School of Computing and Information Systems
COMP30026 Models of Computation
Week 2: Propositional Logic

If you find yourself getting stuck on a particular question in the tutorial, try to move onto other questions until you have a chance to ask your tutor for help.

Exercises

T2.1 For each pair of formulas, write down and compare their truth tables. Are they equivalent? In which rows (if any) are they different?

(i) $\neg P \to Q$ and $P \to \neg Q$

(ii) $\neg(P \land \neg Q)$ and $P \to Q$

(iii) $(P \land Q) \to R$ and $P \to (Q \to R)$

(iv) $P \to (Q \to R)$ and $(P \to Q) \to R$

Solution:

(i) Not logically equivalent:

P	Q	$\neg P$	$\to$	Q	P	$\to$	$\neg Q$
0	0	1	0	0	0	1	1
0	1	1	1	1	0	1	0
1	0	0	1	0	1	1	1
1	1	0	1	1	1	0	0
			✗			✗	

(ii) Logically equivalent:

P	Q	$\neg$	$($ P	$\land$	$\neg Q$ $)$	P	$\to$	Q
0	0	1	0	0	1	0	1	0
0	1	1	0	0	0	0	1	1
1	0	0	1	1	1	1	0	0
1	1	1	1	0	0	1	1	1
		✓					✓	

(iii) Logically equivalent:

P	Q	R	$($ P	$\land$	Q $)$	$\to$	R	P	$\to$	$($ Q	$\to$	R $)$
0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	1	0	0	0	1	1	0	1	0	1	1
0	1	0	0	0	1	1	0	0	1	1	0	0
0	1	1	0	0	1	1	1	0	1	1	1	1
1	0	0	1	0	0	1	0	1	1	0	1	0
1	0	1	1	0	0	1	1	1	1	0	1	1
1	1	0	1	1	1	0	0	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1
						✓			✓			

(iv) Not logically equivalent:

P	Q	R	P	$\to$	$(Q$	$\to$	$R)$	$(P$	$\to$	$Q)$	$\to$	R
0	0	0	0	1	0	1	0	0	1	0	0	0
0	0	1	0	1	0	1	1	0	1	0	1	1
0	1	0	0	1	1	0	0	0	1	1	0	0
0	1	1	0	1	1	1	1	0	1	1	1	1
1	0	0	1	1	0	1	0	1	0	0	1	0
1	0	1	1	1	0	1	1	1	0	0	1	1
1	1	0	1	0	1	0	0	1	1	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1
				✗							✗	

T2.2 Find a formula that is equivalent to $(P \land \neg Q) \lor P$ but shorter.

Solution: $(P \land \neg Q) \lor P$ is logically equivalent to P. You can see this by drawing out the truth tables, or reasoning by cases on the truth value of P.

T2.3 Write down the definitions of *satisfiable*, *valid*, *unsatisfiable* and *non-valid*. Explain why any given propositional formula must have exactly two of these properties, and describe the following propositional formulas using these terms.

(i) $P \lor Q$

(ii) $P \land \neg P$

(iii) $((P \to Q) \to P) \to P$

(iv) $(P \land \neg P) \to P$

Solution:

- A *satisfiable* formula has *at least one* truth assignment that makes it true (it *can* be satisfied by a truth assignment).
- A *valid* formula is one which is made true by *all* truth assignments (it *must* be satisfied by any truth assignment).
- An *unsatisfiable* formula is one which is made false by *all* truth assignments (it *cannot* be satisfied by any truth assignment).
- A *non-valid* formula has *at least one* truth assignment that makes it false (it *can* be made false by a truth assignment)

Consider the truth table of a propositional formula φ.

- If there are some rows of the truth table which make φ true **and** some rows that make φ false, then φ is both *satisfiable* and *non-valid.* φ is not *valid*, since there are rows much make φ false, and φ is not *unsatisfiable*, since there are rows that make φ true.
- If all rows of the truth table make φ true, then φ is *valid* and also *satisfiable.* φ is not *non-valid*, nor *unsatisfiable*, since there are no rows which make φ false.
- If all rows of the truth table make φ false, then φ is *unsatisfiable* and also *non-valid.* φ is not *satisfiable*, nor *valid*, since there are no rows which make φ true.

These are the only possibilites for the rows of any truth table. Note that *valid* is a stronger property than *satisfiable*, and *unsatisfiable* is a stronger property than *non-valid.*

(i) $P \lor Q$ is satisfiable and non-valid.

(ii) $P \land \neg P$ is unsatisfiable and non-valid.

