Fluid Mechanics III

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School of Mathematical Sciences

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No guarantee is made that these notes are free from error! Always check results against original sources and your own independent analysis.

1 Outline

1.1 General information

Course Information

Course Codes	:	APP MTH 3002 Fluid Mechanics III APP MTH 4102 Fluid Mechanics Hons APP MTH 7075 Fluid Mechanics PG
Pre-requisites	:	(MATHS 2101 and MATHS 2102) Multivariable & Complex Calculus II and Differential Equations II or (MATHS 2201 and MATHS 2202) Engineering Mathematics IIA and Engineering Mathematics IIB
Assumed Knowledge	:	MATHS 2104 Numerical Methods II

Learning Outcomes

Understand:

- Basic concepts of fluid mechanics.
- Mathematical description of fluid flow.
- Conservation principles governing fluid flows.

Be able to:

- Solve inviscid flow problems using streamfunctions and velocity potentials.
- Compute forces on bodies in fluid flows.
- Solve (analytical and numerical) viscous flow problems.
- Use mathematical software packages (Maple and Matlab) in solution methods.

Staff

Lecturer / Course coordinator	:	Dr Trent Mattner
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Office Location	:	6.41, Ingkarni Wardli
Office Hours	:	2 pm, Tuesday

Course Organisation

Lectures (5 hours per fortnight):

- Tuesday 10am–11am in Napier, 209.
- Wednesday 2pm–3pm in Napier, G03.
- Friday 2pm–3pm in Napier, 209.
- Lectures will be recorded.

Tutorials (1 hour per fortnight):

- Every second Friday starting in week 2.
- Tutorials will not be recorded.

Resources

MyUni:

- New announcements to all students.
- Lecture materials for download.
- Access to tutorial question sheets and solutions.
- Access to assignment questions and solutions.
- Past exam papers will be available for download.
- Lecture recordings will be available for watching.

Suggested books:

- Introduction to Theoretical and Computational Fluid Dynamics, Pozrikidis, Oxford University Press.
- Elementary fluid dynamics, Acheson, Oxford University Press.
- An introduction to fluid mechanics, Batchelor, Cambridge University Press.

Assessment

Assignments:

- 30% of total assessment.
- Total 5 assignments of equal value.
- Submit electronically via MyUni.

Late policy:

- Late assignments submitted up to 24 hours late will receive 60% credit.
- Assignments will not be accepted more than 24 hours late.
- Any variation to this policy will require medical documentation.

Exam:

- 70% of total assessment.
- For timetables, alternative exam arrangements, and replacement and additional (R/AA) exam information, see:

  `https://www.adelaide.edu.au/student/exams/`

Academic Honesty

We encourage you to:

- Work together.
- Seek help from your lecturer.

However:

- All assignments submitted must be your own work.
- Substantially similar pieces of work from different students are not acceptable.

Therefore we recommend that you:

- Plan how to do the assignment in groups.
- Work on your written assignment separately.
- Be aware of the university academic honesty policy:

  `http://www.adelaide.edu.au/policies/230`

2 Introduction

2.1 Definitions

What is a fluid?

Materials are roughly classified as solid or fluid according to the ease with which they are deformed. There are no precise, universally-accepted definitions.

A simple fluid is a material that deforms continuously when acted on by a shear stress of any magnitude.

Fluids are further classified as:

Liquid A fluid that is difficult to compress. Volume is almost independent of pressure.

Gas A fluid that compresses readily. Volume is dependent on pressure.

What is mechanics?

Mechanics is the branch of applied mathematics that deals with the motion and equilibrium of bodies and the actions of forces (OED).

There are three branches of mechanics:

Statics The study of bodies in equilibrium.

Kinematics The study of bodies in motion without reference to forces.

Dynamics The study of forces that change or produce motion.

What is fluid mechanics?

Fluid Mechanics is the branch of applied mathematics that deals with the motion and equilibrium of fluids and the action of forces. Fluid mechanics has a remarkable diversity of applications including:

Industrial processes: Lubrication, coating processes, glass blowing, oil recovery.

Engineering: Hydraulics, heating and ventilation systems, aircraft design, ship design, traffic flow.

Biology and medicine: Blood flow, air flow in the lungs, swimming and flying.

Geophysical and environmental: Ocean currents, meteorology, lava flows, dispersion of pollutants in the atmosphere.

Astrophysical: Dust, stars, galaxies, accretion disks.

2.2 Assumptions

Continuum model

We shall treat the fluid mathematically as if it formed a complete continuous medium. That is, variables such as temperature, pressure, density and velocity are well-defined for infinitely small points in the fluid, and are continuous in space and time (except on certain surfaces).

In reality, fluids are composed of discrete molecules that are in constant motion. However, when observed at macroscopic scales, the average behaviour of clusters of these molecules appears to be smooth and continuous.

The continuum approximation allows us to create a model that is mathematically tractable.

The continuum approximation is justifiable if molecular variables are averaged over length and time scales that are:

1. Large compared with molecular scales, but
2. Small compared with scales of practical interest.

The approximation is harder to justify when:

1. Densities are tiny (rarefied gas dynamics — space vehicle re-entry).
2. Spatial scales are small (nanotechnology).
3. There are sharp interfaces (shocks in supersonic flow).

3 Notation

3.1 Suffix notation

Gibbs notation

In these notes, vectors will be denoted by bold face fonts. In this notation, a position vector is written as

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k},$$

where x, y and z are the components of $\boldsymbol{x}$ along the coordinate directions denoted by the unit-vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively. This is referred to as Gibbs notation.

The vector $\boldsymbol{x}$ can also be written as a column vector,

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

in which case $\boldsymbol{x}^T = (x, y, z)$. Although, formally, it is important to keep track of the transposition, the superscript T is often omitted for brevity.

Suffix notation

Suffix notation (also known as Cartesian index notation) provides an alternative way of writing vectors. It is widely used in fluid mechanics and other areas of mathematical physics. In general, it is easier to manipulate vectors using suffix notation (particularly when the manipulations involve differential operators).

If x_1, x_2 and x_3 are the components of $\boldsymbol{x}$ in the coordinate directions denoted by the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively, then

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \sum_{i=1}^{3} x_i\boldsymbol{e}_i.$$

In this notation, x_i is the i^{th} component of the vector $\boldsymbol{x}$ and is *scalar*. The vector $\boldsymbol{x}$ can also be written as $\{x_i\}$, and the scalar component x_i can be written as $[\boldsymbol{x}]_i$.

Consider the vector equation $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ in $\mathbb{R}^3$. This is really shorthand for three scalar equations,

$$\begin{aligned} x_1 + y_1 &= z_1, \\ x_2 + y_2 &= z_2, \\ x_3 + y_3 &= z_3. \end{aligned}$$

In suffix notation, we write

$$x_i + y_i = z_i,$$

which is understood to hold for each component i. The suffix i is called a free suffix as it appears exactly once in each term.

There is no need to write $i = 1, 2, 3$. This is understood from the context. For physical problems, by default we have $i = 1, 2, 3$. However, if we are working in a space of dimension N, we can have $i = 1, 2, \ldots, N$.

Einstein summation convention

A suffix may appear twice in a single term, in which case it is known as a repeated suffix or dummy suffix. Where this occurs the Einstein summation convention is that the index should be summed over all possible values of the index.

Using the summation convention,

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = x_i\boldsymbol{e}_i.$$

The dot product of two vectors $\boldsymbol{a} = a_i\boldsymbol{e}_i$ and $\boldsymbol{b} = b_i\boldsymbol{e}_i$ is

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= \sum_{i=1}^{3} a_ib_i \\ &= a_ib_i \end{aligned}$$

Observe that there are no free suffixes on either side of the equation. The dummy suffix is ‘summed out’ of the equation.

Rules for suffix notation

1. Expressions with free suffixes hold for each possible value of the free suffix.
2. The free suffixes on each side of an equation should be the same
3. A repeated suffix (i.e., dummy suffix) implies summation over all values of that suffix.
4. We can rename dummy suffixes at will, hence

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_ib_i = a_kb_k = a_mb_m.$$

5. No suffix should appear more than twice in any term of an equation. If any suffix appears three or more times in any term, the equation is meaningless.

Suffix notation

Example 1. Write the following using suffix notation.

1. Newton's law, $\boldsymbol{F} = m\boldsymbol{a}$.
2. $\boldsymbol{u} + (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{v} = \|\boldsymbol{a}\|^2(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}$.
3. The gradient, $\nabla\phi$.
4. The divergence, $\nabla \cdot \boldsymbol{u}$.
5. The scalar quantity, $\boldsymbol{u} \cdot \nabla\phi$.
6. The Laplacian, $\nabla^2\phi$.

Example 2. Explain why the following equations cannot be correct:

1. $a_i = b_j$,
2. $a_i b_i = c_j d_k$,
3. $a_i = b_i b_j c_j d_j$.

Matrices in suffix notation

Suffix notation is also applicable to matrices. The elements of a matrix $\boldsymbol{A}$ are denoted by A_{ij}, where i is the row index and j is the column index.

For a 3×3 matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \{A_{ij}\},$$

where $i, j = 1, 2, 3$.

Using the Einstein summation convention, the trace of a 3×3 matrix $\boldsymbol{A}$ is

$$\operatorname{Tr}\boldsymbol{A} = A_{11} + A_{22} + A_{33} = A_{ii}.$$

Suffix notation provides a convenient way of denoting matrix-vector products. Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $\boldsymbol{A}$ is a 3×3 matrix. In suffix notation, the components of the product denoted by

$$\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u} \quad \text{are} \quad v_i = A_{ij} u_j.$$

This corresponds to multiplication of the matrix $\boldsymbol{A}$ by a column vector $\boldsymbol{u}$.

Example 3. Verify that $v_i = A_{ij} u_j$ corresponds to the matrix-vector product

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

The components of the product denoted by

$$\boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{A} \quad \text{are} \quad w_j = u_i A_{ij}.$$

This corresponds to multiplication of a row vector $\boldsymbol{u}$ by the matrix $\boldsymbol{A}$.

Example 4. Verify that $w_j = u_i A_{ij}$ corresponds to the matrix-vector product

$$(w_1, w_2, w_3) = (u_1, u_2, u_3) \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

The Kronecker delta

The Kronecker delta is

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

It corresponds to the identity matrix.

Example 5. Using the definition of the Kronecker delta δ_{ij}:

1. Evaluate δ_{ii} in three dimensions.
2. Show that $\delta_{ij} u_j = u_i$.

The last example illustrates the substitution rule for the Kronecker delta, whereby it swaps one of its suffixes for the other on anything it multiplies.

The alternating tensor

The alternating tensor, alternator or Levi-Civita symbol is

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123} \\ 0 & \text{otherwise.} \end{cases}$$

Even permutations are 123, 231, 312. Odd permutations are 321, 132, 213.

The alternating tensor is useful because it allows us to write equations involving vector (cross) products in suffix notation. The components of the vector product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ are

$$w_i = \epsilon_{ijk} u_j v_k.$$

The alternating tensor satisfies the following identities:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji},$$
$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

Example 6. Using the definition of the alternating tensor ϵ_{ijk}:

1. How could we write the curl, $\nabla \times \boldsymbol{u}$, in suffix notation?
2. Show that $\nabla \times \nabla\phi = \boldsymbol{0}$.
3. Show that $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

4 Kinematics

4.1 Eulerian and Lagrangian descriptions

Eulerian description

In the Eulerian specification of a flow field, flow quantities are regarded as functions of position $\boldsymbol{x}$ and time t.

For example, the Eulerian velocity field is written as $\boldsymbol{u}(\boldsymbol{x}, t)$.

Lagrangian description

In the Lagrangian specification of a flow field, we focus on material particles (points) as they move through the flow. Each particle is uniquely identified by a label $\boldsymbol{x}_0$, which is the position of the particle at some instant in time, usually $t = 0$.

Flow quantities are regarded as functions of the particle label $\boldsymbol{x}_0$ and time t.

The position of the particle is given by the function $\boldsymbol{X}(\boldsymbol{x}_0, t)$, where $\boldsymbol{x}_0 = \boldsymbol{X}(\boldsymbol{x}_0, 0)$.

The velocity and acceleration of the particle are

$$\boldsymbol{U}(\boldsymbol{x_0}, t) = \frac{\partial \boldsymbol{X}}{\partial t} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{x_0}, t) = \frac{\partial^2 \boldsymbol{X}}{\partial t^2},$$

respectively.

Example 7. Consider $\boldsymbol{X}(\boldsymbol{x_0}, t) = x_0 e^t \boldsymbol{i} + y_0 e^{-t} \boldsymbol{j}$, where $\boldsymbol{x_0} = (x_0, y_0)$.

1. Verify that $\boldsymbol{x_0} = \boldsymbol{X}(\boldsymbol{x}_0, 0)$.
2. Find $\boldsymbol{x_0}$ for the particle that passes through (e^2, e^{-2}) at the time $t = 2$.
3. Determine the velocity and acceleration of that particle.

Transformation between descriptions

The transformation from a Lagrangian to an Eulerian description is accomplished by considering the particle that is at the Eulerian position $\boldsymbol{x}$ at time t, that is,

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t). \tag{1}$$

Given $\boldsymbol{x}$, t and the function $\boldsymbol{X}$, equation 1 can, in principle, be solved for $\boldsymbol{x}_0$.

Example 8. Consider $\boldsymbol{X}(\boldsymbol{x}_0, t) = x_0 e^t \boldsymbol{i} + y_0 e^{-t} \boldsymbol{j}$, as before.

1. Find $\boldsymbol{x}_0$ for the particle that passes through the arbitrary Eulerian point $\boldsymbol{x} = (x, y)$ at time t.
2. Find the Lagrangian description of the velocity $\boldsymbol{U}(\boldsymbol{x}_0, t)$ and acceleration $\boldsymbol{A}(\boldsymbol{x}_0, t)$.

3. Use the relationship between $\boldsymbol{x}_0$ and $\boldsymbol{x}$ found in part 1 to find the Eulerian description of the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$.
4. Use the expression for the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ to calculate $\partial \boldsymbol{u}/\partial t$. Is it the same as the acceleration?

Steady flow

A flow is steady when the Eulerian description of the velocity is independent of time, that is,

$$\frac{\partial \boldsymbol{u}}{\partial t} = \mathbf{0}.$$

Otherwise, the flow is unsteady.

Example 9. Is the flow in the previous example steady or not?

This concept only applies to the Eulerian description of the flow. A steady flow does not imply that the fluid particle velocity is independent of time.

Whether a flow is steady or not depends on the frame of reference. For example a wake trailing behind a moving boat would appear steady to an observer on the boat, and unsteady to an observer on the shore.

4.2 Flow Visualisation

Pathlines

A pathline is a curve traced out by a particle as it moves through a flow field. It is the curve generated by treating t as a parameter in $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t)$, hence is associated with Lagrangian coordinates.

Suppose instead that $\boldsymbol{u}(\boldsymbol{x}, t)$ is given, where $\boldsymbol{x}$ is the Eulerian position. Then

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x}_0, t), t) = \boldsymbol{U}(\boldsymbol{x}_0, t) = \frac{\partial \boldsymbol{X}}{\partial t},$$

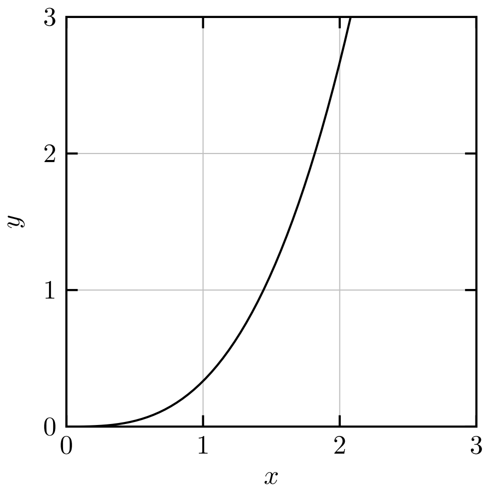
where $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t)$. For a given particle, this is simply

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}, t), \tag{2}$$

that is, a system of ODEs to solve for $\boldsymbol{X}$.

Example 10. Suppose $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} + xt\boldsymbol{j}$ and let $\boldsymbol{X} = X\boldsymbol{i} + Y\boldsymbol{j}$.

1. Write down a system of first-order ODEs for X and Y.
2. Solve this system and hence write down $\boldsymbol{X}(\boldsymbol{x}_0, t)$ and $\boldsymbol{x}_0$.
3. Write out the components of $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} = \boldsymbol{X}(\boldsymbol{x}_0, t)$, writing y in terms of x by eliminating t.
4. Draw the pathline of the particle that starts from the origin at time $t = 0$.



