Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Receding horizon cost optimization for overly constrained nonlinear plants

David Angeli , Rishi Amrit and James B. Rawlings

***Abstract*— A receding horizon control algorithm, originally proposed for tracking best-possible steady-states in the presence of overly stringent state and/or input constraints, is analyzed for the case of nonlinear plant models and possibly non-convex cost functionals. Unlike the linear case (with convex cost functionals), convergence to equilibrium is not always possible and only average performance bounds are guaranteed in general.**

I. Introduction

Current practice in industrial process control systems is to decompose a plant's economic optimization into two levels [2]. The first performs a steady-state optimization and determines the economically optimal plant operating conditions (setpoints). This level is usually referred to as real-time optimization or RTO (see [5] and references therein). The setpoints are then passed on to the second level, the advanced control system, which performs a dynamic optimization. Many advanced process control systems use some form of model predictive control (MPC) for this layer. This paper is concerned with the question of how to use receding horizon control in order to directly optimize process performance. Such an attempt leads to the definition of stage costs which are not necessarily minimal (or zero) at the best possible equilibrium, and as a consequence, to the consideration of unbounded cost functionals (when defined over an infinite horizon) and of average performance indexes. As clarified later, from a purely mathematical point of view, this is reflected in the fact that our cost functional need not be minimal at the best equilibrium point. This is in contrast to standard MPC where, typically, one artificially introduces a designed cost functional which has a global minimum at the desired equilibrium, for instance it is 0 at 0, obviously trading off cost optimality for asymptotic tracking performance. It was recently suggested in [6, 5] that MPC could be directly used to optimize steady-state performance by imposing a terminal constraint at the best feasible equilibrium, with the understanding that such equilibrium need not coincide with the global minimum of the cost functional. For linear systems a convexity-based analysis implies nice convergence properties of the proposed scheme. To date, however, there is no Lyapunov-based interpretation of the above results so that extension to the case of nonlinear systems appears far from trivial. The current note addresses the issue of identifying tools to allow some analysis of this algorithm in the presence of nonlinear plant models and possibly non convex cost functionals. Rather interestingly, convergence to steady-state cannot be guaranteed and potentially need not hold, whereas an asymptotic average performance which is at least as good as that of the best feasible equilibrium results.

II. Problem formulation and basic result

Let us consider nonlinear discrete time systems of the following kind:

$$x(t+1) = f(x(t), u(t)) \tag{1}$$

with $f : X \times U \to X$ a continuous function and $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$ arbitrary closed sets. Together with system (1) we consider a cost functional, $L(x, u)$, that is a continuous function $L : X \times U \to \mathbb{R}_{\geq 0}$. Our goal is to design a receding horizon control strategy which optimizes the cost $L(x, u)$ is some average sense to be defined later. Assume that the system's state is constrained within some set $\mathbb{X} \subseteq X$ and, similarly, the input signal should take value in $\mathbb{U} \subseteq U$ (and let us assume such sets to be compact for the sake of simplicity).

Definition 2.1: Any pair $(x^\star, u^\star) \in \mathbb{X} \times \mathbb{U}$ satisfying

$$L(x^\star, u^\star) = \min \left\{ L(x, u) : x = f(x, u),\ x \in \mathbb{X},\ u \in \mathbb{U} \right\} \tag{2}$$

is an optimal (admissible) steady-state/input pair. □

As no convexity nor linearity conditions are assumed, optimal steady-state/input pairs need not be unique. Nevertheless, the algorithm originally devised in [6], can still be applied by arbitrarily selecting one of them as the terminal constraint.

Let us next describe in detail the receding horizon control algorithm which is implemented in order to optimize L. For given $N > 1$, let the N-steps ahead cost function be defined:

$$J(x(0), u(0), u(1), \ldots, u(N-1)) = \textstyle\sum_{t=0}^{N-1} L(x(t), u(t)) \quad \text{subject to (1)} \tag{3}$$

To it we associate the set of *optimal virtual control sequences* $\mathcal{U}^\star_{x(0)}$, defined as:

$$\mathcal{U}^\star_{x(0)} = \left\{ [u^\star(0), \ldots, u^\star(N-1)] \in U^N : \begin{array}{rl} & J(x(0), u^\star(0), \ldots, u^\star(N-1)) \\ = & \min J(x(0), u(0), \ldots, u(N-1)) \\ & \text{subject to } x(N) = x^\star \\ & (x(1), \ldots, x(N-1)) \in \mathbb{X}^{N-1} \\ & (u(0), \ldots, u(N-1)) \in \mathbb{U}^N \end{array} \right\}$$

A closed-loop feedback system is obtained simply by letting $k : X \to U$ be the map that associates to each $x(0)$ the corresponding value of $u^\star(0)$ (that is the first move of the optimizing virtual control sequence). As the optimal virtual control sequence need not be unique, also k need not be uniquely defined (nor continuous).

