}^{-s}\sum_{n=0}^{\infty}(-s)^n\mathrm{d}s$$

A very useful result in evaluating these integrals is **Watson's Lemma**. This is really what we used in the previous example. For less general integrals (set $a = 0$ and $\phi = -t$) of the form:

$$I(x) = \int_0^b f(t)\mathrm{e}^{-xt}\mathrm{d}t,$$

where $f(t)$ is continuous for $0 \le t \le b$ and represented by the asymptotic series

$$f(t) \sim t^{\alpha}\sum_{n=0}^{\infty} a_n t^{\beta n}, \quad \text{as } t \to 0^+.$$

with $\alpha > -1$ and $\beta > 0$ (ie. convergent at $t = 0$). Additionally, if $b = \infty$ require that $f(t) \ll \mathrm{e}^{ct}$ as $t \to \infty$ with c a positive constant (ie. converges at $t = \infty$). If that all is the case then

$$I(x) \sim \sum_{n=0}^{\infty}\frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}, \quad \text{as } x \to \infty.$$

This is a very user-friendly lemma! To apply it all we need to do is come up with that asymptotic series representation of $f(t)$. Handy in finding such representations is the binomial series, recall that

$$(1+t)^{\alpha} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}x^n$$

Here's a couple of quick examples. Firstly,

$$I(x) = \int_0^5 \frac{\mathrm{e}^{-xt}}{1+t^2}\mathrm{d}t,$$

and (in non-standard form) the modified Bessel function $K_0(x)$,

$$K_0(x) = \mathrm{e}^{-x}\int_0^{\infty}(t^2+2t)^{-1/2}\mathrm{e}^{-xt}\mathrm{d}t.$$

Apply Watson's Lemma to the above integrals.

We now return to the most general possible form of **Laplace's method** for evaluating

$$I(x) = \int_a^b f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t,$$

as $x \to \infty$. The key idea is that that dominant contribution from the integral comes from point in the domain where $\phi(t)$ is largest.

Ie. get different approximations, depending on where the maximum in $\phi(t)$ occurs. To start with let's come up with a leading-order approximation when $\phi(t)$ has a maximum at the endpoint $t = a$.

Laplace's method for a maximum at $t = a$. Here is an outline of how Laplace's method works for a maximum at an end point ($t = a$):

- Split the integral to zoom in on a neighbourhood near the maximum point of $\phi(t)$ in the interval (in this case $t = a$).
- Assume that the 'other bit' is negligible.

  $\Longrightarrow$ Restriction 1: $\epsilon \gg x^{-1}$
- Taylor expand $f(t)$ and $\phi(t)$.
- Rescale the integration variable.

  $\Longrightarrow$ Restriction 2: $\epsilon \ll x^{-1/2}$
- Replace upper limit with ∞ to zoom out.

  Only introduces **exponentially small** errors.
- Evaluate.

Here is an outline of how Laplace's method works for an interior maximum ($t = c$):

- Split the integral to zoom in on a neighbourhood near the maximum point of $\phi(t)$ in the interval (in this case $t = c$).
- Assume that the 'other bits' are negligible.

  $\Longrightarrow$ Restriction 1: $\epsilon \gg x^{-1/2}$
- Taylor expand $f(t)$ and $\phi(t)$.
- Rescale the integration variable.

  $\Longrightarrow$ Restriction 2: $\epsilon \ll x^{-1/3}$
- Replace upper and lower limits with $\pm\infty$ to zoom out.

  Only introduces **exponentially small** errors.
- Evaluate.

There are three possibilities:

- Maximum at $t = a$ (and $\phi'(a) < 0$):

$$I(x) \sim -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)}.$$

- Maximum at $t = b$ (and $\phi'(b) > 0$):

$$I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)}.$$

- Maximum at $t = c$, with $a < c < b$. Then to leading order

$$I(x) \sim \frac{\sqrt{2\pi}f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}.$$

These were all derived via Taylor series for $f(t)$ and $\phi(t)$, higher order approximations may be derived by including more terms in these series.

Applying Laplace's method to leading-order is then just a matter of applying those expressions as appropriate.

Sometimes the maximum is varies with the large parameter (a so-called 'movable maxima'), but this can be dealt with via a change of variables. For instance,

$$\begin{aligned} I(x) &= \int_0^\infty e^{-xt-1/t}dt, \\ \Gamma(x+1) &= \int_0^\infty e^{-t}t^x dt. \quad \text{(Stirling's formula)} \end{aligned}$$

Briefly go through movable maxima examples

Very closely related to Laplace integrals are Fourier integrals, which are of the type

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}dt,$$

for $x \to \infty$. The are approximated by the **method of stationary phase**. Here's how that goes:

For $\psi'(c) = 0$ for $a < c < b$, and $\psi'(t) \neq 0$ for $a \leq t < c$ and $c < t \leq b$, have (similarly technique as Laplace's method):

$$I(x) \sim \frac{\sqrt{2\pi}f(c)e^{ix\psi(c)}e^{\pm i\pi/4}}{x^{1/2}|\psi''(c)|^{1/2}},$$

with the factor of $e^{i\pi/4}$ for $\psi''(c) > 0$ or $e^{-i\pi/4}$ for $\psi''(c) < 0$.

Sketch proof of method of stationary phase. Here is an outline of how method of stationary phase works for an interior stationary point ($\psi'(c) = 0$ at $t = c$):

- Split the integral to zoom in on a neighbourhood near the maximum of $\psi(t)$ in the interval (in this case $t = c$).
- Assume that the 'other bits' are negligible.
- Taylor expand $f(t)$ and $\phi(t)$.
- Rescale the integration variable.

  $\implies$ Restriction 1: $\epsilon \ll x^{-1/3}$
- Replace upper and lower limits with $\pm\infty$ to zoom out.

  Only introduces **algebraically small** errors.
- Evaluate. $\implies$ Restriction 2: $\epsilon \gg x^{-1/2}$

When applying integration by parts to Fourier integrals the **Reimann-Lebesgue lemma** is useful. This tells us that

$$\int_a^b f(t)\mathrm{e}^{ix\psi(t)}\mathrm{d}t \to 0, \text{ as } x \to \infty,$$

if $\int_a^b |f(t)|\mathrm{d}t < \infty$ and $\psi(t)$ is continuously differentiable for $a \leq t \leq b$ and not constant on any subinterval in $a \leq t \leq b$.

