Topic C Assignment 2

Andrew Martin

May 3, 2019

1. (a) Let

$$v(t) = v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \mathcal{O}(\epsilon^3)$$
$$\frac{dv}{dt} = \frac{dv_0(t)}{dt} + \epsilon \frac{dv_1(t)}{dt} + \epsilon^2 \frac{dv_2(t)}{dt} + \mathcal{O}(\epsilon^3)$$

$$\frac{dv}{dt} + \epsilon v^2 + t = 0$$
$$\frac{dv_0(t)}{dt} + \epsilon \frac{dv_1(t)}{dt} + \epsilon^2 \frac{dv_2(t)}{dt} + \mathcal{O}(\epsilon^3) + \epsilon(v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \mathcal{O}(\epsilon^3))^2 + t = 0$$

Subject to the IC:

$$v(0) = 0$$
$$\implies v_0(0) + \epsilon v_1(0) + \epsilon^2 v_2(0) + \mathcal{O}(\epsilon^3) = 0$$

$$\frac{dv_0(t)}{dt} + \epsilon \frac{dv_1(t)}{dt} + \epsilon^2 \frac{dv_2(t)}{dt} + \epsilon \left(v_0(t) + \epsilon v_1(t)\right)^2 + t + \mathcal{O}(\epsilon) = 0$$
$$\frac{dv_0(t)}{dt} + \epsilon \frac{dv_1(t)}{dt} + \epsilon^2 \frac{dv_2(t)}{dt} + \epsilon v_0(t)^2 + \epsilon^2 v_0(t) v_1(t) + t + \mathcal{O}(\epsilon^3) = 0$$

Collecting powers of epsilon gives the set of equations:

$$\frac{dv_0(t)}{dt} + t = 0, \quad v_0(0) = 0$$
$$\frac{dv_1(t)}{dt} + v_0(t)^2 = 0, \quad v_1(0) = 0$$
$$\frac{dv_2(t)}{dt} + v_0(t)v_1(t) = 0, \quad v_2(0) = 0$$

Solve v_0:

$$\frac{dv_0(t)}{dt} + t = 0, \quad v_0(0) = 0$$
$$v_0(t) = \int -t dt$$
$$= -\frac{t^2}{2} + C$$
$$v_0(0) = 0 \implies v_0(t) = -\frac{t^2}{2}$$

Now solve v_1:

$$\begin{aligned}
\frac{dv_1(t)}{dt} + v_0(t)^2 = 0, \quad v_1(0) = 0 \\
v_1(t) &= \int -\frac{t^4}{4} dt \\
&= -\frac{t^5}{20} + C \\
v_1(0) = 0 \implies v_1(t) &= -\frac{t^5}{20}
\end{aligned}$$

Lastly, solve v_2:

$$\begin{aligned}
\frac{dv_2(t)}{dt} + v_0(t)v_1(t) = 0, \quad v_2(0) = 0 \\
v_2(t) &= \int -\left(\frac{t^5}{20}\frac{t^2}{2}\right) dt \\
&= \int -\left(\frac{t^7}{40}\right) dt \\
&= -\frac{t^8}{320} + C \\
v_2(0) = 0 \implies v_2(t) &= -\frac{t^8}{320}
\end{aligned}$$

Thus the expansion of $v(t)$ is

$$\begin{aligned}
v_t(t) &= v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \mathcal{O}(\epsilon^3) \\
&= -\frac{t^2}{2} - \epsilon\frac{t^5}{20} - \epsilon^2\frac{t^8}{320} + \mathcal{O}(\epsilon^3) \\
\therefore v_t(t) &\sim -\frac{t^2}{2} - \epsilon\frac{t^5}{20} - \epsilon^2\frac{t^8}{320}
\end{aligned}$$

as $\epsilon \to 0$

(b) The range of times t for which this expansion is valid are those such that

$$v_0(t) \gg \epsilon v_1(t) \gg \epsilon^2 v_2$$

I.e. t such that

$$\frac{t^2}{2} \gg \epsilon\frac{t^5}{20} \gg -\epsilon^2\frac{t^8}{320}$$

For v_0, ϵv_1:

$\epsilon v_1 \ll v_0$ if:

$$\lim_{\epsilon\to 0}\frac{\epsilon v_1(t)}{v_0(t)} = 0$$

So find t such that this limit doesn't converge to 0

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{\epsilon v_1(t)}{v_0(t)} &= \lim_{\epsilon\to 0}\frac{-\epsilon\frac{t^5}{20}}{-\frac{t^2}{2}} \\
&= \lim_{\epsilon\to 0}\epsilon\frac{t^3}{10} \\
&= \frac{1}{10}\lim_{\epsilon\to 0}\epsilon t^3
\end{aligned}$$

This limit converges to 0 if t^3 is $\mathcal{O}(\epsilon^{-1}$, i.e. we require

$$0 \leq t \leq \mathcal{O}(\epsilon^{-1/3})$$

For ϵv_1, $\epsilon^2 v_2$

$$\begin{aligned}\lim_{\epsilon \to 0} \frac{\epsilon^2 v_2}{\epsilon v_1} &= \lim_{\epsilon \to 0} \frac{\epsilon \frac{t^8}{320}}{\frac{t^5}{20}} \\ &= \lim_{\epsilon \to 0} \frac{\epsilon t^3}{16}\end{aligned}$$

And hence this converges if $t^3 \leq \mathcal{O}(\epsilon)$, and as before

$$0 \leq t \leq \mathcal{O}(\epsilon^{-1/3})$$

I.e. we require $t \leq \mathcal{O}(\epsilon^{-1/3})$ for the asymptotic ordering to hold

2. (a) Want to find

$$\psi = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2$$

Which satisfies

$$\nabla^2 \psi = -1$$

Use a taylor series to expand around the boundary condition

$$\psi_{y=\pm(1+\epsilon \cos kx)} = \psi_{y=\pm 1} \pm \epsilon \cos kx \left.\frac{\partial \psi}{\partial y}\right|_{y=\pm 1} + \frac{1}{2}\epsilon^2 \cos^2 kx \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{y=\pm 1} = 0$$

