PHIL 222
Philosophical Foundations of Computer Science
Week 3, Thursday

Sept. 12, 2024

Recursive Functions

Advice

This is a technical part of the course that will be covered in the technical exercises and the midterm exam.

If anything here does not "click" in your mind,

- *Come to see me in office hours & appointments!*

You are only expected to learn what the rule of the game is like. It is absolutely natural if it does not "click" in your mind for the first time you see it. But to resolve that situation, you need interactive help. Please, please help me help you.

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let's shift our focus from "What computation looks like?"

to "What can be computed?"

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- $\mathrm{double} : n \mapsto 2n$.
- $\mathrm{IsEven}_{\mathrm{semi}} : n \mapsto \begin{cases} n & \text{if } n \text{ is even,} \\ \text{undefined} & \text{if } n \text{ is odd.} \end{cases}$

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But then we know the following are Turing computable, too:

- $\mathrm{double} \circ \mathrm{add} : (m, n) \mapsto 2(m + n)$.
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We start from simple, obviously computable functions, and show more complicated ones to be computable, by showing that they can be built from simple ones by simple operations (e.g. composition).

The Basic Six

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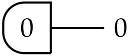

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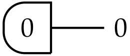

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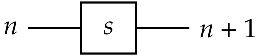

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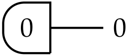

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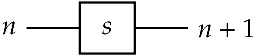

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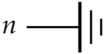

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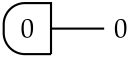

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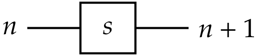

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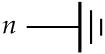

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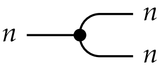

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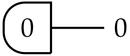

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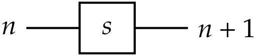

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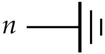

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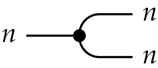

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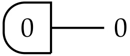

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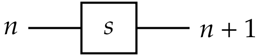

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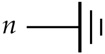

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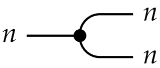

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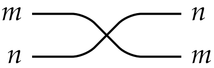

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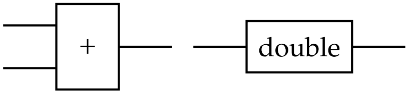
- 

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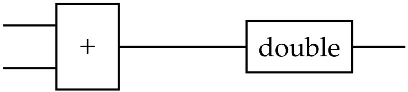
- 

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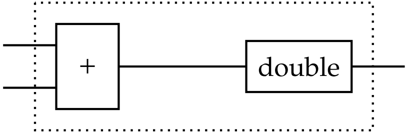
- 

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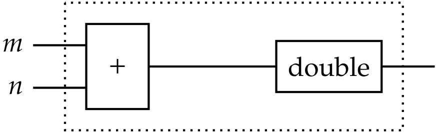
- 

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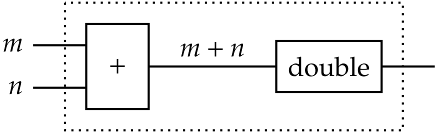
- 

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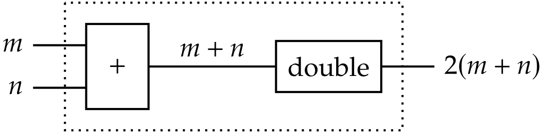
- 

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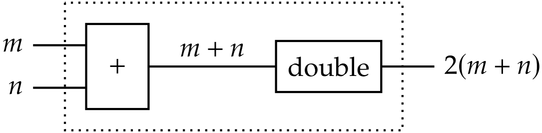
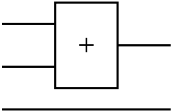
- 

- 

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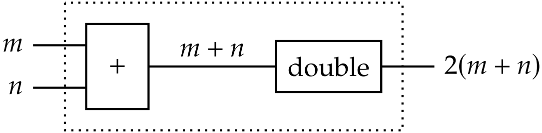
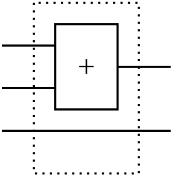
- 

- 

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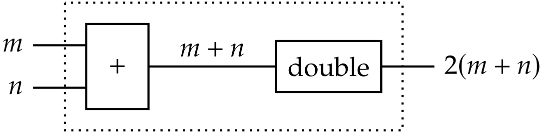
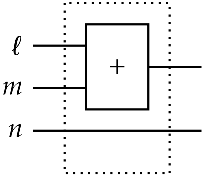
- 

- 

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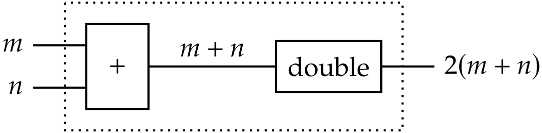
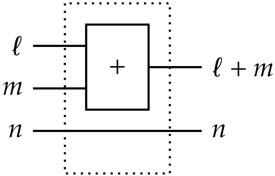
- 

