

# 1 General Topology

**Definition 1.1** (Topology). A topological space is a set  $X$  with a collection of subsets  $\mathcal{U}$ , called open sets, such that

1.  $\emptyset, X \in \mathcal{U}$ .
2. The arbitrary union of open sets is open.
3. The finite union of open sets is open.

The complement  $X - U$  of an open set  $U$  is called closed.

**Definition 1.2.** Let  $(X, \mathcal{U})$ ,  $(X, \mathcal{V})$  be topologies.  $\mathcal{U}$  is called stronger(finer) than  $\mathcal{V}$  if  $\mathcal{V} \subset \mathcal{U}$ , and weaker(coarser) if  $\mathcal{U} \subset \mathcal{V}$ .

**Definition 1.3.** A basis  $\mathcal{B}$  of a topology for  $X$  is a collection of subsets of  $X$  such that

1. For each  $x \in X$  there is at least one  $B \in \mathcal{B}$  with  $x \in B$ .
2. If  $x \in B_1 \cap B_2$  then there exists a  $B_3 \subset B_1 \cap B_2$  with  $x \in B_3$ .

We say that  $\mathcal{B}$  generates the topology  $\mathcal{U}$  if  $U$  is open iff for every  $x \in U$  there exists  $B \in \mathcal{B}$  with  $x \in B \subset U$ .

**Lemma 1.4.** Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{U}$  on  $X$ . Then  $\mathcal{U}$  is equal to the collection of all unions of elements of  $\mathcal{B}$ .

**Lemma 1.5.** If  $\mathcal{C}$  is a collection of open sets, such that for each  $U \subset X$  open,  $x \in U$  there is  $C \in \mathcal{C}$  such that  $x \in C \subset U$ , then  $\mathcal{C}$  is a basis for  $X$ .

**Lemma 1.6.** Let  $\mathcal{B}$ ,  $\mathcal{B}'$  be bases for topologies  $\mathcal{U}$ ,  $\mathcal{U}'$  respectively on  $X$ . Then the following are equivalent:

1.  $\mathcal{U}'$  is finer than  $\mathcal{U}$ .
2. For each  $x \in B \in \mathcal{B}$ , there is a  $B' \in \mathcal{B}'$  with  $x \in B' \subset B$ .

**Definition 1.7.** A subbasis  $\mathcal{S}$  for a topology on  $X$  is a collection of subsets whose union equals  $X$ . The topology generated by the subbasis is defined to be the collection of all unions of finite intersections of elements of  $\mathcal{S}$ .

**Definition 1.8.** The product topology on  $X \times Y$  is defined by the basis consisting of all sets of the form  $U \times V$ ,  $U \subset X$ ,  $V \subset Y$  open.

**Theorem 1.9.** If  $\mathcal{B}_1, \mathcal{B}_2$  are bases of  $X_1, X_2$  respectively, then

$$\mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\} \quad (1)$$

is a basis for  $X_1 \times X_2$ .

**Definition 1.10.** The projections are defined by

$$\pi_n : X_1 \times X_2 \rightarrow X_n; \quad \pi_n(x_1, x_2) = x_n. \quad (2)$$

**Theorem 1.11.** The following is a subbasis for  $X \times Y$ :

$$\{\pi_1^{-1}(U) \mid U \in \mathcal{U} \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \in \mathcal{V} \text{ open}\}. \quad (3)$$