The Dirac spectra of spheres

Stanley Li

November 14, 2023

Abstract

1. Background

Keys: Sphere with Riemannian metric induced by inclusion (as a hypersurface) admits a unique spin structure if $n \geq 2$. Know the Dirac operator admits an orthonormal basis of eigenvectors over compact manifolds. Use Killing spinors to trivialise spinor bundle over sphere

Notation: S_M the spinor bundle over M, $\cdot$ Clifford multiplication, D a fixed Dirac operator, $\mathrm{SO}(TM)$ the principal $\mathrm{SO}(n)$ bundle (frame bundle), $\mathrm{Spin}(TM)$ the principal $\mathrm{Spin}(n)$ bundle associated to M.

[Metric] connection on a vector bundle (compatible with inner products) $X \langle \varphi, \psi \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle$

[[Clifford multiplication on Manifold]] ??

Definition 1. *Let M^n be a Riemannian spin manifold, and ∇ its Levi-Civita connection. We say a connection ∇^S on S_M is* compatible *if*

$$\nabla^S_X (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla^S_X \varphi$$

for all $\varphi \in S_M$ and $X, Y \in TM$.

As in $Cl(\mathbb{R}^n)$, every Riemannian spin manifold admits a metric and compatible connection, and a Hermitian inner product which is compatible in the following sense (and unique up to scale):

Proposition 1. *Let M be a Riemannian spin manifold. Then*

1. *There is a Hermitian inner product $\langle -, - \rangle$ on $\mathcal{S}_M$, such that $\langle X \cdot \varphi, \psi \rangle = - \langle \varphi, X \cdot \psi \rangle$ for all $X \in TM$ and $\varphi, \psi \in \mathcal{S}_M$, and this is pointwise unique up to scale; and*
2. *There is a metric, compatible connection ∇^S on $\mathcal{S}_M$.*

We obtain this Hermitian inner product by averaging a Hermitian inner product over the subgroup on Δ_n, and such a connection by lifting locally the connection 1-form given by the Levi-Civita connection on M. Fixing a positively-oriented local orthonormal frame $\{e_j\}_{j=1}^n$, for a basis $\{\sigma_\alpha\}_{\alpha=1}^{2^{\lfloor n/2 \rfloor}}$ for Δ_n, we get a corresponding local spinorial frame $\varphi_\alpha = [(\tilde{s}, \sigma_\alpha)]$ on $\mathcal{S}_M$, for $\tilde{s}$ the preimage of $(e_1, \dots, e_n)$ under $\mathrm{Spin}(TM) \to \mathrm{SO}(TM)$. This gives the following explicit local expression for this connection:

Proposition 2. *Let (M^n, g) be a Riemannian spin manifold, $\{e_j\}_{j=1}^n$ be a positively-oriented local orthonormal frame for TM, and $\{\varphi_\alpha\}_{\alpha=1}^{2^{\lfloor n/2 \rfloor}}$ be a corresponding spinorial frame. Then*

$$\nabla^S \varphi_\alpha = \frac{1}{4} \sum_{j,k=1}^{n} g(\nabla e_j, e_k) e_j \cdot e_k \cdot \varphi_\alpha$$

A proof of this can be found in [[**put reference**]]. In particular, it follows from this that the curvature R^∇ of ∇^S can be expressed locally in terms of the curvature R of the Levi-Civita connection, as

$$R^\nabla_{X,Y} \varphi = \frac{1}{4} \sum_{j,k=1}^{n} g(R_{X,Y} e_j, e_k) e_j \cdot e_k \cdot \varphi_\alpha$$

This formula gives us the following formula, *which we will use later.*

Lemma 1. *For a Riemannian spin manifold (M^n, g) with Ricci-tensor* Ric*, $X \in TM$ and $\varphi \in \mathcal{S}_M$, we have*

$$\sum_{j=1}^{n} e_j \cdot R^\nabla_{X, e_j} \varphi = \frac{1}{2} \operatorname{Ric}(X) \cdot \varphi$$

The proof of this is just an application of the above formula, using the first Bianchi identity and the antisymmetry of the curvature tensor, and can be found in [**Lemma 1.2.4**].

