Constructive Formalization of Regular Languages

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Goal:
Find simple proofs for the decidability of regular expression equivalence and the Myhill-Nerode theorem.

Roadmap:

1. RE $\Rightarrow$ FA (**DONE**)
2. Emptiness test on FA (**Easy**)
3. RE equivalence (**Follows from 1 and 2**)
4. FA $\Rightarrow$ RE (**Work in progress**)
5. Myhill-Nerode

Definitions:

- We use extended regular Expressions (RE):

$$r, s ::= \emptyset \mid \varepsilon \mid a \mid rs \mid r + s \mid r \,\&\, s \mid r^* \mid \neg r$$

$$\textbf{Def.:} \quad \sum_{x \in X} r_x := r_{x_0} + ... + r_{x_{|X|-1}}$$

- Our Finite Automata (FA) are

$$(\Sigma, Q, q_0, F, \delta)$$

(The transition relation δ of deterministic FA is total).

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Why is that?

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- Converting REs to FAs is done by structural recursion on a **tree**. The result is a **flat structure**.
- Converting FAs to REs **can not be done** by structural recursion. There is **no recursive structure** in FAs.
  But somehow we need to construct a **tree** of REs.

Three methods (+ variations):

1. Transitive Closure
2. State Removal
3. Brzozowski Algebraic Method

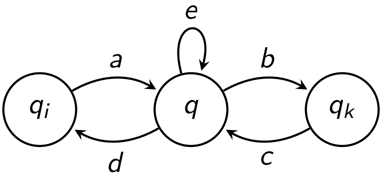
State Removal

Given: NFA $A = (\Sigma, Q, q_0, F, \delta)$.

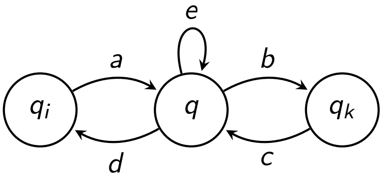
New concept: Automata that have transitions labeled by RE.

Idea: Remove states until there are two or less states remaining. Update the remaining states' transitions by incorporating the "lost" paths.

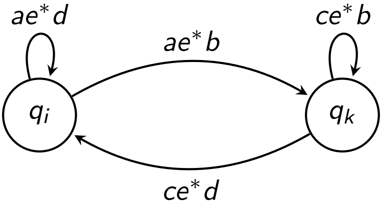
Remove q from



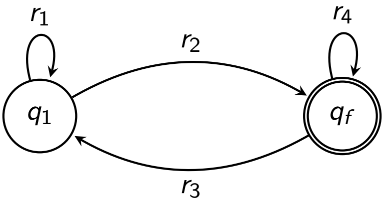
Remove q from



to get



Repeat until A is of this form:



$$\Rightarrow \mathcal{L}(A) = \mathcal{L}(r_1^* r_2 (r_4 + r_3 r_1^* r_2)^*)$$

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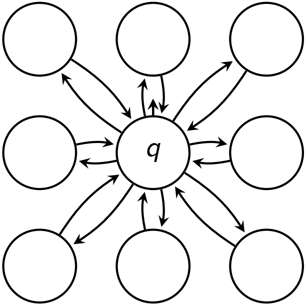

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  In reality, there can be $|Q| - 1$ states connected to q.
- What about final states?
  1. Introduce a new final state without any outgoing edges.
  2. Introduce ε transitions from all other final states to the final new state.
  3. Make all other states non-final.
  4. Never remove the new final state.

Formalization:

- Requires a new kind of finite automaton that has RE transitions.
- Lots of details to consider.
- Induction on the number of states.

Brzozowski Algebraic Method

Given: FA $A = (\Sigma, Q, q_0, F, \delta)$.

Idea: Retrieve a RE for FA by solving a system of equations determined by δ.

