PHIL 222
Philosophical Foundations of Computer Science
Week 2, Thursday

Sept. 5, 2024

From Philosophy to Computer Science:
Entscheidungsproblem

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Hilbert and Ackermann (1928):

> The *Entscheidungsproblem* [decision problem] is solved when we know a procedure that allows for any given logical expression to decide by finitely many operations its validity or satisfiability [. . .]. The *Entscheidungsproblem* must be considered the main problem of mathematical logic.

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- Even though there is no perfect set of axioms, at least can it be mechanically decided whether an inference is valid or not?
  — Church (1936) and Turing (1936): No, with full theories of "mechanically". The *Entscheidungsproblem* is unsolvable.

There are (at least) three approaches to the theory of what can be done “mechanically”, or what computers can do — “computability theory”:

1. Gödel: “recursive functions”.
2. Church: “λ-calculus”.
3. Turing: “Turing machines”.

Turing Machines

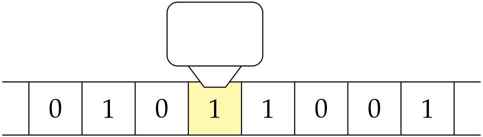
Advice

This is a technical part of the course that will be covered in the technical exercises and the midterm exam.

If anything here does not "click" in your mind,

- *Come to see me in office hours & appointments!*

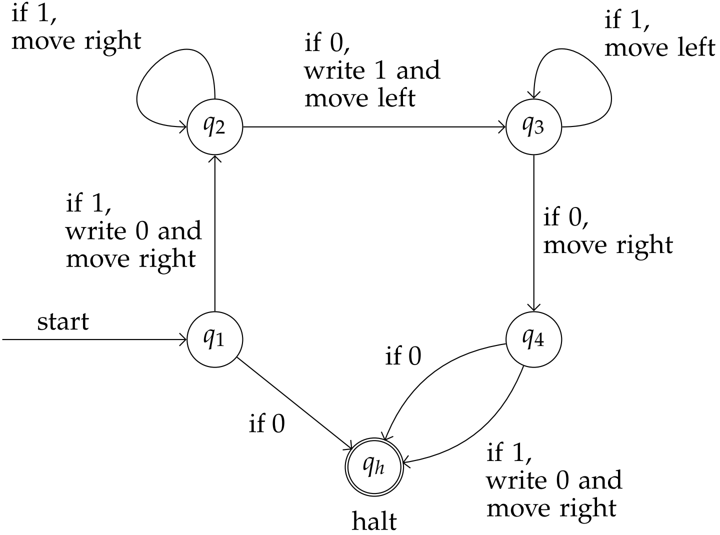
You are only expected to learn what the rule of the game is like. It is absolutely natural if it does not "click" in your mind for the first time you see it. But to resolve that situation, you need interactive help. Please, please help me help you.



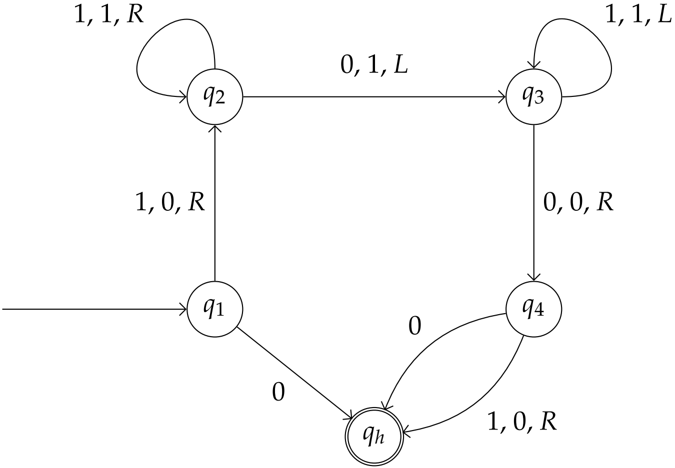
Two components:

