Quantum Mechanics - Solutions for Lecture I and II

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Please note that errors and changes can occur occasionally

"I strongly believe, for all babies and a significant number of grownups, curiosity is a bigger motivator than money" - Elwyn Berlekamp[1]

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1 Pre

Some nice stuff:

Lemma 1.1 *(Symmetric, Positive Definite Matrices)*

$$\langle x, y \rangle = \left\langle \sum_{i=i}^{n} \psi_i, b_i, \sum_{j=1}^{n} \lambda_j b_j \right\rangle = \sum_{i=i}^{n} \sum_{j=1}^{n} \psi_i \langle b_i, b_j \rangle \lambda_j = x^T A y \tag{1}$$

[2][p.73]

Will be updated soon

2 Lecture 1

Theorem 2.1 *Inner Product(Linearity):* $\langle C| \{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle$

Theorem 2.2 *Interchanging bras and kets:* $\langle B|A\rangle = \langle A|B\rangle^*$

2.1 Exercise 1.1

Exercise 1.1a:[3][p.31] Proof:

$$\begin{aligned} \langle A|C\rangle + \langle B|C\rangle &= \{\langle A| + \langle B|\} |C\rangle \\ &= (\langle C| \{|A\rangle + |B\rangle\})^* \\ &= \langle C|A\rangle^* + \langle C|B\rangle^* \\ &= \langle A|C\rangle + \langle B|C\rangle \end{aligned} \tag{2}$$

Exercise 1.1b: Proof:

$$\langle A|A\rangle = \langle A|A\rangle^* \tag{3}$$

must be *real* because: $zz^* = (a + bi)(a - bi) = a^2 + b^2$

Theorem 2.3

$$\langle B|A\rangle = \begin{pmatrix} \beta_1^* & \beta_2^* & \beta_3^* \end{pmatrix} * \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{4}$$

2.2 Exercise 1.2

Exercise 1.2:[3][p.31] Proof - *Interchanging* - :

$$\begin{aligned}\langle A|B\rangle^* &= \langle B|A\rangle \\ &= \begin{pmatrix}\beta_1^* & \beta_2^* & \beta_3^*\end{pmatrix} * \begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{pmatrix} \\ &= \left(\beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \beta_3^*\alpha_3\right) \\ &= \left(\beta_1\alpha_1^* + \beta_2\alpha_2^* + \beta_3\alpha_3^*\right)^* \\ &= \left[\begin{pmatrix}\alpha_1^* & \alpha_2^* & \alpha_3^*\end{pmatrix} * \begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3\end{pmatrix}\right]^*\end{aligned} \tag{5}$$

Exercise 1.2: - *Linearity* -

$$\begin{aligned}\langle C| \{|A\rangle + |B\rangle\} &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^*\end{pmatrix} * \left[\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{pmatrix} + \begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3\end{pmatrix}\right] \\ &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^*\end{pmatrix} * \begin{pmatrix}\alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3\end{pmatrix} \\ &= \gamma_1^*\alpha_1 + \gamma_1^*\beta_1 + \gamma_2^*\alpha_2 + \gamma_2^*\beta_2 + \gamma_3^*\alpha_3 + \gamma_3^*\beta_3 \\ &= \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^*\end{pmatrix} * \begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{pmatrix} + \begin{pmatrix}\gamma_1^* & \gamma_2^* & \gamma_3^*\end{pmatrix} * \begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3\end{pmatrix} \\ &= \langle C|A\rangle + \langle C|B\rangle\end{aligned} \tag{6}$$

3 Lecture 2

3.1 Exercise 2.1

Exercise 2.1: [3][p.42]Proof: Please remind: *Two vectors are orthogonal if their inner product is zero.* Therefore:

$$\langle B|A\rangle = 0 \tag{7}$$

Theorem 3.1 *Probabilty of up and down:* $|u\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ *and* $|d\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Let's begin:

$$|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \tag{8}$$

$$|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle \tag{9}$$

Therefore:

$$\langle l| = \langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{1}{\sqrt{2}} \tag{10}$$

