

1 Algebraic Topology

Definition 1.1 (de Rham complex). Ω^* is the algebra generated over \mathbb{R} by dx_1, \dots, dx_n subject to

1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on \mathbb{R} are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \quad (1)$$

We have $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q -forms on \mathbb{R}^n . We define

$$d : \Omega^q(\mathbb{R}^n) \rightarrow \Omega^{q+1}(\mathbb{R}^n), \quad (2)$$

the exterior differentiation, by

1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f / \partial x_i dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \dots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J dx_I dx_J = (-1)^{\deg \tau \deg \omega} \omega \wedge \tau. \quad (3)$$

Proposition 1.2. d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega. \quad (4)$$

Proposition 1.3. $d^2 = 0$.

Definition 1.4. The q -th de Rham cohomology of \mathbb{R}^n is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\} / \{\text{exact } q\text{-forms}\}, \quad (5)$$

where closed means in the kernel of d and exact means in the image of d . We denote by $[\omega]$ the cohomology class of ω .

Remark. Only the constant functions are relevant for \mathbb{R}^n ,

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Definition 1.5. A differential complex is a direct sum of Vector spaces $C = \bigoplus_{q \in \mathbb{Z}} C^q$ if there are homomorphisms

$$\dots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \dots$$

with $d^2 = 0$. The cohomology of C is given by $H(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C)$, with

$$H^q(C) = (\ker d \cap C^q) / (\text{im } d \cap C^q). \quad (7)$$

A map $f : A \rightarrow B$ between two differential complexes is a chain map if it commutes with the differential operators of A and B , $f d_A = d_B f$. A sequence of vector spaces

$$\dots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \dots$$

is said to be exact if the image of f_{i-1} is the kernel of f_i . An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{array}{ccccc} H^{q+1}(A) & \xrightarrow{f^*} & \dots & & \\ & \searrow d^* & & & \\ H^q(A) & \xrightarrow{f^*} & H^q(B) & \xrightarrow{g^*} & H^q(C) \end{array}$$

f^*, g^* are the naturally induced maps and $d^*[c]$ is obtained through the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{q+1} & \xrightarrow{f} & B^{q+1} & \xrightarrow{g} & C^{q+1} \longrightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \longrightarrow & A^q & \xrightarrow{f} & B^q & \xrightarrow{g} & C^q \longrightarrow 0 \end{array}$$

Since g is surjective there is $b \in B^q$ with $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, there is $a \in A^{q+1}$ with $db = f(a)$. Then $d^*[c] := [a]$. a is closed because f is injective. To see that the sequence is exact, note that if b is closed then $f(a) = 0$, and due to injectivity $a = 0$. On the other hand, $f(a)$ is exact and therefore $[f(a)] = 0$.

Definition 1.6. $\Omega_c^*(\mathbb{R}^n)$ is the de Rham complex for functions of compact support, $H_c^*(\mathbb{R})$ is its cohomology.

Remark. Only the n -forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Definition 1.7. $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ induces a pullback on functions

$$f^*(g) = g \circ f. \quad (9)$$

On forms the pullback is defined as

$$f^*(\sum g_I dy_{i_1} \dots dy_{i_q}) = \sum (g_I \circ f) df_{i_1} \dots df_{i_q}, \quad (10)$$

with $f_i = y_i \circ f$ the i -th component of f , y_i the standard coordinates.

Proposition 1.8. f^* commutes with d . This shows that the cohomology is a diffeomorphism invariant.

Definition 1.9. Let $M = U \cup V$ with U, V open. Then we have the inclusions

$$M \leftarrow U \sqcup V \leftarrow U \cap V \quad (11)$$

where \sqcup is the disjoint union (each element has a label indicating whether it's from U or V). Using the inclusions as pullbacks we get

$$\Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V). \quad (12)$$

The Mayer-Vietoris sequence is given by

$$0 \rightarrow \Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V) \rightarrow 0, \quad (13)$$

with

$$\Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V); (\omega, \tau) \mapsto \tau - \omega. \quad (14)$$