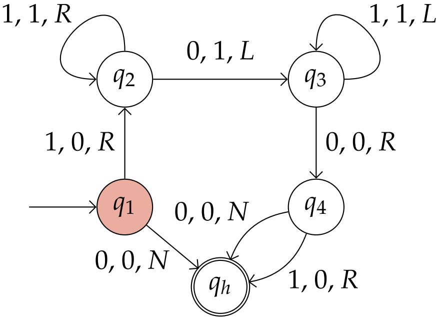
- A tape divided into squares. It may have infinitely many squares. Each square either is blank or contains a symbol from some finite alphabet.
- A machine called a **Turing machine**. At any time it is above one of the squares on the tape, and reads the symbol in it. It may then
  - (re)write a symbol or delete the symbol in the square; and
  - move one square to the left or right; or
  - halt.

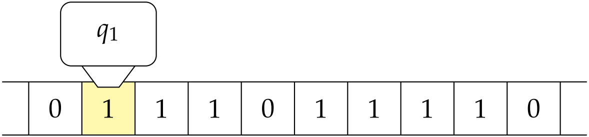
  The machine determines what to do at each step by its "(internal) state", its built-in instruction, and the symbol it reads.



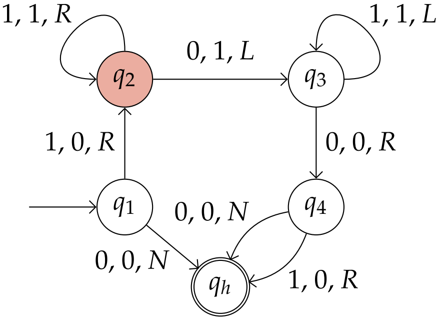
if 1,
move right
if 0,
write 1 and
move left
if 1,
move left
q_2
q_3
if 1,
write 0 and
move right
if 0,
move right
start
q_1
q_4
if 0
if 0
q_h
halt
if 1,
write 0 and
move right

