Adaptive Information Processing
Exercises
for Model complexity and the MDL principle

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Abstract

The exercises are intended to illustrate the results and deepen your understanding. Their level is sometimes higher than expected for the exam. The mark [Hard:] indicates an exercise above the exam level.

Tjalling Tjalkens, 3 March, 2006, Updated for reduced contents, March 2009.

Bayes and the Laplace method

1. The two envelope paradox.

   See `http://www.anc.ed.ac.uk/~amos/doubleswap.html` for a nice paradox that can be solved using the Bayes approach. The answer is also given on the web site.

2. Consider the following integral (similar to the Beta integral)

$$F(\mu_1, \mu_2) = \int_{-\infty}^{\infty} \left(\frac{1}{1+e^{-a}} \right)^{\mu_1} \left(\frac{e^{-a}}{1+e^{-a}} \right)^{\mu_2} da.$$

   (a) Use Laplace's method to approximate this integral.

   (b) Use the Beta integral

$$B(\mu_1, \mu_2) = \int_0^1 p^{\mu_1 - 1}(1-p)^{\mu_2 - 1}\, dp = \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1 + \mu_2)}$$

   with

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x) \\ \Gamma(1) &= 1 \\ \Gamma(0.5) &= \sqrt{\pi} \end{aligned}$$

   and compare your approximation with the actual values in the cases where $\mu_1 = \mu_2 = 0.5$ resp. $\mu_1 = \mu_2 = 1$.

Universal data compression

1. The Shannon-Fano code and Huffman code.

   Consider a binary i.i.d. souce that generates $X_1, X_2, \ldots, X_n$ with the parameter $\theta = \Pr\{X = 1\} = 0.1$.

   Compute, for $n = 1, 2, 3$, the expected code wordlength for the Shannon-Fano code, with lengths

   $$l_C^*(x^n) = \lceil -\log_2 p(x^n) \rceil.$$

   Likewise for the Huffman procedure, see lecture notes Information Theory (5K020/5JJ40).

   Give your comments on this result, (and consider here the source entropy).

2. [Hard] Show that

   $$\bar{p}(x^n) < \sqrt{\frac{\pi}{2n}} e^{\frac{1}{3n}} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k},$$

   where

   $$\bar{p}(x^n) = \int_0^1 (1-\theta)^{N(0|x^n)} \theta^{N(1|x^n)} \, d\theta.$$

   So, we use a *uniform* prior over θ.

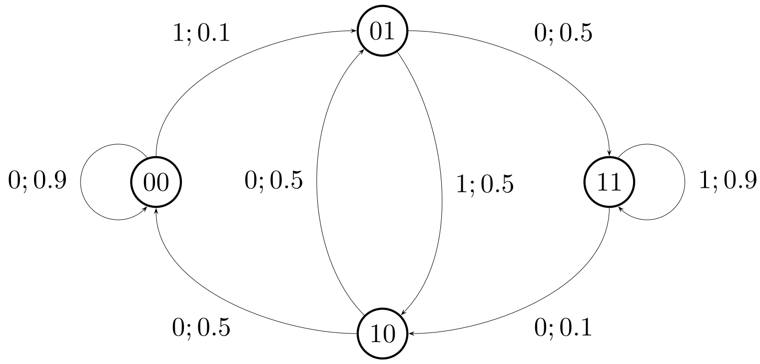
ML and MDL

1. Assume x^n are i.i.d. observations from $\mathcal{N}(\theta, 1)$, so the x_i's are independent Gaussians with unit variance but unknown mean $\theta \in \mathbb{R}$. We test two hypothesis, $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. Otherwise said, we want to choose between the models

   $$\mathcal{M}_0 = \{\mathcal{N}(0,1)\} \text{ and } \mathcal{M}_1 = \{\mathcal{N}(\theta,1) | \theta \neq 0\}$$

   Derive that if we compute the ML probabilities for each model and then choose for the model with the largest ML probability we will never choose for $\mathcal{M}_0$ even if x^n was actually generated by $\mathcal{M}_0$.

2. Consider this following 1^{th}-order binary Markov source. Next to the arrow from state a to state b is written $x; \Pr\{X_i = x, S_i = b | S_{i-1} = a\}$.



(a) Determine the probability $\Pr\{X_i = 1\}$.
Hint: Compute the stationary state distribution and then marginalize $\Pr\{X_i = 1, S_{i-1} = s\}$ to obtain $\Pr\{X_i = 1\}$.

(b) Consider an "ideal" universal datacompression algorithm and we observe a sequence x^n that is typical for the source. How large must i be approximately to select the first order Markov model in stead of the memoryless model.

Hard: can you determine the number of suffix trees with maximal depth not more than D for $D = 0, 1, 2, \ldots, 10$?