1 Algebraic Topology

Definition 1.1 (de Rham complex)**.** Ω^* is the algebra generated over $\mathbb{R}$ by $dx_1, \ldots, dx_n$ subject to

1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on $\mathbb{R}$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \tag{1}$$

We have $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. We define

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n), \tag{2}$$

the exterior differentiation, by

1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f / \partial x_i \, dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \ldots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J \, dx_I dx_J = (-1)^{\deg \tau \deg \omega} \omega \wedge \tau. \tag{3}$$

Proposition 1.2. d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega. \tag{4}$$

Proposition 1.3. $d^2 = 0$.

Definition 1.4. The q-th de Rham cohomology of $\mathbb{R}^n$ is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}, \tag{5}$$

where closed means in the kernel of d and exact means in the image of d. We denote by $[\omega]$ the cohomology class of ω.

Remark. Only the constant functions are relevant for $\mathbb{R}^n$,

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Definition 1.5. A differential complex is a direct sum of Vector spaces $C = \oplus_{q \in \mathbb{Z}} C^q$ if there are homomorpisms

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d} C^i \xrightarrow{d} C^{q+1} \longrightarrow \cdots$$

with $d^2 = 0$. The cohomology of C is given by $H(C) = \oplus_{q \in \mathbb{Z}} H^q(C)$, with

$$H^q(C) = (\ker d \cap C^q)/(\operatorname{im} d \cap C^q). \tag{7}$$

A map $f : A \to B$ between two differential complexes is a chain map it it commutes with the differential operators of A and B, $f d_A = d_B f$. A sequence of vector spaces

$$\cdots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \cdots$$

is said to be exact if the image of f_{i-1} is the kernel of f_i. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{tikzcd} & H^{q+1}(A) \arrow[r, "f^*"] & \ldots \\ H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[ull, bend right, "d^*"'] \end{tikzcd}$$

f^*, g^* are the naturally induced maps and $d^*[c]$ is obtained through the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^q \arrow[r, "f"] \arrow[u, "d"] & B^q \arrow[r, "g"] \arrow[u, "d"] & C^q \arrow[r] \arrow[u, "d"] & 0 \end{tikzcd}$$

Since g is surjective there is $b \in B^q$ with $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, there is $a \in A^{q+1}$ with $db = f(a)$. Then $d^*[c] := [a]$. a is closed because f is injective. To see that the sequence is exact, note that if b is closed then $f(a) = 0$, and due to injectivity $a = 0$. On the other hand, $f(a)$ is exact and therefore $[f(a)] = 0$.

Definition 1.6. $\Omega_c^*(\mathbb{R}^n)$ is the de Rham complex for functions of compact support, $H_c^*(\mathbb{R})$ is its cohomology.

Remark. Only the n-forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Definition 1.7. $f : \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback on functions

$$f^*(g) = g \circ f. \tag{9}$$

On forms the pullback is defined as

$$f^*(\sum g_I dy_{i_1} \ldots dy_{i_q}) = \sum (g_I \circ f) df_{i_1} \ldots df_{i_q}, \tag{10}$$

with $f_i = y_i \circ f$ the i-th component of f, y_i the standard coordinates.

Proposition 1.8. f^* commutes with d. This shows that the cohomology is a diffeomorphism invariant.

Definition 1.9. Let $M = U \cup V$ with U, V open. Then we have the inclusions

$$M \leftarrow U \sqcup V \leftarrow U \cap V \tag{11}$$

where $\sqcup$ is the disjoint union(each element has a label indicating wether it's from U or V). Using the inclusions as pullbacks we get

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V). \tag{12}$$

The Mayer-Vietoris sequence is given by

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0, \tag{13}$$

with

$$\Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V); (\omega, \tau) \mapsto \tau - \omega. \tag{14}$$

