The analytic class number formula and the distribution of ideals in a number field

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Abstract

Outline ACNF, show how distribution of ideals relates

Contents

Chapter 1

Background

Let $K/\mathbb{Q}$ be a finite extension, which we will refer to as a *number field*. To each such field we have a *ring of integers* $\mathcal{O}_K$ which plays a similar role to $\mathbb{Z} \subseteq \mathbb{Q}$, defined as the set (subring) of $x \in K$ which are roots of monic $f \in \mathbb{Z}[x]$.

Definitions: Number field [embeddings] K, ring of integers $\mathcal{O}_K$, field norm N [in terms of embeddings]

Fractional ideals, invertible ideals, class group $\mathscr{C}(K)$, class number h

Unit group $\mathcal{O}_K^*$, Dirichlet's unit theorem (roots of unity denoted μ_K of cardinality ω_K), Log, trace zero hyperplane and regulator (volume induced by taking measure corresponding to a coordinate projection); volume

The Dedekind zeta function (state both formulations)

State the theorem

1.1 Geometry

Geometric interpretation $K \hookrightarrow K_{\mathbb{R}}$ (and volume), discriminant, lattices, covolume of $\mathcal{O}_K$ and ideals

Chapter 2

The analytic class number formula

2.1 Series and continuations

Lemma 1: (a_i) with $\sum a_i = O(t^\sigma)$

Lemma 1.5: Riemann zeta admits a meromorphic continuation to $\mathrm{Re}(s) > 0$, simple pole at $s = 1$ of residue 1.

Lemma 2: (a_i) with $\sum a_i = \rho t + O(t^\sigma)$, $\sigma \in [0, 1)$.

Motivation for why this is relevant

2.2 The distribution of ideals

Motivate by arguing for sufficiently nice boundary

2.2.1 Lipschitz parametrisability

Idea: count points in image of $\mathcal{O}_K$ under Log, but need nice boundary

Lemma + corollary on computing with $(n-1)$-Lipschitz parametrisable boundary.

2.2.2 Integral ideals of bounded norm

The long computation