Name: Janmejay Mohanty Course: CS 556 B Homework2
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Solutions

Ans1: $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Matrix A_1 is not in Reduced Row Echelon Form because of the following reasons which are:

(a) Row of all zeroes are not at the bottom.

$A_2 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

Matrix A_1 is in Reduced Row Echelon Form because of the following reasons which are:

(a) Row of all zeroes are at the bottom.
(b) Zeroes must be above each pivot or pivots are only non-zero entry in each column which are clearly present in the matrix.

$A_3 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix A_1 is not in Reduced Row Echelon Form because of the following reasons which are:

(a) Zeroes must be above each pivot or pivots are only non-zero entry in each column.

Ans2:

(a) $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The rank of matrix A_1 is 3 because it contains 3 independent vectors.

(b) $A_2 = \begin{bmatrix} 3 & -1 & 7 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Making it Reduced row echelon form,

$$A_2 = \begin{bmatrix} 3 & -1 & 7 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} \\ 0 & \frac{13}{3} & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{3}{13}R_2} \begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} \\ 0 & 1 & \frac{5}{13} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & \frac{32}{13} \\ 0 & 1 & \frac{5}{13} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - \frac{5}{13}R_3} \begin{bmatrix} 1 & 0 & \frac{32}{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{32}{13}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is in reduced row echelon form.

The rank of matrix A_2 is 3 because it contains 3 independent vectors.

Ans3: The row space and column space have the same dimension r $(the\ rank\ of\ the\ matrix)$.

We know the rank of matrix A is $r = 5$ and its dimension is 7×9.

$m = 7 \; and \; n = 9.$

The dimensions for four fundamental subspaces are:

The row space, $C(A^T)$ of matrix A has dimension 5, equal to the rank.

The column space, $C(A)$ of matrix A also has dimension 5.

The null space, $N(A)$ of matrix A has dimension $n - r = 9 - 5 = 4$.

The left null space, $N(A^T)$ of matrix A has dimension $m - r = 7 - 5 = 2$.

The sum of all four dimensions= $5 + 5 + 4 + 2 = 16$.

Ans4: We know the rank of matrix A is $r = 3$ and its dimension is 3×4.

$m = 3 \; and \; n = 4.$

The column space, $C(A)$ of matrix A also has dimension 3 and a subspace of $\mathbb{R}^m = \mathbb{R}^3$.

The left null space, $N(A^T)$ of matrix A has dimension $m - r = 3 - 3 = 0$. It means that the left null space is just the zero vector.

Ans5: $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{\frac{R_1}{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 3 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$\xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 - 3R_3} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is in reduced row echelon form.

Now, finding the null space vectors,

$$AX = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Suppose, if we take $x_4 = c$, then $x_1 = 4c$, $x_2 = -7c$, $x_3 = 2c$.

Therefore, $\vec{x} = \begin{bmatrix} 4c \\ -7c \\ 2c \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 1 \end{bmatrix} c$

Ans6: $x_1 - 2x_2 - 2x_3 = b_1$

$2x_1 - 5x_2 - 4x_3 = b_2$

$4x_1 - 9x_2 - 8x_3 = b_3$

$$Ax = b$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 2 & -5 & -4 \\ 4 & -9 & -8 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 4 & -9 & -8 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

$R_3 - 4R_1$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 4b_1 \end{bmatrix}$$

$-R_2$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_1 - b_2 \\ b_3 - 4b_1 \end{bmatrix}$$

$R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 + 4b_1 - 2b_2 \\ 2b_1 - b_2 \\ b_3 - 4b_1 \end{bmatrix}$$

$R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ b_3 - 4b_1 + 2b_1 - b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ -2b_1 - b_2 + b_3 \end{bmatrix}$$

Making Augmented Matrix,

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & 2b_1 - b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right)$$

There are no solutions.

