MATH3342 summary

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0 Notation and conventions

In these notes, all vector spaces and spaces will be over $\mathbb{R}$. Spaces are taken to be Banach spaces unless otherwise noted. We assume that all functions f are smooth (i.e. C^∞) unless otherwise stated.

Let $\mathcal{V}, \mathcal{W}$ be normed vector spaces, $A \subseteq \mathcal{V}$ and $B \subseteq \mathcal{W}$. Let $f, g : A \to B$. We say that f is $o(g)$ (at $x = a$) iff

$$\lim_{x \to a} \frac{\|f(x)\|_{\mathcal{W}}}{\|g(x)\|_{\mathcal{W}}} = 0$$

and that f is $O(g)$ (at $x = a$) if within a neighbourhood of a,

$$\frac{\|f(x)\|_{\mathcal{W}}}{\|g(x)\|_{\mathcal{W}}} \leq M$$

We denote the vector subspace $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$ (the image of the natural inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$) by Δ_k, where n is implicit in this notation.

1 Calculus review

Definition 1 (Banach space)**.** *A* Banach space V *is a complete normed vector space, usually over* $\mathbb{R}$.

Note that over a complete field, all norms on a vector space are equivalent (or comparable).

Definition 2 (Derivative)**.** *Let $\mathcal{V}, \mathcal{W}$ be Banach spaces, and $A \subseteq \mathcal{V}$. We say the function $f : A \to \mathcal{W}$ is* differentiable *at $x \in A$ if there is a linear transformation $L : \mathcal{V} \to \mathcal{W}$ such that*

$$\lim_{y \to x} \frac{\|f(y) - f(x) - L(y - x)\|_{\mathcal{W}}}{\|y - x\|_{\mathcal{V}}} = 0$$

In this case we say that L is the derivative of f at x*, and write $L = Df|_x$*

This can be equivalently formulated as saying that $f(y) = f(x) + L(y - x) + o\left(\|y - x\|_{\mathcal{V}}\right)$. We can rearrange this to give the usual 1-dimensional formulation, which is that for any $v \in \mathcal{V}$ of unit length,

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = L(v)$$

and we usually write $L = Df|_x$ or $f'(x)$, usually read "the derivative of f at x". The above shows that if the derivative of f exists at x, then it is unique. We write $D_v f := L(v)$ for the directional derivative of f in the direction of v.

The above shows that if f is differentiable, then all directional derivatives exist. The converse is not true, but in $\mathcal{V} = \mathbb{R}^n$ we have

Proposition 1. *$f = \sum_{i=1}^n f_i e_i : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $x \in A$ if and only if each f_i is differentiable, $1 \leq i \leq n$, and for any $v \in \mathbb{R}^m$,*

$$Df|_x(v) = \left[\frac{\partial f_i}{\partial x_j}\right]_{1 \leq i \leq m}^{1 \leq j \leq n}$$

That is, a function in Euclidean space is differentiable if and only if the directional derivatives exist and the partial derivatives “are compatible”.

Theorem 1 (Chain rule)**.** *Let $\mathcal{V}, \mathcal{W}, \mathcal{U}$ be Banach spaces, $A \subseteq \mathcal{V}, B \subseteq \mathcal{W}$. If $f : A \to B$ and $g : B \to \mathcal{U}$ are differentiable, then $g \circ f : A \to \mathcal{U}$ is differentiable, with*

$$D(g \circ f)|_x = Dg|_{f(x)} \circ Df|_x$$

Definition 3 (Sym_k, higher derivatives)**.** *Let $\mathcal{V}, \mathcal{W}$ be vector spaces. The set $\mathrm{Sym}_k(\mathcal{V}, \mathcal{W})$ is the set of symmetric k-linear maps $\mathcal{V}^k \to \mathcal{W}$.*

Let $f : A \subseteq \mathcal{V} \to \mathcal{W}$. We say that f is k times differentiable at x if there are $Df|_x \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, $D^2 f|_x \in \mathrm{Sym}_2(\mathcal{V}, \mathcal{W})$, …, $D^k|_x \in \mathrm{Sym}_k(\mathcal{V}, \mathcal{W})$ such that

$$f(y) = f(x) + Df|_x(y-x) + \frac{1}{2}D^2 f|_x(y-x, y-x) + \ldots + \frac{1}{k!}D^k f|_x(y-x, \ldots, y-x) + o\left(\|y-x\|^k\right)$$

In this case we say that the l-linear maps $D^l f$ are the l^{th} derivatives *of f.*

