LECTURE 1

We start by contemplating the nature of real numbers. We're familiar with the following sets of numbers:

$$\mathbb{N} = \{\, 1, 2, 3, \dots \,\} \quad \text{the } \textbf{natural numbers}$$
$$\mathbb{Z} = \{\, \dots, -2, -1, 0, 1, 2, \dots \,\} \quad \text{the } \textbf{integers}$$
$$\mathbb{Q} = \{\, m/n \mid m, n \in \mathbb{N}, n \neq 0 \,\} \quad \text{the } \textbf{rational numbers}$$

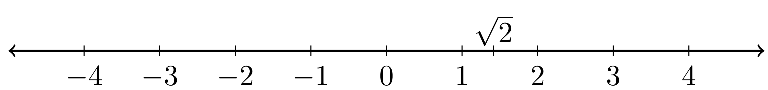
We've also probably built up some intuition for how to think of the set $\mathbb{R}$ of 'real numbers'. For instance, we might have a vague idea that

$$\mathbb{R} = \mathbb{Q} \cup \{\, \text{'the rest', e.g. } \sqrt{2}, \pi, \dots \,\}.$$

In particular, every rational number is a real number, but there are some real numbers that are not rational, 'the rest' of the real numbers are usually referred to as **irrational** numbers. A slightly less vague intuition of the set of real numbers is to view the set $\mathbb{R}$ as the set of infinite decimal expansions, for example

$$\begin{aligned} \sqrt{2} &= 1.414213\cdots \\ \pi &= 3.14159\cdots \\ 1/2 &= 0.50000\cdots \\ &= 0.4999\cdots \end{aligned}$$

Another intuition that we might have about the set $\mathbb{R}$ is to view it as 'the real line', in other words we imagine the points of the set $\mathbb{R}$, i.e. the real numbers, as corresponding to points on the line



We will come to see that both of these are valid intuitions to have about the set $\mathbb{R}$.

When it comes to performing algebraic operations like addition, the set of rational numbers is a little easier to deal with, for instance, in primary school we learn that

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6},$$

but if we're asked, what is the real number

$$\sqrt{2} + \pi?$$

we might feel a little less sure of ourselves. For instance, if we think of real numbers as infinite decimals, then we're supposed to calculate

$$1.414\cdots + 3.14159\cdots = ???$$

How are we supposed to add two infinite decimals together? Where do you even start?? (On the other hand, if we got the answer $\sqrt{2} + \pi$ when computing an integral, we'd be perfectly happy to leave our answer as $\sqrt{2} + \pi$.) This is a slightly mysterious aspect of real numbers.

From another point of view, we might think about applications of mathematics to real world problems. Even the most accurate measurements can only be made to about 15 decimal places, so it seems like contemplating infinite decimals is a purely academic exercise, without any practical benefits.

So we might wonder, with some justification, what is the point of these elusive real numbers? Perhaps we could get away with studying rational numbers — why don't we study a course called *Rational Analysis?*

The answer is that the theorems of Calculus are no longer valid when real numbers are replaced by rational numbers. For example, if $f\colon \mathbb{Q} \to \mathbb{Q}$ is a function defined on the rational numbers and taking rational values, then we can make perfect sense of what it means for f to be differentiable. The problem is that the derivative will no longer give us the useful information that we have come to expect.

For instance, consider the function $f\colon \mathbb{Q} \to \mathbb{Q}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x^2 > 2, \\ -1 & \text{if } x^2 < 2. \end{cases}$$

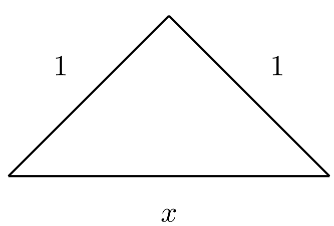
Then f is differentiable at every $x \in \mathbb{Q}$. However, we have the following

- $f'(x) = 0$ for every $x \in \mathbb{Q}$ but f is not a constant function;
- $f(1) = -1$ and $f(2) = 1$, but there is no rational number x between 1 and 2 such that $f(x) = 0$.

This is in stark contrast to what we have come to expect of functions of a real variable.

So what, then, is special about the real numbers that makes the theorems of Calculus work? Let's think a little bit more about the set of real numbers.

In particular, let's think about the first number which was observed to not be rational, the square root of 2. This was first observed by the Pythagoreans in Ancient Greece. The way they might have observed this is as follows. We could imagine constructing a right angled triangle with adjacent sides of unit length:



Of course, such a construction is a virtual exercise that we will have to imagine in our heads — we could not construct such a triangle with perfect accuracy. In this virtual exercise we might ask what is the length x of the hypotenuse?

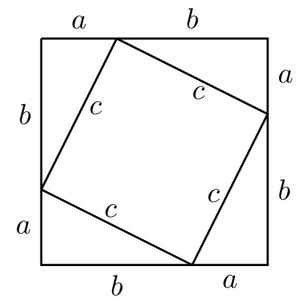
From high-school mathematics we know the answer from:

Theorem (Pythagoras) In a right angled triangle with adjacent sides of length a and b, and hypotenuse of length c, the equation

$$c^2 = a^2 + b^2$$

holds.