2. Killing spinors

Definition 2. *Let (M^n, g) be a Riemannian spin manifold, and $\alpha \in \mathbb{C}$. An α-*Killing spinor *is a section $\psi \in \Gamma(\mathcal{S}_M)$ such that $\nabla_X \psi = \alpha X \cdot \psi$ for all $X \in TM$.*

Note that an α-Killing spinor ψ is an $(-n\alpha)$-eigenvector of any Dirac operator D, as in local coordinates we have

$$D\psi = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \psi = \sum_{j=1}^{n} e_j \cdot \alpha e_j \cdot \psi = -n\alpha\psi$$

We look to compute the Killing spinors on S^n, since this will give a convenient collection of spinors to work with. We compute this using the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$, and for this we will need the following.

Proposition 3. *Let $\tilde{M}^{n+1}$ be a Riemannian spin manifold with connection $\tilde{\nabla}$, and $\iota : M^n \to \tilde{M}^{n+1}$ be an immersed oriented Riemannian hypersurface, with connection ∇ and unit normal $\nu \in \Gamma(T^\perp M)$, so that $\{v_1, \ldots, v_n, \nu\}$ is a positively-oriented orthonormal frame whenever $\{v_1, \ldots, v_n\}$ is. Then M is spin with induced spin structure so that there is a unitary isomorphism*

$$\mathcal{S}_{\tilde{M}}|_M \cong \begin{cases} \mathcal{S}_M & n \text{ even} \\ \mathcal{S}_M \oplus \mathcal{S}_M & n \text{ odd} \end{cases}$$

given by $\varphi \mapsto \varphi$, where the copies of $\mathcal{S}_M$ correspond to the splitting $\mathcal{S}_{\tilde{M}}|_M \cong \mathcal{S}^+_{\tilde{M}}|_M \oplus \mathcal{S}^-_{\tilde{M}}|_M$ for n odd. We further have

$$X \cdot \nu \cdot \varphi = \begin{cases} X \cdot_M \varphi & n \text{ even} \\ X \cdot_M \varphi^+ - X \cdot_M \varphi^- & n \text{ odd} \end{cases}$$

and

$$\tilde{\nabla}_X \varphi = \nabla_X \varphi - \frac{1}{2} \tilde{\nabla}_X \nu \cdot \nu \cdot \varphi$$

for all $X \in TM$ and $\varphi \in \mathcal{S}_{\tilde{M}}|_M$, where $\cdot_M$ denotes Clifford multiplication in M.

Proof. We note that under the map

$$\begin{aligned} i : \mathrm{Fr}_{SO}(M) &\to \mathrm{Fr}_{SO}(\tilde{M})|_M \\ (e_1, \ldots, e_n) &\mapsto (e_1, \ldots, e_n, \nu) \end{aligned}$$

the pullback of $P_{\mathrm{Spin}}(\tilde{M})|_M$ to $\mathrm{Fr}_{SO}(M)$ gives a 2-covering of $\mathrm{Fr}_{SO}(M)$, and this is a principal $\mathrm{Spin}(n)$-bundle as the image of i is closed under the action of $\mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$ on frames, hence so is $i^*(P_{\mathrm{Spin}}(\tilde{M})|_M)$ under the action of $\mathrm{Spin}(n) \subseteq \mathrm{Spin}(n+1)$. The formula relating Clifford multiplication on $\tilde{M}$ to M comes from the usual embedding of $Cl_n(\mathbb{R})$ into $Cl^0_{n+1}(\mathbb{R})$ by $e_i \mapsto e_i \cdot e_{n+1}$.

For the final identity, fixing a positively-oriented local orthonormal frame $\{e_j\}_{j=1}^n$, for the spinorial frame $\{\varphi_\alpha\}$ corresponding to $\{e_1, \ldots, e_n, \nu\}$, in local coordinates we have

$$\tilde{\nabla}_X \varphi_\alpha = \frac{1}{4} \left(\sum_{j,k=1}^{n} g(\tilde{\nabla}_X e_j, e_k) e_j \cdot e_k + \sum_{j=1}^{n} g(\tilde{\nabla}_X e_j, \nu) e_k \cdot \nu + \sum_{k=1}^{n} g(\tilde{\nabla}_X \nu, e_k) \nu \cdot e_k + g(\tilde{\nabla}_X \nu, \nu) \nu \cdot \nu \right) \cdot \varphi_\alpha$$

