Time Series A4

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1. For each of the following processes, where Z_t is white noise with mean zero and variance σ^2, express the model using the backward shift operator notation B and determine whether the process is stationary and/or invertible. Express any stationary ARMA processes as a general linear process.

   (a) $Y_t = Z_t - \frac{4}{3}Z_{t-1} + \frac{4}{9}Z_{t-2}$
   Solution

$$\begin{aligned} Y_t &= Z_t - \frac{4}{3}BZ_t + \frac{4}{9}B^2Z_t \\ &= (1 - \frac{4B}{3} + \frac{B^2 4}{9})Z_t \end{aligned}$$

   This is a moving average process so it is stationary.
   Invertible if $\phi(u)$ has roots outside the unit circle:

$$\phi(u) = 1 - \frac{4}{3}u + \frac{4}{9}u^2 = \left(u - \frac{3}{2}\right)^2$$

$$\implies u = \frac{3}{2}$$

   Since $|u| > 1$ it is invertible! **As required.**

   (b) $Y_t = \frac{1}{4}Y_{t-1} + Z_t - \frac{5}{6}Z_{t-1} + \frac{1}{6}Z_{t-2}$
   Solution

$$\begin{aligned} Y_t - \frac{1}{4}Y_{t-1} &= Z_t - \frac{5}{6}Z_{t-1} + \frac{1}{6}Z_{t-2} \\ (1 - \frac{1}{4}B)Y_t &= (1 - \frac{5}{6}B + \frac{1}{6}B^2)Z_t \end{aligned}$$

   Stationary and invertible if $\phi(B) = (1 - \frac{1}{4}B)$ and $\theta(B) = (1 - \frac{5}{6}B + \frac{1}{6}B^2)$ respectively have roots outside the unit circle Clearly $\phi(B)$ has root $B = 4$ so it is stationary

$$(1 - \frac{5}{6}B + \frac{1}{6}B^2) = 0 \implies \frac{1}{6}(B-3)(B-2) = 0$$

Which has roots $B = 2, 3$ both of which are outside the unit circle, so the process is invertible.

$$\begin{aligned}
(1 - \frac{1}{4}B)Y_t &= (1 - \frac{5}{6}B + \frac{1}{6}B^2)Z_t \\
Y_t &= (1 - \frac{1}{4}B)^{-1}(1 - \frac{5}{6}B + \frac{1}{6}B^2)Z_t \\
&= (1 + \frac{1}{4}B + \frac{1}{16}B^2 + \frac{1}{64}B^3 + \ldots)(1 - \frac{5}{6}B + \frac{1}{6}B^2)Z_t \\
&= (1 - \frac{5}{6}B + \frac{1}{6}B^2 + \frac{1}{4}B - \frac{5}{24}B^2 + \frac{1}{24}B^3 + \frac{1}{16}B^2 - \frac{5}{6 * 16}B^3 + \frac{1}{6 * 16}B^4 + \ldots)Z_t \\
&= (1 + (\frac{1}{4} - \frac{5}{6})B + (\frac{1}{6} - \frac{5}{24} + \frac{1}{16})B^2 + \ldots)Z_t \\
&= \left(1 - \frac{7}{12}B + \frac{1}{48}B^2 + \frac{1}{4 * 48}B^3 + \ldots\right) Z_t \\
\therefore Y_t &= Z_t - \frac{7}{12}Z_{t-1} + \frac{1}{48}\sum_{j=2}^{\infty}(\frac{1}{4})^{j-2}Z_{t-j}
\end{aligned}$$

As required.

