Constructive Formalization of Regular Languages

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Contents

Motivation

We want to develop an elegant formalization of regular languages in Coq based on finite automata.

There are several reasons for choosing this topic and our specific approach:

- Strong interest in formalizations in this area.
- Few formalizations of regular languages in Coq, most of them very long or incomplete.
- Most formalizations avoid finite automata in favor of regular expressions. Regular expressions (with Brzozowski derivatives) lead to more complex but also more performant algorithms.

Quick Recap

We use extended **regular expressions** (regexp):

$$r,s ::= \emptyset \mid \varepsilon \mid a \mid rs \mid r+s \mid r\&s \mid r^* \mid \neg r$$

- $\mathcal{L}(\emptyset) := \{\}$
- $\mathcal{L}(\varepsilon) := \{\varepsilon\}$
- $\mathcal{L}(a) := \{a\}$
- $\mathcal{L}(rs) := \mathcal{L}(r) \cdot \mathcal{L}(s)$
- $\mathcal{L}(r+s) := \mathcal{L}(r) \cup \mathcal{L}(s)$
- $\mathcal{L}(r\&s) := \mathcal{L}(r) \cap \mathcal{L}(s)$
- $\mathcal{L}(r^*) := \mathcal{L}(r)^*$
- $\mathcal{L}(\neg r) := \overline{\mathcal{L}(r)}$

Derivatives of Regular Expressions (1964), *Janusz Brzozowski*:

- der a $\emptyset$= $\emptyset$
- der a ε= $\emptyset$
- der a b = if a = b then ε else $\emptyset$
- der a (r s) = if $\delta(r)$ then (der a s) + ((der a r) s) else (der a r) s
  with $\delta(r) = true \Leftrightarrow \varepsilon \in \mathcal{L}(r)$.
- der a (r + s) = (der a r) + (der a s)
- der a (r & s) = (der a r) & (der a s)
- der a (r*) = (der a r) r*
- der a ($\neg$r) = $\neg$(der a r)

Theorem: $w \in \mathcal{L}(r)$ if and only if the derivative of r with respect to $w_1 .. w_{|w|}$ accepts ε.

Regular languages are also exactly those languages accepted by **finite automata** (FA).

Our definition of FA over an alphabet Σ :

- The finite set of states Q
- The initial state $s_0 \in$ Q
- The (decidable) transition relation Δ $\in$ (Q, Σ, Q)

  Deterministic FA: Δ is functional and **total**.
- The set of finite states F, F $\sqsubseteq$ Q

Let A be a FA.

$$\mathcal{L}(A) := \left\{ w \mid \exists s_1 , \ldots\ s_{|w|} \in Q\ s.t. \forall\ i : 0 < i \le n \rightarrow (s_{i-1}, w_i, s_i) \in \Delta \right\}$$

Finally, regular languages are also characterized by the Myhill-Nerode theorem.

First, we define an equivalence relation on L:

$$x \; R_L \; y \; := \forall z,\; x \cdot z \; \in L \; \Leftrightarrow \; y \cdot z \; \in L$$

Myhill-Nerode theorem: L is regular if and only if R_L divides L into a finite number of equivalence classes.

Previous work

- Constructively formalizing automata theory (2000)

  Robert L. Constable, Paul B. Jackson, Pavel Naumov, Juan C. Uribe

  PA: Nuprl

  The first constructive formalization of MH.

  Based on **FA**.

- Proof Pearl: Regular Expression Equivalence and Relation Algebra (2011)

  Alexander Krauss, Tobias Nipkow

  PA: Isabelle

  Based on **derivatives of regexps**. No proof of termination.

- Deciding Kleene Algebras in Coq (2011)

  Thomas Braibant, Damien Pous

  PA: Coq

  Based on **FA**, matrices. Focus on performance.

- A Decision Procedure for Regular Expression Equivalence in Type Theory (2011)

  Thierry Coquand, Vincent Siles

  PA: Coq

  Based on **derivatives of regexps**.

- A Formalisation of the Myhill-Nerode Theorem based on Regular Expressions (Proof Pearl) (2011)

  Chunhan Wu, Xingyuan Zhang, Christian Urban

  PA: Isabelle

  The first proof of MH based on **derivatives of regexps**.

- Deciding Regular Expressions (In-)Equivalence in Coq (2011)

  Nelma Moreira, David Pereira, Simão Melo de Sousa

  PA: Coq

  Based on Krauss, Nipkow. Proof of termination.

Our Development

- We want to focus on elegance, not performance.
- Our main goals are MH and the decidability of regexp equivalence.
- We use finite automata.
  They are not at all impractical. (Partly thanks to Ssreflect's finType)

Ssreflect

- Excellent support for all things boolean.
- Finite types with all necessary operations and closure properties.
  (very useful for alphabets, FA states, etc.)
- Lots and lots of useful lemmas and functions.

Finite automata

DFA and NFA without e-transitions.

- DFA to prove closure under $\cup$, $\cap$, and $\neg$.
- NFA to prove closure under $\cdot$ and $*$.

Also proven: NFA $\Leftrightarrow$ DFA.

This gives us: regexp $\Rightarrow$ FA.

Roadmap

1. Emptiness test on FA ($\emptyset(A) := \mathcal{L}(A) = \emptyset$)
2. FA $\Rightarrow$ regexp
3. Dedicedability of regexp equivalence using regexp $\Rightarrow$FA, (2) and (1):

$$\mathcal{L}(r) = \mathcal{L}(s)$$

$$\Leftrightarrow$$

$$\emptyset(\mathcal{A}(r) \cap \overline{\mathcal{A}(s)}) \wedge \emptyset(\overline{\mathcal{A}(r)} \cap \mathcal{A}(s))$$

4. Finally, we want to prove the MH theorem

With this we'll have an extensive formalization of regular languages including regular expressions, FA and MH and all corresponding equivalences.