1 Algebraic Topology

Definition 1.1 (de Rham complex)**.** Ω^* is the algebra generated over $\mathbb{R}$ by $dx_1, \dots, dx_n$ subject to
1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on $\mathbb{R}$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \tag{1}$$

We have $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. We define

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n), \tag{2}$$

the exterior differentiation, by
1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i \, dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \dots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J \, dx_I dx_J = (-1)^{\deg\tau \deg\omega} \omega \wedge \tau. \tag{3}$$

Proposition 1.2. d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg\tau} \tau \wedge d\omega. \tag{4}$$

Proposition 1.3. $d^2 = 0$.

Definition 1.4. The q-th de Rham cohomology of $\mathbb{R}^n$ is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}, \tag{5}$$

where closed means in the kernel of d and exact means in the image of d. We denote by $[\omega]$ the cohomology class of ω.

Remark. Only the constant functions are relevant for $\mathbb{R}^n$,

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Definition 1.5. A differential complex is a direct sum of Vector spaces $C = \oplus_{q \in \mathbb{Z}} C^q$ if there are homomorpisms

$$\dots \longrightarrow C^{q-1} \xrightarrow{d} C^i \xrightarrow{d} C^{q+1} \longrightarrow \dots$$

with $d^2 = 0$. The cohomology of C is given by $H(C) = \oplus_{q\in\mathbb{Z}} H^q(C)$, with

$$H^q(C) = (\ker d \cap C^q)/(\text{im } d \cap C^q). \tag{7}$$

A map $f : A \to B$ between two differential complexes is a chain map it it commutes with the differential operators of A and B, $f d_A = d_B f$. A sequence of vector spaces

$$\dots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \dots$$

is said to be exact if the image of f_{i-1} is the kernel of f_i. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{tikzcd} & H^{q+1}(A) \arrow[r, "f^*"] & \dots \\ H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[ull, bend left, "d^*"] \end{tikzcd}$$

f^*, g^* are the naturally induced maps and $d^*[c]$ is obtained through the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^q \arrow[r, "f"] \arrow[u, "d"] & B^q \arrow[r, "g"] \arrow[u, "d"] & C^q \arrow[r] \arrow[u, "d"] & 0 \end{tikzcd}$$

Since g is surjective there is $b \in B^q$ with $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, there is $a \in A^{q+1}$ with $db = f(a)$. Then $d^*[c] := [a]$. a is closed because f is injective. To see that the sequence is exact, note that if b is closed then $f(a) = 0$, and due to injectivity $a = 0$. On the other hand, $f(a)$ is exact and therefore $[f(a)] = 0$.

Definition 1.6. $\Omega_c^*(\mathbb{R}^n)$ is the de Rham complex for functions of compact support, $H_c^*(\mathbb{R})$ is its cohomology.

Remark. Only the n-forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n}, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Definition 1.7. $f : \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback on functions

$$f^*(g) = g \circ f. \tag{9}$$

On forms the pullback is defined as

$$f^*(\sum g_I dy_{i_1} \dots dy_{i_q}) = \sum (g_I \circ f) df_{i_1} \dots df_{i_q}, \tag{10}$$

with $f_i = y_i \circ f$ the i-th component of f, y_i the standard coordinates.

Proposition 1.8. f^* commutes with d. This shows that the cohomology is a diffeomorphism invariant.

Definition 1.9. Let $M = U \cup V$ with U, V open. Then we have the inclusions

$$M \leftarrow U \sqcup V \leftarrow U \cap V \tag{11}$$

where $\sqcup$ is the disjoint union(each element has a label indicating wether it's from U or V). Using the inclusions as pullbacks we get

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V). \tag{12}$$

The Mayer-Vietoris sequence is given by

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0, \tag{13}$$

with

$$\Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V); (\omega, \tau) \mapsto \tau - \omega. \tag{14}$$

Proposition 1.10. The Mayer-Vietoris sequence is exact. This is achieved through partitions of unity ρ,

$$(\rho_U \omega) - (-\rho_V \omega) = \omega. \tag{15}$$

Definition 1.11. The Mayer-Vietoris sequence induces a long exact sequence with the same name:

$$\begin{tikzcd}
H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \\
H^{q}(M) \arrow[r] & H^{q}(U) \oplus H^{q}(V) \arrow[r] & H^{q}(U \cap V) \arrow[llu, "d^*"']
\end{tikzcd}$$

