1 Algebraic Topology

Definition 1.1 (de Rham complex)**.** Ω^* is the algebra generated over $\mathbb{R}$ by $dx_1, \ldots, dx_n$ subject to
1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on $\mathbb{R}$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \tag{1}$$

We have $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. We define

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n), \tag{2}$$

the exterior differentiation, by
1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i \, dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \ldots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J \, dx_I dx_J = (-1)^{\deg\tau \deg\omega} \omega \wedge \tau. \tag{3}$$

Proposition 1.2. *d is an antiderivation,*

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg\tau} \tau \wedge d\omega. \tag{4}$$

Proposition 1.3. *$d^2 = 0$.*

Definition 1.4. The q-th de Rham cohomology of $\mathbb{R}^n$ is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}, \tag{5}$$

where closed means in the kernel of d and exact means in the image of d. We denote by $[\omega]$ the cohomology class of ω.

$$\begin{tikzcd} \ldots \arrow[r] & V_{q-1} \arrow[r, "f_{i-1}"] & V_i \arrow[r, "f_i"] & V_{q+1} \arrow[r] & \ldots \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & H^{q+1}(A) \arrow[r, "f^*"] & ... \\ H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[ul, bend left, "d^*"] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^q \arrow[r, "f"] \arrow[u, "d"] & B^q \arrow[r, "g"] \arrow[u, "d"] & C^q \arrow[r] \arrow[u, "d"] & 0 \end{tikzcd}$$