Definition 4 (C_k)**.** *Let $\mathcal{V}, \mathcal{W}$ be Banach spaces. For $k \in \mathbb{N}_0$, we write $C^k(\mathcal{V}, \mathcal{W})$ for the* set of k-times differentiable maps $f : \mathcal{V} \to \mathcal{W}$ with $f, Df, \ldots, D^k f$ continuous*. We write*

$$C^\infty(\mathcal{V}, \mathcal{W}) = \bigcap_{k \in \mathbb{N}_0} C^k(\mathcal{V}, \mathcal{W})$$

for the set of smooth functions from $\mathcal{V}$ to $\mathcal{W}$

1.1 Intuition for derivatives

For a function $f : A \to \mathcal{W}$ (with $A \subseteq \mathcal{V}$) k-times differentiable at x, its j^{th} derivatives $D^j f|_x \in \mathrm{Sym}_k(\mathcal{V}, \mathcal{W})$ at x can be thought of as carrying the information of all directional j^{th} derivatives in the direction of $y_1, \ldots, y_j$ as the value

$$D^j f|_x(y_1 - x, \ldots, y_j - x) = \frac{\partial^j f}{\partial[y_j - x]\partial[y_{j-1} - x] \ldots \partial[y_1 - x]}(x)$$

and in this sense being a reasonable “j^{th} tangent object” of f at x. The fact that $D^j f|_x \in \mathrm{Sym}_j(\mathcal{V}, \mathcal{W})$ then corresponds to an implicit usage of Clairaut’s theorem, which relies on the implicit assumption that $f \in C^\infty(A, \mathcal{W})$.

The maps $D^j f$ themselves can be viewed as a linear map $A \to \mathrm{Sym}_j(\mathcal{V}, \mathcal{W})$, carrying the tangent information at all points together.

2 Foundations of differential geometry (in the Euclidean case)

Definition 5 (Diffeomorphisms)**.** *Let $\mathcal{V}, \mathcal{W}$ be Banach spaces and $A \subseteq \mathcal{V}, B \subseteq \mathcal{W}$. We say that a map $f : A \to B$ is a C^k diffeomorphism if*

- *f is a bijection;*
- *$f \in C^k(A, B)$; and*
- *$f^{-1} \in C^k(A, B)$*

Diffeomorphisms should be viewed as smooth, invertible maps (of varying degree) in a slightly stronger way than homeomorphisms (which are just C^0 diffeomorphisms).

Theorem 2 (Inverse function theorem)**.** *Let $f \in C^k(A, \mathcal{W})$ and $x \in A$. If $Df|_x$ is an invertible linear map, then there are open sets $B \subseteq A, C \subseteq \mathcal{W}$ such that $f|_B : B \to C$ is a [C^k] diffeomorphism.*

Lemma 1. *Let $C \subseteq \mathcal{V}$, $F : C \to \mathcal{V}$ be such that $F(0) = 0$ and $DF|_0 = \mathrm{id}_{\mathcal{V}}$. Then for $E(z) = F(z) - z$ and R sufficiently small, $\|E(p) - E(q)\| < \frac{1}{2}\|p - q\|$. That is, the error function E is a $\frac{1}{2}$-contraction.*

Some intuition for why **Theorem 2** is true is as follows. Locally, the function is well-approximated by its derivative at x. Since the derivative is injective at x, the function should also be injective very close to x. Explicitly, this "well-approximation" says the error term is negligible in comparison, and in particular a contraction. We can thus use Banach's contraction mapping theorem on the error term to assert the existence of an inverse locally.

From now, we consider only smooth objects (i.e. C^∞ diffeomorphisms, and so on).

Definition 6 (Submanifolds in $\mathbb{R}^n$)**.** *We say that $\Sigma \subseteq \mathbb{R}^n$ is a k-*submanifold in $\mathbb{R}^n$ *if for each $x \in \Sigma$, there is a neighbourhood $x \in A \subseteq \mathbb{R}^n$ and a diffeomorphism $\psi : A \to B \subseteq \mathbb{R}^n$ such that*

$$\psi(\Sigma \cap A) = \Delta_k \cap B$$

That is, a k-submanifold is a space that locally shares the properties of Euclidean space (from a topological and differential point of view). The associated map ψ should also respect the ambient structure. The structure of a k-submanifold Σ thus consists of a *neighbourhood* A for each point x, and an *associated diffeomorphism* $\varphi : A \to B \subseteq \mathbb{R}^n$ (i.e. respecting the ambient space) which *maps the local part $\Sigma \cap A$ of the k-submanifold onto the local part $\Delta_k \cap B$ of the k-space Δ_k.*

Some examples of submanifolds [in $\mathbb{R}^n$] include discrete sets (0-manifolds), disjoint unions of non-overlapping curves (with no endpoints), and graphs of functions $\mathbb{R}^k \to \mathbb{R}^{n-k}$ (with the diffeomorphism $\Delta_k \leftrightarrow (\mathbb{R}^k, f(\mathbb{R}^k))$ given by $(x, 0) \leftrightarrow (x, f(x))$).

