PHIL 222
Philosophical Foundations of Computer Science
Week 6, Tuesday

Oct. 1, 2024

Technical Exercise 2
Review

Use the box-and-wire notation and show that
the projection $\Pi^5_3 : \mathbb{N}^5 \to \mathbb{N} :: (n_1, \ldots, n_5) \mapsto n_3$ is primitive recursive.

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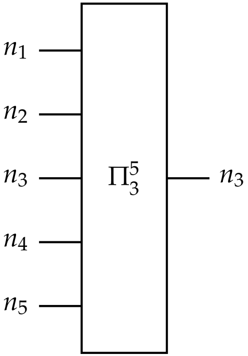

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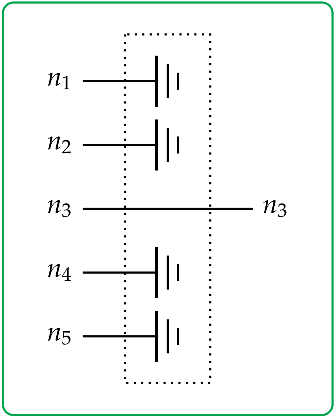

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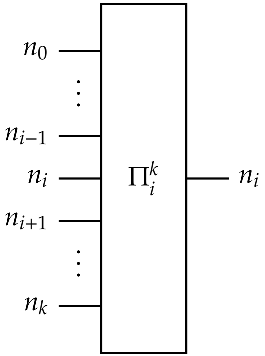

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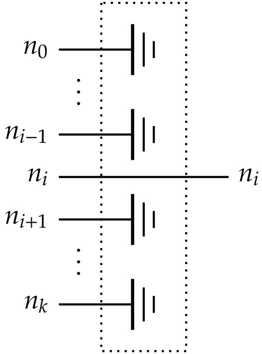

Prove that the following "conditional function" $\mathrm{cond} : \mathbb{N}^3 \to \mathbb{N}$

$$\mathrm{cond}(x, y, n) = \begin{cases} x & \text{if } n > 0, \\ y & \text{if } n = 0 \end{cases}$$

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ⓑ $f = \Pi^2_2$ and $g = \Pi^4_1$, which we have shown to be primitive recursive.

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Assuming neither x nor y occurs in M or N, show that

$$((\overline{\text{True}}\,M)\,N) \xrightarrow{\beta} \cdots \xrightarrow{\beta} M, \qquad ((\overline{\text{False}}\,M)\,N) \xrightarrow{\beta} \cdots \xrightarrow{\beta} N.$$

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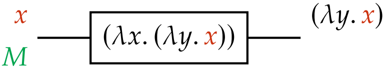

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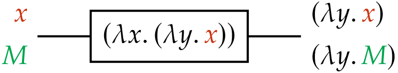

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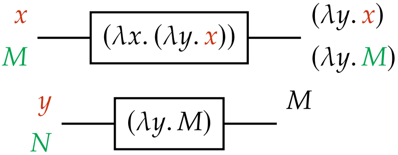

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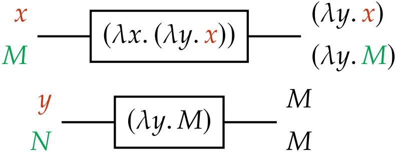

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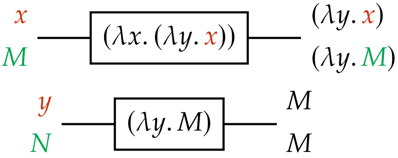

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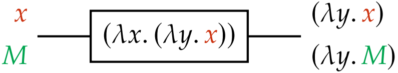

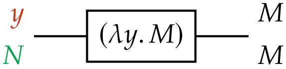

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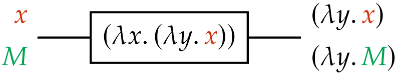

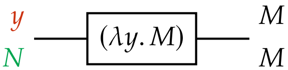

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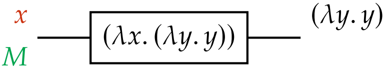

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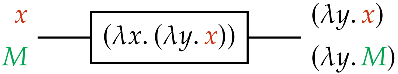

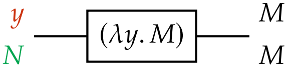

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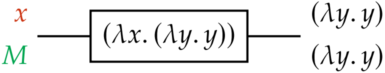