$-2b_1 - b_2 + b_3 = 0$

$x_1 - 2x_3 = 5b_1 - 2b_2$

$x_1 = 5b_1 - 2b_2 + 2x_3$ ------(1)

$-x_2 = 2b_1 - b_2$

$x_2 = -2b_1 + b_2$ ------(2)

Therefore, we get,

$x_1 = 5b_1 - 2b_2 + 2x_3$

$$x_2 = -2b_1 + b_2$$

$$x_3 = x_3$$

General Solutions for $X = \begin{pmatrix} 5b_1 - 2b_2 + 2x_3 \\ -2b_1 + b_2 \\ x_3 \end{pmatrix}$

Ans7: Projection of b onto the column space defined by matrix A,

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$A^T.A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1\times1+0\times0+0\times0 & 1\times1+0\times1+0\times0 \\ 1\times1+1\times0+0\times0 & 1\times1+1\times1+0\times0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T.b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\times2+0\times3+0\times4 \\ 1\times2+1\times3+0\times4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$\hat{\vec{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$(A^T.A).\hat{\vec{x}} = A^T.b$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$
$$x_1 + x_2 = 2 \qquad \text{---------(1)}$$
$$x_1 + 2x_2 = 5 \qquad \text{---------(2)}$$

From equations (1) and (2),

$$x_2 = 3, x_1 = -1$$

$\hat{\vec{x}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\vec{p} = A.\hat{\vec{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1+3 \\ 0+3 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$
$$\vec{e} = b - \vec{p}$$
$$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$
$$\|e\| = \sqrt{0+0+4^2} = \sqrt{4^2} = 16$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$A^T.A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+1\times1+0\times0 & 1\times1+1\times1+0\times1 \\ 1\times1+1\times1+1\times0 & 1\times1+1\times1+1\times1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A^T.b = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 1 \times 4 + 0 \times 6 \\ 1 \times 4 + 1 \times 4 + 1 \times 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$\hat{\vec{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(A^T.A).\hat{\vec{x}} = A^T.b$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$2x_1 + 2x_2 = 8 \qquad \text{----(3)}$$

$$2x_1 + 3x_2 = 14 \qquad \text{----(4)}$$

From equations (3) and (4),

$$x_2 = 6, x_1 = -2$$

$$\hat{\vec{x}} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\vec{p} = A.\hat{\vec{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 + 6 \\ -2 + 6 \\ 0 + 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$$\vec{e} = b - \vec{p}$$

$$= \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\|e\| = \sqrt{0 + 0 + 0} = \sqrt{0} = 0$$

Ans8: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$

Transforming matrix A into reduced row echelon form,

$R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & -2 & 3 \end{bmatrix}$$

$\frac{R_2}{2}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix}$$

$R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_1 - R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\frac{R_3}{3}$

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$R_1 - 2R_2$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Now, A is a 3×3 matrix with 3 pivot columns.

Matrix A has 3 linear independent vectors.

Also, column of matrix A span in $\mathbb{R}^3$.

Therefore, Matrix A basis for the column space of A,

$$\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\} = \left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-2\end{bmatrix}, \begin{bmatrix}3\\-3\\3\end{bmatrix}\right\}$$

Now, using the Gram-Schmidt Process to find an orthogonal basis:

Let's take $X = \overrightarrow{a_1} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$.

Constructing Y, starting with $\overrightarrow{a_2}$ and subtracting its projection along X.