For zeroth order: Since the flow is unperturbed in the x direction, assume

$$\psi_0(x, y) = a + by + cy^2$$

Using the boundary condition:

$$a \pm b + c = 0$$

Hence $b = 0$, and $a = -c$. Using the flow description:

$$\begin{aligned}\nabla^2 \psi_0 = 2c &= -1 \\ \implies c &= -\frac{1}{2} \\ \implies b &= \frac{1}{2}\end{aligned}$$

Hence

$$\boxed{\psi_0 = \frac{1}{2}\left(1 - y^2\right)}$$

Now first order, $\mathcal{O}(\epsilon)$:

$$\nabla^2 \psi_1 = 0$$

With boundary:

$$\begin{aligned}\psi_1 \pm \cos kx \frac{\partial \psi_0}{\partial y} &= 0, \quad y = \pm 1 \\ \psi_1 \pm \cos kx(-y) &= 0, \quad y = \pm 1\end{aligned}$$

Let $\psi_1 = f(x)g(y)$

$$\begin{aligned}f''g + g''f &= 0 \\ \frac{f''}{f} = -\frac{g''}{g} &= \lambda\end{aligned}$$

- $\lambda = 0$

$$f'' = 0$$
$$f = ax + b$$
$$g = cy + d$$

This cannot satisfy the boundaries.

- $\lambda = \mu^2 > 0$

$$f'' = \mu^2 f$$
$$f = a\cosh(\mu x) + b\sinh(\mu x)$$

And for g

$$g'' = -\mu^2 g$$
$$g = c\sin(\mu y) + d\cos(\mu y)$$

$$f(0) = a = a\cosh(\mu 2\pi/k) + b\sinh(\mu 2\pi/k)$$

True if $\mu = k$

$$\psi_1 = (a\cosh kx + b\sinh kx)(c\sin ky + d\cos ky)$$

$$\begin{aligned} \psi_1 \pm \cos kx(-y) = 0, \quad y = \pm 1 \\ (a\cosh kx + b\sinh kx)(c\sin(\pm k) + d\cos(\pm k)) \pm \cos kx(-y) = 0 \end{aligned}$$

Which gives trivial solutions.

- $\lambda = \mu^2 < 0$

$$f = a\sin(\mu x) + b\cos(\mu x)$$
$$g = c\sinh(\mu y) + d\cosh(\mu y)$$

$$\begin{aligned} f(0) = a\sin(0) + b\cos(0) = b = f(2\pi/k) \\ f(2\pi/k) = a\sin(2\mu\pi/k) + b\cos(2\mu\pi/k) \end{aligned}$$

This gives $\mu = k$
Using the boundary condition:

$$\begin{aligned} (a\sin(kx) + b\cos(kx))(c\sinh(\pm k) + d\cosh(\pm k)) \pm (\mp)\cos kx = 0 \\ (a\sin(kx) + b\cos(kx))(\pm c\sinh(k) + d\cosh(k)) = \cos kx \\ (a\tan(kx) + b)(\pm c\sinh(k) + d\cosh(k)) = 1 \\ a\tan kx + b = \frac{1}{\pm c\sinh k + d\cosh k} \end{aligned}$$

Clearly $a = c = 0$ for non-trivial solutions

$$b = \frac{1}{d\cosh(k)}$$

Hence

$$\boxed{\psi_1 = \frac{\cos kx \cosh ky}{\cosh k}}$$

For ψ_2:

$$\nabla^2 \psi_2 = 0$$

Boundary condition:

$$\begin{aligned}
\psi_2 \pm \cos kx \frac{\partial \psi_1}{\partial y} + \cos^2 kx \frac{\partial^2 \psi_0}{\partial y^2} &= 0,\ y = \pm 1 \\
\psi_{2,y=\pm 1} \pm \cos kx \frac{k \cos kx \sinh(\pm k)}{\cosh k} - \cos^2 kx &= 0 \\
\psi_{2,y=\pm 1} + (k \tanh(k) - 1) \cos^2 kx &= 0 \\
\implies \psi_{2,y=\pm 1} &= (1 - k \tanh(k)) \cos^2 kx
\end{aligned}$$

And $\psi_2(0, y) = \psi_2(2\pi/k, y)$,
ψ_2 will have form:

$$\psi_2 = g(y)(1 - k \tanh(k)) \cos^2 kx$$

Where $h(\pm 1) = 0$

$$\begin{aligned}
\nabla^2 \psi_2 = g''(y)(1 - k \tanh(k)) \cos^2 kx + g(y)(1 - k \tanh(k)) 2k^2 (\sin^2 kx - \cos^2 kx) \\
g''(y)(1 - k \tanh(k)) \cos^2 kx - 4k^2 g(y)(1 - k \tanh(k)) \cos^2 kx = 0 \\
g''(y) - 4k^2 g(y) = 0 \\
\frac{g''}{g} = 4k^2 \\
\implies g = a_* e^{2ky} + b_* e^{-2ky} \\
\implies g = a \cosh 2ky + b \sinh 2ky
\end{aligned}$$

Require $g(\pm 1) = 1$

$$\begin{aligned}
g(\pm 1) = a \cosh(\pm 2k) + b \sinh(\pm 2k) &= 1 \\
a \cosh(2k) \pm b \sinh(2k) &= 1 \\
b = 0, \quad a &= \frac{1}{\cosh 2k}
\end{aligned}$$

Hence

$$\boxed{\psi_2 = \frac{(1 - k \tanh(k))}{\cosh 2k} \cos^2 kx \cosh 2ky}$$

Therefore:

$$\boxed{\psi = \frac{1}{2}(1 - y^2) + \epsilon \left(\frac{\cos kx \cosh ky}{\cosh k} \right) + \epsilon^2 \left(\frac{(1 - k \tanh(k)) \cos^2 kx \cosh 2ky}{\cosh 2k} \right)}$$