Proposition 1.10. The Mayer-Vietoris sequence is exact. This is achieved through partitions of unity ρ,

$$(\rho_U \omega) - (-\rho_V \omega) = \omega. \tag{15}$$

Definition 1.11. The Mayer-Vietoris sequence induces a long exact sequence with the same name:

$$\begin{tikzcd} H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \\ H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[llu, "d^*"'] \end{tikzcd}$$

Explicitly

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U, \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \tag{16}$$

Definition 1.12. If $j : U \to M$ is the inclusion of U in M, then let $j_* : \Omega^*_c(U) \to \Omega^*_c(M)$ the map which extends a form to M by zero. Because pullbacks of compact forms are in general not compact, we instead use the inclusions

$$\Omega^*_c(M) \xleftarrow[\text{sum}]{} \Omega^*_c(U) \oplus \Omega^*_c(V) \xleftarrow[\delta]{} \Omega^*_c(U \cap V)$$

$$\delta : \omega \mapsto (-j_* \omega, j_* \omega). \tag{17}$$

We then get the Mayer-Vietoris sequence

$$0 \leftarrow \Omega^*_c(M) \leftarrow \Omega^*_c(U) \oplus \Omega^*_c(V) \leftarrow \Omega^*_c(U \cap V) \leftarrow 0, \tag{18}$$

which induces

$$\begin{tikzcd} H^{q+1}_c(M) & H^{q+1}_c(U) \oplus H^{q+1}_c(V) \arrow[l] & H^{q+1}_c(U \cap V) \arrow[l] \\ H^q_c(M) \arrow[rru, "d^*"] & H^q_c(U) \oplus H^q_c(V) \arrow[l] & H^q_c(U \cap V) \arrow[l] \end{tikzcd}$$

and we now instead get

$$d^*[\omega] = \begin{cases} [-d(\rho_U \omega)] & \text{on } U, \\ [d(\rho_V \omega)] & \text{on } V. \end{cases} \tag{19}$$

Proposition 1.13. The Mayer-Vietoris sequence of forms with compact support is exact.

Proposition 1.14. A manifold of dimension n is orientable iff it has a global nowhere vanishing n-form.

Definition 1.15. Let $\mathbb{H}^n = \{x_n \geq 0\} \subset \mathbb{R}^n$ with standard orientation $dx_1 \dots dx_n$. The induced orientation of $\partial\mathbb{H}^n = \{x_n = 0\}$ is given by the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$. For an orientation-preserving diffeomorphism ϕ we define for manifolds

$$[\partial M] = \phi^*[\partial\mathbb{H}^n]. \tag{20}$$

Remark. This definition is due to $\omega|_{\partial M} := i_{\hat{n}}\omega$ for the normal $\hat{n}$.

Theorem 1.16 (Stokes')**.** Let ω be an $(n-1)$-form with compact support on an oriented manifold M, then

$$\int_M d\omega = \int_{\partial M} \omega. \tag{21}$$

Definition 1.17. Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$; $\pi(x,t) = x$ and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^1$; $s(x) = (x, 0)$. Trivially $\pi \circ s = 1$, $s^* \circ \pi^* = 1$, but $s \circ \pi \neq 1$. K is called a homotopy operator if

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd). \tag{22}$$

$dK \pm Kd$ maps closed forms to exact forms, therefore induces zero in cohomology. If K exists, $\pi^* \circ s^*$ is said to be chain homotopic to the identity. We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$(\pi^*\phi)f(x,t) \mapsto 0, \quad (\pi^*\phi)f(x,t)dt \mapsto (\pi^*\phi)\int_0^t dt\, f \tag{23}$$

with ϕ a form on $\mathbb{R}^n$.

Proposition 1.18. K is a homotopy operator. The maps π^*, s^* on $H^*(\mathbb{R}^n \times \mathbb{R}) \leftrightarrow H^*(\mathbb{R}^n)$ are isomorphisms.

Corollary 1.19 (Poincaré Lemma)**.**

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension 0,} \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

More generally $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$.

