1 Algebraic Topology

Definition 1.1 (de Rham complex)**.** Ω^* is the algebra generated over $\mathbb{R}$ by $dx_1, \dots, dx_n$ subject to

1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on $\mathbb{R}$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \tag{1}$$

We have $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. We define

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n), \tag{2}$$

the exterior differentiation, by

1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i \, dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \dots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J \, dx_I dx_J = (-1)^{\deg\tau \deg\omega} \omega \wedge \tau. \tag{3}$$

Proposition 1.2. d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg\tau} \tau \wedge d\omega. \tag{4}$$

Proposition 1.3. $d^2 = 0$.

Definition 1.4. The q-th de Rham cohomology of $\mathbb{R}^n$ is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}, \tag{5}$$

where closed means in the kernel of d and exact means in the image of d. We denote by $[\omega]$ the cohomology class of ω.

Remark. Only the constant functions are relevant for $\mathbb{R}^n$,

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Definition 1.5. A differential complex is a direct sum of Vector spaces $C = \oplus_{q \in \mathbb{Z}} C^q$ if there are homomorpisms

$$\dots \longrightarrow C^{q-1} \xrightarrow{d} C^i \xrightarrow{d} C^{q+1} \longrightarrow \dots$$

with $d^2 = 0$. The cohomology of C is given by $H(C) = \oplus_{q \in \mathbb{Z}} H^q(C)$, with

$$H^q(C) = (\ker d \cap C^q)/(\mathrm{im}\, d \cap C^q). \tag{7}$$

A map $f : A \to B$ between two differential complexes is a chain map it it commutes with the differential operators of A and B, $f d_A = d_B f$. A sequence of vector spaces

$$\dots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \dots$$

is said to be exact if the image of f_{i-1} is the kernel of f_i. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{tikzcd} H^{q+1}(A) \arrow[r, "f^*"] & \dots & \\ H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[ull, bend left, "d^*"] \end{tikzcd}$$

f^*, g^* are the naturally induced maps and $d^*[c]$ is obtained through the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^q \arrow[r, "f"] \arrow[u, "d"] & B^q \arrow[r, "g"] \arrow[u, "d"] & C^q \arrow[r] \arrow[u, "d"] & 0 \end{tikzcd}$$

Since g is surjective there is $b \in B^q$ with $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, there is $a \in A^{q+1}$ with $db = f(a)$. Then $d^*[c] := [a]$. a is closed because f is injective. To see that the sequence is exact, note that if b is closed then $f(a) = 0$, and due to injectivity $a = 0$. On the other hand, $f(a)$ is exact and therefore $[f(a)] = 0$.

Definition 1.6. $\Omega_c^*(\mathbb{R}^n)$ is the de Rham complex for functions of compact support, $H_c^*(\mathbb{R})$ is its cohomology.

Remark. Only the n-forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Definition 1.7. $f : \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback on functions

$$f^*(g) = g \circ f. \tag{9}$$

On forms the pullback is defined as

$$f^*(\sum g_I dy_{i_1} \dots dy_{i_q}) = \sum (g_I \circ f) df_{i_1} \dots df_{i_q}, \tag{10}$$

with $f_i = y_i \circ f$ the i-th component of f, y_i the standard coordinates.

Proposition 1.8. f^* commutes with d. This shows that the cohomology is a diffeomorphism invariant.

Definition 1.9. Let $M = U \cup V$ with U, V open. Then we have the inclusions

$$M \leftarrow U \sqcup V \leftarrow U \cap V \tag{11}$$

where $\sqcup$ is the disjoint union(each element has a label indicating wether it's from U or V). Using the inclusions as pullbacks we get

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V). \tag{12}$$

The Mayer-Vietoris sequence is given by

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0, \tag{13}$$

with

$$\Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V); (\omega, \tau) \mapsto \tau - \omega. \tag{14}$$

Proposition 1.10. The Mayer-Vietoris sequence is exact. This is achieved through partitions of unity ρ,

$$(\rho_U \omega) - (-\rho_V \omega) = \omega. \tag{15}$$

Definition 1.11. The Mayer-Vietoris sequence induces a long exact sequence with the same name:

$$\begin{tikzcd} H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \\ H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[ull, "d^*"] \end{tikzcd}$$

Explicitly

$$d^*[\omega] = \begin{cases} [-d(\rho_V\omega)] & \text{on } U, \\ [d(\rho_U\omega)] & \text{on } V. \end{cases} \tag{16}$$

Definition 1.12. If $j : U \to M$ is the inclusion of U in M, then let $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ the map which extends a form to M by zero. Because pullbacks of compact forms are in general not compact, we instead use the inclusions

$$\Omega_c^*(M) \xleftarrow{\text{sum}} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{\delta} \Omega_c^*(U \cap V)$$

$$\delta : \omega \mapsto (-j_*\omega, j_*\omega). \tag{17}$$

We then get the Mayer-Vietoris sequence

$$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0, \tag{18}$$

which induces

$$\begin{tikzcd} H_c^{q+1}(M) & H_c^{q+1}(U) \oplus H_c^{q+1}(V) \arrow[l] & H_c^{q+1}(U \cap V) \arrow[l] \\ H_c^q(M) \arrow[urr, "d^*"] & H_c^q(U) \oplus H_c^q(V) \arrow[l] & H_c^q(U \cap V) \arrow[l] \end{tikzcd}$$

and we now instead get

$$d^*[\omega] = \begin{cases} [d(\rho_U\omega)] & \text{on } U, \\ [d(\rho_V\omega)] & \text{on } V. \end{cases} \tag{19}$$

Proposition 1.13. The Mayer-Vietoris sequence of forms with compact support is exact.

