Optimal Functions and Nanomechanics III

APP MTH 3022/7106

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Lecture 27

Last lecture

- Looked at the broad class of problems known as traversals
- Derived a traversality condition
- Looked at a number of problems including a general problem of the form

$$F\{y\} = \int_0^{x_1} K(x,y)\sqrt{1+y'^2}\,dx.$$

  where the traversality condition essentially reduces to the extremal joining the constraint curve at right angles.

Broken Extremals

Until now we have required that extremal curves have at least two well-defined derivatives.

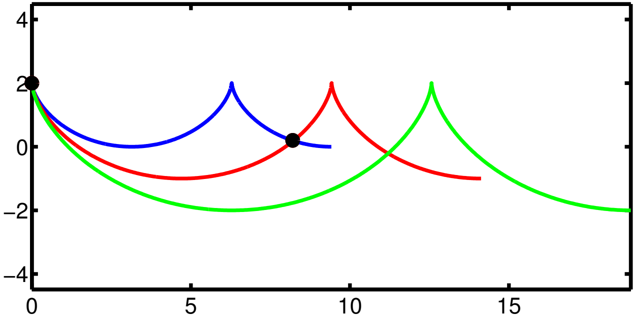
Obviously this is not always true (see for instance Snell's law).

In this lecture we consider the alternatives.

Broken extremals

Broken extremals are continuous extremals for which the gradient has a discontinuity at one of more points.

If a variational problem has a smooth extremal (that therefore satisfies the Euler-Lagrange equations), this will be better than a broken one, e.g. Brachistochrone.



Broken extremals

But some problems don't admit smooth extremals

Example: Find $y(x)$ to minimize

$$F\{y\} = \int_{-1}^{1} y^2 (1 - y')^2 \, dx,$$

subject to $y(-1) = 0$ and $y(1) = 1$.

Broken extremals example

There is no explicit x dependence inside the integral, so we can find

$$\begin{aligned} H(y, y') = y' \frac{\partial f}{\partial y'} - f &= \text{const} \\ y' y^2 (-2)(1 - y') - y^2 (1 - y')^2 &= -c_1 \\ y^2 (1 - y')(-1 + y' - 2y') &= -c_1 \\ y^2 (1 - y')(-1 - y') &= -c_1 \\ y^2 (1 - y'^2) &= c_1 \end{aligned}$$

If $c_1 = 0$ we get the singular solutions

$$y = 0 \quad \text{and} \quad y = \pm x + B$$

Neither of these satisfies both end-points conditions $y(-1) = 0$ and $y(1) = 1$, so we conclude that $c_1 \neq 0$.

Broken extremals example

Given that $c_1 \neq 0$

$$\begin{aligned} y^2(1 - y'^2) &= c_1 \\ y'^2 &= \frac{y^2 - c_1}{y^2} \\ \frac{dy}{dx} &= \pm\frac{1}{y}\sqrt{y^2 - c_1} \\ dx &= \pm\frac{y}{\sqrt{y^2 - c_1}}dy \\ x &= \pm\sqrt{y^2 - c_1} + c_2 \\ (x - c_2)^2 &= y^2 - c_1 \end{aligned}$$

The solution is a rectangular hyperbola

Broken extremals example

Find c_1 and c_2 from

$$(x - c_2)^2 = y^2 - c_1,$$

using the end-points.

$$\begin{array}{rclcrcl} y(-1) & = & 0 & \Rightarrow & (-1 - c_2)^2 & = & -c_1 \\ y(1) & = & 1 & \Rightarrow & (1 - c_2)^2 & = & 1 - c_1 \end{array}$$

Combine the two equations

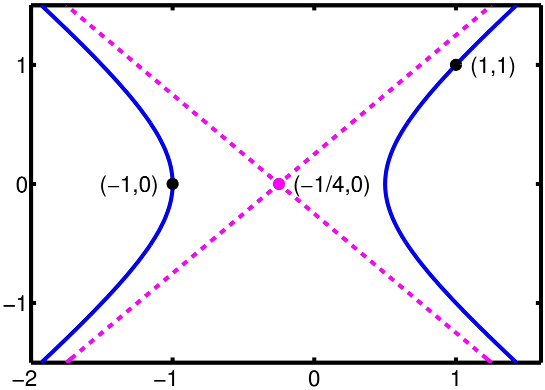
$$(1 - c_2)^2 = 1 + (1 + c_2)^2,$$

which has solutions $c_2 = -1/4$, and so $c_1 = -9/16$ and therefore

$$y^2 = (x + 1/4)^2 - 9/16.$$

Broken extremals example

The end-points are on opposite branches of the hyperbola!