Similar formulas can be derived it the stationary point of $\psi(t)$ is at one of the endpoints.

Finally, let's look at some plots to see heuristically how this method works.

Look at some plots.

We now generalise Laplace integrals to the complex plane. Consider integrals of the form:

$$I(x) = \int_C f(t)\mathrm{e}^{x\phi(t)}\mathrm{d}t,$$

where C is a contour in the complex t-plane, and $f(t)$ and $\phi(t)$ are analytic. We apply the **method of steepest descents** to approximate the behaviour as $x \to \infty$.

The basic principle is similar to Laplace's method - the dominant contribution to the integral comes from some localised bit of the integrand. For Laplace's method this was the maximum of $\phi(t)$. Now that we're in the complex plane this doesn't quite work (there are oscillations as in the method of stationary phase).

Here, (along with endpoints etc.) the dominant contributions to the integral come from **saddle points**. Why?

Let's first review some ideas from applied complex variables. We said that $\phi(t)$ was analytic, that is $\phi(t) = u(\xi,\eta) + \mathrm{i}v(\xi,\eta)$ with $t = \xi + \mathrm{i}\eta$. The Cauchy-Riemann equations tell us that

$$u_\xi = v_\eta, \quad u_\eta = -v_\xi \qquad \Longrightarrow \nabla^2 u = u_{\xi\xi} + u_{\eta\eta} = 0.$$

Recall that a maximum or minimum in $u(\xi,\eta)$ requires $u_{\xi\xi}u_{\eta\eta} > 0$, which can't happen here.

Clearly analytic functions can grow/undulate/do all sorts of things, but the points where $\mathrm{d}\phi/\mathrm{d}t = 0$ are saddles points. Let's look at some surface plots, and see what goes on in the integrand.

Surface plots of the real parts of some analytic functions.

Recall that we can deform contours in the complex plane with wild abandon (provided that we're conscious of branch cuts, poles and so on). Can we deform the contour of our integral in a cunning way?

Ideally we'd like to deform the contour to avoid oscillations in the integrand, that is choose a path such that

$$\mathrm{Im}(\phi) = v(\xi,\eta) = \text{const}.$$

Additionally, we'd like there to be a point on contour that will give us a dominant contribution in the same way as Laplace's method.

We can show that such a contour:

1. Goes through a saddle point (where $\phi'(t) = 0$) ...
2. ... in the direction of **steepest descent**.

Go through how this works.

Method of steepest descents is a bit of a process, here are the steps.

1. Find the saddle points and identify the direction of steepest descents.
2. Deform contour to go through saddle point(s) in that direction of steepest descent.
3. Evaluate the local contribution to the integral from the saddle.
4. Evaluate any contributions from end points (if appropriate), possibly also in a direction of steepest descent.

The usual rules of contour integrals still apply - we might need to use residue calculus if poles have been picked up in deforming the contour.

Unpack that with pictures and an example.

Before we go on to do an example with a saddle point, let's look at an example without a saddle point.

Consider the integral:

$$I(x) = \int_0^1 \log t e^{\mathrm{i}xt} \mathrm{d}t.$$

This can't be evaluated with stationary phase because there is no point where $\psi'(t) = 0$, and integration by parts fails.

Evaluate this integral. The first step is to deform the contour into one that avoid oscillations in the integrand.

[See picture of deformed contour, `Ex8_SteepestDescent1.mlx`]

We have now split the integral into three parts.

$$\begin{aligned} I(x) &= \int_0^1 \log t e^{ixt} \mathrm{d}t, \\ &= \int_{C_1+C_2+C_3} \log t e^{ixt} \mathrm{d}t \\ &= I_1(x) + I_2(x) + I_3(x) \end{aligned}$$

Let's look at each in turn.

C_1 follows the imaginary axis from $t = 0$ to $\mathrm{i}\infty$.

This can be parameterised as $t = \mathrm{i}s$, $0 < s < \infty$.

Making this change of variables gives:

$$\begin{aligned} I_1(x) &= \int_{C_1} \log t e^{\mathrm{i}xt} \mathrm{d}t, \\ &= \int_0^\infty \log(\mathrm{i}s) e^{\mathrm{i}x(\mathrm{i}s)} \mathrm{i}\mathrm{d}s \\ &= \mathrm{i}\int_0^\infty \log(\mathrm{i}s) e^{-xs} \mathrm{d}s \end{aligned}$$

Use the fact that $\log(\mathrm{i}s) = \log\left(e^{\mathrm{i}\pi/2} s\right) = \log s + \mathrm{i}\pi/2$. Then,

$$\begin{aligned} I_1(x) &= \mathrm{i}\int_0^\infty \log(s) e^{-xs} \mathrm{d}s - \frac{\pi}{2}\int_0^\infty e^{-xs} \mathrm{d}s \\ &= I_{1a}(x) + I_{1b}(x) \end{aligned}$$

It turns outs both $I_{1a}(x)$ and $I_{1b}(x)$ can be evaluated exactly (haven't used any asymptotics yet!):

$$\begin{aligned}
I_{1a}(x) &= \mathrm{i}\int_0^\infty \log(s)\mathrm{e}^{-xs}\mathrm{d}s \\
&= \mathrm{i}\int_0^\infty \log(u/x)\mathrm{e}^{-u}(1/x)\mathrm{d}u \qquad (\text{put } u = sx) \\
&= \frac{\mathrm{i}}{x}\int_0^\infty \log(u)\mathrm{e}^{-u}\mathrm{d}u - \frac{\mathrm{i}}{x}\log(x)\int_0^\infty \mathrm{e}^{-u}\mathrm{d}u \\
&= \frac{\mathrm{i}}{x}(-\gamma) - \frac{\mathrm{i}}{x}\log(x)(1) = \boxed{-\frac{\mathrm{i}\gamma}{x} - \frac{\mathrm{i}\log x}{x}}
\end{aligned}$$