Figure 2a shows the contour of ψ for $k = 0.6$ and $\epsilon = 0.1$. The

(b) Streamlines will be

$$\begin{aligned}
u &= \frac{\partial \psi}{\partial y} \\
&= -y + \epsilon k \left(\frac{\cos kx \sinh ky}{\cosh k} \right) + \epsilon^2 2k \left(\frac{(1 - k \tanh(k)) \cos^2 kx \sinh 2ky}{\cosh 2k} \right)
\end{aligned}$$

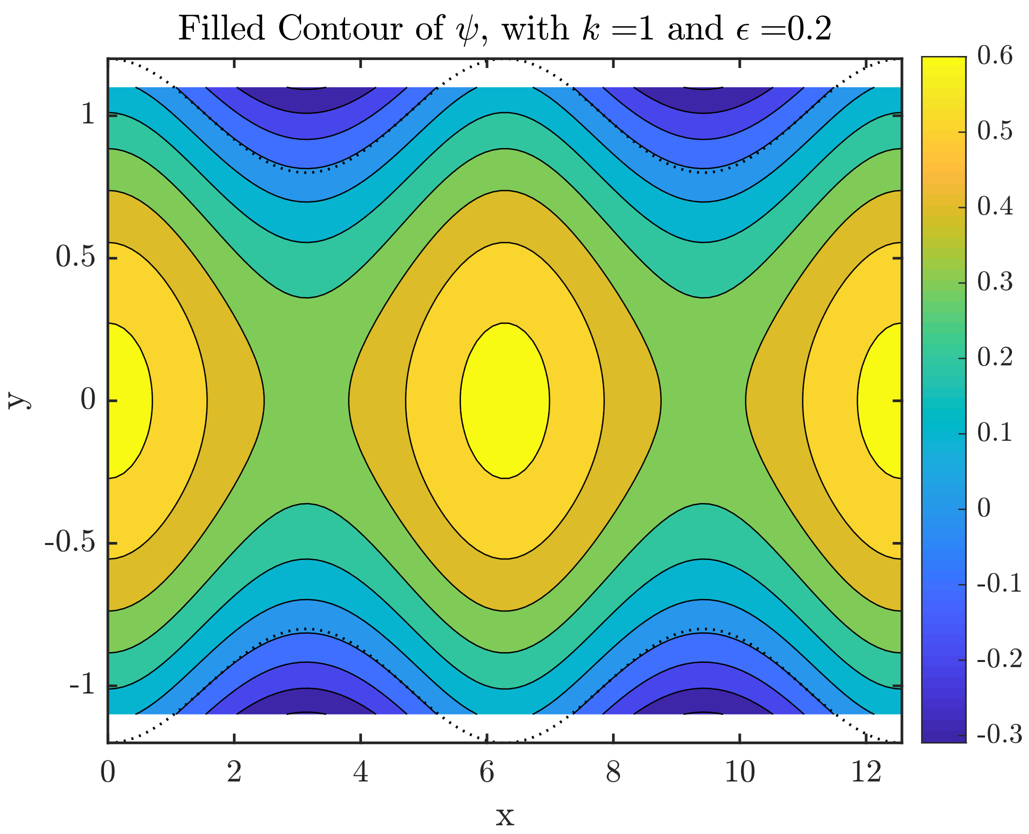


Figure 1: Contour of ψ against a given k and ϵ The dotted line represents the boundary conditions, at $y = \pm(1 + \epsilon \cos kx)$

$$\begin{aligned} v &= -\frac{\partial \psi}{\partial x} \\ &= \epsilon k \left(\frac{\sin kx \cosh ky}{\cosh k} \right) + \epsilon^2 2k \left(\frac{(1 - k\tanh(k)) \sin kx \cos kx \sinh 2ky}{\cosh 2k} \right) \end{aligned}$$

Figure 2b shows the plots of the streamlines. The vertically the plots show a change in k, while across is a change in ϵ. Note when ϵ gets quite large, closed loops can form (the yellow paths for the two left plots).

(c) If $k = \frac{n\pi}{2}$ for $n = 1, 3, 5, \ldots$ there can be issues, Since the ϵ and ϵ^2 terms become infinite. Since the boundary is

$$y = \pm(1 + \epsilon \cos kx)$$

If $\epsilon >= 1$ we will have regions of the flow where the boundaries will overlap I.e. $-h(x; \epsilon) >= h(x; \epsilon)$

The assumption that the channel is infinitely long is also necessary. The formulae used assumed incompressible flow, so we expect the flux of flow to be low when the gap between the boundaries is largest, i.e. for $x = 2nk\pi$ for integer n. Similarly we expect it to be the largest when $x = \frac{\pi}{2} + nk\pi$.

The solution is also questionable as there are these stagnation points which the flow circulates around.

3. (a) I will use $h := h(x, t)$ for shorthand Linearise the conditions about the unperturbed

