Faithfully flat Descent (27.04.17)

Thm Let $\varphi: A \longrightarrow B$ faithfully flat map of rings. Then

$$\mathcal{F}: \{A\text{-modules } M\} \longrightarrow \left\{ \begin{array}{l} B\text{-modules } N + \alpha: N \otimes_A B \cong B \otimes_A N \\ \text{isom of } B \otimes_A B\text{-modules s.t.} \\ \begin{tikzcd} N \otimes_A B \otimes_A B \arrow[r, "\cong"] \arrow[dr, "\cong"'] & B \otimes_A N \otimes B \arrow[d, "\cong"] \\ & B \otimes_A B \otimes_A N \end{tikzcd} \end{array} \right\}$$

is an equivalence.

Geometric reinterpretation $f: Y = \operatorname{Spec} B \longrightarrow X = \operatorname{Spec} A$

faithfully flat. If $\mathcal{M}$ ~~all~~ q.coh. sheaf on X

$\mathcal{N} = f^*\mathcal{M}$ q.coh. sheaf on Y. s.th.

$$\begin{tikzcd} & Y \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Y \arrow[dr, "f"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

$$\begin{array}{ccccc} p_1^* f^* \mathcal{M} & = (p_1 \circ f)^* \mathcal{M} & = & (p_2 \circ f)^* \mathcal{M} \\ \| & & & \| \\ p_1^* \mathcal{N} & \cong p_2^* \mathcal{N} & = & p_2^* f^* \mathcal{M} \end{array}$$

(If $Y = \sqcup U_i$, $U_i \subseteq X$ open, $\alpha \longleftrightarrow$ isomorphism on overlaps $U_i \cap U_j$
cocycle cond. $\longleftrightarrow$ these compose correctly

Why "descent"? Want to descend from cover Y down to X.
Can descend modules (by theorem), schemes, ..., or prop. (e.g. of ~~flatness~~)

For proof of Thm, observe that F has right adjoint

$$G: (N, \alpha) \longmapsto \cancel{eq}\ eq\left(N \overset{n \mapsto 1 \otimes n}{\underset{n \mapsto \alpha(n \otimes 1)}{\rightrightarrows}} B \otimes_A N \right)$$

(in geometric terms, $\{ s \in H^0(Y, \mathcal{N}) \mid p_1^*(s) = p_2^*(s) \}$

$$\left. \begin{array}{l} p_1^*(s) \in H^0(Y \times_X Y, p_1^* \mathcal{N}) \\ \quad \| \\ p_2^*(s) \in H^0(Y \times_X Y, p_2^* \mathcal{N}) \end{array} \right\}$$

To prove thm, suffices: i) $\forall M_\xi$, $M \xrightarrow{\sim} G(F(M))$ ②

ii) $\forall (N, \alpha)$, $F(G(N_\xi, \alpha)) \xrightarrow{\sim} (N, \alpha)$

only ii) requires cocycle condition.

Start with i).

<u>Proposition</u> Let $\varphi: A \to B$ be faithfully flat. Then

$0 \to A \xrightarrow{\varphi} B \to B \otimes_A B$ is exact.

$b \mapsto b \otimes 1 - 1 \otimes b$

<u>Proof</u> <u>1st case</u> $\varphi: A \to B$ has section $\sigma: B \to A$ (of rings, i.e. $\sigma \circ \varphi = \mathrm{Id}$)

(In this case, don't need faithful flatness)

Then clearly φ is injective. Assume $b \otimes 1 = 1 \otimes b \in B \otimes_A B$.

<u>Apply</u> $\sigma \otimes_A 1 : B \otimes_A B \longrightarrow B$

$b \otimes 1 \longmapsto \sigma(b)$

$1 \otimes b \longmapsto b$

Thus, get $\underset{\underset{B}{\cap}}{b} = \underset{\underset{A}{\cap}}{\sigma(b)}$, i.e. $b \in A$ as desired

$B \supseteq A$

<u>General case</u>

Lemma from last time: enough to check after $- \otimes_A B$

Let $A' = B$, $B' = B \otimes_A B$, then the sequence becomes

$0 \to A' \to B' \to B' \otimes_{A'} B'$

$b' \mapsto b' \otimes 1 - 1 \otimes b'$

But $B' = B \otimes_A B \xrightarrow{m} B = A'$ is a section σ of $A' \to B'$.