Proposition 1.14. A manifold of dimension n is orientable iff it has a global nowhere vanishing n-form.

Definition 1.15. Let $\mathbb{H}^n = \{x_n \geq 0\} \subset \mathbb{R}^n$ with standard orientation $dx_1 \dots dx_n$. The induced orientation of $\partial\mathbb{H}^n = \{x_n = 0\}$ is given by the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$. For an orientation-preserving diffeomorphism ϕ we define for manifolds

$$[\partial M] = \phi^*[\partial\mathbb{H}^n]. \tag{20}$$

Remark. This definition is due to $\omega|_{\partial M} := i_{\hat{n}}\omega$ for the normal $\hat{n}$.

Theorem 1.16 (Stokes')**.** Let ω be an $(n-1)$-form with compact support on an oriented manifold M, then

$$\int_M d\omega = \int_{\partial M} \omega. \tag{21}$$

Definition 1.17. Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$; $\pi(x,t) = x$ and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^1$; $s(x) = (x, 0)$. Trivially $\pi \circ s = 1$, $s^* \circ \pi^* = 1$, but $s \circ \pi \neq 1$. K is called a homotopy operator if

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd). \tag{22}$$

$dK \pm Kd$ maps closed forms to exact forms, therefore induces zero in cohomology. If K exists, $\pi^* \circ s^*$ is said to be chain homotopic to the identity. We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$(\pi^*\phi)f(x,t) \mapsto 0, \quad (\pi^*\phi)f(x,t)dt \mapsto (\pi^*\phi)\int_0^t dt\, f \tag{23}$$

with ϕ a form on $\mathbb{R}^n$.

Proposition 1.18. K is a homotopy operator. The maps π^*, s^* on $H^*(\mathbb{R}^n \times \mathbb{R}) \leftrightarrow H^*(\mathbb{R}^n)$ are isomorphisms.

Corollary 1.19 (Poincaré Lemma)**.**

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

More generally $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$.

Corollary 1.20 (Homotopy Axiom for de Rham Cohomology)**.** Homotopic maps induce the same map in cohomology.

Definition 1.21. Two Manifolds M, N have the same homotopy type if there are C^∞ maps $f : M \to N$, $g : N \to M$ such that $g \circ f$, $f \circ g$ are C^∞ homotopic to the identity on M, N respectively. A manifold is called contractible if it has the homotopy type of a point.

Corollary 1.22. Manifolds with the same homotopy type have the same de Rham cohomology.

Definition 1.23. $r : M \to A$ is called a retraction of M onto A if $r \circ i : A \to A$ is the identity, with $i : A \subset M$ the inclusion. If $i \circ r : M \to M$ is homotopic to the identitiy on M, then r is called a deformation retraction of M onto A, and it follows that M, A have the same homotopy type.

Corollary 1.24. If A is a deformation retract of M, then A, M have the same de Rham cohomology.

Corollary 1.25.

$$H^*(S^n) = \begin{cases} R & \text{in dimension 0, n} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Definition 1.26. Let $\pi : M \times \mathbb{R}^1 \to M$ be the projection. We define the push-forward for compact forms $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M)$, called integration along the fiber by

$$\pi^*\phi f(x,t) \mapsto 0, \quad \pi^*\phi f(x,t)dt \mapsto \phi\int_{-\infty}^{\infty} dt\, f \tag{26}$$

where ϕ is a form on M. Let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}^1$ with total integral 1 and define

$$e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R}^1); \quad \phi \to (\pi^*\phi) \wedge e. \tag{27}$$

The homotopy operator $K : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M \times \mathbb{R}^1)$ is defined by

$$\pi^*\phi f \mapsto 0, \quad \pi^*\phi f dt \mapsto \pi^*\phi\int_{-\infty}^t f - \pi^*\phi\int_{-\infty}^t e\int_{-\infty}^{\infty} f. \tag{28}$$

Proposition 1.27. d commutes with both π_* and e_*.

Proposition 1.28. $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$ on $\Omega_c^q(M \times \mathbb{R}^1)$. The maps $H_c^*(M \times \mathbb{R}^1) \leftrightarrow H_d^{*-1}(M)$ are isomorphisms.

Corollary 1.29 (Poincaré Lemma for Compact Supports)**.**

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n} \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

The generator is an n-form of compact support with integral 1.

Definition 1.30. A map is proper if the inverse image of every compact set is compact.

Theorem 1.31 (Sard's)**.** The set of critical values of a smooth map $f : M \to N$ has measure zero.

Definition 1.32. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. Let α be a generator of $H_c^n(\mathbb{R}^n)$(integrates to 1), then

$$\deg f = \int_{\mathbb{R}^n} f^* \alpha \tag{30}$$

is the degree of f.

Proposition 1.33. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. If f is not surjective, then it has degree 0.

Theorem 1.34. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. The degree of f is an integer.