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Solutions

Ans1:

(a) Given:

The normal distribution of weights of the apples picked by Jane are as follows:

$$Mean = \mu = 160,$$
$$Standard\ Deviation = \sigma = 50$$
$$X \in \mathrm{N}(\mu = 160, \sigma = 50)$$

Percentage of apples that Jane picks and doesn't like are:

$$P(X < 95)$$

The Z-score for the raw x value is $Z = \frac{X-\mu}{\sigma}$,

Converting to standard normal variable,

$= P\left(Z < \frac{95-100}{50}\right)$

$= P(Z < -1.3)$

From standard normal tables,

$P(Z < -1.3) = 0.0968$

Hence, the required percentage $= \mathbf{0.0968}$

(b) Now,

Percentage of apples are not liked by Jane $= 9.68\%$

Therefore, 75th percentile of the weights of the apples that Jane likes would have a percentile:

$= (1 - 0.0968) \times 0.75 = 0.6774$

Hence, $67.74th$ is the percentile.

So, the probability of the weights more than equal to the above value:

$= 1 - 0.6774 = 0.3226$

Hence, the required probability $= \mathbf{0.3226}$

Ans2:

(a) Given:

Number of Red Balls $= 10$,	Payoffs when you draw a ball:
Number of Blue Balls $= 10$,	If Yellow Ball is draw $= \$0$,
Number of Green Balls $= 10$,	If Green Ball is draw $= \$0$,
Number of Yellow Balls $= 10$,	If Blue Ball is draw $= \$200$,
Number of Orange Balls $= 3$,	If Red Ball is draw $= \$300$,
Total number of balls $= 10 + 10 + 10 + 10 + 3 = 43$.	If orange ball is draw $= \$500$.

Probability of Red ball, $P(R) = \frac{10}{43} = 0.23$

Probability of Blue ball, $P(B) = \frac{10}{43} = 0.23$

Probability of Green ball, $P(G) = \frac{10}{43} = 0.23$

Probability of Yellow ball, $P(Y) = \frac{10}{43} = 0.23$

Probability of Orange ball, $P(O) = \frac{3}{43} = 0.07$

Balls	Red	Green	Blue	Yellow	Orange
Probabilities	$\frac{10}{43}$	$\frac{10}{43}$	$\frac{10}{43}$	$\frac{10}{43}$	$\frac{3}{43}$
Payoffs	300	0	200	0	500

$E(X) = \frac{10}{43} \times 0 + \frac{10}{43} \times 0 + \frac{10}{43} \times 200 + \frac{10}{43} \times 300 + \frac{3}{43} \times 500$

$= 46.5116 + 69.7674 + 34.8837$

$\boldsymbol{E(X) = 151.1627}$

(b) $E(X^2) = 0^2 \times \frac{10}{43} + 0^2 \times \frac{10}{43} + 200^2 \times \frac{10}{43} + 300^2 \times \frac{10}{43} + 500^2 \times \frac{3}{43}$

$= \frac{400000}{43} + \frac{900000}{43} + \frac{750000}{43}$

$= 9302.3256 + 20930.2326 + 17441.8605$

$E(X^2) = 47674.4187$

$\sigma^2 = E(X^2) - \left(E(X)\right)^2$

$= 47674.4187 - (151.1627)^2$

$= 47674.4187 - 22850.1619$

Variance, $\boldsymbol{\sigma^2 = 24824.2568}$

Standard Deviation, $\boldsymbol{\sigma = \sqrt{24824.2568} = 157.5572}$

Ans3: Let,

$X =$ The event Harry guesses the answer is correct,

Probability of Harry knows the answer, $P(Y) = x$

Probability of Harry guesses the answer, $P(Y') = 1 - P(Y) = 1 - x$

Probability of Harry answered the correctly, given that he knows the answer, $P(X|Y) = 1$

Probability of Harry answered the correctly, given that he guesses the answer, $P(X|Y') = \frac{1}{y}$

Probability of Harry knows the answer, given that he answered it correctly, $P(Y|X)$

Using Bayes' theorem,

$$P(Y|X) = \frac{P(Y)P(X|Y)}{P(Y)P(X|Y) + P(Y')P(X|Y')}$$

$$= \frac{x}{x + \frac{(1-x)}{y}}$$

$$\boldsymbol{P(Y|X) = \frac{xy}{xy + 1 - x}}$$

The probability that Harry knows the correct answer to a problem given that he has answered that problem correctly, $\boldsymbol{P(Y|X) = \frac{xy}{xy+1-x}}$.

Ans4: Let,

$X =$ The event that an email is detected as spam,

$Y =$ The event that an email is spam,

$Y' =$ The event that an email is not spam.

Given:

$P(Y) = 50\% = 0.5$

$P(Y') = 1 - P(Y) = 1 - 0.5 = 0.5$

$P(X|Y) = 99\% = 0.99$

$P(X|Y') = 5\% = 0.05$

Using Bayes' formula:

$$P(Y'|X) = \frac{P(X|Y')P(Y')}{P(X|Y)P(Y)+P(X|Y')P(Y')}$$

$$= \frac{0.05\times0.5}{0.05\times0.5+0.99\times0.5}$$

$$= \frac{0.05}{0.05+0.99}$$

$$= \frac{0.05}{1.04}$$

$$= \frac{5}{104}$$

$\boldsymbol{P(Y'|X) = 0.0480}$

If an email is detected as spam, then the probability that it is in fact a non-spam email, $\boldsymbol{P(Y'|X) = 0.0480}$.

Ans5: Let,

Positive $=$ Mammogram result is positive,

Negative $=$ Mammogram result is negative,

B $=$ Tumor is benign,

M $=$ Tumor is malignant.

