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Instance optimality for hp-type approximation

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Adaptive approximation by piecewise polynomials can be generalized in different ways. One of the most investigated forms of it, often used in finite element methods (FEM), is the *hp-approximation* in which the local size of the elements of the partition and the degree of the polynomials may vary but the total number of degrees of freedom is controlled.

Given a domain D, we partition it using a fixed binary refinement scheme and relate each partition to a binary tree $\mathcal{T}$ with root D and leaves $\mathcal{L}(\mathcal{T})$ that correspond to the elements of the partition. As usual, subdividing the element corresponding to the node $\Delta \in \mathcal{L}(\mathcal{T})$ into two is related to adding two nodes Δ' and Δ'' to the tree and connecting them to Δ. We referred to Δ as "parent" of the new nodes Δ' and Δ'', called "children" of Δ. Note that the binary tree $\mathcal{T}$ should be *admissible*, namely, with the exception of the root for each node Δ of $\mathcal{T}$ its sibling is also in $\mathcal{T}$.

We consider a sequence of polynomial spaces $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset ...$ of orders $p = 1, 2, 3, ...$ and assume that the orders correspond to the numbers of degrees of freedom introduced by the polynomial space. Evidently, in multidimensional settings the order cannot correspond to the degree of the polynomial space. There are two options to recommend in such cases: *(i)* to consider $\mathcal{P}_k$ to be a linear combination of k monomials; or *(ii)* to set $\mathcal{P}_k = \mathcal{P}_{k-1}$ in case k is not high enough to match the number of degrees of freedom of the next polynomial space of choice.

Given a tree $\mathcal{T}$, we define an hp-tree $\mathcal{T}^{\rm hp}$ and an approximating space $\mathcal{A}(\mathcal{T}^{\rm hp})$ by decorating each leave $\Delta \in \mathcal{L}(\mathcal{T})$ with a polynomial space $\mathcal{P}_{p(\Delta)}$ of a *node-specific* order $p(\Delta)$. The complexity $N = \mathcal{N}(\mathcal{T}^{\rm hp})$ of the approximation by functions from $\mathcal{A}(\mathcal{T}^{\rm hp})$ is set to be $\mathcal{N}(\mathcal{T}^{\rm hp}) := \sum_{\Delta\in\mathcal{L}(\mathcal{T})} p(\Delta)$.

Given a function f, we define its best hp-type approximation as

$$\sigma_n(f) := \inf_{\mathcal{N}(\mathcal{T}^{\rm hp})\le n} \inf_{f_n\in\mathcal{A}(\mathcal{T}^{\rm hp})} E_f(f_n) \ ,$$

where $E_f(f_n) = E(\mathcal{T}^{\rm hp})$ is the error of approximating f by f_n.

The goal of this research is to examine hp-adaptive approximation and to establish for it results similar to the ones from [3] used in [2] and several other papers

as an important ingredient in proving optimal rates of convergence for adaptive FEM.

We assume that for any node $\Delta \in \mathcal{T}$ we can calculate the local error of approximation $e_k(\Delta)$ by functions from $\mathcal{P}_k$ at the element of the partition corresponding to Δ. We also assume that these errors satisfy the subadditivity condition that in this setup is as follows:

$$e_k(\Delta) \geq e_{k+1}(\Delta) \text{ for } k \geq 1 \qquad \text{and} \qquad e_1(\Delta) \geq e_1(\Delta') + e_1(\Delta'')\,,$$

where Δ' and Δ'' are the children of Δ. A weaker assumption, similar to the one considered in [3], can be handled using the same general ideas.

The total error $E(\mathcal{T}^{\mathrm{hp}})$ is defined as

$$E(\mathcal{T}^{\mathrm{hp}}) := \sum_{\Delta \in \mathcal{L}(\mathcal{T}^{\mathrm{hp}})} e_{p(\Delta)}(\Delta)\,.$$

Note that because of the subadditivity condition often the total error is not exactly the norm of the difference $f - f_n$. For example, in L_2 we have to define $E_f(f_n) := \|f - f_n\|^2_{L_2}$.

We want to establish a coarse-to-fine algorithm that analyzes the errors at the current tree $\mathcal{T}^{\mathrm{hp}}_N$ and decides how to define the next tree $\mathcal{T}^{\mathrm{hp}}_{N+1}$ with the degrees of freedom increased by one. To this end, we introduce the admissible binary tree $\mathcal{T}^{\mathrm{h}}_N$ with root node D and number of leaves $\#\mathcal{L}(\mathcal{T}^{\mathrm{h}}_N) = N$. For each node $\Delta \in \mathcal{T}^{\mathrm{h}}_N$ define $T^{\mathrm{h}}_N(\Delta)$ to be the maximal subtree of $\mathcal{T}^{\mathrm{h}}_N$ with root Δ. The order $p(\Delta) := \#\mathcal{L}(T^{\mathrm{h}}_N(\Delta))$ of a node $\Delta \in \mathcal{T}^{\mathrm{h}}_N$ is defined as the number of leaves of the tree $T^{\mathrm{h}}_N(\Delta)$. Then for every subtree $\mathcal{T}$ of $\mathcal{T}^{\mathrm{h}}_N$ with root node D, the tree $\mathcal{T}^{\mathrm{hp}}_N$ is defined by assigning the polynomial orders $p(\Delta)$ for $\Delta \in \mathcal{L}(\mathcal{T})$.