As expected, we end up considering the following closed-loop system:

$$x(t+1) = f(x(t), k(x(t))). \tag{4}$$

Our main result for this Section is the following:

Theorem 1: Let $x(0) \in \mathbb{X}$ be a feasible initial condition, viz. such that for at least one admissible control sequence, the state is steered to $x^\star$ at time N without leaving $\mathbb{X}$. Then system (4) has an average performance which is no worse than that of the best admissible steady state. ∎

Proof: Pick $x(0)$, an arbitrary feasible initial state. By assumption, the set:

$$\{[u(0), \dots, u(N-1)] \in U^N : x(N) = x^\star\} \tag{5}$$

is non-empty at time 0. Moreover, it is easily seen that if $[u(0), \dots, u(N-1)]$ is a feasible virtual control at time t and $u(0)$ gets applied, then the sequence $[u(1), \dots, u(N-1), u^\star]$ is also feasible at time $t+1$ so that, by induction, feasibility of our optimization problem follows for all non-negative times. Hence, solutions are globally defined for $t \in \mathbb{N}$. Let $V(x(0)) := J(x(0), u^\star(0), \dots, u^\star(N))$. Then, along solutions of (4) we have:

$$V(x(t+1)) - V(x(t)) \leq L(x^\star, u^\star) - L(x(t), u(t)) \tag{6}$$

Taking averages in both sides of (6) yields:

$$\begin{array}{rl} & \liminf\limits_{T\to+\infty} \dfrac{\sum_{t=0}^{T} V(x(t+1)) - V(x(t))}{T+1} \\ \leq & \liminf\limits_{T\to+\infty} \dfrac{\sum_{t=0}^{T} L(x^\star, u^\star) - L(x(t), u(t))}{T+1} \\ = & L(x^\star, u^\star) - \limsup\limits_{T\to+\infty} \dfrac{\sum_{t=0}^{T} L(x(t), u(t))}{T+1} \end{array} \tag{7}$$

On the other hand, assuming without loss of generality $L(x,u) \geq 0$ for all $(x,u) \in \mathbb{X} \times \mathbb{U}$:

$$\begin{array}{rl} & \liminf\limits_{T\to+\infty} \dfrac{\sum_{t=0}^{T} V(x(t+1)) - V(x(t))}{T+1} \\ = & \liminf\limits_{T\to+\infty} \dfrac{V(x(T+1)) - V(x(0))}{T+1} \\ \geq & \liminf\limits_{T\to+\infty} -\dfrac{V(x(0))}{T+1} = 0 \end{array} \tag{8}$$

Combining (7) and (8) we have shown:

$$\limsup_{T\to+\infty} \frac{\sum_{t=0}^{T} L(x(t), u(t))}{T+1} \leq L(x^\star, u^\star)$$

as claimed. ∎

Remark 2.2: It is well-known in the MPC literature that terminal equality constraint MPC is not competitive with terminal penalty MPC, in several respects [3]. See [5] for an adaptation of the previous algorithm to replace the terminal constraint by a suitable penalty function on the final reached state. □

III. Periodic terminal constraint

We show below that the scheme presented in the previous Section can be modified in order to possibly improve its average performance by taking into account periodic terminal constraints. Let $x_i^\star$ and $u_i^\star$, with $i = 1 \dots Q$ for some $Q \in \mathbb{N}$, denote the optimal (again possibly non-unique) solution of the following minimization problem:

$$\begin{array}{ll} \min\limits_{\substack{x_1, x_2, \dots, x_Q \in \mathbb{X}^Q \\ u_1, u_2, \dots, u_Q \in \mathbb{U}^Q}} & \sum_{i=1}^{Q} L(x_i, u_i) \\ \text{subject to} & \left\{ \begin{array}{rcl} x_2 & = & f(x_1, u_1) \\ \vdots & & \vdots \\ x_{i+1} & = & f(x_i, u_i) \\ \vdots & & \vdots \\ x_1 & = & f(x_Q, u_Q) \end{array} \right. \end{array} \tag{9}$$