Given,

$P(M) = 1\% = 0.01$

$P(B) = 1 - P(M) = 1 - 0.01 = 0.99$

$P(Positive|M) = 80\% = 0.8$

$P(Negative|B) = 90\% = 0.9$

$P(Positive|B) = 1 - 0.9 = 0.1$

$P(M|Positive) = \frac{P(Positive|M)P(M)}{(P(Positive|M)P(M)+P(Positive|B)P(B))}$

$= \frac{0.8\times0.01}{(0.8\times0.01+0.1\times0.99)}$

$\boldsymbol{P(M|Positive)} \cong \mathbf{0.075}$

Therefore, the chances are patient has cancer $= 7.5\%$.

Ans6:

(a) Given: $F(x) = \frac{f(x)}{g(x)}$, where $f(x)$ *and* $g(x)$ are also functions of x.

Limit Definition of $\boldsymbol{F(x)}$:

$$F(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

Prove: $F'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

$F(x) = \frac{f(x)}{g(x)}$

$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

$= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$

$= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}$

$= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}\right)$

Using the basic properties of limits to write this as,

$F'(x) = \frac{1}{\lim_{h\to 0} g(x+h)\lim_{h\to 0} g(x)}\left(\left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) - \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\right)$

The individual limits are,

$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x)$

$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$

$\lim_{h\to 0} g(x+h) = g(x)$

$\lim_{h\to 0} g(x) = g(x)$

$\lim_{h\to 0} f(x) = f(x)$

Putting all the values in the limits,

$F'(x) = \frac{1}{g(x)g(x)}\left(g(x)f'(x) - f(x)g'(x)\right)$

$\boldsymbol{F'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}}$

Hence Proved.

(b) Given: $f(x) = \sin(x)$

Limit Definition of $\boldsymbol{f(x)}$:

$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Using trigonometric formula of $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$,

$f'(x) = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$
$f'(x) = \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h}$

Using the theorem, the limit of the sum of functions is equal to the sum of the limits of these functions to rewrite $f'(x)$ as follows:

$f'(x) = \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$
$f'(x) = \sin(x) \lim_{h\to 0} \frac{(\cos(h)-1)}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$

Using the results of the limits of trigonometric functions,

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$$
$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$$

Putting these values in the $f'(x)$,

$f'(x) = \sin(x)(0) + \cos(x)(1) = \cos(x)$
$\boldsymbol{f'(x) = \cos(x)}$

Ans7: Given: $f(x) = \frac{x^2}{e^x}$

$$f(x) = x^2 e^{-x}$$

Using the product rule,

$m_{tan} = f'(x) = 2xe^{-x} - x^2 e^{-x}$

$m_{tan} = f'(x) = -(x-2)xe^{-x}$

Slope of the tangent line at $x = 1$,

$m_{tan} = f'(x) = -(1-2)1e^{-1} = \frac{1}{e} = 0.37$

Equation of a tangent line,

$m_{tan} = instantaneous\ slope\ at\ x_1 = 0.37$

$f(x=1) = 1^2 e^{-1} = e^{-1} = 0.37$

Using point slope form of line,

$y - y_1 = m(x - x_1)$

$y - 0.37 = 0.37(x - 1)$

$y = 0.37x - 0.37 + 0.37 = 0.37x$

$\boldsymbol{y = 0.37x}$

The equation of a tangent line is $\boldsymbol{y = 0.37x}$.

Ans8: Given: $f(x) = e^x$

$g(x) = \frac{x^2}{x-1}$

$F(x) = f(g(x)) = f\left(\frac{x^2}{x-1}\right) = e^{\left(\frac{x^2}{x-1}\right)} = e^{\frac{x^2}{x-1}}$

Using Chain Rule, $\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{du}(f(u))\frac{d}{dx}(g(x))$,

Let $u = \frac{x^2}{x-1}$,

$F'(x) = \frac{d}{du}(e^u)\frac{d}{dx}\left(\frac{x^2}{x-1}\right)$

Using Derivative of the exponential, $\frac{d}{du}(e^u) = e^u$,

$= e^u \frac{d}{dx}\left(\frac{x^2}{x-1}\right)$

Putting u value back,

$= e^{\frac{x^2}{x-1}} \frac{d}{dx}\left(\frac{x^2}{x-1}\right)$

Using the Quotient Rule, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g(x)^2}$,

$= e^{\frac{x^2}{x-1}} . \frac{\frac{d}{dx}[x^2].(x-1) - x^2.\frac{d}{dx}[x-1]}{(x-1)^2}$

Using both Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},where $n = 2$ and Difference rule,

$= e^{\frac{x^2}{x-1}} . \frac{\left(2x.(x-1) - \left(\frac{d}{dx}[x] - \frac{d}{dx}[1]\right)x^2\right)}{(x-1)^2}$

Derivative of Constant is 0,

$= e^{\frac{x^2}{x-1}} . \frac{(2x.(x-1) - (1+0)x^2)}{(x-1)^2}$

$= \frac{(2(x-1)x - x^2)e^{\frac{x^2}{x-1}}}{(x-1)^2}$

$= \left(\frac{2x}{x-1} - \frac{x^2}{(x-1)^2}\right)e^{\frac{x^2}{x-1}}$

$\boldsymbol{F'(x) = \frac{(x-2)xe^{\frac{x^2}{x-1}}}{(x-1)^2}}$

Ans9:

(a) $F(x) = \frac{\sqrt{x}+2x}{7x-4x^2}$

Using $\sqrt[n]{a^x} = a^{\frac{x}{n}}$,

$$F(x) = \frac{x^{\frac{1}{2}} + 2x}{7x - 4x^2}$$

Using the Quotient Rule, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g(x)^2}$,

$$F'(x) = \frac{(7x-4x^2)\frac{d}{dx}\left[x^{\frac{1}{2}}+2x\right] - (x^{\frac{1}{2}}+2x)\frac{d}{dx}[7x-4x^2]}{(7x-4x^2)^2}$$

