1 General Topology

Definition 1.1 (Topology)**.** A topological space is a set X with a collection of subsets $\mathcal{U}$, called open sets, such that

1. $\emptyset, X \in \mathcal{U}$.
2. The arbitrary union of open sets is open.
3. The finite union of open sets is open.

The complement $X - U$ of an open set U is called closed. If $x \in U$ open, then U is called a neighborhood of x. Sometimes a non open set $A \supset U$ is also referred to as a neighborhood.

Definition 1.2. Let $(X,\mathcal{U})$, $(X,\mathcal{V})$ be topologies. $\mathcal{U}$ is called stronger(finer) than $\mathcal{V}$ if $\mathcal{V} \subset \mathcal{U}$, and weaker(coarser) if $\mathcal{U} \subset \mathcal{V}$.

Definition 1.3. A basis $\mathcal{B}$ of a topology for X is a collection of subsets of X such that

1. For each $x \in X$ there is at least one $B \in \mathcal{B}$ with $x \in B$.
2. If $x \in B_1 \cap B_2$ then there exists a $B_3 \subset B_1 \cap B_2$ with $x \in B_3$.

We say that $\mathcal{B}$ generates the topology $\mathcal{U}$ if U is open iff for every $x \in U$ there exits $B \in \mathcal{B}$ with $x \in B \subset U$.

Lemma 1.4. Let $\mathcal{B}$ be a basis for a topology $\mathcal{U}$ on X. Then $\mathcal{U}$ is equal to the collection of all unions of elements of $\mathcal{B}$.

Lemma 1.5. If $\mathcal{C}$ is a collection of open sets, such that for each $U \subset X$ open, $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis for X.

Lemma 1.6. Let $\mathcal{B}$, $\mathcal{B}'$ be bases for topologies $\mathcal{U}, \mathcal{U}'$ respectively on X. Then the following are equivalent:

1. $\mathcal{U}'$ is finer than $\mathcal{U}$.
2. For each $x \in B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ with $x \in B' \subset B$.

Definition 1.7. A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets whose union equals X. The topology generated by the subbasis is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 1.8. The product topology on $X \times Y$ is defined by the basis consisting of all sets of the form $U \times V$, $U \subset X$, $V \subset Y$ open.

Theorem 1.9. If $\mathcal{B}_1, \mathcal{B}_2$ are bases of X_1, X_2 respectively, then

$$\mathcal{B} = \{B_1 \times B_2 \,|\, B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\} \tag{1}$$

is a basis for $X_1 \times X_2$.

Definition 1.10. The projections are defined by

$$\pi_n : X_1 \times X_2 \to X_n; \quad \pi_n(x_1, x_2) = x_n. \tag{2}$$

Theorem 1.11. The following is a subbasis for $X \times Y$:

$$\{\pi_1^{-1}(U) \,|\, U \in X \text{ open}\} \cup \{\pi_2^{-1}(V) \,|\, V \in Y \text{ open}\}. \tag{3}$$

Definition 1.12. For $(X,\mathcal{U})$ and $Y \subset X$, the subspace topology $\mathcal{V}$ on Y is defined as

$$\mathcal{V} = \{Y \cap U \,|\, U \in \mathcal{U}\}. \tag{4}$$

Lemma 1.13. If $\mathcal{B}$ is a basis for X then the following is a basis for Y:

$$\{B \cap Y \,|\, B \in \mathcal{B}\}. \tag{5}$$

Lemma 1.14. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Theorem 1.15. Let A, B be subspaces of X, Y respectively, then the product topology on $A \times B$ is equal the subspace topology $A \times B$ inherits from $X \times Y$.

Theorem 1.16. Arbitrary intersections and finite unions of closed sets are closed.

Theorem 1.17. If Y is a subspace of X, then a set is closed in Y iff it equals the intersection of a closed set in X with Y.

Theorem 1.18. If Y is a subspace of X, A is closed in Y and Y is closed in X, then A is closed in X.

Definition 1.19. For A a subset of X, the closure $\overline{A}_X$ of A(in X) is defined as the intersection of all closed sets containing A, the interior of A is defined as the union of all open sets contained in A.

Theorem 1.20. Let Y be a subspace of X, A a subset of Y. Then $\overline{A}_Y = \overline{A}_X \cap Y$.

Theorem 1.21. Let A be a subset of X.

1. Then $x \in \overline{A}$ iff every open set containing x intersects A.
2. If X is given by a basis, then $x \in \overline{A}$ iff every basis element containing x intersects A.

Definition 1.22. $x \in X$ is called a limit point of $A \subset X$ if every neighborhood of x intersects A in some other point than x itself, or equally if x belongs to the closure of $A - \{x\}$.

Theorem 1.23. Let A' be the set of all limit points of $A \subset X$, then

$$\overline{A} = A \cup A'. \tag{6}$$

Corollary 1.24. A subset of a topological space is closed iff it contains all its limit points.

Definition 1.25. A sequence x_k converges to $x \in X$ if for each neighborhood U of x there is a positive integer N such that for all $n \leq N$, $x_n \in U$.

Definition 1.26. X is called a Hausdorff space if for each $x_1 \neq x_2$ there are neighborhoods of x_1 and x_2 respectively that are disjoint. It is called T_1 if all finite point sets are closed.

Theorem 1.27. Every finite point set in a Hausdorff space is closed, so it is T_1.

Theorem 1.28. Let X be T_1, then x is a limit point of A iff every neighborhood of x contains infinitely many points of A.

Theorem 1.29. A sequence in a Hausdorff space converges to at most one point.

Theorem 1.30. The product of two Hausdorff spaces is Hausdorff, so is a subspace of a Hausdorff space.

Definition 1.31. A function $f : X \to Y$ is called continuous if for every open $V \subset Y$, $f^{-1}(V)$ is open in X. If f is also a bijection and f^{-1} is also continuous then f is called a homeomorphism.

Theorem 1.32. Let $f : X \to Y$, then the following are equivalent:

1. f is continuous.
2. For every subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
3. For every closed $B \subset Y$, $f^{-1}(B)$ is closed in X.
4. For every $x \in X$ and every neighborhood V of $f(x)$ there is a neighborhood U of x with $f(U) \subset V$.

If (4) holds for some point $x \in X$, then f is called continuous at x.

Definition 1.33. Let $f : X \to Y$ be injective and continuous, then f is called an imbedding if f is a homeomorphism under the subspace topology $f(X) \subset Y$.

Theorem 1.34. Constant functions, inclusions, composites of continuous functions, restriction of the domain or range of a continuous function to a subspace, are continuous. If $f : X \to Y$ is continuous in the subspace topology when restricted to an open cover $f|U_a$, then f is continuous.

Theorem 1.35 (The pasting lemma)**.** Let $X = A \cup B$, with A, B closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If their values agree for every $x \in A \cap B$, then their combination to $X \to Y$ is continuous.

Theorem 1.36. Let $f : A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$. Then f is continuous iff f_1, f_2 are continuous. Those are called the coordinate functions of f.