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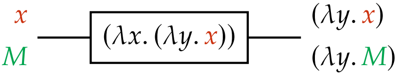

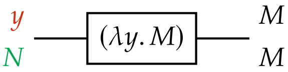

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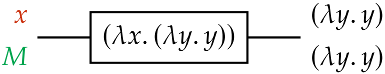

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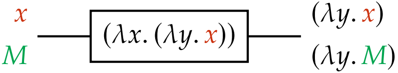

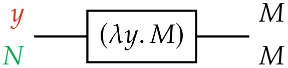

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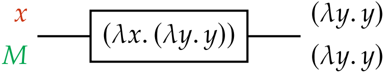

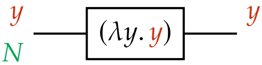

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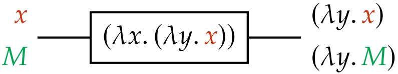

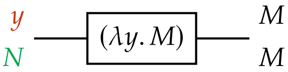

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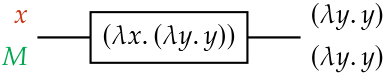

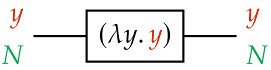

$$\overline{\text{True}} := (\lambda x.\,(\lambda y.\,x)), \qquad \overline{\text{False}} := (\lambda x.\,(\lambda y.\,y)).$$

Assuming neither x nor y occurs in M or N, show that

$$((\overline{\text{True}}\,M)\,N) \xrightarrow{\beta} \cdots \xrightarrow{\beta} M, \qquad ((\overline{\text{False}}\,M)\,N) \xrightarrow{\beta} \cdots \xrightarrow{\beta} N.$$

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$$\begin{array}{c} x \\ M \end{array} \;\rule[0.5ex]{2em}{0.4pt}\; \boxed{(\lambda x.\,(\lambda y.\,x))} \;\rule[0.5ex]{2em}{0.4pt}\; \begin{array}{c} (\lambda y.\,x) \\ (\lambda y.\,M) \end{array}$$

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The Church-Turing Thesis: Other Versions (cont'd)

For Ⅱ, what exactly is an algorithm?

For II, what exactly is an algorithm? We may try:

- E A process (abstractly speaking) that implements (an indendent-of-human version of) an effective method satisfying a–d.
- F A process (abstractly speaking) that satisfies (an indendent-of-human version of) i–vii.

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Let's investigate whether contemporary computer scientists undertand the Church-Turing thesis as A / E / B / F.

There are new paradigms of computers / algorithms that may not satisfy a–d / i–vii.

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One should not confuse the Church-Turing thesis with the (false) claim that every (possible / reasonable) algorithm can be performed by a Turing machine (see Copeland and Shagrir, p. 68). The thesis is about whether Turing machines can compute a given function / task f, rather than perform a given algorithm.

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1. "That's no counterexample to the Church-Turing thesis. That's not the kind of computers / algorithms the thesis is about."
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Indeed, against any modification of A/E/B/F that replaces a–d/i–vii, you may devise a similar argument!

f is Turing computable, λ-definable, partial recursive

CTT-O

i–vii

f can be achieved by an effective method a–d

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CTT-O

i–vii

f can be achieved by an effective method a–d

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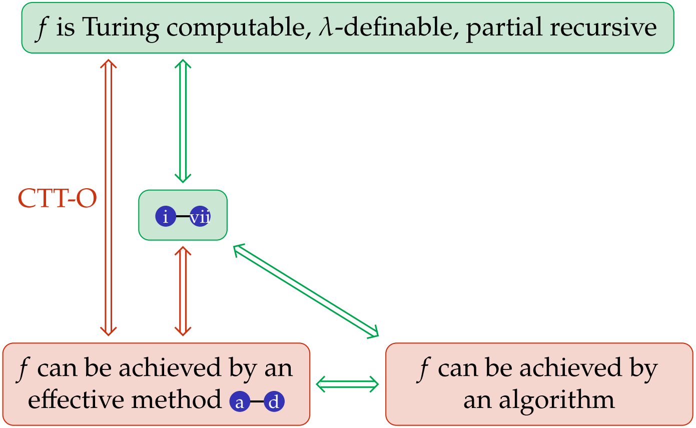

