Sketch of talk

Stanley Li

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Overarching idea: give an example with $K = \mathbb{Q}(i)$, sketching the key ideas behind the proof. Run through an example of computing $\lim_{s\to 1^+}(s-1)\zeta_{\mathbb{Q}(i)}(s)$. Todo:

1. Introduce the $\zeta_{\mathbb{Q}(i)}(s) = \sum_{0\neq I\subseteq \mathbb{Z}[i]} \frac{1}{[\mathbb{Z}[i]:I]^s} = \sum_{a\geq 0, b>0} \frac{1}{(a^2+b^2)^{s/2}}$.
2. Rewrite $\zeta_{\mathbb{Q}(i)}$ in terms of point counting over a cone in $\mathbb{C}$.
3. Bring in theorem 3 (in weak generality) and sketch proof.
4. Compute the volumes v and Δ (quite straightforward with $\mathbb{Z}[i] \hookrightarrow \mathbb{Q}(i) \hookrightarrow \mathbb{C}$) to show
$$\lim_{s\to 1^+}(s-1)\zeta_{\mathbb{Q}(i)}(s) = \frac{\pi}{4}$$
5. Talk a little bit about the generalisation – residues of arbitrary Dedekind zeta functions.

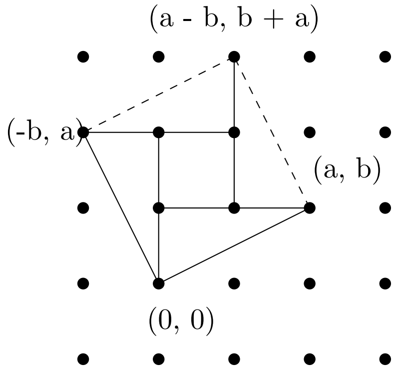
1 Introduction

For a number field K, the analytic class number formula is the residue of the Dedekind zeta function ζ_K at $s = 1$, and carries fundamental data about the number field, and is closely tied to the distribution of ideals in the associated ring of integers $\mathcal{O}_K \subseteq K$ (which plays a similar role to $\mathbb{Z} \subseteq \mathbb{Q}$). We give the idea for computing this residue in the case $K = \mathbb{Q}(i)$, and briefly discuss the generalisation to arbitrary number fields.

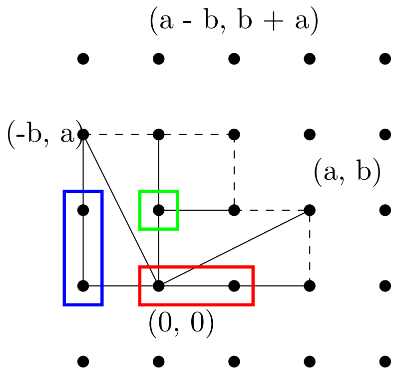
The Dedekind zeta function for $\mathbb{Q}(i)$ is defined by

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{0\neq I\subseteq \mathbb{Z}[i]} \frac{1}{[\mathbb{Z}[i]:I]^s}$$

We note the index of an ideal $I = (a+bi)$ (principal as $\mathbb{Z}[i]$ admits a Euclidean algorithm as every point in $\mathbb{C}$ is within $\sqrt{2}/2$ of one in $\mathbb{Z}[i]$) corresponds to the number of points in the square with corners $(0,0), (a,b), (a-b, b+a)$ and $(-b, a)$ (i.e. the image of the unit square $[0,1]^2$ under multiplication by $a+bi$) excluding the sides away from the origin (i.e. (a,b) to $(a-b, b+a)$ and $(a-b, b+a)$ to $(-b, a)$), which we can count by splitting the square as below (which shows the case where $a \geq b$):



where the dashed lines indicate that we do not count point on these lines (including the corners). We can move the top two rectangles down, giving



The blue and green rectangles in general each have ab points, while the red rectangle will have $|b-a|^2$ points, giving a total of $|b-a|^2 + 2ab = a^2 + b^2$ points, and so $[\mathbb{Z} : (a+bi)] = a^2 + b^2$ (here $a = 2$, $b = 1$). Our zeta function is thus

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{a \geq 0, b > 0} \frac{1}{(a^2 + b^2)^s}$$

where we have used the fact that every ideal has a unique generator in the cone $a \geq 0, b > 0$.

2 Point-counting

To show that $\zeta_{\mathbb{Q}(i)}(s)$ is defined for $s > 1$ (and hence $\text{Re}(s) > 1$) and determine $\lim_{s \to 1^+} \zeta_{\mathbb{Q}(i)}(s)$, we can now view $\zeta_{\mathbb{Q}(i)}$ in terms of point-counting, namely as the sum of norms of Gaussian integers in the upper-right quarter plane, and we look to bound these in terms of the Riemann zeta function. Let $X = \{a + bi \in \mathbb{C} \mid a \geq 0, b > 0\}$ be the corresponding upper-right quarter plane. Then

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{x \in X \cap \mathbb{Z}[i]} \frac{1}{|x|^{2s}}$$

and looking at the set $T = X \cap \overline{B_1(0)}$ (which is the upper-right quarter circle), we look to count the number of points $N(r)$ in $rT \cap \mathbb{Z}[i]$ (which is the same as that of $T \cap \frac{1}{r}\mathbb{Z}[i]$).

