PHIL 222
Philosophical Foundations of Computer Science
Week 4, Tuesday

Sept. 17, 2024

Recursive Functions: Primitive Recursion (cont'd)

All the cases so far can be unified by

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h(x, y, z, 0) := f(x, y, z)
for i in (0, ..., n-1):
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N.B. Keep track of the numbers of inputs / arguments:

- if f takes n inputs, g takes $n+2$ inputs, and h takes $n+1$ inputs.

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This means that + is defined by primitive recursion from these f and g.

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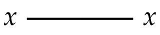

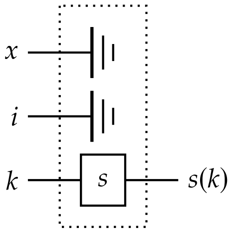

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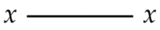

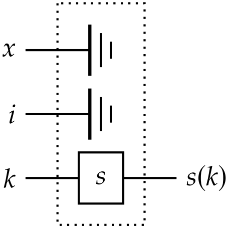

Therefore + is computable!

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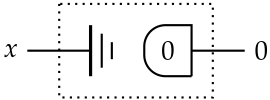

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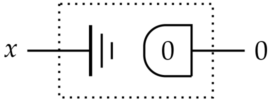

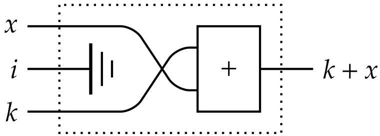

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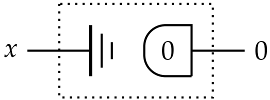

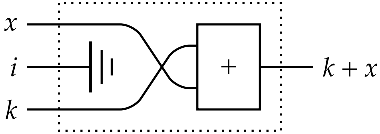

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1. six basic functions and
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We need one more operation to match the power of Turing machines.

Recursive Functions:
Partial Recursive Functions

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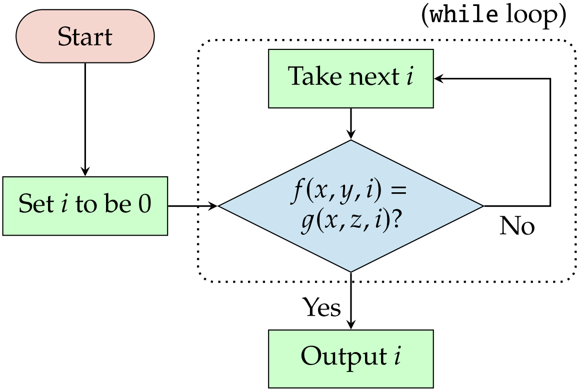

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This search is computable if f and g are!

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To sum up,

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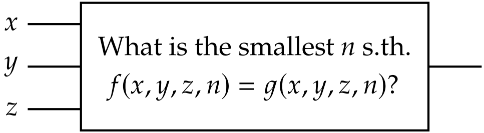

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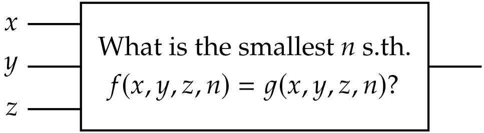

2. But this function may be partial and not total, since depending on inputs, it may fail to output values.

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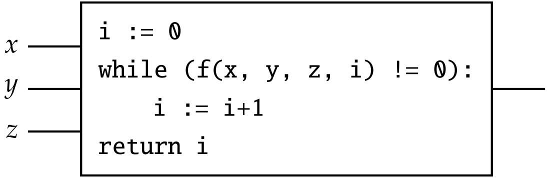

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Definition. We say that a function is "primitive recursive" if (& only if)

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Theorem. Given any partial function f,

$$f \text{ is Turing computable } \iff f \text{ is partial recursive.}$$

Recursive Functions: One Last Bit of Reflection

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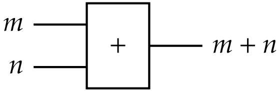

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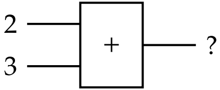

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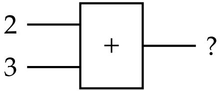

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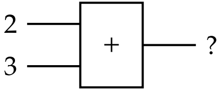

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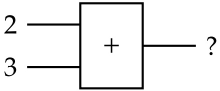

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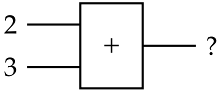

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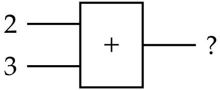

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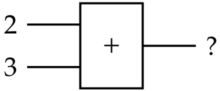

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- You: "110111." A Turing machine: "11111."

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