Remark 3.1: Thanks to the *cyclic* constraints in (9), $x_i^\star$,$u_i^\star$, are the best feasible periodic solution of period Q. Of course, as a special case, for $x_1 = x_2 = \dots = x_Q$ and $u_1 = u_2 = \dots = u_Q$, we end up dealing with the best feasible steady-state, so that in general, the best period Q solution is not worse than the best admissible steady-state. Analogously, whenever $Q_1 = kQ_2$ for some integer k, the optimal solution of period Q_1 is not worse than the optimal solution of period Q_2. □

A periodic control law is designed according to the following definition:

$$k(t,x) = k_{(t \bmod Q)+1}(x) \tag{10}$$

where $k_i(x)$, $i = 1 \dots Q$ are the feedback laws obtained by minimizing the cost functional $J(x(0), u(0), \dots, u(N-1))$ subject to the terminal constraint $x(N) = x_i^\star$; viz, we may assume

$$\begin{array}{rl} & J(x(0), u_i^\star(0), \dots, u_i^\star(N-1)) \\ = & \min J(x(0), u(0), \dots, u(N-1)) \\ & \text{subject to } x(N) = x_i^\star \\ & (x(1), \dots, x(N-1)) \in \mathbb{X}^{N-1} \\ & (u(0), \dots, u(N-1)) \in \mathbb{U}^N \end{array} \tag{11}$$

and letting, as usual in a receding horizon strategy, $k_i(x(0)) = u_i^\star(0)$. If more than one minimizer of (9) exists, we must have care of selecting the values of the terminal constraints in (11), as being part of the same minimizer. Let us consider the closed-loop system obtained by letting $u(t) = k(t, x(t))$, viz.:

$$x(t+1) = f(x(t), k(t, x(t))) \tag{12}$$

We are now ready to state our main result for this Section.

Theorem 2: The closed-loop system (12) has an average performance that is not worse than the best admissible Q periodic solution. ∎

Proof: Let $V_i(x)$ denote the optimal cost relative to the i-th terminal constraint, viz.:

$$\begin{array}{rcl} V_i(x(0)) & = & \min J(x(0), u(0), u(1), \dots, u(N-1)) \\ & & \text{subject to } x(N) = x_i^\star \end{array} \tag{13}$$

Then, we may define the periodic Lyapunov-like function $V(t,x) := V_{t \bmod Q+1}(x)$, and evaluate its increments along solutions of (12). Indeed,

$$\begin{array}{rl} & V(t+1, f(x(t), u(t))) - V(t, x(t)) \\ \leq & L(x^\star_{t \bmod Q+1}, u^\star_{t \bmod Q+1}) - L(x(t), u(t)) \end{array} \tag{14}$$

Taking asymptotic averages of (14) and denoting by $\tau(t) := t \bmod Q+1$ yields:

$$\begin{aligned} 0 &= \liminf_{T\to+\infty} \frac{V(T+1,x(T+1)) - V(0,x(0))}{T+1} \\ &= \liminf_{T\to+\infty} \frac{\sum_{t=0}^{T} V(t+1, f(x(t),u(t))) - V(t,x(t))}{T+1} \\ &\leq \liminf_{T\to+\infty} \frac{\sum_{t=0}^{T} L(x^\star_{\tau(t)}, u^\star_{\tau(t)}) - L(x(t),u(t))}{T+1} \\ &= \frac{\sum_{i=1}^{Q} L(x_i^\star, u_i^\star)}{Q} - \limsup_{T\to+\infty} \frac{\sum_{t=0}^{T} L(x(t),u(t))}{T+1} \end{aligned}$$

The last equality, in particular, follows by taking the following elementary steps:

$$\begin{aligned} &\lim_{T\to+\infty} \frac{\sum_{t=0}^{T} L(x^\star_{\tau(t)}, u^\star_{\tau(t)})}{T+1} \\ =& \lim_{T\to+\infty} \frac{\left(\sum_{k=0}^{\lfloor T/Q \rfloor} \sum_{\tau=1}^{Q} L(x_\tau^\star, u_\tau^\star)\right)}{T+1} \\ &\qquad + \frac{\sum_{\theta=1}^{T \bmod Q} L(x_\theta^\star, u_\theta^\star)}{T+1} \\ =& \lim_{T\to+\infty} \frac{\lfloor T/Q \rfloor \; \sum_{\tau=1}^{Q} L(x_\tau^\star, u_\tau^\star)}{T+1} \\ &\qquad + \frac{\sum_{\theta=1}^{T \bmod Q} L(x_\theta^\star, u_\theta^\star)}{T+1} \\ =& \lim_{T\to+\infty} \frac{(T - T \bmod Q) \; \sum_{\tau=1}^{Q} L(x_\tau^\star, u_\tau^\star)}{Q(T+1)} \\ =& \frac{\sum_{\tau=1}^{Q} L(x_\tau^\star, u_\tau^\star)}{Q} \end{aligned}$$