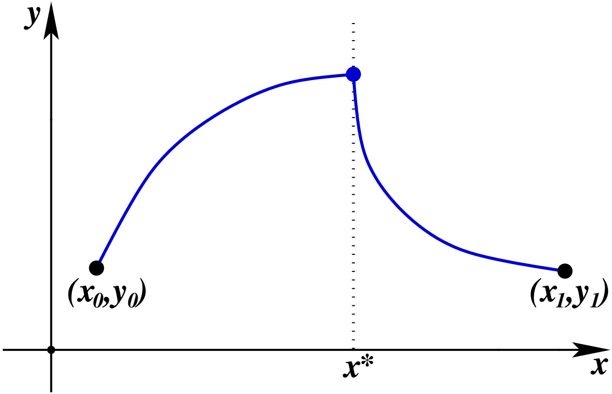
There is **NO** smooth extremal curve that connects $(-1, 0)$ and $(1, 1)$

Broken extremal

- sometimes there is no **smooth** extremal
- we must seek a **broken extremal**
- still want a continuous extremal
- what should we do?
  - previous smoothness results suggest that we should use a smooth extremal when we can, and so we will try to minimize the number of **corners**.
  - We'll start by looking for curves with one corner
  - But can we apply Euler-Lagrange equations?

Broken extremal

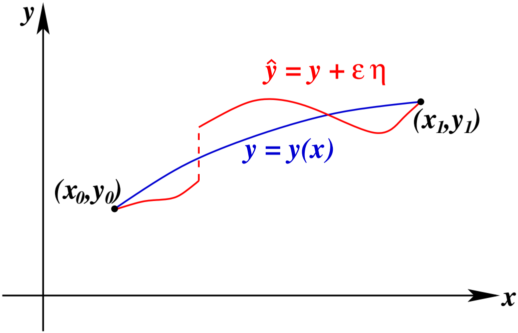
If we have an extremal like this, can we use Euler-Lagrange equations?



Smoothness theorem

Theorem: If the smooth curve $y(x)$ gives an extremal of a functional $F\{y\}$ over the class of all admissible curves in some ϵ neighbourhood of y, then $y(x)$ also gives an extremal of a functional $F\{y\}$ over the class of all **piecewise smooth curves** in the same neighbourhood.

Meaning: we can extend our results to piecewise smooth curves (where a smooth result exists), not just curves with 2 continuous derivatives.



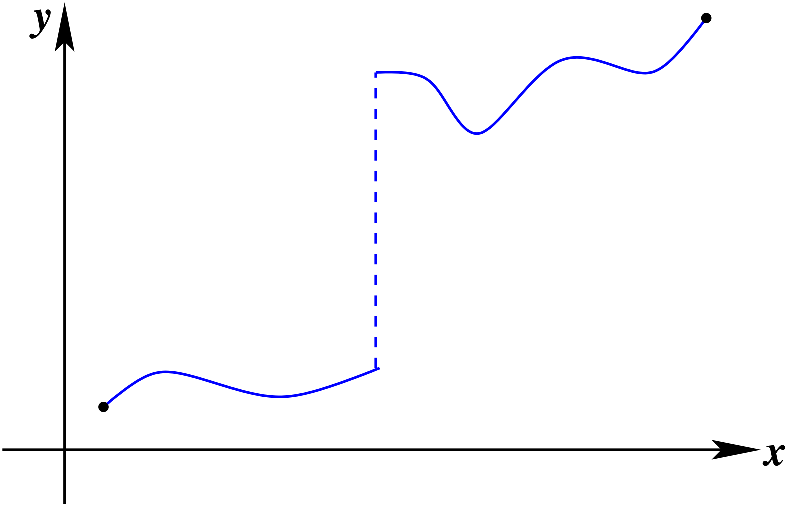
Proof sketch

The theorem assumes that there exists a smooth extremal (in this case a minimum for the purpose of illustration) y, then for any other smooth curve $\hat{y} \in B_\epsilon(y)$ we know $F\{\hat{y}\} > F\{y\}$.