The sphere $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is also a submanifold, as it can be viewed locally as the graph of a function of the form

$$x_i = \pm \sqrt{\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j^2}$$

We further have the following "nice" maps between subsets of Euclidean space.

Definition 7 (Submersion; immersion; local diffeomorphism; embedding)**.** *Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. We say that f is*

1. *a* submersion *if $Df|_x$ is surjective at every $x \in A$;*
2. *an* immersion *if $Df|_x$ is injective at every $x \in A$;*
3. *a* local diffeomorphism *if it is a submersion and immersion; and*
4. *an* embedding *if it is an immersion and a homeomorphism.*

Notice that $n \geq m$ for submersions and $n \leq m$ for immersions. Submersions, immersions and local diffeomorphisms are also local properties of a function, which embeddings are global.

The name given to definition *(3.)* can be thought of as motivated by the inverse function theorem: as $Df|_x$ is invertible at every $x \in A$, it defines a diffeomorphism locally on a neighbourhood around each such x. Submersions look locally like *projections* $\mathbb{R}^m \times \mathbb{R}^{n-m} \twoheadrightarrow \mathbb{R}^m$, while immersions look locally like *inclusions* $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^{n-m}$. Embeddings can intuitively be thought of as inclusions which also respect the ambient space.

An illustration of these concepts is as in the following example.

Example 1. *Consider the singular curve $y^2 = x^2(x+1)$, with the smooth parametrisation*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (t^2 - 1, t^3 - t) \end{aligned}$$

This is an immersion, and never a local diffeomorphism, as $Df|_t(1) = \frac{\partial f}{\partial t} = (2t, 3t^2 - 1)$. It is not an embedding as it is non-injective (with $f(1) = (0,0) = f(-1)$ corresponding to the singular point).

The restriction $f|_{(-\infty,1)}$ (having "just traced out the loop") is still not an embedding, since the topology induced by the inclusion $(-\infty, 1) \hookrightarrow f(-\infty, 1)$ is not the same as the subspace topology under $f(-\infty, 1) \subseteq \mathbb{R}^2$.

For any $a < 1$, the restriction $f|_{(-\infty,a)}$ is however an embedding, and its image is a line.

We have a rather slick relation between k-submanifolds in $\mathbb{R}^n$ and level sets of functions $A \to \mathbb{R}^{n-k}$.

Theorem 3. *A subset $\Sigma \subseteq \mathbb{R}^n$ is a k-submanifold if and only if it is locally the level set of a submersion to $\mathbb{R}^{n-k}$ at 0. That is, if and only if for every $x \in \Sigma$, there is a neighbourhood $U \ni x$ and a submersion $G : U \to \mathbb{R}^{n-k}$ such that*

$$\Sigma \cap U = G^{-1}(0)$$

We can use this to prove that some spaces can be viewed non-trivially as manifolds. By viewing $\mathrm{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^2$, and noting that $G(M) = \det(M) - 1$ is a polynomial for every matrix M (and so in particular smooth), we see that $\mathrm{SL}_n(\mathbb{R})$ is a (n^2-1)-manifold of $\mathbb{R}^{n^2}$ (with ambient space taken to be $\mathrm{Mat}_{n\times n}(\mathbb{R}) \setminus \{0\}$).

By also noting that $G(M) = M^T M - I$ is a polynomial operator and that $DG|_M$ is surjective for $M \in \mathbb{O}(n)$ (with $\frac{1}{2}MC$ mapping to each $C \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ under $DG|_M(A)$), we also see that $\mathbb{O}(n)$ is a $n(n+1)/2$-manifold of $\mathbb{R}^{n^2}$.

We have a further equivalent notion of a submanifold, which leads to the notion of charts.

Theorem 4. *A subset $\Sigma \subseteq \mathbb{R}^n$ is a k-submanifold if and only if it is locally the image of an embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$. That is, if and only if for every $x \in \mathbb{R}^n$, there are neighbourhoods $A \subseteq \mathbb{R}^k$ and $\mathcal{U} \subseteq \mathbb{R}^n$, and an embedding $\varphi : A \to \mathcal{U}$ with $\varphi(A) = \Sigma \cap \mathcal{U}$.*

In the proof of the forward direction of this theorem, we explicitly find that the embedding is $F = \psi^{-1} \circ \iota_k$ where ψ is the associated diffeomorphism and $\iota_k : \mathbb{R}^k \hookrightarrow \Delta_k \subseteq \mathbb{R}^n$. The associated "inverse" to F is then $\pi_k \circ \psi$ (where $\pi_k : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^k$).

Thus submanifolds can be locally parametrised by Euclidean space, motivating the following definition.