Now:

$$\langle l|r\rangle = \left[\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{1}{\sqrt{2}}\right] | \left[\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle\right] \tag{11}$$

$$\begin{aligned}
&= \left[\begin{pmatrix}1 & 0\end{pmatrix} * \frac{1}{\sqrt{2}} - \begin{pmatrix}0 & 1\end{pmatrix} * \frac{1}{\sqrt{2}}\right] * \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\right] \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix} * \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= \begin{pmatrix}0\end{pmatrix}
\end{aligned} \tag{12}$$

3.2 Exercise 2.2

Exercise 2.2:[3][p.44] We have to clearify all conditions where

$$\langle i|o\rangle = 0 \tag{13}$$

$$\begin{aligned}
\langle o|u\rangle \langle u|o\rangle &= \frac{1}{2} \\
\langle o|d\rangle \langle d|o\rangle &= \frac{1}{2} \\
\langle i|u\rangle \langle u|i\rangle &= \frac{1}{2} \\
\langle i|d\rangle \langle d|i\rangle &= \frac{1}{2}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}\langle o|r\rangle \langle r|o\rangle &= \frac{1}{2}\\ \langle o|l\rangle \langle l|o\rangle &= \frac{1}{2}\\ \langle i|r\rangle \langle r|i\rangle &= \frac{1}{2}\\ \langle i|l\rangle \langle l|i\rangle &= \frac{1}{2}\end{aligned} \tag{15}$$

Here we go:

Part a: Please remind the conditions about imaginary numbers.

$$\begin{aligned}\langle i|o\rangle &= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{pmatrix} * \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}\\ &= \left(\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}\right) + \left(\frac{-i}{\sqrt{2}} * -\frac{i}{\sqrt{2}}\right) = \frac{1}{2} + \frac{i^2}{2} = \frac{1}{2} + \frac{-1}{2}\\ &= 0\end{aligned} \tag{16}$$

Condition checked.

Part b: We already know:

$$|u\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

and

$$|d\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

therefore

$$\langle o| = \left[\begin{pmatrix}1 & 0\end{pmatrix} * \frac{1}{\sqrt{2}} - \begin{pmatrix}0 & 1\end{pmatrix} \frac{i}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Action:

$$\begin{aligned}
\langle o|u\rangle \langle u|o\rangle = \\
&= \left\{ \left[\begin{pmatrix} 1 & 0 \end{pmatrix} * \frac{1}{\sqrt{2}} - \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{i}{\sqrt{2}} \right] * \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \\
&* \left\{ \begin{pmatrix} 1 & 0 \end{pmatrix} * \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{i}{\sqrt{2}} \right] \right\} \\
&= \left[\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] * \left[\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right] \\
&= \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} \\
&= \frac{1}{2}
\end{aligned} \tag{17}$$

Next one:

$$\begin{aligned}
\langle o|d\rangle \langle d|o\rangle = \\
&= \left\{ \left[\begin{pmatrix} 1 & 0 \end{pmatrix} * \frac{1}{\sqrt{2}} + \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{i}{\sqrt{2}} \right] * \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \\
&* \left\{ \begin{pmatrix} 0 & 1 \end{pmatrix} * \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{i}{\sqrt{2}} \right] \right\} \\
&= \left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] * \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right] \\
&= \frac{i}{\sqrt{2}} * -\frac{i}{\sqrt{2}} \\
&= \frac{1}{2}
\end{aligned} \tag{18}$$