2. For each of the following AR(2) processes with Z_t white noise, write down the Yule-Walker equations and solve for the autocorrelation function ρ_k

   (a) $Y_t = \frac{\sqrt{3}}{2}Y_{t-1} - \frac{1}{4}Y_{t-2} + Z_t$
   Solution First check for stationarity:

$$\begin{aligned}
\phi(u) &= 1 - \frac{\sqrt{3}}{2}u + \frac{1}{4}u^2 \\
0 &= 4 - 2\sqrt{3} + u^2 \\
u &= \frac{2\sqrt{3} \pm \sqrt{12 - 4(4)(1)}}{2} \\
\implies u &= \sqrt{3} \pm \sqrt{-1} = \sqrt{3} \pm i
\end{aligned}$$

Since $|u| > 1$ we have stationarity.

$$\rho_k - \frac{\sqrt{3}}{2}\rho_{k-1} + \frac{1}{4}\rho_{k-2} = 0$$

Solve auxilliary equation:

$$\begin{aligned}
\lambda^2 - \frac{\sqrt{3}}{2}\lambda + \frac{1}{4} = 0 \\
\lambda &= \frac{\frac{\sqrt{3}}{2} \pm \sqrt{\frac{3}{2} - 4\frac{1}{4}}}{2} \\
\implies \lambda &= \frac{1}{4}\left(\sqrt{3} \pm i\right) \\
&= 0.5e^{\pm i\pi/6}
\end{aligned}$$

Complex roots gives solution of form:

$$\rho_k = 0.5^k(a\cos(k\pi/6) + b\sin(k\pi/6))$$

Use the yule-walker equation for ρ_1 and use $\rho_0 = 1$

$$\begin{aligned}
\rho_0 = 1 &= 0.5^0(a\cos(0) + b\sin(0)) \\
&= a \\
\implies a &= 1
\end{aligned}$$

$$\rho_1 - \frac{\sqrt{3}}{2}\rho_0 + \frac{1}{4}\rho_{-1} = 0$$
$$\rho_1 - \frac{\sqrt{3}}{2} + \frac{1}{4}\rho_1 = 0$$
$$(1 + \frac{1}{4})\rho_1 = \frac{\sqrt{3}}{2}$$
$$\rho_1 = \frac{2\sqrt{3}}{5}$$

$$\rho_1 = \frac{2\sqrt{3}}{5} = 0.5(\cos(\pi/6) + b\sin(\pi/6))$$
$$\frac{2\sqrt{3}}{5} = 0.5(\frac{\sqrt{3}}{2} + \frac{b}{2})$$
$$-\frac{8\sqrt{3}}{5} = \sqrt{3} + b$$
$$b = -\frac{13\sqrt{3}}{5}$$

$$\rho_k = 0.5^k(\cos(k\pi/6) - \frac{13\sqrt{3}}{5}\sin(k\pi/6))$$

As required.

(b) $Y_t = \frac{8}{5}Y_{t-1} - \frac{16}{25}Y_{t-2} + Z_t$

Solution Stationarity:

$$\phi(u) = 1 - \frac{8}{5}u + \frac{16}{25}u^2$$
$$0 = 1 - \frac{8}{5}u + \frac{16}{25}u^2$$
$$0 = (u - \frac{5}{4})^2$$
$$\implies u = \frac{5}{4}$$

Which gives stationarity.
Yule-Walker:

$$\rho_k - \frac{8}{5}\rho_{k-1} + \frac{16}{25}\rho_{k-2}$$

Auxilliary equation:

$$\lambda^2 - \frac{8}{5}\lambda + \frac{16}{25} = 0$$
$$\lambda = \frac{4}{5}$$

Repeated root gives solutions of form:

$$\rho_k = (c_1 + c_2 k)(\frac{4}{5})^k$$

Use $\rho_0 = 1$

$$\rho_0 = 1 = c_1$$
$$\implies \rho_k = (1 + c_2 k)(\frac{4}{5})^k$$

Now use the Yule-Walker equation for ρ_1:

$$\begin{aligned}
\rho_1 - \frac{8}{5}\rho_0 + \frac{16}{25}\rho_{-1} &= 0 \\
\rho_1 - \frac{8}{5} + \frac{16}{25}\rho_1 &= 0 \\
(1 + \frac{16}{25})\rho_1 &= \frac{8}{5} \\
\rho_1 &= \frac{40}{41}
\end{aligned}$$

Put this into the ρ_k equation:

$$\begin{aligned}
\rho_1 = (1 + c_2)(\frac{4}{5}) &= \frac{40}{41} \\
\implies c_2 &= \frac{9}{41}
\end{aligned}$$

Which gives solution:

$$\rho_k = (1 - \frac{9}{41}k)(\frac{4}{5})^k$$

As required.