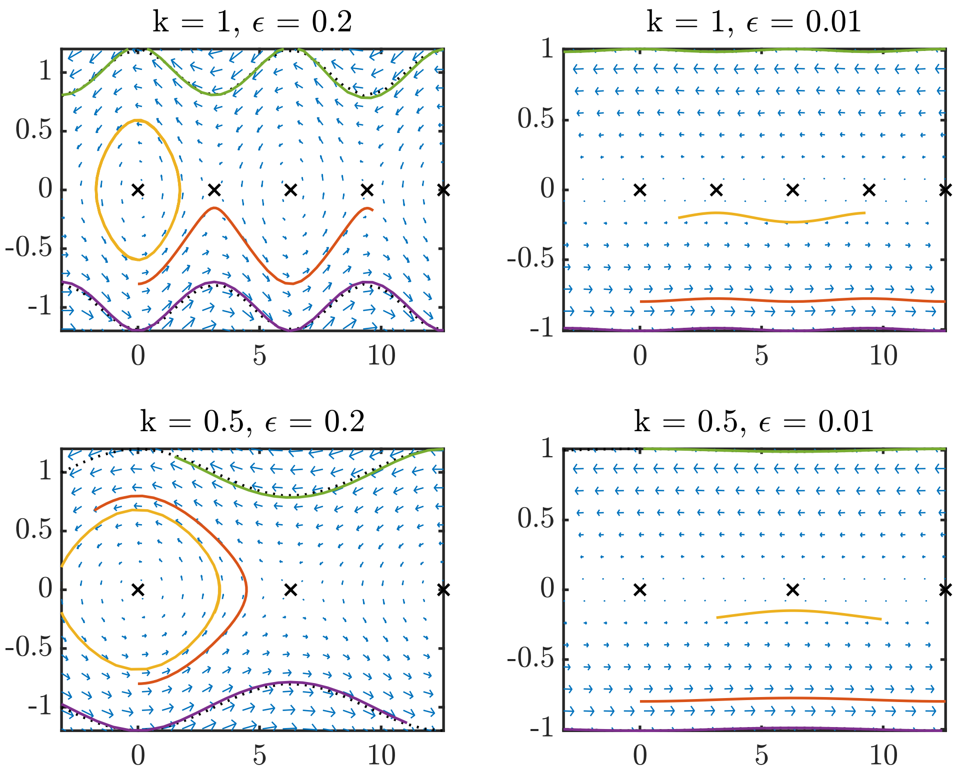


Figure 2: Streamline plots for various k and ϵ with some example paths. $\times$ marks stagnant points in the flow, the purple and green lines are the flows along the negative and positive boundaries respectively, yellow and red are example paths

interface, $y = 0$. Using a taylor series:

$$\begin{aligned}
\frac{\partial \phi}{\partial y}\Big|_{y=\epsilon h} &= \frac{\partial \phi}{\partial y}\Big|_{y=0} + \epsilon h \frac{\partial^2 \phi}{\partial y^2}\Big|_{y=0} + \mathcal{O}(\epsilon^2) \\
\frac{\partial \phi}{\partial x}\Big|_{y=\epsilon h} &= \frac{\partial \phi}{\partial x}\Big|_{y=0} + \epsilon h \frac{\partial^2 \phi}{\partial x \partial y}\Big|_{y=0} + \mathcal{O}(\epsilon^2) \\
\frac{\partial \phi}{\partial t}\Big|_{y=\epsilon h} &= \frac{\partial \phi}{\partial t}\Big|_{y=0} + \epsilon h \frac{\partial^2 \phi}{\partial t \partial y}\Big|_{y=0} + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Since we are only considering powers up to ϵ (neglecting ϵ^2) I will drop all ϵ^2 (and higher) terms.

Kinematic Conditions

$$\frac{\partial \phi}{\partial y} = \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} \right), \quad y = \epsilon h(x,t)$$

$$\frac{\partial \phi}{\partial y} + \epsilon h \frac{\partial^2 \phi}{\partial y^2} = \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \left(\frac{\partial \phi}{\partial x} + \epsilon h \frac{\partial^2 \phi}{\partial x \partial y} \right) \right), \quad y = 0$$

Which is now on $y = 0$, for both ϕ_1 and ϕ_2.

Bernoulli Condition

$$\rho_1\left(\frac{\partial\phi_1}{\partial t}-\frac{1}{2}c_1^2+\frac{1}{2}|\nabla\phi_1|^2+gy\right)=\rho_2\left(\frac{\partial\phi_2}{\partial t}-\frac{1}{2}c_2^2+\frac{1}{2}|\nabla\phi_2|^2+gy\right),\quad y=\epsilon h(x,t)$$

$$\rho_1\left(\frac{\partial\phi_1}{\partial t}\Big|_{y=0}+\epsilon h\frac{\partial\phi_1}{\partial y\partial t}\Big|_{y=0}-\frac{1}{2}c_1^2+\frac{1}{2}|\nabla\phi_{1_{y=0}}|^2+\epsilon gh\right)$$
$$=\rho_2\left(\frac{\partial\phi_2}{\partial t}\Big|_{y=0}+\epsilon h\frac{\partial\phi_2}{\partial y\partial t}\Big|_{y=0}-\frac{1}{2}c_2^2+\frac{1}{2}|\nabla\phi_{2_{y=0}}|^2+\epsilon gh\right)$$

On $y=0$.
Where all of these conditions are true up to $\mathcal{O}(\epsilon^2)$

(b) Introducing

$$\phi_1(x,y,t)=\phi_{10}(x,y,t)+\epsilon\phi_{11}(x,y,t)+\mathcal{O}(\epsilon^2)$$
$$\phi_2(x,y,t)=\phi_{20}(x,y,t)+\epsilon\phi_{21}(x,y,t)+\mathcal{O}(\epsilon^2)$$

Now collect powers for all the conditions:
Kinematic condition:

$$\mathcal{O}(1):\frac{\partial\phi_{\#0}}{\partial y}=0$$
$$\mathcal{O}(\epsilon):\frac{\partial\phi_{\#1}}{\partial y}+h\frac{\partial^2\phi_{\#0}}{\partial y^2}=\frac{\partial h}{\partial t}+\frac{\partial h}{\partial x}\left(\frac{\partial\phi_{\#0}}{\partial x}\right)$$

Bernoulli condition:

$$\mathcal{O}(1):\rho_1\left(\frac{\partial\phi_{10}}{\partial t}-\frac{1}{2}c_1^2+\frac{1}{2}\left(\left(\frac{\partial\phi_{10}}{\partial y}\right)^2+\left(\frac{\partial\phi_{10}}{\partial x}\right)^2+\frac{\partial\psi_{10}}{\partial x}\frac{\partial\psi_{10}}{\partial y}\right)\right)$$
$$=\rho_2\left(\frac{\partial\phi_{20}}{\partial t}-\frac{1}{2}c_2^2+\frac{1}{2}\left(\left(\frac{\partial\phi_{20}}{\partial y}\right)^2+\left(\frac{\partial\phi_{20}}{\partial x}\right)^2+\frac{\partial\psi_{20}}{\partial x}\frac{\partial\psi_{20}}{\partial y}\right)\right)$$

