# 1 Functional Analysis

**Definition 1.1.** A complex vector space $V$ is called an inner product space with an inner product $(\cdot, \cdot)$ on $V \times V$ such that for all $x, y, z \in V$, $a \in \mathbb{C}$:

1. $(x, x) \geq 0$, $(x, x) = 0$ iff $x = 0$
2. $(x, y + z) = (x, y) + (x, z)$
3. $(x, ay) = a(x, y)$
4. $(x, y) = \overline{(y, x)}$

**Definition 1.2.** Two vectors $x$, $y$ are called orthogonal if $(x, y) = 0$. An orthonormal set is a collection of vectors that are mutually orthogonal and such that $\|x\| := \sqrt{(x, x)} = 1$ for all $x$.

**Theorem 1.3** (Pythagorean)**.** Let $\{x_n\}_{n=1}^N$ be orthonormal, then for all $x \in V$

$$\|x\|^2 = \sum_{n=1}^{N} |(x, x_n)|^2 + \left\| x - \sum_{n=1}^{N} (x_n, x) x_n \right\|^2 . \tag{1}$$

**Corollary 1.4** (Bessel's inequality)**.** Let $\{x_n\}_{n=1}^N$ be orthonormal, then for all $x \in V$

$$\|x\|^2 \geq \sum_{n=1}^{N} |(x, x_n)|^2 . \tag{2}$$

**Corollary 1.5** (Schwarz inequality)**.**

$$|(x, y)| \leq \|x\| \, \|y\| . \tag{3}$$

*Remark.* Parallelogram law:

$$\|x + y\| + \|x - y\| = 2 \|x\|^2 + 2 \|y\|^2 . \tag{4}$$

**Theorem 1.6.** Every inner product space is a normed linear space with norm $\|x\| := \sqrt{(x, x)}$.

**Definition 1.7.** A complete inner product space is called a Hilbert space, otherwise a pre-Hilbert space.

**Definition 1.8.** Two Hilbert spaces are called isomorphic if there is a linear operator $U$ from $\mathcal{H}_1$ onto(i.e. surjective) $\mathcal{H}_2$, called unitary, such that $(Ux, Uy)_{\mathcal{H}_2} = (x, y)_{\mathcal{H}_1}$.

**Lemma 1.9.** Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, $x \in \mathcal{H}$. Then there is a unique $z \in \mathcal{M}$ closest to $x$.

**Theorem 1.10** (The projection theorem)**.** Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, $x \in \mathcal{H}$. Then $x$ can be uniquely written as $x = z + w$, where $z \in \mathcal{M}$ and $w \in \mathcal{M}^{\perp}$.