Similalry,

$$I_{1b}(x) = -\frac{\pi}{2}\int_0^\infty \mathrm{e}^{-xs}\mathrm{d}s = \boxed{-\frac{\pi}{2}\frac{1}{x}}$$

Putting that together we have:

$$\boxed{I_1(x) = -\frac{\mathrm{i}\gamma}{x} - \frac{\mathrm{i}\log x}{x} - \frac{\pi}{2}\frac{1}{x}}$$

For completeness (we skipped this in lectures), let's confirm that $I_2 = 0$. The curve C_2 is parametrised as $t = s + \mathrm{i}R$ with $0 < 1 < s$ and $R \to \infty$. So:

$$\begin{aligned}
I_2(x) &= \int_{C_2} \log t\mathrm{e}^{\mathrm{i}xt}\mathrm{d}t, \\
&= \lim_{R\to\infty}\int_0^1 \log(s+\mathrm{i}R)\mathrm{e}^{\mathrm{i}x(s+\mathrm{i}R)}\mathrm{d}s, \quad (\text{put } t = s + \mathrm{i}R) \\
&= \lim_{R\to\infty} \mathrm{e}^{-xR}\int_0^1 \log(s+\mathrm{i}R)\mathrm{e}^{\mathrm{i}xs}\mathrm{d}s = \boxed{0}.
\end{aligned}$$

Finally, C_3 runs parallel to in the imaginary axis, from $t = 1$ to $t = 1 + \mathrm{i}\infty$.

This can be parameterised as $t = 1 + \mathrm{i}s$, $0 < s < \infty$.

Making the change of variables, and introducing a minus sign to account for the direction of the path:

$$\begin{aligned}
I_3(x) &= \int_{C_3} \log t\mathrm{e}^{ixt}\mathrm{d}t \\
&= -\int_0^\infty \log(1+\mathrm{i}s)\mathrm{e}^{ix(1+\mathrm{i}s)}\mathrm{i}\mathrm{d}s, \quad (\text{put } t = 1 + \mathrm{i}s) \\
&= -\mathrm{i}\mathrm{e}^{\mathrm{i}x}\int_0^\infty \log(1+\mathrm{i}s)\mathrm{e}^{-xs}\mathrm{d}s
\end{aligned}$$

This is something we can apply Watson's Lemma to! The Taylor series for the first bit is $\log(1+\mathrm{i}s) = -\sum_{n=1}^\infty \frac{(-\mathrm{i}s)^n}{n}$, applying Watson's gives:

$$\boxed{I_3(x) \sim \mathrm{i}\mathrm{e}^{\mathrm{i}x}\sum_{n=1}^{\infty}\frac{(-i)^n(n-1)!}{x^{n+1}}, \quad x \to \infty}$$

Putting all that together:

$$\boxed{I(x) \sim -\frac{\mathrm{i}\gamma}{x} - \frac{\mathrm{i}\log x}{x} - \frac{\pi}{2}\frac{1}{x} + \mathrm{i}\mathrm{e}^{\mathrm{i}x}\sum_{n=1}^{\infty}\frac{(-i)^n(n-1)!}{x^{n+1}}}$$

as $x \to \infty$.

Note that the only place asymptotics came into this problem was in evaluating that integral with Watson's lemma.

Let's now look an example that **does** have a saddle point.

$$I(x) = \int_0^1 \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t,$$

as $x \to \infty$. As we shall see this is a slightly 'cooked' example since the saddle is also and end point of the integration contour.

Evaluate this integral. We have $f(t) = 1$ and $\phi(t) = it^2$.

$\implies \phi'(t) = \mathrm{i}2t$ so there is a saddle at $t = 0$. This also one of the end points - very handy as it means we don't need to deform the contour to go through the saddle.

Now to find the contours of steepest descent. Let $t = \xi + \mathrm{i}\eta$, then

$$\begin{aligned}\phi(t) &= \mathrm{i}t^2,\\ &= \mathrm{i}(\xi + \mathrm{i}\eta)^2,\\ &= \mathrm{i}\xi^2 - 2\xi\eta - \mathrm{i}\eta^2 = -2\xi\eta + \mathrm{i}(\xi^2 - \eta^2)\end{aligned}$$

So $v(\xi, \eta) = \mathrm{Im}(\phi) = \xi^2 - \eta^2$.Then through the saddle at $t = 0$ the steepest descent/ascent contours are given by:

$$0 = \xi^2 - \eta^2 \implies \eta = \pm\xi.$$

Look at $u(\xi, \eta) = \mathrm{Re}(\phi) = -2\xi\eta$ to find which ascend and which descend. Along $\eta = \xi$ have $u = -2\xi^2$ which decreases away from the saddle point. Along $\eta = -\xi$ have $u = 2\xi^2$ which increases.

We also need a steepest descent path coming into $t = 1$. Through this point v is constant along

$$1 = \xi^2 - \eta^2 \implies \xi = \pm\sqrt{\eta^2 + 1}, \quad \xi \geq 1.$$

Along $\xi = \sqrt{\eta^2+1}$ have $u = -2\eta\sqrt{\eta^2+1}$ which decreases as $\eta \to \infty$ (ie. **descent**).

Along $\xi = -\sqrt{\eta^2+1}$ have $u = 2\eta\sqrt{\eta^2+1}$ which increases as $\eta \to \infty$ (ie. **ascent**).