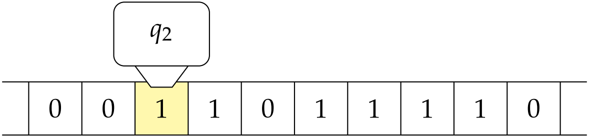

1, 1, R
0, 1, L
1, 1, L
q_2
q_3
1, 0, R
0, 0, R
q_1
q_4
0
0
q_h
1, 0, R



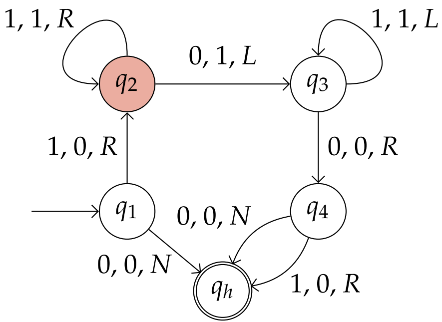


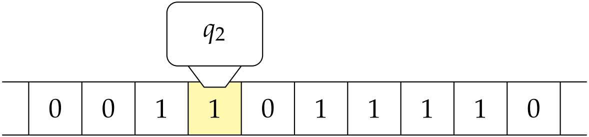
$t = 0$



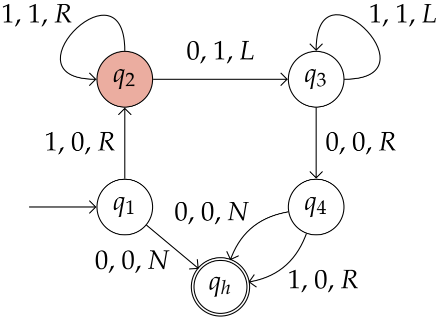


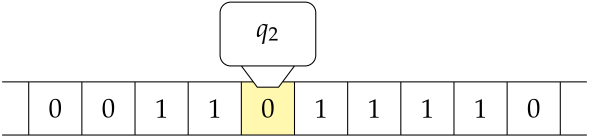
$t = 1$



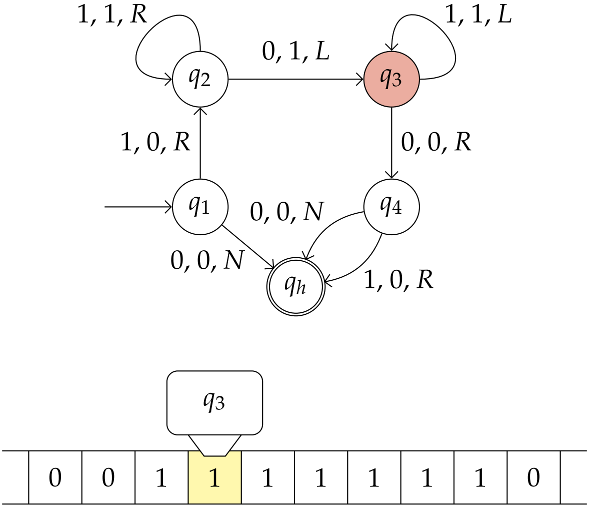


$t = 2$

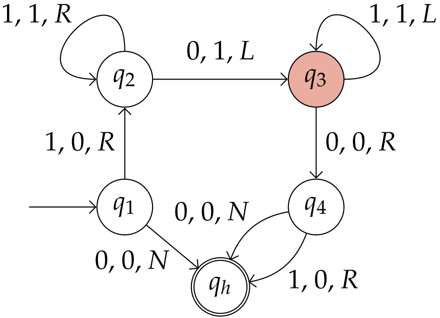


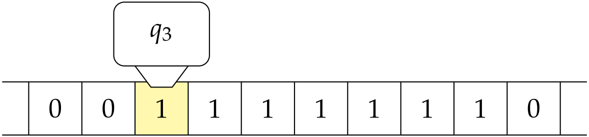


$t = 3$

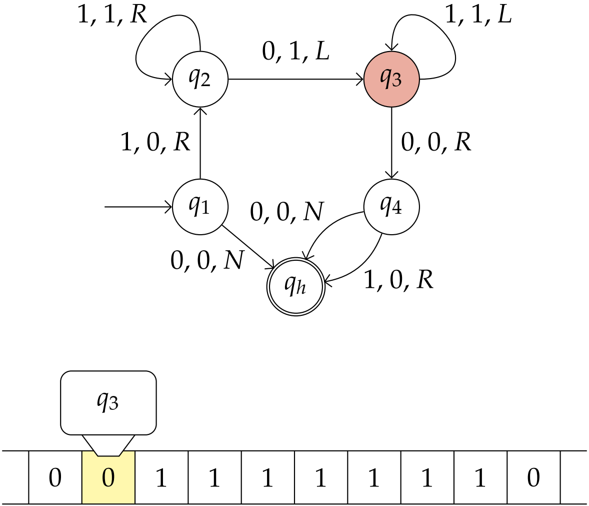


$t = 4$





$t = 5$



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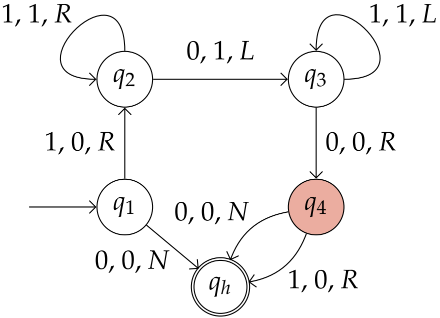
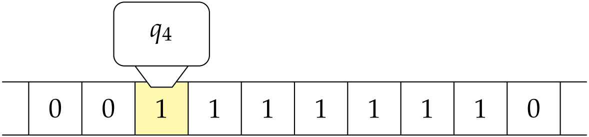
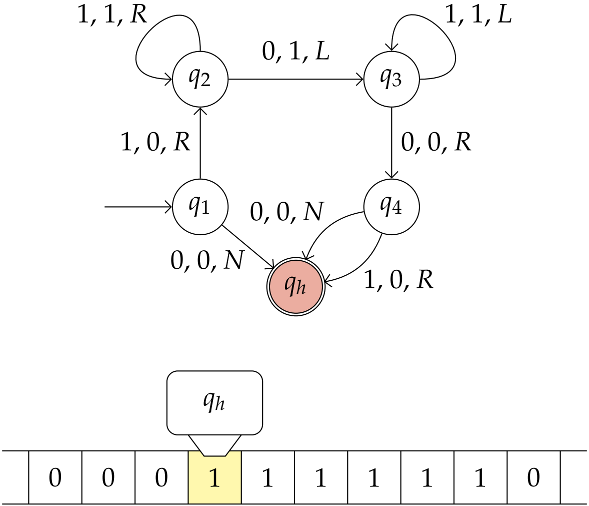




