smoothness $\longleftrightarrow$ $\frac{\text{Smoothness}}{\text{tangent spaces}}$ $\longleftrightarrow$ maps from $\operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2}$

$= \{\ \}$

$= \{pt + \text{Tangentdirection}\}$

first order thickening.

Def A closed immersion $i: S_0 \hookrightarrow S$ of schemes is a nth order thickening if the ~~a~~ $\ker(\mathcal{O}_S \longrightarrow i_* \mathcal{O}_{S_0}) = \mathcal{I}$ satisfies $\mathcal{I}^{n+1} = 0$.

So locally of form $\operatorname{Spec} A_0 \subseteq \operatorname{Spec} A$, $A_0 = \overset{\substack{\text{square zero}\\ \text{extension}\\ \|}}{A/I}$, $I^2 = 0$

We call i split, if there is a section $s: S \longrightarrow S_0$ (so $s \circ i = \mathrm{Id}$)

<u>Remark</u> $\{$Split of first order thickending of $S_0\} \ni S \qquad \underline{\operatorname{Spec}}(\mathcal{O}_{S_0} \oplus \mathcal{M})$

$\updownarrow$

$\{$q-coh. $\mathcal{O}_{S_0}$-module$\} \ni \mathcal{I} \qquad \mathcal{M}$

If $\mathcal{M}$ any q-coh $\mathcal{O}_{S_0}$-module, $\mathcal{O}_{S_0} \oplus \mathcal{M}$ is a q-coh. $\mathcal{O}_{S_0}$-algebra via $(f, m)(g, n) \mapsto (fg, fn + gm)$

(think about $(f + m\varepsilon)(g + n\varepsilon) = fg + (fn + gm)\varepsilon + \underbrace{\varepsilon^2 nm}_{\text{makes no sense}}$)

<u>Remark</u> $\mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ nonsplit square zero extension.

<u>Definition</u> i) A morphism $f: X \longrightarrow S$ of schemes is formally smooth (resp. formally étale, resp. formally unramified) if for all first order thickenings $i: T_0 \subset T$ of affine schemes

$$\begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d, hook] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

there is a map $u : T \longrightarrow X$ making diagram commute (map is unique $\rightsquigarrow$ form. étale resp there is at most one map $\rightsquigarrow$ form. unramified)

"formally smooth = there is no obstruction to extending maps to first order thickenings

<u>Remark</u> Equivalent to ask if for n'th order thickening. $T_0 \hookrightarrow T = T_n$ n^{th} order thickening

$\hookrightarrow T_1 \hookrightarrow T_2 \dots$

$\rightsquigarrow$ composite of n first order thickenings T_i, defined by ideal sheaf $\mathcal{I}^{i+1}$

- ~~closed imm~~ formally étale $\Leftrightarrow$ formally smooth + formally unramified

<u>Examples</u> • open immersions are formally étale

if $f : X \hookrightarrow S$ open immersion $T \to S$ morph. then it factors over $X \Leftrightarrow$ $\begin{array}{l} \mathrm{im}(|T|) \subseteq |X| \subseteq |S| \\ \quad \parallel \\ \mathrm{im}(|T_0|) \end{array}$

- closed immersions are formally unramified
  $(X(T) \hookrightarrow S(T)) \Rightarrow$

- $X = \mathbb{A}^n_S \longrightarrow S$ is formally ~~unramified~~ smooth
  enough for $S = \mathrm{Spec}\,\mathbb{Z}$ (all properties satisfy (BC))

$T_0 = \mathrm{Spec}\, A_0$ ~~$\mathbb{Z}$~~, $T = \mathrm{Spec}\, A$, $A \twoheadrightarrow A_0$

$u_0 : T_0 \longrightarrow \mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[X_1, \dots, X_n]$ $\longleftrightarrow$
elements $x_1, \dots, x_n \in A_0$. Similarly, $u \longleftrightarrow$
elements $\hat{x}_1, \dots, \hat{x}_n \in A$ lifting $x_1, \dots, x_n \in A_0$
possible as $A \twoheadrightarrow A_0$

<u>Proposition</u> i) $P \in \{$form smooth / étale / unramified$\}$
satisfies (BC), (COMP), thus (PROD), also (LOCS), (LOCT)

ii)
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "fg"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$
Assume f formally ~~smooth~~ unramified. Then g formally smooth (resp. étale/unramified) $\Leftrightarrow$ fg is

<u>Proof</u> i) omitted. For "formally smooth"
in (LOCS), (LOCT) need different characterization
of "formally smooth" + <u>Thm</u> Let A ring, $M = A$-module
Then M is projective $\Leftrightarrow$ $\exists \bigcup D(f_i) = A$
s.th. $M[f_i^{-1}]$ are projective $A[f_i^{-1}]$ modules.

e.g. ii) "formally smooth"
$$\begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d, hook] & Y \arrow[d, "g"] \\ T \arrow[r, "v"'] \arrow[ur, "u"] & X \end{tikzcd} \rightsquigarrow \begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d, hook] & Y \arrow[d, "fg"] \\ T \arrow[r] \arrow[ur, "u"] & S \end{tikzcd}$$

Fix such a u
$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, "v"] \arrow[ur, shift right, "gu"'] & S \end{tikzcd}$$
$\Rightarrow v = g \circ u$ because f unramified.