Streaklines

A streakline is a curve made up of all particles that have passed a particular point in space at some earlier time. It is what you would see if dye or smoke was introduced at some point in the flow. It is also associated with Lagrangian coordinates.

Suppose $\boldsymbol{x}^*$ is the point where the dye is being introduced. We want to find all the points $\boldsymbol{x}_0$ such that

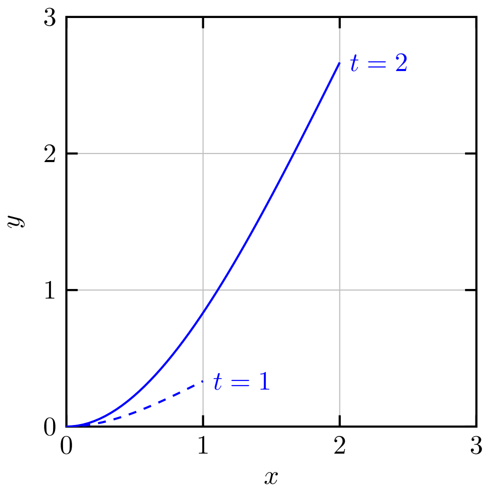
$$\boldsymbol{x}^* = \boldsymbol{X}(\boldsymbol{x}_0, \tau), \tag{3}$$

where τ is any time before the present (that is, $0 \leq \tau \leq t$). If the function $\boldsymbol{X}$ is known, then equation (3) can, in principle, be used to find $\boldsymbol{x}_0$ in terms of $\boldsymbol{x}^*$ and τ, that is, $\boldsymbol{x}_0(\boldsymbol{x}^*, \tau)$. The streakline is the curve given by the set of points

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0(\boldsymbol{x}^*, \tau), t), \quad 0 \leq \tau \leq t. \tag{4}$$

Example 11. Consider the previous example $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} + xt\boldsymbol{j}$, again. Suppose $\boldsymbol{x}^* = x^*\boldsymbol{i} + y^*\boldsymbol{j}$.

1. Using the results from the previous example, find $\boldsymbol{x}_0$ for the particle that passes through $\boldsymbol{x}^*$ at time τ, that is find $\boldsymbol{x}_0(\boldsymbol{x}^*, \tau)$.
2. Write out the components of $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0(\boldsymbol{x}^*, \tau), t)$. Write y in terms of x by eliminating τ.
3. Draw the streakline emanating from the origin at $t = 1$ and $t = 2$.



Streamlines

A streamline is a curve that is everywhere tangent to the velocity field. It is associated with Eulerian coordinates.

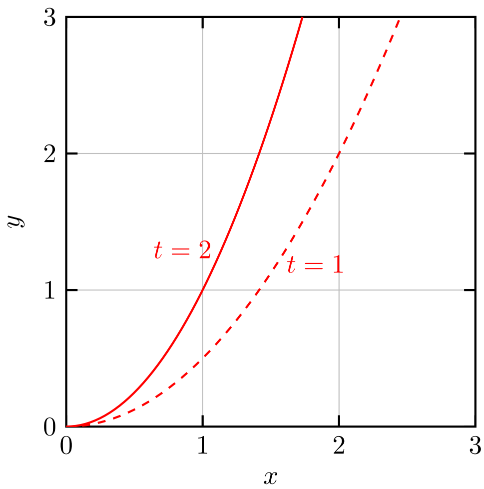
Recall from vector calculus that the tangent to a curve $\boldsymbol{x}(s)$ is given by $\mathrm{d}\boldsymbol{x}/\mathrm{d}s$, where s is a parameter. This must be parallel to the local velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. This is accomplished by setting

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}(\boldsymbol{x}, t), \tag{5}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is expressed in Eulerian coordinates. This is a system of ODEs that can, in principle, be solved for $\boldsymbol{x}$ in terms of s at any instant in time t.

Example 12. Consider the previous example $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} + xt\boldsymbol{j}$, yet again. Suppose $\boldsymbol{x}(s, t) = x(s, t)\boldsymbol{i} + y(s, t)\boldsymbol{j}$.

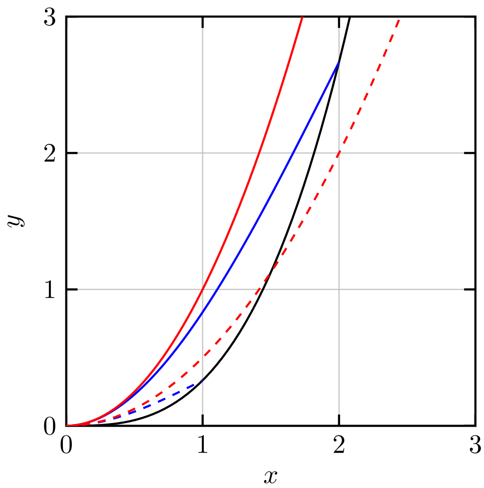
1. Write down a system of first-order ODEs for the dependant variables x and y.
2. Solve this system of ODEs and hence write down $\boldsymbol{x}(s, t)$
3. Find y in terms of x by eliminating s.
4. Draw a streamline that passes through the origin for times $t = 1$ and $t = 2$.



Pathlines, streaklines and streamlines

Pathlines, streaklines and streamlines are coincident when the flow is steady.

Examples 10–12



4.3 The material derivative

The material derivative

The material or convective or substantial derivative is an expression for the rate of change of quantity following a particle (Lagrangian time derivative), written in terms of its Eulerian description.

Consider a particle labelled $\boldsymbol{x}_0$ that passes through the Eulerian position $\boldsymbol{x}$ at time t, so that $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t)$. Suppose that a quantity is expressed in Eulerian coordinates by $f(\boldsymbol{x},t)$ and in Lagrangian coordinates by $F(\boldsymbol{x}_0,t)$. The two functions are related by

$$F(\boldsymbol{x}_0,t) = f(\boldsymbol{X}(\boldsymbol{x}_0,t),t)$$

The rate of change of the quantity following the particle is $\partial F/\partial t$. Using the chain rule,

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial X_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial X_2}{\partial t} + \frac{\partial f}{\partial x_3}\frac{\partial X_3}{\partial t} + \frac{\partial f}{\partial t}.$$

But $\partial \boldsymbol{X}/\partial t = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x}_0,t),t)$, hence

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial t} + u_1\frac{\partial f}{\partial x_1} + u_2\frac{\partial f}{\partial x_2} + u_3\frac{\partial f}{\partial x_3}.$$

The right-hand-side of the above is the material, convective or substantial derivative of f. It is denoted by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f = \frac{\partial f}{\partial t} + u_i\frac{\partial f}{\partial x_i}. \tag{6}$$

Hence, in Eulerian coordinates, the rate of change of a quantity following a fluid particle consists of two components:

1. The local rate of change, $\partial f/\partial t$, due to temporal variation.
2. The convective rate of change, $\boldsymbol{u}\cdot\nabla f$, due to movement of fluid particles through spatial gradients.

The material derivative also applies to vectors. For example, the j-th component of the acceleration of a fluid particle is

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = \frac{\partial u_j}{\partial t} + \boldsymbol{u}\cdot\nabla u_j = \frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i} \tag{7a}$$

which is the j-th component of the acceleration. In vector notation, this is written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}. \tag{7b}$$

Example 13. Let $\boldsymbol{u} = (u,v,w)$ and $\boldsymbol{x} = (x,y,z)$. Write out all three components of the acceleration.

Example 14. In example 8, we found that $\boldsymbol{u}(\boldsymbol{x},t) = u\,\boldsymbol{i} + v\,\boldsymbol{j} = x\,\boldsymbol{i} - y\,\boldsymbol{j}$ and $\boldsymbol{a}(\boldsymbol{x},t) = x\,\boldsymbol{i} + y\,\boldsymbol{j} \neq \partial\boldsymbol{u}/\partial t$, where $\boldsymbol{x} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$. Use the material derivative to verify that

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{a}.$$

4.4 Decomposition of local fluid motion

Decomposition of local fluid motion

Consider the fluid velocity in the vicinity of the point $\boldsymbol{x}$ in Eulerian coordinates. As the velocity is a continuously differentiable function of space and time, it can be expressed using a multivariable Taylor series. The j-th component of the velocity at a nearby point $\boldsymbol{x}+\delta\boldsymbol{x}$ is

$$
\begin{aligned}
u_j(\boldsymbol{x}+\delta\boldsymbol{x},t) &= u_j(\boldsymbol{x},t) + \delta x_1 \left.\frac{\partial u_j}{\partial x_1}\right|_{\boldsymbol{x},t} + \delta x_2 \left.\frac{\partial u_j}{\partial x_2}\right|_{\boldsymbol{x},t} + \delta x_3 \left.\frac{\partial u_j}{\partial x_3}\right|_{\boldsymbol{x},t} + \text{h.o.t.} \\
&= u_j(\boldsymbol{x},t) + \delta x_i \left.\frac{\partial u_j}{\partial x_i}\right|_{\boldsymbol{x},t} + \text{h.o.t.},
\end{aligned}
$$

where 'h.o.t.' stands for 'higher order terms', which involve products of δx_i.

In vector form, this is written as

$$\boldsymbol{u}(\boldsymbol{x}+\delta\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) + \delta\boldsymbol{x}\cdot\nabla\boldsymbol{u}(\boldsymbol{x},t) + \text{h.o.t.},$$

where $\nabla\boldsymbol{u}$ is the velocity gradient matrix (or tensor) with components

$$[\nabla\boldsymbol{u}]_{ij} = \frac{\partial u_j}{\partial x_i}.$$

Example 15. Let $\boldsymbol{x}=(x,y,z)$ and $\boldsymbol{u}=(u,v,w)$. Write out the velocity gradient tensor.

If $|\delta\boldsymbol{x}|$ is sufficiently small, the higher order terms can be neglected. Then the velocity relative to that at $\boldsymbol{x}$ is

$$\varDelta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}+\delta\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t) = \delta\boldsymbol{x}\cdot\nabla\boldsymbol{u},$$

or in suffix notation,

$$\varDelta u_j = \delta x_i \frac{\partial u_j}{\partial x_i}.$$

This is the velocity field that we would observe near the point $\boldsymbol{x}$ if we were to travel with the flow at the point $\boldsymbol{x}$. This local relative motion depends on the velocity gradient tensor.

The velocity gradient can be decomposed into the sum of a symmetric matrix $\boldsymbol{E}$ and an antisymmetric matrix $\boldsymbol{\varOmega}$, such that

$$\nabla\boldsymbol{u} = \boldsymbol{E} + \boldsymbol{\varOmega},$$

where

$$
\begin{aligned}
\boldsymbol{E} &= \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right], \\
\boldsymbol{\varOmega} &= \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^T\right].
\end{aligned}
$$

In suffix notation, this is

$$\frac{\partial u_j}{\partial x_i} = E_{ij} + \varOmega_{ij}$$

where

$$
\begin{aligned}
E_{ij} &= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \\
\varOmega_{ij} &= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).
\end{aligned}
$$

A symmetric matrix $\boldsymbol{E}$ satisfies $E_{ij} = E_{ji}$. An antisymmetric matrix $\boldsymbol{\Omega}$ satisfies $\Omega_{ij} = -\Omega_{ji}$.

$\boldsymbol{E}$ is responsible for deformation of fluid elements and is called the rate-of-strain or rate-of-deformation tensor.

$\boldsymbol{\Omega}$ is responsible for rotation of fluid elements and is called the rate-of-rotation tensor.

Example 16. Use suffix notation to show that:

1. $\boldsymbol{E}$ is symmetric, and
2. $\boldsymbol{\Omega}$ is antisymmetric.

Example 17. Let $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{u} = (u, v, w)$. Write out in matrix form the components of:

1. the rate-of-strain tensor $\boldsymbol{E}$, and
2. the rate-of-rotation tensor $\boldsymbol{\Omega}$.

Strain

To study straining motion, put $\boldsymbol{\Omega} = \mathbf{0}$. Then the relative velocity is

$$\Delta \boldsymbol{u} = \delta \boldsymbol{x} \cdot \boldsymbol{E} \quad \text{or} \quad \Delta u_j = \delta x_i E_{ij}.$$

We consider two cases:

1. Normal strain (given by the diagonal terms).
2. Shear strain (the non-diagonal terms).

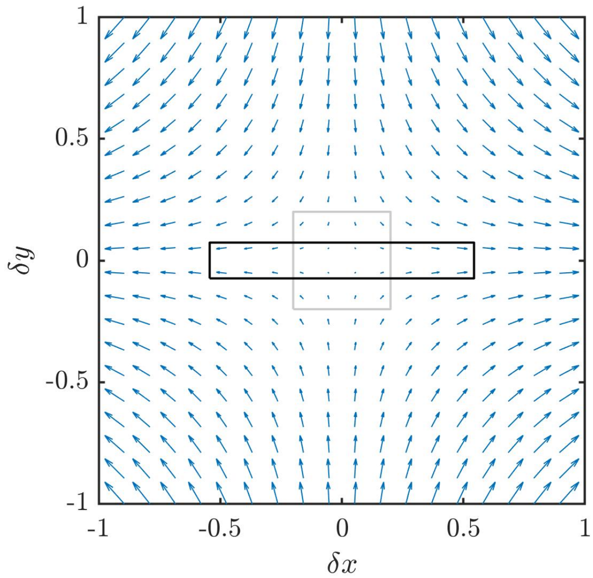
Normal strain

Consider the two-dimensional strain,

$$\boldsymbol{E} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha > 0.$$

The components of the relative velocity are

$$\Delta \boldsymbol{u} = \delta \boldsymbol{x} \cdot \boldsymbol{E} = (\delta x, \delta y, \delta z) \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\alpha\, \delta x, -\alpha\, \delta y, 0)$$



In this example, $E_{11} = \alpha > 0$ and $E_{22} = -\alpha < 0$. Fluid elements are stretched apart in the x-direction (extension) and squeezed together in the y direction (contraction).

Notice that

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_i}{\partial x_i} = E_{ii} = 0$$

in this example. This is a common situation in incompressible flows, as we shall see later.

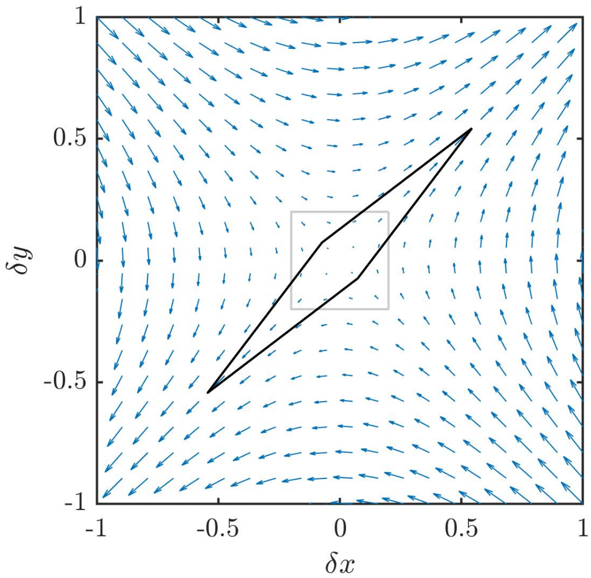
Shear strain

Suppose that

$$\boldsymbol{E} = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma > 0.$$

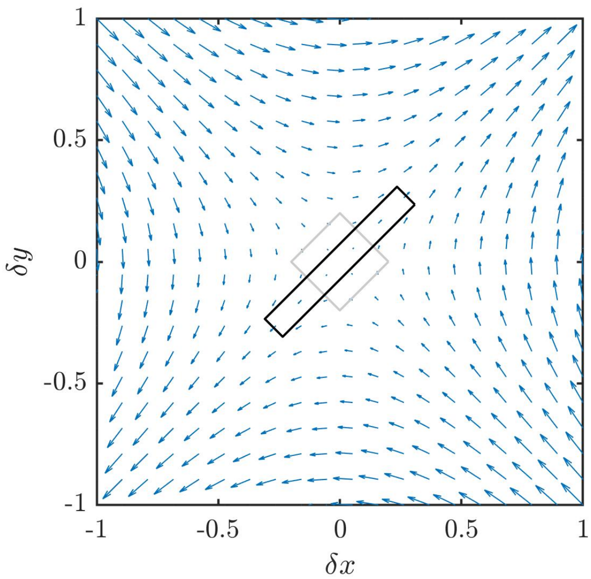
The components of the relative velocity are

$$\boldsymbol{\mathit{\Delta u}} = \delta \boldsymbol{x} \cdot \boldsymbol{E} = (\delta x, \delta y, \delta z) \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\gamma\,\delta y, \gamma\,\delta x, 0).$$



It is always possible to rotate the coordinate system so that the rate-of-strain tensor only has nonzero entries on its diagonal. The axes of that coordinate system are known as principal axes. The diagonal entries are the eigenvalues of $\boldsymbol{E}$ and the directions of the principal axes are the eigenvectors.

