Adaptive Information Processing

Hints to the exercises

for Model complexity and the MDL principle

Bert de Vries and Tjalling Tjalkens
Signal Processing Group

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Bayes and the Laplace method

1. The two envelope paradox. See `http://www.anc.ed.ac.uk/~amos/doubleswap.html` for an answer.

2. Consider the following integral (similar to the Beta integral)

$$F(\mu_1, \mu_2) = \int_{-\infty}^{\infty} \left(\frac{1}{1+e^{-a}}\right)^{\mu_1} \left(\frac{e^{-a}}{1+e^{-a}}\right)^{\mu_2} da.$$

   (a) Use Laplace's method to approximate this integral.

   Hint:

   Just follow the steps in the lecture notes.

   (b) Use the Beta integral

$$B(\mu_1, \mu_2) = \int_0^1 p^{\mu_1 - 1}(1-p)^{\mu_2 - 1}\, dp = \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2)}$$

   with

$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(1) = 1$$
$$\Gamma(0.5) = \sqrt{\pi}$$

   and compare your approximation with the actual values in the cases where $\mu_1 = \mu_2 = 0.5$ resp. $\mu_1 = \mu_2 = 1$.

   Hint:

   Consider the transformation from p to $\frac{1}{1+e^{-a}}$ in the Beta integral.

Universal data compression

1. The Shannon-Fano code and Huffman code.

   Consider a binary i.i.d. souce that generates $X_1, X_2, \ldots, X_n$ with the parameter $\theta = \Pr\{X = 1\} = 0.1$.

   Compute, for $n = 1, 2, 3$, the expected code wordlength for the Shannon-Fano code, with lengths

   $$l_C^*(x^n) = \lceil -\log_2 p(x^n) \rceil.$$

   Likewise for the Huffman procedure, see lecture notes Information Theory (5K020/5JJ40).

   Give your comments on this result, (and consider here the source entropy).

   Hint:

   No hint needed, just do it. For the Huffman procedure you can read the lecture notes of Information Theory I or any basic Information Theory textbook.

2. [Hard] Show that

   $$\bar{p}(x^n) < \sqrt{\frac{\pi}{2n}} e^{\frac{1}{3n}} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k},$$

   where

   $$\bar{p}(x^n) = \int_0^1 (1-\theta)^{N(0|x^n)} \theta^{N(1|x^n)} \, d\theta.$$

   Hint:

   It's hard and you're all on your own. You can make good use of Stirling's formula.

ML and MDL

1. Assume x^n are i.i.d. observations from $\mathcal{N}(\theta, 1)$, so the x_i's are independent Gaussians with unit variance but unknown mean $\theta \in \mathbb{R}$. We test two hypothesis, $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. Otherwise said, we want to choose between the models

   $$\mathcal{M}_0 = \{\mathcal{N}(0, 1)\} \text{ and } \mathcal{M}_1 = \{\mathcal{N}(\theta, 1) | \theta \neq 0\}$$

   Derive that if we compute the ML probabilities for each model and then choose for the model with the largest ML probability we will never choose for $\mathcal{M}_0$ even if x^n was actually generated by $\mathcal{M}_0$.

   Hint:

   The proof is based on the fact that for $\mathcal{M}_1$ we estimate the mean

and only if the estimated mean is equal to zero, $\mathcal{M}_0$ will be selected. This is an event of probability zero.

Compare this to the MDL result of pages 124–125. Doesn't that result seem more satisfying?

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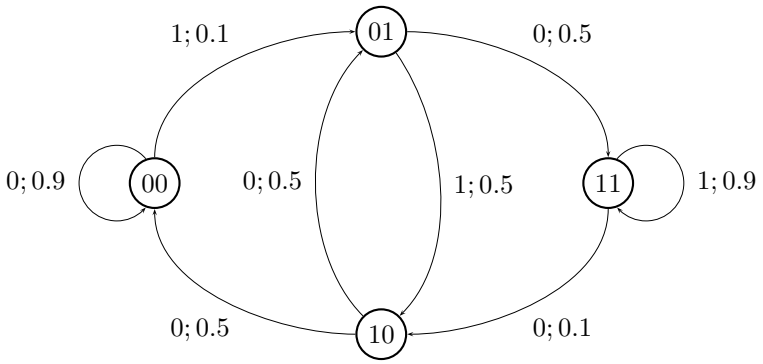
2. Consider this following 1^{th}-order binary Markov source. Next to the arrow from state a to state b is written $x; \Pr\{X_i = x, S_i = b | S_{i-1} = a\}$.



   (a) Determine the probability $\Pr\{X_i = 1\}$.
   Hint: Compute the stationary state distribution and then marginalize $\Pr\{X_i = 1, S_{i-1} = s$ to obtain $\Pr\{X_i = 1\}$.

   (b) Consider an "ideal" universal datacompression algorithm and we observe a sequence x^n that is typical for the source. How large must i be approximately to select the first order Markov model in stead of the memoryless model.

Hint:

Look at the discussion in the notes " Stochastic Complexity (MDL)". This is similar! Also the lecture notes Information Theory (5K020/5JJ40) will be helpful.

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3. [Hard:] can you determine the number of suffix trees with maximal depth not more than D for $D = 0, 1, 2, \ldots, 10$?

Hint:

You should be on your own again but ok.

Let F_i be the number of binary trees of maximum depth at most i. Then

$$F_i = 1 + F_{i-1}^2$$

The 1 comes from the tree containing only a root and otherwise all trees with maximum depth i can be written as the tree of depth 1 having both a tree of depth at most $i-1$ at its zero node and its one node.

D	$\#trees$
0	1
1	2
2	5
3	26
	...
10	$1.437821978 \cdot 10^{181}$

(Unless I made an error in calculating this recursion.)

So the number of models is HUGE for reasonable D. e.g. we use this CTW method for text compression on a byte depth of 12 characters. This is $D = 96$. Still the amount of work is linear in the sequence length times D.