<u>Corollary</u> For all A-modules M, $M \xrightarrow{\sim} F(G(M))$

$m \otimes b \longmapsto m \otimes b \otimes 1$

$= \mathrm{eq}(M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B)$, i.e.

$m \otimes b \longmapsto m \otimes 1 \otimes b$

$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B$ is exact

$m \otimes b \longmapsto m \otimes b \otimes 1 - m \otimes 1 \otimes b$

<u>Proof</u> (would like to take $M \otimes_A$ (Prop'n), but $M \otimes_A -$ may not be exact)

repeat argument: enough after $- \otimes_A B$, so can assume $A \xrightarrow{\varphi} B$ has a section.

Then $M \longrightarrow M \otimes_A B$ still has section $\Rightarrow$ injective

If $\sum m_i \otimes b_i \in M \otimes_A B$ satisfies $\sum m_i \otimes b_i \otimes 1 = \sum m_i \otimes 1 \otimes b_i$

$$\in M \otimes_A B \otimes_A B \xrightarrow[\mathrm{id} \otimes \sigma \otimes \mathrm{id}]{} M \otimes_A A \otimes_A B = M \otimes_A B$$

$$\sum m_i \otimes \sigma(b_i) = \sum m_i \otimes b_i \qquad \square$$

<u>Now step ii)</u>

Let (N, α), $N = B$-module, $\alpha : N \otimes_A B \cong B \otimes_A N$

<u>Prop'n</u> Assume (N, α) satisfies cocycle condition. Then

$$N = \underbrace{(N \otimes_A B)}_{N \otimes_B (B \otimes_A B)} \otimes_{B \otimes_A B} B \underset{\alpha \otimes \mathrm{id}}{\cong} B \otimes_A N \otimes_{B \otimes_A B} B = N$$

is the identity.

<u>Geometrically</u>: $\mathcal{N} / Y$ + $\alpha : p_1^* \mathcal{N} \cong p_2^* \mathcal{N}$

$\rightsquigarrow$

$$\begin{tikzcd}
 & Y \arrow[d] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
 & \cancel{X \times Y}\ Y \times_X Y \arrow[dl, "p_2"'] \arrow[dr, "p_2"] & \\
Y \arrow[dr, "f"'] & & Y \arrow[dl, "f"] \\
 & X &
\end{tikzcd}$$

$\mathrm{id} = \Delta_f^*(\alpha)$:

$\mathcal{N} = \Delta_f^* p_1^* \mathcal{N} \simeq \Delta_f^* p_2^* \mathcal{N} = \mathcal{N}$

<u>Proof</u> Idea: ψ ~~A~~ autom., so suffices to show $\psi(\psi(n)) = \psi(n)$, $\forall n \in N$.

Let $n \in N$. How to compute $\psi(n)$?

Write $\alpha(n \otimes 1) = \sum b_i \otimes x_i$, $b_i \in B$, $x_i \in N$

Then $\psi(n) = \sum b_i x_i \in N$

Now compute $\psi(\psi(n))$: Write $\alpha(x_i \otimes 1)$

$= \sum_j b_{ij} \otimes y_{ij}$, $b_{ij} \in B$, $y_{ij} \in N$.

cocycle cond (applied to $n \otimes 1 \otimes 1$) $\quad N \otimes B \otimes B \quad B \otimes_A N \otimes_A B$

$$\underbrace{\sum_{i,j} b_i \otimes b_{ij} \otimes y_{ij}}_{\sum_{i,j} b_i b_{ij} y_{ij}} = \sum_i b_i \otimes 1 \otimes x_i \in B \otimes_A B \otimes_A N.$$

(the sum on the right maps to $\sum b_i x_i$)

What is $\psi(\psi(n)) = \sum_i b_i \psi(x_i) = \sum_{i,j} b_i b_{ij} y_{ij}$

$= \sum b_i x_i = \psi(n)$ ☐

Prop'n (Step ii) Assume (N, α) satisfies cocycle cond. Let

$$M = \mathrm{eq}\left(N \rightrightarrows B \otimes_A N \right), \quad n \mapsto 1 \otimes n, \quad n \mapsto \alpha(n \otimes 1)$$

Then $M \otimes_A B \xrightarrow{\sim} N$, i.e. $F(G(N,\alpha)) \xrightarrow{\sim} (N, \alpha)$