Using both Sum and Difference Rule,

$$= \frac{(7x-4x^2)\left(\frac{d}{dx}\left[x^{\frac{1}{2}}\right]+\frac{d}{dx}[2x]\right) - \left(x^{\frac{1}{2}}+2x\right)\left(\frac{d}{dx}[7x]-\frac{d}{dx}[4x^2]\right)}{(7x-4x^2)^2}$$

Using Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},

$$= \frac{(7x-4x^2)\left(\frac{1}{2}x^{\frac{1}{2}-1}+2\right) - (x^{\frac{1}{2}}+2x)(7-4[2x])}{(7x-4x^2)^2}$$

$$= \frac{(7x-4x^2)\left(\frac{1}{2}x^{\frac{-1}{2}}+2\right) - (x^{\frac{1}{2}}+2x)(7-8x)}{(7x-4x^2)^2}$$

$$\boldsymbol{F'(x) = \frac{(7x-4x^2)\left(\frac{1}{2\sqrt{x}}+2\right) - (\sqrt{x}+2x)(7-8x)}{(7x-4x^2)^2}}$$

(b) $F(x) = \left(1+\sqrt{x^3}\right)\left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)$

Using Product Rule, $\frac{d}{dx}(u,v) = v\frac{du}{dx} + u\frac{dv}{dx}$,

$$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\frac{d}{dx}\left(1+\sqrt{x^3}\right) + \left(1+\sqrt{x^3}\right)\frac{d}{dx}\left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)$$

Using both Sum and Difference Rule,

$$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\frac{d}{dx}(1) + \frac{d}{dx}\left(\sqrt{x^3}\right)\right) + \left(1+\sqrt{x^3}\right)\left(\frac{d}{dx}\left(\frac{1}{x^3}\right) - \frac{d}{dx}\left(2\sqrt[3]{x}\right)\right)$$

Derivative of Constant is 0,

Using Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},where $n = -3$,

$$\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-4}$$

$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\frac{d}{dx}\left(\sqrt{x^3}\right)\right) + \left(1+\sqrt{x^3}\right)\left(-\frac{3}{x^4} - 2\frac{d}{dx}\left(\sqrt[3]{x}\right)\right)$

Using Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},where $n = \frac{1}{3}$,

$$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\frac{d}{dx}\left(\sqrt{x^3}\right)\right) + \left(1+\sqrt{x^3}\right)\left(-\frac{3}{x^4} - 2\frac{1}{3x^{\frac{2}{3}}}\right)$$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u = x^3$,

$$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\frac{d}{du}\left(\sqrt{u}\right)\frac{d}{dx}(x^3)\right) + \left(1+\sqrt{x^3}\right)\left(-\frac{3}{x^4} - \frac{2}{3x^{\frac{2}{3}}}\right)$$

Using Power Rule, $\frac{d}{du}[u^n]$ is nu^{n-1},where $n = \frac{1}{2}$,

$$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\left(\frac{1}{2\sqrt{u}}\right)\frac{d}{dx}(x^3)\right) + \left(1+\sqrt{x^3}\right)\left(-\frac{3}{x^4} - \frac{2}{3x^{\frac{2}{3}}}\right)$$

Putting u value back,

$$F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\frac{\frac{d}{dx}(x^3)}{2\sqrt{x^3}}\right) + \left(1+\sqrt{x^3}\right)\left(-\frac{3}{x^4} - \frac{2}{3x^{\frac{2}{3}}}\right)$$

Using Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},where $n = 3$,

$$\boldsymbol{F'(x) = \left(\frac{1}{x^3} - 2\sqrt[3]{x}\right)\left(\frac{3x^2}{2\sqrt{x^3}}\right) + \left(1+\sqrt{x^3}\right)\left(-\frac{3}{x^4} - \frac{2}{3x^{\frac{2}{3}}}\right)}$$

(c) $F(x) = (2x+1)^5(3x-2)^7$

Using Product Rule, $\frac{d}{dx}(u,v) = v\frac{du}{dx} + u\frac{dv}{dx}$,

$F'(x) = (3x-2)^7\frac{d}{dx}(2x+1)^5 + (2x+1)^5\ \frac{d}{dx}(3x-2)^7$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,

Let $u = (2x+1)^5$,

$F'(x) = (3x-2)^7\left(\frac{d}{du}(u^5)\frac{d}{dx}(2x+1)\right) + (2x+1)^5\ \frac{d}{dx}(3x-2)^7$

Using Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},where $n = 5$,

$F'(x) = (3x-2)^7\left(5u^4\frac{d}{dx}(2x+1)\right) + (2x+1)^5\ \frac{d}{dx}(3x-2)^7$

Putting back u value,

$$F'(x) = (3x-2)^7\left(5(2x+1)^4\frac{d}{dx}(2x+1)\right) + (2x+1)^5\ \frac{d}{dx}(3x-2)^7$$

Using Sum Rule,

$$F'(x) = (3x-2)^7\left(5(2x+1)^4\left(\frac{d}{dx}(2x)+\frac{d}{dx}(1)\right)\right) + (2x+1)^5\ \frac{d}{dx}(3x-2)^7$$

Derivative of Constant is 0,

$$F'(x) = (3x-2)^7(10(2x+1)^4) + (2x+1)^5\ \frac{d}{dx}(3x-2)^7$$

Using Chain Rule, $\frac{d}{dx}\left(f\big(g(x)\big)\right) = \frac{d}{du}\big(f(u)\big)\frac{d}{dx}\big(g(x)\big)$,

Let $u = (3x-2)^7$,

$$F'(x) = (3x-2)^7(10(2x+1)^4) + (2x+1)^5\ \left(\frac{d}{dx}(u)^7\frac{d}{dx}(3x-2)\right)$$

Using Power Rule, $\frac{d}{dx}[x^n]$ is nx^{n-1},where $n = 7$,

$$F'(x) = (3x-2)^7(10(2x+1)^4) + (2x+1)^5\ \left(7u^6\frac{d}{dx}(3x-2)\right)$$

Putting back u value,

$$F'(x) = (3x-2)^7(10(2x+1)^4) + (2x+1)^5\ \left(7(3x-2)^6\frac{d}{dx}(3x-2)\right)$$