Definition 8. *Let $A \subseteq \mathbb{R}^k$, $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $F : A \to \mathcal{U}$ be an embedding with image Σ. Then the map $\varphi := F^{-1} : \Sigma \cap \mathcal{U} \to A$ is called a* local chart *for Σ.*

If $B \subseteq \mathbb{R}^k$, $\mathcal{V} \subseteq \mathbb{R}^n$ are also open with $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, and $\varphi : \Sigma \cap \mathcal{U} \to A$, $\eta : \Sigma \cap \mathcal{V} \to B$ are also local charts for Σ, then the map $\eta \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{V}) \to \eta(\mathcal{U} \cap \mathcal{V})$ is called a transition map *between the charts φ and ψ.*

We have the following compatibility between local charts (via transition maps).

Proposition 2. *Let $\varphi : A \to \mathcal{U}$, $\eta : B \to \mathcal{V}$ be embeddings. Then the transition map $\eta \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{V}) \to \eta(\mathcal{U} \cap \mathcal{V})$ is smooth.*

The key ingredient in the proof of the above proposition is to note that on $\varphi(\mathcal{U} \cap \mathcal{V})$ and $\mathcal{U} \cap \mathcal{V}$, $\varphi^{-1} = \psi \circ \iota_k$ and $\eta = \pi_k \circ \tilde{\psi}^{-1}$ respectively for some diffeomorphisms $\psi, \tilde{\psi}$ between subsets of $\mathbb{R}^n$.

3 General manifolds

We now describe manifolds in full generality. Continuing on from the intuition that manifolds are "locally like Euclidean space", we have the following topological description.

Definition 9 (Topological manifold)**.** *We say that a topological space M (sometimes denoted M^m where m is as follows) is a* topological manifold *if it is Hausdorff*[1] *and second countable*[2]*, and for every $x \in M$, there is a neighbourhood $x \in \mathcal{U} \subseteq M$ and a homeomorphism $\varphi : \mathcal{U} \to \mathcal{V} \subseteq \mathbb{R}^m$. The homeomorphisms $\varphi : \mathcal{U} \to \mathcal{V}$ are called* local charts*.*

The Hausdorff and second countable properties are required so that the objects considered manifolds are not "too pathological".

Example 2.
1. *A non-Hausdorff (but otherwise satisfying the definition of a manifold) space is the line with two origins, which is explicitly the quotient of $\mathbb{R} \times \{0, 1\}$ by $(x, 0) \sim (x, 1)$ for $x \neq 0$*[3]*.*
2. *A non-second countable (but again otherwise satisfying the definition of a manifold) space is the long line. We can view the normal real line as $|\mathbb{Z}|$ copies of the interval $[0, 1)$, as*

$$\mathbb{R} = \coprod_{n \in \mathbb{Z}} [n, n+1) \text{ “} = \text{” } \mathbb{Z} \times [0, 1)$$

   In the same sense as the above equality, let ω to be an uncountable set of minimal cardinality, with a fixed well-order (by the Well-Ordering Principle). The long line L is then space

$$L = \omega \times [0, 1)$$

   with the lexicographical ordering (in a similar sense to $\mathbb{R}$). L itself does not satisfy the manifold definition, but $L \setminus \{(0_\omega, 0)\}$ (where $0_\omega \in \omega$ is minimal) does satisfy this constraint.

The property in the above definition (at a point $x \in M$) is sometimes referred to as being *locally Euclidean.* Between topological manifolds, we have the following sense of compatibility, which allows us to talk about differentiation locally on abstract manifolds.

Definition 10 (Smooth compatibility)**.** *Let M be a manifold, $\mathcal{U}, \mathcal{V} \subseteq M$, $A, B \subseteq \mathbb{R}^m$ be open, and $\varphi : \mathcal{U} \to A \subseteq \mathbb{R}^m$, $\eta : \mathcal{V} \to B \subseteq \mathbb{R}^m$. We say that φ and η are smoothly compatible if*

$$\eta \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{V}) \to \eta(\mathcal{U} \cap \mathcal{V})$$

is a diffeomorphism.

In this regard, we extend globally with the following definition.

[1] i.e. all distinct $x, y \in M$ have disjoint neighbourhoods $x \in A, y \in B$.
[2] i.e. admits a countable basis.
[3] with the quotient topology, i.e. the minimal topology for which the projection $X \mapsto X/\sim$ is continuous.

Definition 11 (Atlas; smooth atlas)**.** *An* atlas *is a collection of pairwise compatible charts. An atlas is* smooth *if the charts are also pairwise smoothly compatible.*

Leading up to the definition of a differentiable manifold, we have

Definition 12 (Atlas equivalence)**.** *We say that two atlases on a manifold M are* equivalent *if their union is an atlas, or equivalently if all maps are pairwise equivalent between the two atlases.*

Note that this does indeed induce an equivalence relation on the set of atlases, and so the following definition makes sense.