This concludes the proof of Theorem 2. ■

Remark 3.2: It is worth pointing out that the construction preserves its validity also in the presence of non-decoupled state and input constraints, viz: $(x(t),u(t)) \in \mathbb{E} \subset X \times U$. This allows, for instance, to enforce constraints on the increments of x, for instance: $|x(t+1)-x(t)| \leq \Delta$, which indeed reads $|f(x(t),u(t)) - x(t)| \leq \Delta$. This might be especially useful in the case of the periodic algorithm, if we know that high-frequency ripples are not recommended for the system. We can in this way design slowly-varying optimal trajectories which are better than the best-feasible steady-state. □

IV. Handling constraints on average quantities

Shifting the focus from convergence to a desirable steady-state to average performance leads naturally to the consideration of constraints on average values of variables (typically inputs and states), rather than pointwise in time constraints as discussed in the previous Sections and customary in standard MPC. We discuss in the following an adaptation of previous control schemes which, together with a guaranteed average cost, also ensures satisfaction of asymptotic constraints on average quantities. To this end, for any given vector valued bounded signal $v : \mathbb{N} \to \mathbb{R}^{n_v}$, we define the set of asymptotic averages:

$$\mathrm{Av}[v] = \left\{ \lim_{n\to+\infty} \frac{\sum_{t=0}^{t_n} v(t)}{t_n+1}, \; t_n \to +\infty \right\}$$

Notice that, $\mathrm{Av}[v]$ is always non-empty. It need not be a singleton, though, as there may be more than one asymptotic average for each given signal. Also, it is straightforward to verify that, whenever $w(t) = v(t+N)$ for some finite $N \in \mathbb{N}$, we have: $\mathrm{Av}[w,v] \subset \Delta := \{[v_1,v_2] \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_v} : v_1 = v_2\}$. In particular then, $\mathrm{Av}[v] = \mathrm{Av}[w]$. The notation adopted, which does not emphasize time-dependence, is indeed consistent with the above shift-invariance and does not create misunderstandings. It is worth pointing out that the above construction leads to tighter asymptotic averages than those obtained by taking componentwise averages of vector-signals and, for technical reasons which will be clearer later, it appears much more natural in our context.

Consider next $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$, closed sets, satisfying the following nestedness conditions:

$$x^\star \in \bar{\mathbb{X}} \subset \mathbb{X} \qquad u^\star \in \bar{\mathbb{U}} \subset \mathbb{U}. \tag{15}$$

Our goal is to design a receding horizon control strategy which will ensure the following set of constraints:

$$\begin{array}{rcl} \mathrm{Av}[L(x,u)] & \subset & (-\infty, L(x^\star,u^\star)] \\ x(t) & \in & \mathbb{X} \quad \forall t \\ u(t) & \in & \mathbb{U} \quad \forall t \\ \mathrm{Av}[x] & \subset & \bar{\mathbb{X}} \\ \mathrm{Av}[u] & \subset & \bar{\mathbb{U}}. \end{array} \tag{16}$$

The notation introduced so far is not sufficient to treat rigorously the control scheme which we are going to discuss next. Indeed, we need to explicitly make a distinction between *virtual* variables, that is variables which belong to the future and are predicted at time t (these will be denoted by $v(t+k|t)$, and actual variables, that is variables which correspond to the true behavior of the closed loop system. At each time t we solve the following optimization problem:

$$\min_{\substack{u(t|t),\ldots,u(t+N-1|t) \\ x(t|t),\ldots,x(t+N-1|t)}} \sum_{k=0}^{N-1} L(x(t+k|t), u(t+k|t)) \tag{17}$$

subject to the following set of constraints:

$$\begin{array}{rcll} x(t|t) & = & x(t) & \\ x(t+k+1|t) & = & f(x(t+k|t), u(t+k|t)) & \\ & & k \in \{0,1,\ldots,N-1\} & \\ x(t+N|t) & = & x^\star & \\ x(t+k|t) & \in & \mathbb{X} & \forall\, k \in \{0,1,\ldots,N-1\} \\ u(t+k|t) & \in & \mathbb{U} & \forall\, k \in \{0,1,\ldots,N-1\} \\ \sum_{i=0}^{t+N-1} x(i|t) & \in & \mathbb{X}_0 \oplus (t+N)\bar{\mathbb{X}} & \\ \sum_{i=0}^{t+N-1} u(i|t) & \in & \mathbb{U}_0 \oplus (t+N)\bar{\mathbb{U}} & \end{array} \tag{18}$$

where $\mathbb{X}_0$ and $\mathbb{U}_0$ are arbitrary compact sets containing the origin, $\oplus$ denotes the sum of sets, and we adopted the convention that $v(i|t) = v(i)$ for all $i \leq t$, $v = u, x$. The main result for this Section is stated below.