As in the previous example we have found the steepest descent contours coming out of each end point. These can be parameterised as

$$\begin{aligned} &\text{Path 1:} \qquad t = s + \mathrm{i}s = (1+\mathrm{i})s, \quad 0 < s < \infty \\ &\text{Path 3:} \qquad t = \sqrt{s^2+1} + is, \quad 0 < s < \infty \end{aligned}$$

To join these together via (with $R \to \infty$)

$$\text{Path 2:} \qquad t = s + \mathrm{i}R, \quad R < s < \sqrt{R^2+1}$$

Now to evaluate the integral:

$$\begin{aligned} I(x) &= \int_0^1 \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t, \\ &= \int_{C_1+C_2+C_3} \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t, \quad \text{(deform contour)} \\ &= I_1(x) + I_2(x) + I_3(x) \end{aligned}$$

Look at each in turn ... Path for I_1 is $t = (1+\mathrm{i})s$, with $0 < s < \infty$:

$$\begin{aligned} I_1(x) &= \int_{C_1} \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t, \\ &= \int_0^\infty \mathrm{e}^{x\mathrm{i}(1+\mathrm{i})^2s^2}(1+\mathrm{i})\mathrm{d}s, \quad (\text{put } t = (1+\mathrm{i})s) \\ &= (1+i)\int_0^\infty \mathrm{e}^{-2xs^2}\mathrm{d}s, \quad \left(\text{put } u = 2xs^2 \implies \frac{u^{-1/2}\sqrt{2}}{4x^{1/2}}\mathrm{d}u = \mathrm{d}s\right) \\ &= \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{2}}{4x^{1/2}}\int_0^\infty u^{-1/2}\mathrm{e}^{-u}\mathrm{d}u, \end{aligned}$$

Use $\Gamma(1/2) = \sqrt{\pi}$ and get:

$$\boxed{I_1(x) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi}}{2x^{1/2}}}$$

Great! Can evaluate that one exactly.

Path for I_2 is $t = s + \mathrm{i}R$, with $R < s < \sqrt{R^2+1}$, $R \to \infty$.

Reasonably expect $I_2(x)$ to be zero, let's check:

$$\begin{aligned}
I_2(x) &= \int_{C_2} \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t, \\
&= \lim_{R\to\infty} \int_R^{\sqrt{R^2+1}} \mathrm{e}^{\mathrm{i}x(s^2+2\mathrm{i}sR-R^2)}\mathrm{d}s, \\
&= \lim_{R\to\infty} \mathrm{e}^{-\mathrm{i}xR^2} \int_R^{\sqrt{R^2+1}} \mathrm{e}^{\mathrm{i}xs^2-2sR}\mathrm{d}s \\
&= 0.
\end{aligned}$$

Good. Path for I_3 is $t = \sqrt{s^2+1} + \mathrm{i}s$, with $0 < s < \infty$.

$$\begin{aligned}
I_3(x) &= \int_{C_3} \mathrm{e}^{\mathrm{i}xt^2}\mathrm{d}t, \\
&= -\int_0^\infty \mathrm{e}^{\mathrm{i}x(s^2+1+2\mathrm{i}s\sqrt{s^2+1}-s^2)} \left((1/2)2s(s^2+1)^{-1/2} + \mathrm{i}\right)\mathrm{d}s, \\
&= -\mathrm{e}^{\mathrm{i}x} \int_0^\infty \mathrm{e}^{-x2s\sqrt{s^2+1}} \left(s(s^2+1)^{-1/2} + \mathrm{i}\right)\mathrm{d}s.
\end{aligned}$$

That integral can be evaluated with Laplace's method:

$$\begin{aligned}
-I_3(x) &\sim \mathrm{e}^{\mathrm{i}x} \int_0^\epsilon \mathrm{e}^{-x2s}(s+\mathrm{i})\mathrm{d}s, \quad (\text{put } u = 2xs) \\
&= \mathrm{e}^{\mathrm{i}x} \int_0^{2x\epsilon} \mathrm{e}^{-u}\left(\frac{u}{2x} + \mathrm{i}\right)\frac{1}{2x}\mathrm{d}u, \\
&\sim \mathrm{e}^{\mathrm{i}x}\left[\frac{1}{4x^2}\int_0^\infty \mathrm{e}^{-u}u\,\mathrm{d}u + \frac{\mathrm{i}}{2x}\int_0^\infty \mathrm{e}^{-u}\mathrm{d}u\right] \boxed{= \mathrm{e}^{\mathrm{i}x}\left(\frac{1}{4x^2} + \frac{\mathrm{i}}{2x}\right)}
\end{aligned}$$

(Full behaviour) Alternatively, can recognise that near the end point at $t = 1$, the path is locally $t = 1 + \mathrm{i}s$, since we zoomed in on that region anyway on the previous slide, let's do it slightly earlier:

$$\begin{aligned}
-I_3 &\sim \int_0^\epsilon \mathrm{e}^{\mathrm{i}x(1+\mathrm{i}s)^2}\mathrm{i}\mathrm{d}s, \\
&= \int_0^\epsilon \mathrm{e}^{\mathrm{i}x(1+2\mathrm{i}s-s^2)}\mathrm{i}\mathrm{d}s, \\
&= \mathrm{e}^{\mathrm{i}x}\int_0^\epsilon \mathrm{e}^{-x2s}\mathrm{e}^{-\mathrm{i}xs^2}\mathrm{i}\mathrm{d}s, \\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}x}\int_0^\epsilon \mathrm{e}^{-x2s}\sum_{n=0}^\infty \frac{(-\mathrm{i})^n x^n s^{2n}}{n!}\mathrm{d}s, \quad (\text{put } u = 2xs) \\
&\sim \mathrm{i}\mathrm{e}^{\mathrm{i}x}\int_0^\infty \mathrm{e}^{-u}\sum_{n=0}^\infty \frac{(-\mathrm{i})^n x^{-n} u^{2n}}{2^{2n}n!}\frac{\mathrm{d}u}{2x}, \\
&= \mathrm{i}\mathrm{e}^{\mathrm{i}x}\sum_{n=0}^\infty \frac{(-\mathrm{i})^n(2n)!}{2^{2n+1}n!x^{n+1}} \sim \mathrm{e}^{\mathrm{i}x}\left(\frac{\mathrm{i}}{2x} + \frac{1}{4x^2} + \dots\right)
\end{aligned}$$

Putting all of that together, have

$$\boxed{I(x) \sim \frac{e^{i\pi/4}\sqrt{\pi}}{2x^{1/2}} - ie^{ix}\sum_{n=0}^{\infty}\frac{(-i)^n(2n)!}{2^{2n+1}n!x^{n+1}}}$$

Some things to note:

- Contribution from the end point negligible compared to the contribution from the saddle.
- Still got lucky here in that we could evaluate the saddle contribution exactly, this doesn't always happen!

Now let's look at an example where original contour doesn't pass through the saddle.