Using Difference Rule,

$$F'(x) = (3x-2)^7(10(2x+1)^4) + (2x+1)^5\ \left(7(3x-2)^6\left(\frac{d}{dx}(3x)-\frac{d}{dx}(2)\right)\right)$$

Derivative of Constant is 0,

$$F'(x) = (3x-2)^7(10(2x+1)^4) + (2x+1)^5\ \big(7(3x-2)^6(3)\big)$$
$$\boldsymbol{F'(x) = 10(3x-2)^7(2x+1)^4 + 21(2x+1)^5\ (3x-2)^6}$$

(d) $F(x) = \frac{x\sqrt{2x+1}}{e^x\sin^3(x)}$

$$F(x) = \frac{x\sqrt{2x+1}e^{-x}}{\sin^3(x)}$$

Using logarithmic differentiation technique,
Taking logarithm on $F(x)$,

$\ln\left(F(x)\right) = \ln\left(\frac{x\sqrt{2x+1}e^{-x}}{\sin^3(x)}\right)$

$\ln\left(F(x)\right) = -x + \ln(x) + \frac{\ln(2x+1)}{2} - 3\ln(\sin(x))$

$\boldsymbol{\frac{d}{dx}(\ln(F(x)) = \frac{d}{dx}\left(-x + \ln(x) + \frac{\ln(2x+1)}{2} - 3\ln(\sin(x))\right)}$

LHS:

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u = F(x)$,

$\left(\frac{d}{dx}(\ln(F(x)))\right) = \left(\frac{d}{dx}(\ln(u))\frac{d}{dx}\left(F(x)\right)\right)$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,

$\left(\frac{d}{dx}(\ln(F(x)))\right) = \left(\left(\frac{1}{u}\right)\frac{d}{dx}\left(F(x)\right)\right)$

Putting back u value,

$\left(\frac{d}{dx}(\ln(F(x)))\right) = \left(\left(\frac{1}{F(x)}\right)\frac{d}{dx}\left(F(x)\right)\right)$

$\boldsymbol{\left(\frac{d}{dx}(\ln(F(x)))\right) = \frac{\frac{d}{dx}(F(x))}{F(x)}}$

RHS:

$\frac{d}{dx}\left(-x + \ln(x) + \frac{\ln(2x+1)}{2} - 3\ln(\sin(x))\right)$

Using both Sum and Difference rule,

$\frac{d}{dx}\left(-x + \ln(x) + \frac{\ln(2x+1)}{2} - 3\ln(\sin(x))\right) = \left(-\frac{d}{dx}(x) + \frac{d}{dx}(\ln(x)) + \frac{d}{dx}\left(\frac{\ln(2x+1)}{2}\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$

$= \left(-1 + \frac{d}{dx}(\ln(x)) + \left(\frac{1}{2}\right)\frac{d}{dx}(\ln(2x+1)) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,
Let $u = 2x + 1$,

$= \left(-1 + \frac{d}{dx}(\ln(x)) + \left(\frac{1}{2}\right)\left(\frac{d}{du}(\ln(u))\frac{d}{dx}(2x+1)\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$

$= \left(-1 + \frac{d}{dx}(\ln(x)) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{u}\right)\frac{d}{dx}(2x+1)\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$

Putting back u value,

$$= \left(-1 + \frac{d}{dx}(\ln(x)) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{(2x+1)}\right)\frac{d}{dx}(2x+1)\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$$

Using both Sum and Difference rule,

$$= \left(-1 + \frac{d}{dx}(\ln(x)) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{(2x+1)}\right)\left(\frac{d}{dx}(2x) + \frac{d}{dx}(1)\right)\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$$

Derivative of constant is 0.

$$= \left(-1 + \frac{d}{dx}(\ln(x)) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{(2x+1)}\right)2\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,
Let $u = x$,

$$= \left(-1 + \left(\frac{d}{du}(\ln(u))\ \frac{d}{dx}(x)\right) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{(2x+1)}\right)2\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$$
$$= \left(-1 + \left(\frac{1}{u}\right) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{(2x+1)}\right)2\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$$

Putting back u value,

$$= \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{(2x+1)}\right) - 3\frac{d}{dx}(\ln(\sin(x)))\right)$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,
Let $u = \sin(x)$,

$$= \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{(2x+1)}\right) - 3\left(\frac{d}{du}(\ln(u))\frac{d}{dx}(\sin(x))\right)\right)$$

Derivative of the $sine$ is $\frac{d}{dx}(\sin(x)) = \cos(x)$

$$= \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{(2x+1)}\right) - 3\left(\left(\frac{1}{u}\right)\cos(x)\right)\right)$$

Putting back u value,

$$= \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{(2x+1)}\right) - 3\left(\left(\frac{1}{\sin(x)}\right)\cos(x)\right)\right)$$
$$= \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{(2x+1)}\right) - 3\left(\frac{\cos(x)}{\sin(x)}\right)\right)$$
$$= \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{(2x+1)}\right) - \left(\frac{3}{\tan(x)}\right)\right)$$

$$\frac{\frac{d}{dx}(\boldsymbol{F(x)})}{\boldsymbol{F(x)}} = \left(\boldsymbol{-1} + \left(\frac{\boldsymbol{1}}{\boldsymbol{x}}\right) + \left(\frac{\boldsymbol{1}}{(\boldsymbol{2x+1})}\right) - \left(\frac{\boldsymbol{3}}{\boldsymbol{\tan(x)}}\right)\right)$$

$$\frac{d}{dx}(F(x)) = \left(-1 + \left(\frac{1}{x}\right) + \left(\frac{1}{(2x+1)}\right) - \left(\frac{3}{\tan(x)}\right)\right)F(x)$$

$$\boldsymbol{F'(x)} = \left(\boldsymbol{-1} + \left(\frac{\boldsymbol{1}}{\boldsymbol{x}}\right) + \left(\frac{\boldsymbol{1}}{(\boldsymbol{2x+1})}\right) - \left(\frac{\boldsymbol{3}}{\boldsymbol{\tan(x)}}\right)\right)\left(\frac{\boldsymbol{x\sqrt{2x+1}}}{\boldsymbol{e^x \sin^3(x)}}\right)$$