$t = 7$



$t = 8$

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- 3 strokes and 4 strokes (separated by a 0) $\mapsto$ 6 strokes

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   etc., etc. . . .

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ⓑ the function $(n, m) \mapsto n + m$ in the representation ⓑ.

NB. In this way, what function a Turing machine computes depends on coding! (A point which may be relevant to PHIL 223.)

Let's play a bit with
`https://turingmachine.io/`

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It seems correct to say that anything Turing computable is computed mechanically or effectively. How about the opposite? We'll see . . . .

Turing Machines:
What is a Function?

We think of a **function** f as a box that takes a fixed number of numbers as inputs and outputs a fixed number of numbers.

In the mathematician's notation,

$$f : (n_1, \dots, n_k) \mapsto f(n_1, \dots, n_k).$$

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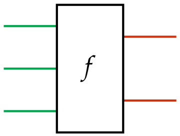
In a picture, a box with input wires and output wires:



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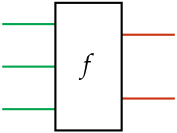

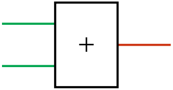

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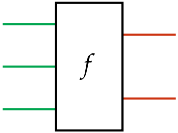

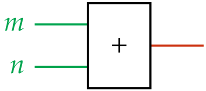

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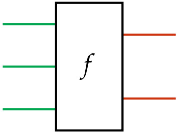

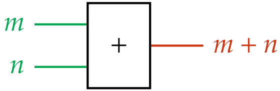

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1. Once you feed in an input (a combination of numbers), the output must be unique. I.e., there cannot be two different outputs for the same input.
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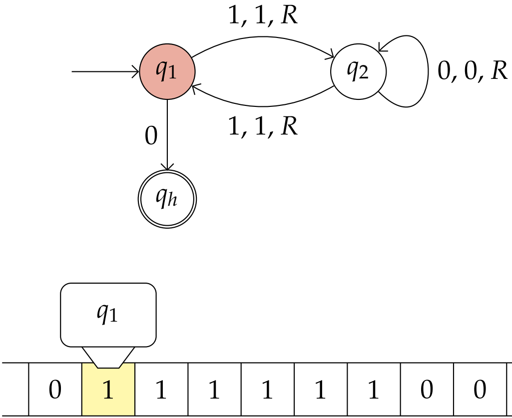
In technical terms:

Definition. A "partial function" f is a mapping of numbers to numbers s.th., although the value $f(n_1, \ldots, n_k)$ may be undefined, if it is defined then it is unique.

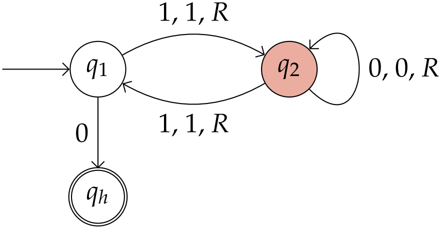
We say a function is "total" to mean that its values are always defined.