Indeed, the last two examples are the same when one is rotated by 45 degrees with respect to the other!



Rotation

To study rotation, put $\boldsymbol{E} = \mathbf{0}$. Then the relative velocity is

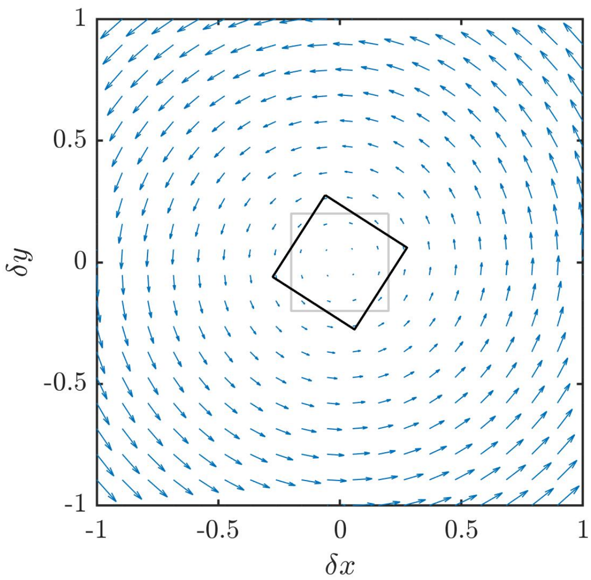
$$\Delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\boldsymbol{\Omega} \quad \text{or} \quad \Delta u_j = \delta x_i \Omega_{ij}.$$

Suppose that

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega > 0.$$

The components of the relative velocity $\Delta\boldsymbol{u}$ are

$$\Delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\boldsymbol{\Omega} = (\delta x, \delta y, \delta z)\begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (-\omega\,\delta y, \omega\,\delta x, 0).$$



This results in rotation about the z-axis.

The three independent components of the rate-of-rotation tensor are associated with a vector known as the vorticity.

Vorticity

The vorticity is defined as the curl of the velocity, that is,

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{8a}$$

In suffix notation, the components of the vorticity are

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \tag{8b}$$

Example 18. Use (8a) and (8b) to write out the components of the vorticity $\boldsymbol{\omega}$.

The rate-of-rotation tensor is related to the vorticity $\boldsymbol{\omega} = \omega_k \boldsymbol{e}_k$ by

$$\Omega_{ij} = \frac{1}{2} \epsilon_{ijk} \omega_k.$$

Explicitly,

$$\boldsymbol{\Omega} = \frac{1}{2} \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

Example 19. Use the definition of the vorticity $\omega_k = \epsilon_{klm} \partial u_m / \partial x_l$ and the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ to show that

$$\frac{1}{2} \epsilon_{ijk} \omega_k = \Omega_{ij}.$$

In suffix notation, the local relative velocity associated with the rate-of-rotation tensor is

$$\Delta u_j = x_i \Omega_{ij} = x_i \left(\frac{1}{2} \epsilon_{ijk} \omega_k \right) = \frac{1}{2} \epsilon_{jki} \omega_k x_i.$$

which is just $\boldsymbol{\Delta u} = \frac{1}{2} \boldsymbol{\omega} \times \boldsymbol{x}$. This corresponds to rotation about the vector $\boldsymbol{\omega}$ with angular velocity $\frac{1}{2} \|\boldsymbol{\omega}\|$.

Example 20. Consider the two-dimensional flow

$$\boldsymbol{u}(\boldsymbol{x}) = \omega y \, \boldsymbol{i} - \omega x \, \boldsymbol{j}.$$

where $\omega > 0$ is a constant.

1. Determine streamlines for the flow.
2. Calculate the components of the rate-of-strain tensor.
3. Find the vorticity vector $\boldsymbol{\omega}$.

In this example, the fluid elements are not deformed. The fluid moves like a rigid body with the same angular velocity everywhere.

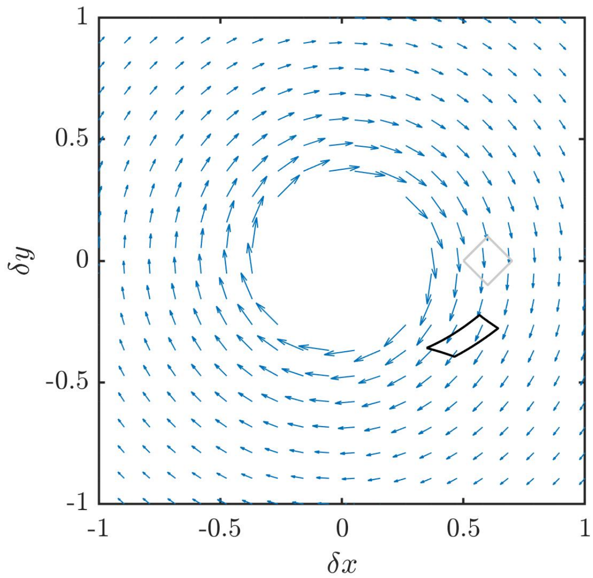
Example 21. Consider the two-dimensional flow

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{y}{x^2 + y^2} \, \boldsymbol{i} - \frac{x}{x^2 + y^2} \, \boldsymbol{j}, \quad x, y \neq 0.$$

1. Determine streamlines for the flow.
2. Calculate the components of the rate-of-strain tensor $\boldsymbol{E}$ at $\boldsymbol{x} = (1, 0)$, $(1, 1)$ and $(0, 1)$.

3. Find the vorticity $\boldsymbol{\omega}$.

In this example, small fluid elements are distorted (strained) but do not rotate, despite the fact that the streamlines are circular. This is an example of an irrotational flow called a potential vortex, which we will see again later.



5 Conservation of mass

5.1 Mass conservation equation

Mass conservation equation

Let $\mathcal{V}$ be an *arbitrary* fixed volume of fluid enclosed by a surface $\mathcal{S}$. Suppose that mass is neither created nor destroyed in $\mathcal{V}$. Then the rate of increase in mass in $\mathcal{V}$ equals the rate at which mass flows into $\mathcal{V}$ through $\mathcal{S}$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{S}} \rho \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S},$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to $\mathcal{S}$.

Recall that the divergence theorem states that

$$\int_{\mathcal{V}} \nabla\cdot\boldsymbol{F}\,\mathrm{d}\mathcal{V} = \int_{\mathcal{S}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S}.$$

Using the divergence theorem to convert the surface integral to a volume integral gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} \nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}\mathcal{V}.$$

Since the volume $\mathcal{V}$ is fixed, we can differentiate under the integral sign on the LHS to obtain

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \, \mathrm{d}\mathcal{V} = 0.$$

This must hold for *any* fixed volume $\mathcal{V}$, hence the integrand must be identically zero. Therefore,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{9}$$

Equation (9) is referred to as the mass conservation equation.

5.2 Incompressible flow

Incompressible flow

A fluid is incompressible if the density of every fluid particle is constant. This means that

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0.$$

The mass conservation equation (9) gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u}.$$

Substituting the first equation into the second, we see that for an incompressible fluid

$$\nabla \cdot \boldsymbol{u} = 0. \tag{10}$$

Equation (10) is known as the continuity equation.

Note that:

1. There are some incompressible flows for which $\nabla \cdot \boldsymbol{u} \neq 0$ (such as variable-density and chemically reacting flows, for example). We will not consider such flows in this course. If a flow is incompressible, then you may assume that $\nabla \cdot \boldsymbol{u} = 0$.
2. In an incompressible flow, it is still possible for the density to vary from fluid particle to fluid particle, but it cannot change on a given particle.
3. If the density is uniform and constant throughout the fluid, then $\nabla \cdot \boldsymbol{u} = 0$ follows immediately from the mass conservation equation.
4. No fluid is perfectly incompressible, but it is an excellent approximation when flow velocities are much smaller than the speed of sound.

5.3 Stream function

The stream function

Recall from vector calculus that the divergence of a curl is always zero

$$\nabla \cdot (\nabla \times \boldsymbol{f}) = 0.$$

Hence, we can automatically satisfy the continuity equation (10) by setting

$$\boldsymbol{u} = \nabla \times \boldsymbol{\Psi},$$

for some vector function $\boldsymbol{\Psi}$. This does not, at first sight, seem to make things much easier. However, for certain types of flow, $\boldsymbol{\Psi}$ takes a particularly simple form.

Consider a *two-dimensional* incompressible flow in the (x, y)-plane. The continuity equation is

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

This can be satisfied by defining a scalar function $\psi(x, y, t)$ such that

$$\boldsymbol{\Psi} = \psi(x, y, t)\, \boldsymbol{k}.$$

Upon taking the curl, we obtain

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{11}$$

The function ψ is called the stream function.

Example 22. Verify that ψ automatically satisfies $\nabla \cdot \boldsymbol{u} = 0$ for the two-dimensional flow $\boldsymbol{u} = (u, v)$.

Example 23. Consider the streamline

$$\boldsymbol{x}(s) = x(s)\, \boldsymbol{i} + y(s)\, \boldsymbol{j},$$

parameterised by s. The rate of change of ψ along the streamline is

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{\partial \psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial \psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}.$$

Show that ψ is constant along a streamline.

Example 24. In a two-dimensional flow, the volume flux (per unit width) Q across any curve $\mathcal{C}$ between two points is

$$Q = \int_{\mathcal{C}} \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}s,$$

wher s is the arc length. Show that the volume flux is equal to the difference in ψ between the two points, that is,

$$Q = \int_{\mathcal{C}} u\, \mathrm{d}y - v\, \mathrm{d}x = \psi_2 - \psi_1.$$

Notes:

1. Lines of constant ψ are tangential to the fluid velocity, hence fluid cannot cross a streamline. Lines of constant ψ *may* represent a fixed impermeable boundary.

2. It will be shown (see tutorial) that ψ is related to the vorticity $\boldsymbol{\omega} = \omega\, \boldsymbol{k}$ by
$$\nabla^2 \psi = -\omega.$$

Example 25. Find the stream function of:

1. the parallel flow $\boldsymbol{u} = U\, \boldsymbol{i}$, for some constant U.
2. the rotating flow $\boldsymbol{u} = \omega y\, \boldsymbol{i} - \omega x\, \boldsymbol{j}$, for some constant ω.

Stream functions in other coordinate systems

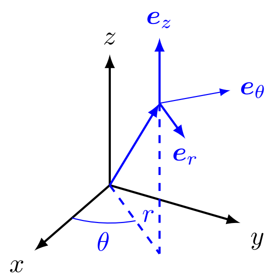
There is no equivalent to the scalar stream function in a general three-dimensional flow. But there are some other special cases with a high degree of symmetry, where a stream function can be defined.

Here we will consider:

1. Cylindrical coordinates: (r, θ, z)
2. Spherical coordinates: (r, θ, ϕ)

Cylindrical coordinates

Consider the cylindrical coordinate system with radial, azimuthal and axial coordinates (r, θ, z) and associated unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$, respectively. These are related to Cartesian coordinates by $x = r\cos\theta$ and $y = r\sin\theta$.



For an incompressible flow,

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0, \tag{12}$$

where $\boldsymbol{u} = u_r\, \boldsymbol{e}_r + u_\theta\, \boldsymbol{e}_\theta + u_z\, \boldsymbol{e}_z$.

Plane flow in cylindrical coordinates

For two-dimensional flow in the r-θ plane, $\partial/\partial z = 0$, hence

$$\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0.$$

The stream function is then defined by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}. \tag{13}$$

Axisymmetric flow in cylindrical coordinates

For axisymmetric flow, $\partial/\partial\theta = 0$, hence

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0.$$

The stream function is then defined by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{14}$$

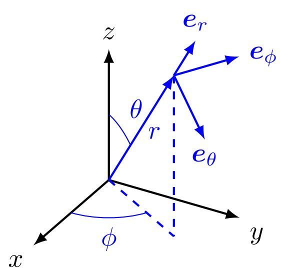
Lines of constant ψ in the r-z plane correspond to stream surfaces in three-dimensions. The volume flux across any surface connecting two stream surfaces is constant.

Example 26. 1. Verify that this stream function automatically satisfies the continuity equation if the flow is axisymmetric.

2. Find the stream function of the parallel flow $\boldsymbol{u} = U\,\boldsymbol{e}_z$.

Spherical coordinates

Consider the spherical coordinate system with radial, latitudinal and longitudinal coordinates (r, θ, ϕ) and associated unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$, respectively. These are related to Cartesian coordinates by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$.



For an incompressible flow,

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0, \tag{15}$$

where $\boldsymbol{u} = u_r\,\boldsymbol{e}_r + u_\theta\,\boldsymbol{e}_\theta + u_\phi\,\boldsymbol{e}_\phi$.

Axisymmetric flow in spherical coordinates

For axisymmetric flow, $\partial/\partial\phi = 0$, hence

$$\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) = 0,$$

The stream function is then defined by

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{16}$$

6 Dynamics

6.1 Forces

Forces

We will consider fluids governed by Newton's Second Law, $\boldsymbol{F} = m\boldsymbol{a}$, written in a form suitable for application to a fluid.

Thus we need to consider the forces acting on a parcel of fluid. These are

1. External forces
2. Internal forces

External forces

External forces are due to some external agent, such as gravity or electromagnetic forces.

For a small element of fluid $\delta\mathcal{V}$, the external force is $\delta\mathcal{V}\boldsymbol{F}$, where $\boldsymbol{F}$ is the force per unit volume. For example, the force per unit volume due to gravity is

$$\boldsymbol{F} = -\rho g\,\boldsymbol{k},$$

where $\boldsymbol{k}$ points vertically upwards.

Note:

- An external force is also referred to as a body force.
- Sometimes, the force is defined per unit mass of fluid ($\hat{\boldsymbol{F}}$). Then, $\boldsymbol{F} = \rho\hat{\boldsymbol{F}}$, where ρ is the density.

Internal forces

Internal forces are due to interactions between fluid particles. They can only be transmitted across contact surfaces.

For a small element of area $\delta\mathcal{S}$, the internal force is $\delta\mathcal{S}\,\boldsymbol{T}$, where $\boldsymbol{T}$ is the force per unit area or stress.

The stress $\boldsymbol{T}$ can be resolved into two components:

1. A normal stress or pressure, acting perpendicular to the surface, and
2. A shear stress acting parallel to the surface.

The stress tensor

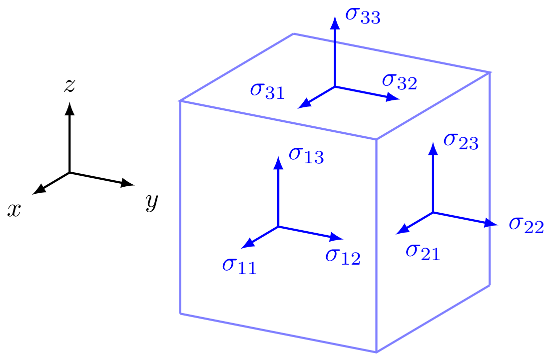
The stress, $\boldsymbol{T}$, acting at a point $\boldsymbol{x}$ on a surface with outward unit normal $\hat{\boldsymbol{n}}$ is

$$T_j(\boldsymbol{x}, t, \hat{\boldsymbol{n}}) = \hat{n}_i \sigma_{ij}(\boldsymbol{x}, t) \quad \text{or} \quad \boldsymbol{T} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma},$$

where σ_{ij} are the components the stress tensor $\boldsymbol{\sigma}$.

The quantity σ_{ij} gives the force per unit area acting in the direction j upon a surface that has its unit outward normal in the direction i. Thus ,

- σ_{11} is the force acting in the x-direction on a surface with normal $\boldsymbol{i}$,
- σ_{21} is the force acting in the x-direction on a surface with normal $\boldsymbol{j}$,
- σ_{31} is the force acting in the x-direction on a surface with normal $\boldsymbol{k}$,
- and so on.



