School of Computing and Information Systems
COMP30026 Models of Computation
Week 2: Propositional Logic

If you find yourself getting stuck on a particular question in the tutorial, try to move onto other questions until you have a chance to ask your tutor for help.

Exercises

T2.1 For each pair of formulas, write down and compare their truth tables. Are they equivalent? In which rows (if any) are they different?

(i) $\neg P \to Q$ and $P \to \neg Q$

(ii) $\neg(P \wedge \neg Q)$ and $P \to Q$

(iii) $(P \wedge Q) \to R$ and $P \to (Q \to R)$

(iv) $P \to (Q \to R)$ and $(P \to Q) \to R$

T2.2 Find a formula that is equivalent to $(P \wedge \neg Q) \vee P$ but shorter.

T2.3 Write down the definitions of *satisfiable*, *valid*, *unsatisfiable* and *non-valid*. Explain why any given propositional formula must have exactly two of these properties, and describe the following propositional formulas using these terms.

(i) $P \vee Q$

(ii) $P \wedge \neg P$

(iii) $((P \to Q) \to P) \to P$

(iv) $(P \wedge \neg P) \to P$

T2.4 Suppose we are scheduling a three month long software project with three stages: Planning, Coding, and Testing. Like any schedule, there are constraints on when stages can be performed.

We can create formulae encoding the constraints on our schedule using 9 propositional variables, one for each combination of stage and month. For example, variable C_1 is true iff the Coding stage will be performed in month 1.

(i) Using the variables P_i, C_i, and T_i, $i \in \{1, 2, 3\}$ for Planning, Coding, and Testing respectively, please translate each of the following sentences into a propositional formula. With each constraint, try to translate its meaning without assuming any of the other constraints hold.

- Each stage must be performed in **at least** one month
- No stage can be performed in **more than** one month
- Coding can only be performed in a month if planning has been performed in a previous month
- Coding cannot be performed in the third month
- Testing can only be performed in a month if Coding is performed in the same month or an earlier month

(ii) Now try to find a solution that satisfies the conjunction of your formulae. This solution will in fact tell us how to assign the stages of our project!

Homework problems

P2.1 Which of the following pairs of formulas are equivalent?

(a) $\neg P \to Q$ and $P \to \neg Q$

(b) $\neg P \to Q$ and $Q \to \neg P$

(c) $\neg P \to Q$ and $\neg Q \to P$

(d) $(P \to Q) \to P$ and P

(e) $P \to (Q \to R)$ and $Q \to (P \to R)$

(f) $P \to (Q \to R)$ and $(P \to Q) \to R$

(g) $(P \wedge Q) \to R$ and $P \to (Q \to R)$

(h) $(P \vee Q) \to R$ and $(P \to R) \wedge (Q \to R)$

P2.2 Define your own binary connective $\Box$ by writing out a truth table for $P \Box Q$ (fill in the middle column however you like). Can you write a formula which has the the same truth table as $P \Box Q$ using only the symbols P, Q, $\neg$, $\wedge$, $\vee$, and $\to$? Repeat the exercise once.

P2.3 How many distinct truth tables are there involving two fixed propositional letters? In other words, how many meaningfully distinct connectives could we have defined in the previous question?

P2.4 Find a formula that is equivalent to $P \leftrightarrow (P \wedge Q)$ but shorter.

P2.5 Find a formula that is equivalent to $(\neg P \vee Q) \wedge R$ using only $\to$ and $\neg$ as logical connectives.

P2.6 Consider the formula $P \to \neg P$. Is that a contradiction (is it *unsatisfiable*)? Can a proposition imply its own negation?

P2.7 By negating a satisfiable proposition, can you get a tautology? A satisfiable proposition? A contradiction? Illustrate your affirmative answers.

P2.8 For each of the following propositional formulas, determine whether it is satisfiable, and if it is, whether it is a tautology:

(a) $P \leftrightarrow ((P \to Q) \to P)$

(b) $(P \to \neg Q) \wedge ((P \vee Q) \to P)$

P2.9 Complete the following sentences, using the words "satisfiable, valid, non-valid, unsatisfiable".

(a) F is satisfiable iff F is not _____

(b) F is valid iff F is not _____

(c) F is non-valid iff F is not _____

(d) F is unsatisfiable iff F is not _____

(e) F is satisfiable iff $\neg F$ is _____

(f) F is valid iff $\neg F$ is _____

(g) F is non-valid iff $\neg F$ is _____

(h) F is unsatisfiable iff $\neg F$ is _____

P2.10 Show that $P \leftrightarrow (Q \leftrightarrow R) \equiv (P \leftrightarrow Q) \leftrightarrow R$. This tells us that we could instead write

$$P \leftrightarrow Q \leftrightarrow R \tag{1}$$

without introducing any ambiguity. Mind you, that may not be such a good idea, because many people (incorrectly) tend to read "$P \leftrightarrow Q \leftrightarrow R$" as

$$P,\ Q, \text{ and } R \text{ all have the same truth value} \tag{2}$$

Show that (1) and (2) are incomparable, that is, neither is a logical consequence of the other.

P2.11 Let F and G be propositional formulas. What is the difference between "$F \equiv G$" and "$F \leftrightarrow G$"? Prove that "$F \leftrightarrow G$" is valid iff $F \equiv G$.

P2.12 Is $(P \wedge Q) \leftrightarrow P$ logically equivalent to $(P \vee Q) \leftrightarrow Q$?

P2.13 Consider this puzzle from Smullyan's Island of Knights and Knaves. On this island there are knights, who always tell the truth, and knaves, who always lie. We meet three people from the island, and we are reliably informed that one of the three is also a magician. They make these statements:

A: B is not both a knave and a magician.

B: Either A is a knave or I am not a magician.

C: The magician is a knave.

Who is the magician? Give a proof, and check that your solution indeed solves the problem.

Hint: If you have a disjunction like $P \vee Q$ or a negated conjunction like $\neg(P \wedge Q)$, negating it (e.g. for the sake of contradiction) produces a conjunction, which is usually much easier to work with.