AlgGeo:

Claim: $f: \operatorname{Spec} A \longrightarrow \operatorname{Spec} R$ formally smooth
$\Rightarrow \Omega^1_{A/R}$ projective
($\rightsquigarrow$ smooth $\Rightarrow$ locally free.)

Proof:
Take $M \twoheadrightarrow M'$ surjection of A-Modules
Have to show:

$$\begin{array}{ccc} \operatorname{Hom}(\Omega^1_{A/R}, M) & \longrightarrow & \operatorname{Hom}(\Omega^1_{A/R}, M') \\ \wr\wr & & \wr\wr \\ \operatorname{Der}_R(A, M) & & \operatorname{Der}_R(A, M') \end{array}$$

is surjective.

$\rightsquigarrow \varphi' : A \longrightarrow A[M'] = A \oplus \varepsilon M \quad (\varepsilon^2 = 0)$
$a \longmapsto a + \varepsilon\delta(a)$

Have

$$\begin{tikzcd} A[M'] & & A \arrow[ll, hook', "\varphi"'] \arrow[dll, dashed] \\ A[M] \arrow[u, two heads] & & R \arrow[ll, hook'] \arrow[u] \end{tikzcd}$$

Sqare-zero extension.

$\rightsquigarrow$ ex lift $\varphi = \mathrm{Id}_A + \varepsilon\delta, \ \delta \in \operatorname{Der}_R(A, M)$

Last time:

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & S \end{tikzcd}$$

i closed im, def by $J \subseteq \mathcal{O}_X$

$*$ $$\begin{array}{ccccccc} J/J^2 & \xrightarrow{d} & i^*\Omega^1_{X/S} & \longrightarrow & \Omega^1_{Z/S} & \longrightarrow & 0 \\ \| & & & & & & \\ i^*J & & & & & & \\ a & \longmapsto & da & & & & \end{array}$$

then
(i) f formally smooth $\Rightarrow$ (*) is exact + locally split
(ii) g formally smooth, * exact + locally split $\Rightarrow$ f form smooth.

Ex:
$S = \operatorname{Spec} R$, $X = \mathbb{A}^n_S = \operatorname{Spec}(\overbrace{R[x_1, \dots, x_n]}^{B})$
$Z = \operatorname{Spec} A = V(I)$ $\quad I = (f_1, \dots, f_r)$
then

$$\begin{tikzcd} \bar{f}_i & I/I^2 \arrow[r, "d"] & A \otimes_B \Omega^1_{B/R} \arrow[r] & \Omega^1_{A/R} \arrow[r] & 0 \quad (**) \\ e_i \arrow[u, mapsto] & \bigoplus_{i=1}^r A e_i \arrow[u] \arrow[r, "d^{ind}", "\mathrm{Mat}_{n\times r}(A)"'] & \bigoplus_{i=1}^n A dx_i \arrow[u, "\wr"] & & \end{tikzcd}$$

$$\rightsquigarrow \quad df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j$$

$$\rightsquigarrow \quad d^{ind} = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \in \mathrm{Mat}_{n\times r}(A)$$

Pick $z \in Z$ with corr prime $\mathfrak{p} \subseteq A$
Assume $\bar{f}_1, \dots, \bar{f}_r \in I/I^2$ form a basis in $I/I^2 \otimes k(z)$

Prop:
Z is smooth in a nbhd of z
iff (**) exact + locally split after $- \otimes_A A_{\mathfrak{p}}$
iff (**) exact after $- \otimes_A k(z)$
iff $J := \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}(z)$ has rank r.

We need a small lemma.

<u>Lemma:</u>
Let (A, m, k) local Ring, $M: A^r \to A^n$ morph. TFAE:

(i) M inj an split

(ii) $M \otimes_A id_k$ injective

<u>Prf:</u>
i) ⇒ ii) ✓

i) ⇒ ii) $M \otimes id_k$ inj ⇒ $r \leq n$

wlog $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ s.th. $\det M_1 \neq 0$ in k

$\underset{A \text{ local}}{\leadsto}$ M is invertable because $\det M_1 \in A^\times$

$$\leadsto \begin{tikzcd} A^r \arrow[r, "M"] & A^n \\ & A^r \arrow[ul, "M_1^{-1}"] \arrow[u, "\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & * & \end{pmatrix}"'] \end{tikzcd}$$ Split + inj

<u>Thm:</u> (Jacobian criterion)

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{A}^n_R \arrow[d, "g"] \\ & \operatorname{Spec} R \end{tikzcd}$$

i closed imm
lofp, $z \in Z$.