Pressure

The pressure is defined as

$$p = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}). \tag{17}$$

It is useful to write the stress tensor as

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}, \tag{18}$$

where $\boldsymbol{\tau}$ is the deviatoric part of $\boldsymbol{\sigma}$.

The pressure part corresponds to the normal stress if the fluid is at rest. The deviatoric part corresponds to stress caused by deformation of the fluid.

6.2 Equation of motion

The Cauchy equation of motion

We now derive a momentum balance equation for a general continuum material with a stress tensor $\boldsymbol{\sigma}$.

Let $\mathcal{V}$ be an arbitrary volume moving with the fluid, enclosed by a surface $\mathcal{S}$, with unit outward normal $\hat{\boldsymbol{n}}$. According to Newton's Second Law ($\boldsymbol{F} = m\boldsymbol{a}$),

$$
\begin{aligned}
\iiint_{\mathcal{V}} \rho \frac{\mathrm{D}u_j}{\mathrm{D}t} \, \mathrm{d}\mathcal{V} &= \iiint_{\mathcal{V}} F_j \, \mathrm{d}\mathcal{V} + \iint_{\mathcal{S}} T_j \, \mathrm{d}\mathcal{S} \\
&= \iiint_{\mathcal{V}} F_j \, \mathrm{d}\mathcal{V} + \iint_{\mathcal{S}} \hat{n}_i \sigma_{ij} \, \mathrm{d}\mathcal{S} \\
&= \iiint_{\mathcal{V}} F_j \, \mathrm{d}\mathcal{V} + \iiint_{\mathcal{V}} \frac{\partial \sigma_{ij}}{\partial x_i} \, \mathrm{d}\mathcal{V}.
\end{aligned}
$$

Since $\mathcal{V}$ is arbitrary,

$$
\rho \frac{\mathrm{D}u_j}{\mathrm{D}t} = F_j + \frac{\partial \sigma_{ij}}{\partial x_i} \tag{19a}
$$

or in vector notation,

$$
\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{F} + \nabla \cdot \boldsymbol{\sigma}. \tag{19b}
$$

This is called the Cauchy equation of motion. It applies to any continuum material (e.g., elastic solids, plastics, as well as fluids like air and water).

Note that the Cauchy equation of motion (19) involves two unknown fields, the velocity field, $\boldsymbol{u}$ (3 components) and the stress field, $\boldsymbol{\sigma}$ (9 components), and so the system cannot be solved without a further set of phenomenological equations relating σ_{ij} and u_i. These equations are called constitutive relations.

Constitutive relations

As in (18), we split the stress into two parts, a normal stress that would exist if the fluid were at rest and a deviatoric part,

$$
\sigma_{ij} = -p \, \delta_{ij} + \tau_{ij}.
$$

The deviatoric part must depend in some way on the velocity gradients. For simplicity, we will assume that the tensor τ_{ij} is linearly dependent on the velocity gradient tensor $\mathcal{D}_{ij} = \partial u_j / \partial x_i$. Hence,

$$
\begin{aligned}
\tau_{11} = {} & A_{1111}\mathcal{D}_{11} + \mathcal{A}_{1112}\mathcal{D}_{12} + A_{1113}\mathcal{D}_{13} \\
& + A_{1121}\mathcal{D}_{21} + A_{1122}\mathcal{D}_{22} + A_{1123}\mathcal{D}_{23} \\
& + A_{1131}\mathcal{D}_{31} + A_{1132}\mathcal{D}_{32} + A_{1133}\mathcal{D}_{33},
\end{aligned}
$$

and so on for the other eight components of $\boldsymbol{\tau}$.

More succinctly,

$$
\tau_{ij} = A_{ijmn} \mathcal{D}_{mn},
$$

where $\boldsymbol{A}$ is a fourth-order tensor with 81 coefficients!

This can be greatly simplified if it assumed that the fluid is isotropic, meaning that it has no preferred direction. In that case, the most general expression for $\boldsymbol{A}$ is of the form

$$A_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm},$$

where λ, μ and γ will in general depend on the physical properties of the fluid, e.g., temperature. Henceforth, we will assume they are constants.

Since $\boldsymbol{\sigma}$ is symmetric, $\boldsymbol{\tau}$ must be also, and so $A_{ijmn} = A_{jimn}$. Interchanging i and j,

$$\begin{aligned} A_{jimn} &= \lambda\delta_{ji}\delta_{mn} + \mu\delta_{jm}\delta_{in} + \gamma\delta_{jn}\delta_{im} \\ &= \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}, \end{aligned}$$

hence,

$$A_{ijmn} - A_{jimn} = (\mu - \gamma)(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) = 0,$$

from which we deduce that $\mu = \gamma$.

Now we have,

$$\begin{aligned} \tau_{ij} &= A_{ijmn}\mathcal{D}_{mn} \\ &= \lambda\delta_{ij}\delta_{mn}\mathcal{D}_{mn} + \mu\delta_{im}\delta_{jn}\mathcal{D}_{mn} + \mu\delta_{in}\delta_{jm}\mathcal{D}_{mn} \\ &= \lambda\delta_{ij}\mathcal{D}_{mm} + \mu\left(\mathcal{D}_{ij} + \mathcal{D}_{ji}\right). \end{aligned}$$

If the flow is incompressible, then $D_{mm} = 0$ and

$$\tau_{ij} = \mu\left(\mathcal{D}_{ij} + \mathcal{D}_{ji}\right) = 2\mu E_{ij},$$

where $E_{ij} = \frac{1}{2}(\mathcal{D}_{ij} + \mathcal{D}_{ji})$ are the components of the rate-of-strain tensor.

The constitutive equation of an incompressible Newtonian fluid is

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu E_{ij}, \tag{20}$$

where μ is a constant called the dynamic viscosity.

The Navier–Stokes equations

Substituting the constitutive relation (20) into the Cauchy equation of motion (19) gives

$$\rho\frac{\mathrm{D}u_j}{\mathrm{D}t} = -\frac{\partial p}{\partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_i \partial x_i} + F_j \tag{21a}$$

or in vector notation,

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}. \tag{21b}$$

These are the Navier–Stokes equations for an incompressible Newtonian fluid.

Example 27. After expanding the material derivative, the Navier–Stokes equations are

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{F_j}{\rho},$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

1. Write this out in vector form.
2. Let $\boldsymbol{x} = (x, y, z)$, $\boldsymbol{u} = (u, v, w)$ and $\boldsymbol{F} = \rho g\,\boldsymbol{k}$. Write out all three components of the Navier–Stokes equations.

The Navier–Stokes equations consist of three partial differential equations. Assuming a known density ρ and external force $\boldsymbol{F}$, there are four unknowns to be solved for: the pressure p and the three components of the velocity field $\boldsymbol{u}$. The incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$, closes the system.

The Navier–Stokes equations are non-linear and it is not possible to construct solutions by superposition. No general solution is known. They are the subject of one of the seven Clay Millenium Prize problems. A solution of one of these problems is worth $1 million. See:

`http://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation`

For an inviscid flow, $\mu = 0$ and the Navier–Stokes equations reduce to the Euler equations,

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{F}. \tag{22}$$

No-slip condition

For an impermeable boundary moving with velocity $\boldsymbol{U}$,

$$(\boldsymbol{u} - \boldsymbol{U}) \cdot \hat{\boldsymbol{n}} = 0, \tag{23}$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the boundary. This boundary condition applies for viscous or inviscid flows.

For viscous flows, additional boundary conditions are required because the Navier–Stokes equations are of higher order than the Euler equations. For an impermeable boundary moving with velocity $\boldsymbol{U}$ in a viscous fluid, the no-slip condition

$$\boldsymbol{u} = \boldsymbol{U} \tag{24}$$

is enforced at the boundary. This automatically satisfies the impermeability constraint (23). The no-slip condition is an empirical observation for normal conditions (it begins to fail for rarefied flows or flows at which the continuum approximation is not appropriate).

In an inviscid flow, the velocity tangent to the surface is not specified.

6.3 Solutions of the Navier–Stokes equations

Incompressible unidirectional flow

Assume that there are no external forces ($\boldsymbol{F} = \mathbf{0}$) and also that flow is only in the z-direction, so that

$$\boldsymbol{u} = w\,\boldsymbol{k} \Rightarrow u = v = 0.$$

For incompressible flow,

$$\nabla \cdot \boldsymbol{u} = \frac{\partial w}{\partial z} = 0,$$

hence $w = w(x, y, t)$.

Componentwise, the Navier-Stokes equations are

$$\begin{aligned}
\frac{\mathrm{D}u}{\mathrm{D}t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u,\\
\frac{\mathrm{D}v}{\mathrm{D}t} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v,\\
\frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w,
\end{aligned}$$

where $\nu = \mu/\rho$ is called the kinematic viscosity. With $u = v = 0$, the first two equations reduce to

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0,$$

hence $p = p(z, t)$.

The third of the Navier–Stokes equations becomes

$$\underbrace{\frac{\partial w}{\partial t} - \nu\nabla^2 w}_{g(x,y,t)} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{h(z,t)} = f(t), \tag{25}$$

where

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

and $f(t)$ is an arbitrary function.

Notes:

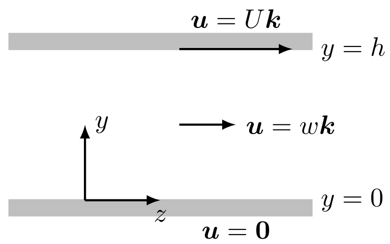
1. If $f \neq 0$, there is a spatially constant pressure gradient. The pressure is

$$p = P(t) - \rho f(t) z.$$

2. If the flow is steady, then $P(t)$ and $f(t)$ are constants and $\partial w/\partial t = 0$.
3. The velocity w is determined by the solution of equation (25) subject to suitable boundary conditions.

Plane Couette flow

Example 28. Consider two-dimensional unidirectional flow between two infinite parallel flat plates separated by a distance h, with one plate moving parallel to the other at speed U. Suppose the flow is steady with zero pressure gradient.



By solving (25), show that

$$w = U\,\frac{y}{h}$$

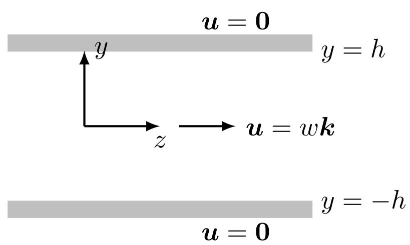
where w is the component of velocity in the direction of the plate motion and y is the wall-normal coordinate.

Plane Poiseuille flow

Example 29. Consider two-dimensional unidirectional flow between two fixed infinite parallel plates separated by a distance $2h$. Suppose the flow is steady and the pressure gradient parallel to the plates is

$$\frac{\partial p}{\partial z} = -\frac{\Delta p}{L},$$

where Δp is the pressure drop over a distance L.

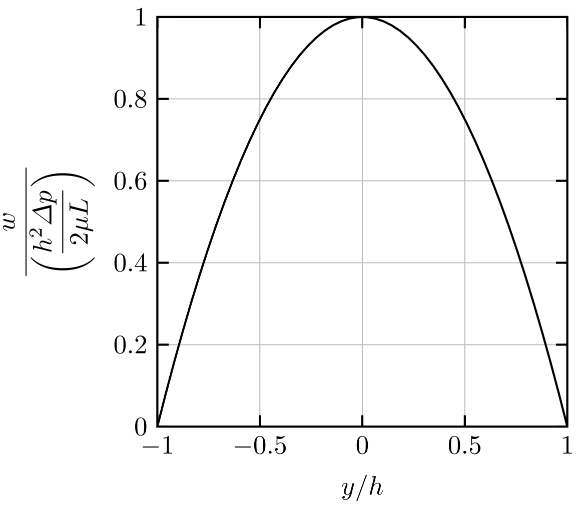


1. By solving (25), show that

$$w = \frac{\Delta p}{2\mu L}(h^2 - y^2)$$

   where w is the component of velocity in the direction of the pressure gradient and y is the wall-normal coordinate.

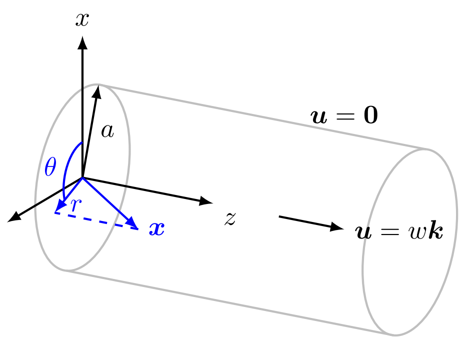
2. Sketch the velocity profile.



3. Use the solution found in part 1 to show that the volume flux per unit width is
$$Q = \frac{2\Delta p}{3\mu L} h^3.$$

Pipe Poiseuille flow

Example 30. Consider unidirectional axisymmetric flow in a pipe of radius a. Suppose the flow is steady and the pressure gradient along the pipe is
$$\frac{\partial p}{\partial z} = -\frac{\Delta p}{L},$$
where Δp is the pressure drop over a distance L.



1. By solving (25), Show that
$$w = \frac{\Delta p}{4\mu L}(a^2 - r^2),$$
where w is the component of velocity along the pipe and r is the radial coordinate measured from the centreline of the pipe.

2. Sketch the velocity profile.

3. Find the volume flux.

7 Fourier spectral methods

7.1 Background

Introduction

Spectral collocation (or pseudospectral) methods are used to find numerical solutions of PDEs, such as the Navier–Stokes equations. In spectral collocation methods, approximate derivatives are obtained by globally interpolating data at discrete grid points and evaluating the derivative of the interpolant at the grid points.

For smooth functions, spectral methods converge very rapidly, which enables high accuracy to be achieved with relatively few grid points.

For periodic problems, trigonometric interpolants and uniform grids are used, while for nonperiodic problems polynomial interpolants and nonuniform grids are used. In this course we will concentrate on periodic problems with trigonometric interpolants—think Fourier series.

Fourier series

Recall that a piecewise continuous function $f(x)$ which is 2π-periodic (i.e., $f(x) = f(x + 2\pi)$) can be represented by its Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{26}$$

where the coefficients a_0, a_n and b_n are given by the Euler formulae

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \tag{27a}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,\mathrm{d}x, \tag{27b}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,\mathrm{d}x. \tag{27c}$$

The derivation of the Euler formulae relies on the set of functions

$$\{1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\},$$

forming an orthogonal set on the interval $-\pi \le x \le \pi$, which is to say that the set satisfies an orthogonality condition

$$\int_{-\pi}^{\pi} \cos nx \cos mx\,\mathrm{d}x = 0, \quad n \ne m,$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx\,\mathrm{d}x = 0, \quad n \ne m,$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx\,\mathrm{d}x = 0.$$

The complex form of the Fourier series is obtained by substituting the identities

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right) \quad \text{and} \quad \sin nx = \frac{1}{2\mathrm{i}}\left(e^{inx} - e^{-inx}\right) \tag{28}$$

into (26) and (27) and defining

$$c_n = \begin{cases} \frac{1}{2}(a_{-n} + \mathrm{i}b_{-n}), & n < 0, \\ a_0, & n = 0, \\ \frac{1}{2}(a_n - \mathrm{i}b_n), & n > 0. \end{cases}$$

The complex form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}nx}, \tag{29}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-\mathrm{i}nx}\,\mathrm{d}x. \tag{30}$$

Example 31. If $f(x)$ is a real-valued function, show that

$$c_{-n} = \overline{c}_n.$$

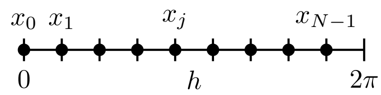
7.2 Discrete Fourier transform

Discretization of periodic functions

Suppose the 2π-periodic function $u(x)$ is evaluated at N uniformly spaced points over the interval $0 \le x < 2\pi$, so that

$$u_j = u(x_j), \quad x_j = jh, \quad j = 0, 1, 2, \ldots, N-1, \tag{31}$$

where $h = 2\pi/N$ is the grid spacing and the points x_j are called grid points or nodes.



Note that $x = 2\pi$ is excluded from the grid because $u(2\pi) = u(0)$. We assume that N is even.