$$Y = \overrightarrow{a_2} - \frac{X^T.\overrightarrow{a_2}}{X^T.X}.X = \begin{bmatrix}2\\0\\-2\end{bmatrix} - \frac{[1 \quad -1 \quad 0].\begin{bmatrix}2\\0\\-2\end{bmatrix}}{[1 \quad -1 \quad 0].\begin{bmatrix}1\\-1\\0\end{bmatrix}}.\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix} - \frac{[1\times2-1\times0+0\times-2]}{[1\times1-1\times-1+0\times0]}.\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$= \begin{bmatrix}2\\0\\-2\end{bmatrix} - \frac{[2-0-0]}{[1+1+0]}.\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix} - \frac{2}{2}.\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}2-1\\0+1\\-2-0\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$Z = \overrightarrow{a_3} - \frac{X^T.\overrightarrow{a_3}}{X^T.X}.X - \frac{Y^T.\overrightarrow{a_3}}{Y^T.Y}.Y = \begin{bmatrix}3\\-3\\3\end{bmatrix} - \frac{[1 \quad -1 \quad 0].\begin{bmatrix}3\\-3\\3\end{bmatrix}}{[1 \quad -1 \quad 0].\begin{bmatrix}1\\-1\\0\end{bmatrix}}.\begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{[1 \quad 1 \quad -2].\begin{bmatrix}3\\-3\\3\end{bmatrix}}{[1 \quad 1 \quad -2].\begin{bmatrix}1\\1\\-2\end{bmatrix}}.\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} - \frac{[1\times3-1\times-3+0\times3]}{[1\times1-1\times-1+0\times0]}.\begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{[1\times3+1\times-3-2\times3]}{[1\times1+1\times1-2\times-2]}.\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} - \frac{[3+3+0]}{[1+1+0]}.\begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{[3-3-6]}{[1+1+4]}.\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}3\\-3\\3\end{bmatrix} - \frac{6}{2}.\begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{-6}{6}.\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} - 3.\begin{bmatrix}1\\-1\\0\end{bmatrix} + 1.\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}3\\-3\\3\end{bmatrix} - \begin{bmatrix}3\\-3\\0\end{bmatrix} + \begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}3-3+1\\-3+3+1\\3-0-2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

The three orthonormal vectors: $q_1 = \frac{X}{\|X\|} = \frac{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\sqrt{1^2+(-1)^2+0^2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$

$q_2 = \frac{Y}{\|Y\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}{\sqrt{1^2+1^2+(-2)^2}} = \frac{1}{\sqrt{6}} . \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$

$q_3 = \frac{Z}{\|Z\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}} . \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

Ans9: $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

Calculating the eigenvalues and eigenvectors of matrix A,

$$A - \lambda I = 0$$

$$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix} = 0 \qquad \text{--------(1)}$$

Finding the determinant of equation (1),

$$\begin{vmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(3-\lambda) - (2 \times -1) = 0$$

$$18 - 6\lambda - 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 9\lambda + 20 = 0$$

$$\lambda^2 - 5\lambda - 4\lambda + 20 = 0$$

$$\lambda(\lambda - 5) - 4(\lambda - 5) = 0$$

$$(\lambda - 5)(\lambda - 4) = 0$$

The roots are $\lambda_1 = 5, \lambda_2 = 4$

These are the eigenvalues.

Next, finding the eigenvectors,

For $\lambda_1 = 5$,

$$\begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 6-5 & -1 \\ 2 & 3-5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

Finding the reduced row echelon form of $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$,

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Computing the null space of reduced row echelon form of matrix $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose, if we take $x_2 = c$, then $x_1 = c$.

Therefore, $\vec{x} = \begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c$.

This is the null space.

Therefore, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector.

For $\lambda_2 = 4$,

$$\begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 6-4 & -1 \\ 2 & 3-4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

Finding the reduced row echelon form of $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$,

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \xrightarrow{\frac{R_1}{2}} \begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Computing the null space of reduced row echelon form of matrix $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose, if we take $x_2 = c$, then $x_1 = \frac{c}{2}$.

Therefore, $\vec{x} = \begin{bmatrix} \frac{c}{2} \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} c$.

This is the null space.

Therefore, $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is the eigenvector.

Form the matrix P, whose column i is eigenvector number i: $P = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$.

Form the diagonal matrix D, whose element are row i, column i is eigenvalue number i:

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

Both matrices P and D are such that the initial matrix $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = PDP^{-1}$.

Ans10: $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

Finding the transpose of matrix A,

$$A^T = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

Now multiplying the matrix A with its transpose A^T,

$$W = A.A^T = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 3 \times 3 & 0 \times 2 + 3 \times 2 \\ 0 \times 2 + 2 \times 3 & 0 \times 0 + 2 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 9 & 0 + 6 \\ 0 + 6 & 0 + 4 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}$$

Now, finding both eigenvalues and eigenvectors of W,

$$W - \lambda I = 0$$

$$\begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 13 - \lambda & 6 \\ 6 & 4 - \lambda \end{bmatrix} = 0 \qquad \text{--------(1)}$$

