

## 5 Myhill-Nerode

In this chapter, we consider three additional characterizations of regular languages:

1. Myhill relations,
2. weak Nerode relations,
3. and Nerode relations.

We will show that these three characterizations can be used to characterize regular languages by proving them equivalent to the existence of a (deterministic) finite automaton.

### 5.1 Definitions

Before we can state the Myhill-Nerode theorem, we need a number of auxiliary definitions. We roughly follow [24].

**Definition 5.1.1.** *Let  $\equiv$  be an equivalence relation. The **equivalence class** of  $u \in \Sigma^*$  with respect to  $\equiv$  is the set of all  $v$  such that  $u \equiv v$ . It is denoted by  $[u]_{\equiv}$ .*

**Definition 5.1.2.** *Let  $\equiv$  be an equivalence relation.  $\equiv$  is of **finite index** if and only if the set of  $\{[u]_{\equiv} \mid u \in \Sigma^*\}$  is finite.*

Due to the lack of native support for quotient types in Coq, we formalize equivalence relations of finite index as surjective functions from  $\Sigma^*$  to a finite type  $X$ .

**Definition 5.1.3.** *Let  $X$  be finite. Let  $f : \Sigma^* \mapsto X$  be surjective. Let  $u, v \in \Sigma^*$ .  $f$  is an **equivalence relation of finite index**.  $u$  and  $v$  are equivalent with respect to  $f$  if and only if  $f(u) = f(v)$ .  $f(u)$  is the equivalence class of  $u$  with respect to  $f$ .*

**Record** Fin\_Eq\_Cls :=  
 { fin\_type : finType;  
 fin\_func :> word -> fin\_type;  
 fin\_surjective : surjective fin\_func }.

**Definition 5.1.4.** *Let  $f$  be as above. Let  $x \in X$ .  $w \in \Sigma^*$  is a **representative** of  $x$  if and only if  $f(w) = x$ . Since  $f$  is surjective, every  $w$  has a representative. We write  $cr(x)$  to denote the **canonical representative** of  $x$ , which we obtain by constructive choice.*

**Definition** cr (f: Fin\_Eq\_Cls) x := xchoose ( fin\_surjective f x).

### 5.1.1 Myhill Relations

**Definition 5.1.5.** Let  $\equiv$  be an equivalence relation of finite index.  $\equiv$  is a **Myhill relation** [24] for  $L$  if and only if

(i)  $\equiv$  is **right congruent**, i.e. for all  $u, v \in \Sigma^*$  and  $a \in \Sigma$ ,

$$u \equiv v \Rightarrow u \cdot a \equiv v \cdot a.$$

(ii)  $\equiv$  **refines**  $L$ , i.e. for all  $u, v \in \Sigma^*$ ,

$$u \equiv v \Rightarrow (u \in L \iff v \in L).$$

Myhill relations are commonly referred to as “Myhill-Nerode relations”. In this thesis, it makes sense to split the concept of a Myhill relation from that of the Nerode relation. Apart from the Nerode relation, which can be seen as the coarsest Myhill relation, we also define weak Nerode relations that have no direction connection to Myhill relations. Thus, we strictly separate the characterizations.

Mathematically, Myhill relations are required to be of finite index. We only formalize equivalence relations of finite index. Thus, proving that a Myhill relation is of finite index mathematically corresponds to constructing a Myhill relation in our formalization.

**Definition** `right_congruent {X} (f: word → X) :=`  
`forall u v a, f u = f v → f (rcons u a) = f (rcons v a).`

**Definition** `refines {X} (f: word → X) :=`  
`forall u v, f u = f v → u \in L = (v \in L).`

**Record** `Myhill_Rel :=`  
`{ myhill_func :> Fin_Eq_Cls;`  
`myhill_congruent : right_congruent myhill_func ;`  
`myhill_refines : refines myhill_func }.`

Myhill relations correspond to the equivalence relations defined as the pairs of words  $(u, v)$  whose runs on a DFA  $A$  end in the same state. These relations are right congruent, refine  $\mathcal{L}(A)$  and are of finite index as  $A$  has finitely many states. We will later give a formal proof of this.

