Exercise 1. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms of rings.

i) Show that there exists an exact sequence

$$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

of C-modules.

ii) Assume that $B \to C$ is surjective with kernel $I := \ker(B \to C)$. Prove that there exists an exact sequence (of C-modules)

$$C \otimes_B I \cong I/I^2 \xrightarrow{\alpha} C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to 0$$

where $\alpha(g) := dg$ for $g \in I$.

Solution. i) Recall from commutative algebra that an exact sequence of C-modules $N' \to N \to N'' \to 0$ is exact if and only if for every C-module M the sequence $0 \to \mathrm{Hom}_C(N'', M) \to \mathrm{Hom}_C(N, M) \to \mathrm{Hom}_C(N', M)$ is exact. Thus it is sufficient to show that for all C-modules M the sequence

$$0 \to \mathrm{Hom}_C(\Omega^1_{C/B}, M) \to \mathrm{Hom}_C(\Omega^1_{C/A}, M) \to \mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, M) \qquad (1)$$

is exact. We have natual isomorphisms

$$\mathrm{Hom}_C(C \otimes_B \Omega^1_{B/A}, M) \cong \mathrm{Hom}_B(\Omega^1_{B/A}, \mathrm{Hom}_C(C, M)) \cong \mathrm{Hom}_B(\Omega^1_{B/A}, M).$$

This and the universal property of the module of differentials implies that the sequence (1) is isomorphic to

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$

where the first map is the natural inlcusion and the second map is given by precomposition with $g : B \to C$. If $D \in \mathrm{Der}_B(C, M)$ is a derivation, then $D(g(b)) = g(b)D1 = 0$. For the other inclusion let now $D \in \mathrm{Der}_A(C, M)$ be a derivation such that $D \circ g = 0$. But then for $b \in B, c \in C$ we have $D(b \cdot c) = D(g(b)c) = D(g(b))c + g(b)D(c) = g(b)D(c)$, so D is in fact B-linear.

ii) Let M be any C-module. By the Tensor-hom adjunction we have a natural isomorphism $\mathrm{Hom}_C(C \otimes_A I, M) \cong \mathrm{Hom}_A(I, M)$ and similarly as in i) we reduce the claim to showing that the sequence

$$0 \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \to \mathrm{Hom}_A(I, M)$$

is exact. The first map is injective because g is surjective. A derivation $D \in \mathrm{Der}_A(B, M)$ vanishes on I if and only if we can consider it as a derivation on $C = B/I$, so the sequence is also exact at $\mathrm{Der}_A(B, M)$ and the claim follows.

Exercise 2. Compute $\Omega^1_{B/A}$ for

i) $B = A[X]/(f(X))$ with $f(X) \in A[X]$ a polynomial.

ii) $B = \mathbb{Z}[i], A = \mathbb{Z}$.

iii) $B = k[x,y]/(y^2 - x^3 - x)$ with $A = k$ not of characteristic 2.

iv) $B = k[x,y]/(xy)$, $A = k$.

Solution. Consider the map of rings $A \to B = A[X_1, \dots X_n] \to C = \frac{A[X_1,\dots,X_n]}{(f_1,\dots,f_k)}$. Then by Exercise 1, ii) there is an exact sequence

$$\frac{(f_1,\dots,f_k)}{(f_1,\dots,f_k)^2} \xrightarrow{\alpha} C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to 0.$$

Hence we obtain

$$\Omega^1_{C/A} \cong \frac{C \otimes_B \bigoplus_{i=1}^n B\mathrm{d}X_i}{(\alpha(f_1),\dots,\alpha(f_k))} \cong \frac{\bigoplus_{i=1}^n C\mathrm{d}X_i}{(\sum_j \frac{\partial f_i}{\partial X_j}\mathrm{d}X_j)}$$

We now apply this to the exercise.

i) Here we get

$$\Omega^1_{B/A} \cong \frac{A[X]/(f(X))}{f'(X)} \cong A[X]/(f(X), f'(X))$$

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ii) In this case we have $\mathbb{Z}[i] = [X]/(X^2+1)$ and we find

$$\Omega^1_{B/A} \cong \mathbb{Z}[X]/(X^2+1, 2X) \cong \mathbb{Z}[i]/(2i) \cong \mathbb{Z}[i]/(2) \cong \mathbb{F}_2[i] \cong \mathbb{F}_2^2$$

iii) We compute that

$$\Omega^1_{B/A} \cong \frac{B\mathrm{d}x + B\mathrm{d}y}{((3x^2+1)\mathrm{d}x - y\mathrm{d}y)}$$

iv) We get

$$\Omega^1_{B/A} \cong \frac{B\mathrm{d}x + B\mathrm{d}y}{(x\mathrm{d}y + y\mathrm{d}x)}$$

Exercise 3. Let A be a perfect $\mathbb{F}_p$-algebra, i.e., the Frobenius $\mathrm{Fr}_A : A \to A, x \mapsto x^p$ of A is bijective . Prove that $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{F}_p)$ is formally etale.