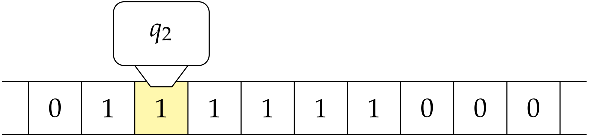
Consider the following Turing machine.



$t = 0$

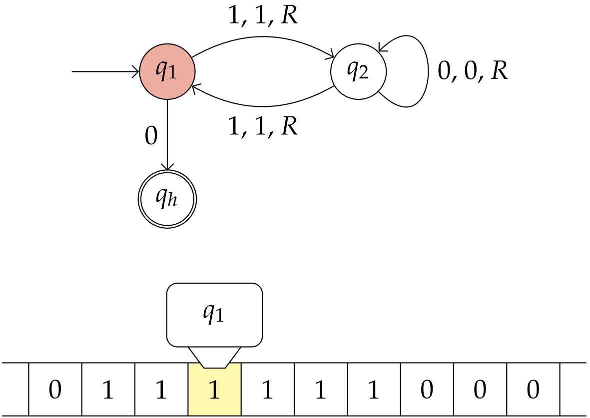
Consider the following Turing machine.





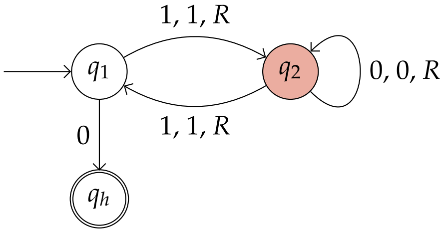
$t = 1$

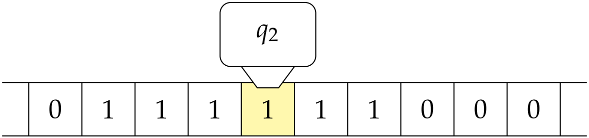
Consider the following Turing machine.



$t = 2$

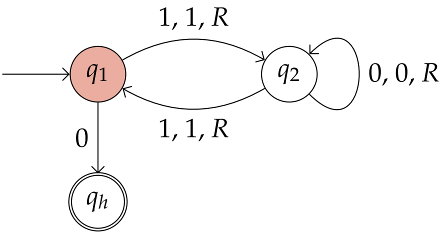
Consider the following Turing machine.

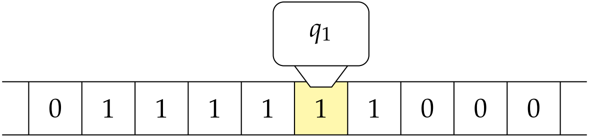




$t = 3$

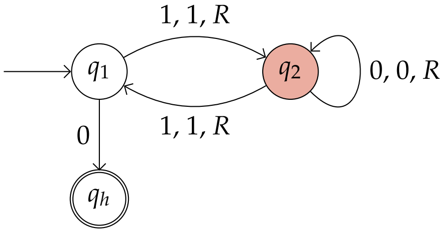
Consider the following Turing machine.

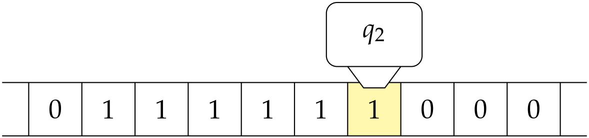




$t = 4$

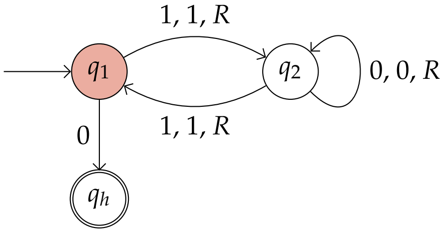
Consider the following Turing machine.

