Modelling With ODEs Tutorials

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1 Tute 1

1. Fishery Model where $N(t)$ is the number of fish

$$\frac{dN}{dt} = f(N) = BN - DN^2 - Y$$

Y fishing yield, B birth rate, D death rate. In lectures we showed that a non-dimensional version of the model is

$$\frac{d\hat{N}}{d\hat{t}} = \hat{N}(1-\hat{N}) - y$$

(a) Show that the steady state is:

$$\hat{N}_*^{\pm} = \frac{1 \pm \sqrt{1-4y}}{2}$$

and which values of y does it exist: Steady state when

$$\begin{aligned}
\hat{N}(1-\hat{N}) - y &= 0 \\
\hat{N} - \hat{N}^2 - y &= 0 \\
\hat{N}^2 - \hat{N} + y &= 0 \\
\hat{N} &= \frac{1 \pm \sqrt{1-4y}}{2}
\end{aligned}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exists if $\Delta \geq 0$, i.e. $1 - 4y \geq 0$.

$$\implies 4y \leq 1 \implies y \leq \frac{1}{4}$$

(b) Stability of the steady states?

$$\begin{aligned}
f'(\hat{N}) &= 1 - 2\hat{N} \\
\implies f'(\hat{N}_+^*) &< 0 \\
\implies f'(\hat{N}_-^*) &> 0
\end{aligned}$$

So $\hat{N}_+^*$ is stable, and $\hat{N}_-^*$ is unstable. When $\hat{N}(0) < \hat{N}_-^*$ the population will go to 0.

(c) When $y = .25$ we get a repeated root $\hat{N}^* = \frac{1}{2}$.

$$f'(N^*) = 0, \quad f''(N^*) = -2$$

So since the slope is 0 and it is a turning point, it is semi-stable (if we shift to the right it will come back, to the left it will continue to the left)

(d) What happens when $y > 0.25$? There are no real roots so there are no steady states...

2. (a) Fixed points are where $C(x)$, $S(x : \mu)$ intersect, treat $S(x)$ as x, so rotate the coordinate system.

   (b) i. Bifurcation points when

$$f(\bar{x} : \bar{\mu}) = 0, \frac{\partial f}{\partial x} = 0$$

$$\begin{aligned} \mu x + x^3 - x^5 &= 0 \\ x(\mu + x^2 - x^4) &= 0 \\ x = 0 \ or \ x^4 - x^2 - \mu &= 0 \\ \implies x^2 &= \frac{1 \pm \sqrt{1+4\mu}}{2} \\ \implies x &= \pm\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}} \end{aligned}$$

The four will exist/cease to exist for values of μ need both square roots to exist

$$x_{\pm+} \implies \mu > -1/4$$

$$x_{\pm-} \implies \mu \in \{-1/4, 1/4\}$$

I.e. bifurcation at $(x, \mu) = (1/2, -1/4), (-1/2, -1/4)$ And $\bar{x} = 0$ works for all μ.

   ii. This was found at the start:

$$x = \pm\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}}, 0$$

   iii.

   iv.

   (c) i.

   ii.

2 Tute 3

1. Trajectories of form

$$\mathbf{x} = e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}$$

$\alpha = 0$ gives centres and $\alpha \neq 0$ gives spirals.

   (a) Effect of β on the direction of trajectory: For $0 \leq t \leq \beta 2\pi$ $-\sin\beta t > 0$ when $\beta t \in (\pi, 2\pi)$ $\cos\beta t > 0$ when $\beta t \in (-\pi/2, \pi/2)$

   (b)

2. Done in matlab

3.

$$\frac{dx}{dt} = 3x - x^2 - xy, \quad \frac{dy}{dt} = 2y - y^2 - xy$$

   (a) Competition model

(b) $n_x = x = 0, x = 3 - y,$
$n_y = y = 0, y = 2 - x$
Biologically relevant where $x, y \geq 0$, steady states are:
$x, y = 0$
$x = 3, y = 0,\ x = 0, y = 2$

(c) Linearisation

$$J(x) = \begin{pmatrix} 3 - 2x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}$$

At the steady states:

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \implies \lambda_{1,2} > 0 \implies \textit{unstable}$$

$$J(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \implies \lambda = -1, -3 \implies \textit{asymptotically stable}$$

$\det(J(3,0)) = 3$ $trace(J(3,0)) = -4$ $\frac{1}{4}tr(J)^2 = 4$ stable node.

$$J(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$$

(d) n ty

(e)

3 Tute 4

1. (a) Saddle node bifurcation if you can create/destroy 2 fixed points by changing the parameter. Nullclines:

$$\dot{x} = -ax + y, \quad and \quad \dot{y} = \frac{x^2}{1+x^2} - y$$

$$\begin{aligned} \eta_x &= x = \frac{y}{a} \implies y = ax \\ \eta_y &= y = \frac{x^2}{1+x^2} \end{aligned}$$

Bifurcation: Hence fixed points if $x, y = 0$ for all a, or

$$\begin{aligned} \frac{x^2}{1+x^2} - ax &= 0 \\ x^2 - ax(1+x^2) &= 0 \\ x - a + ax^2 &= 0 \\ a &= \end{aligned}$$

(b) Show pitchfork for:

$$\dot{x} = -bx + y + \sin x \quad \dot{y} = x - y$$

$$\begin{aligned} \eta_x &= y = bx - \sin x \\ \eta_y &= x = y \end{aligned}$$

$$\begin{aligned} x &= bx - \sin x \\ (1-b)x - \sin x &= 0 \end{aligned}$$

$x = 0, y = 0$ for all b is a solution

2. The ODE

$$2tx^3 + 3t^2x^2\frac{dx}{dt} = 0$$

$$\begin{aligned} 2tx^3 + 3t^2x^2\frac{dx}{dt} &= 0 \\ 2tx^3dt + 3t^2x^2dx &= 0 \end{aligned}$$

Exact if it can be written as $f(x,t)dt + g(x,t)dx = 0$ its exact.

$$\begin{aligned} 2tx^3 + 3t^2x^2\frac{dx}{dt} &= 0 \\ \frac{2}{3tx} + \frac{dx}{dt} &= 0 \end{aligned}$$

Hence linear

$$\frac{2}{3t} + x\frac{dx}{dt} = 0$$

Hence separable All of these show it is homogeneous.

3.

$$f(x) = \log|x|$$

This can't be globally Lipschitz continuous since it is not continuous about $x = 0$. Lipschitz continuous if

$$|f(x) - f(y)| \leq L|x - y|$$

hence

$$\frac{df}{dx} = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ -\frac{1}{x} & \text{if } x < 0 \end{cases}$$

For $x, y > 0$

$$\begin{aligned} |\log|x| - \log|y|| &= |\log|\frac{x}{y}|| \\ &= ||x| - |y|\frac{1}{c}| \quad (mvt) \\ &= \frac{1}{c}|x - y| \end{aligned}$$

However $\frac{1}{c}$ is not bounded above. However for the intervals $(x, y) \in [-\infty, 0)$ or $(x, y) \in (0, \infty]$