Definition $f: X \longrightarrow S$ is smooth (resp étale, resp unramified) if it is formally smooth (resp blah) and f is locally of finite presentation (resp f is locally of finite presentation, resp f is locally of finite type)

Remark In EGA, also ask loc. of f.p., for "unramified". But this excludes closed immersions for which the ideal sheaf is not locally finitely generated (Def from Stacks)

"Recall" Def A morphism $f: X \to S$ is locally of finite type (resp. locally of finite presentation) if equiv

i) for all open affin

$$\begin{array}{ccc} U = \operatorname{Spec} A & \subseteq & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \subseteq & S \end{array}$$

A is fin. gen R-Algebra

$A = R[X_1, \dots, X_n]/(f_1, \dots, f_m)$

ii) same for covering

Remark i) f is of f.t. (resp. of f.pres) if in addition f is q.c. (resp qcqs)

ii) $P \in \{$loc. of f.t., l.f.p$\}$ are (BC), (COMP), (PROD), (LOCS), (LOCT)

Cor $P \in \{$smooth, étale, unram$\}$ satisfies (BC), (COMP), (PROD), (LOCS), (LOCT)

Remark This can be proved without hard work in formally smooth case.

Example k field, $S = \operatorname{Spec} k$

$$f : X \longrightarrow \operatorname{Spec} k \quad \text{scheme } /k$$

$\operatorname{Spec} k = T_0 \subset T = \operatorname{Spec}\left(k[\varepsilon]/_{\varepsilon^2}\right)$

Fix $u_0 : T_0 \longrightarrow X \ /k$

equivalently, $x \in X(k)$

$$\begin{tikzcd} T_0 = \operatorname{Spec} k \arrow[r, hook] \arrow[d, hook] & X \\ T \arrow[ur, dashed, "u"] & \end{tikzcd} \quad /k$$

Def The tangent space of X at x is $T_x X = \{\text{set of all such } u\}$

Lets compute this. wlog $X = \operatorname{Spec} A$, A – k-Algebra

$x \longleftrightarrow \varphi : A \longrightarrow k$

$u \longleftrightarrow \tilde{\varphi} : A \longrightarrow \frac{k[\varepsilon]}{\varepsilon^2} = k \oplus k\varepsilon$

map of k-algebras lifting φ

Write $\tilde{\varphi}(a) = \varphi(a) + d(a)\varepsilon$

$\tilde{\varphi}$ map of k-algebras iff

i) $\tilde{\varphi}(a+b) = \tilde{\varphi}(a) + \tilde{\varphi}(b) \Leftrightarrow d(a+b = d(a) + d(b)$

ii) $\tilde{\varphi}(\lambda a) = \tilde{\lambda}\tilde{\varphi}(a) \Leftrightarrow d(\lambda a) = \lambda d(a)$

} $d : A \to$ is k-lin

is k-linear and iii) $\tilde{\varphi}(ab) = \tilde{\varphi}(a)\tilde{\varphi}(b) \Leftrightarrow d(ab) = ad(b) + bd(a)$

$\tilde{\varphi}(ab) = \varphi(ab) + d(ab)\varepsilon$

$\tilde{\varphi}(a)\tilde{\varphi}(b) = \varphi(ab) + (\varphi(a)d(b) + \varphi(b)d(a))\varepsilon$

Definition Let $R \longrightarrow A$ map of rings, M an A-module

A derivation of A/R with values in M is an R-linear map

$$d : A \longrightarrow M$$

s.th. $d(ab) = a\,d(b) + b\,d(a)$ (Leibniz rule) is satisfied.

<u>Proposition</u> Let $R \longrightarrow A$ be a map of rings. There is a universal derivation of A/R $d : A \longrightarrow \Omega^1_{A/R}$, i.e. for any derivation

$\Omega^1_{A/R}$ = "module of Kähler differentials"

$d' : A \longrightarrow M$ $\exists!$ map $\Omega^1_{A/R} \longrightarrow M$ of A-modules s.t.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega^1_{A/R} \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

i.e. $Der_R(A, M) \equiv Hom_{A\,mod}(\Omega^1_{A/R}, M)$ functorially in A-modules M.