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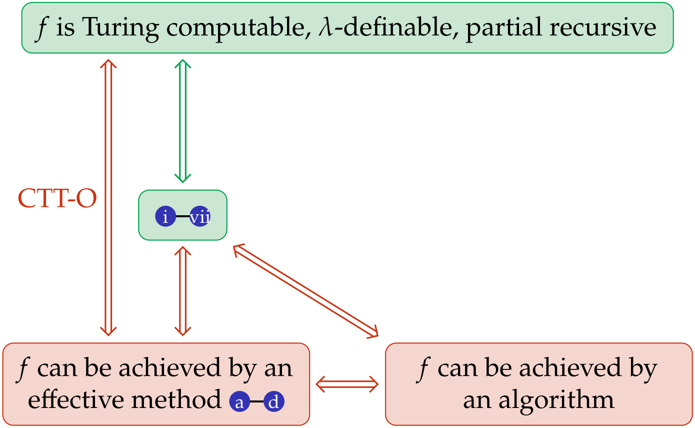

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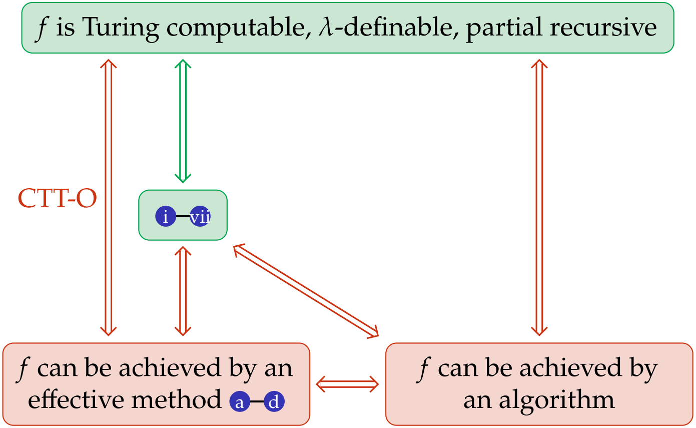
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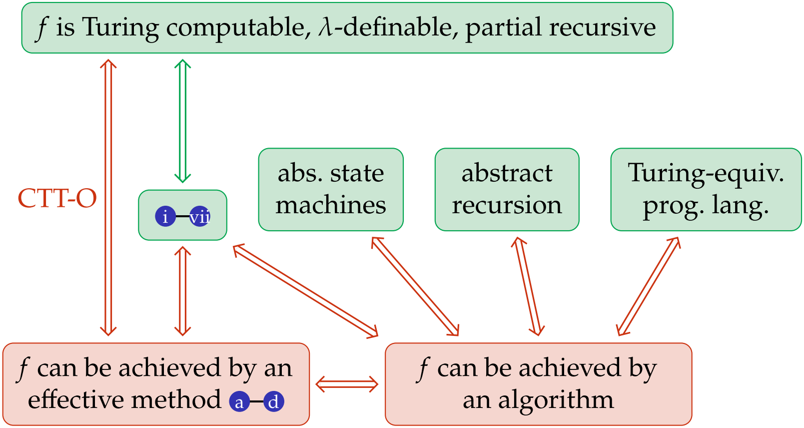
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CTT-O
i—vii
abs. state machines
abstract recursion
Turing-equiv. prog. lang.
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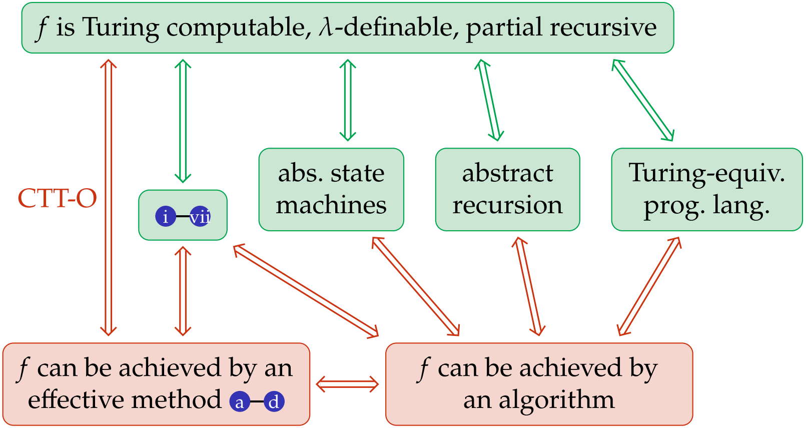
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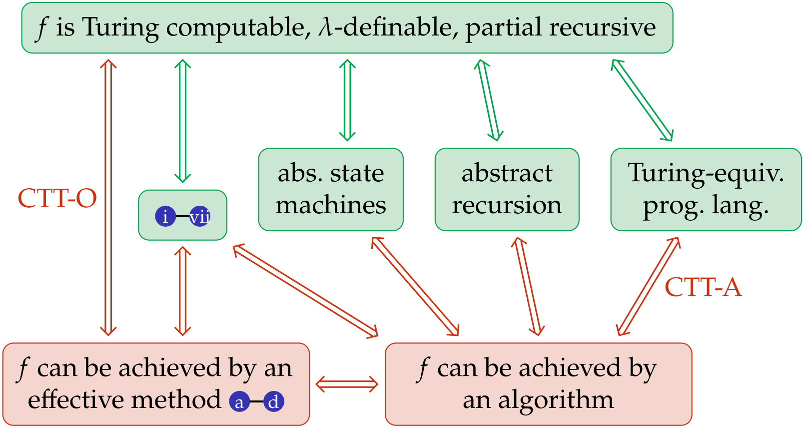