I will write the $\mathcal{O}(\epsilon)$ version below.
The unperterbed problem is described by:

$$\nabla^2\phi_{10}=0,\quad y>0$$
$$\nabla^2\phi_{20}=0,\quad y>0$$

With kinematic conditions:

$$\frac{\partial\phi_{10}}{\partial y}=0,\quad y=0$$
$$\frac{\partial\phi_{20}}{\partial y}=0,\quad y=0$$

The flow equation reduces to

$$\frac{\partial^2\phi_{\#0}}{\partial x^2}=0$$

Which gives

$$\phi_{\#0}=A_{\#}x+B_{\#}$$

And using the condition far awayfrom the interface, it gives

$$\phi_{10} = c_1 x$$

$$\phi_{20} = c_2 x$$

To first power of ϵ, the problem becomes:
Flow description

$$\nabla^2 \phi_{\#1} = 0$$

For ϕ_1 with $y > 0$, and for ϕ_2 around $y < 0$
Kinematic Condition

$$\begin{aligned}\frac{\partial \phi_{\#1}}{\partial y} + h\frac{\partial^2 \phi_{\#0}}{\partial y^2} &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(\frac{\partial \phi_{\#0}}{\partial x}\right) \\ \frac{\partial \phi_{\#1}}{\partial y} &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(\frac{\partial \phi_{\#0}}{\partial x}\right)\end{aligned}$$

Bernoulli Condition

$$\begin{aligned}&\rho_1\Big(\frac{\partial \phi_{11}}{\partial t} + h\frac{\partial \phi_{10}}{\partial y \partial t} + \frac{1}{2}(|\nabla \phi_1|^2 + gh\Big) \\ &= \rho_2\Big(\frac{\partial \phi_{21}}{\partial t} + h\frac{\partial \phi_{20}}{\partial y \partial t} + \frac{1}{2}|\nabla \phi_2|^2 + gh\Big)\end{aligned}$$

On $y = 0$, extracting the $\mathcal{O}(\epsilon)$ part of $|\nabla \phi_{\#}|^2$.
With the far condition that

$$\phi_{11} = 0, \quad y \to \infty$$

$$\phi_{21} = 0, \quad y \to -\infty$$

(c) Assume

$$h(x,t) = ae^{i(kx - \omega t)}$$

Since it is Laplaces' equation, we can assume that $\phi_{\#1}$ has form:

$$\phi_{\#1} = f(y)e^{i(kx-\omega t)}$$

$$\begin{aligned}\nabla^2 \phi_{\#1} = (-k^2 f + f'')e^{i(kx-\omega t})&= 0 \\ -k^2 f + f'' &= 0 \\ f''/f &= k^2 \\ \implies f_{\#} &= be^{k_{\#} y}\end{aligned}$$

To satisfy the far field conditions:

$$\begin{aligned}\phi_{11} = 0, \quad y &\to \infty \\ \implies e^{k_1 \infty} &= 0 \\ \phi_{21} = 0, \quad y &\to -\infty \\ \implies e^{-k_2 \infty} &= 0\end{aligned}$$

And so $k_1 = -k$ and $k_2 = k$. Hence

$$\phi_{11} = b_1 e^{-ky} e^{i(kx-\omega t)}, \quad \phi_{21} = b_2 e^{ky} e^{i(kx-\omega t)}$$

Now using the kinematic condition(s)

$$\begin{aligned}
\frac{\partial \phi_{11}}{\partial y} &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(\frac{\partial \phi_{\#0}}{\partial x}\right), \quad y = 0 \\
-kb_1 e^{-k0} e^{i(kx-\omega t)} &= -i\omega a e^{i(kx-\omega t)} + iac_1 e^{i(kx-\omega t)} \\
-kb_1 &= -i\omega a + iac_1 k \\
b_1 &= \frac{i\omega a - iac_1 k}{k} \\
&= (-ai)\frac{-\omega + c_1 k}{|k|}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \phi_{21}}{\partial y} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(\frac{\partial \phi_{\#0}}{\partial x}\right) \\
kb_2 e^{i(kx-\omega t)} &= -i\omega a e^{i(kx-\omega t)} + iac_2 k e^{i(kx-\omega t)} \\
kb_2 &= -i\omega a + iac_2 k \\
b_2 &= \frac{-i\omega a + iac_2 k}{k} \\
&= (ai)\frac{-\omega + c_2 k}{k}
\end{aligned}$$

Hence

$$\phi_{11} = (-ai)\frac{-\omega + c_1 k}{k} e^{-ky} e^{i(kx-\omega t)}, \quad \phi_{21} = (ai)\frac{-\omega + c_2 k}{k} e^{ky} e^{i(kx-\omega t)}$$

(d) Calculate (at $\mathcal{O}(\epsilon)$):

$$|\nabla \phi|^2, \quad y = 0$$

Expanding $|\nabla \phi|_\epsilon^2$, where the subscript ϵ is the order

$$\begin{aligned}
|\nabla \phi_1|_\epsilon^2 &= |\nabla \phi_{10} + \epsilon \nabla \phi_{11}|_\epsilon^2 \quad (y = 0) \\
&= \left| c_1 + \epsilon ik(-ai)\frac{-\omega + c_1 k}{k} e^{-ky} e^{i(kx-\omega t)} - \epsilon k(-ai)\frac{-\omega + c_1 k}{k} e^{-ky} e^{i(kx-\omega t)} \right|_\epsilon^2 \\
&= \left| c_1 + \epsilon ka\frac{-\omega + c_1 k}{k} e^{i(kx-\omega t)} + \epsilon kai\frac{-\omega + c_1 k}{k} e^{i(kx-\omega t)} \right|_\epsilon^2 \\
&= \left(c_1 + \epsilon ka\frac{-\omega + c_1 k}{k} e^{i(kx-\omega t)} \right)^2 + \left(\epsilon ka\frac{-\omega + c_1 k}{k} e^{i(kx-\omega t)} \right)_\epsilon^2 \\
&= 2c_1 ka\frac{-\omega + c_1 k}{k} e^{i(kx-\omega t)} \\
&= i2kc_1 \phi_{11,y=0}
\end{aligned}$$