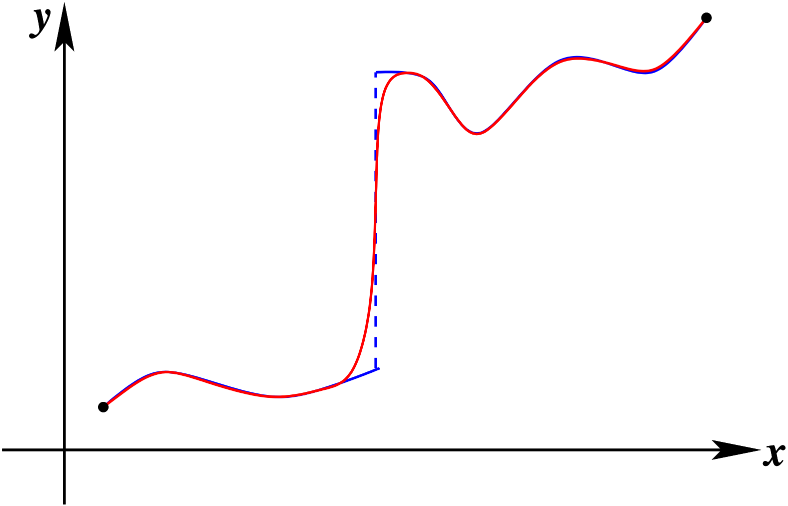
Assume for the moment that for a piecewise smooth function $\tilde{y} \in B_\epsilon(y)$ that $F\{\tilde{y}\} < F\{y\}$. We can approximate $\tilde{y}$ by a smooth curve $\hat{y}_\delta \in B_\epsilon(y)$ by rounding off the edges of the discontinuity.

Given that we can approximate the curve $\tilde{y}$ arbitrarily closely by a smooth curve $\hat{y}_\delta$, for which we already know $F\{\hat{y}_\delta\} > F\{y\}$, we get a contradiction with $F\{\tilde{y}\} < F\{y\}$, and so no such alternative extremal can exist.

Proof sketch



Proof sketch

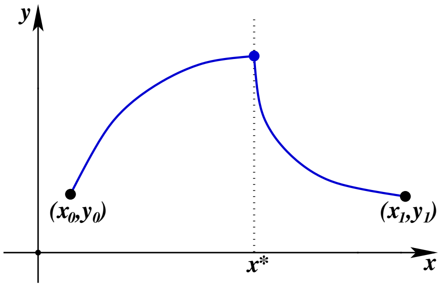


So what do we do?

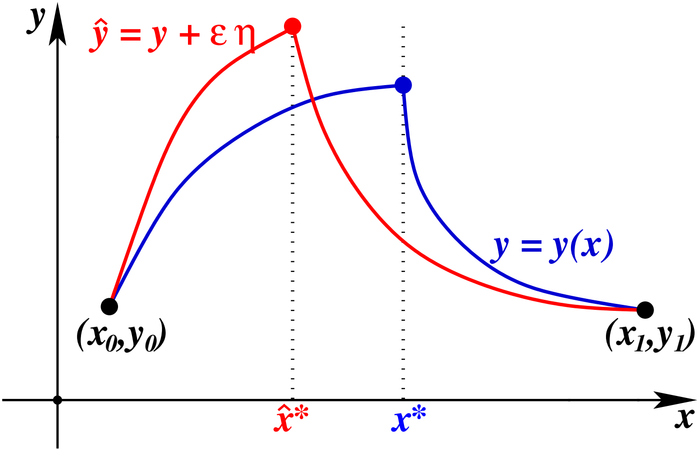
Break the functional into two parts:

$$F\{y\} = F_1\{y\} + F_2\{y\} = \int_{x_0}^{x^\star} f(x, y_1, y_1')\, dx + \int_{x^\star}^{x_1} f(x, y_2, y_2')\, dx,$$

where we require y to have two continuous derivatives everywhere except at $x^\star$, and $y_1(x^\star) = y_2(x^\star)$



Possible perturbations



The location of the “corner” can also be perturbed.

The First Variation: part 1

We get first component of the first variation by considering a problem with only one fixed end-point, and allowing x^* to vary, so that

$$\delta F_1(\eta, y) = \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \left[\int_{x_0}^{\hat{x}^\star} f(x, \hat{y}_1, \hat{y}_1')\, dx - \int_{x_0}^{x^\star} f(x, y_1, y_1')\, dx \right]$$

And as with transversals, we get an integral term which results in the Euler-Lagrange equation, plus the additional term

$$\Big[p_1 \delta y - H_1 \delta x \Big]_{x^\star},$$

where

$$\begin{array}{rclcrcl} \delta x(x^\star) & = & X^\star & \text{and} & \delta y(y_1^\star) & = & Y^\star \\ H_1 & = & y_1' \dfrac{\partial f}{\partial y_1'} - f & \text{and} & p_1 & = & \dfrac{\partial f}{\partial y_1'} \end{array}$$

The First Variation: part 2

Note that, for the second component of the First Variation we get a similar extra term, e.g. $\delta F_2(\eta, y)$ introduces the term

$$\Big[- p_2 \delta y + H_2 \delta x \Big]_{x^\star},$$

the sign is reversed because it corresponds to the x_0 term in the transversal problem (as opposed to the x_1 term for δF_1.