(iii) $((P \to Q) \to P) \to P$ is valid and satisfiable.

(iv) $(P \wedge \neg P) \to P$ is valid and satisfiable.

T2.4 Suppose we are scheduling a three month long software project with three stages: Planning, Coding, and Testing. Like any schedule, there are constraints on when stages can be performed.

We can create formulae encoding the constraints on our schedule using 9 propositional variables, one for each combination of stage and month. For example, variable C_1 is true iff the Coding stage will be performed in month 1.

(i) Using the variables P_i, C_i, and T_i, $i \in \{1, 2, 3\}$ for Planning, Coding, and Testing respectively, please translate each of the following sentences into a propositional formula. With each constraint, try to translate its meaning without assuming any of the other constraints hold.

- Each stage must be performed in **at least** one month
- No stage can be performed in **more than** one month
- Coding can only be performed in a month if planning has been performed in a previous month
- Coding cannot be performed in the third month
- Testing can only be performed in a month if Coding is performed in the same month or an earlier month

(ii) Now try to find a solution that satisfies the conjunction of your formulae. This solution will in fact tell us how to assign the stages of our project!

Solution: Note that there are many possible equivalent formulae for each of these questions. Below is one possible formula for each.

(i)
- $(P_1 \vee P_2 \vee P_3) \wedge (C_1 \vee C_2 \vee C_3) \wedge (T_1 \vee T_2 \vee T_3)$
- $(\neg P_1 \vee \neg P_2) \wedge (\neg P_2 \vee \neg P_3) \wedge (\neg P_1 \vee \neg P_3) \wedge (\neg C_1 \vee \neg C_2) \wedge (\neg C_2 \vee \neg C_3) \wedge (\neg C_1 \vee \neg C_3) \wedge (\neg T_1 \vee \neg T_2) \wedge (\neg T_2 \vee \neg T_3) \wedge (\neg T_1 \vee \neg T_3)$
- $\neg C_1 \wedge (C_2 \to P_1) \wedge (C_3 \to (P_1 \vee P_2))$
- $\neg C_3$
- $(T_1 \to C_1) \wedge (T_2 \to (C_1 \vee C_2)) \wedge (T_3 \to (C_1 \vee C_2 \vee C_3)))$

(ii) One truth assignment that satisfies the formulae is $v = \{P_1 \mapsto 1, P_2 \mapsto 0, P_3 \mapsto 0, C_1 \mapsto 0, C_2 \mapsto 1, C_3 \mapsto 0, T_1 \mapsto 0, T_2 \mapsto 1, T_3 \mapsto 0\}$. This truth assignment corresponds to scheduling Planning in month 1 and both coding and testing in month 2.

You may find in this case it was easy enough to figure out the solution without using propositional logic. However, in a more complicated case with many more stages, constraints, and possibly also involving assigning people to these stages, the problem would quickly become impossible to solve by hand. The propositional formula in this case would span multiple pages, but a SAT solver would be able to near-instantly spit out a solution!

Homework problems

P2.1 Which of the following pairs of formulas are equivalent?

(a) $\neg P \to Q$ and $P \to \neg Q$

(b) $\neg P \to Q$ and $Q \to \neg P$

(c) $\neg P \to Q$ and $\neg Q \to P$

(d) $(P \to Q) \to P$ and P

(e) $P \to (Q \to R)$ and $Q \to (P \to R)$

(f) $P \to (Q \to R)$ and $(P \to Q) \to R$

(g) $(P \wedge Q) \to R$ and $P \to (Q \to R)$

(h) $(P \vee Q) \to R$ and $(P \to R) \wedge (Q \to R)$

Solution:

(a) Not equivalent:

P	Q	$\neg P$	$\to$	Q	P	$\to$	$\neg Q$
0	0	1	0	0	0	1	1
0	1	1	1	1	0	1	0
1	0	0	1	0	1	1	1
1	1	0	1	1	1	0	0
			✗			✗	

We see that the columns for the two implications are different.

