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CS-556-B

Solutions

Ans1: $x = \begin{bmatrix} 2 \\ -3 \\ 2 \\ 4 \\ -4 \end{bmatrix}$

$$\|x\| = \sqrt{2^2 + (-3)^2 + 2^2 + 4^2 + (-4)^2}$$
$$= \sqrt{4 + 9 + 4 + 16 + 16}$$
$$= \sqrt{49}$$
$$= 7$$

Ans2: Given,

Two vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(in\ Vector\ space\ R^2)$.

The span is $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is the set of linear combinations of the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In other terms, span $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \end{bmatrix} : a, b \in R\right\}$.

So, Span = all linear combinations of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

As, the (linear) span is a vector space. By definition it is the smallest vector space that contains all the elements in the set. In particular it includes all linear combinations of those elements (and will in fact contain exactly all linear combinations that can be formed with those elements).

Hence, the two vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ forms the standard basis of the vector space R^2.

Therefore, they span R^2 and so any two linearly dependent vectors in R^2.

Ans3: Given,

$$\|a\| = 3, \|b\| = 5,$$
$$\theta = \frac{\pi}{2}\ radians$$

Dot Product of vectors a and b :

$$a.b = \|a\|\|b\| \cos\theta$$
$$= 3 \times 4 \times \cos\frac{\pi}{2}$$
$$= 12\ \times 0$$
$$= 0$$

Ans4: Given,

$$u = \begin{bmatrix}1\\3\end{bmatrix}, v = \begin{bmatrix}-2\\7\end{bmatrix}$$

$$u.v = (1 \times -2) + (3 \times 7) = -2 + 21 = 19$$

$$\|u\| = \sqrt{1^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$$

$$\|v\| = \sqrt{(-2)^2 + 7^2} = \sqrt{4+49} = \sqrt{53}$$

Using Dot Product formula,

$$\cos\theta = \frac{u.v}{\|u\| \times \|v\|} = \frac{19}{\sqrt{10} \times \sqrt{53}} = \frac{19}{\sqrt{530}} = \frac{19}{23.02} = 0.83^{\circ}$$

$$\theta = \cos^{-1}(0.83) = 34.38^{\circ}$$

Ans5: Given,

$$x = \begin{bmatrix}-1\\0\\2\end{bmatrix},\ y = \begin{bmatrix}3\\-2\\2\end{bmatrix},\ z = \begin{bmatrix}5\\2\\6\end{bmatrix}$$

If $a\begin{bmatrix}-1\\0\\2\end{bmatrix} + b\begin{bmatrix}3\\-2\\2\end{bmatrix} + c\begin{bmatrix}5\\2\\6\end{bmatrix} = 0$ has the only solution $a = b = c = 0$, then a, b, c are linearly independent.

Forming a matrix and row reducing it,

$$\begin{bmatrix}-1 & 3 & 5\\0 & -2 & 2\\2 & 2 & 6\end{bmatrix} \xrightarrow{R1\times(-1)} \begin{bmatrix}1 & -3 & -5\\0 & -2 & 2\\2 & 2 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & -5\\0 & -2 & 2\\2 & 2 & 6\end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix}1 & -3 & -5\\0 & -2 & 2\\0 & 8 & 10\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & -5\\0 & -2 & 2\\0 & 8 & 10\end{bmatrix} \xrightarrow{R2\times\left(\frac{-1}{2}\right)} \begin{bmatrix}1 & -3 & -5\\0 & 1 & -1\\0 & 8 & 10\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & -5\\0 & 1 & -1\\0 & 8 & 10\end{bmatrix} \xrightarrow{R3-8R2} \begin{bmatrix}1 & -3 & -5\\0 & 1 & -1\\0 & 0 & 18\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & -5\\0 & 1 & -1\\0 & 0 & 18\end{bmatrix} \xrightarrow{R1+3R2} \begin{bmatrix}1 & 0 & -8\\0 & 1 & -1\\0 & 0 & 18\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -5\\0 & 1 & -1\\0 & 0 & 18\end{bmatrix} \xrightarrow{R3\times\left(\frac{1}{18}\right)} \begin{bmatrix}1 & 0 & -5\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -5\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix}1 & 0 & -5\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -5\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \xrightarrow{R1+5R1} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Hence, $ax + by + cz = 0$ only $a = b = c = 0$.

Therefore, x, y, z are linearly independent vectors.

Ans6:

Part 1: Suppose there is a subset S of a vector space V is linearly independent. We need to prove that 0 cannot be expressed as a linear combination of elements of S with non-zero coefficients.

By contradiction, assume that

$$x_1 s_1 + x_2 s_2 + x_3 s_3 + \cdots + x_n s_n = 0$$

Where $s_1, s_2, s_3, \dots, s_n$ are elements of S and not all coefficients are equal to 0. Let x_i be a non-zero coefficient.