The combined second variation (minus the terms that result from the Euler-Lagrange equation which must be zero) is

$$\delta F(\eta, y) = \delta F_1(\eta, y) + \delta F_2(\eta, y) = \Big[p_1 \delta y - H_1 \delta x - p_2 \delta y + H_2 \delta x \Big]_{x^\star}.$$

Conditions

We rearrange to give

$$\delta F(\eta, y) = \Big[(p_1 - p_2)\delta y - (H_1 - H_2)\delta x\Big]_{x^\star}.$$

Note that the point of discontinuity may vary freely, so we may independently vary δx and δy or set one or both to zero. Hence, we can separate the condition to get two conditions

$$\Big[p_1 - p_2\Big]_{x^\star} = 0$$

$$\Big[H_1 - H_2\Big]_{x^\star} = 0$$

Weierstrass-Erdman

We can write the conditions as

$$p_1\Big|_{x^\star} = p_2\Big|_{x^\star}, \quad H_1\Big|_{x^\star} = H_2\Big|_{x^\star}.$$

Called the **Weierstrass-Erdman Corner Conditions**

Rather than separating y into y_1 and y_2 we may write the corner conditions in terms of limits from the left and right, e.g.

$$p\Big|_{x^{\star-}} = p\Big|_{x^{\star+}}, \quad H\Big|_{x^{\star-}} = H\Big|_{x^{\star+}}.$$

Solution

So the broken extremal solution must satisfy

- the Euler-Lagrange Equations
- and the Weierstrass-Erdman Corner Conditions

$$p\Big|_{x^{\star-}} = p\Big|_{x^{\star+}}, \quad H\Big|_{x^{\star-}} = H\Big|_{x^{\star+}},$$

must hold at any "corner"

Example 1

In the example considered,

$$p = -2y^2(1 - y'), \quad H = y^2(1 - y'^2).$$

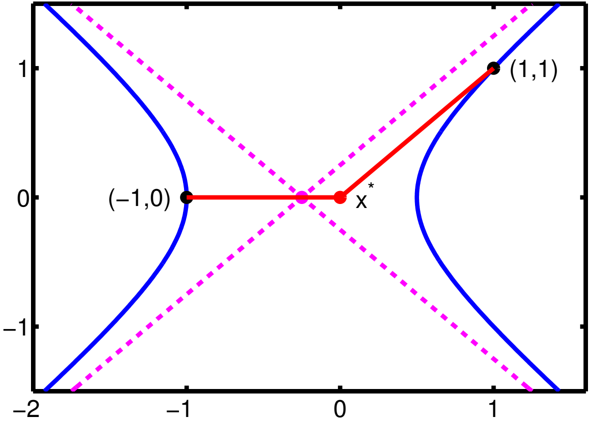
Remember that $y = 0$ and $y = x + A$ are valid solutions to the Euler-Lagrange equations, and that for both of these solutions $p = H = 0$, so we can put a "corner" where needed.

The solution must also satisfy the end-point conditions, so $y(-1) = 0$ and $y(1) = 1$, and therefore, a valid solution has $x^\star = 0$ and

$$\begin{aligned} y_1 &= 0 \text{ for } x \in [-1, x^\star] \\ y_2 &= x \text{ for } x \in [x^\star, 1] \end{aligned}$$

Example 1

The actual extremal (in red)



Obviously, this is only valid if we allow non-smooth solutions.

More insight

- sometimes we have a constraint on where the corner can appear:
    - sometimes the discontinuity arise from the problem itself, e.g., a discontinuous boundary such as in refraction (see Fermat's principle, and Snell's law in earlier lectures)
- in these cases, we need to go back to the condition

$$\delta F(\eta, y) = \Big[(p_1 - p_2)\delta y - (H_1 - H_2)\delta x\Big]_{x^\star} = 0$$

and look at whether δx or δy are forced to be zero, or if there is a relationship between them, and use that to form a constraint such as we had for transversals.

