

1 Functional Analysis

Definition 1.1. A complex vector space V is called an inner product space with an inner product (\cdot, \cdot) on $V \times V$ such that for all $x, y, z \in V, a \in \mathbb{C}$:

1. $(x, x) \geq 0, (x, x) = 0$ iff $x = 0$
2. $(x, y + z) = (x, y) + (x, z)$
3. $(x, ay) = a(x, y)$
4. $(x, y) = \overline{(y, x)}$

Definition 1.2. Two vectors x, y are called orthogonal if $(x, y) = 0$. An orthonormal set is a collection of vectors that are mutually orthogonal and such that $\|x\| := \sqrt{(x, x)} = 1$ for all x .

Theorem 1.3 (Pythagorean). Let $\{x_n\}_{n=1}^N$ be orthonormal, then for all $x \in V$

$$\|x\|^2 = \sum_{n=1}^N |(x, x_n)|^2 + \left\| x - \sum_{n=1}^N (x, x_n) x_n \right\|^2. \quad (1)$$

Corollary 1.4 (Bessel's inequality). Let $\{x_n\}_{n=1}^N$ be orthonormal, then for all $x \in V$

$$\|x\|^2 \geq \sum_{n=1}^N |(x, x_n)|^2. \quad (2)$$

Corollary 1.5 (Schwarz inequality).

$$|(x, y)| \leq \|x\| \|y\|. \quad (3)$$

Remark. Parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \quad (4)$$

Theorem 1.6. Every inner product space is a normed linear space with norm $\|x\| := \sqrt{(x, x)}$.

Definition 1.7. A complete inner product space is called a Hilbert space, otherwise a pre-Hilbert space.

Definition 1.8. Two Hilbert spaces are called isomorphic if there is a linear operator U from \mathcal{H}_1 onto (i.e. surjective) \mathcal{H}_2 , called unitary, such that $(Ux, Uy)_{\mathcal{H}_2} = (x, y)_{\mathcal{H}_1}$.