MATH3228 Summary Notes

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Contents

1 Foundations

Definition (The complex numbers): The complex numbers $\mathbb{C}$ are the set $\mathbb{R}^2$ with the additional multiplication rule $(0,1)^2 = (-1,0)$ (secretly $i^2 = -1$).

This definition is somewhat janky to allow us to use tools from real analysis. We notate $i = (0,1)$ and $r = (r,0)$ for any $r \in \mathbb{R}$ (so $x + yi = (x,y)$). Some other simple functions on $\mathbb{C}$ are $\mathrm{Re}, \mathrm{Im}, |\cdot|$ and $\bar{\cdot}$, defined in the usual way. These satisfy the usual properties.

Proposition ($\mathbb{C}$ is a field): For each $z \in \mathbb{C} \setminus \{0\}$, there is a unique w with $zw = 1$, namely $\frac{1}{z} := \frac{\overline{z}}{|z|^2}$.

Definition (Elementary functions): For every $z \neq 0$, $\arg(z)$ is the unique $\theta \in [0, 2\pi)$ corresponding to the anticlockwise angle between z and the positive real axis. For $z = x+iy$, the *complex exponential* is

$$e^z = e^x(\cos(y) + i\sin(y))$$

For $z \neq 0$, we also define $\ln(z) := \ln|z| + i\arg(z)$, and

$$\begin{aligned}\mathrm{Arg}(z) &:= \left\{\theta \mid |z|\, e^{i\theta} = z\right\} = \{\arg(z) + 2k\pi \mid k \in \mathbb{Z}\}\\ \mathrm{Ln}(z) &:= \ln(z) + i\,\mathrm{Arg}(z) = \{\ln(z) + i\arg(z) + i2\pi k \mid k \in \mathbb{Z}\}\end{aligned}$$

Notice that for $z \neq 0$ we have $e^{\ln(z)} = z$ and in fact $e^{\mathrm{Ln}(z)} = \{z\}$, but $\ln(e^z) \neq z$ in general.

Definition (Complex exponentiation): For $z, w \in \mathbb{C}$, we define $z^w := e^{\ln(z)w}$.

Proposition (n-th roots): For every $w \in \mathbb{C} \setminus \{0\}$ there are n distinct solutions to $z^n - w = 0$.

Lemma: Let $p(z)$ be a non-constant polynomial. Then $|p(z)|$ achieves its minimum in $\mathbb{C}$.

Theorem (Fundamental Theorem of Algebra): Every non-constant polynomial with coefficients in $\mathbb{C}$ has a zero.

Corollary: Every polynomial with coefficients in $\mathbb{C}$ factors into linear terms.

2 Complex Differentiability

Definition (Complex differentiable): We say that a function $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-differentiable if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, and we call the limit $f'(z_0)$. Equivalently, if there exists some $f'(z_0) \in \mathbb{C}$ with

$$\lim_{\Delta z \to 0} \frac{|f(z_0 + \Delta z) - f(z_0) - f'(z_0)\Delta z|}{|\Delta z|} = 0$$

Definition (Dalbeault operators): The *Dalbeault operators* are defined by

$$\begin{aligned}
\frac{\partial}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial \overline{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)
\end{aligned}$$

Theorem (Cauchy-Riemann Equations): The following are equivalent for $f : \mathcal{U} \to \mathbb{C}$, for an open subset $\mathcal{U} \subseteq \mathbb{C}$.

1. f is $\mathbb{C}$-differentiable at $z_0 \in \mathcal{U}$ with $f'(z_0) = a + ib$.
2. f is $\mathbb{R}$-differentiable at $z_0 \in \mathcal{U}$ with $J_f(x_0, y_0) = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$
3. If $f(x + iy) = (u(x, y), v(x, y))$, f is $\mathbb{R}$-differentiable at (x_0, y_0) and

$$\begin{aligned}
\frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0) = a \\
\frac{\partial v}{\partial x}(x_0, y_0) &= -\frac{\partial u}{\partial y}(x_0, y_0) = b
\end{aligned}$$

   Then $f'(z_0) = \frac{\partial f}{\partial x}(z_0)$
4. $\frac{\partial f}{\partial \overline{z}} = 0$, or equivalently $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$.

Definition (Domain): A *domain* $D \subseteq \mathbb{C}$ is an open connected subset.