We have to show now how to define the tree $\mathcal{T}^{\mathrm{h}}_N$ and then how to choose its subtree $\mathcal{T}$ in such a way that the error is as small as possible.

The first part of the algorithm follows the ideas from [3] and [1] and defines the modified error functionals

$$\tilde{e}^{\mathrm{h}}(D) := e_1(D) \qquad \text{and} \qquad \tilde{e}^{\mathrm{h}}(\Delta) := \left(\frac{1}{e_1(\Delta)} + \frac{1}{\tilde{e}^{\mathrm{h}}(\Delta^+)}\right)^{-1}\,,$$

where Δ^+ is the parent of the node $\Delta \neq D$. Although these quantities are good indicators how to grow the tree $\mathcal{T}^{\mathrm{h}}_N$, they have to be further adjusted to take into account the improvements of the approximation provided by the hp-option. The adjusted quantities $\tilde{e}(\Delta)$ are defined below. We then grow $\mathcal{T}^{\mathrm{h}}_N$ to $\mathcal{T}^{\mathrm{h}}_{N+1}$ by subdividing the leaf $\Delta_N \in \mathcal{L}(\mathcal{T}^{\mathrm{h}}_N)$ with the largest $\tilde{e}(\Delta_N) = \tilde{e}_N := \max_{\Delta \in \mathcal{L}(\mathcal{T}^{\mathrm{h}}_N)} \tilde{e}(\Delta)$.

In the second part of the algorithm we check in a fine-to-coarse manner which subtree $\mathcal{T}$ of $\mathcal{T}^{\mathrm{h}}_N$ gives the smallest total error. We start with $\mathcal{T} = \mathcal{T}^{\mathrm{h}}_N$ and set $\tilde{e}(\Delta) := \tilde{e}^{\mathrm{h}}(\Delta)$ for all leaves $\Delta \in \mathcal{L}(\mathcal{T}^{\mathrm{h}}_N)$. In the course of the algorithm we observe the dynamic quantities

$$E^{\mathrm{hp}}(\Delta) := \sum_{\Delta' \in \mathcal{L}(\mathcal{T}^{\mathrm{h}}_N(\Delta) \cap \mathcal{T})} e_{p(\Delta')}(\Delta')\,,$$

where $\mathcal{T}$ is the current subtree. We examine level by level, starting from the finest, the internal nodes Δ of $\mathcal{T}$ and compare this quantity with $e_{p(\Delta)}(\Delta)$. If $e_{p(\Delta)}(\Delta) < E^{\rm hp}(\Delta)$, we make Δ a leaf node of $\mathcal{T}$ by trimming all its descendants and also modify $\tilde{e}(\Delta') := \tilde{e}(\Delta')\frac{e_{p(\Delta)}(\Delta)}{E^{\rm hp}(\Delta)}$ for all $\Delta' \in \mathcal{L}(T_N^{\rm h}(\Delta))$; otherwise we do nothing at this step. It should be clear that after examining all the internal nodes this algorithm will result in a subtree $\mathcal{T}$ and the corresponding decorated tree $\mathcal{T}^{\rm hp}$ with minimal total error $E(\mathcal{T}^{\rm hp})$ among all the subtrees $\mathcal{T}$ of $\mathcal{T}_N^{\rm h}$. We denote this optimal hp-tree by $\mathcal{T}_N^{\rm hp}$.

It is important to note that once the tree $\mathcal{T}_N^{\rm hp}$ is found, it is not necessary to repeat the second part of the algorithm in its entirety to obtain $\mathcal{T}_{N+1}^{\rm hp}$ but only to reexamine the quantities related to the node Δ, subdivided to receive $\mathcal{T}_{N+1}^{\rm h}$ from $\mathcal{T}_N^{\rm h}$, and all its ancestors. Therefore, the complexity of the algorithm varies between $\mathcal{O}(N \log N)$ for well balanced trees to $\mathcal{O}(N^2)$ for highly unbalanced ones.

The analysis of the performance of the algorithm is based on the comparison of the resulting tree $\mathcal{T}_N^{\rm hp}$ with the best possible hp-tree $T_n^\star$ with n degrees of freedom. We set $t := \min_{k\le N} \tilde{e}_k$ and estimate in terms of t the errors $E^{\rm hp}(\mathcal{T}_N^{\rm hp})$ from above and $E^{\rm hp}(T_n^\star)$ from below to obtain the following result.

Theorem. *Let $\mathcal{T}_N^{\rm hp}$ is the tree received by the algorithm. Then for $n \le N$ we have*

$$E^{\rm hp}(\mathcal{T}_N^{\rm hp}) \le \frac{N+n-1}{N-n+1}\, \sigma_n(f)\,.$$

Assuming that the calculation of each local error $e_k(\Delta)$ requires at most constant number of operations, the complexity of the algorithm cannot exceed $\mathcal{O}(N^2)$.

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Low-rank techniques applied to moment equations for the stochastic Darcy problem with lognormal permeability

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(joint work with Fabio Nobile, Daniel Kressner and Christine Tobler)

In many natural phenomena and engineering applications the problem data are either incompletely known or contain a certain level of uncertainty due to the material properties, boundary conditions, loading terms, domain geometry, etc. One way to treat and include this uncertainty in the model is to describe