Proposition 1.10. The Mayer-Vietoris sequence is exact. This is achieved through partitions of unity ρ ,

$$(\rho_U \omega) - (-\rho_V \omega) = \omega. \quad (15)$$

Definition 1.11. The Mayer-Vietoris sequence induces a long exact sequence with the same name:

$$\begin{array}{ccccc} H^{q+1}(M) & \hookrightarrow & H^{q+1}(U) \oplus H^{q+1}(V) & \longrightarrow & H^{q+1}(U \cap V) \\ & & \searrow d^* & & \\ H^q(M) & \longrightarrow & H^q(U) \oplus H^q(V) & \longrightarrow & H^q(U \cap V) \end{array}$$

Explicitly

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U, \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \quad (16)$$

Definition 1.12. If $j : U \rightarrow M$ is the inclusion of U in M , then let $j_* : \Omega_c^*(U) \rightarrow \Omega_c^*(M)$ the map which extends a form to M by zero. Because pullbacks of compact forms are in general not compact, we instead use the inclusions

$$\begin{aligned} \Omega_c^*(M) &\xleftarrow{\text{sum}} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{\delta} \Omega_c^*(U \cap V) \\ \delta : \omega &\mapsto (-j_*\omega, j_*\omega). \end{aligned} \quad (17)$$

We then get the Mayer-Vietoris sequence

$$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0, \quad (18)$$

which induces

$$\begin{array}{ccccc} H_c^{q+1}(M) & \longleftarrow & H_c^{q+1}(U) \oplus H_c^{q+1}(V) & \longleftarrow & H_c^{q+1}(U \cap V) \\ & & \searrow d^* & & \\ H_c^q(M) & \longleftarrow & H_c^q(U) \oplus H_c^q(V) & \longleftarrow & H_c^q(U \cap V) \end{array}$$

and we now instead get

$$d^*[\omega] = \begin{cases} [-d(\rho_U \omega)] & \text{on } U, \\ [d(\rho_V \omega)] & \text{on } V. \end{cases} \quad (19)$$

Proposition 1.13. The Mayer-Vietoris sequence of forms with compact support is exact.

Proposition 1.14. A manifold of dimension n is orientable iff it has a global nowhere vanishing n -form.

Definition 1.15. Let $\mathbb{H}^n = \{x_n \geq 0\} \subset \mathbb{R}^n$ with standard orientation $dx_1 \dots dx_n$. The induced orientation of $\partial\mathbb{H}^n = \{x_n = 0\}$ is given by the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$. For an orientation-preserving diffeomorphism ϕ we define for manifolds

$$[\partial M] = \phi^*[\partial\mathbb{H}^n]. \quad (20)$$

Remark. This definition is due to $\omega|_{\partial M} := i_n^* \omega$ for the normal \hat{n} .

Theorem 1.16 (Stokes'). Let ω be an $(n-1)$ -form with compact support on an oriented manifold M , then

$$\int_M d\omega = \int_{\partial M} \omega. \quad (21)$$

Definition 1.17. Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$; $\pi(x, t) = x$ and $s : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^1$; $s(x) = (x, 0)$. Trivially $\pi \circ s = 1$, $s^* \circ \pi^* = 1$, but $s \circ \pi \neq 1$. K is called a homotopy operator if

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd). \quad (22)$$

$dK \pm Kd$ maps closed forms to exact forms, therefore induces zero in cohomology. If K exists, $\pi^* \circ s^*$ is said to be chain homotopic to the identity. We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \rightarrow \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$(\pi^* \phi)f(x, t) \mapsto 0, \quad (\pi^* \phi)f(x, t)dt \mapsto (\pi^* \phi) \int_0^t dt f \quad (23)$$

with ϕ a form on \mathbb{R}^n .

Proposition 1.18. K is a homotopy operator. The maps π^* , s^* on $H^*(\mathbb{R}^n \times \mathbb{R}) \leftrightarrow H^*(\mathbb{R}^n)$ are isomorphisms.

Corollary 1.19 (Poincaré Lemma).

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

More generally $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$.

Corollary 1.20 (Homotopy Axiom for de Rham Cohomology). Homotopic maps induce the same map in cohomology.

Definition 1.21. Two Manifolds M , N have the same homotopy type if there are C^∞ maps $f : M \rightarrow N$, $g : N \rightarrow M$ such that $g \circ f$, $f \circ g$ are C^∞ homotopic to the identity on M , N respectively. A manifold is called contractible if it has the homotopy type of a point.