Definition (Holomorphic; entire): If D is a domain, a function $f : D \to \mathbb{C}$ is called *holomorphic on* D if it is differentiable at every $z \in D$. We write $H(D)$ for the space of all holomorphic functions on D. We refer to every $f \in H(\mathbb{C})$ as *entire*.

Definition (Conformal equivalence): We say that two domains $D, G \subseteq \mathbb{C}$ are *conformally equivalent* if there exists a bijection $f : D \to G$ with f, f^{-1} holomorphic. In this case we say that f is a *conformal equivalence* or *biholomorphic*.

Proposition ("Dual map" to a conformal equivalence): If $f : D \to G$ is biholomorphic, then

$$\begin{aligned}
\Phi_f : H(G) &\to H(D) \\
g &\mapsto g \circ f
\end{aligned}$$

is an isomorphism of vector spaces and (non-unital) rings.

Definition (Angle and strip domains, logarithm on an angle domain): Let $\alpha, \beta \in \mathbb{R}$. Then $S_{\alpha,\beta} = \{z \in \mathbb{C} \mid \alpha < \mathrm{Im}(z) < \beta\}$, and $e^z|_{S_{\alpha,\beta}}(S_{\alpha,\beta}) = A_{\alpha,\beta}$. We further have $\ln_{\alpha,\beta} := (e^z|_{S_{\alpha,\beta}})^{-1}$.

Definition (Holomorphic branches of the logarithm): If D is a domain, $f : D \to \mathbb{C}$ is a *holomorphic branch of the logarithm* if $f \in H(D)$ and $e^{f(z)} = z$ for all $z \in D$.

Proposition (Conformal equivalences): Let $M \in \mathrm{GL}_2(\mathbb{R})$. Then the following are equivalent.

1. $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$
2. M is a product of a dilation $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and a rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
3. If $u, v \neq 0$, the angle from u to v is the same as the angle from Mu to Mv.

Definition (Path in $\mathbb{C}$): A path is a continuous map $\gamma : [0,1] \to \mathbb{C}$. We say that a path is smooth at t_0 if γ is differentiable at t_0, with $\gamma'(t_0) \neq 0$.

For paths γ and $\tilde{\gamma}$ with $\gamma(t_0) = z_0 = \tilde{\gamma}(\tilde{t}_0)$, we say that the *angle at z_0 from γ to $\tilde{\gamma}$* is the angle between $\gamma'(t_0)$ and $\tilde{\gamma}'(\tilde{t}_0)$.

Definition (Conformal) Let D be a domain. We say that $f : D \to \mathbb{C}$ is conformal at z_0 if one (and hence both) of the following hold

1. f is $\mathbb{C}$-differentiable at z_0 with $f'(z_0) \neq 0$.
2. For all paths $\gamma, \tilde{\gamma}$ which cross at a point z_0, the angle between γ and $\tilde{\gamma}$ at z_0 is the same as the angle at $f \circ \gamma$ and $f \circ \tilde{\gamma}$ at $f(z_0)$.

We say that $f : D \to \mathbb{C}$ is conformal if it is conformal at every $z_0 \in D$.

Proposition (Conformal equivalences are conformal): Let $f : D \to \mathbb{C}$ be a conformal equivalence. Then f is conformal.

3 Mobius transformations and generalised disks, circles

3.1 Stereographic projection

Definition (The Riemann sphere): Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ∞ formally stands in for the point $(0, 0, 1)$, and we identify $\hat{\mathbb{C}}$ with the sphere

$$\mathbb{S}^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + \left(w - \frac{1}{2}\right)^2 = \frac{1}{4} \right\} \subseteq \mathbb{R}^3$$

by the stereographic projection

$$\begin{aligned} \Pi : \mathbb{S}^2 \setminus \{(0, 0, 1)\} &\to \mathbb{C} \\ (x, y, w) &\mapsto \left(\frac{x}{1 - w}, \frac{y}{1 - w} \right) \end{aligned}$$

This has corresponding inverse

$$\begin{aligned} \Pi^{-1} : \mathbb{C} &\to \mathbb{S}^2 \setminus \{(0, 0, 1)\} \\ (x, y) &\mapsto \left(\frac{x}{1 - x^2 - y^2}, \frac{y}{1 - x^2 - y^2}, \frac{x^2 + y^2}{1 - x^2 - y^2} \right) \end{aligned}$$

Note that the map Π is conformal, and so a conformal map $f : \mathbb{S}^2 \to \mathbb{S}^2$ induces a conformal map $\Pi \circ f \circ \Pi^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (and vice versa).