And for ϕ_2:

$$
\begin{aligned}
|\nabla\phi_2|_\epsilon^2 &= |\nabla\phi_{20} + \nabla\epsilon\phi_{21}|_\epsilon^2 \quad (y=0)\\
&= \left| \, c_2 + \epsilon ik(ai)\frac{-\omega + c_2k}{k}e^{-ky}e^{i(kx-\omega t)} - \epsilon k(ai)\frac{-\omega + c_2k}{k}e^{-ky}e^{i(kx-\omega t)} \, \right|_\epsilon^2\\
&= \left| \, c_2 - \epsilon ka\frac{-\omega + c_2k}{k}e^{i(kx-\omega t)} - \epsilon kai\frac{-\omega + c_2k}{k}e^{i(kx-\omega t)} \, \right|_\epsilon^2\\
&= \left(c_2 - \epsilon ka\frac{-\omega + c_1k}{k}e^{i(kx-\omega t)}\right)^2 + \left(\epsilon ka\frac{-\omega + c_2k}{k}e^{i(kx-\omega t)}\right)_\epsilon^2\\
&= -2c_2ka\frac{-\omega + c_2k}{k}e^{i(kx-\omega t)}\\
&= i2kc_2\phi_{21,y=0}
\end{aligned}
$$

Now using the bernoulli condition:

$$
\begin{aligned}
&\rho_1\left(\frac{\partial\phi_{11}}{\partial t} + \frac{1}{2}\left(|\nabla\phi_1|_\epsilon^2\right) + gh\right) = \rho_2\left(\frac{\partial\phi_{21}}{\partial t} + \frac{1}{2}\left(|\nabla\phi_2|_\epsilon^2\right) + gh\right)\\
&\rho_1\left(\frac{\partial\phi_{11}}{\partial t} + \frac{1}{2}\left(i2kc_1\phi_{11}\right) + gh\right) = \rho_2\left(\frac{\partial\phi_{21}}{\partial t} + \frac{1}{2}\left(i2kc_2\phi_{21}\right) + gh\right)
\end{aligned}
$$

$$
\begin{aligned}
&\rho_1\left(-i\omega(-ai)\frac{-\omega + c_1k}{k}e^{i(kx-\omega t)} + ikc_1(-ai)\frac{-\omega + c_1k}{k}e^{i(kx-\omega t)} + gae^{i(kx-\omega t)}\right)\\
&= \rho_2\left(-i\omega(ai)\frac{-\omega + c_2k}{k}e^{i(kx-\omega t)} + ikc_2(ai)\frac{-\omega + c_2k}{k}e^{i(kx-\omega t)} + gae^{i(kx-\omega t)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\rho_1\left(-\omega a\frac{-\omega + c_1k}{k} + kc_1a\frac{-\omega + c_1k}{k} + ga\right) = \rho_2\left(\omega a\frac{-\omega + c_2k}{k} - kc_2a\frac{-\omega + c_2k}{k} + ga\right)\\
&\rho_1\left(\frac{a\omega^2 - c_1ka\omega - \omega kc_1a + c_1^2k^2a}{k} + ga\right) = \rho_2\left(\frac{-a\omega^2 + c_2ka\omega + kc_2a\omega - c_2^2k^2a}{k} + ga\right)\\
&\rho_1\left(\omega^2 - 2c_1k\omega + c_1^2k^2 + gk\right) + \rho_2\left(\omega^2 - 2c_2k\omega + c_2^2k^2 - gk\right) = 0\\
&(\rho_1 + \rho_2)\omega^2 - (\rho_1c_1 + \rho_2c_2)2k\omega + (\rho_1c_1^2 + \rho_2c_2^2)k^2 + (\rho_1 - \rho_2)gk = 0
\end{aligned}
$$

Which is a quadratic in ω

$$
\omega = \frac{2k(\rho_1c_1 + \rho_2c_2) \pm \sqrt{4k^2(\rho_1c_1 + \rho_2c_2)^2 - 4(\rho_1 + \rho_2)((\rho_1c_1^2 + \rho_2c_2^2)k^2 + (\rho_1 - \rho_2)gk)}}{2(\rho_1 + \rho_2)}
$$

Cleaning the square root:

$$
\begin{aligned}
\Delta\omega &= 4k^2(\rho_1c_1 + \rho_2c_2)^2 - 4(\rho_1 + \rho_2)((\rho_1c_1^2 + \rho_2c_2^2)k^2 + (\rho_1 - \rho_2)gk)\\
&= 4k^2(\rho_1^2c_1^2 + \rho_2^2c_2^2 + 2\rho_1\rho_2c_1c_2)\\
&\quad - 4(\rho_1^2c_1^2k^2 + \rho_2^2c_2^2k^2 + \rho_1\rho_2c_2^2k^2 + \rho_2\rho_1c_1^2k^2 + \rho_1^2gk + \rho_2^2gk)\\
\\
&= 4k\left(2k\rho_1\rho_2c_1c_2 - \rho_1\rho_2c_2^2k - \rho_1\rho_2c_1^2k - \rho_1^2g - \rho_2^2g\right)
\end{aligned}
$$

(e) Noting that $\rho_1, \rho_2 > 0$ is a physical constraint. There will be imaginary part (and hence solutions will be unstable) if

$$\begin{aligned}
4k\left(2k\rho_1\rho_2c_1c_2 - \rho_1\rho_2c_2^2k - \rho_1\rho_2c_1^2k - \rho_1^2g - \rho_2^2g\right) &< 0\\
2k\rho_1\rho_2c_1c_2 - \rho_1\rho_2c_2^2k - \rho_1\rho_2c_1^2k &< gk(\rho_1^2+\rho_2^2)\\
\rho_1\rho_2k^2\left(2c_1c_2 - c_2^2 - c_1^2\right) &< gk(\rho_1^2+\rho_2^2)\\
\frac{\rho_1\rho_2}{\rho_1^2+\rho_2^2}\left(c_1+c_2\right)^2 &> -\frac{g}{k}\\
\text{positive thing} &> -\frac{1}{k}
\end{aligned}$$

So positive k (and small negative values) will give unstable solutions, which will blow up. Otherwise, the conditions will be purely oscillatory and will not die out/blow up.