(e) $F(x) = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Using logarithmic differentiation technique,
Taking logarithm on $F(x)$,

$$\ln(F(x)) = \ln\left(\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\right)$$
$$\ln(F(x)) = -\frac{\ln(x-5)}{2} - \frac{\ln(x-4)}{2} - \frac{\ln(x-3)}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x-1)}{2}$$

$$\frac{d}{dx}(\ln(F(x)) = \frac{d}{dx}\left(-\frac{\ln(x-5)}{2} - \frac{\ln(x-4)}{2} - \frac{\ln(x-3)}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x-1)}{2}\right)$$

LHS:

Using Chain Rule, $\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{du}(f(u))\frac{d}{dx}(g(x))$,
Let $u = F(x)$,

$$\left(\frac{d}{dx}(\ln(F(x)))\right) = \left(\frac{d}{dx}(\ln(u))\frac{d}{dx}(F(x))\right)$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,

$$\left(\frac{d}{dx}(\ln(F(x)))\right) = \left(\left(\frac{1}{u}\right)\frac{d}{dx}(F(x))\right)$$

Putting back u value,

$$\left(\frac{d}{dx}(\ln(F(x)))\right) = \left(\left(\frac{1}{F(x)}\right)\frac{d}{dx}(F(x))\right)$$
$$\left(\frac{\boldsymbol{d}}{\boldsymbol{dx}}(\boldsymbol{\ln(F(x))})\right) = \frac{\frac{d}{dx}(\boldsymbol{F(x)})}{\boldsymbol{F(x)}}$$

RHS:

$\frac{d}{dx}\left(-\frac{\ln(x-5)}{2}-\frac{\ln(x-4)}{2}-\frac{\ln(x-3)}{2}+\frac{\ln(x-2)}{2}+\frac{\ln(x-1)}{2}\right)=-\frac{d}{dx}\left(\frac{\ln(x-5)}{2}\right)-\frac{d}{dx}\left(\frac{\ln(x-4)}{2}\right)-\frac{d}{dx}\left(\frac{\ln(x-3)}{2}\right)+\frac{d}{dx}\left(\frac{\ln(x-2)}{2}\right)+\frac{d}{dx}\left(\frac{\ln(x-1)}{2}\right)$

Using Constant multiple rule,

$=\frac{1}{2}\left[-\frac{d}{dx}(\ln(x-5))-\frac{d}{dx}(\ln(x-4))-\frac{d}{dx}(\ln(x-3))+\frac{d}{dx}(\ln(x-2))+\frac{d}{dx}\ln(x-1)\right]$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right)=\frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u=(x-5)$,

$=\frac{1}{2}\left[-\left(\frac{d}{du}(\ln(u))\frac{d}{dx}\left((x-5)\right)\right)-\frac{d}{dx}(\ln(x-4))-\frac{d}{dx}(\ln(x-3))+\frac{d}{dx}(\ln(x-2))+\frac{d}{dx}\ln(x-1)\right]$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u))=\frac{1}{u}$,

$=\frac{1}{2}\left[-\left(\left(\frac{1}{u}\right)\frac{d}{dx}(x-5)\right)-\frac{d}{dx}(\ln(x-4))-\frac{d}{dx}(\ln(x-3))+\frac{d}{dx}(\ln(x-2))+\frac{d}{dx}\ln(x-1)\right]$

Putting back u value,

$=\frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\frac{d}{dx}(x-5)\right)-\frac{d}{dx}(\ln(x-4))-\frac{d}{dx}(\ln(x-3))+\frac{d}{dx}(\ln(x-2))+\frac{d}{dx}\ln(x-1)\right]$

Using Difference Rule,

$=\frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\left(\frac{d}{dx}(x)-\frac{d}{dx}(5)\right)\right)-\frac{d}{dx}(\ln(x-4))-\frac{d}{dx}(\ln(x-3))+\frac{d}{dx}(\ln(x-2))+\frac{d}{dx}\ln(x-1)\right]$

Derivative of Constant is 0,

$=\frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right)-\frac{d}{dx}(\ln(x-4))-\frac{d}{dx}(\ln(x-3))+\frac{d}{dx}(\ln(x-2))+\frac{d}{dx}\ln(x-1)\right]$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right)=\frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u=(x-4)$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\frac{d}{du}(\ln(u))\frac{d}{dx}((x-4))\right) - \frac{d}{dx}(\ln(x-3)) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{u}\right)\frac{d}{dx}((x-4))\right) - \frac{d}{dx}(\ln(x-3)) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Putting back u value,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\frac{d}{dx}((x-4))\right) - \frac{d}{dx}(\ln(x-3)) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Using Difference Rule,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\left(\frac{d}{dx}(x) - \frac{d}{dx}(4)\right)\right) - \frac{d}{dx}(\ln(x-3)) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Derivative of Constant is 0,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \frac{d}{dx}(\ln(x-3)) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u = (x-3)$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\frac{d}{du}(\ln(u))\frac{d}{dx}((x-3))\right) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{u}\right)\frac{d}{dx}((x-3))\right) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Putting back u value,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\frac{d}{dx}((x-3))\right) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Using Difference Rule,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\left(\frac{d}{dx}(x) - \frac{d}{dx}(3)\right)\right) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Derivative of Constant is 0,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \frac{d}{dx}(\ln(x-2)) + \frac{d}{dx}\ln(x-1)\right]$$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u = (x-2)$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\frac{d}{du}(\ln(u))\frac{d}{dx}\left((x-2)\right)\right) + \frac{d}{dx}\ln(x-1)\right]$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{u}\right)\frac{d}{dx}\left((x-2)\right)\right) + \frac{d}{dx}\ln(x-1)\right]$$