We note that $\mathcal{W}(X) = -\nabla_X \nu \in TM$ is the Weingarten map of M, and so $\tilde{\nabla}_X Y = \nabla_X Y - g(\nabla_X \nu, Y)\nu$ and $g(\nabla_X \nu, \nu) = 0$. For the first term we get $g(\tilde{\nabla}_X e_j, e_k) = g(\nabla_X e_j, e_k) - g(\nabla_X \nu, e_j) g(\nu, e_k) = g(\nabla_X e_j, e_k)$, so

$$\frac{1}{4} \sum_{j,k=1}^{n} g(\nabla_X e_j, e_k) e_j \cdot e_k \cdot \varphi_\alpha = \nabla_X \varphi_\alpha$$

using our previous formula on M. We have $g(\tilde{\nabla}_X e_j, \nu) = g(\nabla_X e_j, \nu) - g(\tilde{\nabla}_X \nu, e_j) g(\nu, \nu) = -g(\tilde{\nabla}_X \nu, e_j)$, so

$$\sum_{j=1}^{n} g(\tilde{\nabla}_X e_j, \nu) e_j = \sum_{j=1}^{n} g(\tilde{\nabla}_X \nu, e_j) e_j = \tilde{\nabla}_X \nu - g(\tilde{\nabla}_X \nu, \nu)\nu = \tilde{\nabla}_X \nu$$

For the third term we note $\nu \cdot e_k = -e_k \cdot \nu$ as ν and e_k are orthogonal, and so the third term in the brackets evaluates to $\tilde{\nabla}_X \nu \cdot \nu$. The last term vanishes, and putting this together gives

$$\tilde{\nabla}_X \varphi_\alpha = \nabla_X \varphi_\alpha + \frac{1}{4}\left(-\nabla_X \nu \cdot \nu \cdot \varphi_\alpha - \nabla_X \nu \cdot \nu \cdot \varphi_\alpha + 0\right) = \nabla_X \varphi_\alpha - \frac{1}{2} \tilde{\nabla}_X \nu \cdot \nu \cdot \varphi_\alpha$$

□

Applying this to the case of $S^n \subseteq \mathbb{R}^{n+1}$ for $n \geq 2$, S^n has Weingarten map $\mathcal{W} = -\operatorname{id}_{TS^n}$ with respect to the normal $\nu(x) = x$. We note that the positive (when n is odd) part of $S_{\mathbb{R}^{n+1}} = \mathbb{R}^{n+1} \times \Delta_{n+1}$ is $2^{\lfloor n/2 \rfloor}$-dimensional, and so for any positive constant section $\psi \in \Gamma(\mathbb{R}^{n+1}, S_{\mathbb{R}^{n+1}})$ and $X \in TS^n$ we have

$$0 = \tilde{\nabla}_X \psi = \nabla_X \psi + \frac{1}{2} \mathcal{W}(X) \cdot \nu \cdot \psi = \nabla_X \psi - \frac{1}{2} X \cdot_{S^n} \psi$$

so $\nabla_X \psi = \frac{1}{2} X \cdot_{S^n} \psi$, and ψ is a $\frac{1}{2}$-Killing spinor, and we get $2^{\lfloor n/2 \rfloor}$ pointwise linearly independent $\frac{1}{2}$-Killing spinors $\{\psi_i\}$. For the same class of spinors, the spinors $\nu \cdot \psi$ satisfy

$$-X \cdot_{S^n} (\nu \cdot \psi) = -X \cdot \nu \cdot \nu \cdot \psi = X \cdot \psi = \tilde{\nabla}_X (\nu \cdot \psi) = \nabla_X (\nu \cdot \psi) - \frac{1}{2} X \cdot_{S^n} (\nu \cdot \psi)$$

so $\nu \cdot \psi$ is a $-\frac{1}{2}$-Killing spinor, and we also get $2^{\lfloor n/2 \rfloor}$ pointwise linearly independent $-\frac{1}{2}$-Killing spinors $\eta_i = \nu \cdot \psi_i$. Thus as $\operatorname{rank}_{\mathbb{C}}(\mathcal{S}_{S^n}) = 2^{\lfloor n/2 \rfloor}$, this yields trivialisations of $\mathcal{S}_{S^n}$ by both $\frac{1}{2}$ and $-\frac{1}{2}$-Killing spinors.