Explicitly

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U, \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \tag{16}$$

Definition 1.12. If $j : U \to M$ is the inclusion of U in M, then let $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ the map which extends a form to M by zero. Because pullbacks of compact forms are in general not compact, we instead use the inclusions

$$\Omega_c^*(M) \xleftarrow[\text{sum}]{} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow[\delta]{} \Omega_c^*(U \cap V)$$

$$\delta : \omega \mapsto (-j_* \omega, j_* \omega). \tag{17}$$

We then get the Mayer-Vietoris sequence

$$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0, \tag{18}$$

which induces

$$\begin{tikzcd}
H_c^{q+1}(M) & H_c^{q+1}(U) \oplus H_c^{q+1}(V) \arrow[l] & H_c^{q+1}(U \cap V) \arrow[l] \\
H_c^{q}(M) \arrow[rru, "d^*"] & H_c^{q}(U) \oplus H_c^{q}(V) \arrow[l] & H_c^{q}(U \cap V) \arrow[l]
\end{tikzcd}$$

and we now instead get

$$d^*[\omega] = \begin{cases} [-d(\rho_U \omega)] & \text{on } U, \\ [d(\rho_V \omega)] & \text{on } V. \end{cases} \tag{19}$$

Proposition 1.13. The Mayer-Vietoris sequence of forms with compact support is exact.

Proposition 1.14. A manifold of dimension n is orientable iff it has a global nowhere vanishing n-form.

Definition 1.15. Let $\mathbb{H}^n = \{x_n \geq 0\} \subset \mathbb{R}^n$ with standard orientation $dx_1 \dots dx_n$. The induced orientation of $\partial \mathbb{H}^n = \{x_n = 0\}$ is given by the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$. For an orientation-preserving diffeomorphism ϕ we define for manifolds

$$[\partial M] = \phi^* [\partial \mathbb{H}^n]. \tag{20}$$

Remark. This definition is due to $\omega|_{\partial M} := i_{\hat{n}} \omega$ for the normal $\hat{n}$.

Theorem 1.16 (Stokes'). Let ω be an $(n-1)$-form with compact support on an oriented manifold M, then

$$\int_M d\omega = \int_{\partial M} \omega. \tag{21}$$

Definition 1.17. Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$; $\pi(x,t) = x$ and $s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^1$; $s(x) = (x, 0)$. Trivially $\pi \circ s = 1$, $s^* \circ \pi^* = 1$, but $s \circ \pi \neq 1$. K is called a homotopy operator if

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd). \tag{22}$$

$dK \pm Kd$ maps closed forms to exact forms, therefore induces zero in cohomology. If K exists, $\pi^* \circ s^*$ is said to be chain homotopic to the identity. We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$(\pi^* \phi) f(x,t) \mapsto 0, \quad (\pi^* \phi) f(x,t) dt \mapsto (\pi^* \phi) \int_0^t dt\, f \tag{23}$$

with ϕ a form on $\mathbb{R}^n$.

Remark. K can be used to calculate an inverse of d.

Proposition 1.18. K is a homotopy operator. The maps π^*, s^* on $H^*(\mathbb{R}^n \times \mathbb{R}) \leftrightarrow H^*(\mathbb{R}^n)$ are isomorphisms.

Corollary 1.19 (Poincaré Lemma).

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

More generally $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$.

Corollary 1.20 (Homotopy Axiom for de Rham Cohomology). Homotopic maps induce the same map in cohomology.

Definition 1.21. Two Manifolds M, N have the same homotopy type if there are C^∞ maps $f : M \to N$, $g : N \to M$ such that $g \circ f$, $f \circ g$ are C^∞ homotopic to the identity on M, N respectively. A manifold is called contractible if it has the homotopy type of a point.

Corollary 1.22. Manifolds with the same homotopy type have the same de Rham cohomology.

Definition 1.23. $r : M \to A$ is called a retraction of M onto A if $r \circ i : A \to A$ is the identity, with $i : A \subset M$ the inclusion. If $i \circ r : M \to M$ is homotopic to the identitiy on M, then r is called a deformation retraction of M onto A, and it follows that M, A have the same homotopy type.