(b) Not equivalent:

P	Q	$\neg P$	$\to$	Q	Q	$\to$	$\neg P$
0	0	1	0	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	1	0	1	1	1	0	0
			✗			✗	

(c) Equivalent:

P	Q	$\neg P$	$\to$	Q	$\neg Q$	$\to$	P
0	0	1	0	0	1	0	0
0	1	1	1	1	0	1	0
1	0	0	1	0	1	1	1
1	1	0	1	1	0	1	1
			✓			✓	

(d) Equivalent:

P	Q	$(P$	$\to$	$Q)$	$\to$	P
0	0	0	1	0	0	0
0	1	0	1	1	0	0
1	0	1	0	0	1	1
1	1	1	1	1	1	1
✓					✓	

(e), (f) : the pair in (e) equivalent; but the pair in (f) are not equivalent:

P	Q	R	P	$\to$	$(Q$	$\to$	$R)$	Q	$\to$	$(P$	$\to$	$R)$	$(P$	$\to$	$Q)$	$\to$	R
0	0	0	0	1	0	1	0	0	1	0	1	0	0	1	0	0	0
0	0	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1
0	1	0	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0
0	1	1	0	1	1	1	1	1	1	0	1	1	0	1	1	1	1
1	0	0	1	1	0	1	0	0	1	1	0	0	1	0	0	1	0
1	0	1	1	1	0	1	1	0	1	1	1	1	1	0	0	1	1
1	1	0	1	0	1	0	0	1	0	1	0	0	1	1	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
				✓					✓								
				✗												✗	

Note that the formulas in (e) are both equivalent to $(P \wedge Q) \to R$, which explains why the order of P and Q does not matter here.

(g) Equivalent:

P	Q	R	$(P$	$\wedge$	$Q)$	$\to$	R	P	$\to$	$(Q$	$\to$	$R)$
0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	1	0	0	0	1	1	0	1	0	1	1
0	1	0	0	0	1	1	0	0	1	1	0	0
0	1	1	0	0	1	1	1	0	1	1	1	1
1	0	0	1	0	0	1	0	1	1	0	1	0
1	0	1	1	0	0	1	1	1	1	0	1	1
1	1	0	1	1	1	0	0	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1
						✓			✓			

(h) Equivalent:

P	Q	R	$(P$	$\vee$	$Q)$	$\to$	R	$(P$	$\to$	$R)$	$\wedge$	$(Q$	$\to$	$R)$
0	0	0	0	0	0	1	0	0	1	0	1	0	1	0
0	0	1	0	0	0	1	1	0	1	1	1	0	1	1
0	1	0	0	1	1	0	0	0	1	0	0	1	0	0
0	1	1	0	1	1	1	1	0	1	1	1	1	1	1
1	0	0	1	1	0	0	0	1	0	0	0	0	1	0
1	0	1	1	1	0	1	1	1	1	1	1	0	1	1
1	1	0	1	1	1	0	0	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
						✓					✓			

P2.2 Define your own binary connective $\Box$ by writing out a truth table for $P \Box Q$ (fill in the middle column however you like). Can you write a formula which has the the same truth table as $P \Box Q$ using only the symbols P, Q, $\neg$, $\wedge$, $\vee$, and $\to$? Repeat the exercise once.

Solution: Here's an example

P	Q	P	$\Box$	Q
0	0	0	1	0
0	1	0	1	1
1	0	1	1	0
1	1	1	0	1

With this truth table for $\Box$, $P \Box Q$ is logically equivalent to $\neg(P \wedge Q)$.

P2.3 How many distinct truth tables are there involving two fixed propositional letters? In other words, how many meaningfully distinct connectives could we have defined in the previous question?

Solution: There are four rows in the truth table, so that's 2 choices for each row, which amounts to $2 \times 2 \times 2 \times 2 = 16$ possibilities.

P2.4 Find a formula that is equivalent to $P \leftrightarrow (P \wedge Q)$ but shorter.

Solution: $(P \wedge Q) \leftrightarrow P$ is logically equivalent to $P \to Q$. This is easily checked with a truth table, but how can we simplify $(P \wedge Q) \leftrightarrow P$ when we don't know what it is supposed to be equivalent to? Well, we can just try. Let us expand the biimplication and obtain $(P \to (P \wedge Q)) \wedge ((P \wedge Q) \to P)$. Intuitively, the conjunct on the right is just true, and we can check that with a truth table. So we have found that the original formula is equivalent to $P \to (P \wedge Q)$, which isn't any shorter, but still. We can rewrite the result as $(P \to P) \wedge (P \to Q)$, and now it becomes clear that all we need is $P \to Q$.

P2.5 Find a formula that is equivalent to $(\neg P \lor Q) \land R$ using only $\rightarrow$ and $\neg$ as logical connectives.

Solution: $(\neg P \lor Q) \land R$ is logically equivalent to $\neg((P \rightarrow Q) \rightarrow \neg R)$

P2.6 Consider the formula $P \rightarrow \neg P$. Is that a contradiction (is it *unsatisfiable*)? Can a proposition imply its own negation?

