Flatness

24.4.16 (I)

Recall 1) Let A be a ring, M an A-Module. Then M is flat if $-\otimes_A M$ is exact.

2) If $\varphi: A \longrightarrow B$ ring map, then B is flat if B flat as A-module.

3) φ is faithfully flat if ~~$S \subseteq A$~~ $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. + φ flat.

Example i) $A \longrightarrow S^{-1}A$ is exact

ii) filtered colimits of flat modules are flat. (see Exercise 1 of 1. sheet.)

This implies i) since $S^{-1}A = \lim\limits_{s \in S} A_s$

Theorem An A-module is flat iff it is a filtered colimit of finite free modules.

iii) Completions: If A is noetherian ring and $I \subseteq A$ is some ideal, then the I-adic completion $\hat{A} = \varprojlim A/I^n$ is flat

Proof Use: for finitely gen. A-module M,

$$\hat{M} = \varprojlim M/I^n M \longleftarrow M \otimes_A \hat{A}$$

and $M \longmapsto \hat{M}$ is exact for fin. gen. A-modules.

This is enough: In general, M flat A-module iff for all $I \subseteq A$ the map $I \otimes_A M \longrightarrow M$ is injective.

Use this for $M = \hat{A}$:

enough to check $\hat{I} \hookrightarrow \hat{A}$

which is part of exactness of completion of f.g. A-modules.

Lemma Let $\varphi: A \to B$, M = A-module, N = B-module

1) If M flat A-module, then $N \otimes_A B$ is flat B-module
2) If φ is flat B-module, then N is also flat as an A-module

Proof 1)

$$\begin{tikzcd}
\{B\text{-modules}\} \arrow[r, "-\otimes_B (B\otimes_A M)"] \arrow[d] & \{B\text{-modules}\} \arrow[d] \\
\{A\text{-modules}\} \arrow[r, "-\otimes_A M", "\text{exact}"'] & \{A\text{-modules}\}
\end{tikzcd}$$

$\Rightarrow -\otimes_B (B \otimes_A M)$ is exact.

2)

$$\begin{tikzcd}
\{A\text{-module}\} \arrow[r, "-\otimes_A B", "\text{exact}"'] \arrow[drr, "{-\otimes_A N = -\otimes_A B \otimes_B N}"'] & \{B\text{-modules}\} \arrow[r, "-\otimes_B N", "\text{exact}"'] & \{B\text{-modules}\} \arrow[d, "\text{exact}"] \\
& & \{\cancel{B}A\text{-modules}\}
\end{tikzcd}$$

Proposition (Descent of flatness). If $\varphi: A \to B$ is faithfully flat, M an A-module, then
M flat $\Leftrightarrow$ $M \otimes_A B$ is flat

~~Proof~~

Example If $X = \operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i)$, $A \to B = \prod A_{f_i}$ is faithfully flat, because

- finite direct sum of flat modules
- $\operatorname{Spec} B = \coprod \operatorname{Spec} A_{f_i} \to \operatorname{Spec} A$ surj.

In this case M flat $\Leftrightarrow$ $M[f_i]$ flat $A[f_i]$ module.

<u>Proof of Prop.</u> use <u>Lemma</u> let

$\varphi : A \to B$ be flat and

$C^\bullet : \quad \to C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} C^1$

complex of A-modules (i.e. $dd = 0$)

Then $C^\bullet$ exact $\Rightarrow$ $C^\bullet \otimes_A B$ is exact in fact

$$H^i(C^\bullet \otimes_A B) = H^i(C^\bullet) \otimes_A B$$

Conversely, if φ faithfully flat, then

$C^\bullet \otimes_A B$ is exact $\Rightarrow$ $C^\bullet$ is exact.

<u>Recall</u> If $C^\bullet$ complex of A-modules,

$$\underset{\text{Boundaries}}{B^i} \subseteq \underset{\text{cycles}}{Z^i} \subseteq C^i$$

cycles $= \{x \in C^i \mid d(x) = 0\}$

Boundaries $= \{x \in C^i \mid \exists y \in C^{i-1} : x = d(y)\}$

<u>Def</u>: $C^\bullet$ is exact if all $H^i(C^\bullet) = 0$, i.e. $B^i = Z^i$ $\forall i$

<u>Proof of Lemma</u> First part is formal consequence of $- \otimes_A B$ is exact:

$$\begin{tikzcd}
C^{i-1} \arrow[rr, "d"] \arrow[dr, two heads] & & C^i \arrow[rr, "d"] \arrow[dr, two heads] & & C^{i+1} \\
& B^i \subseteq Z^i \arrow[ur, hook] & & B^{i+1} \subseteq Z^{i+1} \arrow[ur, hook] &
\end{tikzcd}$$

