School of Mathematical Sciences

Mathematical Biology (Honours)

Assignment 4 question sheet

Due: Friday, 1 November, by 5pm (leave in box on office door)

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1. Two chemicals, which control the pigmentation of an animal's skin, have concentrations u and v, diffuse and interact on $-\infty < x < \infty$ according to the (dimensionless) equations

$$\frac{\partial u}{\partial t} = \frac{u^2}{v} - \beta u + \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial v}{\partial t} = u^2 - v + \delta \frac{\partial^2 v}{\partial x^2},$$

where β and δ are positive constants.

   (a) Which chemical plays the role of activator, and which is the inhibitor? [1 marks]

   (b) Show there is a unique spatially uniform steady state with $u, v > 0$. [2 marks]

   (c) Carry out a linear stability analysis of this steady state by considering solutions of the form

$$(u, v) = (u_0, v_0) + \epsilon(u_1, v_1)e^{iqx+\lambda t} + \ldots,$$

   (where the real part is to be understood). Hence, determine the conditions for a diffusion-driven (Turing) instability to occur, and sketch the corresponding region in the (β, δ) plane. [6 marks]

   (d) When a diffusion-driven instability occurs, what are the unstable wave numbers? [4 marks]

   (e) Briefly explain how your results would change if the diffusion coefficient for v was proportional to u, so the second of the two equations would become:

$$\frac{\partial v}{\partial t} = u^2 - v + \delta \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial x}\right),$$

   [3 marks]

2. Tumour cells are grown in a tube, occupying the region $0 < x < L(t)$, where x is the distance along the tube. The region $x > L(t)$ contains nutrient medium, where the nutrient concentration, c, is maintained at a constant level, c_∞. Nutrient diffuses through the tumour cells (whether alive or dead), with diffusion coefficient, $D = 1$. There is no flux of cells or nutrient through the bottom of the tube, $x = 0$. Provided $c > c_n$, tumour cells consume nutrient at a constant rate, λ, and proliferate at a rate proportional to the nutrient concentration. However, if $c < c_n$, the cells become necrotic; necrotic cells do not consume nutrient and decay at a constant rate, β. You may assume that the diffusion of nutrient occurs much faster than the growth of the tumour, and that the density of cells within the tumour region remains constant at all times.

   (a) Assuming that the tumour is small enough not to have developed a necrotic region, show that the length of the tumour region $L(t)$ satisfies

   $$\frac{dL}{dt} = sc_\infty L - \frac{\lambda s L^3}{3}, \tag{1}$$

   where s is constants, whose meaning you should explain. [6 marks]

   (b) Show, from equation (1) above that there are two steady states, $L = 0$, and $L = L^* > 0$, which are unstable and stable, respectively. Explain why, in reality, the $L = L^*$ steady state will not be attained. (*Hint: consider the nutrient level at* $x = 0$.) [4 marks]

   (c) Now consider a tumour which has a necrotic region $0 < x < L_n(t)$. Find the nutrient concentration throughout the entire tumour ($0 < x < L(t)$), assuming that both the nutrient concentration and its derivative are continuous at $x = L_n$. Hence, obtain an equation for the length of the proliferating region, $L - L_n$ in terms of λ, c_∞ and c_n. [5 marks]

   (d) Solve for the cell velocity, and thus show the length of the tumour region satisfies

   $$\frac{dL}{dt} = s(L - L_n)\left(c_n + \frac{\lambda}{6}(L - L_n)^2\right) - \beta L_n.$$

   [3 marks]

Total: 34 marks