3. The Dirac spectrum of the sphere

We compute the Dirac spectrum of the sphere by first relating it to the spectrum of the Laplacian on S^n. We note that $Cl(\mathbb{R}^n) \cong \Lambda^* \mathbb{R}^n$, and that under this linear isomorphism, Clifford multiplication by $x \in \mathbb{R}^n$ is given by $x \cdot y \simeq x \wedge y - x \lrcorner y$, which we can write as $(x^\flat \wedge) - (x \lrcorner)$. Let d and δ denote the exterior differential and codifferential on M, so that $\Delta = \delta d$ is the scalar Laplace operator on $C^\infty(M)$. We also write $\operatorname{grad}(f)$ for the gradient vector field of $f \in C^\infty(M)$, given by $\operatorname{grad}(f) = \sum_{j=1}^n e_j(f) e_j = (df)^\sharp$ for a local orthonormal basis of TM.

Lemma 2. *Let (M, g) be a Riemannian spin manifold, $\varphi \in \Gamma(M, \mathcal{S}_M)$ and $f \in C^\infty(M)$. Then*

1. *$D(f\varphi) = \operatorname{grad}(f) \cdot \varphi + f D\varphi$; and*
2. *$D^2(f\varphi) = f D^2 \varphi - 2\nabla_{\operatorname{grad}(f)} \varphi + (\Delta f)\varphi$*

Proof. Fixing a local orthonormal frame $\{e_j\}_{j=1}^n$ for TM, we find

$$D(f\varphi) = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}(f\varphi) = \sum_{j=1}^{n} e_j \cdot (e_j(f)\varphi + f \nabla_{e_j} \varphi) = \operatorname{grad}(f) \cdot \varphi + f D\varphi$$

In computing $D^2(f\varphi)$, we first have that

$$D(\operatorname{grad}(f) \cdot \varphi) = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}(\operatorname{grad}(f) \cdot \varphi) = \sum_{j=1}^{n} e_j \cdot (\nabla_{e_j} \operatorname{grad}(f)) \cdot \varphi + \sum_{j=1}^{n} e_j \cdot \operatorname{grad}(f) \cdot \nabla_{e_j} \varphi$$

For the first term, identifying vector fields with 1-forms through the metric g gives

$$\sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \operatorname{grad}(f) = \sum_{j=1}^{n} e_j \wedge \nabla_{e_j} df - \sum_{j=1}^{n} e_j \lrcorner \nabla_{e_j} df = (d + \delta) df = \delta df = \Delta f$$

and for the second term, using the Clifford multiplication we find

$$\sum_{j=1}^{n} e_j \cdot \mathrm{grad}(f) \cdot \nabla_{e_j}\varphi = -\,\mathrm{grad}(f) \cdot \left(\sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\varphi \right) - \sum_{j=1}^{n} 2g(\mathrm{grad}(f), e_j)\nabla_{e_j}\varphi = -\,\mathrm{grad}(f) \cdot D\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi$$

We thus have

$$\begin{aligned} D^2(f\varphi) = D(\mathrm{grad}(f) \cdot \varphi + fD\varphi) &= \left(\Delta f \cdot \varphi - \mathrm{grad}(f) \cdot D\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi\right) + \mathrm{grad}(f) \cdot D\varphi + fD^2\varphi \\ &= fD^2\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + (\Delta f)\varphi \end{aligned}$$

□

We recall the spectrum of the scalar Laplacian on the sphere:

Theorem 1. *For $n \geq 1$, the Laplacian Δ_{S^n} on $L^2(S^n)$ admits an orthonormal basis of eigenvectors, with eigenvalues $\lambda_k = k(n+k-1)$ for $k \in \mathbb{Z}_{\geq 0}$, with multiplicity $\frac{n+2k-1}{n+k-1}\binom{n+k-1}{k} = \binom{n+k-1}{k} + \binom{n+k-2}{k-1}$.*

We give a sketch of why each of these values appears as an eigenvalue of the Laplacian, and the exact details of the multiplicities can be found in **berger..**.