Find the leading behaviour as $x \to \infty$ of the Bessel function

$$J_0(x) = \mathrm{Re}\frac{1}{i\pi}\int_C e^{ix\cosh t}dt,$$

where C is any contour that goes from $-\infty - i\pi/2$ to $\infty + i\pi/2$.

Discuss this contour, and evaluate the integral. This is a Sommerfeld contour ... sketch etc.

We have $f(t) = 1$ and $\phi(t) = i\cosh t$.

$\implies \phi'(t) = i\sinh t$ so there is a saddle at $t = 0$. Let's find the path of steepest descent through this point, the figure out how to deform our (pretty arbitrary) contour.

Let $t = \xi + i\eta$:

$$\begin{aligned}
\phi(t) &= i\cosh t, \\
&= i\cosh(\xi + i\eta) \\
&= i(\cosh\xi\cosh(i\eta) + \sinh\xi\sinh(i\eta)) \\
&= i(\cosh\xi\cos\eta + i\sinh\xi\sin\eta) \\
&= -\sinh\xi\sin\eta + i\cosh\xi\cos\eta
\end{aligned}$$

Then $v = \mathrm{Im}\phi = \cosh\xi\cos\eta$, so path of steepest descent/ascent through $t = 1$ is

$$1 = \cosh\xi\cos\eta.$$

Sketch this contour for large ξ and small ξ.

Large ξ: see that contour goes to $\pm i\pi/2$ as $\xi \to \pm\infty$.

Small ξ: Have $1 \approx (1+\xi^2/2+...)(1-\eta^2/2+...)$, so near $t=0$ have $\eta \approx \pm\xi$.

Descent/ascent?: when $\eta \approx -\xi$, then

$$\mathrm{Re}(\phi) \approx -\sinh\xi\sin(-\xi) = \sinh\xi\sin\xi \implies \text{ascent}$$

When $\eta \approx \xi$, then

$$\mathrm{Re}(\phi) \approx -\sinh\xi\sin(\xi) \implies \text{descent}$$

This is great ... the steepest descent contour **is** a Sommerfeld contour! We can deform whatever contour we started with straight onto this special contour.

To MATLAB ... see `Ex10_SteepestDescent3_Bessel.mlx`

So far we have established:

- Location of the saddle.
- Steepest descent path and direction (call this C_S)
- How to deform the contour through the saddle in the right direction (exactly the steepest descent contour).

Near $t=0$ we have:

- steepest descent path is parameterised by $t=(1+\mathrm{i})s$, $0<s<\infty$.
- $\cosh t \sim 1+t^2/2 = 1+(1+\mathrm{i})^2s^2/2 = 1+is^2$, $s \to 0$.

Now use all that to evaluate the integral.

Here we go:

$$J_0(x) = \mathrm{Re}\frac{1}{\mathrm{i}\pi}\int_C \mathrm{e}^{\mathrm{i}x\cosh t}\mathrm{d}t,$$

Look at:

$$\begin{aligned}
I(x) &= \int_C \mathrm{e}^{\mathrm{i}x\cosh t}\mathrm{d}t,\\
&= \int_{C_S} \mathrm{e}^{\mathrm{i}x\cosh t}\mathrm{d}t, \quad \text{(deform contour)}\\
&\sim \int_{-\epsilon}^{\epsilon} \mathrm{e}^{\mathrm{i}x\cosh t}\frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s, \quad \text{(as } \epsilon \to 0\text{, zoom in on saddle)}\\
&\sim \int_{-\epsilon}^{\epsilon} \mathrm{e}^{\mathrm{i}x(1+\mathrm{i}s^2)}(1+\mathrm{i})\mathrm{d}s, \quad \text{(near saddle)}\\
&= (1+\mathrm{i})\mathrm{e}^{\mathrm{i}x}\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-xs^2}\mathrm{d}s
\end{aligned}$$

Continuing:

$$
\begin{aligned}
I(x) &\sim (1+\mathrm{i})\mathrm{e}^{\mathrm{i}x}\int_{-\epsilon}^{\epsilon}\mathrm{e}^{-xs^2}\mathrm{d}s \\
&= (1+\mathrm{i})\mathrm{e}^{\mathrm{i}x}\int_{-x\epsilon^2}^{x\epsilon^2}\mathrm{e}^{-u}\left(\frac{1}{2}u^{-1/2}x^{-1/2}\right)\mathrm{d}u, \quad (\text{put } u = xs^2) \\
&= \frac{(1+\mathrm{i})\mathrm{e}^{\mathrm{i}x}}{2\sqrt{x}}\int_{-x\epsilon^2}^{x\epsilon^2}\mathrm{e}^{-u}u^{-1/2}\mathrm{d}u, \\
&\sim \frac{(1+\mathrm{i})\mathrm{e}^{\mathrm{i}x}}{2\sqrt{x}}\int_{-\infty}^{\infty}\mathrm{e}^{-u}u^{-1/2}\mathrm{d}u, \quad (\text{zoom out, let } x\to\infty) \\
&= \frac{(1+\mathrm{i})\mathrm{e}^{\mathrm{i}x}}{\sqrt{x}}\int_{0}^{\infty}\mathrm{e}^{-u}u^{-1/2}\mathrm{d}u = \frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi}\mathrm{e}^{\mathrm{i}x}}{\sqrt{x}}
\end{aligned}
$$

Then

$$
\begin{aligned}
J_0 &\sim \operatorname{Re}\left(\frac{1}{\mathrm{i}\pi}\frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\pi}\mathrm{e}^{\mathrm{i}x}}{\sqrt{x}}\right) \\
&= \operatorname{Re}\left(\frac{1}{\mathrm{e}^{\mathrm{i}\pi/2}}\frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}x}}{\sqrt{\pi}\sqrt{x}}\right) \\
&= \operatorname{Re}\left(\frac{\sqrt{2}\mathrm{e}^{\mathrm{i}(x-\pi/4)}}{\sqrt{\pi}\sqrt{x}}\right).
\end{aligned}
$$

Which gives:

$$\boxed{J_0(x) \sim \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{x}}\cos(x-\pi/4)}$$

as $x\to\infty$.

