Monash University
Faculty of Information Technology

FIT2014 Theory of Computation

Lecture 22
Undecidability

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Overview

- Halting Problem (or Entscheidungsproblem)
- Proof of its undecidability
- Using mapping reductions to prove undecidability
- Other undecidable problems

Undecidable languages exist

The set of all deciders is countable.

- $\{\text{CWL-encodings of deciders}\} \subseteq \{\text{CWL}\} \subseteq \Sigma^*$
- and Σ^* is countable. (Lecture 5)

The set of all decidable languages is countable.

The set of *all* languages is <u>un</u>countable. (Lecture 5)

Therefore undecidable languages exist.

Halting Problem: Definition

Halting Problem
INPUT: Turing machine P, input x
QUESTION: If P is run with input x, does it eventually halt?

As a language:

$$\text{HaltingProbem} \quad := \quad \{\langle P, x\rangle : \text{when } P \text{ is run with input } x\text{, it eventually halts.}\}$$

Theorem.
The Halting Problem is undecidable.

Proved by:

- Alonzo Church (1936): lambda calculus
- Alan Turing (1936-37): Turing machines

Halting Problem

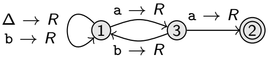
Theorem.
The Halting Problem is undecidable.

Proof ingredients:

- contradiction
- diagonalisation
- a version of the Liar Paradox: "This sentence is false."

Consider what happens when we run Turing machines (encoded as strings) on input strings.
✓ = Halts; ✗ = Doesn't halt.

inputs to TMs

Turing machines	ε	a	b	aa	ab	ba	bb	aaa	aab	…
ε	✓	✗	✗	✓	✗	✓	✓	✓	✗	…
a	✗	✗	✓	✓	✗	✗	✗	✓	✓	…
b	✓	✓	✗	✗	✗	✓	✗	✗	✓	…
aa	✓	✗	✓	✓	✗	✗	✓	✓	✗	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	…
 ababababbababbbbbbabaaabaaaab-aaababababbaaabaabaaaab	✗	✓	✗	✓	✗	✓	✗	✓	✓	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

E: ✗ ✓ ✓ ✗ …

Halting Problem is Undecidable

Proof. (by contradiction)

Assume there is a Decider, D, for the Halting Problem.

So it can tell, for any P and x, whether or not P eventually halts after being given input x.

So it can tell, for any P, whether or not P eventually halts after being given input P!

Construct another program (Turing machine) E as follows . . .

Halting Problem is Undecidable (cont'd)

E

Input: P
Use D to determine what happens if P runs on itself.
If D says, "P halts, with input P" : loop forever.
If D says, "P loops forever, with input P" : Halt.

What happens when E is given itself as input?

If E halts, for input E: **then** E loops forever, for input E.

If E loops forever, for input E: **then** E halts, for input E.

Contradiction! □

YouTube film of proof: `https://www.youtube.com/watch?v=92WHN-pAFCs`

Other Undecidable Problems

DIAGONAL HALTING PROBLEM
INPUT: Turing machine P
QUESTION: Does P eventually halt, for input P?

Above proof already shows this.

HALT FOR INPUT ZERO
INPUT: Turing machine P
QUESTION: Does P eventually halt, for input 0?

Theorem.
HALT FOR INPUT ZERO is undecidable.

We'll prove this by mapping reduction from the Diagonal Halting Problem.

Using mapping reductions

Recall:

If there is a mapping reduction f from K to L, then:

If L is decidable, then K is decidable.
If K is <u>un</u>decidable, then L is <u>un</u>decidable.

Proof. ...that HALT FOR INPUT ZERO is undecidable:

Let M be any program, which we regard as an input to the Diagonal Halting Problem. Define M' as follows:

M'

Input: x Run M on input M.

Observe:

- The construction $M \mapsto M'$ is computable.
- M halts on input M if and only if M' halts on input 0.

So, the function that sends $M \mapsto M'$ is a mapping reduction from DIAGONAL HALTING PROBLEM to HALT FOR INPUT ZERO.

Therefore HALT FOR INPUT ZERO is undecidable. $\square$

Other Undecidable Problems

There's nothing special about zero, here.
So we get a whole lot of undecidability results.

For example:

HALT FOR INPUT 42
INPUT: Turing machine P
QUESTION: Does P eventually halt, for input 42?

Proof of undecidability is virtually identical to the previous one . . .
Use a mapping reduction, with 42 instead of 0.

Other Undecidable Problems

ALWAYS HALTS
INPUT: Turing machine P
QUESTION: Does P always halt eventually, for any input?