### 5.1.2 Nerode Relations

**Definition 5.1.6.** Let  $u, v \in \Sigma^*$ . Let  $L$  be a language. We define the **Nerode relation**  $\dot{=}_L$  for  $L$  such that

$$u \dot{=}_L v \iff \forall w \in \Sigma^*. uw \in L \iff vw \in L.$$

The Nerode relation given above is a propositional statement in COQ. To prove that the Nerode relation is of finite index, we require an equivalence relation, i.e. a function  $f$  from words to a finite type, such that  $f$  is equivalent to  $\dot{=}_L$ .

**Definition** `equiv_suffix {X} (f: word → X) :=`  
**forall** `u v, f u = f v <=> suffix_equal u v.`

**Record** `Nerode_Rel :=`  
`{ nerode_func :> Fin.Eq_Cls;`  
`nerode_equiv: equiv_suffix nerode_func }.`

**Definition 5.1.7.** *Let  $L$  be a language and let  $\equiv$  be an equivalence relation. We say that  $\equiv$  is a **weak Nerode relation** for  $L$  if and only if*

$$\forall u, v \in \Sigma^*. u \equiv v \implies u \dot{=}_L v.$$

**Definition** `suffix_equal u v :=`  
**forall** `w, u ++ w \in L = (v ++ w \in L).`

**Definition** `imply_suffix {X} (f: word → X) :=`  
**forall** `u v, f u = f v → suffix_equal u v.`

**Record** `Weak_Nerode_Rel :=`  
`{ weak_nerode_func :> Fin.Eq_Cls;`  
`weak_nerode_imply: imply_suffix weak_nerode_func }.`

It appears that the notion of a weak Nerode relation is not found in the literature. We will later prove them weaker than Myhill relations, in the sense that every Myhill relation is also a weak Nerode relation.

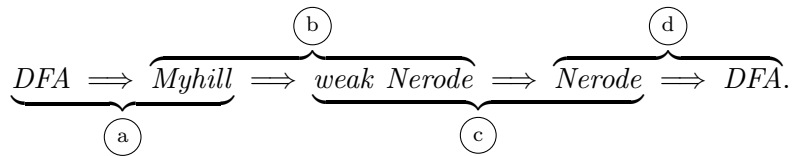
### 5.1.3 Myhill-Nerode Theorem

We can now move on to the statement of the Myhill-Nerode theorem [24].

**Theorem 5.1.8.** (*Myhill-Nerode*) *Let  $L$  be a language. The following four statements are equivalent:*

1. *there exists a deterministic finite automaton that accepts  $L$ ;*
2. *there exists a Myhill relation for  $L$ ;*
3. *there exists a weak Nerode relation for  $L$ ;*
4. *the Nerode relation for  $L$  is of finite index.*

Our proof of equivalence will have the following structure:



We will first show (a), (b), and (d). We will then give a proof of (c), which is the most challenging proof and formalization in this chapter.

## 5.2 Finite Automata to Myhill relations

We assume we are given a DFA  $A$ . We will be using the states of  $A$  as equivalence classes. Our goal is to construct a Myhill relation, for which we will need an equivalence relation of finite index. Therefore, we first need to ensure that the mapping from words to equivalence classes is surjective. Thus, we consider the equivalent, connected automaton  $A_c = (Q_c, s_c, F_c, \delta_c)$  (Definition 4.3.1), which has only reachable states. This enables us to construct a surjective function from words to the states of  $A_c$ .

**Definition 5.2.1.** Let  $u \in \Sigma^*$ . Let  $\sigma$  be the run of  $u$  on  $A_c$ . We define  $f_M : \Sigma^* \mapsto Q_c$  such that  $f_M(u)$  is the last state in  $\sigma$ , i.e.

$$f_M(u) := \sigma_{|\sigma|-1}.$$

Note that  $f_M$  is surjective (follows directly from Lemma 4.3.5) and, thus, an equivalence relation of finite index.

**Definition**  $f\_M := \text{fun } w \Rightarrow \text{last } (\text{dfa\_s } A\_c) (\text{dfa\_run } A\_c w).$

**Lemma**  $f\_M.\text{surjective} : \text{surjective } f\_M.$

**Definition**  $f\_fin : \text{Fin\_Eq\_Cls} :=$   
 $\{ | \text{ fin\_func } := f\_M;$   
 $\text{ fin\_surjective } := f\_M.\text{surjective} \}.$

In order to show that  $f_M$  is a Myhill relation, we prove that it fulfills Definition 5.1.5.

