THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2014

101488 APP MTH 3016 Random Processes III

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 67

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators without remote communications facilities are permitted.
- Tables of Laplace Transforms are provided at the end of the Examination question book

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Consider a continuous time Markov chain $\{X(t), t \geq 0\}$ on the finite state space $\mathcal{S}$. For all $i, j \in \mathcal{S}$ and $s, t \geq 0$, let $P_{ij}(t) = \mathbb{P}(X(t+s) = j \mid X(s) = i)$.

   (a) Define the *infinitesimal generator matrix* Q of this Markov chain.

   (b) Give physical interpretations for the elements of the matrix Q.

   (c) Prove the Chapman-Kolmogorov equation

   $$P_{ij}(t) = \sum_{k \in \mathcal{S}} P_{ik}(u) P_{kj}(t-u) \quad \text{for } 0 \leq u \leq t.$$

   (d) Write down the Kolmogorov forward differential equations and the Kolmogorov backward differential equations of the continuous-time Markov chain.

   [12 marks]

2. Consider a restaurant with three chefs cooking only one dish: mushroom risotto. Assume that customers arrive according to a Poisson process with rate λ, and each chef takes an exponential period of time to serve an individual customer, with mean μ. (When there are two customers in service, one chef remains idle; similarly, when there are only one customer in service, two chefs remain idle.) Moreover, assume that there are $N > 3$ spaces in the restaurant, for both customers who are waiting to be served and customers who are being served.

   (a) Define a suitable state space $\mathcal{S}$ for this system, including a definition of each state.

   (b) Write down the transition rates for this system.

   (c) Write down the Kolmogorov backward differential equations for this Markov chain, only for $P_{0i}(t)$, $i \in \mathcal{S}$ and $t \geq 0$. **DO NOT SOLVE.**

   (d) State the physical meaning of the quantity $P_{0i}(t)$, for $i \in \mathcal{S}$.

   (e) What initial conditions should be satisfied by $P_{0i}(0)$, for $i \in \mathcal{S}$.

   (f) Write down the global balance equations for this Markov chain. **DO NOT SOLVE.**

   [14 marks]

Please turn over for page 3

3. Emma is a bibliophile (someone who has a great love of books). Suppose that Emma buys books (one at a time) according to a Poisson process with rate 5. The time it takes for Emma to finish reading a book is exponentially distributed with rate μ (that is, with mean $1/\mu$). Upon finishing a book, Emma might choose to

   - immediately reread it with probability p_1,
   - give it to a friend with probability p_2, and
   - with probability p_3 Emma donates it to the local library, $p_1 + p_2 + p_3 = 1$.

   In the last two cases, Emma moves on to reading the next book, if there is one. Emma's bookcase, magically, can hold infinitely many books.

   (a) Write down an appropriate state space $\mathcal{S}$ to help Emma keep track of the size of her book collection.

   (b) Write down the transition rates for the system.

   (c) Let $f_i, i \in \mathcal{S}$ be the probability that Emma's bookcase ever gets empty, given that it has i books at the beginning. Write down the appropriate set of equations satisfied by $f_i, i \in \mathcal{S}$.

   (d) What additional conditions do we need in order to determine which solution to this set of equations corresponds to $f_i, i \in \mathcal{S}$?

   (e) Assume that $\mu p_i > 5$ for $i = 1, 2, 3$, and solve the set of equations in part (c) subject to the additional conditions in part (d).

   (f) What does this tell us about the transience, or otherwise, of this Markov chain? Why?

   [15 marks]

4. Consider an open Jackson network of two queues, labelled 1 and 2, with service rates μ_1 and μ_2, respectively. Assume that there are arrivals to queue 1 according to a Poisson process with rate 20, and that the traffic routing probabilities are given by

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

   (a) Write down the state space for this network, including a definition of each state.

   (b) Solve the traffic equations to determine the effective arrival rate at queue i, y_i, for $i = 1, 2$.

   (c) Assuming that both queues are infinite-server queues, write down the equilibrium distribution for this network of queues.

   [6 marks]

5. At the local clinic, each doctor has their own room where they treat a single patient. Outside that room, they have their own designated waiting room for 2 patients. There is also a common waiting room for up to 20 patients, which is there for those patients who cannot not fit into their doctor's designated waiting room. Any patient who arrives to find the common waiting room full, leaves the facility.

   Every time a person leaves their place in the designated waiting room, a patient waiting for that doctor would move from the common waiting room to that designated waiting room, if such a patient was waiting.

   Assume that there are 3 doctors in the facility and that each doctor takes an exponentially distributed period of time to treat a patient with mean 20 minutes. Further, assume that patients arrive for each doctor as a Poisson process of rate λ per hour.

   (a) Assuming that the common waiting room has unlimited capacity, write down the equilibrium distribution for the number of patients in the entire facility. State any restrictions on the value of λ in order for this equilibrium distribution to hold.

   (b) Now, impose the restricted common waiting room of size 20. Write down the restrictions on the state space.

   (c) Write down the form of the equilibrium distribution for the number of patients in the waiting room and an expression for the appropriate normalising constant.

   (d) Write 2 sentences explaining the argument that justifies your result in (c).

   [9 marks]

6. Let $\{N(t), t \geq 0\}$ be a counting process, with inter-event time distribution given by the distribution function $F(t), t \geq 0$. Define the renewal function $M(t)$ by

$$M(t) = \mathbb{E}[N(t)].$$

   (a) Prove that the renewal function $M(t)$ satisfies the renewal equation

$$M(t) = F(t) + \int_{x=0}^{t} M(t-x)dF(x).$$

   (b) Hence, show that $M(t) = \sum_{n=0}^{\infty} F_n(t)$, where $F_n(t)$ is the distribution function for the convolution of n random variables with distribution function $F(t)$.

   [11 marks]

Table of Laplace Transforms

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$1/s$	1
$1/s^2$	t
$1/s^n \quad (n = 1, 2, \ldots)$	$t^{n-1}/(n-1)!$
$1/\sqrt{s}$	$1/\sqrt{\pi t}$
$1/s^{3/2}$	$2\sqrt{t/\pi}$
$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^2}$	te^{at}
$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \ldots)$	$\dfrac{1}{(n-1)!}\, t^{n-1}e^{at}$
$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}\, t^{k-1}e^{at}$
$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(e^{at} - e^{bt})$
$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(ae^{at} - be^{bt})$

$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\,\sin\omega t$
$\dfrac{s}{s^2+\omega^2}$	$\cos\omega t$
$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\,\sinh at$
$\dfrac{s}{s^2-a^2}$	$\cosh at$
$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}\, e^{at}\sin\omega t$
$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1-\cos\omega t)$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}\,(\omega t - \sin\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}\,(\sin\omega t - \omega t\cos\omega t)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\,\sin\omega t$
$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}\,(\sin\omega t + \omega t\cos\omega t)$
$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}\,(\cos at - \cos bt)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$

Basic General Formulas for the Laplace Transformation

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\,\mathcal{L}(f)$	Differentiation of Function Integration of Function
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (1st Shifting Theorem)
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (2nd Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s}$	Differentiation of Transform Integration of Transform
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1-e^{-\ell s}} \int_0^\ell e^{-st}f(t)dt$	f Periodic with Period ℓ