Kähler-differentials:

$R \to A$ morphism of rings

$\rightsquigarrow \Omega^1_{A/R} = \sum_{a \in A} A\,da$

$d: A \to \Omega^1_{A/R}$ universal derivation

$a \mapsto da$

M A-Module $\rightsquigarrow \mathrm{Hom}_A(\Omega^1_{A/R}, M) \xrightarrow{\sim} \mathrm{Der}_R(A, M)$

$(\varphi: \Omega^1_{A/R} \to M) \longmapsto \varphi \circ d$

If
$$\begin{array}{ccc} A & \to & A' \\ \uparrow & & \uparrow \\ R & \to & R' \end{array}$$
$\rightsquigarrow A' \otimes_A \Omega^1_{A/R} \to \Omega^1_{A'/R'}$

canonical morph

$a' \otimes da \longmapsto a' \cdot d(f(a))$

Lemma: ("derivation and liftings")

given
$$\begin{array}{ccc} R & \to & A \\ \downarrow & \swarrow & \downarrow \\ B & \to & B/I \end{array}$$
of rings with $I^2 = 0$ $\rightsquigarrow$ I naturally a B/I-module

(lifts have to be R-lin)

(Lifting situation
$$\begin{array}{ccc} T_0 = \mathrm{Spec}(B/I) & \longrightarrow & \mathrm{Spec}\, A \\ \downarrow & \dashrightarrow & \downarrow \\ T = \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}\, R \end{array}$$
)

(i) $\varphi_1, \varphi_2 : A \to B$ lifts of f, i.e. $\pi \circ \varphi_i = f$

$\implies \delta = \varphi_1 - \varphi_2 \to I$ is a $\underset{\vee}{R\text{-lin.}}$ derivation

(ii) $\varphi: A \to B$ lifts of f, $\delta: A \to I$ R-lin derivation, $\rightsquigarrow \varphi + \delta : A \to B$ is another lift

($\Rightarrow$ If there ex a lift, then $\mathrm{Der}_R(A, I)$ acts freely, transitively on the set of liftings)

Prf:

(i) $\delta(ab) = \varphi_1(ab) - \varphi_2(ab)$

$= \varphi_1(a)\varphi_1(b) - \varphi_1(a)\varphi_2(b) + \varphi_1(a)\varphi_2(b) - \varphi_2(a)\varphi_2(b)$

$= \underbrace{\varphi_1(a)\,\delta(b)}_{\overset{\shortparallel}{a\,\delta(b)}} + \underset{\overset{\shortparallel}{b\,\delta(a)}}{\varphi_2(b)\cdot\delta(a)}$ $\rightsquigarrow$ Derivation

(ii) $(\varphi + \delta)(ab) = \varphi(ab) + \delta(ab) = \varphi(a)\varphi(b)$
$+ a\,\delta(b) + b\,\delta(a)$

$= \varphi(a)\varphi(b) + \varphi(a)\delta(b) + \varphi(b)\delta(a) + \delta(a)\delta(b)$

$= (\varphi+\delta)(a) \cdot (\varphi+\delta)(b)$

$\rightsquigarrow$ claim $\square$

Conj $\mathrm{Der}_R(A, M) \simeq \{(\varphi_1, \varphi_2) : A \longrightarrow A[M] \mid \varphi_i$ R-lin sections $A \to A[M]\}$

where $A[M] = A \oplus \varepsilon M$, $(a+\varepsilon m)(b+\varepsilon n) := ab + \varepsilon bm + \varepsilon an$

$M \longrightarrow A[M] \longrightarrow A$ square zero.

Cor: $\mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ formally unramified (at most one lift)

$\Leftrightarrow \Omega^1_{A/R} = 0$

Now Localize:

Lemma: $S \subseteq A$ mult.

$\rightsquigarrow \Omega^1_{A/R} \otimes A[S^{-1}] \simeq \Omega^1_{A[S^{-1}]/R}$.