Corollary 1.20 (Homotopy Axiom for de Rham Cohomology)**.** Homotopic maps induce the same map in cohomology.

Definition 1.21. Two Manifolds M, N have the same homotopy type if there are C^∞ maps $f : M \to N$, $g : N \to M$ such that $g \circ f$, $f \circ g$ are C^∞ homotopic to the identity on M, N respectively. A manifold is called contractible if it has the homotopy type of a point.

Corollary 1.22. Manifolds with the same homotopy type have the same de Rham cohomology.

Definition 1.23. $r : M \to A$ is called a retraction of M onto A if $r \circ i : A \to A$ is the identity, with $i : A \subset M$ the inclusion. If $i \circ r : M \to M$ is homotopic to the identitiy on M, then r is called a deformation retraction of M onto A, and it follows that M, A have the same homotopy type.

Corollary 1.24. If A is a deformation retract of M, then A, M have the same de Rham cohomology.

Corollary 1.25.

$$H^*(S^n) = \begin{cases} R & \text{in dimension 0, n} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Definition 1.26. Let $\pi : M \times \mathbb{R}^1 \to M$ be the projection. We define the push-forward for compact forms $\pi_* : \Omega^*_c(M \times \mathbb{R}^1) \to \Omega^{*-1}_c(M)$, called integration along the fiber by

$$\pi^*\phi f(x,t) \mapsto 0, \quad \pi^*\phi f(x,t)dt \mapsto \phi \int_{-\infty}^{\infty} dt\, f \tag{26}$$

where ϕ is a form on M. Let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}^1$ with total integral 1 and define

$$e_* : \Omega^*_c(M) \to \Omega^{*+1}_c(M \times \mathbb{R}^1); \quad \phi \to (\pi^*\phi) \wedge e. \tag{27}$$

The homotopy operator $K : \Omega^*_c(M \times \mathbb{R}^1) \to \Omega^{*-1}_c(M \times \mathbb{R}^1)$ is defined by

$$\pi^*\phi f \mapsto 0, \quad \pi^*\phi f dt \mapsto \pi^*\phi \int_{-\infty}^t f - \pi^*\phi \int_{-\infty}^t e \int_{-\infty}^{\infty} f. \tag{28}$$

Proposition 1.27. d commutes with both π_* and e_*.

Proposition 1.28. $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$ on $\Omega^q_c(M \times \mathbb{R}^1)$. The maps $H^*_c(M \times \mathbb{R}^1) \leftrightarrow H^{*-1}_d(M)$ are isomorphisms.

Corollary 1.29 (Poincaré Lemma for Compact Supports)**.**

$$H^*_c(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n} \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

The generator is an n-form of compact support with integral 1.

Definition 1.30. A map is proper if the inverse image of every compact set is compact.

Theorem 1.31 (Sard's)**.** The set of critical values of a smooth map $f : M \to N$ has measure zero.

Definition 1.32. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. Let α be a generator of $H_c^n(\mathbb{R}^n)$(integrates to 1), then

$$\deg f = \int_{\mathbb{R}^n} f^*\alpha \tag{30}$$

is the degree of f.

Proposition 1.33. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. If f is not surjective, then it has degree 0.

Theorem 1.34. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. The degree of f is an integer.

Definition 1.35. An open cover is called a good cover if all nonempty finite intersections $U_{p_1} \cap \cdots \cap U_{p_n}$ are diffeomorphic to $\mathbb{R}^n$. A manifold with a finite good cover is said to be of finite type. If $\mathcal{U}$, $\mathcal{V}$ are two covers, $\mathcal{U}$ is called the refinement of $\mathcal{V}$ if every U_p is contained in some V_q and write $\mathcal{V} < \mathcal{U}$.

Theorem 1.36. Every manifold has a good cover. If it is compact, the cover can be chosen to be finite. Every open cover has a refinement which is a good cover.

Corollary 1.37. The good covers are cofinal in the set of all covers of a manifold(For directed sets, $J \subset I$ is cofinal in I if for every $i \in I$ there is $j \in J$ with $i < j$).