Putting back u value,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\frac{d}{dx}\left((x-2)\right)\right) + \frac{d}{dx}\ln(x-1)\right]$$

Using Difference Rule,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\left(\frac{d}{dx}(x) - \frac{d}{dx}(2)\right)\right) + \frac{d}{dx}\ln(x-1)\right]$$

Derivative of Constant is 0,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\right) + \frac{d}{dx}\ln(x-1)\right]$$

Using Chain Rule, $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{d}{du}\left(f(u)\right)\frac{d}{dx}\left(g(x)\right)$,
Let $u = (x-1)$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\right) + \left(\frac{d}{du}(\ln(u))\frac{d}{dx}\left((x-1)\right)\right)\right]$$

Derivative of natural logarithm is $\frac{d}{du}(\ln(u)) = \frac{1}{u}$,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\right) + \left(\left(\frac{1}{u}\right)\frac{d}{dx}((x-1))\right)\right]$$

Putting back u value,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\right) + \left(\left(\frac{1}{(x-1)}\right)\frac{d}{dx}((x-1))\right)\right]$$

Using Difference Rule,

$$= \frac{1}{2}\left[-\left(\left(\frac{1}{(x-5)}\right)\right) - \left(\left(\frac{1}{(x-4)}\right)\right) - \left(\left(\frac{1}{(x-3)}\right)\right) + \left(\left(\frac{1}{(x-2)}\right)\right) + \left(\left(\frac{1}{(x-1)}\right)\left(\frac{d}{dx}(x) - \frac{d}{dx}(1)\right)\right)\right]$$

Derivative of Constant is 0,

$$= \frac{1}{2}\left[-\left(\frac{1}{(x-5)}\right) - \left(\frac{1}{(x-4)}\right) - \left(\frac{1}{(x-3)}\right) + \left(\frac{1}{(x-2)}\right) + \left(\frac{1}{(x-1)}\right)\right]$$

$$\boldsymbol{\frac{\frac{d}{dx}(F(x))}{F(x)} = \frac{1}{2}\left[-\left(\frac{1}{(x-5)}\right) - \left(\frac{1}{(x-4)}\right) - \left(\frac{1}{(x-3)}\right) + \left(\frac{1}{(x-2)}\right) + \left(\frac{1}{(x-1)}\right)\right]}$$

$$\frac{d}{dx}(F(x)) = \frac{1}{2}\left[-\left(\frac{1}{(x-5)}\right) - \left(\frac{1}{(x-4)}\right) - \left(\frac{1}{(x-3)}\right) + \left(\frac{1}{(x-2)}\right) + \left(\frac{1}{(x-1)}\right)\right]F(x)$$

$$F'(x) = \frac{1}{2}\left[-\left(\frac{1}{(x-5)}\right) - \left(\frac{1}{(x-4)}\right) - \left(\frac{1}{(x-3)}\right) + \left(\frac{1}{(x-2)}\right) + \left(\frac{1}{(x-1)}\right)\right]F(x)$$

$$\boldsymbol{F'(x) = \frac{1}{2}\left[-\left(\frac{1}{(x-5)}\right) - \left(\frac{1}{(x-4)}\right) - \left(\frac{1}{(x-3)}\right) + \left(\frac{1}{(x-2)}\right) + \left(\frac{1}{(x-1)}\right)\right]\left(\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\right)}$$

Ans10:

(a) $F(x,y) = \frac{x^3y^2-2x+5}{e^x}$

Taking partial derivative with respect to x,

Using Product Rule, $\frac{\partial}{\partial x}(u,v) = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$,

$$\frac{\partial}{\partial x}((x^3y^2-2x+5)e^{-x}) = (x^3y^2-2x+5)\frac{\partial}{\partial x}(e^{-x}) + e^{-x}\frac{\partial}{\partial x}(x^3y^2-2x+5)$$

Using Chain Rule, $\frac{\partial}{\partial x}\left(f(g(x))\right) = \frac{\partial}{\partial u}(f(u))\frac{\partial}{\partial x}(g(x))$,
Let $u = -x$,

$$\frac{\partial}{\partial x}((x^3y^2-2x+5)e^{-x}) = (x^3y^2-2x+5)\left(\frac{\partial}{\partial u}(e^u)\frac{\partial}{\partial x}(-x)\right) + e^{-x}\frac{\partial}{\partial x}(x^3y^2-2x+5)$$

Using Derivative of the exponential, $\frac{\partial}{\partial u}(e^u) = e^u$,

$\frac{\partial}{\partial x}((x^3y^2 - 2x + 5)e^{-x}) = (x^3y^2 - 2x + 5)(-e^u) + e^{-x}\frac{\partial}{\partial x}(x^3y^2 - 2x + 5)$

Using both Sum and Difference rule,
Putting back u value,

$\frac{\partial}{\partial x}((x^3y^2 - 2x + 5)e^{-x}) = (x^3y^2 - 2x + 5)(-e^{-x}) + e^{-x}\frac{\partial}{\partial x}(x^3y^2) - \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial x}(5)$

Derivative of Constant is 0,

$$\frac{\partial}{\partial x}((x^3y^2 - 2x + 5)e^{-x}) = (x^3y^2 - 2x + 5)(-e^{-x}) + e^{-x}\frac{\partial}{\partial x}(x^3y^2) - 2\frac{\partial}{\partial x}(x)$$

