# Analytic class number formula talk

Stanley Li

June 8, 2023

We outline the proof of the analytic class number formula and how it relates to the distribution of integral ideals in the case $K = \mathbb{Q}(i)$, and briefly state the generalisation to general finite extensions of $\mathbb{Q}$.

## 1 The Dedekind zeta function and some motivation

For any finite extension $K/\mathbb{Q}$ we have a generalisation of the usual Riemann zeta function: for $\mathbb{Q}$ we can write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{0 \neq I \subseteq \mathbb{Z}} \frac{1}{[\mathbb{Z} : I]^s}$$

and for any $K/\mathbb{Q}$ we have a natural analogue $\mathcal{O}_K \subseteq K$ of the integers $\mathbb{Z} \subseteq \mathbb{Q}$, the "ring of integers of $K$". Taking the natural generalisation of the above, the Dedekind zeta function is

$$\zeta_K(s) = \sum_{0 \neq I \subseteq \mathcal{O}_K} \frac{1}{[\mathcal{O}_K : I]^s} = \sum_{n=1}^{\infty} \frac{\#\{0 \neq I \subseteq \mathcal{O}_K \mid [\mathcal{O}_K : I] = n\}}{n^s}$$

and we can think of the index $N(I) := [\mathcal{O}_K : I]$ as the "norm" of the ideal $I$, which we can view as a measure of how large the generators are. We are interested in the convergence, poles and their residues of this (a priori formal) function. It turns out we have a continuation to the whole complex plane except a simple pole at $s = 1$, whose residue encodes many invariants about the field $K$. For the rest of the talk we will only consider the case $K = \mathbb{Q}(i)$.

When $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$ is a principal ideal domain (in effect as the unit balls centred at points in $\mathbb{Z}[i]$ cover $\mathbb{C}$), so every ideal is uniquely represented as $I = (a + bi)$ for $a \geq 0, b > 0$ (since the units of $\mathbb{Z}[i]$ are $\pm 1, \pm i$), and its index is $a^2 + b^2$ (this is just the volume of the square with $0, a + bi$ and $i(a + bi)$ as 3 of its vertices), so we have

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{a \geq 0, b > 0} \frac{1}{(a^2 + b^2)^s}$$

## 2 Distribution of ideals

More generally, for a sequence $(a_n)$ with $\sum_{n=1}^{t} a_n = \rho t + O(t^\sigma)$ where $\sigma \in [0, 1)$, the associated "Dirichlet series"

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

admits a continuation to $\mathrm{Re}(s) > \sigma$ holomorphic everywhere except for a simple pole at $s = 1$ of residue $\rho$ (effectively bootstrapped off the statement for the Riemann zeta function). For the Dedekind zeta function, $a_n$ is the number of ideals of norm $n$, so we want to argue about asymptotically about the distribution of ideals of norm at most $t$ as $t \to \infty$.

To count principal ideals in $\mathbb{Z}[i]$, we note that $(\alpha) = (\alpha')$ if and only if $\alpha/\alpha' \in \mathbb{Z}[i]^*$ (with $\mathbb{Z}[i]^* = \langle i \rangle$)[1]. The principal ideals in $\mathbb{Z}[i]$ thus correspond to the points of $\mathbb{Z}[i]$ in some suitable "multiplicative complement" $S$ of $\mathbb{Z}[i]^*$ in $\mathbb{C}^*$, i.e. a set $S$ so that every $x \in \mathbb{C}^*$ can be written uniquely in the form $x_1 x_2$ for $x_1 \in S$, $x_2 \in \mathbb{Z}[i]^*$.

For $\mathbb{Z}[i]$, $S = \left\{ re^{i\theta} \mid r \in (0,1], \theta \in [0, \pi/2) \right\}$ is the upper-right quarter plane excluding the positive imaginary line. Since $N((a+bi)) = \|a+bi\|^2$, the points in $S_{\leq \sqrt{t}} = \overline{B(0,\sqrt{t})} \cap S = \sqrt{t}(\overline{B(0,1)} \cap S) = \sqrt{t} S_{\leq 1}$ correspond exactly to the principal ideals of norm at most $t$.

We now look to count the number of points of the lattice $\Lambda = \mathbb{Z}[i]$ inside $rS$ for a reasonably shaped set $S$, as $r \to \infty$. Heuristically this should be asymptotic to $\mu(S)r^2$, divided by the volume $\mathrm{covol}(\mathbb{Z}[i])$ of a unit grid square of $\mathcal{O}_K$, with an $O(r)$ error term corresponding to the boundary:

$$\#(rS \cap \Lambda) = \frac{\mu(S)}{\mathrm{covol}(\mathbb{Z}[i])} r^2 + O(r)$$

Applying this to our case, $S_{\leq 1}$ is the top-right quarter circle for $\mathbb{Z}[i]$. We thus compute

$$\# \left\{ 0 \neq (\alpha) \subseteq \mathbb{Z}[i] \mid N(\alpha) \leq t \right\} = \frac{\pi}{4} t + O(\sqrt{t})$$

and so in this case we have convergence on $\mathrm{Re}(s) > 1/2$, except for a pole at $s = 1$ with residue $\pi/4$.

## 3 The general case

The procedure outlined above will, in general, compute the distribution of principal ideals, and yield a value dependent on

1. The degree of $K/\mathbb{Q}$, more specifically the number of embeddings $K \hookrightarrow \mathbb{C}$ with real or complex embeddings
2. The unit group – in general this will be a product of a free abelian part and the roots of unity
3. The "size" of $\mathcal{O}_K$, usually written $\sqrt{|\Delta_K|}$

In a general extension $K/\mathbb{Q}$ there are finitely many "classes" of ideals up to principal ideals. More formally, we can extend the ideals with multiplication to a group by "allowing denominators", where the set of principal ideals "allowing denominators" is a subgroup of finite index. Given an ideal class, we can multiply by any ideal in its inverse class (to map it to the principal ideals in $\mathcal{O}_K$) and do a similar point-counting argument. The principal term in the distribution of ideals in each class is actually independent of ideal class, so this ultimately just gives us a factor corresponding to the number of ideal classes. Put together, ideals of index at most $t$ are distributed by

$$\frac{2^r (2\pi)^s h R}{\omega \sqrt{|\Delta_K|}} t + O(t^{1-1/n})$$

where $r$ is the number of real embeddings, $s$ the number of complex embeddings. The $2^r$, $\pi^s$ correspond to the sets of norm at most 1 being the interval $[-1, 1]$ and the unit disk, $R$ is a term corresponding to

[1] Here we can check this by noting $a + bi \mapsto a^2 + b^2$ is multiplicative from $\mathbb{Z}[i] \to \mathbb{Z}$

the density of the free units in $\mathcal{O}_K$, and $\sqrt{|\Delta_K|}/2^s$ is the covolume of $\mathcal{O}_K$, and the error term comes comes from $(t^{1/n})^{n-1}$.

The usefulness of this formula comes in computing the number of ideal classes – if we know all the other values (of which only $R$ may be hard to find), then by approximating the residue of the zeta function at $s = 1$, we can find the class number (noting also that it is necessarily an integer value).