Definition 13 (Differentiable structure; differentiable manifold)**.** *Let M be a manifold. A* differentiable structure *on M is an equivalence class of manifolds under the above equivalence relation.*

We say that M is a differentiable manifold *if it is equipped with a differentiable structure.*

We then have the following examples of differentiable manifolds.

Example 3.

1. *$M = \mathbb{R}$ with the atlases $\{\mathrm{id}_{\mathbb{R}}\}$ and $\{(\cdot)^{1/3} : x \mapsto x^{1/3}\}$. Note that these induce different differentiable structures on M, with the latter structure being "flat" at $x = 0$.*
2. *Any submanifold of Euclidean space $\Sigma \subseteq \mathbb{R}^m$ is a differentiable manifold with local charts as the inverses of the associated embeddings.*
3. *Products of manifolds: Let M^m and N^n be differentiable manifolds. Then $(M \times N)^{m+n}$ is a manifold with differentiable structure given by maps (φ, ψ) where φ and ψ are local maps from M to $\mathbb{R}^m$ and N to $\mathbb{R}^n$ respectively (with the associated product topology).*
4. *The real projective space $\mathbb{RP}^n$ defined as the equivalence classes of $\mathbb{R}^{n+1} \setminus \{0\}$ under $x \sim \lambda x$ for $\lambda \neq 0$, or equivalently*
$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\mathbb{R} \setminus \{0\}}$$

Note that the topology in the last example is not specified. We will show that the real projective space satisfies conditions for which the associated maps define a reasonable topology.

We have the following result, which gives sufficient conditions for a unique differentiable structure on an atlas.

Lemma 2 (Smooth chart lemma)**.** *Let M be a set, Λ be an index set and for each $\alpha \in \Lambda$, $\mathcal{U}_\alpha \subseteq M$ and $\varphi_\alpha : \mathcal{U}_a \to \mathbb{R}^n$ be a map. If*

1. *For all α, φ_α is injective, and $\mathcal{V}_a = \varphi_\alpha(\mathcal{U}_\alpha)$ is open in $\mathbb{R}^n$.*
2. *For all α, β, $\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is open in $\mathbb{R}^n$.*
3. *For all α, β, $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is a diffeomorphism.*
4. *M is the union of countably many $\mathcal{U}_\alpha$.*
5. *For all distinct $x, y \in M$, either there is α with $x, y \in \mathcal{U}_\alpha$, or there are α, β with $x \in \mathcal{U}_\alpha, y \in \mathcal{U}_\beta$ and $\mathcal{U}_\alpha \cap \mathcal{U}_\beta = \emptyset$.*

Then there is a unique smooth manifold structure on M with φ_α as charts.

There is *probably* a way to write down countably many $\mathcal{U}_\alpha$, possibly by choosing associated lines in $\mathbb{RP}^n$ and working with projections killing those (by looking at the induced action of $\mathrm{GL}_{n+1}(\mathbb{R})$). In any case, the real projective space *is* a smooth manifold in some way.

We continue defining the notions of differentation through the lens of the associated Euclidean spaces.

Definition 14 (Smooth map)**.** *Let M^m and N^n be manifolds. A map $f : M^m \to \mathbb{R}^n$ is* smooth *if for each chart φ of M, the map $f \circ \varphi^{-1}$ is smooth.*

More generally, we say that a map $f : M^m \to N^n$ is smooth *if for each chart φ of M and η of N, the map $\eta \circ f \circ \varphi^{-1}$ (between Euclidean space) is smooth.*

To check that a map is smooth, by the smoothness of transition maps it suffices to check the maps in any given smooth atlas, and we also only check pairs of charts φ of M and η of N for which the composition $\eta \circ f \circ \varphi^{-1}$ makes sense, that is for when there is some $x \in M$ for which $x \in U = \mathrm{dom}(\varphi)$ and $f(x) \in V = \mathrm{dom}(\eta)$.

We have the following diagram for a smooth map f between manifolds.

$$\begin{tikzcd} \mathcal{U}_\alpha \subseteq M^m \arrow[r, "f"] \arrow[d, "\varphi^{-1}"] & N^n \\ V_\alpha \subseteq \mathbb{R}^m \arrow[r] & \mathbb{R}^n \arrow[u, "\eta"'] \end{tikzcd}$$

In this sense, like all other notions described, the notion of a smooth manifold descends to the Euclidean case. For a diffeomorphism, we have

Definition 15 (Diffeomorphism)**.** *Let M^m and N^n be smooth manifolds. A map $f : M \to N$ is a* diffeomorphism *is a smooth map with smooth inverse.*

As per usual, we say that two manifolds are diffeomorphic if there is a diffeomorphism between them. This is an equivalence relation, and so it makes sense to talk about manifolds up to diffeomorphism.