Using Power Rule, $\frac{\partial}{\partial x}[x^n]$ is nx^{n-1},where $n = 3$,

$$\boldsymbol{\frac{\partial}{\partial x}\left((x^3y^2 - 2x + 5)e^{-x}\right) = (x^3y^2 - 2x + 5)(-e^{-x}) + e^{-x}(3x^2y^2) - 2}$$

Taking partial derivative with respect to y,

$\frac{\partial}{\partial y}((x^3y^2 - 2x + 5)e^{-x}) = e^{-x}\frac{\partial}{\partial y}(x^3y^2 - 2x + 5)$
Using both Sum and Difference Rule,
$= e^{-x}\left(\frac{\partial}{\partial y}(x^3y^2) - \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial y}(5)\right)$
Derivative of Constant is 0,
$= e^{-x}\left(\frac{\partial}{\partial y}(x^3y^2)\right)$
$= e^{-x}x^3\left(\frac{\partial}{\partial y}(y^2)\right)$
Using Power Rule, $\frac{\partial}{\partial y}[y^n]$ is nx^{n-1},where $n = 2$,

$= e^{-x}x^3(2y)$
$\boldsymbol{\frac{\partial}{\partial y}\left((x^3y^2 - 2x + 5)e^{-x}\right) = 2e^{-x}x^3y}$

(b) $F(x, y, z) = y^2\ln(x + 2y) - \ln(3z)(x^3 + y^2 - 4z)$

Taking partial derivative with respect to x,
Using Difference Rule,

$\frac{\partial}{\partial x}(y^2\ln(x + 2y) - \ln(3z)(x^3 + y^2 - 4z)) = y^2\frac{\partial}{\partial x}(\ln(x + 2y)) - \ln(3z)\left(\frac{\partial}{\partial x}\left((x^3 + y^2 - 4z)\right)\right)$

Using both Sum and Difference rule,

$$\frac{\partial}{\partial x}(y^2\ln(x+2y)-\ln(3z)(x^3+y^2-4z)) = y^2\frac{\partial}{\partial x}(\ln(x+2y)) - \ln(3z)\left(\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial x}(4z)\right)$$

Using Power Rule, $\frac{\partial}{\partial x}[x^n]$ is nx^{n-1},where $n=3$, $\frac{\partial}{\partial x}(x^3)=3x^2$,
Derivative of constant is 0,

$$\frac{\partial}{\partial x}(y^2\ln(x+2y)-\ln(3z)(x^3+y^2-4z)) = y^2\frac{\partial}{\partial x}(\ln(x+2y)) - \ln(3z)(3x^2)$$

Using Chain Rule, $\frac{\partial}{\partial x}\left(f(g(x))\right) = \frac{\partial}{\partial u}(f(u))\frac{\partial}{\partial x}(g(x))$,
Let $u = x+2y$,

$$\frac{\partial}{\partial x}(y^2\ln(x+2y)-\ln(3z)(x^3+y^2-4z)) = y^2\left(\frac{\partial}{\partial u}(\ln(u))\frac{\partial}{\partial x}(x+2y)\right) - \ln(3z)(3x^2)$$

Derivative of natural logarithm is $\frac{\partial}{\partial u}(\ln(u)) = \frac{1}{u}$,
Derivative of constant is 0,

$$\frac{\partial}{\partial x}(y^2\ln(x+2y)-\ln(3z)(x^3+y^2-4z)) = y^2\left(\left(\frac{1}{u}\right)\left(\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial x}(2y)\right)\right) - \ln(3z)(3x^2)$$
$$\frac{\partial}{\partial x}(y^2\ln(x+2y)-\ln(3z)(x^3+y^2-4z)) = y^2\left(\frac{1}{u}\right) - \ln(3z)(3x^2)$$

Putting back u value,

$$\boldsymbol{\frac{\partial}{\partial x}\left(y^2\ln(x+2y)-\ln(3z)\left(x^3+y^2-4z\right)\right) = \frac{y^2}{(x+2y)} - 3x^2\ln(3z)}$$

Taking partial derivative with respect to y,
Using Difference Rule,

$$\frac{\partial}{\partial y}(y^2\ln(x+2y)-\ln(3z)(x^3+y^2-4z)) = \frac{\partial}{\partial y}(y^2\ln(x+2y)) - \ln(3z)\frac{\partial}{\partial y}(x^3+y^2-4z)$$

Using both Sum and Difference Rule,

$$= \frac{\partial}{\partial y}(y^2\ln(x+2y)) - \ln(3z)\left(\frac{\partial}{\partial y}(x^3)+\frac{\partial}{\partial y}(y^2)-\frac{\partial}{\partial y}(4z)\right)$$

Derivative of Constant is 0,

$$= \frac{\partial}{\partial y}(y^2\ln(x+2y)) - \ln(3z)\left(\frac{\partial}{\partial y}(y^2)\right)$$

Using Power Rule, $\frac{\partial}{\partial y}[y^n]$ is nx^{n-1},where $n=3$, $\frac{\partial}{\partial y}(y^2)=2y$,

$= \frac{\partial}{\partial y}(y^2 \ln(x+2y)) - \ln(3z)\,(2y)$

Using Product Rule, $\frac{\partial}{\partial y}(u, v) = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$,

$= (\ln(x+2y)\frac{\partial}{\partial y}(y^2) + y^2\frac{\partial}{\partial y}(\ln(x+2y)) - \ln(3z)\,(2y)$

Using Power Rule, $\frac{\partial}{\partial y}[y^n]$ is nx^{n-1},where $n = 3$, $\frac{\partial}{\partial y}(y^2) = 2y$,

$= (\ln(x+2y)(2y) + y^2\frac{\partial}{\partial y}(\ln(x+2y)) - \ln(3z)\,(2y)$

Derivative of natural logarithm is $\frac{\partial}{\partial u}(\ln(u)) = \frac{1}{u}$,
Let $u = x + 2y$