On a very similar theme, let's evaluate this definition of the Gamma function:

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{e}^t t^{-x}\mathrm{d}t,$$

with a contour C that starts at $t=-\infty-\mathrm{i}a$ ($a>0$) avoids the branch cut along the negative real axis (due to the t^{-x} term), and ends at $t=-\infty+\mathrm{i}b$ ($b>0$).

This is a movable saddle problem - but this can be fixed via a change of variables.

Evaluate the integral. Start by making a change of variables (just like we did a few lectures ago) of $t = xs$. This gives:

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi\mathrm{i}x^{x-1}}\int_{C'}\mathrm{e}^{x(s-\log s)}\mathrm{d}s.$$

Here C' is just a tranformed version of the original contour.

We have $\phi(s) = s - \log s$.

Then $\phi'(s) = 1 - 1/s$, and there is a saddle when $\phi'(s) = 0$, that is at $s = 1$.

Now let $s = \xi + \mathrm{i}\eta = \sqrt{\xi^2 + \eta^2}\mathrm{e}^{\mathrm{i}\arctan(\eta/\xi)}$ and we have:

$$\begin{aligned}\phi(s) &= \xi + \mathrm{i}\eta - \log\left(\sqrt{\xi^2+\eta^2}\mathrm{e}^{\mathrm{i}\arctan(\eta/\xi)}\right),\\ &= \xi - \log\left(\sqrt{\xi^2+\eta^2}\right) + \mathrm{i}\left(\eta - \arctan(\eta/\xi)\right)\end{aligned}$$

Have $u(\xi, \eta) = \xi - \log(\sqrt{\xi^2 + \eta^2})$ and $v(\xi, \eta) = \eta - \arctan(\eta/\xi)$.

Then paths with constant imaginary part going through $s = 1$ (note $v(1, 0) = 0$):

$$0 = \eta - \arctan(\eta/\xi)$$

Which has solution $\eta = 0$ and $\xi = \eta \cot \eta$.

Easy see that $\eta = 0$ increases as it leaves the saddle $\implies$ ascent.

... slightly less easy to see that $\xi = \eta \cot \eta$ is a descent path (will leave this as an exercise).

We're really interested in the direction of the path at the saddle. Note that on the curve:

$$\begin{aligned}\xi = \eta\cot\eta &\implies \frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \cot\eta - \eta(1 + \cot^2\eta)\\ &\implies \left.\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right|_{\eta=0} = 0\end{aligned}$$

The path is vertical near the saddle!

Right. Parameterise the path near the saddle as $s = 1 + \mathrm{i}v$.

Taylor expansion of $\phi(s)$ near $s = 1$ is:

$$\phi(s) \sim 1 + \frac{(s-1)^2}{2} - \frac{(s-1)^3}{3} + \ldots$$

Then we have:

$$
\begin{aligned}
\frac{1}{\Gamma(x)} &= \frac{1}{2\pi \mathrm{i} x^{x-1}} \int_{C'} \mathrm{e}^{x(s-\log s)} \mathrm{d}s, \\
&= \frac{1}{2\pi \mathrm{i} x^{x-1}} \int_{C_S} \mathrm{e}^{x(s-\log s)} \mathrm{d}s, \quad \text{(deform)} \\
&\sim \frac{1}{2\pi \mathrm{i} x^{x-1}} \int_{-\epsilon}^{\epsilon} \mathrm{e}^{x(1+\frac{(s-1)^2}{2}-\frac{(s-1)^3}{3}+\dots)} \mathrm{i}\mathrm{d}v, \quad \text{(zoom in, use above)} \\
&= \frac{\mathrm{e}^x}{2\pi x^{x-1}} \int_{-\epsilon}^{\epsilon} \mathrm{e}^{x(\frac{(\mathrm{i}v)^2}{2}-\frac{(\mathrm{i}v)^3}{3}+\dots)} \mathrm{d}v, \\
&= \frac{\mathrm{e}^x}{2\pi x^{x-1}} \int_{-\epsilon}^{\epsilon} \mathrm{e}^{-xv^2/2} \mathrm{e}^{x(\mathrm{i}v^3/3+\dots)} \mathrm{d}v,
\end{aligned}
$$

Continuing:

$$
\begin{aligned}
\frac{1}{\Gamma(x)} &\sim \frac{\mathrm{e}^x}{2\pi x^{x-1}} \int_{-\epsilon}^{\epsilon} \mathrm{e}^{-xv^2/2} \mathrm{e}^{x(\mathrm{i}v^3/3+\dots)} \mathrm{d}v, \\
&= \frac{\mathrm{e}^x}{2\pi x^{x-1}} \int_{-x^{1/2}\epsilon}^{x^{1/2}\epsilon} \mathrm{e}^{-u^2/2} \mathrm{e}^{\mathrm{i}u^3/(3\sqrt{x})+\dots} \frac{\mathrm{d}u}{\sqrt{x}}, \quad \text{(put } u = \sqrt{x}v\text{)} \\
&\sim \frac{\mathrm{e}^x}{2\pi x^{x-1}\sqrt{x}} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/2} \mathrm{d}u, \\
&= \frac{\mathrm{e}^x}{2\pi x^{x-1}\sqrt{x}} \int_{-\infty}^{\infty} \mathrm{e}^{-q} q^{-1/2} \frac{1}{\sqrt{2}} \mathrm{d}q, \quad \text{(put } q = u^2/2\text{)} \\
&= \frac{2\mathrm{e}^x}{2\pi x^{x-1}\sqrt{x}\sqrt{2}} \int_{0}^{\infty} \mathrm{e}^{-q} q^{-1/2} \mathrm{d}q \\
&= \frac{\sqrt{\pi}\mathrm{e}^x}{\pi x^{x-1}\sqrt{x}\sqrt{2}} = \frac{\mathrm{e}^x}{x^{x-1/2}\sqrt{2\pi}}
\end{aligned}
$$

Which gives:

$$
\boxed{\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} \mathrm{e}^{-x}, \quad x \to \infty.}
$$

For our final example we return to the Airy equation (recall we studied its large x behaviour back in chapter 1).