Theorem 3: Let $u^\star(t+k|t)$ and $x^\star(t+k|t)$ ($k = 0, 1, \dots, N-1$) be any minimizer of (17) subject to (18). Consider the closed-loop system obtained by letting $u(t) = u^\star(t|t)$ at each time step, that is:

$$x(t+1) = f(x(t), u^\star(t|t)), \tag{19}$$

then, provided $x(0)$ is a feasible initial condition, feasibility is ensured for all subsequent times and (16) holds for the actual variables $x(t)$ and $u(t)$. ∎

Proof: The proof can be divided in 4 steps, some of which closely follow the arguments used in Section II.

1) *Feasibility*: The proof is by induction, showing that feasibility at time t implies feasibility at time $t+1$. Let $x(t)$ be such that there exist $u(t+k|t)$ and $x(t+k|t)$ for $k = 0, \dots, N-1$ fulfilling (18). We claim that, $x(t+1) = f(x(t), u(t|t))$ is again a feasible state (obviously letting $u(t) = u(t|t)$). The proof of the claim is constructive. In particular, let $u(t+k|t+1) = u(t+k|t)$ for $k = 1, \dots N-1$ and $u(t+N|t+1) = u^\star$. As a consequence, $x(t+k|t+1) = x(t+k|t)$ for all $k = 1, \dots, N$; moreover

$$x(t+1+N|t+1) = f(x(t+N|t+1), u(t+N|t+1))$$

$$= f(x(t+N|t), u^\star) = f(x^\star, u^\star) = x^\star$$

This shows that point-wise in time constraints on x and u are indeed feasible and that the terminal constraint on x can also be fulfilled at time $t+1$.

We are left to show feasibility of integral constraints. Let us deal with u first. By the induction hypothesis we know that:

$$\sum_{i=0}^{t+N-1} u(i|t) \in \mathbb{U}_0 \oplus (t+N)\bar{\mathbb{U}}$$

At time $t+1$ we have again:

$$\begin{aligned} & \textstyle\sum_{i=0}^{t+N} u(i|t+1) = u^\star + \sum_{i=0}^{t+N-1} u(i|t+1) \\ = \quad & u^\star + \textstyle\sum_{i=0}^{t+N-1} u(i|t) \in \mathbb{U}_0 \oplus (t+N+1)\bar{\mathbb{U}} \end{aligned}$$

where the last inclusion follows because $u^\star \in \bar{\mathbb{U}}$. A similar argument can be used to show feasibility of the integral constraint on x.

2) *Average performance*: The proof of the inclusion $\text{Av}[L(x,u)] \in (-\infty, L(x^\star, u^\star)]$ can be performed exactly along the lines of Section II, once feasibility of the shifted virtual control sequence concatenated to $u^\star$ is proved.

3) *Average constraints*: We show next that $\text{Av}[u] \subset \bar{\mathbb{U}}$ and $\text{Av}[x] \subset \bar{\mathbb{X}}$. Indeed, by construction,

$$\sum_{i=0}^{t+N-1} u(i|t) \in \mathbb{U}_0 \oplus (t+N)\bar{\mathbb{U}}$$

for all $t \in \mathbb{N}$. Moreover,

$$\sum_{i=0}^{t+N-1} u(i|t) = \sum_{i=0}^{t} u(i) + \sum_{i=t+1}^{t+N-1} u(i|t)$$

holds for all $t \in \mathbb{N}$. Let us divide both sides of the previous inclusion by $t+1$; this yields:

$$\frac{\sum_{i=0}^{t} u(i)}{t+1} + \frac{\sum_{i=t+1}^{t+N-1} u(i|t)}{t+1} = \frac{\sum_{i=0}^{t+N-1} u(i|t)}{t+1}$$

Notice that the second term in the left-hand side of the previous equation only involves a finite number of addenda, irrespectively of t, and each one of them can be bounded by a quantity independent of t, thanks to compactness of $\mathbb{U}$. Hence, by letting t grow to infinity along any subsequence t_n such that $\sum_{i=0}^{t_n} u(i)/(t_n+1)$ admits a limit we achieve:

$$\begin{aligned} \lim_{n \to +\infty} \frac{\sum_{i=0}^{t_n} u(i)}{t_n+1} \quad &= \quad \lim_{n \to +\infty} \frac{\sum_{i=0}^{t_n+N-1} u(i|t_n)}{t_n+1} \\ &\in \quad \lim_{n \to +\infty} \frac{\mathbb{U}_0 \oplus (t_n+N)\bar{\mathbb{U}}}{t_n+1} = \bar{\mathbb{U}} \end{aligned}$$

This shows indeed that: $\text{Av}[u] \subset \bar{\mathbb{U}}$.