Proof: 1. Pass:

$da \otimes \frac{a'}{s} \longmapsto \frac{a'}{s}\, d(\frac{a}{1})$

$da \otimes \frac{1}{s} + ds \otimes (-\frac{a}{s^2}) \longleftrightarrow d(\frac{a}{s})$

Prop:

$X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, $\Delta : X \to X \times_S X$

with ideal $I = \ker(A \otimes A \xrightarrow{\sim} A)$

$a \otimes b \to a \cdot b$

$$\rightsquigarrow \quad \Omega^1_{A/R} \simeq I/I^2 = \Delta^*(I).$$

~~Note~~:

/Prop $\Rightarrow$ lemma 8

$$I \otimes_{A \otimes_R A} (A[S^{-1}] \otimes_R A[S^{-1}]) \simeq \ker(A[S^{-1}] \otimes_R A[S^{-1}] \to A[S^{-1}]) =: J$$

$$\rightsquigarrow \quad I/I^2 \otimes_A A[S^{-1}] \simeq J/J^2$$

$$\begin{array}{ccc} I/I^2 \otimes_A A[S^{-1}] & \simeq & J/J^2 \\ \wr\vert & & \wr\vert \\ \Omega^1_{A/R} \otimes_A A[S^{-1}] & & \Omega^1_{A[S^{-1}]/R} \end{array}$$

~~Proof~~ of Prop:

$\delta: A \to I/I^2 \simeq \cancel{I \otimes_A A/I}$ $\quad \ker(\cdot\, A \otimes A \to A)/I$ $\quad I = \ker(A \otimes A \xrightarrow{\sim} A)$

$a \mapsto 1 \otimes a - a \otimes 1$ $\quad I \otimes_{A \otimes A} \frac{A \otimes A}{I}$

R-lin derivation $\quad 1 \otimes a - a \otimes 1$

$$\delta(ab) = 1 \otimes ab - ba \otimes 1$$
$$= 1 \otimes ab - a \otimes b + a \otimes b - ab \otimes 1.$$
$$= 1 \otimes b\,(1 \otimes a - a \otimes 1) + (a \otimes 1)(\delta(b))$$
$$= b\,\delta(a) + a \otimes \delta(b).$$

$\rightsquigarrow$ get

$$\Omega^1_{A/R} \to I/I^2$$

$$da \mapsto 1 \otimes a - a \otimes 1$$

Conversely, $d: A \to \Omega^1_{A/R}$ universal derivation

$\rightsquigarrow \varphi_1 : A \to A[\Omega^1_{A/R}]$ $\quad a \mapsto (a, 0)$

$\rightsquigarrow A \otimes_R A \longrightarrow A[\Omega^1_{A/R}]$

$a \otimes b \longmapsto a(b + db)$

$$\varphi_2 : A \to A[\Omega^1_{A/R}].$$
$$b \longmapsto (b, db)$$

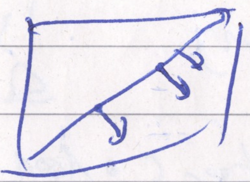
Check:
Both are inverse to each other

$$\rightsquigarrow \quad I/I^2 \longrightarrow \Omega^1_{A/R}$$

$$\sum a_i \otimes b_i \mapsto \sum a_i (b_i + d(b_i)) = \sum a_i \, d(b_i).$$

<u>Motivation</u>:

$(I/I^2)^\vee$ "=" normal bundle of diagonal"



$(\Omega^1_{A/R})^{\vee (2)}$ = "tangent bundle of X".

<u>Corollary</u>: $D(f) \longmapsto \Omega^1_{A[f^{-1}]/R} \simeq \Omega^1_{R/A} \otimes A[f^{-1}]$
$\cap$
$X = \operatorname{Spec} A$

defines a sheaf. (q.coh.) on X.
Moreover
$$\Omega^1_{X/S} \simeq \Delta^*(I_\Delta)$$
$$\Delta : X \longrightarrow X \times_S X \quad \text{diagonal}$$

<u>Def</u>: If $X \xrightarrow{f} S$ any morphism,
$$\Delta_f : X \xrightarrow[\Delta_f^u]{\text{closed}} U \overset{\text{open}}{\subseteq} X \times_S X$$
def $\Omega^1_{X/S} = (\Delta_f^u)^*(I_{\Delta_f^u})$, with $I_{\Delta_f^u}$ ideal sheaf def X in U.

Pf:
$$\begin{tikzcd} U = \operatorname{Spec} A \arrow[d] & \subseteq X \\ \operatorname{Spec} R \arrow[r] & S \end{tikzcd}$$

$$\rightsquigarrow \Omega_{X/S|U} \cong \widehat{\Omega^1_{A/R}}$$

Get (exercise)

Prop: $X \xrightarrow{f} Y \xrightarrow{g} S$ morph

(i) $\exists$ can $d: \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$ derivation, restricting to $d: A \rightarrow \Omega^1_{A/R}$

(ii) $\Omega^1_{X/S}$ commutes with b.c. in S.

