Topic C Project

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Uniform Asymptotics for the Linearized Boltzmann Equation Describing Sound Wave Propagation

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Abstract — The multiple-scale expansion method is used for constructing a uniformly applicable asymptotic approximation of the solution of the linearized Boltzmann equation for small Knudsen numbers. The asymptotic expansion is constructed for the particular example of a sound wave generated by a plane oscillation source and dissipating in a half-space. The simplicity of the problem makes it possible clearly to demonstrate the appearance of secular terms in the expansion and the introduction of multiple scales opens the way to eliminating them.

Keywords: Boltzmann equation, multiple-scale expansion method, small Knudsen numbers, uniformly applicable expansions.

The fullest description of rarefied gas behavior over the entire region of Knudsen numbers of interest is given by the integro-differential Boltzmann equation [1, 2]. Of considerable theoretical interest is the case of small Knudsen numbers for which it is natural to expect that the solutions of the kinetic Boltzmann equation and the Navier-Stokes equations of continuum mechanics to lead to identical results. In this connection, the problem of solving the Boltzmann equation in the limit of small Knudsen numbers acquires fundamental significance. In this area the first basic result was obtained by D. Hilbert who showed that it is possible to go over from the kinetic description to the description using hydrodynamic variables.

The presence of a small parameter in the dimensionless Boltzmann equation opens up broad opportunities for applying asymptotic methods. We must, however, remember that solutions in the form of simple power series expansions are poor approximations on large intervals of variation of the independent variables. This defect is usually manifested in the form of growing secular terms. As a result, such solutions give a poor description of weak dissipative processes.

Moreover, due to the complexity of the Boltzmann equation itself, most theoretical results known in this area are based on certain simplifying assumptions which, in their turn, lead to various difficulties. As an example, we note the nonphysical instabilities in higher approximations of the Chapman-Enskog method [3, 4].

Thus, it remains important to construct an asymptotic approximation uniformly applicable at large values of the independent variables. Use of the multiple-scale expansion method for this purpose was first proposed in [5–7]. In those studies the need to detect and eliminate sources of secular behavior in the expansion constructed was mentioned and the problem of self-consistency of the expansions for the macroscopic variables was indicated.

Due to the complexity of finding a solution in general form, investigating simple problems continues to be of interest. In the case considered, such a test problem is the theory of propagation of small perturbations in a gas, where, firstly, we can restrict ourselves to the linearized one-dimensional Boltzmann equation and, secondly, there is no need to take the detailed initial and boundary conditions into account [8]. Here, two basic formulations are possible [9]: the time relaxation of a wave perturbation initially specified in the entire space and the dissipation of a sound wave emitted by a source oscillating at a fixed frequency. Recently, it has been shown [10] that in the first case, by using the multiple-scale expansion technique, it is possible to construct an asymptotic approximation uniformly applicable at large times.

In this study, with reference to the dissipation of a sound wave propagating in a half-space from a plane oscillation source, the second formulation is investigated. In this case, instead of a sequence of time scales, a sequence of linear scales is introduced. This makes it possible to obtain a self-consistent asymptotic solution which also holds at a large distance from the source. The procedure is brought to the third approximation, the stability of all the approximations considered being obvious from the analytical form of the solution. A similar problem is considered using the Navier-Stokes equations. The solutions obtained are compared.

We will describe the evolution of small perturbations in a stationary monatomic gas with constant parameters on the basis of the one-dimensional linearized Boltzmann equation [1, 2]

$$\varepsilon\left(\frac{\partial\varphi}{\partial t} + c_z\frac{\partial\varphi}{\partial z}\right) = L\varphi \tag{1}$$

$$L\varphi = \int f_0(\mathbf{c}_1)[\varphi(\mathbf{c}_1') + \varphi(\mathbf{c}') - \varphi(\mathbf{c}_1) - \varphi(\mathbf{c})]gb\,db\,d\beta\,d\mathbf{c}_1$$

$$f_0(\mathbf{c}) = \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{c^2}{2}\right)$$

Here, ε is the Knudsen number, equal to the ratio of the effective molecular free path to the characteristic macroscopic length, and the hydrodynamic perturbation parameters are determined in terms of the distribution function φ by the relations

$$n = \int f_0\varphi\,d\mathbf{c}, \qquad v = \int c_z f_0\varphi\,d\mathbf{c}, \qquad u = \int\frac{c^2}{2}f_0\varphi\,d\mathbf{c}, \tag{2}$$

$$u = \frac{3}{2}p, \qquad p = n + T$$

We note that as a scale for the disturbed density the characteristic perturbation density is used.