- 

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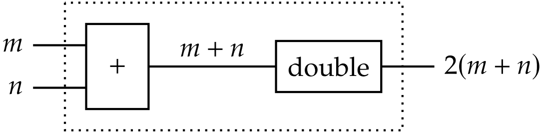
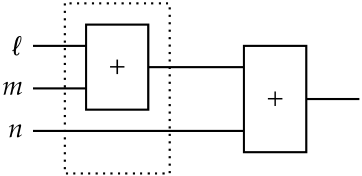
- 

- 

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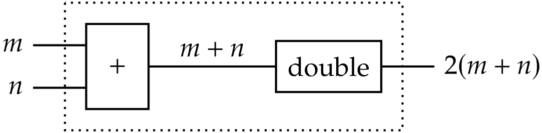
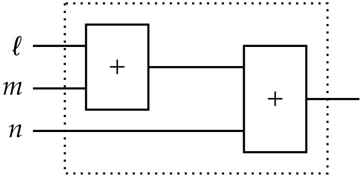
- 

- 

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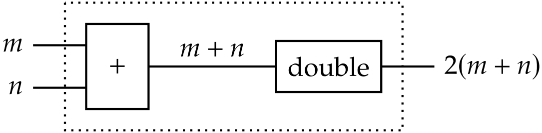
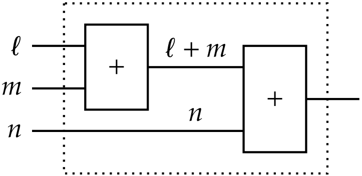
- 

- 

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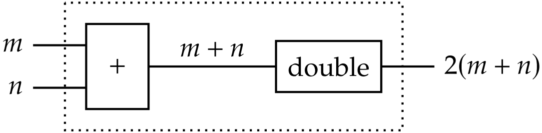
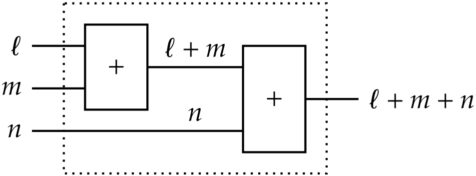
- 

- 

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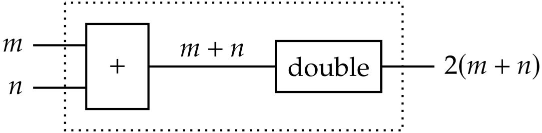
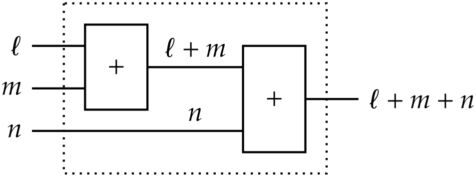
- 

- 

If parts are computable so is their composition!

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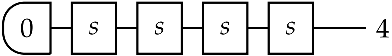

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  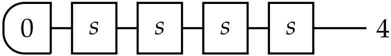

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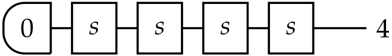

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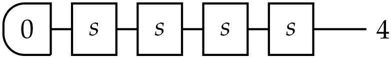

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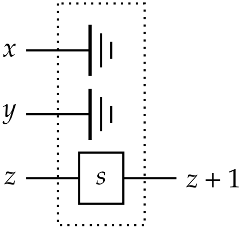

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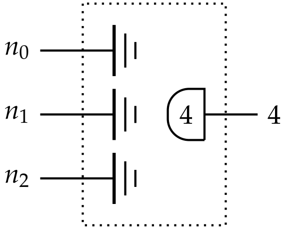

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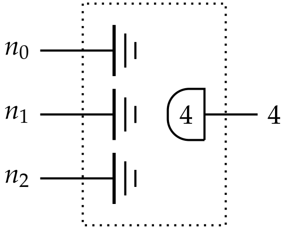

Indeed, every constant function $n : \mathbb{N}^k \to \mathbb{N} :: (n_0, \dots, n_{k-1}) \mapsto n$ is computable.

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Our next order of business is to introduce another way of constructing new functions, and to use it to create + or $\times$.