(iii) $f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$

locally

$$d(a) \longmapsto d(f^\#(a))$$

(iv) if f is closed immersion with ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$

$$f^*(\mathcal{I}) \xrightarrow{d} f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

$$f^*(\mathcal{I}) \cong \mathcal{I}/\mathcal{I}^2$$

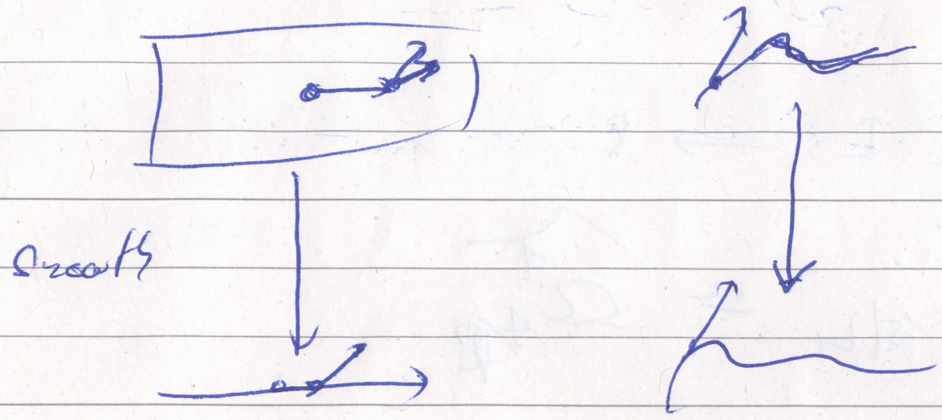
Recall:

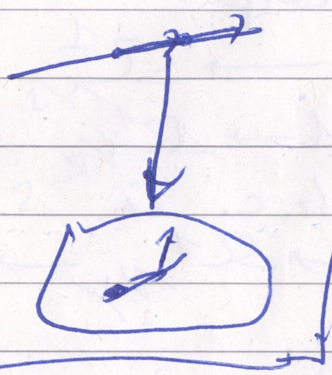
$f: X \longrightarrow S$ formally smooth / etale / unram.

if $\forall$ 1st order $T_0 \subseteq T$ of affine schemes

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed, "u"] & S \end{tikzcd}$$

there exists a / $\exists !$ / at most one u.





Rem.: formally smooth, étale, unram. is loc.

Cor.:
$f: X \longrightarrow S$ morph.
TFAE:
a) f formally unramified
b) $\Omega^1_{X/S} = 0$
If f is locally of finite type
$\Leftrightarrow$ c) $\Delta_f : X \longrightarrow X \times_S X$ open immersion

($\Rightarrow$ locally closed immersions are unramified)

Proof:
Claim local.
$X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$, $I = \ker(A \otimes_R A \to A)$
a) $\Leftrightarrow$ b) was proven
c) $\Rightarrow$ b)
Assume b): f loft. $\rightsquigarrow$ Δ_f loft.
I gen by $1 \otimes a_i - a_i \otimes 1$ $a_1, \dots, a_n$ gen of A as R algebra.)
$I/I^2 = 0 \rightsquigarrow I_x = 0 \quad \forall x \in \operatorname{Spec} A$)
$\rightsquigarrow \Delta$ flat closed immersion

⇝ Δ open immersion

Exercise

$\cancel{f^*\Omega_{Y/S} \to \Omega^1_{X/S} \to \Omega}$

Prop:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & S \end{tikzcd}$$

(i) If f is formally smooth, then

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact and locally split.

(ii). If gf formally smooth +

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

locally split + exact, then

f is formally smooth

Proof:

Claims are local. $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$

$S = \operatorname{Spec} A$

Show: $0 \to C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$

exact on the left + locally split

⇝ produce g $\quad c \otimes db \mapsto c \cdot d(f^{\#}(b))$

$$\begin{tikzcd} C \arrow[r, equal] & C \arrow[dl, dashed, "\exists u"'] \\ C[C \otimes_B \Omega^1_{B/A}] \arrow[u] & B \arrow[l] \arrow[u, "f^{\#}"'] \end{tikzcd}$$

$f^{\#}(b) + 1 \otimes db \longleftarrow b$

⇝ for formally smooth:

⇝ $\exists u : C \to C[C \otimes_B \Omega^1_{B/A}] = C \oplus C \otimes_B \Omega^1_{B/A}$

$u = \delta : C \longrightarrow C \otimes_B \Omega^1_{B/A}$

$\rightsquigarrow \delta' : \Omega^1_{C/A} \longrightarrow C \otimes \Omega^1_{B/A}$

$d(c) \longmapsto \delta(c)$

<u>check</u>: δ' is a retraction

$$\delta'(c \cdot d(f^\#(b))) = c\,\delta'(d(f^\#(b))) = c\, d(f^\#(b))$$

$\rightsquigarrow$ claim

(ii) let $\delta : \Omega^1_{C/A} \longrightarrow C \otimes_B \Omega^1_{B/A}$

retraction

$$\begin{tikzcd}
R/I & C \arrow[l] \arrow[dl, dashed, "u'"] \\
R \arrow[u] & B \arrow[l] \arrow[u, "f^\#"] \\
& A \arrow[u, "g^\#"] \arrow[ul]
\end{tikzcd}$$

$I^2 = 0$

~~$f^\#$~~ g of formally smooth

$\rightsquigarrow \exists\, u' : C \to R$ s.t.

