1 Functional Analysis

Theorem 1.1 (Zorn's lemma)**.** Let X be a nonenpty partially ordered set such that every linearly ordered(i.e. $x \prec y$ or $y \prec x$) subset has an upper bound in X. Then every linearly ordered set has some upper bound that is also maximal in X(i.e. for x maximal, $m \prec x$ implies $x = m$ for all m).

Definition 1.2. A metric space is a set M with a real-valued function $d(\cdot, \cdot)$ on $M \times M$, called metric, such that

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

The topology is generated by the base of open balls,

$$B_r(x) = \{y \in M : d(x, y) < r\}. \tag{1}$$

Definition 1.3. A sequence $\{x_n\}$ is called Cauchy if

$$\forall \epsilon > 0 \, \exists N : N \leq n, m \implies d(x_n, x_m) < \epsilon. \tag{2}$$

Proposition 1.4. Any convergent sequence is Cauchy.

Definition 1.5. A complex vector space V is called an inner product space with an inner product $(\cdot, \cdot)$ on $V \times V$ such that for all $x, y, z \in V$, $a \in \mathbb{C}$:

1. $(x, x) \geq 0$, $(x, x) = 0$ iff $x = 0$
2. $(x, y + z) = (x, y) + (x, z)$
3. $(x, ay) = a(x, y)$
4. $(x, y) = \overline{(y, x)}$

Definition 1.6. Two vectors x, y are called orthogonal if $(x, y) = 0$. An orthonormal set is a collection of vectors that are mutually orthogonal and such that $\|x\| := \sqrt{(x, x)} = 1$ for all x.

Theorem 1.7 (Pythagorean)**.** Let $\{x_n\}_{n=1}^N$ be orthonormal, then for all $x \in V$

$$\|x\|^2 = \sum_{n=1}^{N} |(x, x_n)|^2 + \left\| x - \sum_{n=1}^{N} (x_n, x) x_n \right\|^2 . \tag{3}$$

Corollary 1.8 (Bessel's inequality)**.** Let $\{x_n\}_{n=1}^N$ be orthonormal, then for all $x \in V$

$$\|x\|^2 \geq \sum_{n=1}^{N} |(x, x_n)|^2 . \tag{4}$$

Corollary 1.9 (Schwarz inequality)**.**

$$|(x, y)| \leq \|x\| \, \|y\| . \tag{5}$$

Remark. Parallelogram law:

$$\|x + y\| + \|x - y\| = 2 \|x\|^2 + 2 \|y\|^2 . \tag{6}$$

Theorem 1.10. Every inner product space is a normed linear space with norm $\|x\| := \sqrt{(x, x)}$.

Definition 1.11. A complete inner product space is called a Hilbert space, otherwise a pre-Hilbert space.

Definition 1.12. Two Hilbert spaces are called isomorphic if there is a linear operator U from $\mathcal{H}_1$ onto(i.e. surjective) $\mathcal{H}_2$, called unitary, such that $(Ux, Uy)_{\mathcal{H}_2} = (x, y)_{\mathcal{H}_1}$.

Lemma 1.13. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, $x \in \mathcal{H}$. Then there is a unique $z \in \mathcal{M}$ closest to x.

Theorem 1.14 (The projection theorem)**.** Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, $x \in \mathcal{H}$. Then x can be uniquely written as $x = z + w$, where $z \in \mathcal{M}$ and $w \in \mathcal{M}^\perp$.