Corollary 1.22. Manifolds with the same homotopy type have the same de Rham cohomology.

Definition 1.23. $r : M \rightarrow A$ is called a retraction of M onto A if $r \circ i : A \rightarrow A$ is the identity, with $i : A \subset M$ the inclusion. If $i \circ r : M \rightarrow M$ is homotopic to the identity on M , then r is called a deformation retraction of M onto A , and it follows that M , A have the same homotopy type.

Corollary 1.24. If A is a deformation retract of M , then A , M have the same de Rham cohomology.

Corollary 1.25.

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, n \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

Definition 1.26. Let $\pi : M \times \mathbb{R}^1 \rightarrow M$ be the projection. We define the push-forward for compact forms $\pi_* : \Omega_c^*(M \times \mathbb{R}^1) \rightarrow \Omega_c^{*-1}(M)$, called integration along the fiber by

$$\pi^* \phi f(x, t) \mapsto 0, \quad \pi^* \phi f(x, t)dt \mapsto \phi \int_{-\infty}^{\infty} dt f \quad (26)$$

where ϕ is a form on M . Let $e = e(t)dt$ be a compactly supported 1-form on \mathbb{R}^1 with total integral 1 and define

$$e_* : \Omega_c^*(M) \rightarrow \Omega_c^{*+1}(M \times \mathbb{R}^1); \quad \phi \rightarrow (\pi^* \phi) \wedge e. \quad (27)$$

The homotopy operator $K : \Omega_c^*(M \times \mathbb{R}^1) \rightarrow \Omega_c^{*-1}(M \times \mathbb{R}^1)$ is defined by

$$\pi^* \phi f \mapsto 0, \quad \pi^* \phi f dt \mapsto \pi^* \phi \int_{-\infty}^t f - \pi^* \phi \int_{-\infty}^t e \int_{-\infty}^{\infty} f. \quad (28)$$

Proposition 1.27. d commutes with both π_* and e_* .

Proposition 1.28. $1 - e_* \pi_* = (-1)^{q-1}(dK - Kd)$ on $\Omega_c^q(M \times \mathbb{R}^1)$. The maps $H_c^*(M \times \mathbb{R}^1) \leftrightarrow H_d^{*-1}(M)$ are isomorphisms.

Corollary 1.29 (Poincaré Lemma for Compact Supports).

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

The generator is an n -form of compact support with integral 1.

Definition 1.30. A map is proper if the inverse image of every compact set is compact.

Theorem 1.31 (Sard's). The set of critical values of a smooth map $f : M \rightarrow N$ has measure zero.

Definition 1.32. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be proper. Let α be a generator of $H_c^n(\mathbb{R}^n)$ (integrates to 1), then

$$\deg f = \int_{\mathbb{R}^n} f^* \alpha \quad (30)$$

is the degree of f .

Proposition 1.33. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be proper. If f is not surjective, then it has degree 0.

Theorem 1.34. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be proper. The degree of f is an integer.

Definition 1.35. An open cover is called a good cover if all nonempty finite intersections $U_{p_1} \cap \dots \cap U_{p_n}$ are diffeomorphic to \mathbb{R}^n . A manifold with a finite good cover is said to be of finite type. If \mathcal{U}, \mathcal{V} are two covers, \mathcal{U} is called the refinement of \mathcal{V} if every U_p is contained in some V_q and write $\mathcal{V} < \mathcal{U}$.

Theorem 1.36. Every manifold has a good cover. If it is compact, the cover can be chosen to be finite. Every open cover has a refinement which is a good cover.

Corollary 1.37. The good covers are cofinal in the set of all covers of a manifold (For directed sets, $J \subset I$ is cofinal in I if for every $i \in I$ there is $j \in J$ with $i < j$).

Proposition 1.38. If a manifold has a finite good cover, then its (compact) cohomology is finite dimensional.

Lemma 1.39. The pairing $\langle \cdot, \cdot \rangle : V \otimes W \rightarrow \mathbb{R}$ for finite vector spaces is nondegenerate iff $v \mapsto \langle v, \cdot \rangle$ is an isomorphism $V \rightarrow W^*$.