Then f is smooth at z
⇔ locally $Z = V(f_1, \ldots, f_r)$, s.th.
$\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}\right)(z) = r$.

<u>Last time:</u>

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & S \end{tikzcd}$$

(*) $f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$

(i) formally smooth ⇒ (*) exact on the left + loc split

(ii) gf formally smooth + (*) exact on left + loc split ⇒ f formally smooth

Cor: $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$ (i) f formally ~~smooth~~ étale

$$f^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S}$$

(ii) if $X \to S$ formally smooth + $f^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S}$ then f is formally étale.

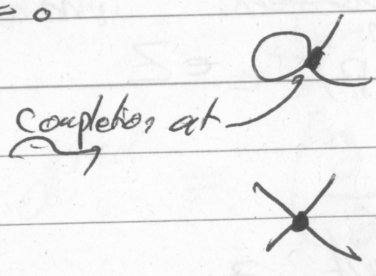
(i) $\Omega^1_{X/Y} = 0$ as unramified + smooth $\Rightarrow$ exact seq.

(ii) ?

Lemma: $X \xrightarrow{f} S$ formally étale, $x \in X$, $s = f(x)$, assume $k := k(s) \cong k(x)$

then $\hat{\mathcal{O}}_{S,s} := \varprojlim_n \mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} = \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n$.

Picture:



algebraically: completion of $\frac{k[x,y]}{y^2 - x^3 - x^2}$ at (x,y) $\overset{?}{\cong}$ $k[[u,v]]/u \cdot v$.

Prf: Consider $\mathcal{C}_{Sn} := \left\{ \begin{array}{l} (A, m_A, k_A) \text{ local rings, s.t. } \\ m_A^n = 0 + \text{iso } i : k \cong k_A \\ + \text{ local morphs resp } i \end{array} \right\}$

$\rightsquigarrow \mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n,\ \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n \in \mathcal{C}_{Sn}$.

$\rightsquigarrow \quad F_n := \mathrm{Hom}_{\mathcal{S}_{\le n}}(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n, -)$

$\quad\quad G_n := \mathrm{Hom}_{\mathcal{S}_{\le n}}(\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n, -)$

Have: $\eta : F_n \longrightarrow G_n$
$\quad\quad\quad \varphi \longmapsto \varphi \circ f^{\#}$

suff η bij:

Induction on n:

$\underline{n=1}$: ✓ $(k(s) \cong k(x))$

Let $A \in \mathcal{S}_{\le n} \rightsquigarrow \bar{A} = A/\mathfrak{m}_A^{n-1} \in \mathcal{S}_{\le (n-1)}$

$\overset{\text{ind}}{\rightsquigarrow} \quad \eta : F_n(\bar{A}) \xrightarrow{\sim} G_n(\bar{A})$

But $\quad F_n(A) \cong F_n(\bar{A}) \times_{G_n(\bar{A})} G_n(A)$

by formal étaleness

$$\left(\begin{tikzcd} \bar{A} & \mathcal{O}_{X,x} \arrow[l] \\ A \arrow[u] & \mathcal{O}_{S,s} \arrow[l] \arrow[u] \end{tikzcd}\right) \rightsquigarrow F_n(A) \xrightarrow{\sim} G_n(A).$$

Now study smooth schemes over fields

<u>Prop</u> (exercise):

k field, $X \xrightarrow{f} \mathrm{Spec}(k)$, loft

~~Ex~~ <u>TFAE</u>:

(i) f étale

(ii) f unramified

(iii) f smooth + locally quasi-finite

($g : Y \longrightarrow X$ locally quasi-finite if $\forall x \in X$: $Y_x := Y \times_X \mathrm{Spec}\, k(x)$ are discrete topological spaces + g loft.)

Thm: k field, $X \xrightarrow{f} \mathrm{Spec}(k)$ loft.
TFAE:
(a) f is smooth
(b) f is geom regular
(c) $\exists$ K/k algebraically closed X_K regular.