An integral representation of Airy functions of the first kind is:

$$
\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{C'} \mathrm{e}^{\mathrm{i}(t^3/3+xt)} \mathrm{d}t
$$

where the contour C' start at infinity with $2\pi/3 < \arg(t) < \pi$ and ends at infinity with $0 < \arg(t) < \pi/3$.

We could use method of steepest descents to again find the behaviour as $x \to \infty$, but instead let's find see what happens when $x \to -\infty$.

First, let's see where this integral representation comes from.

Obtain integral representation, then approximate this integral for $x \to -\infty$. Obtain integral representation: Recall that the Airy equation is

$$y'' - xy = 0$$

Let

$$y(x) = \frac{1}{2\pi \mathrm{i}} \int_C q(t) \mathrm{e}^{tx} \mathrm{d}t.$$

We then need to find $q(t)$ and C to satisfy the DE.Substitution gives:

$$\frac{1}{2\pi \mathrm{i}} \int_C t^2 q(t) \mathrm{e}^{tx} \mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \int_C x q(t) \mathrm{e}^{tx} \mathrm{d}t = 0$$

The second term can be integrated by parts to give:

$$\frac{1}{2\pi \mathrm{i}} \int_C \left[t^2 q(t) + q'(t)\right] \mathrm{e}^{tx} \mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \left[q(t)\mathrm{e}^{tx}\right]_{t=a}^{t=b} = 0$$

where a and b are the end points of the contour. This tells us what we need from the solution:

1. Need $t^2 q(t) + q'(t) = 0 \implies q(t) = \mathrm{e}^{-t^3/3}$ (up to a constant).
2. Need $q(t)\mathrm{e}^{tx}$ to vanish at the end points.

Look at the function

$$q(t)\mathrm{e}^{tx} = \mathrm{e}^{tx - t^3/3}$$

This will only ever vanish as $|t| \to \infty$ (since it's an exponential), and (for any x) when $-t^3/3$ has a negative real part. That happens when

- $0 < \arg(t) < \pi/3$;
- or $2\pi/3 < \arg(t) < \pi$;
- or $-2\pi/3 < \arg(t) < -\pi/3$.

Thus we need our the endpoints of the contour to be at infinity in one of these sectors.

Further, we need it to start in one and end in another (otherwise it hasn't gone anywhere).

This leads to two independent solutions (sketch).

The choice for $\mathrm{Ai}(x)$ is a contour starting in $2\pi/3 < \arg(t) < \pi$ and ending in $0 < \arg(t) < \pi/3$.

We could proceed as per the last few examples to approximate this integral as $x \to \infty$, but let's do something a bit different. We can investigate the behaviour for $x \to -\infty$ by looking at:

$$\mathrm{Ai}(-x) = \frac{1}{2\pi} \int_{C'} e^{i(t^3/3 - xt)} dt$$

Here the saddle moves, depending on the value of x, but we can avoid this by letting $t = x^{1/2} z$. This gives:

$$\mathrm{Ai}(-x) = \frac{x^{1/2}}{2\pi} \int_{C'} e^{ix^{3/2}(z^3/3 - z)} dz$$

That is, $\phi(z) = i(z^3/3 - z)$.

$\implies \phi'(z) = i(z^2 - 1)$ so there are saddles at $z = \pm 1$.

Let's find the steepest descent contours! Let $z = \xi + i\eta$, then

$$\begin{aligned}
\phi(z) &= i(z^3/3 - z), \\
&= i((\xi + i\eta)^3/3 - (\xi + i\eta)) \\
&= i((\xi^3 + 3i\xi^2\eta - 3\xi\eta^2 - i\eta^3)/3 - (\xi + i\eta)) \\
&= -\xi^2\eta + \eta^3/3 + \eta + i(\xi^3/3 - \xi\eta^2 - \xi)
\end{aligned}$$

That is $u(\xi, \eta) = -\xi^2\eta + \eta^3/3 + \eta$ and $v(\xi, \eta) = \xi^3/3 - \xi\eta^2 - \xi$.

As yet, we don't know which saddle to deform through - need to figure out paths for both.

At $z = \pm 1$, $v(\pm 1, 0) = \pm 1/3 \mp 1 = \mp 2/3$.

For the saddle at $z = 1$, steepest descent/ascent curves are:

$$-2/3 = \xi^3/3 - \xi\eta^2 - \xi \implies \eta = \pm\sqrt{\xi^2/3 - 1 + 2/(3\xi)}$$

For the saddle at $z = -1$, steepest descent/ascent curves are:

$$2/3 = \xi^3/3 - \xi\eta^2 - \xi \implies \eta = \pm\sqrt{\xi^2/3 - 1 - 2/(3\xi)}$$

It's nice to know the full path, but we really only care about the local direction through the saddles (and whether this is ascent or descent). Here's a quick way of determining that.

Say we have a (simple) saddle at $z = z_0$ (here $z = \pm 1$). Then have:

$$\begin{aligned}
\phi(z) - \phi(z_0) &\sim \frac{(z - z_0)^2}{2} \left.\frac{d^2\phi}{dz^2}\right|_{z=z_0} \\
&= \frac{\rho^2 e^{i2\theta}}{2} a e^{i\alpha} = \frac{\rho^2 a}{2} (\cos(\alpha + 2\theta) + i\sin(\alpha + 2\theta))
\end{aligned}$$

where $z - z_0 = \rho e^{i\theta}$ (ie. θ the direction we're interested in) and $\phi''(z_0) = a e^{i\alpha}$.

Steepest descent paths when the imaginary part is zero, (ie. $\sin(\alpha + 2\theta) = 0$) and decreases (ie. $\cos(\alpha + 2\theta) < 0$). This happens when:

$$\theta = -\frac{\alpha}{2} + \frac{\pi}{2} \quad \text{and} \quad -\frac{\alpha}{2} + \frac{3\pi}{2}$$

For our problem, $\phi''(z) = i2z$ so for the saddle at $z = 1$,

$$\phi''(1) = 2i = 2e^{i\pi/2} \implies \alpha = \pi/2.$$

The directions of steepest descent are then

$$\theta = \frac{\pi}{4} \quad \text{and} \quad \frac{5\pi}{4} \equiv -\frac{3\pi}{4}$$

For the saddle at $z = -1$,

$$\phi''(1) = -2i = 2e^{-i\pi/2} \implies \alpha = -\pi/2.$$

The directions of steepest descent are then

$$\theta = -\frac{\pi}{4} \quad \text{and} \quad \frac{3\pi}{4}$$

That was (relatively) easy.