For submanifolds of Euclidean space, we have the following nice characterisations (or properties).

Lemma 3. *Let $\Sigma \subseteq \mathbb{R}^n$ be a submanifold.*

1. *(Restriction) If $f : \mathcal{U} \subseteq \mathbb{R}^n \to N$ is smooth with $\Sigma \subseteq \mathcal{U}$, then the restriction $f|_\Sigma : \Sigma \to N$ is smooth.*
2. *(Projection) If $f : M \to \mathbb{R}^n$ is smooth with $f(M) \subseteq \Sigma$, then the map $f : M \to \Sigma$ with restricted codomain is also smooth.*

More generally, we define submanifolds of arbitrary smooth manifolds by the following.

Definition 16 (Submanifolds)**.** *A subset $\Sigma^k \subseteq M^m$ is a* smooth k-submanifold *if for every $x \in \Sigma$, there is a neighbourhood $x \in \mathcal{U} \subseteq M$ and a diffeomorphism $\varphi : \mathcal{U} \to \mathcal{V} \subseteq \mathbb{R}^m$ with*

$$\varphi(\Sigma \cap \mathcal{U}) = \Delta_k \cap \mathcal{V}$$

We have an almost exact generalisation of the previous lemma.

Lemma 4. *Let $\Sigma \subseteq M$ and $\Sigma' \subseteq N$ be a submanifolds.*

1. *(Restriction) If $f : M \to N$ is smooth with $\Sigma \subseteq \mathcal{U}$, then the restriction $f|_\Sigma : \Sigma \to N$ is smooth.*
2. *(Projection) If $f : M \to N$ is smooth with $f(M) \subseteq \Sigma'$, then the map $f : M \to \Sigma'$ with restricted codomain is also smooth.*

Additionally, we have a notion of a (smooth) manifold with a group structure, captured by the following definition.

Definition 17 (Lie group)**.** *A* Lie group G *is a group with a smooth manifold structure, such that composition* $(a, b) \mapsto ab$ *and inversion* $a \mapsto a^{-1}$ *are smooth maps.*

For examples of Lie groups, we have $\mathrm{GL}(n), \mathrm{SL}(n)$ and $\mathbb{O}(n)$. We also have the following relationship between submanifolds and subgroups of a Lie group.

Lemma 5. *Let M be a Lie group and $\Sigma \subseteq M$ be a submanifold. Then Σ is a Lie subgroup of M.*

4 Tangent spaces

We define explicitly what it means for a function to be differentiable, independent of chart. We do this by treating all the charts as "the same" up to some reasonable identification. In the following definition, $\mathscr{C}_x$ denote the set of charts φ with $x \in \mathrm{dom}(\varphi)$.

Definition 18 (Tangent space; tangent vector)**.** *Let M^m be a manifold and $x \in M$. The* tangent space T_xM at x *is the quotient*[4] *of $\mathscr{C}_x \times \mathbb{R}^m$ by the relation*

$$(\varphi, v) \sim (\eta, w) \iff w = D(\eta \circ \varphi^{-1})|_{\varphi(x)}(v)$$

We denote the class of (φ, v) by $[\varphi, v]$, and call such an element a tangent vector*.*

We should think of each class $[\varphi, v]$ as a well-defined "direction" on the manifold. The equivalence relation can be thought of as motivated by the case where $w = D\eta|_x$ and $v = D\varphi|_x$ in the Euclidean case (where these make sense).

The tangent space at a point x has a natural vector space structure by addition in the second coordinate, and by noting that

$$[\varphi, v] = [\eta, D(\eta \circ \varphi^{-1})|_{\varphi(x)}(v)]$$

exactly by the above relation. We also have a natural isomorphism

$$\begin{aligned} T_X M &\to \mathbb{R}^m \\ [\varphi, v] &\mapsto v \end{aligned}$$

which is well-defined as $[\varphi, v] = [\varphi, w]$ exactly when $w = D(I)_{\varphi(x)}(v) = v$.

We have some alternate formulations or viewpoints on tangent spaces. For one, we can view the tangent space as the collection of tangent vectors of curves through x.

More formally, for $x \in M$, let $\sigma \in C^\infty(I, M)$ where I is an interval containing 0 and $\sigma(0) = x$. We set

$$\sigma' := [\varphi, (\varphi \circ \sigma)'(0)]$$

where our definition is independent of choice of chart φ by the chain rule, since for $\sigma, \tau \in C^\infty(I, M)$, we have

$$\sigma' = \tau' \iff (\varphi \circ \sigma)'(0) = (\varphi \circ \tau)'(0)$$

[4] i.e. set of equivalence classes under the relation

for all charts φ, i.e. the tangent at 0 is the same in every chart.