For $x \in S^n$, we can compute the Laplacian in geodesic normal coordinates around x, by extending x to an orthonormal basis $\{x_1, x_2, \ldots, x_{n+1}\}$ for $\mathbb{R}^{n+1}$ (with $x = x_1$), and taking the corresponding geodesics $\gamma_i(s) = \cos(s)x_1 + \sin(s)x_{j+1}$ for $1 \leq j \leq n$. For $f \in C^\infty(\mathbb{R}^{n+1})$, in this case the Laplacian is just

$$\Delta_{S^n}(f|_{S^n})(x) = -\sum_{j=2}^{n+1} \frac{d^2}{ds^2}(f \circ \gamma_j)(0)$$

The chain rule gives $\frac{d(f\circ\gamma_i)}{ds}(s) = -\sin(s)\frac{\partial f}{\partial x_1}(\gamma(s)) + \cos(s)\frac{\partial f}{\partial x_{j+1}}(\gamma(s))$. On taking the second partial derivative and evaluating at zero, we find

$$\frac{d^2(f \circ \gamma_i)}{ds^2}(0) = -\cos(0)\frac{\partial f}{\partial x_1}(\gamma(0)) + \cos^2(0)\frac{\partial^2 f}{\partial x_{j+1}^2}(\gamma(0)) = -\frac{\partial f}{\partial x_1}(x) + \frac{\partial^2 f}{\partial x_j^2}(x)$$

and in particular noting that x_1 is the radial direction r, we see

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}(f)|_{S^n}(x) = -\sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}(x) &= -\sum_{j=2}^{n+1} \left(\frac{\partial^2 f}{\partial x_j^2}(x) - \frac{\partial f}{\partial x_1}(x) \right) - \frac{\partial^2 f}{\partial x_1^2}(x) - n\frac{\partial f}{\partial x_1}(x) \\ &= \Delta_{S^n}(f|_{S^n})(x) - \frac{\partial^2 f}{\partial r^2}(x) - n\frac{\partial f}{\partial r}(x) \end{aligned}$$

so that $\Delta_{\mathbb{R}^{n+1}}(f)|_{S^n} = \Delta_{S^n}(f|_{S^n})(x) - \frac{\partial^2 f}{\partial r^2} - n\frac{\partial f}{\partial r}$ Now for a homogeneous polynomial f of degree k we have $r\frac{\partial f}{\partial r} = kf$ and $r^2\frac{\partial^2 f}{\partial r^2} = k(k-1)$, so on restricting to S^n (where $r = 1$) we find

$$\Delta_{\mathbb{R}^{n+1}}(f)|_{S^n} = \Delta_{S^n}(f|_{S^n}) - k(n+k-1)f$$

and in particular if we choose f to be harmonic (i.e. such that $\Delta_{\mathbb{R}^{n+1}}(f) = 0$), we get

$$\Delta_{S^n}(f|_{S^n}) = k(n+k-1)f|_{S^n}$$

so that $f|_{S^n}$ is a $k(n+k-1)$-eigenvector of Δ_{S^n}.

We are now ready to compute the Dirac spectrum of S^n, which we will split into a few small steps. In the following results, μ is understood to be either $1/2$ or $-1/2$. We first need to relate the spectra of Dirac operators to the spectrum of the Laplacian, which the following result allows us to do nicely.

Lemma 3. *Let $n \geq 2$, and D be a Dirac operator on S^n. Then for any μ-Killing spinor φ and $f \in C^\infty(S^n)$, we have*

$$(D + \mu I)^2 (f\varphi) = \left(\Delta f + \left(\frac{n-1}{2} \right)^2 f \right) \varphi$$

Proof. Recall that $D\varphi = -n\mu\varphi$ as φ is a μ-Killing spinor, so using the formulas in **Lemma 2** we compute

$$\begin{aligned} D^2(f\varphi) &= fD^2\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + (\Delta f)\varphi \\ &= \frac{n^2}{4} f\varphi - 2\mu\,\mathrm{grad}(f) \cdot \varphi + (\Delta f)\varphi \\ &= \frac{n^2}{4} f\varphi - 2\mu(D(f\varphi) - fD\varphi) + (\Delta f)\varphi \\ &= \left(\frac{n^2}{4} - \frac{n}{2} \right) f\varphi - 2\mu D(f\varphi) + (\Delta f)\varphi \end{aligned}$$

and thus

$$(D + \mu I)^2 (f\varphi) = D^2(f\varphi) + 2\mu D(f\varphi) + \frac{1}{4} f\varphi = \left((\Delta f) + \left(\frac{n-1}{2} \right)^2 \right) \varphi$$

$\square$

Combining this with the spectrum of the Laplacian, we have the following for the Dirac spectrum of the sphere.