<u>Proof</u> $\Omega^1_{A/R}$ = (free A-module on $d(a)$, $a \in A$) $\Big/ \left(\begin{array}{l} d(a+b) = d(a) + d(b) \\ d(ra) = r\,d(a) \\ d(ab) = a\,d(b) + d(a)\,b \end{array} \right)$

<u>Cor</u> $T_x X = Der_k(A, k) = Hom(\Omega^1_{A/k}, k)$

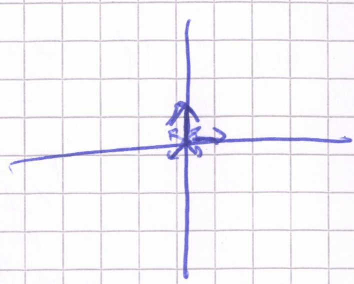
$$= Hom_k(\Omega^1_{A/k} \otimes_{A,\varphi} k, k)$$

$$= (\Omega_{A/k} \otimes_{A,\varphi} k)^*$$

In part. $T_x X$ is canonically a k-VS.

<u>Example</u> $Z = \operatorname{Spec} \frac{k[x,y]}{xy}$

$x=(0,0)$. What is $T_x Z$?



$$\varphi : \frac{k[X,Y]}{XY} \longrightarrow \frac{k[\varepsilon]}{\varepsilon^2}$$

$$X \longmapsto a\varepsilon$$

$$Y \longmapsto b\varepsilon$$

$$XY \longmapsto ab\varepsilon^2 = 0 .$$

Let's consider $\tilde{x} : \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2} \longrightarrow Z$

$$\begin{tikzcd} \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^2} = T_0 \arrow[r, "\tilde{x}"] \arrow[d, hook] & Z \\ T = \operatorname{Spec} \frac{k[\varepsilon]}{\varepsilon^3} \arrow[ur, dashed] & \end{tikzcd}$$

given $X \longmapsto \varepsilon$, $Y \longmapsto \varepsilon$

This would be given by map $k[X,Y] \longrightarrow \frac{k[\varepsilon]}{\varepsilon^3}$

This would be given by map $X \longmapsto \cancel{\varepsilon + a}\ \varepsilon + a\varepsilon^2$

$$Y \longmapsto \varepsilon + b\varepsilon^2$$

$$XY \longmapsto 0 \neq \varepsilon^2$$

This shows that Z is not smooth over k.

How to compute $\Omega^1_{A/R}$?

<u>Proposition</u> $R \longrightarrow A$, $R \longrightarrow S$ maps of rings.

$A \longrightarrow B$

i) $\Omega^1_{A \otimes_R S / S} \cong \Omega^1_{A/R} \otimes_R S$

ii) $\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \twoheadrightarrow \Omega^1_{B/A} \longrightarrow 0$

exact

iii) If $A \twoheadrightarrow B$ surj, $I = \ker(A \to B)$

$$\underset{I \otimes_A B}{\overset{\shortparallel}{I/I^2}} \xrightarrow{d} \Omega^1_{A/R} \otimes_A B \twoheadrightarrow \Omega^1_{B/R} \longrightarrow \Omega_{B/A} \longrightarrow 0$$

iv)
If $A = R[X_i, i \in I]$ free polynomial ~~ring~~ algebra
then $\Omega^1_{A/R} \xleftarrow{\cong} \bigoplus_{i \in I} A \cdot dX$

<u>Cor</u> If $A = \dfrac{R[X_i]}{(f_j)}$ then $\Omega^1_{A/R} = \left(\bigoplus_{i \in I} A dX_i \right) \Big/ (df_j)$

<u>Sketch</u> i) Universal prop

ii), iii) use presentation

iv) The map $\bigoplus_{i \in I} A dX_i \longrightarrow \Omega^1_{A/R}$ is surj.

Need to see that for all $f = f(X_i) \in A = R[X_i]$, df lies in image. $f(X_i) = \sum_{(n_i) \in \mathbb{N}^I} r_{(n_i)} \prod_{i \in I} X_i^{n_i}$

Then $df = \sum r_{(n_i)} d(\prod X_i^{n_i})$. What is

$$d(\prod X_i^{n_i}) = \sum n_i X_i^{n_i - 1} \prod X_j^{n_j} dX_i$$

For injectivity, construct derivation

$$d : A \longrightarrow \bigoplus A dX_i$$

via $f \longmapsto \sum_{i \in I} \frac{\partial f}{\partial X} dX_i = \sum_{i \in I} r_{(n_i)} n_i X_i^{n_i - 1} \prod_{j \neq i} X_j^{n_j}$

Check: This is derivation

Hence ~~ex~~

$$\begin{tikzcd} & \Omega^1_{A/R} \\ A \arrow[ur] \arrow[r] & \bigoplus A dX_i \arrow[u, "\exists!\text{ inverse.}"'] \end{tikzcd}$$