∎

V. On systems which are optimal at steady state

Let us consider the problem of how to define and identify systems which are best operated at steady state.

Definition 5.1: We say that a system $x(t+1) = f(x(t), u(t))$ is optimally operated at steady-state with respect to the cost functional $L(x,u)$, if for any bounded input $u(t) \in \mathbb{U}$ and any $x(0) \in \mathbb{X}$, such that $x(t) \in \mathbb{X}$ for all $t \in \mathbb{N}$, it holds:

$$\text{Av}[L(x,u)] \subset [L(x^\star, u^\star), +\infty)$$

where $x^\star$ is the best admissible steady state defined in (2). If, in addition, for each solution (corresponding to a bounded input) at least one of the conditions below holds:

1) $\text{Av}[L(x,u)] \subset (L(x^\star, u^\star, +\infty)$
2) $\liminf_{t \to \infty} |x(t) - x^\star| = 0$

we say that the system is sub-optimally operated off steady-state. □

Characterizing such a property for general nonlinear systems and non-convex cost functions appears to be a rather formidable task. Some techniques for claiming optimality at steady-state in certain classes of nonlinear systems and linear cost functionals are currently under investigation. For this reason we will focus in the following on linear systems, which appear to be amenable to explicit analysis. The case of linear systems is of course especially interesting. As intuition suggests, the best performance is always achieved at steady-state.

Theorem 4: Linear systems are *optimally operated at steady state* with respect to arbitrary convex cost functionals. Moreover, for strictly convex cost functionals, they are sub-optimally operated off steady-state. ∎

Proof: It is a straightforward consequence of convexity, that for any signal $v(t)$, and any convex function $L(v)$, the following holds:

$$\forall \bar{L} \in \text{ Av } [L(v)],\, \exists\, \bar{v} \in \text{Av}[v] : \bar{L} \geq L(\bar{v}). \tag{20}$$

Consider now, an arbitrary bounded state-input pair, $(x(t), u(t))$, satisfying:

$$x(t+1) = Ax(t) + Bu(t) \tag{21}$$

for real matrices A and B of appropriate dimensions. It holds:

$$\text{Av}[x] = [A, B]\text{Av}\begin{bmatrix} x \\ u \end{bmatrix}. \tag{22}$$

Equation (22), shows that every asymptotic average of a bounded state-input pair, also fulfills the equation which characterizes equilibria and their corresponding input value. This property will be crucial in establishing the desired optimality. Indeed, by virtue of (20), for all $\bar{L} \in \text{Av}[L(x,u)]$, there exists $(\bar{x}, \bar{u}) \in \text{Av}\begin{bmatrix} x \\ u \end{bmatrix}$, such that:

$$\bar{L} \geq L(\bar{x}, \bar{u}) \tag{23}$$

However, by virtue of (22), $L(\bar{x}, \bar{u}) \geq L(x^\star, u^\star)$ which completes the proof of optimality at steady state. Next, consider the case of a strictly convex $L(x,u)$, and assume that $\liminf_{t \to +\infty} |x(t) - x^\star| > 0$. Then, for some $\bar{\varepsilon} > 0$ and some $T > 0$, it holds

$$|x(t) - x^\star| \geq \bar{\varepsilon} \qquad \forall\, t \geq T. \tag{24}$$

Since time-averages are shift-invariant, we may assume $T = 0$ without loss of generality. Moreover, by strict convexity of L, there exists a $\mathcal{K}_\infty$ function α, such that $L(x,u) \geq L(x^\star, u^\star) + \alpha(|x - x^\star| + |u - u^\star|)$ for all x and u. Hence, exploiting monotonicity of averages: for all $\bar{L} \in \text{Av}[L(x,u)]$ it holds $\bar{L} \geq L(x^\star, u^\star) + \alpha(\bar{\varepsilon})$, so that in the end $\bar{L} > L(x^\star, u^\star)$. This shows that indeed linear systems with strictly convex cost functionals are sub-optimally operated off steady-state. ■