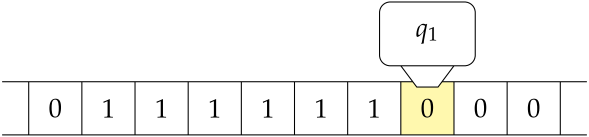




$t = 5$

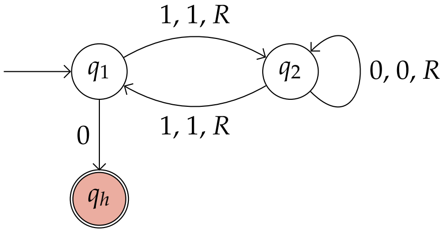
Consider the following Turing machine.

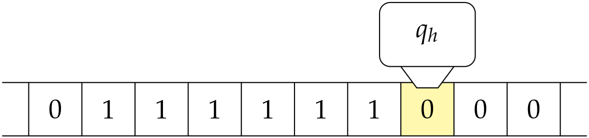




$t = 6$

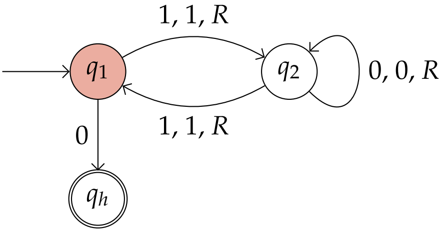
Consider the following Turing machine.

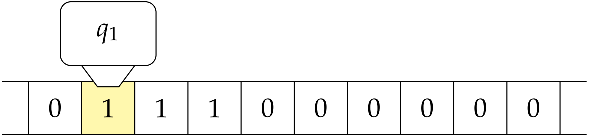




$t = 7$

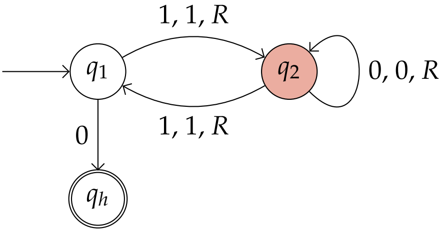
Consider the following Turing machine.

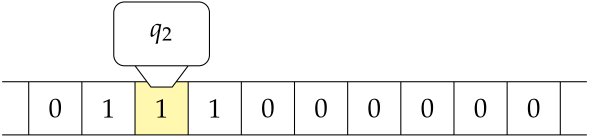




$t = 0$

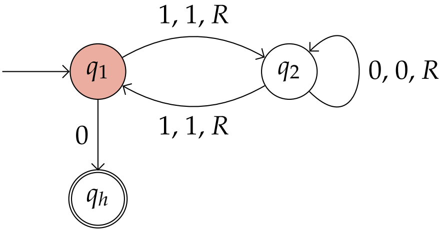
Consider the following Turing machine.

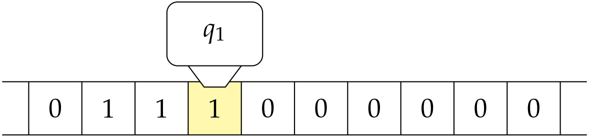




$t = 1$

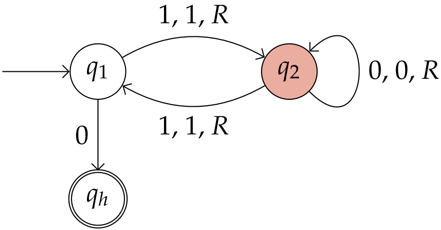
Consider the following Turing machine.

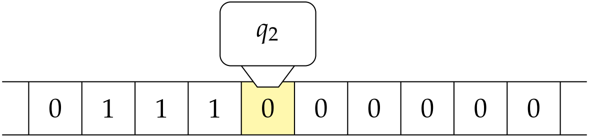




$t = 2$

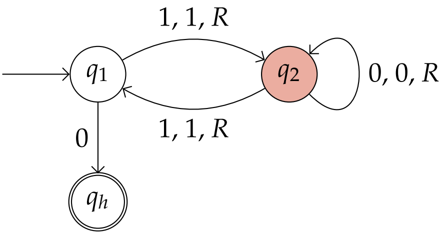
Consider the following Turing machine.