$0 \to Z^i \to C^i \to B^{i+1} \to 0$ is exact for all i,

$0 \to B^i \to Z^i \to H^i \to 0$ is exact for all i,

Apply $- \otimes_A B$: preserves all these short exact sequence

For converse, note that $H^i(C^\bullet \otimes_A B) = H^i(C^\bullet) \otimes_A B$, so if we set $M = H^i(C^\bullet)$, then suffices

$$M \otimes_A B = 0 \Rightarrow M = 0.$$

Assume not, let $0 \neq x \in M$, $I = \mathrm{Ann}(x) \subsetneq A$ ideal

$$A/I \hookrightarrow M \xrightarrow{B \text{ flat}} B/IB \hookrightarrow M \otimes_A B$$

But
$$\begin{tikzcd} \mathrm{Spec}(B/IB) \arrow[r, two heads] \arrow[d, phantom, "\subseteq" sloped] \arrow[dr, phantom, "\square"] & \mathrm{Spec}(A/I) \arrow[d, phantom, "\subseteq" sloped] \\ \mathrm{Spec}\, B \arrow[r] & \mathrm{Spec}\, A \end{tikzcd}$$

so $\mathrm{Spec}\, B/IB \twoheadrightarrow \mathrm{Spec}(A/I) \neq 0$

So $B/IB \neq 0 \hookrightarrow M \otimes_A B$ ☒

Proof of Prop'n

Let $0 \to N \to N' \to N'' \to 0$ exact A-modules

to show $0 \to N \otimes M \to N' \otimes M \to N'' \otimes M \to 0$ is exact.

φ faithfully flat: suffices to check after $- \otimes_A B$, i.e that
$$0 \to N \otimes_A (M \otimes_A B) \to N' \otimes_A (M \otimes_A B) \to N'' \otimes_A (M \otimes_A B) \to 0$$
is exact. But $M \otimes_A B$ is flat as B-module

$\overset{\varphi \text{ flat}}{\Rightarrow}$ flat as A-module, so get result. ☒

<u>Def'n / Prop'n</u> Let X be a scheme and $\mathcal{M}$ q.c. sheaf. Then $\mathcal{M}$ is flat if one of the following equiv. cond. holds:

i) For all open $U = \mathrm{Spec}\, A \subseteq X$, $\mathcal{M}(U)$ is flat $A = \mathcal{O}_X(U)$-module

ii) For a cover $\bigcup U_i = \mathrm{Spec}(A_i)$ of X, "

<u>Proof</u> i) $\Rightarrow$ ii) ✓

ii) $\Rightarrow$ i) If $\mathcal{M}(U)$ flat $\mathcal{O}_X(U)$-module, $U = \mathrm{Spec}\, A$ affine and $f \in A$, then for $V = D(f) \subseteq U$ have $\mathcal{M}(V) = \mathcal{M}(U) \otimes_A A[f^{-1}]$ by quasi coherence so $\mathcal{M}(V)$ flat $A[f^{-1}] = \mathcal{O}(V)$ module

Such V form basis for topology; in fact if $U' \subseteq X$ any affine open can find $f_1', \ldots, f_n' \in A'$ s.th. $D_{\mathrm{Spec} A'}(f_i')$ are one of the above V's. Thus $\mathcal{M}(D_{U'}(f_i'))$ flat. By Descent of Flatness along $A' \to A'[f_i'^{-1}] \Rightarrow \mathcal{M}(U')$ flat A'-module. ☐

<u>Def'n / Prop'n</u> Let $f: Y \to X$ be morphism of schemes. Then f is flat if the following equiv. conditions hold.

i) For all open affine

$$\begin{array}{ccc} V = \operatorname{Spec} B & \subseteq & Y \\ \downarrow & & \downarrow \\ U = \operatorname{Spec} A & \subseteq & X \end{array}$$

the map $A \to B$ is flat.

ii) There is a cover Y by open affine $\operatorname{Spec} B \subseteq Y$ mapping into some $\operatorname{Spec} A \subseteq X$, s.th. $A \to B$ is flat.

~~B~~ <u>Proof</u> i) $\Rightarrow$ ii) ✓

ii) $\Rightarrow$ i): Again two steps: i) Shrink V, U

ii) If Y, X affine, true for open cover then true globally (+ composites of flat ring maps are flat)

For i) use that localizations are flat (+ comp. of flat ring maps are flat)

For ii) $\operatorname{Spec} B \to \operatorname{Spec} A$ affine and assume ~~prope~~ ii) holds. $\rightsquigarrow$ $Y = \bigcup_{i=1}^{n} D(g_i)$ s.th. $A \to B[g_i^{-1}]$ is flat. (a priori only get flat $A[f_i^{-1}] \to B[g_i^{-1}]$ but $A \to A[f_i^{-1}]$ is flat)

If $0 \to M \to M' \to M''$ exact

want ~~an~~ $0 \to M \otimes_A B \to M' \otimes_A B \to M'' \otimes_A B \to 0$ exact.