Finding the determinant of equation (1),

$$\begin{vmatrix} 13 - \lambda & 6 \\ 6 & 4 - \lambda \end{vmatrix} = 0$$

$$(13 - \lambda)(4 - \lambda) - (6 \times 6) = 0$$

$$52 - 13\lambda - 4\lambda + \lambda^2 - 36 = 0$$

$$\lambda^2 - 17\lambda + 16 = 0$$

$$\lambda^2 - 16\lambda - 1\lambda + 16 = 0$$

$$\lambda(\lambda - 16) - 1(\lambda - 16) = 0$$

$$(\lambda - 16)(\lambda - 1) = 0$$

The roots are $\lambda_1 = 16, \lambda_2 = 1$

These are the eigenvalues.

Next, finding the eigenvectors,

For $\lambda_1 = 16$,

$$\begin{bmatrix} 13 - \lambda & 6 \\ 6 & 4 - \lambda \end{bmatrix} = \begin{bmatrix} 13 - 16 & 6 \\ 6 & 4 - 16 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix}$$

Finding the reduced row echelon form of $\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix}$,

$$\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} \xrightarrow{-\frac{R_1}{3}} \begin{bmatrix} 1 & -2 \\ 6 & -12 \end{bmatrix} \xrightarrow{R_2 - 6R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Computing the null space of reduced row echelon form of matrix $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose, if we take $x_2 = c$, then $x_1 = 2c$.

Therefore, $\vec{x} = \begin{bmatrix} 2c \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} c.$

This is the null space.

Therefore, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigenvector.

For $\lambda_2 = 1$,

$$\begin{bmatrix} 13-\lambda & 6 \\ 6 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 13-1 & 6 \\ 6 & 4-1 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$$

Finding the reduced row echelon form of $\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$,

$$\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix} \overset{\frac{R_1}{12}}{\rightarrow} \begin{bmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{bmatrix} \xrightarrow{R_2-6R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Computing the null space of reduced row echelon form of matrix $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose, if we take $x_2 = c$, then $x_1 = -\frac{c}{2}$.

Therefore, $\vec{x} = \begin{bmatrix} -\frac{c}{2} \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} c.$

This is the null space.

Therefore, $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is the eigenvector.

The columns of the U matrix is the normalized $(unit)$ vectors:

$$U = \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{2}.\frac{\sqrt{4}}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 1.\frac{\sqrt{4}}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{2}.\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 1.\frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\Sigma^2 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

The Σ matrix is a zero matrix with σ_i on its diagonal:

$$\Sigma = \begin{bmatrix} \sqrt{16} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, $v_i = \frac{1}{\sigma_i}.A^T.u_i$

$$v_1 = \frac{1}{\sigma_1}.A^T.u_1 = \frac{1}{4}.\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}^T.\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \frac{1}{4}.\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}.\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}.\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times \frac{2}{\sqrt{5}} + 0 \times \frac{1}{\sqrt{5}} \\ \frac{3}{4} \times \frac{2}{\sqrt{5}} + \frac{1}{2} \times \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{3}{2\sqrt{5}} + \frac{1}{2\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{4}{2\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{bmatrix}$$

$$v_2 = \frac{1}{\sigma_2}.A^T.u_2 = \frac{1}{1}.\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}^T.\begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}.\begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 \times -\frac{1}{\sqrt{5}} + 0 \times \frac{2}{\sqrt{5}} \\ 3 \times -\frac{1}{\sqrt{5}} + 2 \times \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} + \frac{4}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix}$$

$$V = [v_1 \quad v_2] = \begin{bmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

Therefore, all three matrices U, Σ and V are such way that the initial matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = U\Sigma V^T$$

Ans11:

(a) R(User1, Amelie) = UUser1.S.VT = 1.5
(b) The Strength of concept 1 in SVD decomposition: 13.27
(c) Average rating of Harry Potter: [[2.41666667]]
(d) Casablanca has overall highest rating: 2.5833333333333335

Ans12:

(b) Total percentage of variance captured by the first 2 components of PCA: 99.99999999999999

(c) The strength of each of the two principal components are: 1.006711409395973 , 1.0067114093959741

(d) The magnitude of each of the two principal components are: 1.0136452424516673 , 1.0136452424516684