

1 Algebraic Topology

Definition 1.1 (de Rham complex). Ω^* is the algebra generated over \mathbb{R} by dx_1, \dots, dx_n subject to

1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on \mathbb{R} are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \quad (1)$$

We have $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q -forms on \mathbb{R}^n . We define

$$d : \Omega^q(\mathbb{R}^n) \rightarrow \Omega^{q+1}(\mathbb{R}^n), \quad (2)$$

the exterior differentiation, by

1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f / \partial x_i dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \dots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J dx_I dx_J = (-1)^{\deg \tau \deg \omega} \omega \wedge \tau. \quad (3)$$

Proposition 1.2. d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge d\omega. \quad (4)$$

Proposition 1.3. $d^2 = 0$.

Definition 1.4. The q -th de Rham cohomology of \mathbb{R}^n is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\} / \{\text{exact } q\text{-forms}\}, \quad (5)$$

where closed means in the kernel of d and exact means in the image of d . We denote by $[\omega]$ the cohomology class of ω .

Remark. Only the constant functions are relevant for \mathbb{R}^n ,

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Definition 1.5. A differential complex is a direct sum of Vector spaces $C = \bigoplus_{q \in \mathbb{Z}} C^q$ if there are homomorphisms

$$\dots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \dots$$

with $d^2 = 0$. The cohomology of C is given by $H(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C)$, with

$$H^q(C) = (\ker d \cap C^q) / (\text{im } d \cap C^q). \quad (7)$$

A map $f : A \rightarrow B$ between two differential complexes is a chain map if it commutes with the differential operators of A and B , $f d_A = d_B f$. A sequence of vector spaces

$$\dots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \dots$$

is said to be exact if the image of f_{i-1} is the kernel of f_i . An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{array}{ccccc} H^{q+1}(A) & \xrightarrow{f^*} & \dots & & \\ & \searrow d^* & & & \\ H^q(A) & \xrightarrow{f^*} & H^q(B) & \xrightarrow{g^*} & H^q(C) \end{array}$$

f^*, g^* are the naturally induced maps and $d^*[c]$ is obtained through the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{q+1} & \xrightarrow{f} & B^{q+1} & \xrightarrow{g} & C^{q+1} \longrightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \longrightarrow & A^q & \xrightarrow{f} & B^q & \xrightarrow{g} & C^q \longrightarrow 0 \end{array}$$

Since g is surjective there is $b \in B^q$ with $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, there is $a \in A^{q+1}$ with $db = f(a)$. Then $d^*[c] := [a]$. a is closed because f is injective. To see that the sequence is exact, note that if b is closed then $f(a) = 0$, and due to injectivity $a = 0$. On the other hand, $f(a)$ is exact and therefore $[f(a)] = 0$.

Definition 1.6. $\Omega_c^*(\mathbb{R}^n)$ is the de Rham complex for functions of compact support, $H_c^*(\mathbb{R})$ is its cohomology.

Remark. Only the n -forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Definition 1.7. $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ induces a pullback on functions

$$f^*(g) = g \circ f. \quad (9)$$

On forms the pullback is defined as

$$f^*(\sum g_I dy_{i_1} \dots dy_{i_q}) = \sum (g_I \circ f) df_{i_1} \dots df_{i_q}, \quad (10)$$

with $f_i = y_i \circ f$ the i -th component of f , y_i the standard coordinates.

Proposition 1.8. f^* commutes with d . This shows that the cohomology is a diffeomorphism invariant.

Definition 1.9. Let $M = U \cup V$ with U, V open. Then we have the inclusions

$$M \leftarrow U \sqcup V \leftarrow U \cap V \quad (11)$$

where \sqcup is the disjoint union (each element has a label indicating whether it's from U or V). Using the inclusions as pullbacks we get

$$\Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V). \quad (12)$$

The Mayer-Vietoris sequence is given by

$$0 \rightarrow \Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V) \rightarrow 0, \quad (13)$$

with

$$\Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V); (\omega, \tau) \mapsto \tau - \omega. \quad (14)$$