Discrete Fourier modes

Consider the complex Fourier mode

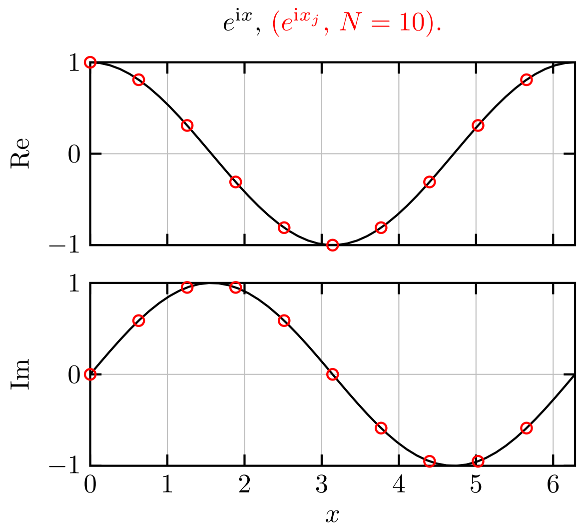
$$e^{\mathrm{i}kx} = \cos kx + \mathrm{i}\sin kx$$

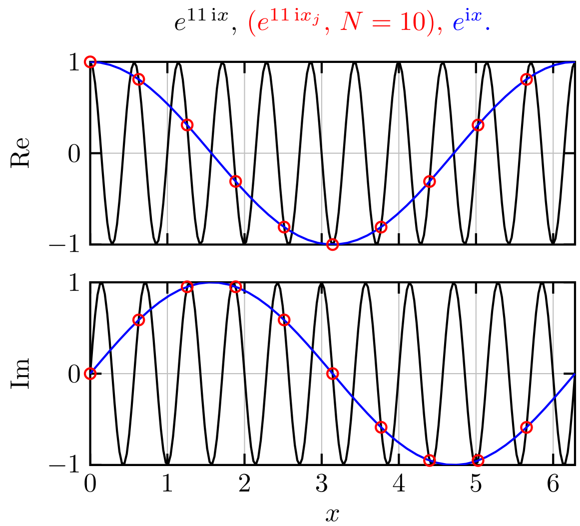
The integer k is the called the wavenumber. The wavelength is $2\pi/|k|$, so high wavenumbers correspond to short wavelengths, and vice versa.

Suppose e^{ikx} is discretized as described above. We cannot expect to successfully resolve waves whose wavelength is shorter than the grid spacing.

Waves with wavenumber $k+mN$, $m \in \mathbb{Z}$, are indistinguishable from those with wavenumber k when evaluated on the grid. This is called aliasing.

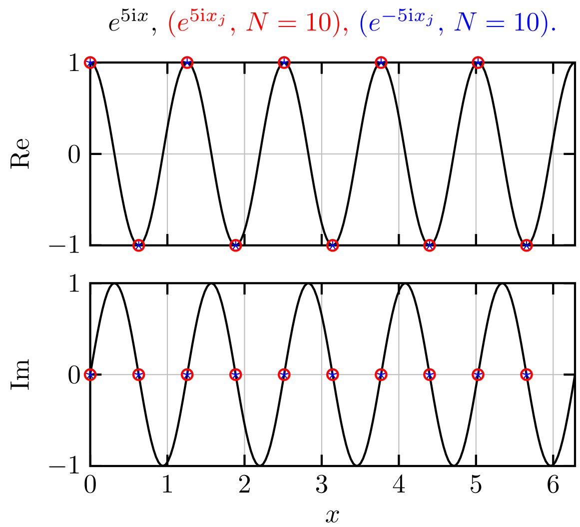
Example 32. Show that $e^{\mathrm{i}kx_j} = e^{\mathrm{i}(k+mN)x_j}$ for $m \in \mathbb{Z}$.





Waves with wavenumber $k = \pm N/2$ are indistinguishable from $\cos\left(\frac{N}{2}x\right)$ when evaluated on the grid.

Example 33. Show that $e^{\mathrm{i}(N/2)x_j} = e^{-\mathrm{i}(N/2)x_j} = (-1)^j = \cos\left(\frac{N}{2}x_j\right)$.



Trigonometric interpolation

Consider a truncated Fourier series that consists of the N smallest-$|k|$ modes that can be represented on the grid,

$$v(x) = \sum_{k=-N/2+1}^{N/2-1} \hat{u}_k e^{\mathrm{i}kx} + \hat{u}_{N/2} \cos\left(\tfrac{N}{2}x\right). \tag{32}$$

If this is to interpolate $u_j = u(x_j)$, then

$$\begin{aligned} u_j = v(x_j) &= \sum_{k=-N/2+1}^{N/2-1} \hat{u}_k e^{\mathrm{i}kx_j} + \hat{u}_{N/2} \cos\left(\tfrac{N}{2}x_j\right) \\ &= \sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{\mathrm{i}kx_j}. \end{aligned} \tag{33}$$

The coefficients $\hat{u}_k$ are found by using a discrete analogue of the orthogonality condition.

Multiplying (33) by $e^{-\mathrm{i}\tilde{k}x_j}$, $\tilde{k} \in \mathbb{Z}$, and summing from $j = 0$ to $N - 1$,

$$\begin{aligned} \sum_{j=0}^{N-1} u_j e^{-\mathrm{i}\tilde{k}x_j} &= \sum_{j=0}^{N-1} \sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{\mathrm{i}(k-\tilde{k})x_j} \\ &= \sum_{k=-N/2+1}^{N/2} \hat{u}_k \sum_{j=0}^{N-1} e^{\mathrm{i}(k-\tilde{k})x_j} \end{aligned} \tag{34}$$

Consider the sum

$$\sum_{j=0}^{N-1} e^{\mathrm{i}(k-\tilde{k})x_j} = \sum_{j=0}^{N-1} e^{\mathrm{i}(k-\tilde{k})jh} = \sum_{j=0}^{N-1} r^j$$

where $r = e^{2\pi \mathrm{i}(k-\tilde{k})/N}$.

For a geometric sum,

$$\sum_{j=0}^{N-1} ar^j = ar^0 + ar^1 + \cdots + ar^{n-1} = a\frac{1-r^N}{1-r}, \quad r \neq 1.$$

This corresponds to our sum when $a = 1$.

The case $r = 1$ occurs when $k = \tilde{k} + mN$, $m \in \mathbb{Z}$ because

$$r = e^{2\pi \mathrm{i}(k-\tilde{k})/N} = e^{2\pi \mathrm{i} m} = \cos(2\pi m) + \mathrm{i}\sin(2\pi m) = 1.$$

Therefore,

$$\sum_{j=0}^{N-1} r^j = \sum_{j=0}^{N-1} 1 = N, \quad k = \tilde{k} + mN.$$

For $k \neq \tilde{k} + mN$, $r \neq 1$ and

$$r^N = e^{2\pi i(k-\tilde{k})} = \cos 2\pi(k-\tilde{k}) + \mathrm{i}\sin 2\pi(k-\tilde{k}) = 1.$$

Using the formula for the geometric sum,

$$\sum_{j=0}^{N-1} r^j = 0, \quad k \neq \tilde{k} + mN.$$

The discrete analogue of the orthogonality condition is

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{\mathrm{i}(k-\tilde{k})x_j} = \delta_{k,\tilde{k}+mN} = \begin{cases} 1, & k = \tilde{k} + mN, \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

Substituting (35) into (34),

$$\frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-\mathrm{i}\tilde{k}x_j} = \sum_{k=-N/2+1}^{N/2} \hat{u}_k \delta_{k,\tilde{k}+mN} = \hat{u}_{\tilde{k}}.$$

The coefficients $\hat{u}_{\tilde{k}}$ can thus be determined from

$$\hat{u}_{\tilde{k}} = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-\mathrm{i}\tilde{k}x_j}.$$

Discrete Fourier transform

The discrete Fourier transform (DFT) is defined by

$$\hat{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-\mathrm{i}kx_j}, \quad k = -N/2+1, \ldots, N/2. \tag{36}$$

The inverse discrete Fourier transform (IDFT) is defined by

$$u_j = \sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{\mathrm{i}kx_j}, \quad j = 0, \ldots, N-1. \tag{37}$$

Example 34. Determine the discrete Fourier transform of

1. $u_j = \cos(3x_j)$,
2. $u_j = \sin(3x_j)$,

where $x_j = jh$, $h = 2\pi/N$ and $N = 8$.

Example 35. Determine the discrete Fourier transform of $u_j = \sin(5x_j)$, where $x_j = jh$, $h = 2\pi/N$ and $N = 8$.

Example 36. Show that $\hat{u}_{-k} = \overline{\hat{u}}_k$ when $u_j \in \mathbb{R}$.

Implementation

To understand the implementation of the discrete Fourier transform and its inverse in MATLAB, we shall write it in an alternative form. From (37),

$$\begin{aligned} u_j &= \sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{\mathrm{i}kx_j} \\ &= \sum_{k=0}^{N/2} \hat{u}_k e^{\mathrm{i}kx_j} + \sum_{k=-N/2+1}^{-1} \hat{u}_k e^{\mathrm{i}kx_j} \end{aligned}$$

Since $e^{\mathrm{i}kx_j} = e^{\mathrm{i}(k+N)x_j}$,

$$u_j = \sum_{k=0}^{N/2} \hat{u}_k e^{\mathrm{i}kx_j} + \sum_{k=-N/2+1}^{-1} \hat{u}_k e^{\mathrm{i}(k+N)x_j}.$$

Shifting indices with $m = k + N$,

$$u_j = \sum_{k=0}^{N/2} \hat{u}_k e^{\mathrm{i}kx_j} + \sum_{m=N/2+1}^{N-1} \hat{u}_{m-N} e^{\mathrm{i}mx_j}.$$

Defining

$$\hat{w}_k = \begin{cases} N\hat{u}_k, & k = 0, \ldots, N/2, \\ N\hat{u}_{k-N}, & k = N/2+1, \ldots, N-1, \end{cases} \tag{38}$$

the IDFT becomes

$$u_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{w}_k e^{\mathrm{i}kx_j}.$$

For $k = 0, \dots, N/2$, the DFT (36) gives

$$\hat{w}_k = N\hat{u}_k = \sum_{j=0}^{N-1} u_j e^{-\mathrm{i}kx_j}.$$

For $k = N/2+1, \dots, N-1$, we use $e^{-\mathrm{i}kx_j} = e^{-\mathrm{i}(k-N)x_j}$ to obtain

$$\hat{w}_k = N\hat{u}_{k-N} = \sum_{j=0}^{N-1} u_j e^{-\mathrm{i}(k-N)x_j} = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-\mathrm{i}kx_j}.$$

An equivalent form of the DFT is

$$\hat{w}_k = \sum_{j=0}^{N-1} u_j e^{-\mathrm{i}kx_j}, \quad k = 0, \dots, N-1. \tag{39}$$

An equivalent form of the IDFT is

$$u_j = \frac{1}{N}\sum_{k=0}^{N-1} \hat{w}_k e^{\mathrm{i}kx_j}, \quad j = 0, \dots, N-1. \tag{40}$$

The coefficients $\hat{u}_k$ are related to $\hat{w}_k$ by (38).

The DFT and IDFT can be implemented as a matrix-vector product. Define the $N \times 1$ column vectors

$$\begin{aligned} \boldsymbol{w} &= (\hat{w}_0, \dots, \hat{w}_{N-1})^T = N(\hat{u}_0, \dots, \hat{u}_{N/2}, \hat{u}_{-N/2+1}, \dots, \hat{u}_{-1})^T \\ \boldsymbol{u} &= (\hat{u}_0, \dots, \hat{u}_{N-1})^T \end{aligned}$$

and the $N \times N$ matrices $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ with components

$$A_{kj} = e^{-\frac{2\pi\mathrm{i}}{N}(j-1)(k-1)}, \quad A^{-1}_{jk} = \frac{1}{N} e^{\frac{2\pi\mathrm{i}}{N}(j-1)(k-1)} = \frac{1}{N}\overline{A}_{kj},$$

for $j = 1, \dots, N$ and $k = 1, \dots, N$.

Then the DFT and IDFT are

$$\boldsymbol{w} = \boldsymbol{A}\,\boldsymbol{u} \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{A}^{-1}\,\boldsymbol{w},$$

respectively. This costs $O(N^2)$ floating point operations (N multiplications for each of the N components).

However, it is usually more convenient and quicker to compute the discrete Fourier transform using the MATLAB functions `fft` and `ifft`, respectively.

These functions use the Fast Fourier Transform (FFT) algorithm to do this calculation in $O(N \log_2 N)$ operations, which is much faster than a matrix-vector product for large N. These functions are an interface to a highly optimised numerical library called the "Fastest Fourier Transform in the West" (FFTW, `http://www.fftw.org/`).

Example 37. Check that (39) and (40) correspond to the definition used by the MATLAB function `fft`. To see the definition, type `doc fft` in MATLAB and scroll to the bottom.

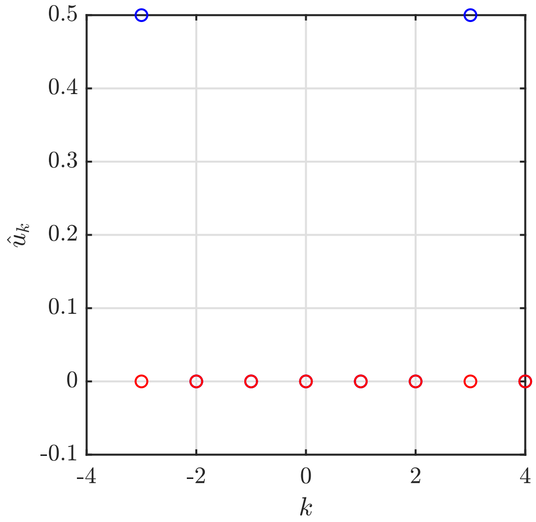
Example 38. Use MATLAB to calculate the the DFT and IDFT of $\cos 3x_j$, $\sin 3x_j$ and $\sin 5x_j$ for $x_j = jh$, $h = 2\pi/N$ and $N = 8$.

1. Plot the Fourier coefficients $\hat{u}_k$ for $k = -N/2+1, \ldots, N/2$. Check that the Fourier coefficients agree with the answers obtained from the previous examples.
2. Plot the IDFT of the Fourier coefficients. Check that this recovers the original function.

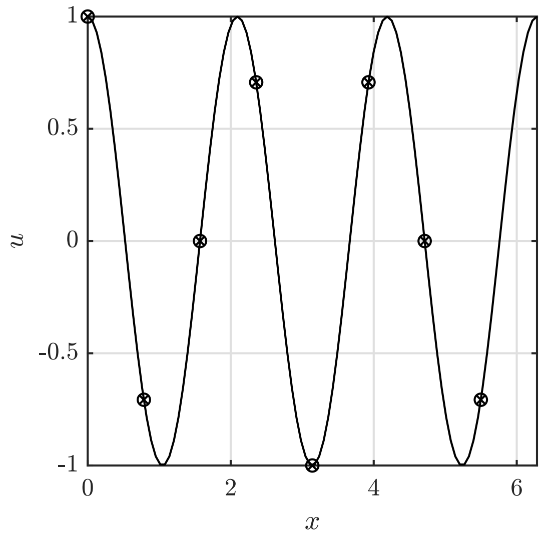
See `spectral_fft.m`.

```
% Set up grid and data.

N = 8;
dx = 2*pi/N;
x = dx*(0:N-1);
u = cos(3*x);
```

```
% Plot the Fourier coefficients vs wavenumber. The
% wavenumbers need to be in the same order as the
% Fourier coefficients:
%
% uh_0, ..., uh_N/2, uh_-N/2+1, ..., uh_-1.
%
% We also need to be mindful that fft calculates
% N times the Fourier coefficients.

k = [0:N/2, -N/2+1:-1];
uh = fft(u);
figure
plot(k, real(uh)/N, 'ob', k, imag(uh)/N, 'or')
```




```
% Recover the original data by taking the IDFT of uh.
% Since the original u is conjugate symmetric, we
% pass the optional argument 'symmetric' to IFFT,
% which is faster and ensures the output is real.

u = ifft(uh, 'symmetric');
figure
plot(x, u, 'ok', x, cos(3*x), 'xk')
norm(u - cos(3*x))
```



7.3 Partial differential equations

Differentiation

The derivative of the interpolant (32) is

$$v'(x) = \sum_{-N/2+1}^{N/2-1} (\mathrm{i}k)\hat{u}_k e^{\mathrm{i}kx} - \left(\tfrac{N}{2}\right) \hat{u}_{N/2} \sin\left(\tfrac{N}{2}x\right).$$

Since $\sin(Nx_j/2) = \sin(\pi j) = 0$,

$$v'(x_j) = \sum_{-N/2+1}^{N/2-1} (\mathrm{i}k)\hat{u}_k e^{\mathrm{i}kx_j}.$$

The second derivative of the interpolant (32) is

$$v''(x) = \sum_{-N/2+1}^{N/2-1} (\mathrm{i}k)^2\hat{u}_k e^{\mathrm{i}kx} - \left(\tfrac{N}{2}\right)^2 \hat{u}_{N/2} \cos\left(\tfrac{N}{2}x\right).$$

Since $\cos(Nx_j/2) = e^{\mathrm{i}(N/2)x_j}$ and $-(N/2)^2 = (\mathrm{i}N/2)^2$,

$$\begin{aligned} v''(x_j) &= \sum_{-N/2+1}^{N/2-1} (\mathrm{i}k)^2\hat{u}_k e^{\mathrm{i}kx_j} - \left(\tfrac{N}{2}\right)^2 \hat{u}_{N/2} e^{\mathrm{i}(N/2)x_j} \\ &= \sum_{-N/2+1}^{N/2} (\mathrm{i}k)^2\hat{u}_k e^{\mathrm{i}kx_j}. \end{aligned}$$

The derivative of $u(x)$ at the nodes $x_j = jh$, $j = 0, \ldots, N-1$ is estimated from the derivative of the interpolant that passes through the points $u_j = u(x_j)$. For example,

$$\begin{aligned} u'(x_j) \approx u'_j &= \sum_{-N/2+1}^{N/2-1} (\mathrm{i}k)\hat{u}_k e^{\mathrm{i}kx_j}, \\ u''(x_j) \approx u''_j &= \sum_{-N/2+1}^{N/2} (\mathrm{i}k)^2\hat{u}_k e^{\mathrm{i}kx_j}, \end{aligned}$$

where $\hat{u}_k$ are the Fourier coefficients obtained from the DFT (36) of u_j.