Corollary 1.24. If A is a deformation retract of M, then A, M have the same de Rham cohomology.

Corollary 1.25.

$$H^*(S^n) = \begin{cases} R & \text{in dimension 0, n} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Definition 1.26. Let $\pi : M \times \mathbb{R}^1 \to M$ be the projection. We define the push-forward for compact forms $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M)$, called integration along the fiber by

$$\pi^* \phi f(x,t) \mapsto 0, \quad \pi^* \phi f(x,t) dt \mapsto \phi \int_{-\infty}^{\infty} dt\, f \tag{26}$$

where ϕ is a form on M. Let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}^1$ with total integral 1 and define

$$e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R}^1); \quad \phi \to (\pi^* \phi) \wedge e. \tag{27}$$

The homotopy operator $K : \Omega_c^*(M \times \mathbb{R}^1) \to \Omega_c^{*-1}(M \times \mathbb{R}^1)$ is defined by

$$\pi^* \phi f \mapsto 0, \quad \pi^* \phi f dt \mapsto \pi^* \phi \int_{-\infty}^t f - \pi^* \phi \int_{-\infty}^t e \int_{-\infty}^{\infty} f. \tag{28}$$

Proposition 1.27. d commutes with both π_* and e_*.

Proposition 1.28. $1 - e_* \pi_* = (-1)^{q-1}(dK - Kd)$ on $\Omega_c^q(M \times \mathbb{R}^1)$. The maps $H_c^*(M \times \mathbb{R}^1) \leftrightarrow H_d^{*-1}(M)$ are isomorphisms.

Corollary 1.29 (Poincaré Lemma for Compact Supports).

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n} \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

The generator is an n-form of compact support with integral 1.

Definition 1.30. A map is proper if the inverse image of every compact set is compact.

Theorem 1.31 (Sard's)**.** The set of critical values of a smooth map $f : M \to N$ has measure zero.

Definition 1.32. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. Let α be a generator of $H_c^n(\mathbb{R}^n)$(integrates to 1), then

$$\deg f = \int_{\mathbb{R}^n} f^*\alpha \tag{30}$$

is the degree of f.

Remark. This is independent of the generator, as changing the generator from α_1 to α_2 can be done by adding $\alpha_2 - \alpha_1$. As this term is exact, it is equal to some $d\beta$, whose integral vanishes by Stokes' theorem(because β is compact). Therefore, α_2 also has an integral of 1, i.e. compact n-forms in $\mathbb{R}^n$ represent the same vector in the compact de Rahm cohomology iff their integral is the same.

Proposition 1.33. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. If f is not surjective, then it has degree 0.

Theorem 1.34. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be proper. The degree of f is an integer.

Definition 1.35. An open cover is called a good cover if all nonempty finite intersections $U_{p_1} \cap \cdots \cap U_{p_n}$ are diffeomorphic to $\mathbb{R}^n$. A manifold with a finite good cover is said to be of finite type. If $\mathcal{U}$, $\mathcal{V}$ are two covers, $\mathcal{U}$ is called the refinement of $\mathcal{V}$ if every U_p is contained in some V_q and write $\mathcal{V} < \mathcal{U}$.

Theorem 1.36. Every manifold has a good cover. If it is compact, the cover can be chosen to be finite. Every open cover has a refinement which is a good cover.

Corollary 1.37. The good covers are cofinal in the set of all covers of a manifold(For directed sets, $J \subset I$ is cofinal in I if for every $i \in I$ there is $j \in J$ with $i < j$).

Proposition 1.38. If a manifold has a finite good cover, then its (compact) cohomology is finite dimensional.

Lemma 1.39. The pairing $\langle\, , \,\rangle : V \otimes W \to \mathbb{R}$ for finite vector spaces is nondegenerate iff $v \mapsto \langle v, \,\rangle$ is an isomorphism $V \to W^*$.

Lemma 1.40. Given a commutative diagram of Abelian groups and group homomorphisms

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

where the rows are exact, if a, b, d, e are isomorphisms, so is c.