We will seek an asymptotic approximation of the solution of Eq. (1) in the limiting case of small Knudsen numbers in the form of the expansion

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \ldots \tag{3}$$

in which the sequence of functions φ_m must be such that the truncated expansion (3) is uniformly applicable. For this it is necessary that for each m the term $\varepsilon^m\varphi_m$ be a small correction to the previous term $\varepsilon^{m-1}\varphi_{m-1}$ for all z considered. In order to satisfy these requirements, we introduce the sequence of new variables $z_k = \varepsilon^k z$ [11, 12]. Then

$$\varphi_k = \varphi_k(t, z_0, \ldots, z_m), \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \varepsilon\frac{\partial}{\partial z_1} + \ldots + \varepsilon^m\frac{\partial}{\partial z_m} \tag{4}$$

where the number m determines the region in which expansion (3) holds.

The substitution of (3) and (4) in (1) leads to a chain of equations of the form:

$$L\varphi_k = Q_k \tag{5}$$

where on the right side all the inhomogeneous terms known from the previous approximations are collected.

The substitution of (3) in (2) yields

$$n = \sum\varepsilon^k n_k, \qquad v = \sum\varepsilon^k v_k, \qquad u = \sum\varepsilon^k u_k \tag{6}$$

$$u_k = \frac{3}{2}p_k, \qquad p_k = n_k + T_k$$

$$n_k = \int f_0\varphi_k\,d\mathbf{c}, \quad v_k = \int c_z f_0\varphi_k\,d\mathbf{c}, \qquad u_k = \int\frac{c^2}{2}f_0\varphi_k\,d\mathbf{c}$$

Equations (5) have solutions only if the following solvability condition is satisfied:

$$\int \psi_r f_0 Q_k \, d\mathbf{c} = 0 \tag{7}$$

where ψ_r are the eigenfunctions of the corresponding homogeneous equation.

The solutions of Eqs. (5) can be represented in the form:

$$\varphi_k = g_k + h_k, \quad g_k = p_k + \left(\frac{c^2}{2} - \frac{5}{2}\right) T_k + c_z v_k$$

where g_k is the solution of the corresponding homogeneous equation and h_k is a particular solution of the initial Eq. (5). When writing the expression for g_k, we used the Hilbert conditions [1, 2]

$$\int \psi_r f_0 h_k \, d\mathbf{c} = 0$$

We will now turn to the successive consideration of the approximations.

The zeroth approximation can be reduced to the relation

$$\varphi_0 = p_0 + \left(\frac{c^2}{2} - \frac{5}{2}\right) T_0 + c_z v_0 \tag{8}$$

In the next step, Eq. (5) has the form:

$$L\varphi_1 = \frac{\partial \varphi_0}{\partial t} + c_z \frac{\partial \varphi_0}{\partial z_0} \tag{9}$$

Calculating integrals (7), we obtain the integration conditions for (9)

$$\frac{\partial n_0}{\partial t} + \frac{\partial v_0}{\partial z_0} = 0, \quad \frac{\partial v_0}{\partial t} + \frac{\partial p_0}{\partial z_0} = 0, \quad \frac{\partial p_0}{\partial t} + \frac{5}{3}\frac{\partial v_0}{\partial z_0} = 0 \tag{10}$$