Proof It is enough to check this after $- \otimes_A B$

(Note equalizer commutes with $- \otimes_A B$ by flatness)

As before, reduce to a case that $\varphi: A \to B$ has section $\sigma: B \to A$

injectivity: Assume $m_i \in M$, $b_i \in B$, $\sum b_i m_i = 0 \in N$.
(of $M \otimes_A B \to N$) $\quad \Downarrow \quad \alpha(m_i \otimes 1) = 1 \otimes m_i$ $\quad$ (i.e. $\sum m_i \otimes b_i$ is in the kernel)

~~Then~~ 0

Then $0 = \alpha(\sum b_i m_i \otimes 1) = \cancel{\alpha(\sum b_i \otimes} \sum b_i \otimes m_i$,

so $\sum b_i \otimes m_i = 0 \in B \otimes_A M$ ☐

<u>Surjectivity</u> Let $n \in N$

Write $\alpha(n \otimes 1) = \sum b_i \otimes x_i \in B \otimes_A N$, $b_i \in B, x_i \in N$.

Then as $\psi(n) = n$, $n = \sum_i b_i x_i$.

$\alpha(x_i \otimes 1) = \sum b_{ij} \otimes y_{ij}$ as before

By cocycle condition:

$$\begin{array}{ccccc} & \sum b_i \otimes b_{ij} \otimes y_{ij} & = & \sum b_i \otimes 1 \otimes x_i & \in B \otimes_A B \otimes_A N \\ \alpha\left(\sum_i \sigma(b_i) x_i \otimes 1\right) & \downarrow & & \downarrow & \downarrow {\scriptstyle \sigma \otimes id \otimes id} \\ \wr\wr & & & & \\ \sum_i \sigma(b_i)\alpha(x_i \otimes 1) = & \sum_i \sigma(b_i)\left(\sum b_{ij} \otimes y_{ij}\right) & = & \sum \sigma(b_i) \otimes x_i & B \otimes_A N \end{array}$$

$$\Rightarrow \sum_i \underbrace{\sigma(b_i)}_{\in A} x_i \in M \subseteq N$$

Now specialize along $B \otimes_A B \otimes_A N \xrightarrow{id \otimes \sigma \otimes id} B \otimes_A N \to N$

$$\sum_i b_i \underbrace{\left(\sum \sigma(b_{ij}) y_{ij}\right)}_{\in M} = \sum_i b_i \otimes x_i = n.$$

<u>Thus</u> $\forall n \in N$, if $\alpha(n \otimes 1) = \sum b_i \otimes x_i$, then $\sum \sigma(b_i) x_i \in M$

(Apply previous argument to x's instead of n) $\square$

<u>Remarks</u> One can make the proof much more abstract.

"Barr-Beck Monadicity theorem"

<u>Generalizations</u>: Given any map of rings $\varphi: A \to B$, when is the functor of A-modules $F: \{A\text{-Modules}\} \to \left\{\begin{array}{l} B\text{-modules } N \\ + \text{cocycle cond} \end{array}\right\}$ fully faithfull / essentially surj. ?

Necessary cond.: for fully faithful (or just faithful)

For all A-modules $M \cong \mathrm{Hom}_A(A, M) \hookrightarrow \mathrm{Hom}_B(B, B \otimes_A M)$

equivalently to asking $\forall I \subseteq A$, $A/I \hookrightarrow B/IB = B \otimes_A M$.

"φ universally injective".

<u>Thm</u> If φ is universally inj, then F is an equivalence.

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~~Theorem If φ is an~~ This is ~~as~~ satisfied for example if $A \to B$ has a splitting as A-modules.

Direct summand conjecture (Hochster, 1973). If A is regular $A \to B$ finite + injective, then it splits as A-modules.

($\Rightarrow$ descent
$\Leftarrow$ not hard)

Now a Thm of Y. André. Colloquium talk next Wednesday!

Comments: easy if $\mathbb{Q} \subseteq A$. there is a "trace map" $tr : B \to A$ (for any finite extension of normal rings)

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "\text{mult by degree } B/A = d \in A^*"'] & B \arrow[r, "tr"] & A \end{tikzcd}$$

So $\frac{1}{d} tr$ is a split.

If $\mathbb{F}_p \subseteq A$, proved in 80's - 70's. Hochster developed big theory around this. ~~But mixed~~