By subtracting $x_i s_i$ from both sides and dividing the resulting equality by $-x_i$ we get:

$$\frac{1}{x_i}(x_1 s_1 + x_2 s_2 + x_3 s_3 + \cdots + x_n s_n) = s_i$$

where s_i is missing in the left-hand sum. This equality implies that s_i is a linear combination of other elements of S. We obtained a contradiction with our assumption that S is linearly independent, which completes the proof.

Part 2: Now we need to prove that if 0 cannot be expressed as a linear combination of elements of S with non-zero coefficients then S is linearly independent. By contradiction, suppose that S is not linearly independent. Then there exists an element s in S which is equal to a linear combination of other elements of S:

$$x_1 s_1 + x_2 s_2 + x_3 s_3 + \cdots + x_n s_n = s$$

By subtracting s from both sides, we get:

$$x_1 s_1 + x_2 s_2 + x_3 s_3 + \cdots + x_n s_n - s = 0$$

Thus, we have a linear combination of distinct elements of S which is equal to 0 and not all coefficients of this linear combination are equal to zero (for example, the coefficient of s is -1).

Hence Proved.

Ans7: Given,

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$$

Demonstrate: $A(B + C) = AB + AC$

LHS:

$$\begin{aligned} A(B + C) &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 6+1 & 2-2 \\ 3+4 & 2-1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 7 & 0 \\ 7 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} (3\times 7)+(4\times 7) & (3\times 0)+(4\times 1) \\ (1\times 7)+(2\times 7) & (1\times 0)+(2\times 1) \end{bmatrix}$$

$$= \begin{bmatrix} 21+28 & 0+4 \\ 7+14 & 0+2 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 4 \\ 21 & 2 \end{bmatrix}$$

RHS:

$$AB + AC = \left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix} \right) + \left(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} (3\times 6)+(4\times 3) & (3\times 2)+(4\times 2) \\ (1\times 6)+(2\times 3) & (1\times 2)+(2\times 2) \end{bmatrix} \right) + \left(\begin{bmatrix} (3\times 1)+(4\times 4) & (3\times -2)+(4\times -1) \\ (1\times 1)+(2\times 4) & (1\times -2)+(2\times -1) \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 18+12 & 6+8 \\ 6+6 & 2+4 \end{bmatrix} \right) + \left(\begin{bmatrix} 3+16 & -6-4 \\ 1+8 & -2-2 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 30 & 14 \\ 12 & 6 \end{bmatrix} \right) + \left(\begin{bmatrix} 19 & -10 \\ 9 & -4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 30+19 & 14-10 \\ 12+9 & 6-4 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & 4 \\ 21 & 2 \end{bmatrix}$$

As LHS = RHS,

Hence Proved.

Ans8: Given,

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -4 & 1 \\ 5 & -3 & 2 \end{bmatrix}$$

Calculating Minors,

$$a_{11} = (-4\times 2) - (-3\times 1) = -8+3 = -5$$

$$a_{12} = (-2\times 2) - (5\times 1) = -4-5 = -9$$

$$a_{13} = (-2\times -3) - (5\times -4) = 6+20 = 26$$

$$a_{21} = (1\times 2) - (-3\times 2) = 2+6 = 8$$

$$a_{22} = (3\times 2) - (5\times 2) = 6-10 = -4$$

$$a_{23} = (3\times -3) - (5\times 1) = -9-5 = -14$$

$$a_{31} = (1\times 1) - (-4\times 2) = 1+8 = 9$$

$$a_{32} = (3\times 1) - (-2\times 2) = 3+4 = 7$$

$$a_{33} = (3\times -4) - (-2\times 1) = -12+2 = -10$$

$$Minor = \begin{bmatrix} -5 & -9 & 26 \\ 8 & -4 & -14 \\ 9 & 7 & -10 \end{bmatrix}$$

$$Cofactors = \begin{bmatrix} -5 & 9 & 26 \\ -8 & -4 & 14 \\ 9 & -7 & -10 \end{bmatrix}$$

$$Adjugate = \begin{bmatrix} -5 & -8 & 9 \\ 9 & -4 & -7 \\ 26 & 14 & -10 \end{bmatrix}$$

Calculating Determinant,

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & -4 & 1 \\ 5 & -3 & 2 \end{vmatrix}$$

$= (-1)^{1+1} \times 3[(-4 \times 2) - (-3 \times 1)] + (-1)^{1+2} \times 1[(-2 \times 2) - (5 \times 1)]$