From the first and third equations of system (10), there follow the adiabatic relations between the zeroth-approximation parameters

$$p_0 = \frac{5}{3} n_0, \qquad p_0 = \frac{5}{2} T_0$$

From the second and third equations we obtain the homogeneous wave equations

$$\frac{\partial^2 p_0}{\partial t^2} - a_0^2 \frac{\partial^2 p_0}{\partial z_0^2} = 0, \qquad \frac{\partial^2 v_0}{\partial t^2} - a_0^2 \frac{\partial^2 v_0}{\partial z_0^2} = 0, \qquad a_0^2 = \frac{5}{3} \tag{11}$$

Obviously, system (11) describes the propagation of a plane wave

$$v_0 = C_0 \exp\left[i(k_0 z_0 - \omega t)\right], \qquad p_0 = a_0 v_0, \qquad \omega = a_0 k_0 \tag{12}$$

where ω is a given oscillation frequency and C_0 is a function of $z_1, \ldots, z_m$.

Returning to Eq. (9) and using (10) to eliminate the time derivative, we obtain

$$L\varphi_1 = \left(\frac{c^2}{2} - \frac{5}{2}\right) c_z \frac{\partial T_0}{\partial z_0} + \left(c_z^2 - \frac{c^2}{3}\right) \frac{\partial v_0}{\partial z_0} \tag{13}$$

Solution (13) is known from [1, 2]:

$$\varphi_1 = p_1 + \left(\frac{c^2}{2} - \frac{5}{2}\right)T_1 + c_z v_1 - A\frac{\partial T_0}{\partial z_0} - B\frac{\partial v_0}{\partial z_0} \tag{14}$$

where A and B are the solutions of the integral equations obtained by substituting (14) in (13).

The wave solution obtained in form (12) neglects the dissipative effects and in this respect is unsatisfactory. However, by using the multiple-scale expansion method, it can be refined. To do this, we must consider the second approximation

$$L\varphi_2 = \frac{\partial \varphi_1}{\partial t} + c_z\frac{\partial \varphi_1}{\partial z_0} + c_z\frac{\partial \varphi_0}{\partial z_1} \tag{15}$$

For (15) the integrability conditions (7) can be reduced to the system

$$\begin{aligned}
&\frac{\partial n_1}{\partial t} + \frac{\partial v_1}{\partial z_0} = -\frac{\partial v_0}{\partial z_1}\\
&\frac{\partial v_1}{\partial t} + \frac{\partial p_1}{\partial z_0} = -\left(\frac{\partial p_0}{\partial z_1} - \frac{4}{3}\mu\frac{\partial^2 v_0}{\partial z_0^2}\right)\\
&\frac{\partial p_1}{\partial t} + \frac{5}{2}\frac{\partial v_1}{\partial z_0} = -\left(\frac{5}{2}\frac{\partial v_0}{\partial z_1} - \frac{4}{15}\lambda\frac{\partial^2 p_0}{\partial z_0^2}\right)
\end{aligned} \tag{16}$$

where μ and λ are the shear viscosity and thermal conductivity [1, 2].

From the first and third equations of system (16) we find

$$T_1 = \frac{2}{5}p_1 - \frac{4}{25}\lambda\frac{\partial v_0}{\partial z_0}$$

where for the first-approximation parameters the adiabaticity conditions are no longer satisfied. From the second and third equations in (16), for the quantities p_1 and v_1 there follow the inhomogeneous wave equations

$$\begin{gathered}
\frac{3}{5}\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial z_0^2} = 2\frac{\partial}{\partial z_0}F_1, \quad \frac{3}{5}\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial^2 v_1}{\partial z_0^2} = 2\frac{\partial}{\partial z_0}F_2\\
F_1 = \frac{\partial p_0}{\partial z_1} - \left(\frac{2}{3}\mu + \frac{2}{15}\lambda\right)\frac{\partial^2 v_0}{\partial z_0^2}, \quad F_2 = \frac{\partial v_0}{\partial z_1} - \frac{3}{5}\left(\frac{2}{3}\mu + \frac{2}{15}\lambda\right)\frac{\partial^2 p_0}{\partial z_0^2}
\end{gathered} \tag{17}$$

