

1 General Topology

Definition 1.1 (Topology). A topological space is a set X with a collection of subsets \mathcal{U} , called open sets, such that

1. $\emptyset, X \in \mathcal{U}$.
2. The arbitrary union of open sets is open.
3. The finite union of open sets is open.

The complement $X - U$ of an open set U is called closed. If $x \in U$ open, then U is called a neighborhood of x . Sometimes a non open set $A \supset U$ is also referred to as a neighborhood.

Definition 1.2. Let $(X, \mathcal{U}), (X, \mathcal{V})$ be topologies. \mathcal{U} is called stronger(finer) than \mathcal{V} if $\mathcal{V} \subset \mathcal{U}$, and weaker(coarser) if $\mathcal{U} \subset \mathcal{V}$.

Definition 1.3. A basis \mathcal{B} of a topology for X is a collection of subsets of X such that

1. For each $x \in X$ there is at least one $B \in \mathcal{B}$ with $x \in B$.
2. If $x \in B_1 \cap B_2$ then there exists a $B_3 \subset B_1 \cap B_2$ with $x \in B_3$.

We say that \mathcal{B} generates the topology \mathcal{U} if U is open iff for every $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subset U$.

Lemma 1.4. Let \mathcal{B} be a basis for a topology \mathcal{U} on X . Then \mathcal{U} is equal to the collection of all unions of elements of \mathcal{B} .

Lemma 1.5. If \mathcal{C} is a collection of open sets, such that for each $U \subset X$ open, $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$, then \mathcal{C} is a basis for X .

Lemma 1.6. Let $\mathcal{B}, \mathcal{B}'$ be bases for topologies $\mathcal{U}, \mathcal{U}'$ respectively on X . Then the following are equivalent:

1. \mathcal{U}' is finer than \mathcal{U} .
2. For each $x \in B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ with $x \in B' \subset B$.

Definition 1.7. A subbasis \mathcal{S} for a topology on X is a collection of subsets whose union equals X . The topology generated by the subbasis is defined to be the collection of all unions of finite intersections of elements of \mathcal{S} .

Definition 1.8. The product topology on $X \times Y$ is defined by the basis consisting of all sets of the form $U \times V$, $U \subset X$, $V \subset Y$ open.

Theorem 1.9. If $\mathcal{B}_1, \mathcal{B}_2$ are bases of X_1, X_2 respectively, then

$$\mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\} \quad (1)$$

is a basis for $X_1 \times X_2$.

Definition 1.10. The projections are defined by

$$\pi_n : X_1 \times X_2 \rightarrow X_n; \quad \pi_n(x_1, x_2) = x_n. \quad (2)$$

Theorem 1.11. The following is a subbasis for $X \times Y$:

$$\{\pi_1^{-1}(U) \mid U \in \mathcal{U}\} \cup \{\pi_2^{-1}(V) \mid V \in \mathcal{V}\}. \quad (3)$$

Definition 1.12. For (X, \mathcal{U}) and $Y \subset X$, the subspace topology \mathcal{V} on Y is defined as

$$\mathcal{V} = \{Y \cap U \mid U \in \mathcal{U}\}. \quad (4)$$

Lemma 1.13. If \mathcal{B} is a basis for X then the following is a basis for Y :

$$\{B \cap Y \mid B \in \mathcal{B}\}. \quad (5)$$

Lemma 1.14. Let Y be a subspace of X . If U is open in Y and Y is open in X , then U is open in X .

Theorem 1.15. Let A, B be subspaces of X, Y respectively, then the product topology on $A \times B$ is equal the subspace topology $A \times B$ inherits from $X \times Y$.

Theorem 1.16. Arbitrary intersections and finite unions of closed sets are closed.

Theorem 1.17. If Y is a subspace of X , then a set is closed in Y iff it equals the intersection of a closed set in X with Y .

Theorem 1.18. If Y is a subspace of X , A is closed in Y and Y is closed in X , then A is closed in X .

Definition 1.19. For A a subset of X , the closure \overline{A}_X of A (in X) is defined as the intersection of all closed sets containing A , the interior of A is defined as the union of all open sets contained in A .

Theorem 1.20. Let Y be a subspace of X , A a subset of Y . Then $\overline{A}_Y = \overline{A}_X \cap Y$.

Theorem 1.21. Let A be a subset of X .

1. Then $x \in \overline{A}$ iff every open set containing x intersects A .
2. If X is given by a basis, then $x \in \overline{A}$ iff every basis element containing x intersects A .