**Lemma 5.2.2.**  $f_M$  is right congruent.

*Proof.* Let  $u, v \in \Sigma^*$  such that  $f_M(u) = f_M(v)$ . Let  $a \in \Sigma$ . Since  $A$  is deterministic, we get  $f_M(ua) = f_M(va)$ .  $\square$

**Lemma 5.2.3.**  $f_M$  refines  $\mathcal{L}(A_c)$ .

*Proof.* Let  $u, v \in \Sigma^*$  such that  $f_M(u) = f_M(v)$ . By definition of  $f_M$ , the runs  $u$  and  $v$  on  $A$  end in the same state. Thus, either  $u$  and  $v$  are both accepted, or both not accepted.  $\square$

**Theorem 5.2.4.**  $f_M$  is a Myhill relation for  $\mathcal{L}(A)$ .

*Proof.* By Lemma 4.3.3, we have  $\mathcal{L}(A_c) = \mathcal{L}(A)$ . Thus, it suffices to show that  $f_M$  is a Myhill relation for  $\mathcal{L}(A_c)$ . This follows with Lemma 5.2.2 and Lemma 5.2.3.  $\square$

We only have extensional equality on  $\mathcal{L}(A_c)$  and  $\mathcal{L}(A)$  in Coq. Thus, we first show that  $f_M$  is a Myhill relation for  $\mathcal{L}(A_c)$ . Then, we show that we can get a Myhill relation for  $\mathcal{L}(A)$  from a Myhill relation for  $\mathcal{L}(A_c)$ .

**Definition**  $\text{dfa\_to\_myhill}' : \text{Myhill\_Rel } (\text{dfa\_lang } A\_c) :=$   
 $\{ | \text{ myhill\_func } := f\_fin ;$   
 $\text{ myhill\_congruent } := f\_M.\text{right\_congruent} ;$   
 $\text{ myhill\_refines } := f\_M.\text{refines} \}.$

**Lemma**  $\text{myhill\_lang\_eq } L1 L2 : L1 =_i L2 \rightarrow \text{Myhill\_Rel } L1 \rightarrow \text{Myhill\_Rel } L2.$

**Lemma**  $\text{dfa\_to\_myhill} : \text{Myhill\_Rel } (\text{dfa\_lang } A).$

This concludes the proof of step (a).

### 5.3 Myhill relations to weak Nerode relations

We show that if we are given a Myhill relation for a language, we can also give a weak Nerode relation for that language. In fact, we will prove that any Myhill relation *is* a weak Nerode relation.

**Theorem 5.3.1.** *Let  $f$  be a Myhill relation for a language  $L$ . Then  $f$  is a weak Nerode relation for  $L$ .*

*Proof.* Let  $u, v \in \Sigma^*$  such that  $u =_f v$ . We have to show that for all  $w \in \Sigma^*$ ,  $uw \in L \Leftrightarrow vw \in L$ . By induction on  $w$ .

1. For  $w = \varepsilon$ , we get  $u \in L \Leftrightarrow v \in L$  as  $f$  refines  $L$ .
2. For  $w = aw'$ , we get  $ua =_f va$  by congruence of  $f$  and thus, by inductive hypothesis,  $uaw' \in L \Leftrightarrow vaw' \in L$ .

□

**Lemma** myhill\_suffix: imply\_suffix  $L$   $f$ .

**Lemma** myhill\_to\_weak\_nerode: Weak\_Nerode\_Rel  $L$ .

*Proof.* exact

{| weak\_nerode\_func :=  $f$ ;  
weak\_nerode\_imply := myhill\_suffix |}.

Defined.

This concludes step (b) of Theorem 5.1.8.

### 5.4 Nerode relations to Finite Automata

We prove step (d) of Theorem 5.1.8. If the Nerode relation for a language  $L$  is of finite index, we can construct a DFA that accepts  $L$ . The construction is very straightforward and uses the equivalence classes of the Nerode relation as the set of states for the automaton.