Recursive Functions:
Primitive Recursion

Recursion

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This is to compute exp2(n) by

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We can regard this as the following program.

```
exp2(0) := 1
for i in (0, ..., n-1):
    exp2(i+1) := double(exp2(i))
return exp2(n)
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h(0) := f
for i in (0, ..., n-1):
    h(i+1) := g(h(i))
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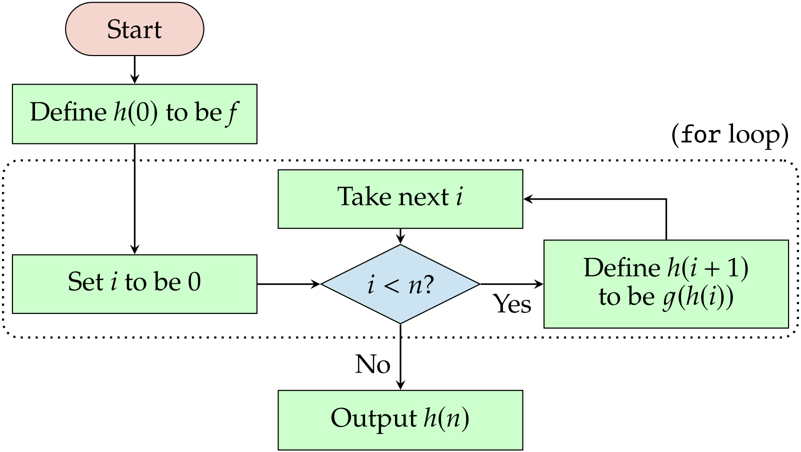
return h(n)
```



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$$\text{factorial} :: n \mapsto n! = 1 \times 2 \times 3 \times \cdots \times n.$$

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$$\text{factorial} :: n \mapsto n! = 1 \times 2 \times 3 \times \cdots \times n.$$

$$\begin{aligned} 0! &= & & & 1 \\ 1! &= 0! \times 1 &= 1 \times 1 &= 1 \\ 2! &= 1! \times 2 &= 1 \times 2 &= 2 \\ 3! &= 2! \times 3 &= 2 \times 3 &= 6 \end{aligned}$$

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$$\begin{aligned}
0! &= & & & 1 \\
1! &= 0! \times 1 &= 1 \times 1 &= 1 \\
2! &= 1! \times 2 &= 1 \times 2 &= 2 \\
3! &= 2! \times 3 &= 2 \times 3 &= 6 \\
4! &= 3! \times 4 &= 6 \times 4 &= 24
\end{aligned}$$

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- **base step**: start with 1, and
- **inductive step**: in the ith iteration, apply the function $- \times (i+1)$ to the previous value $i!$ — iterate this n times.

New: for each $i < n$, the ith iteration of the ind. step also depends on i.

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```
factorial(0) := 1
for i in (0, ..., n-1):
    factorial(i+1) := multiply(factorial(i), i+1)
return factorial(n)
```

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10 + 0 &= & 10 \\
10 + 1 &= (10 + 0) + 1 = 10 + 1 = & 11 \\
10 + 2 &= (10 + 1) + 1 = 11 + 1 = & 12 \\
10 + 3 &= (10 + 2) + 1 = 12 + 1 = & 13
\end{aligned}$$

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- **base step**: start with x, and
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New: add takes a parameter x as an argument in addition to n.

```
add(x, 0) := x
for i in (0, ..., n-1):
    add(x, i+1) := add(x, i)+1
return add(x, n)
```

$$\exp :: (x, n) \mapsto x^n.$$

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$$\begin{aligned} 10^0 &= & & & 1 \\ 10^1 &= 10^0 \times 10 &=& \quad 1 \times 10 &= 10 \\ 10^2 &= 10^1 \times 10 &=& \; 10 \times 10 &= 100 \end{aligned}$$

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Again let's take $x = 10$ for instance.

$$\begin{aligned}
10^0 &= & & & 1 \\
10^1 &= 10^0 \times 10 &= & 1 \times 10 &= 10 \\
10^2 &= 10^1 \times 10 &= & 10 \times 10 &= 100 \\
10^3 &= 10^2 \times 10 &= & 100 \times 10 &= 1000
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New: each inductive step depends on the parameter x.

```
exp(x, 0) := 1
for i in (0, ..., n-1):
    exp(x, i+1) := multiply(exp(x, i), x)
return exp(x, n)
```

All the cases so far can be unified by

```
h(x, y, z, 0) := f(x, y, z)
for i in (0, ..., n-1):
    h(x, y, z, i+1) := g(x, y, z, i, h(x, y, z, i))
```

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```

Definition. We say that a function $h(\bar{x}, n)$ is defined from two other functions $f(\bar{x})$ and $g(\bar{x}, n, k)$ by "primitive recursion" if it satisfies

$$h(\bar{x}, 0) = f(\bar{x}), \qquad h(\bar{x}, s(i)) = g(\bar{x}, i, h(\bar{x}, i)).$$

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If f and g are computable so is h!

N.B. Keep track of the numbers of inputs / arguments:

- if f takes n inputs, g takes $n + 2$ inputs, and h takes $n + 1$ inputs.