The solution of Eqs. (17) can be written in the general form:

$$p_1 = p_1^* - z_0F_1, \qquad v_1 = v_1^* - z_0F_2 \tag{18}$$

where the solutions of the corresponding homogeneous equations are denoted by an asterisk. For $z_0 = O(\varepsilon^{-1})$, since in (18) the particular solutions grow proportionally to z_0, regularity conditions, such as $\varepsilon p_1 \ll p_0$, are violated and expansions (3) and (6) lose their meaning. However, the introduction of a scale sequence in (4) makes it possible to eliminate the secular terms. In fact, although on the right sides of Eqs. (17) only previous-approximation terms are present, they contain a quantity C_0 which is an arbitrary function of the variables $z_1, \dots, z_m$. Choosing the function C_0 provides an opportunity to regularize the expansion within the framework of the second approximation considered. As can be seen from (18), in order to eliminate the secular behavior, it is necessary to set

$$\frac{\partial p_0}{\partial z_1} = \left(\frac{2}{3}\mu + \frac{2}{15}\lambda\right)\frac{\partial^2 v_0}{\partial z_0^2}, \qquad \frac{\partial v_0}{\partial z_1} = \frac{3}{5}\left(\frac{2}{3}\mu + \frac{2}{15}\lambda\right)\frac{\partial^2 p_0}{\partial z_0^2} \tag{19}$$

Conditions (19) make it possible to find a dependence of C_0 on the variable z_1 that ensures the regularity of the first two expansion terms. Substituting (2) in (19) and integrating the equation obtained, we find

$$v_0 = C_1(z_2,\ldots)\exp\left[i(k_0 z_0 - \omega t) - \frac{3}{5}\left(\frac{2}{3}\mu + \frac{2}{15}\lambda\right)k_0^2 a_0 z_1\right] \tag{20}$$

where the new function C_1 now depends on the next variables z_k.

Thus, the second approximation makes it possible to take the effect of dissipative wave damping into account. Using (19), it is easy to identify on the right sides of the second and third equations (16) both the "bad" combinations of terms responsible for the secular behavior of the solution, which should be eliminated, and the "good" terms that remain after the secularity sources have been removed [7]. Hence, even after the secular terms are eliminated, Eqs. (16) remain inhomogeneous and may have nontrivial solutions. We note that Eqs. (16) can be reduced to form (10) by introducing the function w_1

$$\begin{gathered} w_1 = p_1 - \left(\frac{2}{3}\mu - \frac{2}{15}\lambda\right)\frac{\partial v_0}{\partial z_0} \\ \frac{\partial v_1}{\partial t} + \frac{\partial w_1}{\partial z_0} = 0, \qquad \frac{\partial w_1}{\partial t} + \frac{5}{3}\frac{\partial v_1}{\partial z_0} = 0 \end{gathered} \tag{21}$$

the solution of this system describing the propagation of a plane wave for the first-order macroscopic variables

$$v_1 = D_0(z_1,\ldots,z_m)\exp[i(k_0 z_0 - \omega t)], \qquad w_1 = a_0 v_1 \tag{22}$$

From (21) and (22) it also follows that for zero boundary conditions for v_1 the quantity p_1 is non-zero. Thus, in the case considered, strictly speaking, the known Enskog closing conditions cannot be satisfied.

Returning to Eq. (15), we note that by using (8), (10), (14), (16), and (19) and approximation [1], its right side can be transformed to a form similar to (13):

$$A = \frac{2}{5}\lambda\left(\frac{c^2}{2} - \frac{5}{2}\right)c_z, \qquad B = \mu\left(c_z^2 - \frac{c^2}{3}\right), \qquad \lambda = \frac{15}{4}\mu$$

A further expansion of the applicability region for the expansion terms already obtained calls for the investigation of the next approximation. Since for the third approximation the solvability conditions can be obtained without solving (15) in detail, we go directly over to the equation

$$L\varphi_3 = \frac{\partial \varphi_2}{\partial t} + c_z\frac{\partial \varphi_2}{\partial z_0} + c_z\frac{\partial \varphi_1}{\partial z_1} + c_z\frac{\partial \varphi_0}{\partial z_2}$$