$= (\ln(x+2y)(2y) + y^2\left(\frac{\partial}{\partial u}(\ln(u))\frac{\partial}{\partial y}(x+2y)\right) - \ln(3z)\,(2y)$
$= (\ln(x+2y)(2y) + y^2\left(\left(\frac{1}{u}\right)\frac{\partial}{\partial y}(x+2y)\right) - \ln(3z)\,(2y)$

Putting u value back,

$= (\ln(x+2y)(2y) + y^2\left(\left(\frac{1}{(x+2y)}\right)\frac{\partial}{\partial y}(x+2y)\right) - \ln(3z)\,(2y)$

Using Sum Rule,

$= (\ln(x+2y)(2y) + y^2\left(\left(\frac{1}{(x+2y)}\right)\left(\frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(2y)\right)\right) - \ln(3z)\,(2y)$

Derivative of Constant is 0,

$= (\ln(x+2y)(2y) + y^2\left(\left(\frac{1}{(x+2y)}\right)2\right) - \ln(3z)\,(2y)$

$\boldsymbol{\frac{\partial}{\partial y}\left(y^2 \ln(x+2y) - \ln(3z)\left(x^3 + y^2 - 4z\right)\right) = (2y)\ln(x+2y) + \frac{2y^2}{(x+2y)} - \ln(3z)\,(2y)}$

Taking partial derivative with respect to z,
Using Difference Rule,

$\frac{\partial}{\partial z}(y^2 \ln(x+2y) - \ln(3z)(x^3 + y^2 - 4z)) = \frac{\partial}{\partial z}(y^2 \ln(x+2y)) - \frac{\partial}{\partial z}(\ln(3z)(x^3 + y^2 - 4z))$

Derivative of Constant is 0,

$= -\frac{\partial}{\partial z}(\ln(3z)(x^3 + y^2 - 4z)$

Using Product Rule, $\frac{\partial}{\partial z}(u, v) = v\frac{\partial u}{\partial z} + u\frac{\partial v}{\partial z}$,

$= -\left(\ln(3z)\frac{\partial}{\partial z}(x^3 + y^2 - 4z) + (x^3 + y^2 - 4z)\frac{\partial}{\partial z}(\ln(3z))\right)$

Using both Sum and Difference Rule,

$= -\left(\ln(3z)\left(\frac{\partial}{\partial z}(x^3) + \frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial z}(4z)\right) + (x^3 + y^2 - 4z)\frac{\partial}{\partial z}(\ln(3z))\right)$

Derivative of Constant is 0,

$= -\ln(3z)\left(-\frac{\partial}{\partial z}(4z)\right) - (x^3 + y^2 - 4z)\frac{\partial}{\partial z}(\ln(3z))$
$= -\ln(3z)(-4) - (x^3 + y^2 - 4z)\frac{\partial}{\partial z}(\ln(3z))$
$= 4\ln(3z) - (x^3 + y^2 - 4z)\frac{\partial}{\partial z}(\ln(3z))$

Derivative of natural logarithm is $\frac{\partial}{\partial u}(\ln(u)) = \frac{1}{u}$,
Let $u = 3z$,

$= 4\ln(3z) - (x^3 + y^2 - 4z)\left(\frac{\partial}{\partial u}(\ln(u))\frac{\partial}{\partial z}(3z)\right)$
$= 4\ln(3z) - (x^3 + y^2 - 4z)\left(\left(\frac{1}{u}\right)\frac{\partial}{\partial z}(3z)\right)$

Putting u value back,

$= 4\ln(3z) - (x^3 + y^2 - 4z)\left(\left(\frac{1}{3z}\right)\frac{\partial}{\partial z}(3z)\right)$
$= 4\ln(3z) - (x^3 + y^2 - 4z)\left(\left(\frac{1}{3z}\right)3\right)$
$= 4\ln(3z) - \frac{(x^3+y^2-4z)}{z}$

$$\boldsymbol{\frac{\partial}{\partial z}\left(y^2\ln(x + 2y) - \ln(3z)(x^3 + y^2 - 4z)\right) = 4\ln(3z) - \frac{(x^3+y^2-4z)}{z}}$$

Ans 11: Given:

Hours after 7 PM	0	5	10	15	20	25	30
Rainfall Depth (mm)	250	290	330	240	300	350	400

(a) Central difference for approximating the slope of rainfall depth using numerical differentiation at hour 12.5 and at hour 22.5,

$f'(a) = \frac{f(a+h)-f(a-h)}{2h}$

$$f'(12.5) = \frac{f(12.5+2.5)-f(12.5-2.5)}{2\times 2.5}$$
$$= \frac{f(15)-f(10)}{5}$$
$$= \frac{240-330}{5}$$
$$= \frac{-90}{5}$$
$$\boldsymbol{f'(12.5) = -18\ mm}$$

$$f'(22.5) = \frac{f(22.5+2.5)-f(22.5-2.5)}{2\times 2.5}$$
$$= \frac{f(25)-f(20)}{5}$$
$$= \frac{350-300}{5}$$
$$= \frac{50}{5}$$
$$\boldsymbol{f'(22.5) = 10\ mm}$$

(b) The slope of rainfall depth at hour 15 using forward numerical differentiation,

$$f'(a) = \frac{f(a+h)-f(a)}{h}$$
$$f'(15) = \frac{f(15+5)-f(15)}{5}$$
$$= \frac{f(20)-f(15)}{5}$$
$$= \frac{300-240}{5}$$
$$= \frac{60}{5}$$
$$\boldsymbol{f'(15) = 12\ mm}$$

(c) The slope of rainfall depth at hour 15 using backward numerical differentiation,

$$f'(a) = \frac{f(a)-f(a-h)}{h}$$
$$f'(15) = \frac{f(15)-f(15-5)}{5}$$
$$= \frac{f(15)-f(10)}{5}$$
$$= \frac{240-330}{5}$$
$$= \frac{-90}{5}$$
$$\boldsymbol{f'(15) = -18\ mm}$$