Matlab Code

```matlab
%Better plots
set(groot, 'DefaultLineLineWidth', 1, ...
    'DefaultAxesLineWidth', 1, ...
    'DefaultAxesFontSize', 12, ...
    'DefaultTextFontSize', 12, ...
    'DefaultTextInterpreter', 'latex', ...
    'DefaultLegendInterpreter', 'latex', ...
    'DefaultColorbarTickLabelInterpreter', 'latex', ...
    'DefaultAxesTickLabelInterpreter','latex');

%%Q2b
close all
clear all
epsilon = 0.2;
k =1;
[x,y] = meshgrid(linspace(0,4*pi/k),linspace(-1.1,1.1));
hplus = 1 + epsilon*cos(k*x);
hminus = -hplus;
psi0 = 0.5*(1-y.^2);
psi1 = cos(k*x).*cosh(k*y)/(cosh(k));
psi2 =((1-k*tanh(k))/cosh(2*k))*cosh(2*k*y).*(cos(k*x).^2) ;
psi = psi0 + epsilon*psi1 + epsilon^2*psi2;
contourf(x,y,psi)%,'Edgecolor','none')
hold on
plot(x(1,:), hplus(1,:), ':k')
plot(x(1,:), hminus(1,:), ':k')
xlabel("x")
ylabel("y")
 titlestr = "Filled Contour of $$\psi$$, with $$k=$$"+num2str(k) ...
    + " and $$\epsilon=$$" +num2str(epsilon);
 title( titlestr )
colorbar
hold off
saveas(gcf,"TopicCA2Q2a.eps",'epsc')
%%
```

```
%%streamlines
close all

figure
title("Streamline plots")
subplot(2,2,1)
bunchaplots(1,0.2,gcf);
subplot(2,2,2)
bunchaplots(1,0.01,gcf);
subplot(2,2,3)
bunchaplots(0.5,0.2,gcf);
subplot(2,2,4)
bunchaplots(0.5,0.01,gcf);
saveas(gcf,"TopicCA2Q2b.eps",'epsc')

function out = bunchaplots(k,epsilon,fig)
[x,y] = meshgrid(linspace(-pi,4*pi,20),linspace(-1.5,1.5,20));
u0 =@(x,y) -y;
u1 =@(x,y) k*(cos(k*x).*sinh(k*y)/(cosh(k)));
u2 =@(x,y) 2*k* ((1-k*tanh(k))/cosh(2*k))*sinh(2*k*y).*(cos(k*x).^2);

u=@(x,y) u0(x,y) + epsilon*u1(x,y) + epsilon^2 *u2(x,y);

v1 =@(x,y) -k *(sin(k*x).*cosh(k*y)/(cosh(k)));
v2 =@(x,y) -2*k*(((1-k*tanh(k))/cosh(2*k))*cosh(2*k*y).*sin(k*x).*cos(k*x));
v =@(x,y) -(epsilon*v1(x,y) + epsilon^2 *v2(x,y));

quiver(x,y,u(x,y),v(x,y),"HandleVisibility",'off')
hold on
hplus = 1 + epsilon*cos(k*x);
hminus = -hplus;

plot(x(1,:), hplus(1,:), ':k',"HandleVisibility",'off')
plot(x(1,:), hminus(1,:), ':k',"HandleVisibility",'off')

%random normal path
[temp1,temp2] = ode45(@(t,a)odesys(t,a,k,epsilon),[0,25],[0,-0.8]);
plot(temp2(:,1),temp2(:,2))
%a closed loop
[temp1,temp2] = ode45(@(t,a)odesys(t,a,k,epsilon),[0,40],[0.5* pi/k,-0.2]);
plot(temp2(:,1),temp2(:,2))
%edge case1
[temp1,temp2] = ode45(@(t,a)odesys(t,a,k,epsilon),[0,25],[-2*pi/k,-1-epsilon]);
plot(temp2(:,1),temp2(:,2))
%edge case2
[temp1,temp2] = ode45(@(t,a)odesys(t,a,k,epsilon),[0,25],[4* pi/k,1 + epsilon]);
plot(temp2(:,1),temp2(:,2))
%stagnant point
plot(0:pi/k:4*pi/k,zeros(size(0:pi/k:4*pi/k)),'xk')
title("k = "+num2str(k) + ", $\epsilon$ = " +num2str(epsilon))
axis([-pi,4*pi,-1-epsilon,1+epsilon])
```

```
hold off
out=gcf;
end
function uv = odesys(t,a,k,epsilon)
x = a(1);
y = a(2);
u0 =@(x,y) -y;
u1 =@(x,y) k*(cos(k*x).*sinh(k*y)/(cosh(k)));
u2 =@(x,y) 2*k* ((1-k*tanh(k))/cosh(2*k))*sinh(2*k*y).*(cos(k*x).^2);

u= u0(x,y) + epsilon*u1(x,y) + epsilon^2 *u2(x,y);
v1 =@(x,y) -k *(sin(k*x).*cosh(k*y)/(cosh(k)));
v2 =@(x,y) -2*k*(((1-k*tanh(k))/cosh(2*k))*cosh(2*k*y).*sin(k*x).*cos(k*x));

v = -(epsilon*v1(x,y) + epsilon^2 *v2(x,y));

uv = [u;v];
end
```

Practical Asymptotics (APP MTH 4048/7044)
Assignment 2 (5%)

Due 12 April 2019

1. (Bowen & Witelski) Consider the problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \epsilon v^2 + t = 0, \quad v(0) = 0, \quad \epsilon \to 0.$$

