Lecture 14: Expected Hitting Times – A very useful performance measure

Concepts checklist

At the end of this lecture, you should be able to:

- *derive a system of linear equations* that the expected first hitting times of a state satisfy;
- *state a theorem* regarding the desired solution to this system of equations; and,
- *evaluate* expected first hitting times for simple CTMCs, both analytically and with the assistance of a computer.

Expected Hitting Time

We have shown how to calculate the probability f_i that a continuous-time Markov chain ever reaches state j given that it starts in state i. For the case $f_i = 1$ for all $i \in \mathcal{S}$, it can be of interest to calculate the expected time that this will take.

Let T the time to first reach state j, and $t_i = \mathbb{E}(T|X(0) = i)$ be the expected time (t_i might be infinite) until the process is absorbed in state j given that it starts in state i. Then assuming $i \neq j$, using a first step analysis we can write

$$t_i = -\frac{1}{q_{ii}} + \sum_{\substack{k\neq i\\ k\in\mathcal{S}}} \frac{q_{ik}}{-q_{ii}} t_k, \qquad \text{with } t_j = 0,$$

where

$$\begin{aligned} -\frac{1}{q_{ii}} &= \text{expected time until the next transition,}\\ \frac{q_{ik}}{-q_{ii}} &= \text{probability of jumping to state } k \text{ at the next transition,}\\ t_k = \mathbb{E}(T|X(0) = k) &= \text{expected time to reach } j \text{ given that the process starts in state } k. \end{aligned}$$

Theorem 12. *The expected time to first reach state j starting from state i, t_i, is given by the minimal non-negative solution to the equations*

$$\sum_{k\in\mathcal{S}} q_{ik} t_k = -1\ , \qquad i \in \mathcal{S} \setminus \{j\},$$

subject to $t_j = 0$. If no non-negative solution exists, then t_i is infinite for all i.

The proof of this result follows by using similar but more complicated methods to those used in the proof of the hitting probability result.

Note, defining Q_{-j} as the generator Q with the jth row and column removed, we have $Q_{-j}t = -\mathbf{1}$ where $t = (t_i)_{i\in\mathcal{S}\setminus\{j\}}$ and $\mathbf{1}$ is a vector of ones.

Example 3. M/M/1 Queue

When $\mu \geq \lambda$, the probability that the single server queue ever visits state 0 given that it starts in state $i > 0$ is equal to 1. Let us consider the expected time t_i until the Markov chain reaches state 0 given that it starts in state i.

$$t_i = \frac{1}{\lambda+\mu} + \left(\frac{\lambda}{\lambda+\mu}\right) t_{i+1} + \left(\frac{\mu}{\lambda+\mu}\right) t_{i-1}, \text{ for } i > 0, \text{ with } t_0 = 0.$$

Step 1. The homogeneous version of this equation has a solution of the form

$$t_i = \begin{cases} A\left(\frac{\mu}{\lambda}\right)^i + B & \text{for } \mu > \lambda, \\ Ai + B & \text{for } \mu = \lambda. \end{cases}$$

In the non-homogeneous case we try a particular solution of the form

$$t_i = \begin{cases} Ci & \text{for } \mu > \lambda, \\ Ci^2 & \text{for } \mu = \lambda. \end{cases}$$

- If $\mu > \lambda$:

$$\begin{aligned} iC &= \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}(i+1)C + \frac{\mu}{\lambda+\mu}(i-1)C \\ \Rightarrow \quad 0 &= \frac{1}{\lambda+\mu} + \frac{C\lambda}{\lambda+\mu} - \frac{C\mu}{\lambda+\mu} \\ \Rightarrow \quad (\mu-\lambda)C &= 1 \\ \Rightarrow \quad C &= \frac{1}{\mu-\lambda}. \end{aligned}$$

Note: A solution in the non-homogeneous case = a general solution to the homogeneous case + a particular solution to the non-homogeneous case.

Therefore, the general solution of the non-homogeneous equation is

$$t_i = \frac{i}{\mu-\lambda} + A\left(\frac{\mu}{\lambda}\right)^i + B.$$

Note: We do not use the boundary conditions until we have the entire form of solution, and then after that we choose the minimal non-negative solution, if necessary.

Step 2. Using the boundary condition $t_0 = 0$ gives us that $B = -A$; thus,

$$t_i = \frac{i}{\mu-\lambda} + A\left[\left(\frac{\mu}{\lambda}\right)^i - 1\right].$$

Step 3. We now need to find the minimal non-negative solution. Since $\mu > \lambda$, the term in square brackets is always positive and grows much quicker in i than does the first term.

$\Rightarrow$ the minimal non-negative solution occurs when $A = 0$, as we cannot guarantee $t_i > 0$ for any $A < 0$. Hence,

$$t_i = \frac{i}{\mu-\lambda},$$

which tells us that the time until absorption in state 0 is linear in the initial number i of customers present.

- If $\mu = \lambda$: We try a solution of the form $T_i = i^2 C$.

$$i^2 C = \frac{1}{2\mu} + \left(\frac{1}{2}\right)(i+1)^2 C + \left(\frac{1}{2}\right)(i-1)^2 C = \frac{1}{2\mu} + C\left(i^2+1\right)$$
$$\Rightarrow C = -\frac{1}{2\mu}.$$

Therefore the general solution is $t_i = Ai + B - \frac{1}{2\mu} i^2$.

Step 2. Now we use the boundary condition, $t_0 = 0$ to show that $B = 0$ and thus

$$t_i = Ai - \frac{1}{2\mu} i^2.$$

There exists no A such that this is always non-negative (since i^2 grows faster than i) and hence there is no non-negative solution and $t_i = \infty$ for all i.

$\Rightarrow$ When $\mu = \lambda$ the probability of reaching state 0 is 1, but the expected time that this takes is infinity; hence the CTMC is null recurrent.