COMP30026 Models of Computation

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Lecture 3

Consequence and Satisfaction

Recap: Models

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$$\{P \mapsto \mathbf{0}\} \models \neg P$$

Non-examples:

$$\{P \mapsto \mathbf{1}, Q \mapsto \mathbf{0}\} \not\models P \rightarrow Q$$
$$\{P \mapsto \mathbf{1}\} \not\models \neg P$$

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Non-examples:

$$P \rightarrow Q \not\equiv R \rightarrow S$$
$$P \wedge Q \not\equiv P \vee Q$$

Semantic Consequence

Definition

G is a *semantic consequence* of F if and only if every model of F is a model of G.

For short, we write "$F \models G$".

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Note: $F \equiv G$ iff $F \models G$ and $G \models F$.

Consequence and Implication

models used before defined later?

Let F and G be formulas.

Theorem

$F \models G$ if and only if $\models F \rightarrow G$.

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As an immediate corollary:

Corollary

$F \equiv G$ if and only if $\models F \leftrightarrow G$.

Poll

Of the following formulas, which allow us to conclude $P \rightarrow Q$?

1. P
2. $\neg P$
3. Q
4. $P \rightarrow (Q \wedge R)$
5. $(P \vee R) \rightarrow Q$
6. $\neg P \vee Q$
7. $\neg Q \rightarrow \neg P$
8. $P \rightarrow (Q \vee R)$
9. $(P \rightarrow Q) \vee R$

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Of the following formulas, which allow us to conclude $P \rightarrow Q$?

1. P
2. $\neg P$
3. Q
4. $P \rightarrow (Q \land R)$
5. $(P \lor R) \rightarrow Q$
6. $\neg P \lor Q$
7. $\neg Q \rightarrow \neg P$
8. $P \rightarrow (Q \lor R)$
9. $(P \rightarrow Q) \lor R$

- No
- Yes
- Yes
- Yes
- Yes
- Yes
- Yes
- No
- No

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Which of the formulas are consequences of $P \rightarrow Q$?

Tautology

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$(\neg P \land Q) \rightarrow (P \rightarrow R)$ is a tautology:

P	Q	R	(	¬	P	∧	Q	)	→	(	P	→	R	)
1	1	1		0	1	0	1		**1**		1	1	1	
1	1	0		0	1	0	1		**1**		1	0	0	
1	0	1		0	1	0	0		**1**		1	1	1	
1	0	0		0	1	0	0		**1**		1	0	0	
0	1	1		1	0	1	1		**1**		0	1	1	
0	1	0		1	0	1	1		**1**		0	1	0	
0	0	1		1	0	0	0		**1**		0	1	1	
0	0	0		1	0	0	0		**1**		0	1	0	

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If either P or Q are false, $\wedge$ makes whole formula false.

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When P and Q are both true:

P	Q	(	P	$\land$	Q	)	$\land$	(	$\neg$	Q	$\leftrightarrow$	(	$\neg$	P	$\lor$	Q	))
1	1		1	1	1		**0**		0	1	0		0	1	1	1	

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When P and Q are both true:

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Negating a contradiction gives a tautology and vice versa.

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It is true *regardless* of what “program” or “works” mean.

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Valid: Always true.

Non-valid: Sometimes false.

$\models F$ is short for "F is valid".

Contradictions Are Unsatisfiable

Consider:
"the application is good and the application is not good."

It is false *regardless* of what "the application" or "good" mean.

Contradictions Are Unsatisfiable

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"the application is good and the application is not good."

It is false *regardless* of what "the application" or "good" mean.

Unsatisfiable: Never true.

Satisfiable: Sometimes true.

Most Statements Are Contingent

Consider: “It is currently raining.”

It is true if and only if it is currently raining.

Most Statements Are Contingent

Consider: “It is currently raining.”

It is true if and only if it is currently raining.

Contingent: Sometimes true, sometimes false.

Poll

Classify the following formulas as valid, contingent, or unsatisfiable:

1. P
2. $P \leftrightarrow \neg P$
3. $P \rightarrow (\neg Q \vee P)$
4. $\neg Q \vee \neg(P \wedge \neg Q)$

Poll

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1. P
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- contingent
- unsatisfiable
- valid
- valid

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Result: "$(Q \wedge R) \rightarrow (Q \wedge R)$".

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Example

Consider "$P \rightarrow P$".

Substitute P with "$(Q \wedge R)$".

Result: "$(Q \wedge R) \rightarrow (Q \wedge R)$".

Substitution preserves validity! Students do not know what preserves MEANS?

Poll

Does substitution preserve unsatisfiability?

Poll

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Yes!

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Yes!

Negation of contradiction is a tautology.

Poll

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No — a counterexample is easy:

Take P (which is clearly satisfiable).

Then substitute P by $Q \wedge \neg Q$.

Substitution Preserves Logical Equivalence

Denote by $F[A := B]$ the result of substituting A with B in F.

Example: $(P \to P)[P := Q]$ is $(Q \to Q)$.

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Theorem

Let F, G, H be formulas and P be any propositional variable.

If $F \equiv G$, then $F[P := H] \equiv G[P := H]$.

Interchange of Equivalents

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If $F \equiv G$, then $H \equiv H'$.

Result is equivalent: all semantic properties preserved.

Rewrite formulas algebraically!

Some Equivalences

Absorption: $P \land P \equiv P$

$P \lor P \equiv P$

Commutativity: $P \land Q \equiv Q \land P$

$P \lor Q \equiv Q \lor P$

Associativity: $P \land (Q \land R) \equiv (P \land Q) \land R$

$P \lor (Q \lor R) \equiv (P \lor Q) \lor R$

Distributivity: $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$

$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$

More Equivalences

Double negation:	$P \equiv \neg\neg P$
De Morgan:	$\neg(P \land Q) \equiv \neg P \lor \neg Q$ $\neg(P \lor Q) \equiv \neg P \land \neg Q$
Implication:	$P \rightarrow Q \equiv \neg P \lor Q$
Contraposition:	$\neg P \rightarrow \neg Q \equiv Q \rightarrow P$ $P \rightarrow \neg Q \equiv Q \rightarrow \neg P$ $\neg P \rightarrow Q \equiv \neg Q \rightarrow P$
Biimplication:	$P \leftrightarrow Q \equiv (P \land Q) \lor (\neg P \land \neg Q)$

Last Equivalences

$\bot$ is equivalent to any unsatisfiable formula
$\top$ is equivalent to any valid formula

Duality:	$\neg\top \equiv \bot$ $\neg\bot \equiv \top$
Negation from absurdity:	$P \to \bot \equiv \neg P$
Identity:	$P \lor \bot \equiv P$ $P \land \top \equiv P$
Dominance:	$P \land \bot \equiv \bot$ $P \lor \top \equiv \top$
Contradiction:	$P \land \neg P \equiv \bot$
Excluded middle:	$P \lor \neg P \equiv \top$

Poll

Which of these claims hold?

1. $P \to Q \equiv (Q \leftrightarrow (P \vee Q))$
2. $(P \to Q) \wedge (P \to R) \equiv P \to (Q \wedge R)$
3. $(P \to R) \wedge (Q \to R) \models (P \wedge Q) \to R$