Evaluating integrals (7), we obtain the system

$$\begin{aligned} &\frac{\partial n_2}{\partial t} + \frac{\partial v_2}{\partial z_0} = -\frac{\partial v_1}{\partial z_1} - \frac{\partial v_0}{\partial z_2} \\ &\frac{\partial v_2}{\partial t} + \frac{\partial p_2}{\partial z_0} = -\left(\frac{\partial p_1}{\partial z_1} - \frac{4}{3}\mu\frac{\partial^2 v_1}{\partial z_0^2}\right) - \left(\frac{\partial p_0}{\partial z_2} - \frac{12}{5}\mu^2\frac{\partial^3 p_0}{\partial z_0^3}\right) \\ &\frac{\partial p_2}{\partial t} + \frac{5}{3}\frac{\partial v_2}{\partial z_0} = -\left(\frac{5}{3}\frac{\partial v_1}{\partial z_1} - \mu\frac{\partial^2 p_1}{\partial z_0^2}\right) - \left(\frac{5}{3}\frac{\partial v_0}{\partial z_2} - \frac{7}{3}\mu^2\frac{\partial^3 v_0}{\partial z_0^3}\right) \end{aligned}$$

From the second and third equations we find

$$\frac{3}{5}\frac{\partial^2 p_2}{\partial t^2} - \frac{\partial^2 p_2}{\partial z_0^2} = 2\frac{\partial}{\partial z_0}G_1, \quad \frac{3}{5}\frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_2}{\partial z_0^2} = 2\frac{\partial}{\partial z_0}G_2 \tag{23}$$

$$G_1 = \left(\frac{\partial p_1}{\partial z_1} - \frac{7}{6}\mu\frac{\partial^2 v_1}{\partial z_0^2}\right) + \left(\frac{\partial p_0}{\partial z_2} - \frac{241}{120}\mu^2\frac{\partial^3 p_0}{\partial z_0^3}\right)$$

$$G_2 = \left(\frac{\partial v_1}{\partial z_1} - \frac{7}{10}\mu\frac{\partial^2 p_1}{\partial z_0^2}\right) + \left(\frac{\partial v_0}{\partial z_2} - \frac{213}{120}\mu^2\frac{\partial^3 v_0}{\partial z_0^3}\right)$$

The solutions of Eqs. (23) can be represented in the form:

$$p_2 = p_2^* - z_0 G_1, \qquad v_2 = v_2^* - z_0 G_2$$

From this it can be seen that for $z_0 = O(\varepsilon^{-1})$ the regularity conditions $\varepsilon p_2 \ll p_1$ and $\varepsilon v_2 \ll v_1$ are satisfied only if we require that $G_1 = 0$ and $G_2 = 0$.

Eliminating by turns the quantities p_1 and v_1 from these relations, we obtain

$$\begin{aligned}
&\frac{\partial^2 p_1}{\partial z_1^2} - \frac{49}{60}\mu^2\frac{\partial^4 p_1}{\partial z_0^4} = -2\frac{\partial}{\partial z_1}Hp_0\\
&\frac{\partial^2 v_1}{\partial z_1^2} - \frac{49}{60}\mu^2\frac{\partial^4 v_1}{\partial z_0^4} = -2\frac{\partial}{\partial z_1}Hv_0\\
&H = \frac{\partial}{\partial z_2} - \frac{227}{120}\mu^2\frac{\partial^3}{\partial z_0^3}
\end{aligned} \tag{24}$$

