Analytic class number formula talk

Stanley Li

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We outline the proof of the analytic class number formula and how it relates to the distribution of integral ideals in the case $K = \mathbb{Q}(i)$, and briefly state the generalisation to general finite extensions of $\mathbb{Q}$.

1 The Dedekind zeta function and some motivation

For any finite extension $K/\mathbb{Q}$ we have a generalisation of the usual Riemann zeta function: for $\mathbb{Q}$ we can write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{0 \neq I \subseteq \mathbb{Z}} \frac{1}{[\mathbb{Z} : I]^s}$$

and for any $K/\mathbb{Q}$ we have a natural analogue $\mathcal{O}_K \subseteq K$ of the integers $\mathbb{Z} \subseteq \mathbb{Q}$, the "ring of integers of K". Taking the natural generalisation of the above, the Dedekind zeta function is

$$\zeta_K(s) = \sum_{0 \neq I \subseteq \mathcal{O}_K} \frac{1}{[\mathcal{O}_K : I]^s} = \sum_{n=1}^{\infty} \frac{\#\{0 \neq I \subseteq \mathcal{O}_K \mid [\mathcal{O}_K : I] = n\}}{n^s}$$

and we can think of the index $N(I) := [\mathcal{O}_K : I]$ as the "norm" of the ideal I, which we can view as a measure of how large the generators are. We are interested in the convergence, poles and their residues of this (a priori formal) function. It turns out we have a continuation to the whole complex plane except a simple pole at $s = 1$, whose residue encodes many invariants about the field K. For the rest of the talk we will only consider the case $K = \mathbb{Q}(i)$.

When $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$ is a principal ideal domain (in effect as the unit balls centred at points in $\mathbb{Z}[i]$ cover $\mathbb{C}$), so every ideal is uniquely represented as $I = (a + bi)$ for $a \geq 0, b > 0$ (since the units of $\mathbb{Z}[i]$ are $\pm 1, \pm i$), and its index is $a^2 + b^2$ (this is just the volume of the square with $0, a + bi$ and $i(a + bi)$ as 3 of its vertices), so we have

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{a \geq 0, b > 0} \frac{1}{(a^2 + b^2)^s}$$

2 Distribution of ideals

More generally, for a sequence (a_n) with $\sum_{n=1}^{t} a_n = \rho t + O(t^\sigma)$ where $\sigma \in [0, 1)$, the associated "Dirichlet series"

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

admits a continuation to $\mathrm{Re}(s) > \sigma$ holomorphic everywhere except for a simple pole at $s = 1$ of residue ρ (effectively bootstrapped off the statement for the Riemann zeta function). For the Dedekind zeta function, a_n is the number of ideals of norm n, so we want to argue about asymptotically about the distribution of ideals of norm at most t as $t \to \infty$.

To count principal ideals in $\mathbb{Z}[i]$, we note that $(\alpha) = (\alpha')$ if and only if $\alpha/\alpha' \in \mathbb{Z}[i]^*$ (with $\mathbb{Z}[i]^* = \langle i \rangle$)[1]. The principal ideals in $\mathbb{Z}[i]$ thus correspond to the points of $\mathbb{Z}[i]$ in some suitable "multiplicative complement" S of $\mathbb{Z}[i]^*$ in $\mathbb{C}^*$, i.e. a set S so that every $x \in \mathbb{C}^*$ can be written uniquely in the form $x_1 x_2$ for $x_1 \in S$, $x_2 \in \mathbb{Z}[i]^*$.

For $\mathbb{Z}[i]$, $S = \left\{ re^{i\theta} \mid r \in (0,1], \theta \in [0, \pi/2) \right\}$ is the upper-right quarter plane excluding the positive imaginary line. Since $N((a+bi)) = \|a+bi\|^2$, the points in $S_{\leq \sqrt{t}} = \overline{B(0,\sqrt{t})} \cap S = \sqrt{t}(\overline{B(0,1)} \cap S) = \sqrt{t} S_{\leq 1}$ correspond exactly to the principal ideals of norm at most t.

We now look to count the number of points of the lattice $\Lambda = \mathbb{Z}[i]$ inside rS for a reasonably shaped set S, as $r \to \infty$. Heuristically this should be asymptotic to $\mu(S) r^2$, divided by the volume $\mathrm{covol}(\mathbb{Z}[i])$ of a unit grid square of $\mathcal{O}_K$, with an $O(r)$ error term corresponding to the boundary:

$$\#(rS \cap \Lambda) = \frac{\mu(S)}{\mathrm{covol}(\mathbb{Z}[i])} r^2 + O(r)$$

Applying this to our case, $S_{\leq 1}$ is the top-right quarter circle for $\mathbb{Z}[i]$. We thus compute

$$\# \left\{ 0 \neq (\alpha) \subseteq \mathbb{Z}[i] \mid N(\alpha) \leq t \right\} = \frac{\pi}{4} t + O(\sqrt{t})$$

and so in this case we have convergence on $\mathrm{Re}(s) > 1/2$, except for a pole at $s = 1$ with residue $\pi/4$.

3 The general case

The procedure outlined above will, in general, compute the distribution of principal ideals, and yield a value dependent on

1. The degree of $K/\mathbb{Q}$, more specifically the number of embeddings $K \hookrightarrow \mathbb{C}$ with real or complex embeddings
2. The unit group – in general this will be a product of a free abelian part and the roots of unity
3. The "size" of $\mathcal{O}_K$, usually written $\sqrt{|\Delta_K|}$

In a general extension $K/\mathbb{Q}$ there are finitely many "classes" of ideals up to principal ideals. More formally, we can extend the ideals with multiplication to a group by "allowing denominators", where the set of principal ideals "allowing denominators" is a subgroup of finite index. Given an ideal class, we can multiply by any ideal in its inverse class (to map it to the principal ideals in $\mathcal{O}_K$) and do a similar point-counting argument. The principal term in the distribution of ideals in each class is actually independent of ideal class, so this ultimately just gives us a factor corresponding to the number of ideal classes. Put together, ideals of index at most t are distributed by

$$\frac{2^r (2\pi)^s h R}{\omega \sqrt{|\Delta_K|}} t + O(t^{1-1/n})$$

where r is the number of real embeddings, s the number of complex embeddings. The 2^r, π^s correspond to the sets of norm at most 1 being the interval $[-1,1]$ and the unit disk, R is a term corresponding to the density of the free units in $\mathcal{O}_K$, and $\sqrt{|\Delta_K|}/2^s$ is the covolume of $\mathcal{O}_K$, and the error term comes comes from $(t^{1/n})^{n-1}$.

[1] Here we can check this by noting $a + bi \mapsto a^2 + b^2$ is multiplicative from $\mathbb{Z}[i] \to \mathbb{Z}$