Just for fun, let's look at one of the ways to prove this theorem. Place four copies of the triangle as shown to make a square with sides of length $a + b$:



The area of this triangle is $(a + b)^2$. On the other hand, we can calculate the area of the square from the area of the inscribed square with sides of length c, and the four copies of the right angled triangle. The combined area of a pair of triangles is equal to the area of a rectangle with sides of length a and b. Therefore the area of the big square is also equal to $c^2 + 2ab$. Therefore

$$(a + b)^2 = c^2 + 2ab$$

from which the desired statement readily follows.

Returning to our question above, we observe that the length of the hypotenuse in this example satisfies $x^2 = 2$. It was known to the Pythagoreans that such a number x could not be rational.[1]

We might hope to enlarge the set of rational numbers by including $\sqrt{2}$ — this would lead us to consider numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$, which at high school you might have called *surds*. You can add, multiply and take reciprocals surds: for instance

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2},$$

and

$$(a + b\sqrt{2})^{-1} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$$

if $a, b \neq 0$. In mathematics the set of all such surds is denoted $\mathbb{Q}(\sqrt{2})$ — it is an example of something called an *algebraic number field*. The set $\mathbb{Q}(\sqrt{2})$ does not contain every real number however, for instance it does not contain

$$\sqrt{3}, \sqrt{\sqrt{2} + \sqrt{3}}, \sqrt[5]{29}, \ldots$$

We might try and enlarge $\mathbb{Q}$ even further by adding all of the numbers which we could form by the operations of addition, subtraction, multiplication, division and extracting n-th roots. The resulting set of numbers is still not equal to the set $\mathbb{R}$. For instance, we cannot solve a general quintic polynomial equation of the form

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

where $a, b, c, d, e, f \in \mathbb{Q}$ and $a \neq 0$ with these numbers! This is a very famous old theorem ('the insolubility of quintics by radicals') due to Niels Abel. Later Evariste Galois was able to explain

[1] This, it is believed, was a fact that was highly disturbing to the Pythagoreans, so much so that Hippasus of Mentapontum was murdered at sea for revealing this.

exactly which quintic equations could be solved by radicals. Abel and Galois are two of the tragic heroes of mathematics, Abel died in poverty at the age of 24 and Galois was killed in a duel at the age of 21. We have human tragedy to rival Shakespeare in Real Analysis.

Let's contemplate an even bigger set — let's look at the set of all (real) numbers x which are solutions of equations $p(x) = 0$, where p is a polynomial with rational coefficients (such a number x is called *algebraic*). Could this set be $\mathbb{R}$? Not even close! $\mathbb{R}$ is far bigger, in fact there is a precise sense in which almost every real number is not one of these algebraic numbers.

Instead, the majority of real numbers are what are known as *transcendental* numbers. These are the real numbers which are not the solutions of any polynomials equations with rational coefficients. These are numbers like e and π. Paradoxically, even though there are so many of these numbers they are very difficult to describe. Even though they constitute almost all of the real numbers they're so rare they're usually named after the people who discovered them:

- The Gelfond-Schneider constant $\sqrt{2}^{\sqrt{2}}$,
- Liouville's Number $\sum_{n=1}^{\infty} 10^{-n!} = 0 \cdot 110001000000000000000000010000 \cdots$,
- Champernowne's Constant $0 \cdot 123456789101112131415 \cdots$
- $\ldots$

Hopefully this discussion has given you the idea that the real numbers are not quite as straightforward as you might have thought — in fact there are many things that are still unknown about them. Let me mention just one such thing which you might find startling: it is an open problem as to whether $e + \pi$ is *rational* or not!!

Our approach to the real numbers will be an axiomatic one. We will list all of the properties that we want the real numbers to have as a list of axioms, and then postulate that there exists a set with these desired properties. Later we will see that indeed there is such a set and that moreover it is uniquely determined by these properties. We will then proceed to make deductions from these axioms using the laws of logic rather than any preconceived intuitions that we may have about the real numbers.

We finish this lecture by making some comments on the methods of proof that we shall use. There are four main methods of proof that we shall use.

Sometimes we will give a *direct proof.* In this method of proof we start from a known statement and proceed to show that the desired conclusion is true using the laws of logic. Sometimes we shall give a *proof by contradiction.* A proof by contradiction works in the following way. If we want to prove a statement of the form $A \implies B$ we suppose that the conclusion B is false, i.e. that the statement 'not B' is true. We then proceed to draw a series of conclusions from this starting point until we reach a statement which we know to be false.

A good example is the proof that $\sqrt{2}$ is irrational. This can be quickly established as follows. First, we suppose that $\sqrt{2}$ is not irrational, i.e. that $\sqrt{2}$ is rational. Therefore there exist m and n (which we may suppose are positive) such that $\sqrt{2} = m/n$. We may suppose that the fraction m/n is in lowest terms, if it is not then we may cancel some factors until it is. The assumption that m/n is in lowest terms is the same as the assumption that m is as small as possible. Next, we observes that

$$\sqrt{2} = \frac{2n - m}{m - n}$$

and that $2n - m$ is a natural number which is smaller than m. This contradicts our assumption that m/n is in lowest terms and hence the assumption that $\sqrt{2}$ is rational. Therefore $\sqrt{2}$ is irrational.