The inhomogeneous terms on the right side of (24) are responsible for the secular behavior since they give particular solutions of the form:

$$p_1 = p_1 - z_1 Hp_0, \qquad v_1 = v_1 - z_1 Hv_0$$

Consequently, now even for $z_1 = O(\varepsilon^{-1})$, that is, for $z_0 = O(\varepsilon^{-2})$, the conditions $\varepsilon p_1 \ll p_0$ and $\varepsilon v_1 \ll v_0$ are not satisfied. Therefore, in order for the first terms of expansions (6) to remain for any z up to $O(\varepsilon^{-2})$ small corrections to the zeroth terms, it is necessary to eliminate the sources of their nonuniformity by requiring that

$$\frac{\partial p_0}{\partial z_2} - \frac{227}{120}\mu^2\frac{\partial^3 p_0}{\partial z_0^3} = 0, \qquad \frac{\partial v_0}{\partial z_2} - \frac{227}{120}\mu^2\frac{\partial^3 v_0}{\partial z_0^3} = 0 \tag{25}$$

From Eqs. (25) we find the dependence of the quantity C_1 on the variable z_2. Finally, for the velocity we obtain

$$\begin{aligned}
&v_0 = C_2(z_3, \ldots)\exp\left[i(kz_0 - \omega t) - \frac{7}{10}\mu k_0^2 a_0 z_1\right]\\
&k = \left(1 - \frac{227}{120}\mu^2 k_0^2\varepsilon^2\right)k_0
\end{aligned} \tag{26}$$

Then, for the phase velocity we have

$$a = \frac{\omega}{k} = \left(1 + \frac{227}{120}\mu^2 k_0^2\varepsilon^2\right)a_0$$

Comparing the expression obtained for the phase velocity with the results of [10], we can see that for the problem considered the correction to the velocity is an order greater. We can also refine the solution for the first-order macroscopic parameters. In fact, relations (24), with account for (25), determine the dependence of the quantity D_0 on the variable z_1.

On the basis of the fundamental role of the Boltzmann equation, we can regard the self-consistent, uniformly applicable asymptotic solution obtained above as a standard for evaluating the approximations used in the limiting case of small Knudsen numbers. Since in this region the gas behavior is usually described within the framework of the gas-dynamic approximation, it is of interest to obtain the solution for a sound wave using the phenomenological Navier-Stokes equations. In the linear one-dimensional approximation these equations have the form:

$$\frac{\partial n}{\partial t} + \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} = \frac{4}{3}\varepsilon\mu\frac{\partial^2 v}{\partial z^2}$$
$$\frac{\partial u}{\partial z} + \frac{5}{2}\frac{\partial v}{\partial z} = \varepsilon\lambda\frac{\partial^2 T}{\partial z^2}, \qquad u = \frac{3}{2}p, \qquad p = n + T \tag{27}$$

As before, we seek the solution of system (27) in form (4) and (6). Omitting the analogous intermediate results, we note that for the zeroth-approximation parameters we also arrive at the system of inviscid equations (10). In the next approximation we obtain a system identical to system (16). Therefore, from the solution boundedness conditions there follows the same wave damping decrement as in (20). Differences appear only in the third step. Finally, for the velocity we obtain the following formula, similar to (26) but with another wave number and phase velocity:

$$a = \left(1 + \frac{141}{120}\mu^2 k_0^2 \varepsilon^2\right) a_0$$

Thus, the asymptotics for the linearized Boltzmann equations and the Navier-Stokes system are completely identical up to terms $O(\varepsilon)$. The discrepancies at the Burnett level are natural, since the Navier-Stokes theory is based on linear approximations for the stress tensor and the heat flux.

Summary. Asymptotics for the linearized Boltzmann equation, uniformly applicable in a half-space, are constructed using the multiple-scale technique. It is shown that in order to do this it is necessary separately to seek and eliminate the secular terms in each approximation considered. With reference to the second approximation it is shown that in the inhomogeneous parts of the differential equations for the macroscopic variables it is possible to distinguish a "good" part and a "bad" secular part to be eliminated. Since, after the secular combinations have been eliminated, the differential equations retain the "good" inhomogeneities, they have nontrivial solutions. Hence, in the case considered, the Enskog closing conditions cannot be satisfied. Finally, an investigation of the third approximation shows that the multiple-scale expansion method makes it possible to obtain a stable solution at the Burnett level.

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