General strategy

- solve Euler-Lagrange equations
- look for solutions for each end condition
- match up the solutions at a corner $x^\star$ so that
  - $y_1(x^\star) = y_2(x^\star)$
  - the Weierstrass-Erdman Corner Conditions are satisfied
- in theory can allow more than one corner, but this would get very painful!

Newton's aerodynamical problem

Find extremal of "air resistance"

$$F\{y\} = \int_0^R \frac{x}{1+y'^2}\,dx,$$

subject to $y(0) = L$ and $y(R) = 0$ with solutions

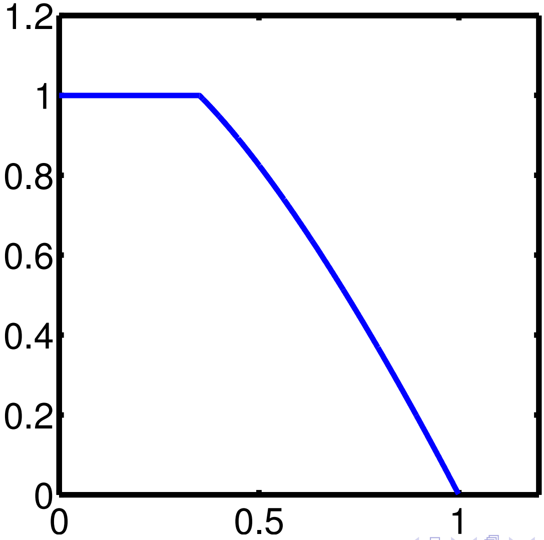
1. $y =$const for $x \in [0, x_1]$
2. $u \in [u_1, u_2]$

$$x(u) = \frac{c}{u}(1+u^2)^2 = c\left(\frac{1}{u} + 2u + u^3\right),$$

$$y(u) = L - c\left(-\log u - A + u^2 + \frac{3}{4}u^4\right).$$

Tricky bit is working out u_1 which sets the location of the "corner", and fixes A, c and u_2.

Newton's aerodynamical problem



Newton's aerodynamical problem

- we could find u_1 by trying to minimize F as a function of u_1, but this is hard because we only have a numerical solution to get u_2.
- alternative is to use corner conditions
  1. at the corner
     1. $x^\star = x(u_1)$ is free
     2. $y = L$ is fixed
  2. corner condition of interest is

$$H\Big|_{x^{\star -}} = H\Big|_{x^{\star +}}$$

Newton's aerodynamical problem

Calculating H

$$
\begin{aligned}
H &= y' \frac{\partial f}{\partial y'} - f \\
&= \frac{-2y'^2 x}{(1+y'^2)^2} - \frac{x}{(1+y'^2)} \\
&= \frac{-x}{(1+y'^2)^2} \left[2y'^2 + (1+y'^2) \right] \\
&= \frac{-x}{(1+y'^2)^2} \left[3y'^2 + 1 \right].
\end{aligned}
$$

Newton's aerodynamical problem

Corner condition

$$H = \frac{-x}{(1+y'^2)^2}\left[2y'^2+1\right].$$

Now on the LHS of $x_1 = x^\star$ we have $y' = 0$, so

$$H\Big|_{x^{\star -}} = -x^\star.$$

On the RHS, remember $y' = -u$ (from a previous lecture)

$$H\Big|_{x^{\star +}} = \frac{-x^\star}{(1+u^2)^2}\left[3u^2+1\right].$$

Newton's aerodynamical problem

$$
\begin{aligned}
H\Big|_{x^{\star -}} &= H\Big|_{x^{\star +}} \\
-x^\star &= \frac{-x^\star}{(1+u^2)^2}\left[3u^2+1\right] \\
(1+u^2)^2 &= 3u^2+1 \\
u^4-u^2 &= 0 \\
u^2(u^2-1) &= 0 \\
u &= 0 \quad \text{or} \quad \pm 1
\end{aligned}
$$

but $-y' = u > 0$ so $u = 1$ is the only valid solution, hence

$$u_1 = 1$$

and the rest of the solution follows from there.

Newton's aerodynamical problem

- real rockets don't look like this
  1. resistance functional is only approximate
     1. ignores friction
     2. ignores shock waves
  2. rockets must pass through multiple layers of atmosphere, at varying speeds
- additional constraints:
  1. nose cone is tangent to rocket at joint

$$y'(R) = -\infty$$

  2. nose is easy to build
- really, we need to do CFD++