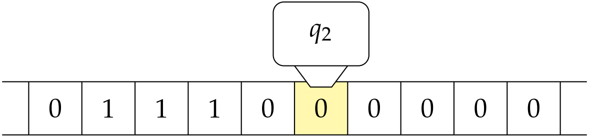




$t = 3$

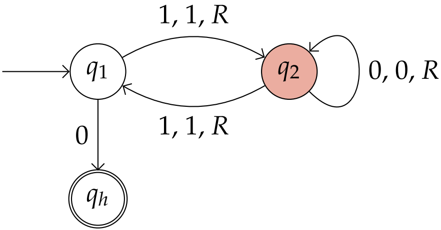
Consider the following Turing machine.

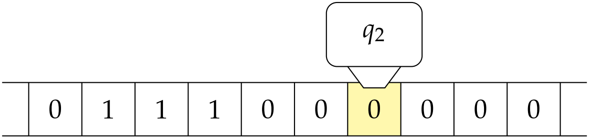




$t = 4$

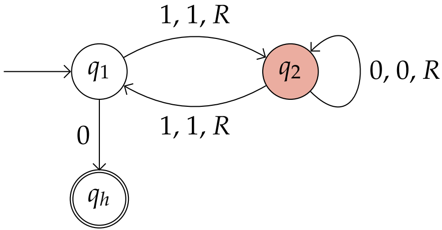
Consider the following Turing machine.

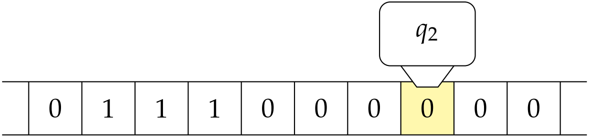




$t = 5$

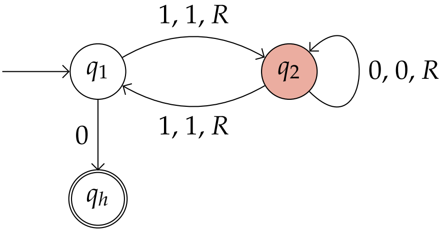
Consider the following Turing machine.

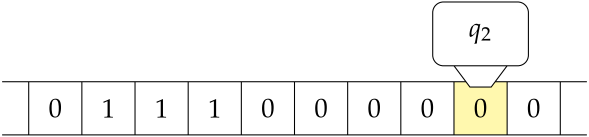




$t = 6$

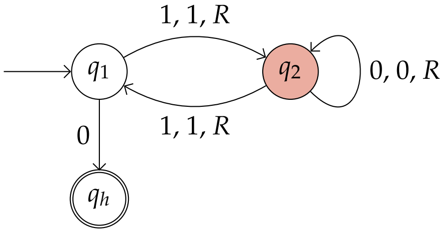
Consider the following Turing machine.

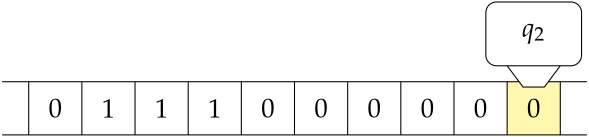




$t = 7$

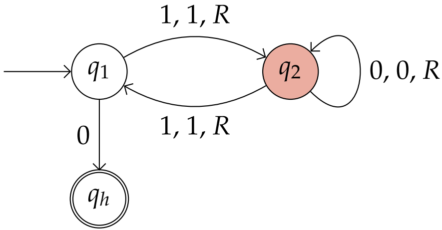
Consider the following Turing machine.

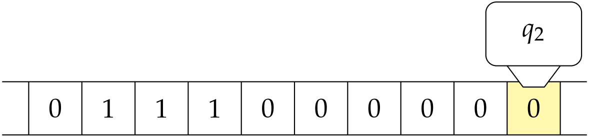




$t = 8$

Consider the following Turing machine.





$t = 8$

Some Turing machines, depending on the input, may never halt!