Definition (Continuity/differentiability/conformality in $\hat{\mathbb{C}}$): Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \hat{\mathbb{C}}$. We say that f *is continuous/differentiable/conformal at* z_0 if

1. For $z_0, f(z_0) \in \mathbb{C}$, the regular definition holds.
2. For $z_0 = \infty, f(z_0) \in \mathbb{C}$, the regular definition holds for $g(z) = f\left(\frac{1}{z}\right)$ at $z = 0$.
3. For $z_0 \in \mathbb{C}, f(z_0) = \infty$, the regular definition holds for $h(z) = \frac{1}{f(z)}$ at $z = z_0$.
4. For $z_0 = f(z_0) = \infty$, the regular definition holds for $k(z) = \frac{1}{f\left(\frac{1}{z}\right)}$ at $z = 0$.

Heuristic: If $z_0 = \infty$ we ***pre****compose with $\frac{1}{z}$, and if $f(z_0)$, we* ***post****compose with $\frac{1}{z}$.*

Definition (Möbius transformations): For $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{C})$, if $c \neq 0$, we define the corresponding Möbius transformation by

$$\begin{aligned} \lambda_X : \hat{\mathbb{C}} &\to \hat{\mathbb{C}} \\ z &\mapsto \begin{cases} \dfrac{az + b}{cz + d} & z \neq -\dfrac{d}{c}, \infty \\ \infty & z = -\dfrac{d}{c} \\ \dfrac{a}{c} & z = \infty \end{cases} \end{aligned}$$

When $c = 0$, we instead set

$$\begin{aligned} \lambda_X : \hat{\mathbb{C}} &\to \hat{\mathbb{C}} \\ z &\mapsto \begin{cases} \dfrac{az + b}{d} & z \neq \infty \\ \infty & z = \infty \end{cases} \end{aligned}$$

Proposition (Möbius transformations are biholomorphic): Let $X \in \mathrm{GL}_2(\mathbb{C})$ and $\lambda_X : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the corresponding Möbius transformation. Then λ_X is biholomorphic.

Thus to check things for Möbius transformations it ultimately suffices to check them for the main formula (where $z \neq -\frac{d}{c}, \infty$), as it is are continuous in $\mathbb{C} \setminus \left\{-\frac{d}{c}, \infty\right\}$, which is dense in $\hat{\mathbb{C}}$.

Definition (Set of Möbius transformations): We define

$$\text{Möb} := \left\{\lambda_X : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid X \in \mathrm{GL}_2(\mathbb{C})\right\}$$

Proposition (Compatibility of Möb with matrix multiplication): Let $X, X\tilde{X} \in \mathrm{GL}_2(\mathbb{C})$. Then $\lambda_X \circ \lambda_{\tilde{X}} = \lambda_{X\tilde{X}}$.

Corollary: We have $\text{Möb} \cong \mathbb{P}\,\mathrm{GL}_2(\mathbb{C})$, and so $\lambda_X^{-1} = \lambda_{X^{-1}}$ and $\lambda_X = \lambda_{\tilde{X}}$ if and only if $X = r\tilde{X}$ for $r \in \mathbb{C} \setminus \{0\}$.

Lemma (Fixed points of Möbius transformations): Let $\lambda \in \text{Möb}$ with $\lambda \neq \mathrm{id}_{\mathbb{C}}$. Then λ has exactly 1 or exactly 2 fixed points.

Proposition (Determining Möbius transformations): For triples of distinct $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and distinct $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, there is a unique Möbius transformation $\lambda \in \text{Möb}$ with $\lambda : z_i \mapsto w_i$ for $i = 1, 2, 3$.

Definition (Generalised circle): A *generalised circle* in $\hat{\mathbb{C}}$ is a circle or a line with ∞.

Lemma (Forms of generalised circles): A subset $C \subseteq \hat{\mathbb{C}}$ is a

1. Line if and only if it is the solution to $Az + \overline{A}\overline{z} + c = 0$ for some $A \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{R}$
2. Circle if and only if it is the solution to $|z|^2 + Az + \overline{A}z + c = 0$ for some $A \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{R}$, $c < |A|^2$

Proposition (Generalised circles and Möbius transformations): If $C \subseteq \hat{\mathbb{C}}$ is a generalised circle and $\lambda \in \text{Möb}$, then so is $\lambda(C)$.