We have another viewpoint on tangent vectors, for which we first introduce the notion of a "local function".

Definition 19 (Function germs)**.** *Let $x \in M$. We say that two smooth functions $f : U(\subseteq M) \to \mathbb{R}$ and $g : U'(\subseteq M) \to \mathbb{R}$ are equivalent if there is a neighbourhood $V \subseteq U \cap U'$ with $f|_V = g|_V$. An equivalence class $[f]$ (usually just denoted as the function f) is called a C^∞* function germ *at x, and the* set of function germs at x *is denoted $C^\infty_x(M)$.*

Definition 20 (Derivation)**.** *Let M be a manifold and $x \in M$. A* derivation *$V : C^\infty_x(M) \to \mathbb{R}$ is a linear map such that $V(fg) = f(x)V(g) + V(f)g(x)$. The space of derivations at x on M is denoted D^M_x.*

This identity should be familiar as it is the analogue of the 1-dimensional product rule. Indeed, for a manifold M^m, a point $x \in M$ and tangent vector $[\varphi, v] \in T_xM$, we have a natural derivation

$$[\varphi, v]f = \frac{\partial(f \circ \varphi^{-1})}{\partial v}(\varphi(x)) = \frac{d(f \circ \varphi^{-1})}{ds}(\varphi(x) + sv)|_{s=0}$$

It turns out these are all such derivations, as given in the following result.

Theorem 5. *The space of derivations at x is naturally isomorphic to T_xM.*

To prove this we have two lemmas. The first gives a Taylor-like expansion to a smooth function at a manifold.

Lemma 6. *Let $g \in C^\infty_0(\mathbb{R}^n)$. Then we have function germs $h_i \in C^\infty_0(\mathbb{R}^n)$ with*

$$g(z) = g(0) + \sum_{i=1}^{n} h_i(z)z^i$$

and $h_i(0) = \frac{\partial g}{\partial z^i}\big|_0$

This can be reformulated for an arbitrary manifold M^n and $x \in M$ by composing with a chart $\varphi : U(\subseteq M) \to \mathbb{R}^n$ with $\varphi(x) = 0$. The second lemma says that derivations are zero at constants.

Lemma 7. *Let $V : C^\infty_x(M) \to \mathbb{R}$ be a derivation. Then $V(1) = 0$.*

In particular, in the proof of **Theorem 5**, we explicitly find that

$$V(f) = \left[\varphi, \sum_{i=1}^{n}(V\varphi^i)e_i\right] f$$

Having established this, we can now define what it means for a function on a manifold to be differentiable, independent of chart.

Definition 21. *Let $f \in C^\infty(M^m, N^n)$. Its* derivative *$Df|_x : T_xM \to T_{f(x)}N$ is given by*

$$Df|_x([\varphi, v]) = [\eta, D(\eta \circ f \circ \varphi^{-1})|_{\varphi(x)}(v)]$$

where φ is a chart with $x \in \mathrm{dom}(\varphi)$ and η a chart with $f(x) \in \mathrm{dom}(\eta)$.

We have referred to charts in the definition, but as tangent vectors are ultimately independent of chart, this definition remains independent of chart. Note that as $[\varphi, v] \mapsto v$ is a linear isomorphism between T_xM and $\mathbb{R}^m$ for any $x \in \mathrm{dom}(\varphi)$, this definition ultimately reduces to the Euclidean case: to check injectivity, surjectivity or any other property on tangent vectors, we simply check it in the Euclidean case.

(Perspective from the curve point-of-view) When we view our tangent vector as $[\varphi, v] = \sigma'$ for some curve σ, we see that $Df|_x(\sigma') = (f \circ \sigma)'$, that is, the derivative of f acting on the tangent vector of a curve σ is ju, st the tangent vector of the image curve.

Alternatively, for a derivation $V \in D_x^M$, and $DF|_x(V) \in D_{F(x)}^N$, we have $(DF|_x(V))f = V(f \circ F)$ (where $f \circ F \in C_x^\infty(M)$). We can see this by writing $V = \sigma'$, so that $Vf = (\sigma \circ f)'(0)$.

As we have now defined differentiation independent of the associated charts, we have the following definitions for general submanifolds.

