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CS 556-B: Mathematical Foundations of Machine Learning
Homework 1: Linear Algebra (100 points)

Note: Calculators allowed for trigonometric operations & arithmetic operations
(i.e., addition, subtraction, multiplication or division of *scalars*). All solutions methods must be full explained.

Vectors

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1. (5 points) Find the magnitude of the vector $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 2 \\ 4 \\ -4 \end{bmatrix}$

2. (5 points) Consider two vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (in vector space $\mathbb{R}^2$), what is their span? Briefly explain your reasoning leveraging the definition of the span of a set of vectors.

Dot Product

3. (10 points) If two vectors $\mathbf{a}$, $\mathbf{b}$ have magnitudes 3 and 5 respectively and the angle between them is $\frac{\pi}{2}$ radians, what is their dot product?

4. (10 points) Let vector $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, calculate the dot product of $\mathbf{u}$ and $\mathbf{v}$ also calculate the angle between (i.e., not the cosine of the angle but the actual angle in radians or degrees) $\mathbf{u}$ and $\mathbf{v}$.

Linear Independence

5. (15 points) Check if the vectors $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix}$ are linearly independent. Note: The condition for linear independence is that given a set S of vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and coefficients a, b, c, $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = 0$ if and only if $a = b = c = 0$.

6. (15 points) Given a subset of vectors $S = \{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k\}$ for $k \in \mathbb{N}$ of a vector space V, prove that S is linearly independent iff a linear combination of elements of S with non-zero coefficients does not yield $\mathbf{0}$. **Hint**: To prove *iff* statements, i.e., A **iff** B (A $\iff$ B), first prove A $\rightarrow$ B, then prove B $\leftarrow$ A.

Matrices

7. (10 points) Demonstrate the distributive property of matrix multiplication over addition.
Given $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$, demonstrate: A(B+C) = AB + AC

8. (15 points) Calculate the inverse of matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -4 & 1 \\ 5 & -3 & 2 \end{bmatrix}$. Note: It is acceptable to leave the final solution with fractional entities in the matrix (i.e., no requirement to convert fractions to decimal numbers).

Change of Bases

9. (15 points) Consider the three columns in matrix A (problem 8) to be our new basis of interest in R^3. If a vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ defined on the natural basis in R^3 (i.e., $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$), how would vector $\mathbf{x}$ be represented in the basis defined by the matrix A in problem 8.