**Definition 5.4.1.** *Let  $L$  be a language. Let  $X$  be a finite type. Let  $f : \Sigma^* \mapsto X$  be the equivalence relation which proves that the Nerode relation for  $L$  is of finite index. We construct DFA  $A$  such that*

$$\begin{aligned} s &:= f(\varepsilon) \\ F &:= \{x \mid x \in X \wedge cr(x) \in L\} \\ \delta &:= \{(x, a, f(cr(x)a)) \mid x \in X, a \in \Sigma\} \\ A &:= (X, s, F, \delta). \end{aligned}$$

**Definition** `nerode_to_dfa` :=

```
{| dfa_s := f [|];
  dfa_fin := [pred x | cr f x \in L ];
  dfa_step := [fun x a => f (rcons (cr f x) a)] |}.
```

In order to show that  $A$  accepts the language  $L$ , we first need to connect runs on  $A$  to the equivalence classes, i.e. the range of  $f$ . The following lemma gives a direct connection.

**Lemma 5.4.2.** *Let  $w \in \Sigma^*$ . Let  $\sigma$  be the run of  $w$  on  $A$  starting in  $s$ . We have that the last state of  $\sigma$  is the equivalence class of  $w$ , i.e.*

$$\sigma_{|\sigma|-1} = f(w).$$

*Proof.* We proceed by induction on  $w$  from right to left.

1. For  $w = \varepsilon$  we have  $s = f(\varepsilon)$ .
2. For  $w = w'a$  we know that the run of  $w'$  on  $A$  starting in  $s$  ends in  $f(w')$ . It remains to show that  $(f(w'), a, f(w)) \in \delta$ . We have  $cr(f(w'))a =_f w$ , i.e.  $f(cr(f(w'))a) = f(w)$  by definition of  $f$ . Thus, it suffices to show  $(f(w'), a, f(cr(f(w'))a)) \in \delta$ , which holds by definition of  $\delta$ .

□

**Theorem 5.4.3.**  *$A$  accepts  $L$ , i.e.  $\mathcal{L}(A) = L$ .*

*Proof.* Let  $w \in \Sigma^*$ . Let  $\sigma$  be the run of  $w$  on  $A$  starting in  $s$ .  $w$  is accepted if and only if  $\sigma_{|\sigma|-1} \in F$ , i.e. if and only if  $cr(\sigma_{|\sigma|-1}) \in L$ . We have  $w =_f cr(\sigma_{|\sigma|-1})$  and therefore  $w \in L \Leftrightarrow cr(\sigma_{|\sigma|-1}) \in L$ . Thus  $w$  is accepted if and only if  $w \in L$ . □

The resulting automaton is minimal, i.e. there exists no other automaton that accepts the same language and has less states than  $A$ .

This concludes step (d) of Theorem 5.1.8.

## 5.5 Minimizing Equivalence Classes

Finally, we prove that if there is a weak Nerode relation for a language  $L$ , the Nerode relation is of finite index. For this purpose, we employ a table-filling algorithm [21] to find states indistinguishable under the Myhill-Nerode relation. This algorithm was originally intended to be used on automata. It identifies (un)distinguishable states based on their successors. We use the finite type  $X$ , i.e. the equivalence classes, instead of states.

For the remainder of this section, we assume we are given a language  $L$  and a weak Nerode relation  $f_0$ .

We employ a fixed-point construction to find equivalence classes that are  $\dot{=}_L$ -equivalent. In each step, we add those equivalence classes that are distinguishable based on equivalence classes that were distinguishable in the previous step. The initial set of distinguishable equivalence classes are distinguishable by the inclusion of their canonical representative in  $L$ . We denote this initial set  $dist_0$ .

$$dist_0 := \{(x, y) \in F \times F \mid cr(x) \in L \Leftrightarrow cr(y) \notin L\}.$$

**Definition** `distinguishable` := [ fun x y => (cr f\_0 x) \in L != ((cr f\_0 y) \in L) ].

**Definition** `dist0` := [ set x | distinguishable x.1 x.2 ].

To find more distinguishable equivalence classes, we have to identify equivalence classes that “lead to” distinguishable equivalence classes. In analogy to the minimization procedure on automata, we define successors of equivalence classes with respect to a single character. The intuition is that two states are distinguishable if they are succeeded by a pair of distinguishable states. Conversely, if a pair of states is not distinguishable, then their predecessors can not be distinguished by those states. Thus, two states are indistinguishable if none of their succeeding pairs of states are distinguishable.