Solution. We have to show that for every commutative diagram

$$\begin{tikzcd} R/I & A \arrow[l, "f"'] \\ R \arrow[u] & \mathbb{F}_p \arrow[l] \arrow[u] \end{tikzcd}$$

such that $R \to R/I$ is a square zero extension, there exists a unique map $u : R \to A$ making the diagram commute.
Assume that for $x, y \in R$ we have $x = y \mod I$. Then $x - y \in I$ and hence $x^p - y^p = (x-y)^p = 0$ because $I^2 = 0$. Consider the map $s : R/I \to R, \overline{x} \mapsto$

x^p. This is well defined by the previous observation and since everything is of characteristic p, s is in fact a ringhomomoprhism. Now we define a map $u = s \circ f \circ \mathrm{Fr}_A^{-1}$. The upper left triangle commutes by construction and the lower right triangle commutes for ever ringhomomorphism $R \to A$. Thus $\mathbb{F}_p \to A$ is formally smooth.
Since A is perfect, for any $a \in A$ there is a p'th root $b \in A$ and therefore $\mathrm{d}a = \mathrm{d}b^p = pb^{p-1} \cdot \mathrm{d}b = 0$. So $\Omega^1_{A/\mathbb{F}_p} = 0$, which shows that A is formally unramified over $\mathbb{F}_p$.

Exercise 4. For a ring A we set $A[\epsilon] := A[t]/t^2$. Let $X \to S$ be a morphism of schemes. We define a functor, the "tangent bundle" $\mathcal{T}_{X/S}$ of X/S, by sending an affine scheme $\mathrm{Spec}(A)$ over S to the set $X(A[\epsilon])$. Prove that $\mathcal{T}_{X/S}$ is representable by the relative spectrum $\underline{\mathrm{Spec}}_{\mathcal{O}_X}(\mathrm{Sym}^\bullet(\Omega^1_{X/S}))$.

Solution. I omit the reduction to the affine case.

Assume that $X = \mathrm{Spec}(R), S = \mathrm{Spec}(B)$ are affine.
Then the scheme $\underline{\mathrm{Spec}}_{\mathcal{O}_X}(\mathrm{Sym}^\bullet(\Omega^1_{X/S}))$ represents the functor

$$(f : \mathrm{Spec}(A) \to S) \mapsto \mathrm{Hom}_{\mathcal{O}_S-\mathrm{alg}}(\mathrm{Sym}^\bullet(\Omega^1_{X/S}), f_*(\mathcal{O}_{\mathrm{Spec}(A)})) \tag{1}$$

Let M be an R'-module. Recall that $\mathrm{Sym}^\bullet(M)$ is the smallest R'-algebra such that there is a map of R'-modules $M \to \mathrm{Sym}^\bullet(M)$, or in other words, the symmetric algebra of M satisfies the universal property $\mathrm{Hom}_{R'-\mathrm{alg}}(\mathrm{Sym}^\bullet(M), A) = \mathrm{Hom}_{R'}(M, A)$ for all R'-algebras A. Using this we can compute the Hom-set in (1) as follows:

$$\mathcal{T}_{X/S}(A) = \mathrm{Hom}_S(\mathrm{Spec}(A[\epsilon]), X) \cong \mathrm{Hom}_B(R, A[\epsilon]) \tag{2}$$

Let $\varphi \in \mathrm{Hom}_B(R, A[\epsilon])$. Then we can write $\varphi = \varphi_1 + \epsilon\varphi_2$ and one easily verifies that φ is a B-algebra homomorphism if and only if φ_1 is a B-algebra homomorphism and if we consider A as B-algebra via φ_1, then φ_2 is B-linear derivation. Hence

$$(1) = \{(\varphi_1, \varphi_2) : \varphi_1 : R \to A \text{ algebra homomorphism }, \varphi_2 \in \mathrm{Hom}_{R,\varphi_1}(\Omega^1_{R/B}, A)\}$$