Remark 5.2: It is worth pointing out that for systems which are sub-optimally operated off steady-state the algorithm discussed in Section 2 provides convergence in the $\liminf$ sense to the best admissible equilibrium $x^\star$. Combined with local asymptotic stability this finally yields asymptotic stability of $x^\star$. For instance, linearization-based arguments can be employed in order to claim local stability on the basis of the main result in [6]. Typically, for f and L sufficiently smooth (of class $\mathcal{C}^1$ and $\mathcal{C}^2$ respectively) and admitting a controllable linearized model at $x^\star$, the resulting closed-loop system is also linearizable and locally asymptotically stable. □

VI. Example: Asymmetric temperature control

Consider a room heating system in which the temperature of the room is controlled by venting hot air in the room. Consider the following heat balance

$$\rho C_p V \frac{dT}{dt} = Q_{in} - hA(T - T_a)$$

where ρ and C_p are the density and heat capacity of air in the room of volume V. hA is the resistance to heat transfer from inside the room to outside. The heat loss term describes the heat lost due to colder temperature (T_a) outside. Q_{in} accounts for the heat gained from the hot air coming in. Assuming that heat input (Q_{in}) is a linear function of position of the valve controlling the hot air flow into the room, we have a linear system with temperature of the room as the state and outside temperature as disturbance. Hence the discrete time system can be modeled as:

$$x(t+1) = Ax(t) + Bu(t) + B_d d(t)$$

The room temperature is required to be maintained at a value set by the user, using least amount of input possible (least cost). We define deviation variables for state and disturbance with respect to the desired temperature value and hence the system is required to settle at $x_s = 0$. Consider the following system

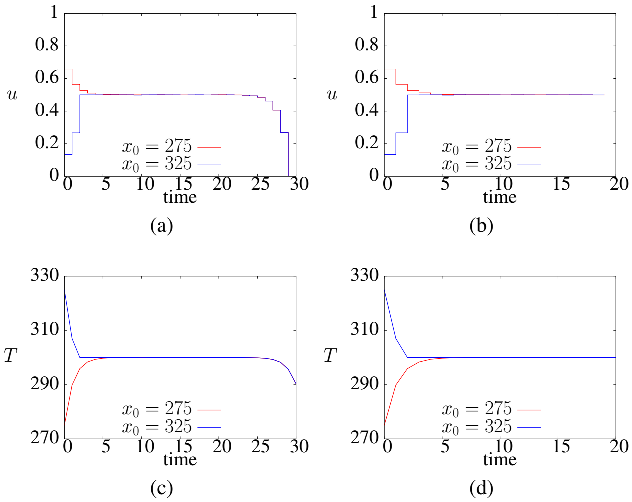
$$x(t+1) = 0.5x(t) + 15u(t) - 7.5$$

where the input $u(t)$ varies between 0 and 1 signifying a fully closed or open valve respectively. For this system, the steady state state input corresponding to $x_s = 0$ is $u_s = 0.5$. For minimizing the cost, which is proportional to the input, the ideal choice for a convex cost function is $L = u^2$, which subject to the model, has an optimal steady state at $x = -15$. The system being linear, when subject to a convex cost, is expected to settle at the optimal steady state according Theorem 4. It was shown in [5] that given a long enough horizon, for systems with inconsistent set points, the optimal open loop dynamic solution exhibits turnpike [1, 4] and the system settles at the optimal steady state in the closed loop sense. Since our desired final steady state is $x_s = 0$, we chose the following piecewise quadratic cost function instead.

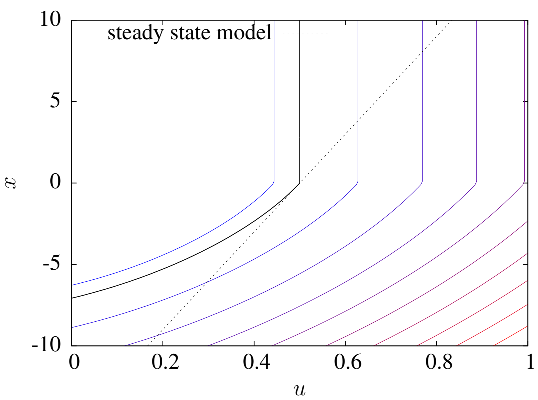
$$L(x,u) = \begin{cases} 50u^2 + \delta x^2 & x \geq 0 \\ 0.25x^2 + 50u^2 + 3.34xu & x < 0 \end{cases}$$

Here a small penalty for x^2 is added to the cost to make the cost strictly convex. Hence δ is chosen to be a very small number. This is a continuous and a convex function since the Hessian is positive definite for all feasible states and inputs. Figure 2 shows the cost contours, and the steady state line of the system. The optimal steady state of this cost, subject to the model, is $x_s = 0$, and hence we expect the system to settle there.