But $B \to \prod B[g_i^{-1}]$ is faithfully flat, so can check after $- \otimes \prod_{i=1}^{n} B[g_i^{-1}]$, then follows from $B[g_i^{-1}] / A$ flatness.

In EGA, $f : Y \to X$ + q.coh. sheaf $\mathcal{N}$ on Y, then defines "flatness" of $\mathcal{N}$ over $\mathcal{O}_X$"

- If $Y = X$, recovers flatness of q.coh. sheaves
- If $\mathcal{N} = \mathcal{O}_Y$ — " — of f.

<u>Prop'n</u> $P = \{$flat morphism of schemes$\}$ then P satisfies (BC), (PROD), (LOCS), (LOCT), (COMP).

Faithfully flat Descent

<u>Last semester</u> had following statement:

If $X = \operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i)$, then

$$\{A \text{ modules } M\} \cong \left\{ \begin{array}{l} (M_i \ A[f_i^{-1}]\text{-modules} \ + \\ \alpha_{ij} : M_i[f_j^{-1}] \simeq M_j[f_i^{-1}] \\ \text{s.th. } \alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \end{array} \right\}$$

This is the special case for $A \to \prod A[f_i^{-1}]$ of a more general statement true for all faithfully flat maps

<u>Thm</u> Let $\varphi : A \to B$ be faithfully flat. Then

$$\{A\text{-modules } M\} \xrightarrow{\cong} \{B\text{-modules } N \ +$$

$\alpha : N \otimes_A B \cong B \otimes_A N$ isom of $B \otimes_A B$-modules s.th.

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[r, "\alpha_{01}", "\simeq"'] & B \otimes_A N \otimes_A B \arrow[r, "\alpha_{12}", "\cong"'] & B \otimes_A B \otimes_A N \arrow[ll, bend left, "\alpha_{02}", "\sim"'] \end{tikzcd}$$

commutes $\}$

$$M \longmapsto (N = B \otimes_A M, \ \alpha_{can} : N \otimes_A B = B \otimes_A M \otimes B \simeq B \otimes_A N)$$

The functor $M \mapsto (M \otimes_A B, \alpha_{can})$ has right adjoint

$$(N, \alpha) \longmapsto eq\Big(N \underset{n \mapsto \alpha(n \otimes 1)}{\overset{n \mapsto 1 \otimes n}{\rightrightarrows}} B \otimes_A N \Big) = \{ n \in N : \underset{= 1 \otimes n}{\alpha(n \otimes 1)} \}$$

Indeed given any M and (N,α)

$$\mathrm{Hom}((M\otimes_A B, \alpha_{can}), (N,\alpha)) = \left\{ f \in \underbrace{\mathrm{Hom}_B(M\otimes_A B, N)}_{=\mathrm{Hom}_A(M,N)} \;\middle|\; \begin{array}{ccc} M\otimes_A B\otimes_A B & \xrightarrow{f\otimes 1} & N\otimes_A B \\ \Vert\,\alpha_{can} & \circlearrowleft & \Vert\,\alpha \\ B\otimes_A M\otimes_A B & \xrightarrow{1\otimes f} & B\otimes_A N \end{array} \right\}$$

$$= \left\{ f_0 \in \mathrm{Hom}_A(M,N) \;\middle|\; \begin{array}{ccccc} & & n & \mapsto & n\otimes 1 \\ M & \xrightarrow{f_0} & N & \longrightarrow & N\otimes_A B \\ \Vert & & & & \downarrow\alpha \\ M & \xrightarrow{f_0} & N & \longrightarrow & B\otimes_A N \\ & & n & \mapsto & 1\otimes n \end{array} \text{ commutes} \right\}$$

$$= \mathrm{Hom}_A(M, \mathrm{eq}(N \rightrightarrows B\otimes_A N))$$

Then to prove theorem, need to prove that unit + ____ of adjunction are equivalences, i.e. 1) For all A-modules M, $M = \mathrm{eq}(M\otimes_A B \rightrightarrows M\otimes_A B\otimes_A B)$

2) For all (N,α) ~~satisfying~~ satisfying cocycle condition, if $M = \mathrm{eq}(N \rightrightarrows B\otimes_A N)$, then $M\otimes_A B \cong N$.