Lemma 1.41. The Mayer-Vietoris sequences may be paired together to form a sign-commutative diagram

$$\begin{array}{ccccccc}
H^q(U\cup V) & \longrightarrow & H^q(U)\oplus H^q(V) & \longrightarrow & H^q(U\cap V) & \xrightarrow{d^*} & H^{q+1}(U\cup V) \\
\otimes & & \otimes & & \otimes & & \otimes \\
H_c^{n-q}(U\cup V) & \leftarrow & H_c^{n-q}(U)\oplus H_c^{n-q}(V) & \leftarrow & H_c^{n-q}(U\cap V) & \underset{d_*}{\leftarrow} & H_c^{n-q-1}(U\cup V) \\
\downarrow{\scriptstyle \int_{U\cup V}} & & \downarrow{\scriptstyle \int_U + \int_V} & & \downarrow{\scriptstyle \int_{U\cap V}} & & \downarrow{\scriptstyle \int_{U\cup V}} \\
\mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{array}$$

where sign-commutativity means that

$$\int_{U\cap V} \omega \wedge d_*\tau = \pm \int_{U\cup V} (d^*\omega) \wedge \tau, \tag{31}$$

with $\omega \in H^q(U\cap V), \tau \in H_c^{n-q-1}(U\cup V)$. This is equivalent to the sign-commutative diagram

$$\begin{array}{ccccc}
H^q(U\cup V) & \longrightarrow & H^q(U)\oplus H^q(V) & \longrightarrow & H^q(U\cap V) \\
\downarrow & & \downarrow & & \downarrow \\
H_c^{n-q}(U\cup V)^* & \to & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* & \to & H_c^{n-q}(U\cap V)^*
\end{array}$$

Theorem 1.42 (Poincaré Duality)**.** If M is orientable and has a finite good cover, then

$$\int H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R} \tag{32}$$

is nondegenerate, or equivalently

$$H^q(M) \simeq (H_c^{n-q}(M))^*. \tag{33}$$

Corollary 1.43. If M is connected, oriented with dimension n, then $H_c^n(M) \simeq \mathbb{R}$. If M is also compact then $H^n(M) \simeq \mathbb{R}$.

Theorem 1.44. Let $f : M \mapsto N$ be a map between two compact, oriented manifolds. Its degree is an integer, and is equal to the number of points in the inverse image of any regular point in N, counted with multiplicity ± 1.

Definition 1.45. Let G be a topological group acting effectively(only $1 \in G$ acts trivially) on a space F on the left. A surjection $\pi : E \to B$ between topological spaces is a fiber bundle with fiber F and structure group G(also called a G-bundle) if B has an open cover $\{U_a\}$ such that there are fiber-preserving homeomorphisms

$$\phi_a : E|_{U_a} \mapsto U_a \times F, \tag{34}$$

called trivialisations, and continuous transition functions with values in G

$$g_{ab} = \phi_a \phi_b^{-1}|_{\{x\}\times F} \in G. \tag{35}$$

$E_x = \pi^{-1}(x)$ is called the fiber at x. E, B are called total and base space respectively. If G is not explicitely defined then it is understood to be the group of diffeomorphisms of F.

Remark. The transition functions satisfy the cocycle condition

$$g_{ab} \cdot g_{bc} = g_{ac}. \tag{36}$$

Given a cocycle $\{g_{ab}\}$ we can construct a fiber bundle with $\{g_{ab}\}$ as its transition function,

$$E = \left(\coprod U_a \times F\right) / (x, y) \sim (x, g_{ab}(x)y). \tag{37}$$

Theorem 1.46 (Künneth Formula)**.** For two manifolds M, F, with at least one having a good finite cover, we have

$$H^*(M \times F) = H^*(M) \otimes H^*(F), \tag{38}$$

which means

$$H^n(M \times F) = \oplus_{p+q=n} H^p(M) \otimes H^q(F). \tag{39}$$

Also holds in the compact cohomology case.

Theorem 1.47 (Leray-Hirsch)**.** Let E be a fiber bundle over M with fiber F. Suppose M has a finite good cover. If there are global cohomology classes $e_1, \ldots, e_r$ on E which restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \ldots, e_r\}$ meaning that

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, \ldots, e_r\} \simeq H^*(M) \otimes H^*(F). \quad (40)$$

Definition 1.48. Let M be an oriented manifold of dimension n, S a closed oriented submanifold of dimension k, and let ω be a closed k-form with compact support on M. Integration over S induces a linear functional on $H_c^k(M)$. The Poincaré dual $[\mu_S]$ of S is then the Poincaré dual of the element in $H_c^k(M)$ dual to this functional. By definition

$$\int_S i^*\omega = \int_M \omega \wedge \mu_S. \quad (41)$$

One can demand the Poincaré dual of a compact S to have compact support.