Need facts:
A local Noeth, $m \subset A$ max.
i) A is regular $\Leftrightarrow$ $\hat{A} = \varprojlim_n A/m^n$ is regular
ii) $A \to B$ local flat + B regular $\Rightarrow$ A regular
($\leadsto$ "$\Leftarrow$" in i) follows from ii as $A \to \hat{A}$ is flat)
(for A Noeth
iii) $p \subset A$ prime $\leadsto$ A_p regular if A regular.
Sketch:
i) lengths are the same
ii) Serres criterion for regularity.
iii) ? (also follows?).
Pf of thm:
a) $\Rightarrow$ b): Let $\bar{k}$ be an algebraic closure of k.
wlog: As smooth is (BC) can assume
$k = \bar{k}$
locally $\leadsto$, $X \xrightarrow{g} \mathbb{A}^n_k$ étale
Pick $x \in X$, wlog $x \in X$ closed (Fact (iii))
$\leadsto$ $k \cong k(x) \cong k(g(x))$
$\Rightarrow$ $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{\mathbb{A}^n_k, g(x)} \cong k[[t_1, \dots, t_n]]$ regular
Fact (i) $\leadsto$ $\mathcal{O}_{X,x}$ regular
~~Assume~~ b) $\Rightarrow$ c) ✓
c) $\Rightarrow$ a):

~~Claim~~ Claim: f smooth $\Leftrightarrow f_K : X_K \to \mathrm{Spec}(K)$ smooth

Prf: $X = \mathrm{Spec}\, A$, $A_K = A \otimes_k K$

$\rightsquigarrow \Omega^1_{A/k} \otimes_k K = \Omega^1_{AK/K}$

In part. $\Omega^1_{A/k}$ locally free of finite rk iff $\Omega^1_{AK/K}$ is loc free of finite rk.

Lemma:
$B \to C$ faithfully flat, M B-Module, then

i) M f.t. $\Leftrightarrow M \otimes_B C$ f.t.

ii) M flat$/B$ $\Leftrightarrow M \otimes_B C$ flat$/C$.

~~Pf of Lemma~~:

i) "$\Leftarrow$" Write $M = \varinjlim_{\substack{N \subseteq M \\ f.g}} N$

$\Rightarrow M \otimes C = \varinjlim_{\substack{N \subseteq M \\ f.g}} (N \otimes C) = N' \otimes_B C$ for some $N \subseteq M$ f.g.

(f.g)

f faithfully flat.

$\rightsquigarrow M = N'$

(ii) "$\Leftarrow$" $M \otimes_B -$ exact $\Leftrightarrow C \otimes_B M \otimes_B -$ exact

$\Leftrightarrow (C \otimes_B M) \otimes_C (C \otimes_B -)$ exact. $\square$

Let $g : X \hookrightarrow \mathbb{A}^n_k$ closed im with ideal I, f smooth $\Leftrightarrow$

$0 \to I/I^2 \xrightarrow{d} g^* \Omega^1_{k[x_1, \ldots, x_n]/k} \to \Omega^1_{A/k} \to 0$

exact + locally split

$\Rightarrow \Omega^1_{A/k}$ loc free iff f_k smooth . d inj $\Leftrightarrow$ $I_k/I_k^2 \xrightarrow{d_k} g_k^* \Omega^1_{k[x_1,\dots,x_n]/K}$ inj.

$\leadsto$ claim.

wlog: $k = K$

Thm: k field, X/k loft, $x \in X$ closed, st $k(x)/k$ sep. and $\mathcal{O}_{X,x}$ regular

$\Rightarrow$ X is smooth in a nbh of x.

Pf: $k(x)/k$ sep $\leadsto$ $\mathrm{Spec}\, k(x) \longrightarrow \mathrm{Spec}(k)$ is étale

$$\begin{tikzcd}
\mathrm{Spec}\, k(x) \arrow[r] \arrow[dr] & X \arrow[d] \\
& \mathrm{Spec}(k)
\end{tikzcd}$$

$\leadsto$ $0 \longrightarrow m_{X,x}/m_{X,x}^2 \xrightarrow{\sim} k(x) \otimes (\Omega^1_{X/k})_x \longrightarrow \underset{\substack{\| \\ 0}}{\Omega^1_{k(x)/k}} \to 0$

pick k: $g: X \hookrightarrow \mathbb{A}^n_k$ with ideal I

get

$$I/I^2 \longrightarrow g^* \Omega^1_{\mathbb{A}^n_k/k} \longrightarrow \Omega^1_{X/k} \longrightarrow 0$$

let $d = \dim m_{X,x}/m_{X,x}^2 = \dim \mathcal{O}_{X,x}$

get $f_1, \dots, f_d \in I$, sth $df_i \otimes k(x)$ lin ind.

then $X \subseteq X_0$, X_0 smooth at dim d (Jacobi)

Have $\mathcal{O}_{X_0,x} \twoheadrightarrow \mathcal{O}_{X,x}$

regular $\Rightarrow$ integral $\qquad$ at dim d.