Definition 22 (Immersion, submersion, embedding)**.** *Let M^m, N^n be manifolds. A map $f : M \to N$ is a / an*

1. submersion *if $Df|_x$ is surjective for every $x \in M$.*
2. immersion *if $Df|_x$ is injective for every $x \in M$.*
3. embedding *if it is an immersion and a homeomorphism to its image.*

Hence as these definitions inherit from the ultimately inherit from the Euclidean case, we have the following properties

Lemma 8 (Properties of the derivative)**.** *Let M, N, P be smooth manifolds, $\Sigma \subseteq M$ and $f : M \to N$, $g : N \to P$ be smooth maps. Then*

1. *(Chain rule) $D(g \circ f)|_x = Dg|_{f(x)} \circ Df|_x$*
2. *(Inverse function theorem) If $Df|_x$ is invertible, then there are neighbourhoods $U \subseteq M$, $V \subseteq N$ for which $f|_U : U \to V$ is a diffeomorphism.*
3. *Σ is a submanifold if and only if it is (locally) the level set of a submersion.*
4. *Σ is a submanifold if and only if it is (locally) the image of an embedding.*

We also have the following result, which gives a condition for which an injective smooth map is an embedding.

Lemma 9. *If M is a compact manifold and $f : M \to N$ is an injective smooth map, then f is an embedding.*

A submanifold $\Sigma \subseteq M^n$ has an associated tangent space $T_x\Sigma$, which is naturally isomorphic to a k-dimensional subspace $Di(T_x\Sigma)$ of T_xM, given by

1. (Tangents of curves) $\{\sigma'(0) \mid \sigma \in C\infty(I, M), \sigma(0) = x\}$
2. (Images of smooth embeddings) $DF|_{F^{-1}(x)}(\mathbb{R}^k)$ for an embedding $F : A(\subseteq \mathbb{R}^n) \to B(\subseteq M^n)$ with $F(\Delta_k \cap A) = \Sigma \cap B$.
3. (Level sets of submersions) $\ker(DG|_x)$ for a submersion $G : U(\subseteq M^n) \to N^{n-k}$ with $G^{-1}(p) = \Sigma \cap \mathcal{U}$ for some $p \in N$.
4. (The natural inclusion) $Di|_x(T_xM)$ for the inclusion $T_x\Sigma \hookrightarrow T_xM$.

From the third perspective, we compute that $T_xS^n = \left\{v \in \mathbb{R}^{n+1} \mid \langle x, v\rangle = 0\right\} = \mathrm{span}(x)^\perp$ by noting $\frac{\partial}{\partial v} \|x\|^2 = 2\langle x, v\rangle$. To compute T_xS^n, we use the following lemma.

Lemma 10. *Let $A(s)$ be a 1-parameter family of matrices. Then*

$$\frac{d}{ds}\det(A(s)) = \det(A(s))\operatorname{tr}\left(A(s)^{-1}\frac{d}{ds}(A(s))\right)$$

From this we see that $T_A SL(n) = \left\{B \in M_{n\times n} \mid \operatorname{tr}(A^{-1}B) = 0\right\}$. We also have that $T_A \mathbb{O}(n) = \left\{B \in M_{n\times n} \mid B^T + B = 0\right\}$.

We have the following construction "lying over" M^m, in the sense that every $x \in M$ has a m-dimensional vector space lying over it.

Definition 23. *Let M be a smooth manifold. Its* tangent bundle *TM is given by*

$$TM = \{(x, v) \mid x \in M, v \in T_x M\} = \coprod_{x\in M} (\{x\} \times T_x M)$$

The tangent bundle has an associated smooth manifold structure, given below.

Theorem 6. *Let M^m be a manifold. Then TM is a manifold of dimension $2m$, with charts given by*

$$\begin{aligned}\tilde{\varphi} : TU &\to V \times \mathbb{R}^n\\ (x, [\varphi, v]) &\mapsto (\varphi(x), v)\end{aligned}$$

where $\varphi : U \to V$ is a chart of M and TU is the preimage of U under the projection $TM \twoheadrightarrow M$ onto first coordinates.

The tangent bundle is a specific example of a general construction called a vector bundle.

Definition 24. *Let M be a smooth manifold. A smooth manifold E is a vector bundle of rank k if it has a smooth submersion $\pi : E \to M$ where*

1. *$\pi^{-1}(m)$ is a k-dimensional vector space for each $m \in M$; and*
2. *Each $m \in M$ has a neighbourhood $U \ni m$ and a diffeomorphism $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that*
   - *(a) $\psi(x, v) \in \{x\} \times \mathbb{R}^k$; and*
   - *(b) $\psi|_{\pi^{-1}(y)} : \pi^{-1}(y) \to \{y\} \times \mathbb{R}^k$ is a linear isomorphism for each $y \in U$.*

In this sense a vector bundle is initially a disjoint union (coproduct) of M and associated k-dimensional vector spaces $E(m)$:

$$E = \coprod_{m\in M} (\{m\} \times E(m))$$

and the second constraint says that this coproduct becomes a direct product locally, with

$$\coprod_{m\in U} (\{m\} \times E(m)) \xrightarrow{\sim} U \times \mathbb{R}^k$$

as manifolds, and

$$\pi^{-1}(y) \cong \{y\} \times \mathbb{R}^k$$

as vector spaces.