The process can be extended to higher-order derivatives. In general,

$$u^{(n)}(x_j) \approx u_j^{(n)} = \sum_{-N/2+1}^{N/2} \hat{w}_k e^{\mathrm{i}kx_j},$$

where $\hat{w}_k = (\mathrm{i}k)^n \hat{u}_k$, except that $\hat{w}_{N/2} = 0$ when n is odd.

To obtain an estimate of the nth derivative:

1. Calculate $\hat{u}_k$ from u_j using the DFT.
2. Calculate $\hat{w}_k = (\mathrm{i}k)^n \hat{u}_k$, except that $\hat{w}_{N/2} = 0$ when n is odd.

3. Calculate u_j' from $\hat{w}_k$ using the IDFT.

Example 39. Use the DFT to estimate $u'(x_j)$ for

1. $u(x) = \cos(3x)$,
2. $u(x) = \sin(4x)$,

where $x_j = jh$, $h = 2\pi/N$ and $N = 8$.

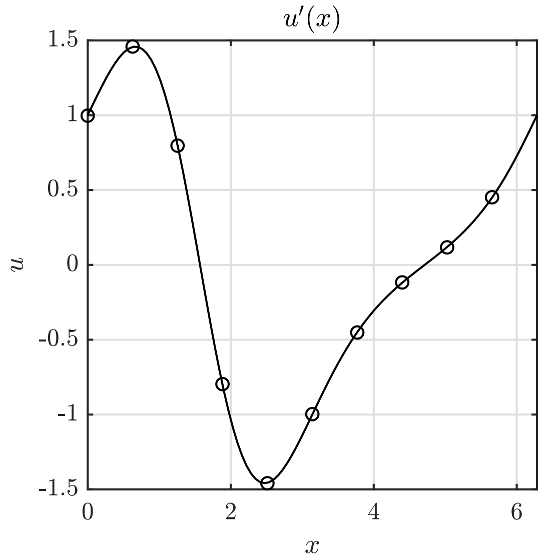
Example 40. Use MATLAB's `fft` and `ifft` to estimate the first and second derivatives of $u(x) = \exp(\sin x)$ for $x_j = jh$, $h = 2\pi/N$ and $N = 10$.

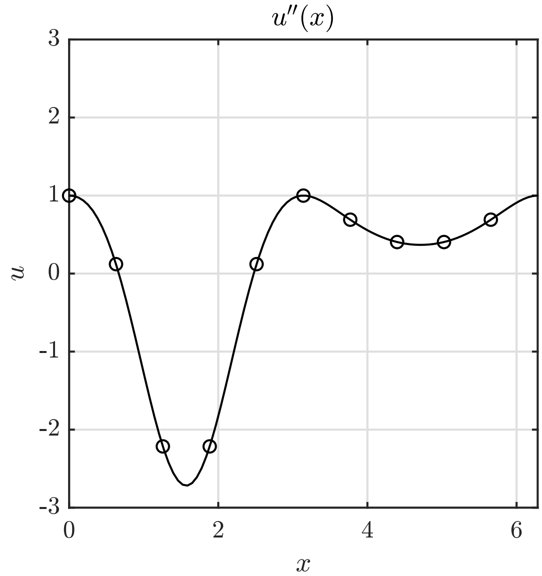
See `spectral_diff.m`.

```
% Set up grid and data.

N = 10;
dx = 2*pi/N;
x = dx*(0:N-1);
u = exp(sin(x));
```

```
% Create the vector i*k for wavenumbers
% from k = -N/2+1 ... N/2-1. Replace wavenumber
% N/2 with zero for first (odd) derivative.

ik = 1i*[0:N/2-1 0 -N/2+1:-1];

% Create the vector (ik)^2 = -k^2 for wavenumbers
% from k = -N/2+1 ... N/2.

k2 = [0:N/2 -N/2+1:-1].^2;
```

```
% First derivative.

uh = fft(u);
uxh = ik.*uh;
ux = ifft(uxh, 'symmetric');

% Second derivative.

uh = fft(u);
uxxh = -k2.*uh;
uxx = ifft(uxxh, 'symmetric');
```





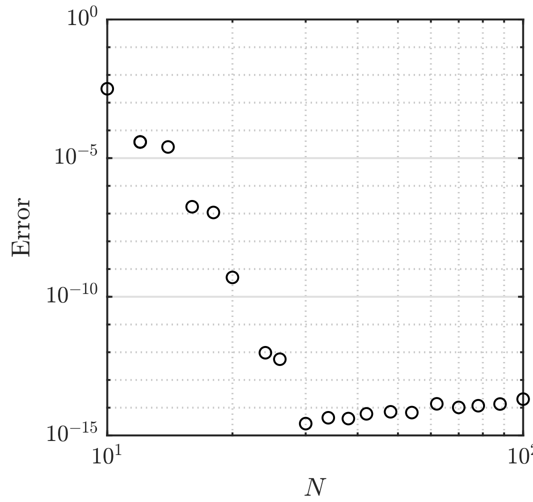
Spectral accuracy

Accuracy depends on how quickly the Fourier coefficients of the original function decay with k, which is related to how smooth the function is.

If $u(x)$ has infinitely many continuous derivatives, then the error is $O(N^{-m})$ for every $m \geq 0$. If $u(x)$ is analytic, then the error is $O(c^N)$ for $0 < c < 1$.

Example 41. Plot the maximum absolute error of the spectral derivative of $u(x) = \exp(\sin x)$ for $10 \leq N \leq 100$.

See `spectral_differr.m`.



Constant-coefficient linear PDEs

In pseudospectral or collocation methods, time-dependent PDEs are solved by applying spectral differentiation in space and enforcing the PDE at the grid points x_j.

Example 42. Find a pseudospectral solution of the one-dimensional advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

subject to the 2π-periodic boundary condition $u(x,t) = u(x+2\pi,t)$ and the initial condition

$$u(x,0) = f(x).$$

Plot the solution for $f(x) = e^{-10(x-\pi)^2}$.

First, we shall find a solution in wavenumber space. Let $u_j(t)$ be the numerical approximation of the solution evaluated at the grid points,

$$u(x_j,t) \approx u_j(t) = \sum_{k=-N/2+1}^{N/2} \hat{u}_k(t) e^{\mathrm{i}kx_j}.$$

The temporal derivative evaluated at the grid points is

$$\left.\frac{\partial u}{\partial t}\right|_{x_j} \approx \frac{\mathrm{d}u_j}{\mathrm{d}t} = \sum_{k=-N/2+1}^{N/2} \frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} e^{\mathrm{i}kx_j}.$$

The spatial derivative evaluated at the grid points is

$$\left.\frac{\partial u}{\partial x}\right|_{x_j} \approx \sum_{k=-N/2+1}^{N/2-1} \mathrm{i}k\hat{u}_k e^{\mathrm{i}kx_j}.$$

Substitution into the PDE yields

$$\sum_{k=-N/2+1}^{N/2} \frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} e^{\mathrm{i}kx_j} + c \sum_{k=-N/2+1}^{N/2-1} \mathrm{i}k\hat{u}_k e^{\mathrm{i}kx_j} = 0.$$

Using the discrete orthogonality condition, or equating like terms,

$$\frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} = -\mathrm{i}ck\hat{u}_k, \quad k = -N/2+1, \ldots, N/2-1,$$
$$\frac{\mathrm{d}\hat{u}_{N/2}}{\mathrm{d}t} = 0.$$

This is subject to the initial conditions $\hat{u}_k(0) = \hat{f}_k$, where $\hat{f}_k$ are the Fourier coefficients of the initial condition

$$u_j(0) = f(x_j) = \sum_{k=-N/2+1}^{N/2} \hat{f}_k e^{\mathrm{i}kx_j}.$$

In this case, the system of ODEs can be solved analytically. The solution is

$$\hat{u}_k(t) = \hat{u}_k(0)e^{-\mathrm{i}ckt}, \quad k = -N/2+1, \ldots, N/2-1,$$
$$\hat{u}_{N/2}(t) = \hat{u}_{N/2}(0),$$

where $\hat{u}_k(0) = \hat{f}_k$.

Once $\hat{f}_k$ is determined from the DFT of $f_j = f(x_j)$, the solution can be obtained by calculating the IDFT of $\hat{u}_k(t)$. See `spectral_adv.m`.

```
% SOLVE_ADV solves the advection equation on a
% 2*pi-periodic domain.
%
% Inputs:
%   N - number of collocation points.
%   nt - number of times for output.
%   c - advection velocity.
%   T - final time.
%   f - function handle specifying IC.
```

```
% Set up grid and initial condition.

dx = 2*pi/N;
x = dx*(0:N-1);
ik = 1i*[0:N/2-1 0 -N/2+1:-1];
t = linspace(0, T, nt);
fh = fft(f(x));
```

```
% Serial calculation of analytic solution.

```

```
u = zeros(nt, N);
for j = 1:nt
    uh = fh.*exp(-c*ik*t(j));
    u(j,:) = ifft(uh, 'symmetric');
end
```

```
% Vectorized calculation of analytic solution. By
% default, IFFT acts along columns of uh. But the
% Fourier coefficients at time t(j) are stored in
% the row uh(j,:). The optional arguments "[], 2"
% ensure the IFFT is calculated along the rows.

uh = exp(-c*(t')*ik).*fh(ones(nt,1),:);
u = ifft(uh, [], 2, 'symmetric');
```

```
% Numerical solution in wavenumber space. ODE45
% requires the Fourier coefficients uh and
% derivatives duhdt to be column vectors. The row
% vector ik is transposed into a column vector
% using (ik.'). Warning: (ik') calculates the
% *conjugate* transpose! ODE45 outputs the solution
% at time t(j) in row uh(j,:).

duhdt = @(t, uh) -c*(ik.').*uh;
[~, uh] = ode45(duhdt, t, fh);
u = ifft(uh, [], 2, 'symmetric');
```

We can also solve the problem in physical space. Evaluating the PDE at the grid points,

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + c\left.\frac{\partial u}{\partial x}\right|_{x_j} = 0,$$

where the spectral approximation of the spatial derivative is

$$\left.\frac{\partial u}{\partial x}\right|_{x_j} = \sum_{k=-N/2+1}^{N/2-1} \mathrm{i}k\hat{u}_k e^{\mathrm{i}kx_j}.$$

This gives a coupled system of ODEs,

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -c\sum_{k=-N/2+1}^{N/2-1} \mathrm{i}k\hat{u}_k e^{\mathrm{i}kx_j}, \quad j = 0, \ldots, N-1,$$

where the $\hat{u}_k$ are obtained from u_j using the DFT,

$$\hat{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-\mathrm{i}kx_j}, \quad k = -N/2+1, \ldots, N/2.$$

This is subject to the initial conditions $u_j(0) = f(x_j)$.

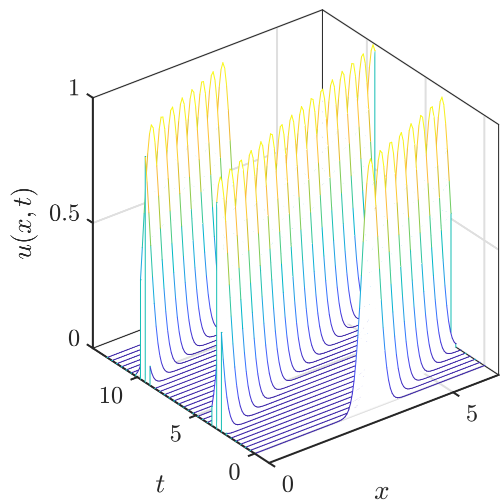
```
% Numerical solution in physical space.

dudt = @(t, u) ifft(-c*(ik.').*fft(u), 'symmetric');
[~, u] = ode45(dudt, t, f(x));
```

The exact solution of the original problem is

$$u(x,t) = f(x - ct),$$

which is just the initial condition $f(x)$ translated in the positive x-direction at speed c.

```
% Compare with exact solution u(x,t) = f(x-c*t).

[xx, tt] = meshgrid(x, t);
err = u - f(mod(xx - c*tt, 2*pi));
fprintf('The error is %g.\n', norm(err, 2))
```



Nonlinear PDEs

Nonlinear and variable-coefficient terms are most easily calculated in physical space.

Example 43. Find a pseudospectral solution of Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2},$$

subject to the 2π-periodic boundary condition $u(x,t) = u(x + 2\pi, t)$ and the initial condition

$$u(x,0) = f(x).$$

Plot the solution for $f(x) = \sin x$.

Let's solve this in physical space. Discretizing as for the advection equation,

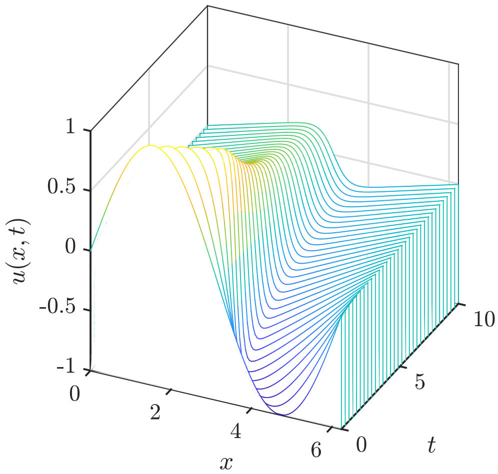
$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -u_j \left.\frac{\partial u}{\partial x}\right|_{x_j} + \nu \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_j}, \quad j = 0, \ldots, N-1,$$

where the spectral approximation of the spatial derivatives are

$$\left.\frac{\partial u}{\partial x}\right|_{x_j} = \sum_{-N/2+1}^{N/2-1} (\mathrm{i}k)\hat{u}_k e^{\mathrm{i}kx_j}, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_j} = \sum_{-N/2+1}^{N/2} (-k^2)\hat{u}_k e^{\mathrm{i}kx_j}$$

and $\hat{u}_k$ are the Fourier coefficients from the DFT of u_j. See `spectral_burgers.m`.

```
% Numerical solution in physical space.

[~, u] = ode45(@burgers, t, f(x));

    function dudt = burgers(t, u)

        uh = fft(u);
        ux = ifft(ik.*uh, 'symmetric');
        uxx = ifft(-k2.*uh, 'symmetric');
        dudt = -u.*ux + nu*uxx;
```



Nonlinear terms are a source of aliasing.

Example 44. Consider $u_j = \cos(2x_j)$ and $v_j = \cos(3x_j)$ for $x_j = jh$, $h = 2\pi/N$ and $N = 8$.

1. Show that $u_j v_j = \frac{1}{2}\cos(5x_j) + \frac{1}{2}\cos x_j$.
2. Find the discrete Fourier transform of $u_j v_j$.

Sometimes, aliasing may lead to numerical instability.