$u' \circ f^\# \circ g^\# = v \circ g^\#$

s.th outer square commutes, want inner square

have to find $\delta \in \mathrm{Der}_A(C, I)$, s.t.

$(u' + \delta) \circ f^\# = v$

$\Leftrightarrow v - u' \circ f^\# = \delta \circ f^\# \in \mathrm{Der}_A(B, I)$

$\mathrm{Der}_A(C, I) \longrightarrow \mathrm{Der}_A(B, I) = \mathrm{Hom}_B(\Omega^1_{B/A}, I)$

$\mathrm{Der}_A(B,I) \cong \mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, I)$

$$\begin{tikzcd} \mathrm{Der}_A(C,I) \arrow[r, "\cong"] & \mathrm{Hom}_C(\Omega^1_{C/A}, I) \arrow[u, "\text{section}"] \end{tikzcd}$$

$\rightsquigarrow$ find δ

Cor: ("uniformizing parameters")

$g: X \to S$ morph

$\rightsquigarrow$ g smooth $\Leftrightarrow$ $\forall x \in X$ $\exists$ $x \in U \subseteq X$ open and sections $t_1, \dots, t_n \in \Gamma(U, \mathcal{O}_U)$ s.t.

$$\begin{tikzcd} U \arrow[r, "f = (t_1, \dots, t_n)"] \arrow[dr, "g|_U"'] & \mathbb{A}^n_S \arrow[d] \\ & S \end{tikzcd}$$

with f étale and $\Omega^1_{X/S}|_U \cong \bigoplus_{i=1}^n \mathcal{O}_U dt_i$

Proof:

"$\Leftarrow$" g smooth as both morph smooth,

by $0 \to f^* \Omega^1_{\mathbb{A}^n_S/S} \xrightarrow{\sim} \Omega_{U/S} \to \Omega^1_{U/\mathbb{A}^n_S} \to 0$

$dT_i \longmapsto dt_i$, $\Omega^1_{U/\mathbb{A}^n_S} = 0$

(if $\mathbb{A}^n_S = \mathrm{Spec}\, \mathcal{O}_S[T_1, \dots, T_n]$)

"$\Rightarrow$": g smooth $\rightsquigarrow$ $\Omega^1_{X/S}$ loc free of finite rank gen by dt_i, $t_i \in \Gamma(U, \mathcal{O}_U)$

$\rightsquigarrow$ $\exists$ nbhd U of x, $t_1, \dots, t_n \in \Gamma(U, \mathcal{O}_U)$ s.th. $\Omega^1_{X/S}|_U \bigoplus \mathcal{O}_U dt_i$

claim: $f = (t_1, \dots, t_n): U \to \mathbb{A}^n_S$ is étale

$$0 \to f^*\Omega^1_{A^n_S/S} \xrightarrow{\sim} \Omega^1_{U/S} \to \Omega^1_{U/A^n_S} \to 0$$

prev prop ⇒ smooth U

but also unram (as comp is smooth?)

⇒ étale

<u>Prop:</u>

$$\begin{tikzcd} & S \\ Z \arrow[r, hook, "i"] \arrow[ur, "f"] & X \arrow[u, "g"'] \end{tikzcd}$$

comm, i closed immersion

let $J \subseteq \mathcal{O}_X$

$$(*) \quad 0 \to J/J^2 \xrightarrow{d} i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$$

(i) If f formally smooth, then $(*)$ is exact on the left and locally split

(ii) If g is formally smooth ~~then~~ and $(*)$ is exact and locally split, then f is formally smooth.

(i)

$$\begin{tikzcd} B/J \arrow[d] & B/J \arrow[l] \arrow[dl] & B \arrow[l, "\pi"'] \\ B/J^2 & & A \arrow[ll] \arrow[ul] \arrow[u, "u'"'] \end{tikzcd}$$

$s - u'$

f formally smooth

⇝ derivation $\partial = \pi' - u \circ \pi,\ B \to J/J^2$

(ii)