Quantum Mechanics - Notes & Solutions for Lecture III

Jan-Philipp Hils

October 5, 2020

Contents

Please note that errors and changes can occur occasionally

"I strongly believe, for all babies and a significant number of grownups, curiosity is a bigger motivator than money" - Elwyn Berlekamp[1]

Please note that errors and changes can occur occasionally

Part I

1 Lecture III - Notes

1.1 Math Interlude

1.1.1 Orthonormal Bases

Review from lecture I: Recall: we are working in a three dimensional space. Therefore we have three unit vectors along all three axes. In QM defined as

$$\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

Orthonormal bases mean that they are normalized and orthogonal to each other. Mathematically:
Normalized

$$\langle A|A\rangle = 1 \tag{2}$$

Orthogonal

$$\langle B|A\rangle = 0 \tag{3}$$

We can write this as a sum where both sides are bases. Therefore

$$\langle j|A\rangle = \sum_i \alpha_i \langle j|i\rangle \tag{4}$$

where α is a complex number. Remember the orthonormal condition. Therefore

$$\langle j|i\rangle = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = 1 \end{cases} \tag{5}$$

This fact leads to

$$\langle j|A\rangle = \alpha_j \tag{6}$$

Interlude: Operators in Quantum Mechanics

This is a short interlude. You can think about operators like a transformation where

$$\hat{A}|\Psi\rangle = |\Psi^{‘}\rangle \tag{7}$$

or

$$\langle\Phi|\hat{A} = \langle\Phi^{\text{`}}| \tag{8}$$

$\hat{A}$ is here an operator. Sometimes operators are written as matrices, but in function space they are defined as

$$\hat{A} = \frac{\partial}{\partial x} \tag{9}$$

Theorem 1.1 *Properties of operators: i) order matters:* $\hat{A}\hat{B} \neq \hat{B}\hat{A}$*, ii)* $\hat{A}\hat{B}\hat{C} = \hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$

1.1.2 Machines and Matrices

Please note: here we are talking about **linear operators**. Therefore we have to know the difference between *states* and *observables*. States are described as vectors besides obs. are described by linear operators. Observables are things we can measure. Later we will see why this funny fact is very important. Therefore assume **M** is a operator - call it machine or whatever.

$$M|A\rangle = |B\rangle \tag{10}$$

Very important: not every machine is a **linear operator**.

Properties of linear operators

$$Mz|B\rangle = z|b\rangle \tag{11}$$

and

$$M\left\{|A\rangle + |B\rangle\right\} = M|A\rangle + M|B\rangle \tag{12}$$

where z is a complex number.

1.1.3 Hermitian Conjugation

We are interested in the following question: is there a corresponding M to get the following:

$$M|A\rangle = |B\rangle \tag{13}$$

to

$$\langle A|M = \langle B| \tag{14}$$

Answer: yes. Let's find out. Therefore we have to know we need the complex conjugation of M, because we are deailing with complex numbers. There is no work to to when dealing with real numbers. Remember this is QM, therefore complex numbers are everywhere. To conjugate a matrix, there is a matrice-so called Hermitian. Therefore

$$\langle A|M^\dagger = \langle B| \tag{15}$$

What did the dagger? Be careful: first it changed the rows and the columns, secondly took the complex conjugate. Therefore

$$M^\dagger = (M^T)^* \tag{16}$$

1.1.4 Hermitian Operator

Recall: $z = z^*$ is only valid for real numbers. A Hermitian is

$$M = M^\dagger \tag{17}$$

or another form

$$M_{ij} = Mji^* \tag{18}$$

Recall: the diagonal elements are always real.

1.1.5 Eigenvalues and Eigenvectors

This will be short, because it's linear algebra stuff.

$$M|\lambda\rangle = \lambda|\lambda\rangle \tag{19}$$

via this equation we can solve every eigenvalue and therefore the eigenvector.

1.2 Principles of Quantum Mechanics

1.2.1 First Principle

The observable or measurable quantities of quantum mechanics are represented by linear operators L.[2][p.69]. Therefore L must be hermitian. It's they have to be real.

1.2.2 Second Principle

Results of the measurement are the eigenvalues of the operator that represents the observables.

1.2.3 Third Principle

Unambiguously distinguishable states are represented by orthogonal vectors. [2][p.70]

1.2.4 Fourth Principle

Probability of an observable is defined as:

$$P(\lambda_i) = \langle A|\lambda_i\rangle\langle\lambda_i|A\rangle \tag{20}$$

Part II

2 Lecture III - Solutions

2.1 Exercise 3.1

under construction.