One remedy is to write the nonlinear terms so that they conserve discrete energy. For example, for Burgers equation the nonlinear terms are written as

$$u\frac{\partial u}{\partial x} = \frac{\theta}{2}\frac{\partial u^2}{\partial x} + (1-\theta)u\frac{\partial u}{\partial x},$$

where $\theta = 2/3$.

Another remedy is to set the Fourier coefficients of all modes in the product whose wavenumber $k > \frac{2}{3}k_{\max} = \frac{N}{3}$ to zero (called the 2/3 rule). This removes the aliased modes.

In many problems, there is sufficient physical dissipation to control aliasing, providing that the solution is sufficiently well resolved.

8 Inviscid flow

8.1 Euler equations

The Euler equations

For an inviscid fluid, the dynamic viscosity $\mu = 0$. In that case, the Navier–Stokes equations reduce to the Euler equations

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \boldsymbol{F}.$$

For a solid boundary moving with velocity $\boldsymbol{U}$, the Euler equations are subject to the impermeability condition

$$(\boldsymbol{u} - \boldsymbol{U}) \cdot \hat{\boldsymbol{n}} = 0,$$

where $\hat{\boldsymbol{n}}$ is the normal to the boundary. The no-slip condition does not apply to inviscid flows.

8.2 Hydrostatics

Hydrostatics

Hydrostatics refers to the special case of stationary fluid subject to gravity.

With $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{F} = -\rho g \boldsymbol{k}$, the Euler equations reduce to

$$\nabla p = -\rho g \boldsymbol{k}. \tag{41}$$

Componentwise,

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g,$$

The first two equations tell us that p is not a function of x or y, hence $p = p(z)$.

If ρ is constant, integrating the third equation gives

$$p = -\rho g z + p_0.$$

Example 45.

1. Substitute equation $\nabla p = -\rho g \boldsymbol{k}$ into the identity $\nabla \times \nabla p = 0$ to show that if ρ is not constant, it can at most depend on z, that is, $\rho = \rho(z)$.
2. Find an expression for the pressure p if the density ρ is not constant.

8.3 Bernoulli equation

Alternative form of the Euler equations

Using the vector identity

$$\boldsymbol{u} \times \boldsymbol{\omega} = \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u} \cdot \nabla \boldsymbol{u},$$

the Euler equations can be written as

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u} \times \boldsymbol{\omega}\right) = -\nabla p + \boldsymbol{F},$$

where $|\boldsymbol{u}|^2 = \boldsymbol{u} \cdot \boldsymbol{u}$ and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$.

Rearranging,

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla|\boldsymbol{u}|^2 + \hat{\boldsymbol{F}}, \tag{42}$$

where $\hat{\boldsymbol{F}} = \boldsymbol{F}/\rho$ is the external force per unit mass.

Conservative external forces

If $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, then $\boldsymbol{F}$ is said to be conservative and there exists a scalar potential f such that

$$\boldsymbol{F} = -\nabla f.$$

The function f is the potential energy per unit volume. Equivalently, we can define a potential energy per unit mass V such that

$$\hat{\boldsymbol{F}} = -\nabla V.$$

Example 46. For gravity,

$$\boldsymbol{F} = -\rho g \boldsymbol{k}.$$

Since $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ gravity is a conservative external force and therefore

$$-\nabla f = -\rho g \boldsymbol{k} \Rightarrow f = \rho g z.$$

Similarly, the force per unit mass is

$$\hat{\boldsymbol{F}} = -g\boldsymbol{k},$$

hence

$$-\nabla V = -g\boldsymbol{k} \Rightarrow V = gz.$$

For a constant-density fluid subject to conservative external forces, the Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla\left(V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2\right). \tag{43}$$

Bernoulli equation for steady flow

If the flow is steady, then the Euler equations become

$$\boldsymbol{u} \times \boldsymbol{\omega} = \nabla \left(V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 \right).$$

Now $\boldsymbol{u} \times \boldsymbol{\omega}$ is perpendicular to $\boldsymbol{u}$, hence

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{u} \cdot \nabla \left(V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 \right) = 0.$$

This means that $\nabla \left(V + p/\rho + \frac{1}{2}|\boldsymbol{u}|^2 \right)$ is perpendicular to $\boldsymbol{u}$. Integrating along a streamline, which is in the direction of $\boldsymbol{u}$,

$$V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 = \text{constant}$$

along streamlines. The constant may be different from streamline to streamline. This is the Bernoulli equation for steady flow.

Bernoulli equation for steady irrotational flow

If the flow is both steady and irrotational, then $\boldsymbol{\omega} = \boldsymbol{0}$ and the Euler equations become

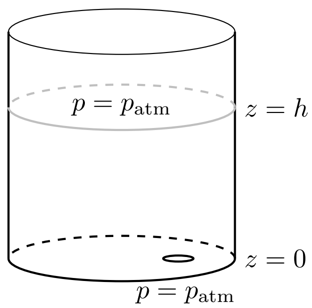
$$\nabla \left(V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 \right) = 0.$$

Integrating over space,

$$V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 = \text{constant}.$$

When the flow is both steady and irrotational, the constant is the same everywhere, unlike the previous case.

Example 47. A large rainwater tank has a small hole at its base. Taking the density of water to be constant, show that the water leaks out with approximate speed $\sqrt{2gh}$, where h is the height of the water above the hole.



Bernoulli equation for unsteady irrotational flow

If the flow is unsteady and irrotational, the Euler equations become

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\left(V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2\right).$$

We will see in a moment that when the flow is irrotational $\boldsymbol{u} = \nabla\phi$, hence

$$\nabla\left(\frac{\partial \phi}{\partial t} + V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2\right) = 0.$$

Integrating over space,

$$\frac{\partial \phi}{\partial t} + V + \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 = C(t),$$

where $C(t)$ does not depend on spatial coordinates but may vary with time.

8.4 Potential flow

The velocity potential

A flow is said to be irrotational if the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \boldsymbol{0}$ everywhere.

An irrotational velocity field possesses a scalar function $\phi(x, y, z, t)$ such that

$$\boldsymbol{u} = \nabla\phi.$$

The function ϕ is called the velocity potential.

Example 48. Show that an incompressible irrotational flow has a velocity potential ϕ that satisfies Laplace's equation.

Example 49. Find the velocity potential of the two-dimensional irrotational flow given by

$$\boldsymbol{u} = U\,\boldsymbol{i}.$$

Stream function and velocity potential

For two-dimensional incompressible irrotational flow, we may define both a stream function ψ and a velocity potential ϕ.

For such a flow

$$\boldsymbol{u} = u\,\boldsymbol{i} + v\,\boldsymbol{j} = \frac{\partial \phi}{\partial x}\,\boldsymbol{i} + \frac{\partial \phi}{\partial y}\,\boldsymbol{j} = \frac{\partial \psi}{\partial y}\,\boldsymbol{i} - \frac{\partial \psi}{\partial x}\,\boldsymbol{j},$$

hence

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}. \tag{44}$$

Example 50. Show that lines of constant potential ϕ (equipotential lines) are perpendicular to lines of constant ψ (streamlines).

Example 51. Use the relationship between vorticity and the stream function to show that the stream function for an irrotational flow satisfies Laplace's equation.

Solutions of Laplace's equation

The beauty of working with Laplace's equation is that it is linear and homogeneous, hence any linear combination of solutions is also a solution.

We can find solutions using separation of variables in different coordinate systems.

Plane Cartesian coordinates

For incompressible irrotational flow, the velocity potential satisfies Laplace's equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

Substituting $\phi = F(x)\,G(y)$,

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = \lambda,$$

where λ is a constant.

Example 52. 1. For $\lambda = \alpha^2$, show that solutions are of the form

$$\phi = \left(Ae^{\alpha x} + Be^{-\alpha x}\right)\cos\alpha y + \left(Ce^{\alpha x} + De^{-\alpha x}\right)\sin\alpha y. \tag{45}$$

2. For $\lambda = 0$, show that

$$\phi = (Ax + B)\,(Cy + D)\,. \tag{46}$$

Uniform flow

Putting $B = C = 0$ in equation (46) gives

$$\phi = Ux,$$

where $U = AD$. From a previous exercise, we know that this corresponds to uniform flow in the x-direction.

Cylindrical coordinates

In cylindrical coordinates (r, θ, z),

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\boldsymbol{e}_\theta + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z \tag{47}$$

and

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{48}$$

Plane polar coordinates

For two-dimensional flow in plane polar coordinates (cylindrical coordinates wih $\partial/\partial z = 0$), Laplace's equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

Substituting $\phi = F(r)\,G(\theta)$,

$$r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = \lambda,$$

where λ is a constant.

Example 53. 1. For $\lambda = \alpha^2$, show that solution are of the form

$$\phi = \left(Ar^{\alpha} + Br^{-\alpha}\right)\cos\alpha\theta + \left(Cr^{\alpha} + Dr^{-\alpha}\right)\sin\alpha\theta. \tag{49}$$

2. For $\lambda = 0$, show that

$$\phi = \left(A\ln r + B\right)\left(C\theta + D\right). \tag{50}$$

Source or Sink

Putting $B = C = 0$ in equation (50) gives

$$\phi = K\ln r, \tag{51}$$

where $K = AD$. The velocity is

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta = \frac{K}{r}\boldsymbol{e}_r,$$

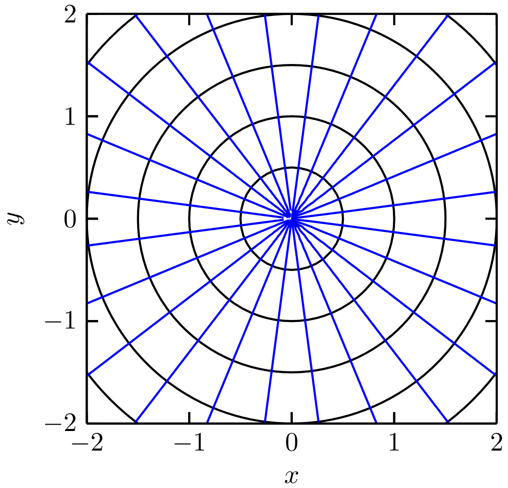
hence streamlines are radial lines emanating from (or converging to) the origin. For $K > 0$, this is referred to as a source. For $K < 0$, this is referred to as a sink.

The volume flux per unit length through a circle of radius $a > 0$ due to a source/sink at the origin is given by

$$\begin{aligned} Q &= \int_{\mathcal{C}} \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,ds \\ &= \int_0^{2\pi} \frac{K}{a}a\,d\theta \\ &= 2\pi K. \end{aligned}$$

Example 54. Find the corresponding stream function for a source/sink.

$\phi = K \ln r$, $\psi = K\theta$.



Line Vortex

Putting $A = D = 0$ in equation (50) gives

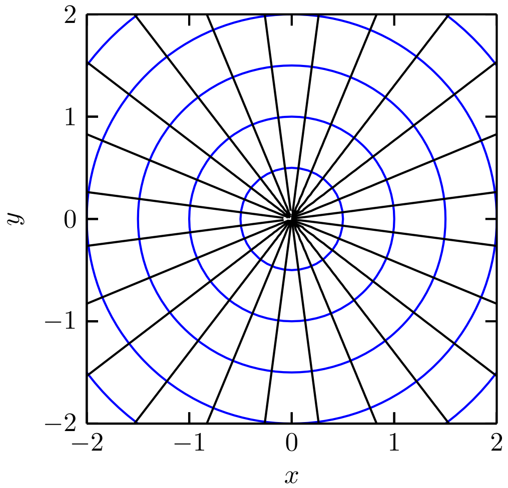
$$\phi = K\theta, \tag{52}$$

where $K = BC$. The velocity is

$$\boldsymbol{u} = \nabla\phi = \frac{\partial \phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\boldsymbol{e}_\theta = \frac{K}{r}\boldsymbol{e}_\theta.$$

Example 55. Find the corresponding stream function for a line vortex.

$\phi = K\theta$, $\psi = -K \ln r$.



Dipole/Doublet

Putting $\alpha = 1$, $B = 1$ and $A = C = D = 0$ in equation (49) gives

$$\phi = \frac{\cos\theta}{r} = \frac{r\cos\theta}{r^2} = \frac{x}{x^2+y^2}. \qquad (53)$$

To see how this is related to sources and sinks, write

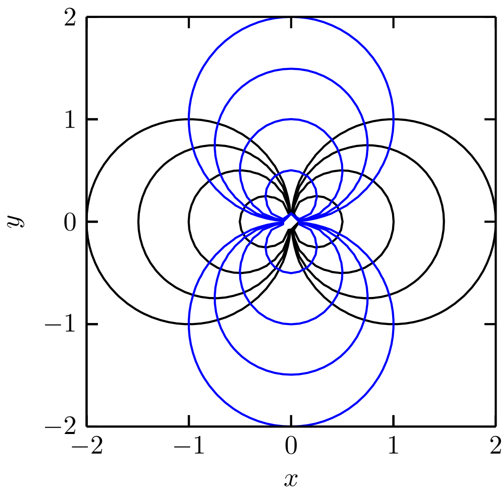
$$\phi = \frac{x}{x^2+y^2} = \frac{\partial}{\partial x}\left(\ln(x^2+y^2)^{1/2}\right) = \frac{\partial}{\partial x}\ln r.$$

Note that $\ln r$ is the velocity potential of a source. From the definition of the partial derivative we can rewrite ϕ as

$$\phi = \frac{\partial}{\partial x}\ln r = \lim_{\epsilon\to 0}\frac{\ln((x+\epsilon)^2+y^2)^{1/2} - \ln((x-\epsilon)^2+y^2)^{1/2}}{2\epsilon}.$$

For finite epsilon the terms on the right represent a linear superposition of a source at $(-\epsilon, 0)$ with strength of $1/2\epsilon$ and a sink at $(\epsilon, 0)$ with strength $-1/2\epsilon$.

$\phi = \cos\theta/r$, $\psi = -\sin\theta/r$.



Dipole in a uniform flow

Putting $\alpha = 1$, $A = U$, $B = a^2U$ and $C = D = 0$ in equation (49) gives

$$\phi = Ur\cos\theta + a^2U\frac{\cos\theta}{r} = Ux + a^2U\frac{x}{x^2+y^2}.$$

This is a linear combination of uniform flow and a dipole. The velocity is

$$\boldsymbol{u} = \nabla\phi = \left(U - \frac{a^2U}{r^2}\right)\cos\theta\,\boldsymbol{e}_r - \left(U + \frac{a^2U}{r^2}\right)\sin\theta\,\boldsymbol{e}_\theta.$$

Example 56. Consider the velocity potential

$$\phi = Ur\cos\theta + a^2U\frac{\cos\theta}{r}.$$

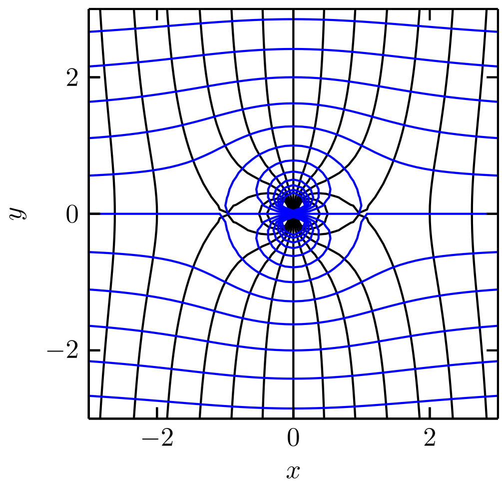
1. Show that far from the origin, the velocity potential approaches that of uniform flow.
2. Calculate the radial and tangential velocity components, u_r and u_θ, respectively on the circle $r = a$.

This velocity potential represents irrotational incompressible flow past a cylinder of radius a in a uniform flow of velocity U in the positive x-direction.

Example 57. Find the stream function corresponding to the dipole in a uniform flow which is given by the velocity potential

$$\phi = Ur\cos\theta + a^2U\frac{\cos\theta}{r}.$$

ϕ, ψ.



Any streamline could be an impermeable boundary.

Assuming that the fluid is everywhere except the region $r < a$ we have uniform flow around a circular cylinder.

We could also exclude the region $y < 0$ and then we would be considering uniform flow over a semicircular hump on a planar boundary.

Axisymmetric flow in cylindrical coordinates

For axisymmetric flow ($\partial/\partial\theta = 0$) in cylindrical coordinates, Laplace's equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

Substituting $\phi = F(r)\,G(z)$,

$$-\frac{F''(r)}{F(r)} - \frac{1}{r}\frac{F'(r)}{F(r)} = \frac{G''(z)}{G(z)} = \lambda$$

where λ is a constant.