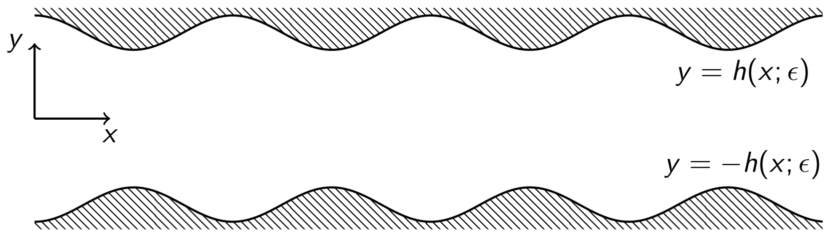
   (a) Find the first three terms in the expansion of the solution $v(t) \sim v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t)$, as $\epsilon \to 0$

   (b) Determine the range of times $0 \le t \le \mathcal{O}(\epsilon^\alpha)$, for which the terms in the expansion retain asymptotic ordering, i.e. $v_0 \gg \epsilon v_1 \gg \epsilon^2 v_2$.

2. (Hinch, adapted) The (shear) flow along a corrugated channel is described by a streamfunction $\psi(x, y)$ (the x and y components of velocity are given by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$). The streamfunction satisfies

$$\nabla^2\psi = -1, \quad \text{in } |y| < h(x, \epsilon) \equiv 1 + \epsilon \cos kx,$$

subject to the boundary condition $\psi = 0$ on the walls at $y = \pm h(x; \epsilon)$, and is periodic in x so that $\psi(0, y) = \psi(2\pi/k, y)$.

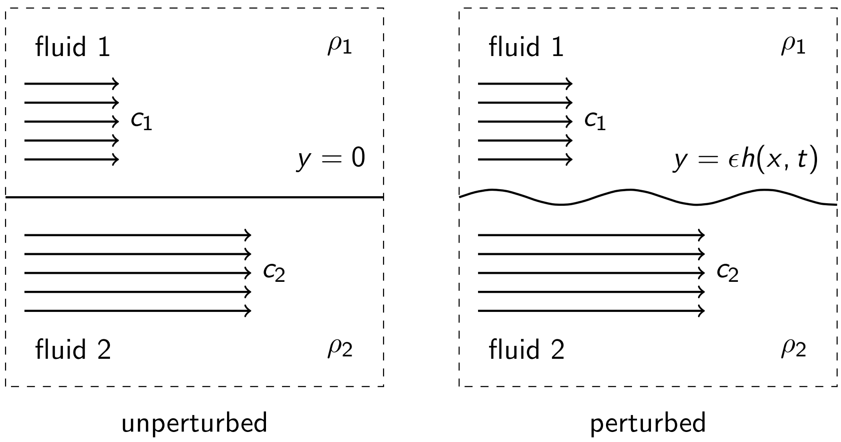


   (a) Obtain the first three terms in the perturbation expansion for ψ.

   (b) Plot a few streamlines in MATLAB for a different values of ϵ and k.

   (c) Comment on the validity of this solution.

3. (Kelvin-Helmholtz instability) Consider two fluid layers moving in parallel:



In the unperturbed state, the upper layer moves with speed c_1 and the lower layer moves with speed c_2. The upper layer is of density ρ_1 and the lower layer is of density ρ_2.

Assume the shape of the interface between the two layers is small in amplitude by writing $y = \epsilon h(x, t)$, where $\epsilon \ll 1$ and $h = \mathcal{O}(1)$.

The flow of the two layers is described by

$$\begin{aligned} \nabla^2 \phi_1 &= 0, \qquad \text{for } y > \epsilon h(x, t), \\ \nabla^2 \phi_2 &= 0, \qquad \text{for } y < \epsilon h(x, t). \end{aligned}$$

Far from the interface the layers are at their unperturbed speeds, namely

$$\begin{aligned} \phi_1 &= c_1 x, \quad y \to \infty, \\ \phi_2 &= c_2 x, \quad y \to -\infty. \end{aligned}$$

The kinematic conditions in each fluid are

$$\begin{aligned} \frac{\partial \phi_1}{\partial y} &= \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi_1}{\partial x} \frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x, t), \\ \frac{\partial \phi_2}{\partial y} &= \epsilon \left(\frac{\partial h}{\partial t} + \frac{\partial \phi_2}{\partial x} \frac{\partial h}{\partial x} \right), \quad \text{on } y = \epsilon h(x, t). \end{aligned}$$

Finally, the Bernoulli condition is

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} - \frac{1}{2} c_1^2 + \frac{1}{2} |\nabla \phi_1|^2 + g y \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} - \frac{1}{2} c_2^2 + \frac{1}{2} |\nabla \phi_2|^2 + g y \right),$$

on $y = \epsilon h(x, t)$.

(a) Expand the various quantities about the unperturbed state to rewrite the kinematic and Bernoulli conditions on $y = 0$.

(b) Introduce perturbation series for the velocity potentials where

$$\begin{aligned} \phi_1(x, y, t) &= \phi_{10}(x, y, t) + \epsilon \phi_{11}(x, y, t) + \mathcal{O}(\epsilon^2), \\ \phi_2(x, y, t) &= \phi_{20}(x, y, t) + \epsilon \phi_{21}(x, y, t) + \mathcal{O}(\epsilon^2), \end{aligned}$$

as $\epsilon \to 0$. Write down a solution for ϕ_{10} and ϕ_{20} (the unperturbed problem), and then write down a problem for ϕ_{11} and ϕ_{21}.

(c) Assume the interface shape is a travelling wave, so that

$$h(x, t) = a e^{\mathrm{i}(kx - \omega t)}.$$

The stability of the system will be determined by ω. Use separation of variables to find ϕ_{11} and ϕ_{21} (each up to a multiplicative constant).

(d) Substitute $h(x, t)$, $\phi_{11}(x, y, t)$ and $\phi_{21}(x, y, t)$ into the boundary conditions to obtain an expression for ω.

(e) Use your expression for ω to determine the stability/instability of the interface in terms of the parameters c_1, c_2, ρ_1, ρ_2 and k. Briefly interpret the physical significance of this result.