Proposition (Determining generalised circles): For every $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ there is a *unique* generalised circle containing them.

Corollary (Generalised circles under Möbius transformations): If $z_1, z_2, z_3 \in C$ for a generalised circle C and $\lambda \in \text{Möb}$, then $\lambda(C)$ is the generalised circle containing $\lambda(z_1), \lambda(z_2), \lambda(z_3)$.

Definition (Generalised disk): A *generalised disk* is a connected component of $\hat{\mathbb{C}} \setminus C$, where C is a generalised circle.

Proposition (Generalised disks and Möbius transformations): If D is a generalised disk and λ is a Möbius transformation, then so is $\lambda(D)$.

4 Complex integration

Definition (Complex integral): Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set, $f : \mathcal{U} \to \mathbb{C}$ be an continuous function and $\gamma : [a, b] \to \mathcal{U}$ be a piecewise-C^1 path. We define the integral over γ to be

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

Definition (Extra definitions): For $z_1, z_2 \in \mathbb{C}$, we define the *line segment*

$$\begin{aligned}[z_1, z_2] : [0, 1] &\to \mathbb{C} \\ t &\mapsto z_1(1-t) + z_2 t\end{aligned}$$

If $\gamma : [0, 1] \to \mathbb{C}$ is a path, we define the reverse path by $\gamma^-(t) = \gamma(1-t)$. We say that a path is *closed* if $\gamma(0) = \gamma(1)$.

We have $[z, w]^- = [w, z]$, $|[z, w]| = |w - z|$ and $\int_{[z,w]} c d\tau = c(w - z)$.

Definition (Length of a path): Let $\gamma : [a, b] \to \mathbb{C}$ be a path. Its length is

$$|\gamma| := \int_a^b \left|\gamma'(t)\right| dt$$

Definition (Primitive): If $D \subseteq \mathbb{C}$ is a domain and $f \in C(D)$, a *primitive* of f is a function $F \in H(D)$ with $F' = f$.

Lemma (ML-estimate): If $\mathcal{U} \subseteq \mathbb{C}$ is open, $f \in C(\mathcal{U})$ and γ is a path in $\mathcal{U}$, then

$$\left|\int_\gamma f(z)dz\right| \leq ML$$

where $L = |\gamma|$ is the length of the path and $|f(\gamma(t))| \leq M$ for all $t \in [0, 1]$ (i.e. M is a bound on the size of f).

Usually we set $M = \max_{z \in D} |f(z)|$ or some variant of this. This is analogous to the triangle inequality for integrable functions in $\mathbb{R}$ (which is false in $\mathbb{C}$).

Theorem (Primitives on disks part 1): If Δ is a disk and $f \in C(\Delta)$, and for all oriented triangles $T \subseteq \Delta$ we have

$$\int_{\partial T} f(z)dz = \int_{[z_1,z_2]} f(z)dz + \int_{[z_2,z_3]} f(z)dz + \int_{[z_3,z_1]} f(z)dz = 0$$

then f has a primitive on Δ.

Theorem (Equivalent conditions for existence of a primitive): Let D be a domain, and $f \in C(D)$. Then the following are equivalent.

1. $\int_\gamma f(z)dz = 0$ for all closed paths $\gamma : [0, 1] \to D$.
2. $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$ for all pairs of paths $\gamma, \tilde{\gamma} : [0, 1] \to D$ with $\gamma(0) = \tilde{\gamma}(0)$ and $\gamma(1) = \tilde{\gamma}(1)$.
3. f has a primitive on D.

Theorem (Primitives on disks part 2; Goursat's Lemma): If $f \in H(\Delta(a, r))$, then $\int_{\partial T} f(z)dz = 0$ for all (oriented) triangles T.

4.1 Homotopy and integrals

Definition (Homotopy and homotopy types): Let $D \subseteq C$ be a domain, and $\gamma, \tilde{\gamma} : [0,1] \to D$ be (piecewise C^1 paths). A homotopy from γ to $\tilde{\gamma}$ in D is a continuous function $\Gamma : [0,1] \times [0,1] \to D$ with $\Gamma(t,0) = \gamma(t)$ and $\Gamma(t,1) = \tilde{\gamma}(t)$.