$+(-1)^{1+3} \times 2[(-2 \times -3) - (5 \times -4)]$

$= 1 \times 3(-8 + 3) - 1 \times 1(-4 - 5) + 1 \times 2(6 + 20)$

$= (3 \times -5) - (-9) + (2 \times 26)$

$= -15 + 9 + 52$

$= 46$

Calculating Inverse of Matrix A,

$$A^{-1} = \frac{1}{|A|} \times Adjugate\ of\ Matrix\ A$$

$$= \frac{1}{46} \times \begin{bmatrix} -5 & -8 & 9 \\ 9 & -4 & -7 \\ 26 & 14 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{46} & \frac{-8}{46} & \frac{9}{46} \\ \frac{9}{46} & \frac{-4}{46} & \frac{-7}{46} \\ \frac{26}{46} & \frac{14}{46} & \frac{-10}{46} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{46} & \frac{-4}{23} & \frac{9}{46} \\ \frac{9}{46} & \frac{-2}{23} & \frac{-7}{46} \\ \frac{13}{23} & \frac{7}{23} & \frac{-5}{23} \end{bmatrix}$$

Ans9: Given,

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -4 & 1 \\ 5 & -3 & 2 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Calculating Minors,

$$a_{11} = (-4 \times 2) - (-3 \times 1) = -8 + 3 = -5$$

$$a_{12} = (-2 \times 2) - (5 \times 1) = -4 - 5 = -9$$

$$a_{13} = (-2 \times -3) - (5 \times -4) = 6 + 20 = 26$$

$$a_{21} = (1 \times 2) - (-3 \times 2) = 2 + 6 = 8$$

$$a_{22} = (3 \times 2) - (5 \times 2) = 6 - 10 = -4$$

$$a_{23} = (3 \times -3) - (5 \times 1) = -9 - 5 = -14$$

$$a_{31} = (1 \times 1) - (-4 \times 2) = 1 + 8 = 9$$

$$a_{32} = (3 \times 1) - (-2 \times 2) = 3 + 4 = 7$$

$$a_{33} = (3 \times -4) - (-2 \times 1) = -12 + 2 = -10$$

$$Minor = \begin{bmatrix} -5 & -9 & 26 \\ 8 & -4 & -14 \\ 9 & 7 & -10 \end{bmatrix}$$

$$Cofactors = \begin{bmatrix} -5 & 9 & 26 \\ -8 & -4 & 14 \\ 9 & -7 & -10 \end{bmatrix}$$

$$Adjugate = \begin{bmatrix} -5 & -8 & 9 \\ 9 & -4 & -7 \\ 26 & 14 & -10 \end{bmatrix}$$

Calculating Determinant,

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & -4 & 1 \\ 5 & -3 & 2 \end{vmatrix}$$

$$= (-1)^{1+1} \times 3[(-4 \times 2) - (-3 \times 1)] + (-1)^{1+2} \times 1[(-2 \times 2) - (5 \times 1)]$$

$$+(-1)^{1+3} \times 2[(-2 \times -3) - (5 \times -4)]$$

$$= 1 \times 3(-8 + 3) - 1 \times 1(-4 - 5) + 1 \times 2(6 + 20)$$

$$= (3 \times -5) - (-9) + (2 \times 26)$$

$$= -15 + 9 + 52$$

$$= 46$$

Calculating Inverse of Matrix A,

$$A^{-1} = \frac{1}{|A|} \times Adjugate\ of\ Matrix\ A$$

$$= \frac{1}{46} \times \begin{bmatrix} -5 & -8 & 9 \\ 9 & -4 & -7 \\ 26 & 14 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{46} & \frac{-8}{46} & \frac{9}{46} \\ \frac{9}{46} & \frac{-4}{46} & \frac{-7}{46} \\ \frac{26}{46} & \frac{14}{46} & \frac{-10}{46} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{46} & \frac{-4}{23} & \frac{9}{46} \\ \frac{9}{46} & \frac{-2}{23} & \frac{-7}{46} \\ \frac{13}{23} & \frac{7}{23} & \frac{-5}{23} \end{bmatrix}$$

Representing changed basis for defined matrix A,

$$x_e = \begin{bmatrix} \frac{-5}{46} & \frac{-4}{23} & \frac{9}{46} \\ \frac{9}{46} & \frac{-2}{23} & \frac{-7}{46} \\ \frac{13}{23} & \frac{7}{23} & \frac{-5}{23} \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{46} - \frac{4(2)}{23} + \frac{9(3)}{46} \\ \frac{9}{46} - \frac{2(2)}{23} - \frac{7(3)}{46} \\ \frac{13}{23} + \frac{7(2)}{23} - \frac{5(3)}{23} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{23} \\ \frac{-10}{23} \\ \frac{12}{23} \end{bmatrix}$$