Theorem.
ALWAYS HALTS is undecidable.

Proof is virtually identical to the previous one ...

Proof. ...that ALWAYS HALTS is undecidable:

Let M be any program, which we regard as an input to the Diagonal Halting Problem. Define M' as follows:

M'

Input: x Run M on input M.

Observe:

- The construction $M \mapsto M'$ is computable.
- M halts on input M if and only if M' *always* halts.

So, the function that sends $M \mapsto M'$ is a mapping reduction from DIAGONAL HALTING PROBLEM to ALWAYS HALTS.

Therefore ALWAYS HALTS is undecidable. $\square$

Other Undecidable Problems

SOMETIMES HALTS
Input: Turing machine P
Question: Is there some input for which P eventually halts?

Theorem.
SOMETIMES HALTS is undecidable.

Proof is virtually identical to the previous one ...

Proof. ...that SOMETIMES HALTS is undecidable:

Let M be any program, which we regard as an input to the Diagonal Halting Problem. Define M' as follows:

M'

Input: x
Run M on input M.

Observe:

- The construction $M \mapsto M'$ is computable.
- M halts on input M if and only if M' halts for *some* input.

So, the function that sends $M \mapsto M'$ is a mapping reduction from DIAGONAL HALTING PROBLEM to SOMETIMES HALTS.

Therefore SOMETIMES HALTS is undecidable. □

Other Undecidable Problems

NEVER HALTS
INPUT: Turing machine P
QUESTION: Does P always loop forever, for any input?

Theorem.
NEVER HALTS is undecidable.

Proof. by a more general type of reduction, from SOMETIMES HALTS.

If D is a decider for NEVER HALTS, then switching Accept and Reject gives a decider for SOMETIMES HALTS.
But we now know that SOMETIMES HALTS is undecidable.
Contradiction.
So NEVER HALTS is undecidable too. $\square$

Other Undecidable Problems

INPUT: Turing machine P and Q
QUESTION: Do P and Q always both halt, or both loop?
i.e., is it the case that:

$$\forall x : P \text{ halts on input } x \iff Q \text{ halts on input } x \qquad \ldots ?$$

INPUT: Turing machine P
QUESTION: If P is run on the input "What's the answer?", does it output "42"?

Decidable or Undecidable?

INPUT: Turing machine P, input x.
QUESTION: Does P accept x?

INPUT: Turing machine P, input x, positive integer t
QUESTION: When P is run on x, does it halt in $\leq t$ steps?

INPUT: Turing machine P, positive integer s.
QUESTION: Does P have $\leq s$ states?

INPUT: Turing machine P, positive integer k.
QUESTION: Does P halt for some input of length $\leq k$.

Other Undecidable Problems

INPUT: a Turing machine P
QUESTION: Is Accept(P) regular?
i.e., is P equivalent to a Finite Automaton?

INPUT: a CFG
QUESTION: is the language it generates regular?

INPUT: a CFG
QUESTION: is there any string that it doesn't generate? (over same alphabet)

INPUT: two CFGs.
QUESTION: Do they define the same language?

Other Undecidable Problems

INPUT: a polynomial (in several variables)
QUESTION: Does it have an integer root?

(Y. Matiyasevich, 1970)

https://mathshistory.st-andrews.ac.uk/Biographies/Matiyasevich/

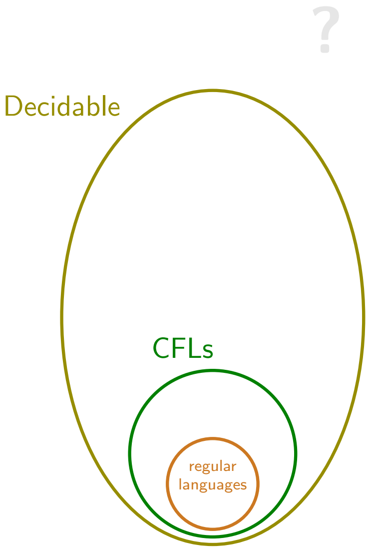
Yuri Matiyasevich (b. 1947)

Post Correspondence Problem
(a problem about string matching;
see Sipser, Section 5.2)

https://mathshistory.st-andrews.ac.uk/Biographies/Post/

Emil Post (1897–1954)

Language classes



Revision

- Know and understand the Halting Problem
- Prove its undecidability
- Be able to use mapping reductions to prove undecidability
- Know examples of undecidable problems.

Reading: Sipser, pp. 201–209, 215–220, 234–236.

Preparation: Sipser, pp. 170, 209–210.