Solution: There is no contradiction at all. The formula is true if (and only if) P is false. The point of a conditional formula is to make a claim about the scenario where the premise (P) is true. If the premise of $\rightarrow$ is false, the formula is satisfied. For the same reason, $\neg P \rightarrow P$ is satisfiable; it is not a contradiction. But $P \leftrightarrow \neg P$ is clearly a contradiction. (If you disagree with any of these statements, draw truth tables.)

P2.7 By negating a satisfiable proposition, can you get a tautology? A satisfiable proposition? A contradiction? Illustrate your affirmative answers.

Solution: If you negate a satisfiable proposition, you can never get a tautology, since at least one truth table row will yield false.

You will get another satisfiable proposition iff the original proposition is not valid. For example, P is satisfiable (but not valid), and indeed $\neg P$ is satisfiable.

Finally, if we have a satisfiable formula which is also valid, its negation will be a contradiction. Example: $P \lor \neg P$.

P2.8 For each of the following propositional formulas, determine whether it is satisfiable, and if it is, whether it is a tautology:

(a) $P \leftrightarrow ((P \rightarrow Q) \rightarrow P)$

(b) $(P \rightarrow \neg Q) \land ((P \lor Q) \rightarrow P)$

Solution: Let us draw the truth tables.

(a)

P	Q	P	$\leftrightarrow$	$((P$	$\rightarrow$	$Q)$	$\rightarrow$	$P)$
0	0	0	1	0	1	0	0	0
0	1	0	1	0	1	1	0	0
1	0	1	1	1	0	0	1	1
1	1	1	1	1	1	1	1	1
			$\Uparrow$					

Hence satisfiable, and in fact valid (all 1).

(b)

P	Q	$(P$	$\rightarrow$	$\neg$	$Q)$	$\land$	$((P$	$\lor$	$Q)$	$\rightarrow$	$P)$
0	0	0	1	1	0	1	0	0	0	1	0
0	1	0	1	0	1	0	0	1	1	0	0
1	0	1	1	1	0	1	1	1	0	1	1
1	1	1	0	0	1	0	1	1	1	1	1
						$\Uparrow$					

Hence satisfiable (at least one 1), but not valid (not all 1). The truth table shows the formula is equivalent to $\neg Q$.

P2.9 Complete the following sentences, using the words "satisfiable, valid, non-valid, unsatisfiable".

(a) F is satisfiable iff F is not ______
(b) F is valid iff F is not ______
(c) F is non-valid iff F is not ______
(d) F is unsatisfiable iff F is not ______
(e) F is satisfiable iff $\neg F$ is ______
(f) F is valid iff $\neg F$ is ______
(g) F is non-valid iff $\neg F$ is ______
(h) F is unsatisfiable iff $\neg F$ is ______

Solution:

(a) F is satisfiable iff F is not unsatisfiable
(b) F is valid iff F is not non-valid
(c) F is non-valid iff F is not valid
(d) F is unsatisfiable iff F is not satisfiable
(e) F is satisfiable iff $\neg F$ is non-valid
(f) F is valid iff $\neg F$ is unsatisfiable
(g) F is non-valid iff $\neg F$ is satisfiable
(h) F is unsatisfiable iff $\neg F$ is valid

P2.10 Show that $P \leftrightarrow (Q \leftrightarrow R) \equiv (P \leftrightarrow Q) \leftrightarrow R$. This tells us that we could instead write

$$P \leftrightarrow Q \leftrightarrow R \tag{1}$$

without introducing any ambiguity. Mind you, that may not be such a good idea, because many people (incorrectly) tend to read "$P \leftrightarrow Q \leftrightarrow R$" as

$$P, Q, \text{ and } R \text{ all have the same truth value} \tag{2}$$

Show that (1) and (2) are incomparable, that is, neither is a logical consequence of the other.

Solution: Even with three variables the truth table is manageable, so let us construct it.

				(1A)						(1B)			(2)	
P	Q	R	P	$\leftrightarrow$	$(Q$	$\leftrightarrow$	$R)$	$(P$	$\leftrightarrow$	$Q)$	$\leftrightarrow$	R	$P \wedge Q \wedge R$	$\vee$	$\neg P \wedge \neg Q \wedge \neg R$
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1
0	0	1	0	1	0	0	1	0	1	0	1	1	0	0	0
0	1	0	0	1	1	0	0	0	0	1	1	0	0	0	0
0	1	1	0	0	1	1	1	0	0	1	0	1	0	0	0
1	0	0	1	1	0	1	0	1	0	0	1	0	0	0	0
1	0	1	1	0	0	0	1	1	0	0	0	1	0	0	0
1	1	0	1	0	1	0	0	1	1	1	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
				✓							✓			✗	

B is a logical consequence of A (we write $A \models B$) iff B is true for any assignment of variables which makes A true. So we see that (1) $\not\models$ (2), because cases like 1 0 0 that make (1) true but (2) false. Similarly, (2) $\not\models$ (1), because the case 0 0 0 makes (2) true but (1) false.