Figure 1 shows the open loop and closed loop input and state profiles for two different starting temperatures. The required settling temperature is 300K. We see that when we solve the open-loop dynamic control problem, the system moves from the initial state to the vicinity of the optimal steady state, stays in the vicinity of the steady state and then makes a transient away in the end to make more profit. The solution when implemented in closed loop sense, settles to the optimal steady state. When the system starts at a higher temperature than the set value, the controller implements control moves which are small in magnitude to save cost. On the other hand, when the system starts at a lower temperature,



1: Open loop **??**, **??** and closed loop **??**, **??** input and state profiles for two different initial conditions



2: Contours of piecewise quadratic stage cost

the magnitude of input does not go very high, and hence the system approaches the steady state at a slower rate, giving an economically good performance.

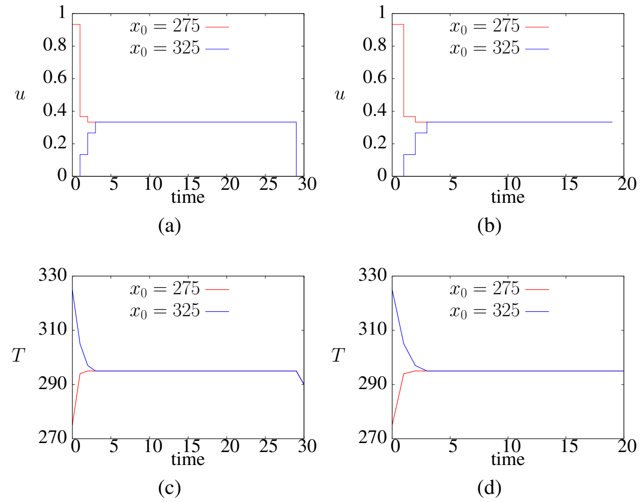
An alternate approach to this problem is to relax the requirement of settling at a user set temperature. Instead, a lower bound on the state is imposed as a soft constraint [7] which ensures feasibility of states below the lower bound. We use the following cost function

$$L(x, u, \epsilon_l.\epsilon_u) = 50u^2 + 5\epsilon_l + 5\epsilon_u$$

Here ϵ_l, ϵ_u are slack variables in the following soft constraints.

$$x \geq -5 - \epsilon_l \qquad x \leq 5 + \epsilon_u \qquad \epsilon_l, \epsilon_u \geq 0$$

The slack variables are augmented with the decision variables in the problem. Now the state is constrained to lie between -5 and 5, but the constraint is imposed as a soft constraint to ensure feasibility of all states. Since we are considering the heating problem, the system is expected to settle at the lower temperature bound.



3: Open loop (a), (c) and closed loop (b), (d) input and state profiles for two different initial conditions

As in the previous case, we add a small penalty for x^2 in the cost to make the cost strictly convex. Figure 3 shows open loop and closed loop profiles for different starting temperatures. The open loop profiles show the turnpike property because of inconsistent setpoints. The closed loop profile settles to the temperature lower bound (which is the optimal steady state).

VII. Acknowledgment

This material is based upon work partially supported by the National Science Foundation under Grant No. 0825306.

References

[1] D. A. Carlson, A. B. Haurie, and A. Leizarowitz. *Infinite Horizon Optimal Control*. Springer Verlag, second edition, 1991.

[2] T. E. Marlin and A. N. Hrymak. Real-time operations optimization of continuous processes. In J. C. Kantor, C. E. García, and B. Carnahan, editors, *Chemical Process Control–V*, pages 156–164. CACHE, AIChE, 1997.

[3] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[4] L. W. McKenzie. Turnpike theory. *Econometrica*, 44(5): 841–865, September 1976.

[5] J. B. Rawlings and R. Amrit. Optimizing process economic performance using model predictive control. In *Proceedings of the International Workshop on Assessment and Future Directions of NMPC*, Pavia, Italy, September 2008.

[6] J. B. Rawlings, D. Bonné, J. B. Jørgensen, A. N. Venkat, and S. B. Jørgensen. Unreachable setpoints in model predictive control. *IEEE Trans. Auto. Cont.*, 53(9):2209–2215, October 2008.

[7] P. O. M. Scokaert and J. B. Rawlings. Feasibility issues in linear model predictive control. *AIChE J.*, 45(8):1649–1659, August 1999.