Proposition 1.38. If a manifold has a finite good cover, then its (compact) cohomology is finite dimensional.

Lemma 1.39. The pairing $\langle \, , \, \rangle : V \otimes W \to \mathbb{R}$ for finite vector spaces is nondegenerate iff $v \mapsto \langle v, \, \rangle$ is an isomorphism $V \to W^*$.

Lemma 1.40. Given a commutative diagram of Abelian groups and group homomorphisms

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

where the rows are exact, if a, b, d, e are isomorphisms, so is c.

Lemma 1.41. The Mayer-Vietoris sequences may be paired together to form a sign-commutative diagram

$$\begin{tikzcd}
H^q(U \cup V) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[r, "d^*"] & H^{q+1}(U \cup V) \\
\otimes & \otimes & \otimes & \otimes \\
H_c^{n-q}(U \cup V) \arrow[d, "\int_{U \cup V}"] & H_c^{n-q}(U) \oplus H_c^{n-q}(V) \arrow[l] \arrow[d, "\int_U + \int_V"] & H_c^{n-q}(U \cap V) \arrow[l] \arrow[d, "\int_{U \cap V}"] & H_c^{n-q-1}(U \cup V) \arrow[l, "d_*"] \arrow[d, "\int_{U \cup V}"] \\
\mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R}
\end{tikzcd}$$

where sign-commutativity means that

$$\int_{U \cap V} \omega \wedge d_*\tau = \pm \int_{U \cup V} (d^*\omega) \wedge \tau, \tag{31}$$

with $\omega \in H^q(U \cap V), \tau \in H_c^{n-q-1}(U \cup V)$. This is equivalent to the sign-commutative diagram

$$\begin{tikzcd}
H^q(U \cup V) \arrow[r] \arrow[d] & H^q(U) \oplus H^q(V) \arrow[r] \arrow[d] & H^q(U \cap V) \arrow[d] \\
H_c^{n-q}(U \cup V)^* \arrow[r] & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* \arrow[r] & H_c^{n-q}(U \cap V)^*
\end{tikzcd}$$

Theorem 1.42 (Poincaré Duality)**.** If M is orientable and has a finite good cover, then

$$\int H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R} \tag{32}$$

is nondegenerate, or equivalently

$$H^q(M) \simeq (H_c^{n-q}(M))^*. \tag{33}$$

Corollary 1.43. If M is connected, oriented with dimension n, then $H_c^n(M) \simeq \mathbb{R}$. If M is also compact then $H^n(M) \simeq \mathbb{R}$.

Definition 1.44. Let G be a topological group acting effectively(only $1 \in G$ acts trivially) on a space F on the left. A surjection $\pi : E \to B$ between topological spaces is a fiber bundle with fiber F and structure group G(also called a G-bundle) if B has an open cover $\{U_a\}$ such that there are fiber-preserving homeomorphisms

$$\phi_a : E|_{U_a} \mapsto U_a \times F, \tag{34}$$

called trivialisations, and continuous transition functions with values in G

$$g_{ab} = \phi_a \phi_b^{-1}|_{\{x\} \times F} \in G. \tag{35}$$

$E_x = \pi^{-1}(x)$ is called the fiber at x. E, B are called total and base space respectively. If G is not explicitely defined then it is understood to be the group of diffeomorphisms of F.

Remark. The transition functions satisfy the cocycle condition

$$g_{ab} \cdot g_{bc} = g_{ac}. \tag{36}$$

Given a cocycle $\{g_{ab}\}$ we can construct a fiber bundle with $\{g_{ab}\}$ as its transition function,

$$E = \left(\coprod U_a \times F\right) / (x, y) \sim (x, g_{ab}(x)y). \tag{37}$$

Theorem 1.45 (Künneth Formula)**.** For two manifolds M, F, with at least one having a good finite cover, we have

$$H^*(M \times F) = H^*(M) \otimes H^*(F), \tag{38}$$

which means

$$H^n(M \times F) = \oplus_{p+q=n} H^p(M) \otimes H^q(F). \tag{39}$$

Also holds in the compact cohomology case.

