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Compact 3D Grid Drawings of Trees

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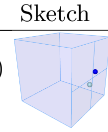
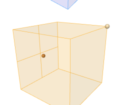
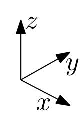
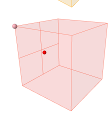
Abstract. We show that perfect binary trees with $n-1$ vertices can be optimally embedded in 3D, that is, they admit a straight-line drawing on a $\sqrt[3]{n}$ by $\sqrt[3]{n}$ by $\sqrt[3]{n}$ grid without intersecting edges. To show it, we adapt a recursive approach used in 2D by Akitaya *et al.* [GD'18] to construct compact embeddings of perfect binary trees on a square grid.

Recently, Dujmović *et al.* [3] showed that planar graphs admit a 3D straight-line grid drawing in linear volume. This was previously shown for planar graphs of bounded degree [2]. The bound in particular applies to perfect binary trees. In this work we show that this class of graphs in fact admits a *compact* embedding in 3D with optimal aspect ratio. In a compact embedding all points of a given grid except for maybe one are used. We follow a similar strategy to that in Akitaya *et al.* [1]. The authors recursively construct two types of compact embeddings of perfect binary trees in 2D.

Our construction in 3D uses three recursive blocks F_k, G_k, and H_k. Every block is a compact embedding of a perfect binary tree T_k of height k, where on one grid point the root of T_k is located and one grid point is empty and does not correspond to any vertex of T_k. The critical difference between the blocks is the placement of the root and empty nodes. The coordinates of these nodes in the three blocks can be found in Table 1.

The trivial blocks, illustrated in the poster, correspond to drawings of a perfect binary tree of height 2 in a $2 \times 2 \times 2$ grid. Each non-trivial block can be

Table 1. Root node and free node placement for F, G, and H.

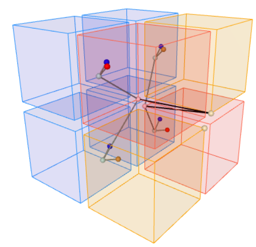
Block	Formulas	Sketch
F_k	**Root node**: $(2 \cdot 2^{(k-2)/3}, 2^{(k-2)/3}+1, 2^{(k-2)/3}+1)$ **Empty node**: $(2 \cdot 2^{(k-2)/3}, 2^{(k-2)/3}, 2^{(k-2)/3})$	
G_k	**Root node**: $(1, 2^{(k-2)/3}+1, 2^{(k-2)/3}$ **Empty node**: $(2 \cdot 2^{(k-2)/3}, 2 \cdot 2^{(k-2)/3}, 2 \cdot 2^{(k-2)/3})$	 
H_k	**Root node**: $(1, 2^{(k-2)/3}+1, 2^{(k-2)/3})$ **Empty node**: $(1, 1, 2 \cdot 2^{(k-2)/3})$	

recursively constructed from eight smaller blocks, which are then connected by non-intersecting edges to complete the drawing of the perfect binary tree. More precisely, to draw a perfect binary tree T_k with height k, the eight leaf subtrees with height $k-3$ are recursively drawn using smaller blocks, and the top three levels of T_k are carefully drawn connecting the blocks.

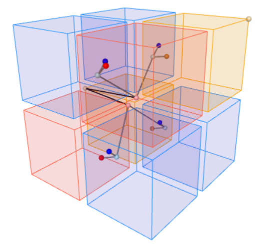
We next present the recursive definition of the three types of blocks.

The eight sub-blocks are listed from bottom to top ($-z$ to $+z$), from front to back ($-y$ to $+y$), and from left to right ($-x$ to $+x$).

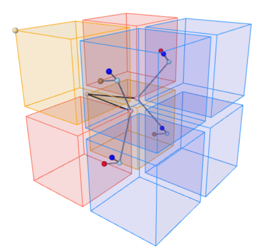
Block F_k
Sub-blocks: F_{k-3}, G_{k-3}, F_{k-3}, H_{k-3}, F_{k-3}, H_{k-3} (rotated π around the x-axis), F_{k-3}, and G_{k-3} (rotated π around the x-axis).



Block G_k
Sub-blocks: H_{k-3} (rotated π around the x-axis), F_{k-3} (rotated π around the z-axis), G_{k-3} (rotated π around the z-axis), F_{k-3} (rotated π around the z-axis), F_{k-3}, H_{k-3} (rotated π around the x-axis), F_{k-3}, G_{k-3}.



Block H_k
Sub-blocks: H_{k-3} (rotated π around the z-axis), F_{k-3} (rotated π around the z-axis), G_{k-3} (rotated π around the z-axis), F_{k-3} (rotated π around the z-axis), G_{k-3} (rotated π around the x-axis), F_{k-3} (rotated π around the x-axis), H_{k-3} (rotated π around the x-axis, rotated π around the z-axis), and F_{k-3} (rotated π around the z-axis).



The connections (top three levels of the tree) are all similar in all blocks: the third level connects the root nodes of the sub-blocks with the empty node of the same or an adjacent F sub-block; the second level connects these empty nodes with the two empty nodes of the two H sub-blocks on a different yz plane. Finally, these two nodes are connected to the empty node of an H sub-block.

All blocks define straight-line drawings without crossings. To see it is enough to focus on the edges that are not recursively defined as part of a sub-block. The projection of these edges on the yz plane defines a crossing-free drawing. Moreover, these edges either connect the root and empty nodes of an F sub-block or they connect the two sides of one of the three planes that split the sub-blocks. From these observations it follows that each of the blocks F_k, G_k, and H_k defines a crossing-free compact drawing of the perfect binary tree of height k in a cubic grid. Thus, we obtain our main result:

Theorem 1. *The perfect binary tree with height $k = 3x - 1$ for $x \in \mathbb{N}$ with $n = 2^{k+1} - 1$ vertices has a compact embedding on the $\sqrt[3]{n} \times \sqrt[3]{n} \times \sqrt[3]{n}$ grid.*

References

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