We provide hints for part (a) of ESL 5.16. No hints are provided for (b),(c),(d) but you also need to solve them. The solution will be easier if one follows the hints for part (a), but any other correct solutions will also receive full credit.

ESL 5.16: Consider the ridge regression problem (5.53), and assume $M \geq N$. Assume you have a kernel K that computes the inner product $K(x, y) = \sum_{m=1}^{M} h_m(x)h_m(y)$.

Part (a):

• Show that $h(x) = VD_\gamma^{\frac{1}{2}}\phi(x)$ where $D_\gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_M)$ and V is $M \times M$ and orthogonal.

Hints:

1. Note both $\{h_m(x)\}$ and $\{\phi_m(x)\}$ are sets of basis functions, so we can express $\{h_m(x)\}$ as an expansion of $\{\phi_m(x)\}$
$$h_m(x) = \sum_{i=1}^{M} v_{mi}\gamma_i^{1/2}\phi_i(x) \quad m = 1, \ldots, M \text{ and } \gamma_i \geq 0 \text{ and } v_{mi} \in \Re. \tag{1}$$
We will be done if we prove V is orthonormal.
2. Using Equation (1) express $\sum_{m=1}^{M} h_m(x)h_m(y)$ in terms of v_{ij}, γ_i, ϕ_i.
3. Recall
$$K(x, y) = \sum_m h_m(x)h_m(y) = \sum_i \gamma_i\phi_i(x)\phi_i(y).$$
Equate the expression of $\sum_{m=1}^{M} h_m(x)h_m(y)$ from step 2 with $\sum_i \gamma_i\phi_i(x)\phi_i(y)$ and use uniqueness of representation to show V is orthonormal.

• Compute V and D_γ theoretically. (slight modification from the text question)

Hints: Check that
$$\int_{x\in\Re^p} h(x)h^T(x)\, dx = VD_\gamma D_\gamma V^T$$
as $\{\phi_i(x)\}$ is an orthonormal basis in the R.K.H.S i.e $\int_{x\in\Re^p} \phi_i(x)\phi_j(x)\, dx = \delta_{ij}$.
Now, an eigen value decomposition of $\int hh^T$ will give us V and D_γ.

• Show (5.63) is equivalent to (5.53).

Hints: Use the representation $h(x) = VD_\gamma^{\frac{1}{2}}\phi(x)$.