2.2 Exercise 3.2

Exercise[2][p.77] Proof:
We know:

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{21}$$

and

$$\begin{aligned} \sigma_z|u\rangle &= \ u\rangle \\ \sigma_z|d\rangle &= -|d\rangle \end{aligned} \tag{22}$$

Therefore

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned} \tag{23}$$

where $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|d\rangle = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

2.3 Exercise 3.3

Exercise[2][p.86] Proof: We know:

$$\sigma_n = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix} \tag{24}$$

Therefore

$$A\vec{x} = \lambda x \tag{25}$$

leads to

$$\begin{aligned} \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix} \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix} &= \lambda \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix} \\ \begin{bmatrix} cos\theta cos\alpha + sin\theta sin\alpha \\ sin\theta cos\alpha - cos\theta sin\alpha \end{bmatrix} &= \begin{pmatrix} \lambda cos\alpha \\ \lambda sin\alpha \end{pmatrix} \end{aligned} \tag{26}$$

Theorem 2.1 *Trigonometric Addition Formulas:* $cos\alpha cos\beta + sin\alpha sin\beta = cos(\alpha - \beta)$ *and* $sin\alpha cos\beta - cos\alpha sin\beta = sin(\alpha - \beta)$

Therefore

$$\begin{pmatrix} cos(\theta - \alpha) \\ sin(\theta - \alpha) \end{pmatrix} = \begin{pmatrix} \lambda cos\alpha \\ \lambda sind\alpha \end{pmatrix} \tag{27}$$

via

$$\lambda = \begin{Bmatrix} \frac{cos(\theta-\alpha)}{cos\alpha} \\ \frac{sin(\theta-\alpha)}{sin\alpha} \end{Bmatrix} \tag{28}$$

leads to

$$\begin{aligned} \frac{cos(\theta - \alpha)}{cos\alpha} &= \frac{sin(\theta - \alpha)}{sin\alpha} \\ \frac{cos(\theta - \alpha)cos\alpha sin\alpha}{cos\alpha} &= \frac{sin(\theta - \alpha)cos\alpha sin\alpha}{sin\alpha} \\ \frac{cos(\theta - \alpha)sin\alpha}{1} &= \frac{sin(\theta - \alpha)cos\alpha}{1} \\ cos(\theta - \alpha)sin(\alpha) - sin(\theta - \alpha)cos(\alpha) &= 0 \\ sin(\alpha - \theta + \alpha) &= 0 \\ sin(2\alpha - \theta) &= 0 \end{aligned} \tag{29}$$

Therefore - from above:

$$\begin{aligned} 2\alpha - \theta &= 0 \\ 2\alpha- &= \theta \\ \alpha_1 &= \frac{\theta}{2} \end{aligned} \tag{30}$$

and

$$\begin{aligned} 2\alpha - \theta &= \pi \\ 2\alpha &= \pi + \theta \\ \alpha_2 &= \frac{\pi}{2} + \frac{\theta}{2} \end{aligned} \tag{31}$$

Plug in leads to

$$\lambda_1 = 1 \tag{32}$$

and

$$|\lambda_1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{pmatrix} \tag{33}$$

Same way leads to

$$\lambda_2 = -1 \tag{34}$$

and

$$|\lambda_2\rangle = \begin{pmatrix} -sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \tag{35}$$

Theorem 2.2 *Pythagorean Identity:* $sin^2\alpha + cos^2\alpha = 1$

Another approach - from algebra I -

$$\begin{aligned} 0 = det(\sigma_n - I\lambda) &= \begin{bmatrix} cos\theta - \lambda & sin\theta \\ sin\theta & -cos\theta - \lambda \end{bmatrix} \\ &= -cos^2 + \lambda^2 - sin^2\theta \\ cos^2\theta + sin^2\theta &= \lambda^2 \\ 1 &= \lambda^2 \\ \pm 1 &= \lambda \end{aligned} \tag{36}$$

2.4 Exercise 3.4

Exercise[2][p.88]

Theorem 2.3 $n_z^2 + n_y^2 + n_x^2 = 1$

Therefore

$$0 = det(\sigma_n - I\lambda) = \begin{pmatrix} n_z - \lambda & (n_x - in_y) \\ (n_x + in_y) & -n_z - \lambda \end{pmatrix} \tag{37}$$

leads to

$$\begin{aligned} 0 &= \lambda^2 - n_z^2 - n_x^2 - in_y^2 \\ -\lambda^2 &= -n_z^2 - n_x^2 - in_y^2 \\ \lambda^2 &= n_z^2 + n_x^2 + in_y^2 \\ \lambda^2 &= 1 \\ \lambda &= \pm 1 \end{aligned} \tag{38}$$