Construct system of equations:

$$\begin{array}{lcll} r_0 & = & \displaystyle\sum_{\substack{a\in\Sigma \\ 0\leq i<|Q|}} \{\, a\, r_i \mid (q_0, a, q_i) \in \delta\} & (+\,\varepsilon \text{ if } r_0 \in F) \\ \vdots & = & \vdots & \\ r_{|Q|-1} & = & \displaystyle\sum_{\substack{a\in\Sigma \\ 0\leq i<|Q|}} \{\, a\, r_i \mid (q_{|Q|-1}, a, q_i) \in \delta\} & (+\,\varepsilon \text{ if } r_{|Q|-1} \in F) \end{array}$$

$$\Rightarrow \mathcal{L}(A) = \mathcal{L}(r_0)$$

Solve the system by substitution and **Arden's Lemma** which states that for all regular languages X, Y and Z the equation

$$X = YX + Z \tag{1}$$

has the unique solution

$$X = Y^*Z \tag{2}$$

Formalization:

- Requires a formalization of these equations and operations on them.
- We would need to prove Arden's Lemma.
- We would also need to prove that Arden's Lemma (and substitution) is enough to solve these systems of equations.

Transitive Closure

Given: FA $A = (\Sigma, Q, q_0, F, \delta)$.

Idea: Construct regexps r_f for every final states $f \in F$ s.t. r_f matches all words which A accepts with final state f.

$$\Rightarrow \mathcal{L}(A) = \mathcal{L}(\sum_{f \in F} r_f)$$

How do we construct r_f?

We generalize the idea of r_f to R_{ij}^k which matches all words which lead from state i to j while passing only through states with index smaller than k.

1. Merge multiple edges between states to one unified edge.
2. Construct regexp R_{ij}^k recursively:

$$R_{ij}^0 := \begin{cases} r & \text{if } i \neq j \wedge i \text{ has edge } r \text{ to j} \\ \varepsilon + r & \text{if } i = j \wedge i \text{ has edge } r \text{ to j} \\ \emptyset & \text{otherwise} \end{cases}$$

$$R_{ij}^k := R_{ik}^{k-1} R_{kk}^{k-1} R_{kj}^{k-1} + R_{ij}^{k-1}$$

$$\Rightarrow \mathcal{L}(A) = \mathcal{L}(\sum_{f \in F} r_f) = \mathcal{L}(\sum_{f \in F} R_{0f}^{|Q|})$$

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- Easier than the other methods.
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This appears to be the simplest formalization.

Our Approach:
There are different ways of formalizing R_{ij}^{k} itself, especially its parameters. Most practical so far:
k is of type nat, i and j are ordinals from $[0..|Q|-1]$.

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k is of type nat, i and j are ordinals from $[0..|Q|-1]$.
This gives us easy recursion and matching on k.
But we have to use $min\,k\,(|Q|-1)$ to map k to the corresponding ordinal (and then to a state).

To get a FA counterpart to R_{ij}^k, we introduce A_{ij}^k s.t.

$$\mathcal{L}(A_{ij}^k) = \mathcal{L}(R_{ij}^k).$$

A_{ij}^k is similar to A. It has one additional state, which has all incoming edges of state j and no outgoing edges. It only leaves states that are i or $< k$. It only enters states less than $< k$ or the new state.

We can then show that

$$\mathcal{L}(\bigcup_{f \in F} A_{q_0 f}^{|Q|}) = \mathcal{L}(A).$$

Lemmas:

1. The language of an automaton is the union of the words with runs on A that end in any final state, i.e. $\mathcal{L}(A) = \bigcup_{f \in F} \mathcal{L}(A_f)$, where A_f accepts only in f. (Not difficult)
2. A path in A_{ij}^{k+1} that is not in A_{ij}^{k} can be decomposed into:
   - a sub path from index 0 to the first occurrence of k
   - a sub path from the first to the last occurrence of k
   - the remaining sub path from the last occurrence of k to the end.

   (Complex; requires a new operations on lists)

More lemmas:

3. A path on A which passes through a final state more than once is in $\mathcal{L}(A)^*$ (Relies on the same operations we need for lemma 2)
4. Any path on A_{ij}^k that ends in the new accepting state can be mapped to an equivalent path that ends state j (and vice-versa).
5. Any path on A_{ij}^k that does not end in the new accepting state is also a path on A.

Thank you for your attention