Proposition 1.10. The Mayer-Vietoris sequence is exact. This is achieved through partitions of unity ρ ,

$$(\rho_U \omega) - (-\rho_V \omega) = \omega. \quad (15)$$

Definition 1.11. The Mayer-Vietoris sequence induces a long exact sequence with the same name:

$$\begin{array}{ccccc} H^{q+1}(M) & \longrightarrow & H^{q+1}(U) \oplus H^{q+1}(V) & \longrightarrow & H^{q+1}(U \cap V) \\ & & \searrow d^* & & \\ H^q(M) & \longrightarrow & H^q(U) \oplus H^q(V) & \longrightarrow & H^q(U \cap V) \end{array}$$

Explicitly

$$d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U, \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \quad (16)$$

Definition 1.12. If $j : U \rightarrow M$ is the inclusion of U in M , then let $j_* : \Omega_c^*(U) \rightarrow \Omega_c^*(M)$ the map which extends a form to M by zero. Because pullbacks of compact forms are in general not compact, we instead use the inclusions

$$\begin{aligned} \Omega_c^*(M) &\xleftarrow{\text{sum}} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{\delta} \Omega_c^*(U \cap V) \\ \delta : \omega &\mapsto (-j_*\omega, j_*\omega). \end{aligned} \quad (17)$$

We then get the Mayer-Vietoris sequence

$$0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0, \quad (18)$$

which induces

$$\begin{array}{ccccc} H_c^{q+1}(M) & \longleftarrow & H_c^{q+1}(U) \oplus H_c^{q+1}(V) & \longleftarrow & H_c^{q+1}(U \cap V) \\ & & \searrow d^* & & \\ H_c^q(M) & \longleftarrow & H_c^q(U) \oplus H_c^q(V) & \longleftarrow & H_c^q(U \cap V) \end{array}$$

and we now instead get

$$d^*[\omega] = \begin{cases} [d(\rho_U \omega)] & \text{on } U, \\ [d(\rho_V \omega)] & \text{on } V. \end{cases} \quad (19)$$

Proposition 1.13. The Mayer-Vietoris sequence of forms with compact support is exact.

Proposition 1.14. A manifold of dimension n is orientable iff it has a global nowhere vanishing n -form.

Definition 1.15. Let $\mathbb{H}^n = \{x_n \geq 0\} \subset \mathbb{R}^n$ with standard orientation $dx_1 \dots dx_n$. The induced orientation of $\partial\mathbb{H}^n = \{x_n = 0\}$ is given by the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$. For an orientation-preserving diffeomorphism ϕ we define for manifolds

$$[\partial M] = \phi^*[\partial\mathbb{H}^n]. \quad (20)$$

Remark. This definition is due to $\omega|_{\partial M} := i_{\hat{n}}\omega$ for the normal \hat{n} .

Theorem 1.16 (Stokes'). Let ω be an $(n-1)$ -form with compact support on an oriented manifold M , then

$$\int_M d\omega = \int_{\partial M} \omega. \quad (21)$$

Definition 1.17. Let $\pi : \mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$; $\pi(x, t) = x$ and $s : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^1$; $s(x) = (x, 0)$. Trivially $\pi \circ s = 1$, $s^* \circ \pi^* = 1$, but $s \circ \pi \neq 1$. K is called a homotopy operator if

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd). \quad (22)$$

$dK \pm Kd$ maps closed forms to exact forms, therefore induces zero in cohomology. If K exists, $\pi^* \circ s^*$ is said to be chain homotopic to the identity. We define $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \rightarrow \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$ by

$$(\pi^* \phi)f(x, t) \mapsto 0, \quad (\pi^* \phi)f(x, t)dt \mapsto (\pi^* \phi) \int_0^t dt f \quad (23)$$

with ϕ a form on \mathbb{R}^n .

Proposition 1.18. K is a homotopy operator. The maps π^*, s^* on $H^*(\mathbb{R}^n \times \mathbb{R}) \leftrightarrow H^*(\mathbb{R}^n)$ are isomorphisms.

Corollary 1.19 (Poincaré Lemma).

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

More generally $H^*(M \times \mathbb{R}^1) \simeq H^*(M)$.

Corollary 1.20 (Homotopy Axiom for de Rham Cohomology). Homotopic maps induce the same map in cohomology.