Definition 1.49. A (real) vector bundle of rank n is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL(n, \mathbb{R})$. A section of the vector bundle E over $U \subset B$ open is a map $s : U \to E$ such that $\pi \circ s$ is the identity on U. The space of all sections over U is denoted by $\Gamma(U, E)$. A frame on U is a collection of sections $s_1, \ldots, s_n$ over U such that they form a basis for each $E_x = \pi^{-1}(x)$, $x \in U$.

Lemma 1.50. If the cocycle $\{g'_{ab}\}$ comes from another trivialization, then there are maps $\lambda_a : U_a \to GL(n, \mathbb{R})$ such that

$$g_{ab} = \lambda_a g'_{ab} \lambda_b^{-1}. \quad (42)$$

The two cocycles are then said to be equivalent.

Proposition 1.51. Two vector bundles on M are isomorphic iff their cocycles are equivalent.

Definition 1.52. A bundle map is a fiber-preserving smooth map $f : E \to E'$ of vector bundles which is linear on the fibers. If it is possible to find an equivalent cocycle with values in a subgroup $H \subset GL(n, \mathbb{R})$ we say that the structure group of E may be reduced to H. A vector bundle is orientable if its structure group may be reduced to $GL^+(n, \mathbb{R})$(determinants are positive). A trivialisation is said to be oriented if the transition functions have positive determinant.

Proposition 1.53. The structure group of a real vector bundle of rank n can be reduced to $O(n)$. It can be reduced to $SO(n)$ iff the bundle is orientable.

Definition 1.54. The direct sum of two vector bundles over M is the direct sum of the local vector spaces and trivialisations, likewise for the tensor product. If

$$\phi_a : E|_{U_a} \mapsto U_a \times \mathbb{R}^n \quad (43)$$

is a trivialisation for E, then

$$(\phi_a^T)^{-1} : E^*|_{U_a} \mapsto U_a \times (\mathbb{R}^n)^* \quad (44)$$

is a trivialisation for E^*. The transition functions for E^* are then

$$(\phi_a^T)^{-1}\phi_b^T = ((\phi_a \phi_b^{-1})^T)^{-1} = (g_{ab}^T)^{-1}. \quad (45)$$

Definition 1.55. Let M and N be manifolds and $\pi : E \to M$ a vector bundle over M. A map $f : N \to M$ induces a vector bundle $f^{-1}E$ on N, called the pullback of E by f. This bundle is defined to be the subset of $N \times E$ given by

$$\{(n, e) \mid f(n) = \pi(e)\}. \quad (46)$$

$\mathrm{Vect}_k(M)$ are the isomorphism classes of rank k real vector bundles over M.

Remark. The transition functions for $f^{-1}E$ are the pullback functions f^*g_{ab}. For the composition of two maps g, f between three manifolds we have

$$(f \circ g)^{-1}E = g^{-1}(f^{-1}E). \quad (47)$$

Theorem 1.56 (Homotopy Property of Vector Bundles)**.** Let Y be a compact manifold. If f_0, f_1 are homotopic maps from Y to a manifold X and E is a vector bundle on X, then $f_0^{-1}E$ and $f_1^{-1}E$ are isomorphic.

Corollary 1.57. A vector bundle over a contractible manifold is trivial.

Remark. Also holds if compact is replaced by paracompact, and also for general topological spaces and continuous maps instead of manifolds and smooth maps.

Remark. Let E be a vector bundle over M, then the zero section $x \mapsto (x, 0)$ embeds M diffeomorphically in E. Since $M \times \{0\}$ is a deformation retract of E, it follows from corollary 1.20 that

$$H^*(E) \simeq H^*(M). \quad (48)$$

Lemma 1.58. An orientable vector bundle over a manifold is an orientable manifold.

Proposition 1.59. If $\pi : E \to M$ is an orientable vector bundle of rank n and M is orientable of finite type, then $H_c^*(E) \simeq H_c^{*-n}(M)$.