To MATLAB: see `Ex12_SteepestDescent5_Airy.mlx`

Which of these saddles should we deform through? Both!

Sketch deformed contour.

We now have:

$$\begin{aligned}\mathrm{Ai}(-x) &= \frac{x^{1/2}}{2\pi} \int_{C'} e^{ix^{3/2}(z^3/3 - z)} dz \\ &= \frac{x^{1/2}}{2\pi} \int_{C_{S1}+C_{S2}} e^{ix^{3/2}(z^3/3 - z)} dz\end{aligned}$$

where C_{S1} and C_{S2} are the steepest descent paths (being careful with direction).

Need to evaluate the contribution from each separately.

For the saddle at $z = -1$ the path is parameterised by $z = -1 + e^{-i\pi/4} s$, and

$$\phi(z) \sim 2i/3 - i(z+1)^2 + \ldots$$

Now look at the integral along the path through $z=-1$:

$$\begin{aligned}
I_1(x) &= \frac{x^{1/2}}{2\pi}\int_{C_{S1}} \mathrm{e}^{\mathrm{i}x^{3/2}(z^3/3-z)}\mathrm{d}z \\
&\sim \frac{x^{1/2}}{2\pi}\int_{-\epsilon}^{\epsilon} \mathrm{e}^{x^{3/2}(2\mathrm{i}/3-\mathrm{i}(z+1)^2+\ldots)}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{d}s \quad \text{(zoom in, use above)} \\
&\sim \frac{x^{1/2}\mathrm{e}^{(2\mathrm{i}/3)x^{3/2}}\mathrm{e}^{-\mathrm{i}\pi/4}}{2\pi}\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-x^{3/2}s^2}\mathrm{d}s \\
&= \frac{x^{1/2}\mathrm{e}^{(2\mathrm{i}/3)x^{3/2}}\mathrm{e}^{-\mathrm{i}\pi/4}}{2\pi}\int_{-x^{3/4}\epsilon}^{x^{3/4}\epsilon} \mathrm{e}^{-u^2}\frac{\mathrm{d}u}{x^{3/4}} \quad \text{(use } u=x^{3/4}s\text{)} \\
&\sim \frac{\mathrm{e}^{\mathrm{i}[(2/3)x^{3/2}-\pi/4]}}{2\pi x^{1/4}}\int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u = \boxed{\frac{\mathrm{e}^{\mathrm{i}[(2/3)x^{3/2}-\pi/4]}}{2\sqrt{\pi}x^{1/4}}}
\end{aligned}$$

For the saddle at $z=1$ the path is parameterised by $z=1+\mathrm{e}^{\mathrm{i}\pi/4}s$, and

$$\phi(z)\sim -2\mathrm{i}/3+\mathrm{i}(z-1)^2+\ldots$$

Now look at the integral along the path through $z=1$:

$$\begin{aligned}
I_2(x) &= \frac{x^{1/2}}{2\pi}\int_{C_{S2}} \mathrm{e}^{\mathrm{i}x^{3/2}(z^3/3-z)}\mathrm{d}z \\
&\sim \frac{x^{1/2}}{2\pi}\int_{-\epsilon}^{\epsilon} \mathrm{e}^{x^{3/2}(-2\mathrm{i}/3+\mathrm{i}(z-1)^2+\ldots)}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{d}s \quad \text{(zoom in, use above)} \\
&\sim \frac{x^{1/2}\mathrm{e}^{-(2\mathrm{i}/3)x^{3/2}}\mathrm{e}^{\mathrm{i}\pi/4}}{2\pi}\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-x^{3/2}s^2}\mathrm{d}s \\
&= \frac{x^{1/2}\mathrm{e}^{-(2\mathrm{i}/3)x^{3/2}}\mathrm{e}^{\mathrm{i}\pi/4}}{2\pi}\int_{-x^{3/4}\epsilon}^{x^{3/4}\epsilon} \mathrm{e}^{-u^2}\frac{\mathrm{d}u}{x^{3/4}} \quad \text{(use } u=x^{3/4}s\text{)} \\
&\sim \frac{\mathrm{e}^{-\mathrm{i}[(2/3)x^{3/2}-\pi/4]}}{2\pi x^{1/4}}\int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u = \boxed{\frac{\mathrm{e}^{-\mathrm{i}[(2/3)x^{3/2}-\pi/4]}}{2\sqrt{\pi}x^{1/4}}}
\end{aligned}$$

Putting that together we have:

$$\begin{aligned}
\mathrm{Ai}(-x) &= I_1(x)+I_2(x) \\
&\sim \frac{\mathrm{e}^{\mathrm{i}[(2/3)x^{3/2}-\pi/4]}}{2\sqrt{\pi}x^{1/4}}+\frac{\mathrm{e}^{-\mathrm{i}[(2/3)x^{3/2}-\pi/4]}}{2\sqrt{\pi}x^{1/4}} \\
&= \frac{1}{\sqrt{\pi}x^{1/4}}\left(\frac{1}{2}\mathrm{e}^{\mathrm{i}[(2/3)x^{3/2}-\pi/4]}+\frac{1}{2}\mathrm{e}^{-\mathrm{i}[(2/3)x^{3/2}-\pi/4]}\right)
\end{aligned}$$

Which gives

$$\boxed{\mathrm{Ai}(-x) \sim \frac{\cos\left((2/3)x^{3/2} - \pi/4\right)}{\sqrt{\pi}x^{1/4}}, \quad x \to \infty.}$$

Note: the Wikipedia article on Airy functions is a bit snarky about this representation (but strictly correct). The representation it refers to is of the form $\mathrm{Ai}(-x) = M(x)\cos(\theta(x))$ with asymptotic expansions given for the amplitude and phase.

$$^{\sim}\backslash\sim(\,\hat{}\sim\hat{}\,)\sim|^{\sim}$$