P2.11 Let F and G be propositional formulas. What is the difference between "$F \equiv G$" and "$F \leftrightarrow G$"? Prove that "$F \leftrightarrow G$" is valid iff $F \equiv G$.

Solution: The connective $\leftrightarrow$ is part of the language that we study, namely the language of propositional logic. So $A \leftrightarrow B$ is just a propositional formula.

The symbol $\equiv$ belongs to a *meta-language*. The meta-language is a language which we use when we reason *about* some language. In this case we use $\equiv$ to express whether a certain relation holds between formulas in propositional logic.

More specifically, $F \equiv G$ means that we have both $F \models G$ and $G \models F$. In other words, F and G have the same value for every possible assignment of truth values to their variables. The two formulas are logically equivalent.

On the other hand $F \leftrightarrow G$ is just a propositional formula (assuming F and G are propositional formulas). For some values of the variables involved, $F \leftrightarrow G$ may be false, for other values it may be true. By the definition of validity, $F \leftrightarrow G$ is *valid* iff it is true for *every* assignment of propositional variables in F and G.

We want to show that $F \equiv G$ iff $F \leftrightarrow G$ is valid.

(a) Suppose $F \equiv G$. Then F and G have the same values for each truth assignment to their variables[1]. But that means that, when we construct the truth table for $F \leftrightarrow G$, it will have a t in every row, that is, $F \leftrightarrow G$ is valid.

(b) Suppose $F \leftrightarrow G$ is valid. That means we find a t in each row of the truth table for $F \leftrightarrow G$. But we get a t for $F \leftrightarrow G$ iff the values for F and G agree, that is, either both are f, or both are t. In other words, F and G agree for every truth assignment. Hence $F \equiv G$.

You may think that this relation between validity and biimplication is obvious and should always be expected, and indeed we will see that it carries over to first-order predicate logic. But there are (still useful) logics in which it does not hold.

P2.12 Is $(P \wedge Q) \leftrightarrow P$ logically equivalent to $(P \vee Q) \leftrightarrow Q$?

Solution: Yes, $(P \wedge Q) \leftrightarrow P \equiv (P \vee Q) \leftrightarrow Q$, and both of these formulas are logically equivalent to $P \to Q$

P	Q	(P	$\wedge$	Q)	$\leftrightarrow$	P	(P	$\vee$	Q)	$\leftrightarrow$	Q
0	0	0	0	0	1	0	0	0	0	1	0
0	1	0	0	1	1	0	0	1	1	1	1
1	0	1	0	0	0	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
					✓					✓	

P2.13 Consider this puzzle from Smullyan's Island of Knights and Knaves. On this island there are knights, who always tell the truth, and knaves, who always lie. We meet three people from the island, and we are reliably informed that one of the three is also a magician. They make these statements:

A: B is not both a knave and a magician.

B: Either A is a knave or I am not a magician.

C: The magician is a knave.

Who is the magician? Give a proof, and check that your solution indeed solves the problem.

Hint: If you have a disjunction like $P \vee Q$ or a negated conjunction like $\neg(P \wedge Q)$, negating it (e.g. for the sake of contradiction) produces a conjunction, which is usually much easier to work with.

Solution: Suppose to the contrary that A is a knave. Then B is both a knave and a magician. Thus, from B's utterance, A is not a knave and B is a magician. Contradiction! Therefore, A is a knight.

Suppose to the contrary that B is a knave. Then A is not a knave and B is a magician. Hence B is both a knave and a magician. But since A is a knight, we know from A's utterance that this is not the case. Contradiction! Hence B is a knight. Thus, by B's utterance, since A is not a knave, B is not a magician.

Suppose to the contrary that C is a knight. Then the magician is a knave. But A, B and C are all knights, so this is a contradiction. Hence C is a knave, and so the magician is not a knave. Hence the magician must be a knight.

[1] We should perhaps be more careful here, because F and G can be logically equivalent without F having the exact same set of variables as G—can you see how? So we should say that we consider both of F and G to be functions of the *union* of their sets of variables.

Since B is not a magician and C is not a knight, by elimination it follows that A is the magician.