Theorem 1.46 (Leray-Hirsch)**.** Let E be a fiber bundle over M with fiber F. Suppose M has a finite good cover. If there are global cohomology classes $e_1, \ldots, e_r$ on E which restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \ldots, e_r\}$ meaning that

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, \ldots, e_r\} \simeq H^*(M) \otimes H^*(F). \tag{40}$$

Definition 1.47. A (real) vector bundle of rank n is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL(n, \mathbb{R})$. A section of the vector bundle E over $U \subset B$ open is a map $s : U \to E$ such that $\pi \circ s$ is the identity on U. The space of all sections over U is denoted by $\Gamma(U, E)$. A frame on U is a collection of sections $s_1, \ldots, s_n$ over U such that they form a basis for each $E_x = \pi^{-1}(x)$, $x \in U$.

Lemma 1.48. If the cocycle $\{g'_{ab}\}$ comes from another trivialization, then there are maps $\lambda_a : U_a \to GL(n, \mathbb{R})$ such that

$$g_{ab} = \lambda_a g'_{ab} \lambda_b^{-1}. \tag{41}$$

The two cocycles are then said to be equivalent.

Proposition 1.49. Two vector bundles on M are isomorphic iff their cocycles are equivalent.

Definition 1.50. A bundle map is a fiber-preserving smooth map $f : E \to E'$ of vector bundles which is linear on the fibers. If it is possible to find an equivalent cocycle with values in a subgroup $H \subset GL(n, \mathbb{R})$ we say that the structure group of E may be reduced to H. A vector bundle is orientable if its structure group may be reduced to $GL^+(n, \mathbb{R})$(determinants are positive). A trivialisation is said to be oriented if the transition functions have positive determinant.

Proposition 1.51. The structure group of a real vector bundle of rank n can be reduced to $O(n)$. It can be reduced to $SO(n)$ iff the bundle is orientable.

Definition 1.52. The direct sum of two vector bundles over M is the direct sum of the local vector spaces and trivialisations, likewise for the tensor product. If

$$\phi_a : E|_{U_a} \mapsto U_a \times \mathbb{R}^n \tag{42}$$

is a trivialisation for E, then

$$(\phi_a^T)^{-1} : E^*|_{U_a} \mapsto U_a \times (\mathbb{R}^n)^* \tag{43}$$

is a trivialisation for E^*. The transition functions for E^* are then

$$(\phi_a^T)^{-1}\phi_b^T = ((\phi_a \phi_b^{-1})^T)^{-1} = (g_{ab}^T)^{-1}. \tag{44}$$

Definition 1.53. Let M and N be manifolds and $\pi : E \to M$ a vector bundle over M. A map $f : N \to M$ induces a vector bundle $f^1 E$ on N, called the pullback of E by f. This bundle is defined to be the subset of $N \times E$ given by

$$\{(n, e) \mid f(n) = \pi(e)\}. \tag{45}$$

$\mathrm{Vect}_k(M)$ are the isomorphism classes of rank k real vector bundles over M.

Remark. The transition functions for $f^{-1}E$ are the pullback functions $f^* g_{ab}$. For the composition of two maps g, f between three manifolds we have

$$(f \circ g)^{-1} E = g^{-1}(f^{-1}E). \tag{46}$$

Theorem 1.54 (Homotopy Property of Vector Bundles)**.** Let Y be a compact manifold. If f_0, f_1 are homotopic maps from Y to X and E is a vector bundle on X, then f_0^{-1} and f_1^{-1} are isomorphic.

Corollary 1.55. A vector bundle over a contractible manifold is trivial.

Remark. Also holds if compact is replaced by paracompact, and also for general topological spaces and continuous maps instead of manifolds and smooth maps.