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CTT-O
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CTT-A
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The Church-Turing Thesis: What the Thesis Does Not Say

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- "Accelerating Turing machines" are not physically implementable:

$$\begin{aligned}
1 \text{ second on step } 0,\\
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  So they can complete infinitely many steps in 2 seconds, and can compute Turing uncomputable functions.

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"Finite" in the criteria of effectiveness / computability is essential.

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To see that, let's recall the mathematical proof (from the final section of the Turing Machines chapter) that there is a Turing uncomputable real number.

Theorem. There are Turing uncomputable real numbers.

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Proof. By comparing the natural-number codes of Turing machines, let's enumerate, as in

$$M_0, M_1, M_2, \ldots,$$

all the Turing machines M_n that compute a real number x_n.

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Then write $x_{n,0}$ for the integer part of x_n and $x_{n,m}$ ($m > 0$) for the digit in the mth decimal place of x_n. Visually, we have the following table:

$$\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & \cdots \\
\hline
x_0 & x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & \cdots \\
x_1 & x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\
x_2 & x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
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Now define a real number d by $d_m = x_{m,m} \pm 1$. Then d differs from every x_n. Therefore d is not Turing computable. $\square$

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— No. No Turing machine can compute such an f,

which brings us to . . .

Computability Theory

Advice

This is a very technical part of the course.

- The concepts and facts (theorems, etc.) covered here may be relevant to both midterm and final exams.
- The proofs of the facts may be tough, but do not worry too much: you will not be tested for the understanding of them, except maybe in extra-credit problems in the midterm exam.

If you find the material hard,

- *Come to see me in office hours & appointments!*

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We proved this by mathematically constructing a Turing uncomputable number d.

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Answer. Mathematicians have no problem saying this enumeration exists, but it is something that Turing machines can never do!

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In addition, we also adopt:

Notation. We write U for the function computed by a universal Turing machine, and M_n for the Turing machine whose code is n, so that

$$U(n, m) = \text{the output of the Turing machine } M_n \text{ on the input } m.$$

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2. This gives a table whose rows list all the computable numbers.

$$\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & \cdots \\
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x_2 & x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\
x_3 & x_{3,0} & x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\
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3. Define a new number d by shifting the numbers on the diagonal.
4. d differs from every row of the table, and hence is uncomputable.
5. It follows that the enumeration above is not computable — because if it were, d would be computable, too.

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Definition. We say that a function $f : \mathbb{N} \to \mathbb{N}$ enumerates (all the) *such-and-such* Turing machines, as in

$$f(0), f(1), f(2), \ldots, f(n), \ldots,$$

if f is a total function s.th.

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Then, by a computable enumeration of *such-and-such* Turing machines, we mean a computable function f enumerating them.