Theorem 2.4 *Complex Trigonometric Identity:* $e^{i\phi} = cos(\phi) + isin(\phi)$

In the next part we want to find the eigenvectors, therefore we fill in our initial values. To compute the eigenvalues we can do the same.
Initial matrix:

$$\sigma_n = \begin{pmatrix} cos\theta & sin\theta(cos\phi - sin\phi) \\ sin\theta(cos\phi + sin\phi) & -cos\theta \end{pmatrix} = \begin{pmatrix} cos\theta & sin(\theta)e^{-i\phi} \\ sin(\theta)e^{i\phi} & -cos\theta \end{pmatrix} \tag{39}$$

We assume like above - here with a face value -

$$|\lambda\rangle = \begin{pmatrix} cos\alpha \\ e^{i\phi}sin(\alpha) \end{pmatrix} \tag{40}$$

leads to

$$\begin{pmatrix} cos\theta & sin(\theta)e^{-i\phi} \\ sin(\theta)e^{i\phi} & -cos\theta \end{pmatrix} \begin{pmatrix} cos\alpha \\ e^{i\phi}sin\alpha \end{pmatrix} = \begin{pmatrix} cos\alpha \\ e^{i\phi}sin\alpha \end{pmatrix} \tag{41}$$

results in

$$\begin{vmatrix} cos\theta cos\alpha + e^{i\phi}e^{-i\phi}sin\alpha = cos\alpha \\ e^{i\phi}cos\alpha - cos\theta e^{i\phi}sin\alpha = e^{i\phi}sin\alpha \end{vmatrix} \tag{42}$$

via trig identities

$$\begin{aligned} cos\theta cos\alpha + sin\theta sin\alpha &= cos\alpha \\ sin\theta cos\alpha - cos\theta sin\alpha &= sin\alpha \end{aligned} \tag{43}$$

results in

$$\begin{aligned} cos(\theta - \alpha) &= cos\alpha \\ sind(\theta - \alpha) &= sin\alpha \end{aligned} \tag{44}$$

therefore

$$\begin{aligned} \theta - \alpha &= \alpha \\ \theta &= 2\alpha \\ \frac{\theta}{2} &= \alpha \end{aligned} \tag{45}$$

Result:

$$\begin{aligned} \lambda_1 &= 1 \\ |\lambda_1\rangle &= \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix} \end{aligned} \tag{46}$$

For the next one we have to take a look at our kowledge.
Recall:

$$\langle\lambda_1|\lambda_2\rangle = 0 \tag{47}$$

This fact leads to

$$\begin{pmatrix} cos\theta & sin(\theta)e^{-i\phi} \\ sin(\theta)e^{i\phi} & -cos\theta \end{pmatrix} \begin{pmatrix} cos\alpha \\ -e^{i\phi}sin\alpha \end{pmatrix} = - \begin{pmatrix} cos\alpha \\ -e^{i\phi}sin\alpha \end{pmatrix} \tag{48}$$

Same way here gives us

$$|\lambda_2\rangle = \begin{pmatrix} sin\frac{\theta}{2} \\ -e^{i\phi}cos\frac{\theta}{2} \end{pmatrix} \tag{49}$$

2.5 Exercise 3.5

Exercise[2][p.90] Proof: From initial condition we know

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{50}$$

We need λ_1 therefore

$$|\lambda_1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix} \tag{51}$$

Therefore

$$P(1) = \langle u|\lambda_1\rangle\langle\lambda_1|u\rangle \tag{52}$$

leads to

$$
\begin{aligned}
P(1) &= \left(\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}cos\frac{\theta}{2}\\ e^{i\phi}sin\frac{\theta}{2}\end{pmatrix}\right)(\begin{pmatrix}cos\frac{\theta}{2} & e^{-i\phi}sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}) \\
&= cos\frac{\theta}{2}cos\frac{\theta}{2} \\
&= cos^2\frac{\theta}{2}
\end{aligned} \tag{53}
$$

3 References

References

[1] Gregory Zuckerman. *The Man Who Solved the Market*. Penguin LCC US, 2019.

[2] Art Friedman Leonard Susskind. *Quantum Mechanics: The Theoretical Minimum*. Penguin Books Ltd (UK), 2015.