**Definition 5.5.1.** Let  $x, y \in X$  and  $a \in \Sigma$ . We define  $succ_a$  and  $psucc_a$ .  $succ_a(x) := f_0(cr(x) \cdot a)$  and  $psucc_a(x, y) := (succ_a(x), succ_a(y))$ .

**Definition** `succ` := [ fun x a => f\_0 ((cr f\_0 x) ++ [:a]) ].

**Definition** `psucc` := [ fun x y => [ fun a => (succ x a, succ y a) ] ].

The fixed-point algorithm tries to extend the set of distinguishable equivalence classes by looking for a pair of equivalence classes that transitions to a pair of distinguishable equivalence classes. Given a set of pairs of equivalence classes  $dist$ , we define the set of pairs of distinguishable equivalence classes that have successors in  $dist$  as

$$dist_S(dist) := \{(x, y) \mid \exists a. psucc_a(x, y) \in dist\}.$$

**Definition** `distS` (`dist` : {set  $X \times X$ }) :=  
[ set (x,y) | x in X, y in X & [ exists a, psucc x y a \in dist ] ].

**Definition 5.5.2.** Let  $dist$  be a subset of  $X \times X$ . We define *one-step-dist* such that

$$one\_step\_dist(dist) := dist_0 \cup dist \cup dist_S(dist).$$

**Definition** `one_step_dist` `dist` := `dist0` :|: `dist` :|: (`distS` `dist`).

**Lemma 5.5.3.** *one-step-dist* is monotone and has a fixed-point.

*Proof.* Monotonicity follows directly from the monotonicity of  $\cup$ . The number of sets in  $X \times X$  is finite. Therefore, *one-step-dist* has a fixed point. We iterate *one-step-dist*  $|X \times X| + 1 = |X|^2 + 1$  times on the empty set. Since there can only ever be  $|X \times X|$  items

in the result of *one-step-dist*, we will find the fixed point in no more than  $|X * X| + 1$  steps.

Let *distinct* be the fixed point of *one-step-dist* and let it be denoted by  $\not\cong$ . We write *equiv* for the complement of *distinct* and denote it  $\cong$ .  $\square$

**Definition**  $\text{lf}p := \text{iter } \#|T|. + 1 \text{ F set0.}$

**Definition**  $\text{distinct} := \text{lf}p \text{ one\_step\_dist.}$

We now show that  $\cong$  is equivalent to the Nerode relation. Formally, this means constructing a function that fulfills our definition of an equivalence relation of finite index. Then, we prove that this equivalence relation is equivalent to the the Nerode relation. First, we will prove that  $\cong$  is an equivalence relation. Then, we will prove it equivalent to the Nerode relation in two separate steps, since the two directions require different strategies.

**Lemma 5.5.4.**  *$\cong$  is an equivalence relation.*

*Proof.* We first state transitivity of  $\cong$  in terms of  $\not\cong$ :

$$\forall x, y, z \in X. \neg(x \not\cong y) \implies \neg(y \not\cong z) \implies \neg(x \not\cong z). \quad (*)$$

It suffices to show that *distinct* is anti-reflexive, symmetric and fulfills (\*). Note that complementary transitivity, anti-reflexivity and symmetry are closed under union. We proceed by fixed-point induction.

1. For  $\text{one-step-dist}(\text{dist}) = \emptyset$  we have anti-reflexivity, symmetry and (\*) by the properties of  $\emptyset$ .
2. For  $\text{one-step-dist}(\text{dist}) = \text{dist}'$  we have *dist* anti-reflexive, symmetric and (\*). It remains to show that  $\text{dist}_0$  and  $\text{distinct}_S(\text{dist})$  are anti-reflexive, symmetric and fulfill (\*).

$\text{dist}_0$  is anti-reflexive, symmetric and fulfills (\*) by definition.

$\text{distinct}_S(\text{dist})$  can be seen as an intersection of a symmetric subset of  $X \times X$  defined by  $\text{psucc}_a$  and *dist*, the latter of which is anti-reflexive, symmetric and fulfills (\*). Thus,  $\text{distinct}_S(\text{dist})$  is anti-reflexive, symmetric and fulfills (\*).