Lemma 1.40. Given a commutative diagram of Abelian groups and group homomorphisms

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where the rows are exact, if a, b, d, e are isomorphisms, so is c .

Lemma 1.41. The Mayer-Vietoris sequences may be paired together to form a sign-commutative diagram

$$\begin{array}{ccccccc} H^q(U \cup V) & \longrightarrow & H^q(U) \oplus H^q(V) & \longrightarrow & H^q(U \cap V) & \xrightarrow{d^*} & H^{q+1}(U \cup V) \\ \otimes & & \otimes & & \otimes & & \otimes \\ H_c^{n-q}(U \cup V) & \longleftarrow & H_c^{n-q}(U) \oplus H_c^{n-q}(V) & \longleftarrow & H_c^{n-q}(U \cap V) & \xleftarrow{d_*} & H_c^{n-q-1}(U \cup V) \\ \downarrow \int_{U \cup V} & & \downarrow \int_U + \int_V & & \downarrow \int_{U \cap V} & & \downarrow \int_{U \cup V} \\ \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \end{array}$$

where sign-commutativity means that

$$\int_{U \cap V} \omega \wedge d_* \tau = \pm \int_{U \cup V} (d^* \omega) \wedge \tau, \quad (31)$$

with $\omega \in H^q(U \cap V), \tau \in H_c^{n-q-1}(U \cup V)$. This is equivalent to the sign-commutative diagram

$$\begin{array}{ccccccc} H^q(U \cup V) & \longrightarrow & H^q(U) \oplus H^q(V) & \longrightarrow & H^q(U \cap V) \\ \downarrow & & \downarrow & & \downarrow \\ H_c^{n-q}(U \cup V)^* & \rightarrow & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* & \rightarrow & H_c^{n-q}(U \cap V)^* \end{array}$$

Theorem 1.42 (Poincaré Duality). If M is orientable and has a finite good cover, then

$$\int H^q(M) \otimes H_c^{n-q}(M) \rightarrow \mathbb{R} \quad (32)$$

is nondegenerate, or equivalently

$$H^q(M) \simeq (H_c^{n-q}(M))^*. \quad (33)$$

Corollary 1.43. If M is connected, oriented with dimension n , then $H_c^n(M) \simeq \mathbb{R}$. If M is also compact then $H^n(M) \simeq \mathbb{R}$.

Definition 1.44. Let G be a topological group acting effectively (only $1 \in G$ acts trivially) on a space F on the left. A surjection $\pi : E \rightarrow B$ between topological spaces is a fiber bundle with fiber F and structure group G (also called a G -bundle) if B has an open cover $\{U_a\}$ such that there are fiber-preserving homeomorphisms

$$\phi_a : E|_{U_a} \xrightarrow{\sim} U_a \times F \quad (34)$$

and continuous transition functions with values in G

$$g_{ab} = \phi_a \phi_b^{-1}|_{\{x\} \times F} \in G. \quad (35)$$

$E_x = \pi^{-1}(x)$ is called the fiber at x . E, B are called total and base space respectively. If G is not explicitly defined then it is understood to be the group of diffeomorphisms of F .

Remark. The transition functions satisfy the cocycle condition

$$g_{ab} \cdot g_{bc} = g_{ac}. \quad (36)$$

Given a cocycle $\{g_{ab}\}$ we can construct a fiber bundle with $\{g_{ab}\}$ as its transition function,

$$E = \left(\coprod U_a \times F \right) / (x, y) \sim (x, g_{ab}(x)y). \quad (37)$$

Theorem 1.45 (Künneth Formula). For two manifolds M, F , with at least one having a good finite cover, we have

$$H^*(M \times F) = H^*(M) \otimes H^*(F), \quad (38)$$

which means

$$H^n(M \times F) = \oplus_{p+q=n} H^p(M) \otimes H^q(F). \quad (39)$$

Also holds in the compact cohomology case.

Theorem 1.46 (Leray-Hirsch). Let E be a fiber bundle over M with fiber F . Suppose M has a finite good cover. If there are global cohomology classes e_1, \dots, e_r on E which restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \dots, e_r\}$ meaning that

$$H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_r\} \simeq H^*(M) \otimes H^*(F). \quad (40)$$