Exercise 1. Let A be a Dedekind domain and set $S = Spec(A)$. Let X be an integral scheme. Prove that a morphism $f : X \to S$ is flat if and only if it maps the generic point of X to the generic point of S.

Solution. Since flatness is local on the source we may assume that $X = SpecR$ for some integral domain R. Let f be induced by the ring homomorphism $\phi : A \to R$. Then ϕ is flat if and only if R is torsionfree as an A-module. But this happens if and only if ϕ is inective, i.e. f maps the generic point of X to the generic point of S.

Exercise 2. i) Let $\phi : A \to B$ be a flat, local morphism of local rings. Prove that the morphism

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is surjective

ii) Let $f : X \to S$ be a flat morphism of schemes. Prove that f is universally generalising.

Solution. i)
Let $\mathfrak{p} \subset A$ be a prime ideal of A. Since B is a flat A-module, tensoring the injection $A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p})$ with B yields an injection

$$B/\mathfrak{p}B \cong A/\mathfrak{p} \otimes_A B \hookrightarrow \kappa(\mathfrak{p}) \otimes_A B.$$

ϕ is a local homomorphism of rings, hence it maps $\mathfrak{p}$ into the maximal ideal of B. Thus $B/\mathfrak{p}B$ is not the zero ring and therefore the fibre of $\mathfrak{p}$ is not zero either, i.e. the map on Spectra is surjective.
ii) Observe that it is enough to show that f is generalising, because the property of being flat is stable under base change. Let $x \in X$ and $y = f(x)$ and choose affine neighborhoods $x \in \mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ of X, y such that $f^{-1}(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Note that the open subschemes $\mathrm{Spec}(A_{\mathfrak{p}_x}) \subset \mathrm{Spec}(A)$ and $\mathrm{Spec}(B_{\mathfrak{p}_y}) \subset \mathrm{Spec}(B)$ are exactly the points which specialize to x and y respectively. Now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(A_{\mathfrak{p}_x}) \arrow[r] \arrow[d] & \mathrm{Spec}(B_{\mathfrak{p}_y}) \arrow[d] \\
\mathrm{Spec}(A) \arrow[r, "f"] & \mathrm{Spec}(B)
\end{tikzcd}$$

Then by i) the upper horizontal arrow is surjective and hence for every point y' that specializes to y there is a point x' that specializes to x and such that $f(x') = y$.

Exercise 3. Let $f : X \to S$ be a closed immersion. Then f is flat and locally of finite presentation if and only if f is an open immersion.

Solution. "$\Rightarrow$": We show that f is open and induces isomorphisms on all stalks. For every $x \in X$ we have morphism $\phi : \mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is a flat and finitely presented module over a local ring, it is finite free, say $\mathcal{O}_{X,x} \cong \mathcal{O}^k_{S,f(x)}$. Also f is a closed immersion, so ϕ is surjective. Hence ϕ is an isomorphism.
Now let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of $f(x)$. Then f restricts to a map $\mathrm{Spec}(A/\mathfrak{a}) = f^{-1}(U) \to U$ for some ideal $\mathfrak{a} \subset A$. If we tensor the SES

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

with the flat A-module $A/\mathfrak{a}$ we obtain the SES

$$0 \to \mathfrak{a}/\mathfrak{a}^2 \to A/\mathfrak{a} \to A/\mathfrak{a} \otimes_A A/\mathfrak{a} \to 0$$

Note that since

$$A/\mathfrak{a} \otimes_A A/\mathfrak{a} \cong \frac{A/\mathfrak{a}}{\mathfrak{a} \cdot A/\mathfrak{a}} \cong A/\mathfrak{a},$$

the last map is an isomorphism, thus $\mathfrak{a} = \mathfrak{a}^2$. Since $A/\mathfrak{a}$ is finitely presented, $\mathfrak{a}$ is finitely generated. Hence Nakayamas Lemma implies that $\mathfrak{a} = (e)$ for some idempotent element $e \in A$. Now by the chinese remainder theorem $A = A/(1-e)e \cong A/e \oplus A/(1-e)$ and therefore $V(\mathfrak{a}) = V(e) = D(1-e)$ is open. This shows that f is open.

"$\Leftarrow$": This is true, because an isomorphism of rings is flat and of finite presentation.