Therefore,  $\text{dist}'$  is anti-reflexive, symmetric and fulfills (\*).  $\square$

**Lemma**  $\text{equiv\_refl } x: x \sim x.$

**Lemma**  $\text{equiv\_sym } x \ y: x \sim y \rightarrow y \sim x.$

**Lemma**  $\text{equiv\_trans } x \ y \ z: x \sim y \rightarrow y \sim z \rightarrow x \sim z.$

**Lemma 5.5.5.** *Let  $u, v \in \Sigma^*$ .  $f_0(u) \cong f_0(v) \implies u \dot{=}_L v$ .*



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*Proof.* Let  $w \in \Sigma^*$ . We then show the contraposition of the claim:

$$uw \in L \not\Rightarrow vw \in L \implies f_0(u) \not\equiv f_0(v).$$

By induction on  $w$ .

1. For  $w = \varepsilon$  we have  $u \in L \not\Rightarrow v \in L$  which gives us  $(f_0(u), f_0(v)) \in \text{dist}_0$ . Thus,  $f_0(u) \not\equiv f_0(v)$ .
2. For  $w = aw'$  we have  $uaw \in L \not\Rightarrow vaw \in L$ . We have to show  $f_0(u) \not\equiv f_0(v)$ , i.e.  $(f_0(u), f_0(v)) \in \text{distinct}$ . By definition of *distinct*, it suffices to show  $(f_0(u), f_0(v)) \in \text{one-step-dist}(\text{distinct})$ .

For this, we prove  $(f_0(u), f_0(v)) \in \text{distinct}_S(\text{distinct})$ . By  $uaw \in L \not\Rightarrow vaw \in L$  we have  $(f_0(cr(u)a), f_0(cr(v)a)) \in \text{dist}_0$ .

It remains to show that  $f_0(cr(u)a) \not\equiv f_0(cr(v)a)$  which we get by inductive hypothesis. For this, we need to show that  $cr(u)aw \in L \not\Rightarrow cr(v)aw \in L$ .

By the properties of  $f$ , we get  $cr(u)aw \in L \Leftrightarrow uaw \in L$  and  $cr(v)aw \in L \Leftrightarrow vaw \in L$ . Thus,  $cr(u)aw \in L \not\Rightarrow cr(v)aw \in L$ .

□

**Lemma 5.5.6.** *Let  $u, v \in \Sigma^*$ . If  $f_0(u) \not\equiv f_0(v)$ , then  $u$  and  $v$  are **not** equivalent with respect to the Nerode relation, i.e.  $f_0(u) \not\equiv f_0(v) \implies u \not\equiv_L v$ .*

*Proof.* We do a fixed-point induction.

1. For  $\text{one-step-dist}(\text{dist}) = \emptyset$  we have  $(f_0(u), f_0(v)) \in \emptyset$  and thus a contradiction.
2. For  $\text{one-step-dist}(\text{dist}) = \text{dist}'$  we have  $(f_0(u), f_0(v)) \in \text{dist}'$ . We do a case distinction on  $\text{dist}'$ .
  - a)  $(f_0(u), f_0(v)) \in \text{dist}_0$ . We have  $u \in L \not\Rightarrow v \in L$ . Thus,  $u \not\equiv_L v$  as witnessed by  $w = \varepsilon$ .
  - b)  $(f_0(u), f_0(v)) \in \text{dist}$ . By inductive hypothesis,  $u \not\equiv_L v$ .
  - c)  $(f_0(u), f_0(v)) \in \text{distinct}_S(\text{dist})$ . We have  $a \in \Sigma$  with  $\text{psucc}_a(f_0(u), f_0(v)) \in \text{dist}$ . By inductive hypothesis, we get  $cr(u)a \not\equiv_L cr(v)a$  as witnessed by  $w \in \Sigma^*$  such that  $cr(u)aw \in L \not\Rightarrow cr(v)aw \in L$ .

By the properties of  $f$ , we get  $cr(u)aw \in L \Leftrightarrow uaw \in L$  and  $cr(v)aw \in L \Leftrightarrow vaw \in L$ . Thus, we have  $u \not\equiv_L v$  as witnessed by  $aw$ .

