PHIL 222
Philosophical Foundations of Computer Science
Week 4, Thursday

Sept. 19, 2024

The Lambda Calculus
(cont'd)

A term of the lambda calculus is either

- a basic or "atomic" term, i.e., one of the variables $x, y, z, \ldots$ or the constants $a, b, c, \ldots$,
- (MN), obtained by the "application" of other terms M and N, or
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$$((\lambda x.\, (x + 3))\, 2) = 2 + 3.$$

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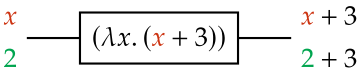

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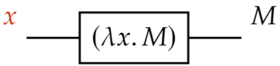

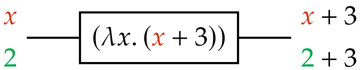

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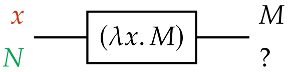

$$((\lambda x.\,M)\,N) = ?$$

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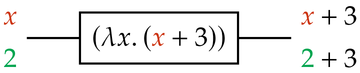
In general,

$$\begin{array}{c} x \\ N \end{array} \;—\; \boxed{(\lambda x.\,M)} \;—\; \begin{array}{c} M \\ ? \end{array}$$

$$((\lambda x.\,M)\,N) = ?$$

where we write $M[N/x]$ for the result of substituting N for x in M (e.g., $(x+3)[2/x] = 2+3$).



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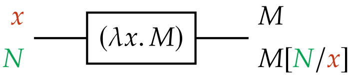

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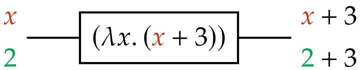

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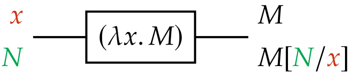

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where we write $M[N/x]$ for the result of substituting N for x in M (e.g., $(x+3)[2/x] = 2+3$).

***β*-reduction** is an application of this transformation

$$((\lambda x.\,M)\,N) \xrightarrow{\beta} M[N/x]$$

to terms and subterms. Computation means to keep applying this until it no longer applies.

(From now on, we omit outermost parens.)

E.g.,

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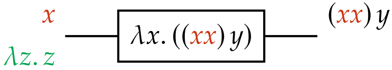

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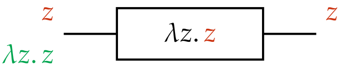

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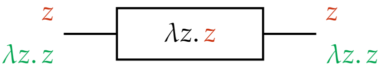

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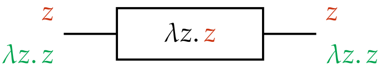

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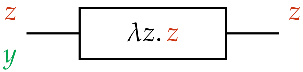

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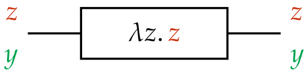

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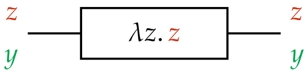

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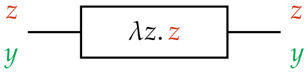

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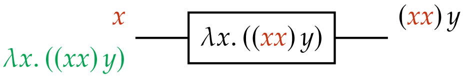

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2. $(\lambda x.\,((xx)\,y))(\lambda x.\,((xx)\,y)) \longrightarrow ((\lambda x.\,((xx)\,y))(\lambda x.\,((xx)\,y)))\,y$

$$\begin{array}{r} x \\ \lambda x.\,((xx)\,y) \end{array} \;\text{—}\; \boxed{\lambda x.\,((xx)\,y)} \;\text{—}\; \begin{array}{l} (xx)\,y \\ ((\lambda x.\,((xx)\,y))(\lambda x.\,((xx)\,y)))\,y \end{array}$$

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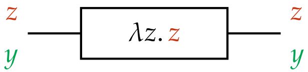

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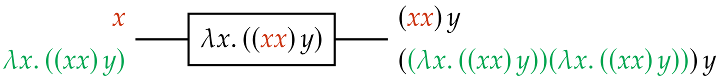

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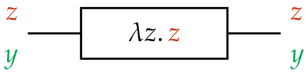
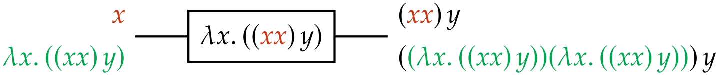