Example 58. For $\lambda = \alpha^2$:

1. Use the transformation $\eta = \alpha r$ to transform the ODE for $F(r)$ into Bessels' equation.
2. Hence show that solutions are of the form

$$\phi = \left(Ae^{\alpha z} + Be^{-\alpha z}\right) J_0(\alpha r) + \left(Ce^{\alpha z} + De^{-\alpha z}\right) Y_0(\alpha r), \qquad (54)$$

   where $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kinds, respectively.

Spherical coordinates

In spherical coordinates (r, θ, φ),

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{e}_\varphi \qquad (55)$$

and

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}. \qquad (56)$$

Axisymmetric flow in spherical coordinates

For axisymmetric flow ($\partial/\partial\varphi = 0$) in spherical coordinates, Laplace's equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} = 0.$$

Substituting $\phi = F(r)\,G(\theta)$,

$$r^2\frac{F''(r)}{F(r)} + 2r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} - \frac{\cos\theta}{\sin\theta}\frac{G'(\theta)}{G(\theta)} = \lambda,$$

where λ is a constant.

The substitution $x = \cos\theta$ transforms the ODE for $G(\theta)$ into

$$(1-x^2)G''(x) - 2xG'(x) + \lambda G(x) = 0.$$

If we *choose* $\lambda = n(n+1)$, then this is Legendre's equation. For n a positive integer, Legendre's equation has polynomial solutions $P_n(x)$, of which the first few are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1). \end{aligned}$$

There is another linearly independent solution of Legendre's equation, $Q_n(x)$, but we will not use it.

Putting $\lambda = n(n+1)$ into the ODE for $F(r)$ yields the Euler–Cauchy equation

$$r^2 F''(r) + 2rF'(r) - n(n+1)F(r) = 0,$$

which has the general solution

$$F(r) = Ar^n + Br^{-n-1}.$$

Axisymmetric solutions of the Laplace equation in spherical coordinates include

$$\phi = \left(Ar^n + Br^{-n-1}\right) P_n(\cos\theta). \tag{57}$$

Source/sink

Putting $n = 0$, $A = 0$ and $B = -K$ in equation (57) gives

$$\phi = -\frac{K}{r}. \tag{58}$$

The velocity is

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta = \frac{K}{r^2}\boldsymbol{e}_r,$$

hence the streamlines are radial lines emanating from (or converging to) the origin. For $K > 0$, this is a source. For $K < 0$, this is a sink.

This source/sink is different to the source/sink found earlier for plane flows. In three dimensions, the source/sink found in plane flows is really a line source/sink, whereas the axisymmetric source/sink is a point source/sink.

The volume flux through a sphere of radius $a > 0$ due to a source at the origin is given by

$$Q = \int_{\mathcal{S}} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_0^\pi \frac{K}{a^2}\,2\pi a^2\sin\theta\,\mathrm{d}\theta = 4\pi K.$$

Dipole/Doublet

Putting $n = 1$, $A = 0$ and $B = 1$ in equation (57) gives

$$\phi = \frac{\cos\theta}{r^2} = \frac{r\cos\theta}{r^3} = \frac{z}{(x^2+y^2+z^2)^{3/2}}. \tag{59}$$

Dipole in uniform flow

Putting $n = 1$, $A = U$ and $B = a^3U/2$ in equation (57) gives

$$\phi = Ur\cos\theta\left(1 + \frac{a^3}{2r^3}\right).$$

This is the linear combination of uniform flow and a dipole. It produces uniform flow past a sphere of radius a.

Forces in inviscid irrotational flow

Recall that the stress acting on a surface with unit normal $\hat{\boldsymbol{n}}$ is

$$T_j = \hat{n}_i \sigma_{ij},$$

where σ_{ij} are the components of the stress tensor. For an inviscid fluid, $\sigma_{ij} = -p\delta_{ij}$, hence

$$T_j = -\hat{n}_i p \delta_{ij} = -p\hat{n}_j \Rightarrow \boldsymbol{T} = -p\hat{\boldsymbol{n}}.$$

The internal force on a small element of area δS is

$$\boldsymbol{T}\,\delta S = -p\,\hat{\boldsymbol{n}}\,\delta S.$$

The total force on the body with surface $\mathcal{S}$ is

$$\boldsymbol{F} = -\int_{\mathcal{S}} p\,\hat{\boldsymbol{n}}\,\mathrm{d}S.$$

Example 59. Calculate the force on a cylinder of radius a embedded in a steady uniform inviscid irrotational flow at speed U.

D'Alembert's paradox

A cylinder immersed in an inviscid potential flow experiences zero drag (which is the component of the force in the streamwise direction). In reality, measurements show that the drag on a cylinder is quite substantial, even when the viscosity is tiny. This disparity between theory and experiment was recognised by D'Alembert in 1752, and is often referred to as D'Alembert's paradox.

The paradox was more or less resolved by Prandtl in 1904, who developed viscous boundary-layer theory. No matter how small the viscosity, there is always a region close to the boundaries in which the effects of viscosity cannot be neglected. These effects directly and indirectly lead to drag forces.

8.5 Circulation

Circulation

The circulation is defined as

$$\Gamma = \oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x},$$

where $\mathcal{C}$ is a simple closed path in the flow and $\mathrm{d}\boldsymbol{x}$ is a vector representing the differential of an infinitesimal element of the curve $\mathcal{C}$.

Example 60. Calculate the circulation of a line vortex around a circular path of radius a, both centred at the origin.

Example 61. Calculate the circulation of the velocity field

$$\boldsymbol{u} = y\,\boldsymbol{i} + 2x\,\boldsymbol{j}$$

around a circular path of radius a, centred at the origin.

Stokes' theorem

Stokes' theorem states that for a differentiable vector field $\boldsymbol{u}$,

$$\oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \iint_{\mathcal{S}} (\nabla \times \boldsymbol{u}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S$$

where $\mathcal{S}$ is an arbitrary surface enclosed by the simple curve $\mathcal{C}$, $\hat{\boldsymbol{n}}$ is a unit normal vector to the surface and $\mathrm{d}S$ is the surface element for the surface $\mathcal{S}$.

For irrotational flow, $\nabla \times \boldsymbol{u} = \boldsymbol{\omega} = \boldsymbol{0}$, hence

$$\Gamma = \oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \iint_{\mathcal{S}} \boldsymbol{\omega} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = 0.$$

This does not necessarily apply if the flow domain is not simply connected (that is, curves cannot be continuously deformed to a point while remaining within the flow domain). For example, two-dimensional irrotational flow past an infinitely long cylinder.

Even if $\Gamma \neq 0$, the value of Γ is the same for all closed paths $\mathcal{C}$ that can be deformed into each other while remaining in a simply connected region of an irrotational flow.

Example 62. The point vortex considered in the example above is irrotational. Why is the circulation not zero?

8.6 Complex potential

Complex potential

For two-dimensional incompressible irrotational flow,

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{60}$$

These are the Cauchy–Riemann equations.

Example 63. Show that functions satisfying the Cauchy–Riemann equations both satisfy the Laplace equation.

The complex potential is

$$w(z) = \phi(x, y) + \mathrm{i}\psi(x, y) \tag{61}$$

where $z = x + \mathrm{i}y$. It is analytic because ϕ and ψ satisfy the Cauchy–Riemann equations.

When it is applicable the complex potential is very convenient.

Complex velocity

The complex velocity is

$$
\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}z} &= \lim_{\Delta z \to 0} \frac{w(z+\Delta z) - w(z)}{\Delta z} \\
&= \lim_{\Delta x \to 0} \frac{\phi(x+\Delta x, y) - \phi(x,y) + \mathrm{i}\psi(x+\Delta x, y) - \mathrm{i}\psi(x,y)}{\Delta x} \\
&= \frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x} = u - \mathrm{i}v \\
&= \lim_{\Delta y \to 0} \frac{\phi(x, y+\Delta y) - \phi(x,y) + \mathrm{i}\psi(x, y+\Delta y) - \mathrm{i}\psi(x,y)}{\mathrm{i}\Delta y} \\
&= \frac{\partial \psi}{\partial y} - \mathrm{i}\frac{\partial \phi}{\partial y} = u - \mathrm{i}v.
\end{aligned}
$$

The magnitude of the velocity is

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = u^2 + v^2 = |\boldsymbol{u}|^2.$$

Uniform flow

For uniform flow at speed U in the positive x direction,

$$\phi = Ux, \quad \psi = Uy.$$

Hence the complex potential is

$$w(z) = \phi + i\psi = Ux + \mathrm{i}Uy = U(x + \mathrm{i}y) = Uz.$$

The complex velocity is

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}(Uz) = U,$$

that is $u = U$ and $v = 0$.

Example 64. Determine the flow that is represented by the complex potential

$$w(z) = Uze^{-i\alpha}.$$

Source/Sink

For a line source with volume flux per unit length Q,

$$\phi = \frac{Q}{2\pi}\ln r \quad \text{and} \quad \psi = \frac{Q}{2\pi}\theta,$$

hence

$$w = \frac{Q}{2\pi}(\ln r + \mathrm{i}\theta) = \frac{Q}{2\pi}(\ln r + \ln e^{i\theta}) = \frac{Q}{2\pi}\ln(re^{i\theta}) = \frac{Q}{2\pi}\ln z \tag{62}$$

Line vortex

For a line vortex of circulation $\mathit{\Gamma}$,

$$\phi = \frac{\mathit{\Gamma}}{2\pi}\theta \quad \text{and} \quad \psi = -\frac{\mathit{\Gamma}}{2\pi}\ln r,$$

hence

$$w = \frac{\mathit{\Gamma}}{2\pi}(\theta - \mathrm{i}\ln r) = -\mathrm{i}\frac{\mathit{\Gamma}}{2\pi}(\ln r + \mathrm{i}\theta) = -\mathrm{i}\frac{\mathit{\Gamma}}{2\pi}\ln z \tag{63}$$

Dipole

For a dipole with strength μ,

$$\phi = \mu\frac{\cos\theta}{r} \quad \text{and} \quad \psi = -\mu\frac{\sin\theta}{r},$$

hence

$$w = \mu\frac{\cos\theta - \mathrm{i}\sin\theta}{r} = \mu\frac{e^{-\mathrm{i}\theta}}{r} = \frac{\mu}{re^{\mathrm{i}\theta}} = \frac{\mu}{z}. \tag{64}$$

Uniform flow past a cylinder with circulation

Example 65. Consider the flow given by the complex potential

$$w = U\left(z + \frac{a^2}{z}\right) - \mathrm{i}\frac{\mathit{\Gamma}}{2\pi}\ln z,$$

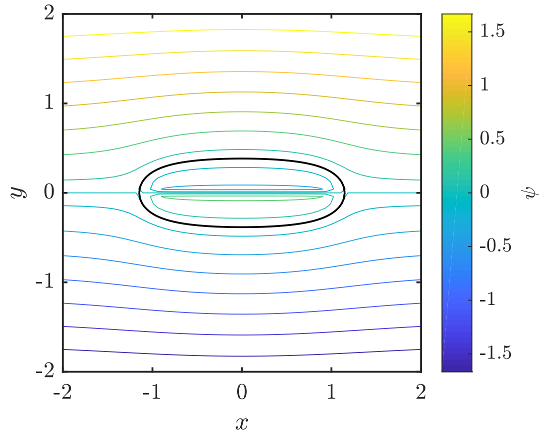
in other words, uniform flow around a dipole and vortex centred at the origin.

1. Show that this complex potential corresponds to flow past a circular cylinder of radius a.
2. Locate any stagnation points.

Flow past a Rankine body

Example 66. Plane flow past a Rankine body is given by uniform flow at speed U in the positive x-direction combined with a line source at $(-a, 0)$ and a line sink at $(a, 0)$ of equal strength Q.

1. Write down the complex potential for this flow.
2. Locate any stagnation points.
3. Use the complex potential to find the velocity potential and stream function for this flow.
4. Using the stream function found in the previous part, find the equation of the streamline that passes through the stagnation points, and hence the equation that defines the shape of the Rankine body.



9 Extra topics

9.1 Dynamic similarity

Nondimensionalisation

The Navier–Stokes equations are

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{F}.$$

Suppose that the flow is characterised by a length scale L and a velocity scale U. Define dimensionless variables as

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \hat{t} = \frac{Ut}{L}, \quad \hat{p} = \frac{p}{\rho U^2}.$$

The derivatives are related by

$$\frac{\partial}{\partial t} = \frac{U}{L}\frac{\partial}{\partial \hat{t}} \quad \text{and} \quad \frac{\partial}{\partial x_i} = \frac{1}{L}\frac{\partial}{\partial \hat{x}_i}.$$

Applying these operators to the Navier–Stokes equations and setting $\boldsymbol{F} = -g\boldsymbol{k}$ yields

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\hat{\nabla}\hat{p} + \left(\frac{\mu}{\rho U L}\right)\hat{\nabla}^2\hat{\boldsymbol{u}} - \left(\frac{gL}{U^2}\right)\boldsymbol{k}.$$

The continuity equation reads

$$\hat{\nabla} \cdot \hat{\boldsymbol{u}} = 0.$$

These equations are nondimensional and governed by two parameters.

The Reynolds number is defined as

$$\mathrm{Re} = \frac{\rho U L}{\mu}.$$

The Froude number is defined as

$$\mathrm{Fr} = \frac{U}{\sqrt{gL}}.$$

The nondimensional Navier–Stokes equations are

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\boldsymbol{u}} - \frac{1}{\mathrm{Fr}^2}\boldsymbol{k}.$$

Dynamic similarity

Two flows are geometrically similar if all lengths scale with a single characteristic length L, that is,

$$\frac{\boldsymbol{x}_1}{L_1} = \frac{\boldsymbol{x}_2}{L_2}$$

Two flows are kinematically similar if all velocities scale with a single characteristic speed U, that is,

$$\frac{\boldsymbol{u}_1}{U_1} = \frac{\boldsymbol{u}_2}{U_2}$$

Two flows are dynamically similar if

$$\mathrm{Re}_1 = \mathrm{Re}_2, \quad \text{and} \quad \mathrm{Fr}_1 = \mathrm{Fr}_2.$$

The dimensional variables can be recovered from

$$\boldsymbol{u} = U\hat{\boldsymbol{u}}, \quad p = \rho U^2 \hat{p}.$$

There is no need to recalculate dynamically similar flows. This is why we nondimensionalise problems!

Experimental results from a small model can be used to predict flow past a full-scale object, provided the experiment is geometrically, kinematically and dynamically similar. In practice, it is often difficult to obtain complete dynamic similarity.

High Reynolds-number flow

At high Reynolds number, $1/\mathrm{Re}$ is small. One might expect that the viscous terms in the Navier-Stokes equations could be dropped, yielding the Euler equations. This is true if the magnitude of $\hat{\nabla}^2\hat{\boldsymbol{u}}$ doesn't change too much.

However, it is possible for $\hat{\nabla}^2\hat{\boldsymbol{u}}$ to become as large as $1/\mathrm{Re}$ is small, so that the viscous terms are not negligible. This is what happens in a boundary layer, which is a thin layer of fluid close to a solid boundary where the velocity changes from its free-stream value to the boundary's velocity (often zero) over short distance $\delta \ll L$.

Many high Reynolds-number flows are unstable. In such flows, small perturbations grow, leading to an unsteady irregular three-dimensional state of fluid motion called turbulence. Turbulence in all manner of situtions, including:

- Astrophysical flows, including super novae.
- Atmospheric and oceanic circulation.
- Flow past aircraft, ships, motor vehicles.
- Combustion devices.
- Bushfires.

Turbulent flows are characterised by irregular three-dimensional motion over a wide range of temporal and spatial scales. At high Reynolds number, it is impossible to resolve the entire range of scales in a direct numerical simulation of the Navier–Stokes equations, even on the largest supercomputers. It is therefore necessary to resort to turbulence models to predict such flows. The development and refinement of such models is an active area of research.

Low Reynolds-number flow

When Re is small, the inertial terms in the Navier-Stokes equations may be neglected, yielding the Stokes equation,

$$\hat{\nabla}\hat{p} = \frac{1}{\mathrm{Re}}\hat{\nabla}^2\hat{\boldsymbol{u}}.$$

Situations where the Reynolds number is small occur in a number of important applications:

- Biological flows with small length scales - e.g., swimming micro-organisms, flows in small blood vessels or airways.
- Thin-film flows - e.g., coating flows or lubrication of machine parts.
- Other industrial applications - e.g., glass manufacturing, optical fibre production.