□

**Corollary 5.5.7.** *Let  $u, v \in \Sigma^*$ . We have that*

$$f_0(u) \cong f_0(v) \iff u \dot{\equiv}_L v.$$

*Proof.* Follows immediately with Lemma 5.5.5 and Lemma 5.5.6. □

**Lemma** `equiv_suffix_equal u v`:  $u \sim_{f_0} v \rightarrow \text{suffix\_equal } L \ u \ v$ .

**Lemma** `distinct_not_suffix_equal u v`:

$u \not\sim_{f_0} v \rightarrow$

**exists**  $w, u ++ w \notin L \neq (v ++ w \notin L)$ .

**Lemma** `equivP u v`:

`reflect (suffix_equal L u v)`

`(u  $\sim_{f_0}$  v).`

**Definition 5.5.8.** Let  $w \in \Sigma^*$ . We define

$$f_{min}(w) := \{x \mid x \in X, f_0(w) \cong x\}.$$

Note that the range of  $f_{min}$  is finite (since  $X$  is finite) and does not contain the empty set (due to reflexivity of  $\cong$ ).

**Lemma 5.5.9.**  $f_{min}$  is surjective.

*Proof.* Let  $s \in \text{range}(f_{min})$ . Since  $s \neq \emptyset$ , there exists  $x \in X$  such that  $x \in s$ . We have  $f_0(x) = f_0(cr(x))$  and therefore  $f_0(x) \cong f_0(cr(x))$  by reflexivity of  $\cong$ . Thus,  $f_0(cr(x)) = s$  since  $f_{min}(x) = f_{min}(cr(x)) = s$ .  $\square$

**Lemma 5.5.10.** For all  $u, v \in \Sigma^*$  we have

$$f_{min}(u) = f_{min}(v) \iff f_0(u) \cong f_0(v).$$

*Proof.* “ $\Rightarrow$ ” We have  $f_{min}(u) = f_{min}(v)$  and thus  $f_0(u) \cong f_0(v)$ .

“ $\Leftarrow$ ” We have  $f_0(u) \cong f_0(v)$ . Let  $x \in X$ . It suffices to show that  $f_0(u) \cong x$  if and only if  $f_0(v) \cong x$ . This follows with symmetry and transitivity of  $\cong$ .  $\square$

**Lemma 5.5.11.**  $f_{min}$  is equivalent to the Nerode relation, i.e.  $f_{min}$  is surjective and for all  $u, v \in \Sigma^*$  we have

$$f_{min}(u) = f_{min}(v) \iff u \dot{=}_L v.$$

*Proof.* We have proven surjectivity in Lemma 5.5.9. By Lemma 5.5.10 we have  $f_{min}(u) = f_{min}(v)$  if and only if  $f_0(u) \cong f_0(v)$ . By corollary 5.5.7 we have  $f_0(u) \cong f_0(v)$  if and only if  $u \dot{=}_L v$ . Thus,  $f_{min}(u) = f_{min}(v)$  if and only if  $u \dot{=}_L v$ .  $\square$

The formalization of  $f_{min}$  is slightly more involved than the mathematical construction. We first need to define the finite type of  $f_{min}$ ’s range, which we do by enumerating all possible values of  $f_{min}$ .

**Definition** `equiv_repr x` := `[set y | x  $\sim_{f_0}$  y]`.

**Definition** `X_min` := `map equiv_repr (enum (fin_type f_0))`.

**Definition** `f_min w` := `SeqSub (equiv_repr_mem (f_0 w))`.

We then prove lemmas 5.5.9, 5.5.10 and Theorem 5.5.11 which are consequential and straight-forward.

**Lemma** `f_min_surjective`: `surjective f_min`.

**Lemma** `f_minP` `u v`:  
`reflect (f_min u = f_min v)`  
`(u ~=f.0 v)`.

**Lemma** `f_min_correct`: `equiv_suffix L f_min`.

**Definition** `f_min_fin` : `Fin_Eq_Cls` :=  
`{| fin_surjective := f_min_surjective |}`.

We can now state the result of this section.

**Theorem 5.5.12.** *The Nerode relation is of finite index.*

*Proof.* This follows directly from Lemma 5.5.4 and Lemma 5.5.11. □

**Lemma** `weak_nerode_to_nerode`: `Nerode_Rel L`.

This concludes step ③ of Theorem 5.1.8 and, thus, this chapter.

## Remarks

The characterizations presented here are very compact, mathematically. Interestingly, they also lend themselves very well to formalization. Even with the fixed-point algorithm, this entire chapter is formalized in less than 530 lines of code. This is a very reasonable size, considering that we introduce three different characterizations and prove them all equally expressive to finite automata.