1 Algebraic Topology

Definition 1.1 (de Rham complex)**.** Ω^* is the algebra generated over $\mathbb{R}$ by $dx_1, \dots, dx_n$ subject to
1. $(dx_i)^2 = 0$,
2. $dx_i dx_j = -dx_j dx_i, i \neq j$.

The C^∞ differential forms on $\mathbb{R}$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*. \tag{1}$$

We have $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. We define

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n), \tag{2}$$

the exterior differentiation, by
1. if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i \, dx_i$,
2. if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$, where $dx_I = dx_i dx_j \dots$.

The wedge product is defined by

$$\tau \wedge \omega = \sum \tau_I \omega_J \, dx_I dx_J = (-1)^{\deg\tau \deg\omega} \omega \wedge \tau. \tag{3}$$

Proposition 1.2. d is an antiderivation,

$$d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg\tau} \tau \wedge d\omega. \tag{4}$$

Proposition 1.3. $d^2 = 0$.

Definition 1.4. The q-th de Rham cohomology of $\mathbb{R}^n$ is the vector space

$$H^q(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}, \tag{5}$$

where closed means in the kernel of d and exact means in the image of d. We denote by $[\omega]$ the cohomology class of ω.

Remark. Only the constant functions are relevant for $\mathbb{R}^n$,

$$H^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension 0,} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Definition 1.5. A differential complex is a direct sum of Vector spaces $C = \oplus_{q \in \mathbb{Z}} C^q$ if there are homomorpisms

$$\dots \longrightarrow C^{q-1} \xrightarrow{d} C^i \xrightarrow{d} C^{q+1} \longrightarrow \dots$$

with $d^2 = 0$. The cohomology of C is given by $H(C) = \oplus_{q\in\mathbb{Z}} H^q(C)$, with

$$H^q(C) = (\ker d \cap C^q)/(\operatorname{im} d \cap C^q). \tag{7}$$

A map $f : A \to B$ between two differential complexes is a chain map it it commutes with the differential operators of A and B, $f d_A = d_B f$. A sequence of vector spaces

$$\dots \longrightarrow V_{q-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{q+1} \longrightarrow \dots$$

is said to be exact if the image of f_{i-1} is the kernel of f_i. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence. Note that f is injective and g surjective. If f, g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{tikzcd}
H^{q+1}(A) \arrow[r, "f^*"] & \dots \\
H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[ull, bend left, "d^*"]
\end{tikzcd}$$

f^*, g^* are the naturally induced maps and $d^*[c]$ is obtained through the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^{q+1} \arrow[r, "f"] & B^{q+1} \arrow[r, "g"] & C^{q+1} \arrow[r] & 0 \\
0 \arrow[r] & A^q \arrow[r, "f"] \arrow[u, "d"] & B^q \arrow[r, "g"] \arrow[u, "d"] & C^q \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

Since g is surjective there is $b \in B^q$ with $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, there is $a \in A^{q+1}$ with $db = f(a)$. Then $d^*[c] := [a]$. a is closed because f is injective. To see that the sequence is exact, note that if b is closed then $f(a) = 0$, and due to injectivity $a = 0$. On the other hand, $f(a)$ is exact and therefore $[f(a)] = 0$.

Definition 1.6. $\Omega_c^*(\mathbb{R}^n)$ is the de Rham complex for functions of compact support, $H_c^*(\mathbb{R})$ is its cohomology.

Remark. Only the n-forms whose integrals are different from zero are relevant, since if it was zero then its antiderivative can have compact support,

$$H_c^*(\mathbb{R}^n) = \begin{cases} R & \text{in dimension n,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Definition 1.7. $f : \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback on functions

$$f^*(g) = g \circ f. \tag{9}$$

On forms the pullback is defined as

$$f^*(\sum g_I dy_{i_1} \dots dy_{i_q}) = \sum (g_I \circ f) df_{i_1} \dots df_{i_q}, \tag{10}$$

with $f_i = y_i \circ f$ the i-th component of f, y_i the standard coordinates.

Proposition 1.8. f^* commutes with d.

Definition 1.9. Let $M = U \cup V$ with U, V open. Then we have the inclusions

$$M \leftarrow U \sqcup V \leftarrow U \cap V \tag{11}$$

where $\sqcup$ is the disjoint union(each element has a label indicating wether it's from U or V). Using the inclusions as pushforwards we get

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V). \tag{12}$$

The Mayer-Vietoris sequence is given by

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0, \tag{13}$$

with

$$\Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V); (\omega, \tau) \mapsto \tau - \omega. \tag{14}$$

Proposition 1.10. The Mayer-Vietoris sequence is exact. This is achieved through partitions of unity ρ,

$$(\rho_U \omega) - (-\rho_V \omega) = \omega. \tag{15}$$

Definition 1.11. The Mayer-Vietoris sequence induces a long exact sequence with the same name:

$$
\begin{tikzcd}
H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \\
H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[llu, "d^*"]
\end{tikzcd}
$$

Explicitly

$$
d^*[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U, \\ [d(\rho_U \omega)] & \text{on } V. \end{cases} \tag{16}
$$

Definition 1.12. If $j : U \to M$ is the inclusion of U in M, then let $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ the map which extends a form to M by zero. Because pullbacks of compact forms are in general not compact, we instead use the inclusions

$$
\Omega_c^*(M) \xleftarrow[\text{sum}]{} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow[\delta]{} \Omega_c^*(U \cap V)
$$

$$
\delta : \omega \mapsto (-j_*\omega, j_*\omega). \tag{17}
$$

We then get the Mayer-Vietoris sequence

$$
0 \leftarrow \Omega_c^*(M) \leftarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \leftarrow \Omega_c^*(U \cap V) \leftarrow 0, \tag{18}
$$

which induces

$$
\begin{tikzcd}
H_c^{q+1}(M) & H_c^{q+1}(U) \oplus H_c^{q+1}(V) \arrow[l] & H_c^{q+1}(U \cap V) \arrow[l] \\
H_c^q(M) \arrow[rru, "d^*"] & H_c^q(U) \oplus H_c^q(V) \arrow[l] & H_c^q(U \cap V) \arrow[l]
\end{tikzcd}
$$

and we now instead get

$$
d^*[\omega] = \begin{cases} [d(\rho_U \omega)] & \text{on } U, \\ [d(\rho_V \omega)] & \text{on } V. \end{cases} \tag{19}
$$

Proposition 1.13. The Mayer-Vietoris sequence of forms with compact support is exact.

Proposition 1.14. A manifold of dimension n is orientable iff it has a global nowhere vanishing n-form.

Definition 1.15. Let $\mathbb{H}^n = \{x_n \geq 0\} \subset \mathbb{R}^n$ with standard orientation $dx_1 \dots dx_n$. The induced orientation of $\partial\mathbb{H}^n = \{x_n = 0\}$ is given by the equivalence class of $(-1)^n dx_1 \dots dx_{n-1}$. For an orientation-preserving diffeomorphism ϕ we define for manifolds

$$
[\partial M] = \phi^*[\partial \mathbb{H}^n]. \tag{20}
$$

Remark. This definition is due to $\omega|_{\partial M} := i_{\hat{n}}\omega$ for the normal $\hat{n}$.

Theorem 1.16 (Stokes'). Let ω be an $(n-1)$-form with compact support on an oriented manifold M, then

$$
\int_M d\omega = \int_{\partial M} \omega. \tag{21}
$$