Another one:

$$\begin{aligned}\langle i|u\rangle \langle u|i\rangle &= \left\{\left[\begin{pmatrix}1 & 0\end{pmatrix} * \frac{1}{\sqrt{2}} \begin{pmatrix}0 & 1\end{pmatrix} \frac{-i}{\sqrt{2}}\right] * \begin{pmatrix}1\\0\end{pmatrix}\right\} \\ &* \left\{\begin{pmatrix}1 & 0\end{pmatrix} * \left[\begin{pmatrix}1\\0\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}0\\1\end{pmatrix} \frac{i}{\sqrt{2}}\right]\right\} \\ &= \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}\right] * \left[\begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\right] \\ &= \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} \\ &= \frac{1}{2}\end{aligned} \tag{19}$$

Last one:

$$\begin{aligned}\langle i|d\rangle \langle d|i\rangle &= \left\{\left[\begin{pmatrix}1 & 0\end{pmatrix} * \frac{1}{\sqrt{2}} \begin{pmatrix}0 & 1\end{pmatrix} \frac{-i}{\sqrt{2}}\right] * \begin{pmatrix}0\\1\end{pmatrix}\right\} \\ &* \left\{\begin{pmatrix}0 & 1\end{pmatrix} * \left[\begin{pmatrix}1\\0\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}0\\1\end{pmatrix} \frac{i}{\sqrt{2}}\right]\right\} \\ &= \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}\right] * \left[\begin{pmatrix}0 & 1\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\right] \\ &= \frac{-i}{\sqrt{2}} * \frac{i}{\sqrt{2}} \\ &= \frac{1}{2}\end{aligned} \tag{20}$$

Conditions checked.
Part c: We know already:

$$\begin{aligned}|r\rangle &= \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \\ |l\rangle &= \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle\end{aligned} \tag{21}$$

Please remind to construct - if it's needed - the conjugate.

$$\begin{aligned}
&\langle o|r\rangle \langle r|o\rangle \\
&= \left\{ \left[\begin{pmatrix}1 & 0\end{pmatrix} * \frac{1}{\sqrt{2}} \begin{pmatrix}0 & 1\end{pmatrix} * \frac{-i}{\sqrt{2}} \right] * \left[\frac{1}{\sqrt{2}} * \begin{pmatrix}1\\0\end{pmatrix} \frac{1}{\sqrt{2}} * \begin{pmatrix}0\\1\end{pmatrix} \right] \right\} \\
&* \left\{ \left[\begin{pmatrix}1 & 0\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}0 & 1\end{pmatrix} \frac{1}{\sqrt{2}} \right] * \left[\frac{1}{|\sqrt{2}} \begin{pmatrix}1\\0\end{pmatrix} \frac{-i}{\sqrt{2}} \begin{pmatrix}0\\1\end{pmatrix} \right] \right\} \\
&= \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \right] \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{pmatrix} \right] \\
&= \left(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right) \left(\tfrac{1}{\sqrt{2}} - \tfrac{i}{\sqrt{2}}\right) \\
&= \frac{1}{4} - \frac{i^2}{4} = \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\langle o|l\rangle \langle l|o\rangle \\
&= \left\{ \left[\begin{pmatrix}1 & 0\end{pmatrix} * \frac{1}{\sqrt{2}} \begin{pmatrix}0 & 1\end{pmatrix} * \frac{i}{\sqrt{2}} \right] * \left[\frac{1}{\sqrt{2}} * \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{\sqrt{2}} * \begin{pmatrix}0\\1\end{pmatrix} \right] \right\} \\
&* \left\{ \left[\begin{pmatrix}1 & 0\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}0 & 1\end{pmatrix} - \frac{1}{\sqrt{2}} \right] * \left[\frac{1}{|\sqrt{2}} \begin{pmatrix}1\\0\end{pmatrix} \frac{-i}{\sqrt{2}} \begin{pmatrix}0\\1\end{pmatrix} \right] \right\} \\
&= \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix} \right] \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{pmatrix} \right] \\
&= \left(\tfrac{1}{2} - \tfrac{i}{2}\right) \left(\tfrac{1}{2} + \tfrac{i}{2}\right) = \frac{1}{4} - \frac{i^2}{4} \\
&= \frac{1}{2}
\end{aligned} \tag{23}$$

$$\langle i|r\rangle \langle r|i\rangle$$

and

$$\langle i|l\rangle \langle l|i\rangle$$

will be proved in the same way.
Proof *unique vector*:

3.3 Exercise 2.3

Exercise2.3: [3][p.45] Assume:

$$|i\rangle = \alpha|u\rangle + \beta|d\rangle$$

$$|o\rangle = \gamma|u\rangle + \delta|d\rangle$$

where the *greeks* are unknown.
Part a: We already know from above:

$$\langle o|u\rangle \langle u|o\rangle = \frac{1}{2}$$

We substitude, than straightforward algebra - again remind the conjugates -

$$\begin{aligned} &= \left\{ \left[\begin{pmatrix}1 & 0\end{pmatrix} \gamma^* \begin{pmatrix}0 & 1\end{pmatrix} \delta^* \right] \begin{pmatrix}1 \\ 0\end{pmatrix} \right\} \left\{ \begin{pmatrix}1 & 0\end{pmatrix} \left[\gamma \begin{pmatrix}1 \\ 0\end{pmatrix} \delta \begin{pmatrix}0 \\ 1\end{pmatrix} \right] \right\} \\ &= \begin{pmatrix}\gamma^* & \delta^*\end{pmatrix} \begin{pmatrix}1 \\ 0\end{pmatrix} \begin{pmatrix}1 & 0\end{pmatrix} \begin{pmatrix}\gamma \\ \delta\end{pmatrix} \qquad (24) \\ \frac{1}{2} &= \gamma^*\gamma \end{aligned}$$

Therefore(other greeks are proved in the same way) $\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$.
Part b: We have to prove:

$$\gamma^*\delta + \gamma\delta^* = 0$$

Therefore we know from above:

$$\langle 0|r\rangle\langle r|o\rangle = \frac{1}{2}$$

We substitude by some greeks:

$$= \left[\begin{pmatrix}\gamma^* & \delta^*\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} \right] \left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\gamma \\ \delta\end{pmatrix} \right] = \left(\frac{\gamma^*}{\sqrt{2}} + \frac{\delta^*}{\sqrt{2}} \right) \left(\frac{\gamma}{\sqrt{2}} + \frac{\delta}{\sqrt{2}} \right) = \left(\frac{\gamma^* + \delta^*}{\sqrt{2}} \right) \left(\frac{\gamma + \delta}{\sqrt{2}} \right)$$

Therefore

$$\frac{\gamma^*\gamma + \gamma^*\delta + \delta^*\gamma + \delta^*\delta}{2} \qquad (25)$$

Remind: $\gamma^*\gamma = \delta^*\delta = \frac{1}{2}$

$$\frac{0.5 + \gamma^*\delta + \delta^*\gamma + 0.5}{2} = \frac{1 + \gamma^*\delta + \delta^*\gamma}{2}$$

where (from above)

$$\frac{1}{2} = \frac{1 + \gamma^*\delta + \delta^*\gamma}{2}$$

After easy algebra we have:

$$\gamma^*\delta + \delta^*\gamma = 0$$

Part c:

$$\begin{aligned} \gamma^*\delta + \gamma\delta^* &= 0 \\ \gamma^*\delta &= -\gamma\delta^* \end{aligned} \tag{26}$$

Therefore we substitude - conjugate! - : $a + bi = -(a - bi)$ where

$$a = -a$$

. Therefore a must be zero. Thus the equation above is pure imaginary.

4 References

References

[1] Gregory Zuckerman. *The Man Who Solved the Market.* Penguin LCC US, 2019.

[2] Marc Peter (University College London) Deisenroth, A. Aldo (Imperial College London) Faisal, and Cheng Soon Ong. *Mathematics for Machine Learning.* Cambridge University Press, 2020.

[3] Art Friedman Leonard Susskind. *Quantum Mechanics: The Theoretical Minimum.* Penguin Books Ltd (UK), 2015.