## 1 Algebraic Topology

**Definition 1.1** (de Rham complex).  $\Omega^*$  is the algebra generated over  $\mathbb{R}$  by  $dx_1, \ldots, dx_n$  subject to

- 1.  $(dx_i)^2 = 0$ ,
- 2.  $dx_i dx_j = -dx_i dx_i, i \neq j$ .

The  $C^{\infty}$  differential forms on  $\mathbb{R}$  are elements of

$$\Omega^*(\mathbb{R}^n) = \{ C^{\infty} \text{ functions on } \mathbb{R}^n \} \otimes_{\mathbb{R}} \Omega^*.$$
 (1)

We have  $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ , where  $\Omega^q(\mathbb{R}^n)$  consists of the  $C^{\infty}$  q-forms on  $\mathbb{R}^n$ . We define

$$d: \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n), \tag{2}$$

the exterior differentiation, by

- 1. if  $f \in \Omega^0(\mathbb{R}^n)$ , then  $df = \sum \partial f/\partial x_i dx_i$ , 2. if  $\omega = \sum f_I dx_I$ , then  $d\omega = \sum df_I dx_I$ , where  $dx_I = \int dx_I dx_I$

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J \, dx_I dx_J = (-1)^{\deg \tau \, \deg \omega} \omega \wedge \tau. \tag{3}$$

**Proposition 1.2.** d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega. \tag{4}$$

Proposition 1.3.  $d^2 = 0$ .

**Definition 1.4.** The q-th de Rham cohomology of  $\mathbb{R}^n$  is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\},$$
 (5)

where closed means in the kernel of d and exact means in the image of d. We denote by  $[\omega]$  the cohomology class of  $\omega$ .

*Remark.* Only the constant functions are relevant for  $\mathbb{R}^n$ .

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension 0,} \\ 0 & \text{otherwise.} \end{cases}$$
 (6)

**Definition 1.5.** A differential complex is a direct sum of Vector spaces  $C = \bigoplus_{q \in \mathbb{Z}} C^q$  if there are homomorpisms

$$\ldots \longrightarrow C^{q-1} \stackrel{d}{\longrightarrow} C^i \stackrel{d}{\longrightarrow} C^{q+1} \longrightarrow \ldots$$

with  $d^2 = 0$ . The cohomology of C is given by H(C) = $\bigoplus_{q\in\mathbb{Z}}H^q(C)$ , with

$$H^{q}(C) = (\ker d \cap C^{q})/(\operatorname{im} d \cap C^{q}). \tag{7}$$

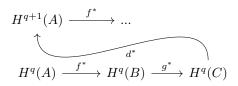
A map  $f: A \to B$  between two differential complexes is a chain map it it commutes with the differential operators of A and B,  $fd_A = d_B f$ . A sequence of vector spaces

$$\ldots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \ldots$$

is said to be exact if the image of  $f_{i-1}$  is the kernel of  $f_i$ . An exact sequence of the form

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups



 $f^*$ ,  $g^*$  are the naturally induced maps and  $d^*[c]$  is obtained through the commutative diagram

$$0 \longrightarrow A^{q+1} \stackrel{f}{\longrightarrow} B^{q+1} \stackrel{g}{\longrightarrow} C^{q+1} \longrightarrow 0$$

$$\downarrow d \qquad \downarrow 0$$

$$0 \longrightarrow A^{q} \stackrel{f}{\longrightarrow} B^{q} \stackrel{g}{\longrightarrow} C^{q} \longrightarrow 0$$

Since g is surjective there is  $b \in B^q$  with g(b) = c. Because g(db) = d(gb) = dc = 0, there is  $a \in A^{q+1}$  with db = f(a). Then  $d^*[c] := [a]$ . a is closed because f is injective. To see that the sequence is exact, note that if b is closed then f(a) = 0, and due to injectivity a = 0. On the other hand, f(a) is exact and therefore [f(a)] = 0.

**Definition 1.6.**  $\Omega_c^*(\mathbb{R}^n)$  is the de Rham complex for functions of compact support,  $H_c^*(\mathbb{R})$  is its cohomology.

Remark. Only the n-forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n,} \\ 0 & \text{otherwise.} \end{cases}$$
 (8)