(extension of integral d dim schemes?)
$\Rightarrow \mathcal{O}_{X_0,x} \cong \mathcal{O}_{X,x}$ iso

$\rightsquigarrow X = X_0$ near x.

Thm:
$X \xrightarrow{f} S$ lofp.
TFAE:
(i) f smooth
(ii) f flat + smooth fibres
(iii) f flat + smooth geometric fibres.

Proof:
(ii) $\Leftrightarrow$ (iii) decent statement of smoothness

(i) $\Rightarrow$ (ii) Claim is local, $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$
$A = B/I$, $B = R[x_1, \ldots, x_n]$ $I_{f,g}$.

Thm: (local criterion for flatness)
(C, m) local Noeth, M C-Mod.
$\rightsquigarrow$ M flat $\Leftrightarrow$ $m \otimes M \to M$ is inj
$(\Leftrightarrow \operatorname{Tor}_1^C(C/m, M) = 0)$

smoothness at fibres is clear by (BC)
so only have to show flatness.
locally $X = V(f_1, \ldots, f_i) \subseteq \mathbb{A}^n_S$
smooth/S.

Suffices to show:
$X \xrightarrow{f} S$ smooth + flat, $x \in X$, $g \in k(x, \mathcal{O}_x)$,
$g(x) = 0$, st $dI \neq 0$ in $\Omega^1_{X/S} \otimes k(x)$
then $Z(g) \to S$ flat (+ smooth).

smoothness follows from Jacobian criterion.

For flat, let $s = f(x)$

Have

$$0 \to \mathcal{O}_{X,x} \xrightarrow{\cdot g} \mathcal{O}_{X,x} \to \mathcal{O}_{Z,z} \to 0$$

Show $m_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Z,z} \to \mathcal{O}_{Z,z}$.

But $\mathcal{O}_{X,x}/\mathcal{O}_{S,s}$ flat + snake lemma + local crit of flatness surfices to show

$$\begin{array}{ccc}
X/m_{S,s}X & \longrightarrow & \mathcal{O}_{X,x}/m_{S,s}\mathcal{O}_{X,x} \quad \text{inj} \\
\uparrow & \nearrow g & \| \\
\mathcal{O}_{X,x}/m_{S,s}\mathcal{O}_{X,x} & \xrightarrow{\bar{g}} & \mathcal{O}_{X_s,x}
\end{array}$$

inj because $\bar{g} \neq 0$.

($\mathcal{O}_{X_s,x}$ is regular $\Rightarrow$ integral by smoothness)

(ii) ⇒ (i):

Have to show

$$A \otimes_B I = I/I^2 \longrightarrow A \otimes_B \Omega^1_{B/R} \longrightarrow \Omega^1_{A/R} \to 0$$

(exact on left) + (locally split)

$$\bigoplus_{i=1}^{r} A f_i \qquad \left(\frac{\partial f_i}{\partial x_j}\right) =: M \text{ has rank } r$$

for transcendence basis

but for $s = f(y) \in S$

$$0 \to J/J^2 \longrightarrow A_s \otimes_{B_s} \Omega^1_{B_s/k(s)} \to \Omega^1_{A_s/k(s)} \to 0$$

exact on left + locally split.

where $A_s = A \otimes_R k(s)$

$B_s = B \otimes_R k(s) = k(s)[x_1, \dots, x_n]$

$J = \ker(B_s \to A_s)$

But $J = I \otimes k(s)$

(Have $0 \to I \to B \to A \to 0$ now $- \otimes_R k(s)$)

$\leadsto$

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] & A_s \otimes_{B_s} \Omega_{B_s/k(s)} \arrow[r] & \Omega^1_{A_s/k(s)} \arrow[r] & 0 \\
& \bigoplus_{i=1}^{n} A_s f_i \arrow[u] \arrow[ur, "M_s"'] & & &
\end{tikzcd}$$

M_s has rank r and $I/I^2 \otimes k(s) \cong J/J^2$

$\to M$ has rank r

$\leadsto$ claim. ∎

[Forgot reduction to Noetherian case:
A/R f.p. $\leadsto \exists R'$ Noeth., A'/R' f.p. and
$R' \to R$, s.t. $A \cong A' \otimes_{R'} R$.
(finite limits + finitely many equations)