Theorem 2. *Let $n \geq 2$. Then the spectrum of the Dirac operator D on S^n is $\left\{ \pm \left(\frac{n}{2} + k \right) \mid k \in \mathbb{Z}_{\geq 0} \right\}$, with the eigenvalues $\lambda_k^\pm = \pm \left(\frac{n}{2} + k \right)$ having multiplicity $2^{\lfloor n/2 \rfloor} \binom{n+k-1}{k}$.*

Proof. Fix an orthonormal basis of eigenfunctions $\{f_m\}_{m \in \mathbb{N}}$ for Δ on S^n, and a trivialisation $\{\varphi_j\}_{j=1}^{2^{\lfloor n/2 \rfloor}}$ of $\mathcal{S}_{S^n}$ by μ-Killing spinors which is pointwise an orthonormal basis. Then by our previous lemma it follows that $\left\{ f_m \varphi_j \mid m \in \mathbb{N}, 1 \leq j \leq 2^{\lfloor n/2 \rfloor} \right\}$ is an orthonormal basis of $L^2(\mathcal{S}_{S^n})$ consisting of eigenvectors of $(D + \mu I)^2$, of eigenvalues $k(n + k - 1) + \left(\frac{n-1}{2} \right)^2 = \left(\frac{n-1}{2} + k \right)^2$, and multiplicity $2^{\lfloor n/2 \rfloor} \frac{n+2k-1}{n+k-1} \binom{n+k-1}{k}$. Thus any eigenvalue λ of D is of the form $-\mu \pm \left(\frac{n-1}{2} + k \right)$ for some $k \in \mathbb{Z}_{\geq 0}$ and both $\mu = 1/2$ and $\mu = -1/2$, so $\lambda \in \left\{ \pm \left(\frac{n}{2} + k \right) \mid k \in \mathbb{Z}_{\geq 0} \right\}$.

To show the claimed eigenvalues all appear with the claimed multiplicities, we use induction. Writing $\lambda_k^\pm = \pm \left(\frac{n}{2} + k \right)$, since we know D admits an orthonormal basis of eigenvectors, by the factorisation $(D + \mu I)^2 - \left(\frac{n-1}{2} + k \right)^2 I = \left(D + \left(\mu + \frac{n-1}{2} + k \right) I \right) \left(D + \left(\mu - \frac{n-1}{2} - k \right) I \right)$ we have

$$\ker \left((D + \mu I)^2 - \left(\frac{n-1}{2} + k \right)^2 I \right) = \ker \left(D + \left(\left(\mu + \frac{1}{2} \right) - \frac{n}{2} - k \right) I \right) \oplus \ker \left(D + \left(\left(\mu - \frac{1}{2} \right) + \frac{n}{2} + k \right) I \right)$$

Letting $m(\lambda_k^\pm)$ be the multiplicity of $\lambda_k^\pm$, taking dimensions in the above equation gives

$$2^{\lfloor n/2 \rfloor} \left(\binom{n+k-2}{k-1} + \binom{n+k-1}{k} \right) = m(\lambda_{k-1}^\mp) + m(\lambda_k^\pm)$$

Now when $k = 0$, the left-hand side of the above is $2^{\lfloor n/2 \rfloor}$. The μ-Killing spinors yield a $2^{\lfloor n/2 \rfloor}$-dimensional space of $n\mu$-eigenvectors, so the space of $\pm\frac{n}{2}$-eigenvectors has the claimed dimension. For $k > 0$, the above formula immediately gives us this result, since we have $m(\lambda_{k-1}^\mp) = 2^{\lfloor n/2 \rfloor} \binom{n+k-2}{k-1}$ if and only if $m(\lambda_k^\pm) = \binom{n+k-1}{k}$. $\square$

References

Ginoux

Christian Bar

Berger

Chavel