Considering $T \cap \frac{1}{r}\mathbb{Z}[i]$, the ratio $N(r)/r^2$ gives the proportion of points in the upper-right unit square part of the lattice $\frac{1}{r}\mathbb{Z}[i] \cap [0,1]^2$ in the set T, which will tend towards $\frac{\pi}{4}$ (the volume of the upper-right quarter-circle).

Viewing $N(r)$ instead as the number of points in $rT \cap \mathbb{Z}[i]$, ordering the points $(x_n)_{n\in\mathbb{N}}$ of $X \cap \mathbb{Z}[i]$ in increasing length, we have that $x_1, \ldots, x_n \in |x_n|T$ but $x_n \notin (|x_n| - \varepsilon)T$ for any $\varepsilon > 0$. This shows that $N(|x_n| - \varepsilon) < n \leq N(|x_n|)$, so

$$\frac{N(|x_n| - \varepsilon)}{(|x_n| - \varepsilon)^2} \frac{(|x_n| - \varepsilon)^2}{|x_n|^2} < \frac{n}{|x_n|^2} \leq \frac{N(|x_n|)}{|x_n|^2}$$

and taking $n \to \infty$ (hence $|x_n| \to \infty$) yields $\frac{n}{|x_n|^2} \to \frac{\pi}{4}$ by the squeeze theorem. This shows by the comparison test (with the Riemann zeta function) that $\zeta_{\mathbb{Q}(i)}$ converges for $\mathrm{Re}(s) > 1$. We note now that $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$ since for $s > 1$ we have

$$\frac{1}{s-1} = \int_1^\infty \frac{1}{x^s}dx < \sum_{n=1}^\infty \frac{1}{n^s} < 1 + \int_1^\infty \frac{1}{x^s}dx = \frac{s}{s-1}$$

Now for each $\varepsilon > 0$, for all sufficiently large n we find

$$\frac{1}{n}\left(\frac{\pi}{4} - \varepsilon\right) < \frac{1}{|x_n|^2} < \frac{1}{n}\left(\frac{\pi}{4} + \varepsilon\right)$$

For $s > 1$, taking s^{th} powers, summing up across sufficiently large n and multiplying by $s - 1$, we find

$$\left(\frac{\pi}{4} - \varepsilon\right)^s (s-1)\sum_{n>N} \frac{1}{n^s} < (s-1)\sum_{n>N} \frac{1}{|x_n|^{2s}} < \left(\frac{\pi}{4} + \varepsilon\right)^s (s-1)\sum_{n>N} \frac{1}{n^s}$$

As $s \to 1^+$ the heads of the corresponding sequences are negligible (as they are finite), so we find

$$\frac{\pi}{4} - \varepsilon \leq \liminf_{s\to 1^+} \sum_{x\in X\cap\mathbb{Z}[i]} \frac{1}{|x_n|^s} \leq \limsup_{s\to 1^+} \sum_{x\in X\cap\mathbb{Z}[i]} \frac{1}{|x_n|^s} \leq \frac{\pi}{4} + \varepsilon$$

That is, $\lim_{s\to 1^+} \zeta_{\mathbb{Q}(i)}(s) = \frac{\pi}{4}$.

3 Generalisation

This argument generalises to the Dedekind zeta function ζ_K of any number field, with some slight caveats. First, point-counting argument given works only for principal ideals. In a general number field, not all ideals of the ring of integers will be principal. However, up to multiplication by a principal ideal there are finitely many classes (and these ideal classes form a finite abelian group in some sense), and so we account for this by splitting the sum by "ideal class" applying the argument to each of these, multiplied by an ideal in the "inverse class", which will be a sum over principal ideals. The values end up being independent of the ideal class, so this just introduces a term corresponding to the number of ideal classes.

The exact geometric interpretation is also not so simple - in general we view a number field K by embedding it into $n = [K : \mathbb{Q}]$-dimensional space in a way which corresponds to the embeddings $K = \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ (namely $K_\mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$ where r is the number of real embeddings, s the number of complex embeddings), and the region X we count points on is such that every $x \in K_\mathbb{R} \setminus \{0\}$ splits uniquely as the product of some $x_1 \in X$ and $x_2 \in \mathcal{O}_K^*$ (under this embedding). In general the volume of $X \cap \overline{B_1(0)}$ depends on the density of the (in general infinitely many) units, and the general formula takes the form

$$\frac{2^r(2\pi)^s hR}{\omega V}$$

where V corresponds to the volume of a single grid square in viewing $\mathcal{O}_K$ as a lattice under this geometric interpretation (in our case $\mathbb{Z}[i]$ and the usual unit square of volume 1), R is a term corresponding to the density of units $\mathcal{O}_K^*$ (from X), ω is the number of roots of unity in K and h is the number of ideal classes.

4 Application or purpose

Besides being a somewhat nice formula, the analytic class number formula allows for numerical computation of the class number h (which in general is the hardest of these quantities to compute), made slightly easier by the fact that h is necessarily an integer value for each number field K. A slightly more general argument relates the distribution of ideals in $\mathcal{O}_K$ of index at most t, and by a similar point-counting argument it follows that ideals in $\mathcal{O}_K$ with index at most t are distributed according to $\rho t + O(t^{1-1/n})$, where the coefficient ρ of the principal term is exactly this formula (and as a direct consequence, the zeta function extends to $\mathrm{Re}(s) > 1 - 1/n$ except for a simple pole at $s = 1$). The analytic class number formula thus also describes the asymptotic distribution of ideals contained in $\mathcal{O}_K$.