2. $$\begin{aligned}(\lambda x.\, ((xx)\, y))(\lambda x.\, ((xx)\, y)) &\longrightarrow ((\lambda x.\, ((xx)\, y))(\lambda x.\, ((xx)\, y)))\, y \\ &\longrightarrow (((\lambda x.\, ((xx)\, y))(\lambda x.\, ((xx)\, y)))\, y)\, y\end{aligned}$$



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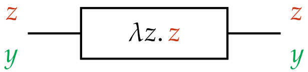

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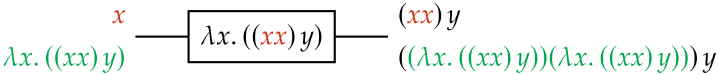

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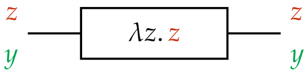

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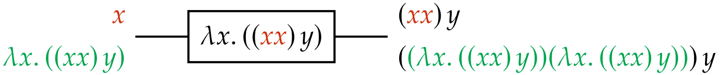

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E.g.,

❶ $$(\lambda x.\, ((xx)\, y))(\lambda z.\, z) \longrightarrow ((\lambda z.\, z)(\lambda z.\, z))\, y \longrightarrow (\lambda z.\, z)\, y \longrightarrow y$$

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$$\begin{array}{r} x \\ \lambda x.\, ((xx)\, y) \end{array} \;—\; \boxed{\lambda x.\, ((xx)\, y)} \;—\; \begin{array}{l} (xx)\, y \\ ((\lambda x.\, ((xx)\, y))(\lambda x.\, ((xx)\, y)))\, y \end{array}$$

Thus, according to the model "computation = β-reduction", some computation halts (like ❶) but other computation fails to (like ❷).

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— We obtain $2 + 3$ in two steps, using

$$\text{plus} := \lambda x.\, (\lambda y.\, (x + y)).$$

First apply this to 2 and obtain "$2 +$ (blank)", and then apply it to 3 and obtain $2 + 3$.

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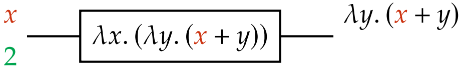

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$$(\text{plus}\,2)\,3 = ((\lambda x.\,(\lambda y.\,(x + y)))\,2)\,3 \longrightarrow (\lambda y.\,(2 + y))\,3$$

$$\begin{matrix} x \\ 2 \end{matrix} \;\text{---}\; \boxed{\lambda x.\,(\lambda y.\,(x + y))} \;\text{---}\; \begin{matrix} \lambda y.\,(x + y) \\ \lambda y.\,(2 + y) \end{matrix}$$

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$$(\text{plus}\,2)\,3 = ((\lambda x.\,(\lambda y.\,(x + y)))\,2)\,3 \to (\lambda y.\,(2 + y))\,3 \to 2 + 3$$

$$\begin{array}{c} x \\ 2 \end{array} \;\text{———}\; \boxed{\lambda x.\,(\lambda y.\,(x + y))} \;\text{———}\; \begin{array}{l} \lambda y.\,(x + y) \\ \lambda y.\,(2 + y) \end{array}$$

In this way, we technically do not need multi-input functions.

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Definition. We say that a term F of the lambda calculus "λ-defines" a k-argument partial function f of natural numbers if

- whenever $f(n_1, \ldots, n_k)$ is defined and equals n,

$$((((F\overline{n_1})\,\overline{n_2})\cdots)\,\overline{n_k}) \to \cdots \to \bar{n},$$

- but when $f(n_1, \ldots, n_k)$ is undefined, the β-reduction of $((((F\overline{n_1})\,\overline{n_2})\cdots)$ never terminates.

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Theorem. For any partial function f of natural numbers,

$$\begin{tikzcd}
 & f \text{ is Turing computable} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
f \text{ is partial recursive} \arrow[rr, Leftrightarrow] & & f \text{ is } \lambda\text{-definable.}
\end{tikzcd}$$