We say that Γ is a *CP-homotopy* (closed-path) if $\gamma_s(t) := \Gamma(t,s)$ is a closed path for each $0 \leq s \leq 1$, and that Γ is a *PCE-homotopy* (paths with common endpoints) if $\Gamma(0,s) = \Gamma(0,0)$ and $\Gamma(1,s) = \Gamma(1,0)$ for all s.

Lemma: Let $K \subseteq \mathcal{U} \subseteq \mathbb{C}$ be such that K is compact and $\mathcal{U}$ is open. Then there is some $\varepsilon > 0$ such that $\Delta(z, \varepsilon)$ for all $z \in K$.

Theorem (Integrals and CP/PCE homotopy): Let $f \in H(D)$, and $\gamma, \tilde{\gamma}$ be two piecewise C^1 paths in D. If Γ is a homotopy from γ to $\tilde{\gamma}$, then if $\gamma^i(s) = \Gamma(0,s)$ and $\gamma^f = \Gamma(1,s)$, then

$$\int_\gamma f(z)dz + \int_{\gamma^f} f(z)dz = \int_{\gamma^i} f(z)dz + \int_{\tilde{\gamma}} f(z)dz$$

Moreover, if Γ is a CP-homotopy or PCE-homotopy, then the integral is invariant, i.e.

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$$

Definition (Simply connected domain): We say that a domain $D \subseteq \mathbb{C}$ is *simply connected* if every path in D is homotopic to a constant path.

Theorem (Determining simple connectedness): Let D be a domain. Then the following are equivalent.

1. D is simply connected.
2. $\partial D \subseteq \hat{\mathbb{C}}$ is connected.
3. $\hat{\mathbb{C}} \setminus D$ is connected.

If D is bounded, these are equivalent to

1. $\mathbb{C} \setminus D$ is connected.
2. ∂D is connected.

Definition (Holomorphic on a closure): Let $D \subseteq \mathbb{C}$ be a domain. We say that $f \in H(\overline{D})$ if there is an open set $\overline{D} \subseteq \mathcal{U} \subseteq \mathbb{C}$ with $f \in H(\mathcal{U})$.

Theorem (Cauchy's integral formula for disks): Let $f \in H(\overline{\Delta(a,r)})$. Then

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{\zeta - a} d\zeta$$

Lemma: Let $a, b \in \mathbb{C}$ and $k \in \mathbb{N}$. Then there is a polynomial p such that

$$\frac{1}{(a-b)^k} - \frac{1}{a^k} - \frac{kb}{a^{k+1}} = \frac{b^2 p(a,b)}{(a-b)^k a^{k+1}}$$

Proposition: Let $g \in C(\partial\Delta(a,r))$. For $z \in D$, let

$$f(z) = \int_{|\zeta-a|=r} \frac{g(\zeta)}{\zeta - z} d\zeta$$

Then $f \in H(\Delta(a,r))$ is infinitely many times differentiable, with

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta-a|=r} \frac{g(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

Corollary (Extended Cauchy's integral formula): Let $f \in H(\overline{\Delta(a,r)})$. Then f is infinitely many times differentiable, and

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

Corollary (Holomorphic functions infinitely differentiable): If $f \in H(D)$, then $f' \in H(D)$.

5 Jordan domains

Definition (Simple path, Jordan arc, curve): A path $\gamma : [0,1] \to \mathbb{C}$ is simple if it does not overlap itself. That is, $\gamma(s) \neq \gamma(t)$ for $s, t \in (0,1)$. A *Jordan arc* is the image of an injective path $\gamma : [0,1] \to \mathbb{C}$, and a *Jordan curve* is the image of a closed simple path.

Definition (Jordan parametrisation): A *Jordan parametrisation* of a Jordan curve Γ is a map $\gamma : [0,1] \to \mathbb{C}$ with $\gamma([0,1]) = \Gamma$.

Definition (Jordan domain): We say that a domain D is a *Jordan domain* if ∂D is a disjoint union of Jordan curves.

Lemma (Integrals across Jordan parametrisations):

Defintion (Standard orientation): Let γ be a Jordan curve in the boundary of a Jordan domain D. The *standard orientation* of γ is the orientation with D on the left of the path γ.

Theorem (Cauchy's integral theorem): Let D be a Jordan domain, and $f \in H(\overline{D})$. Then

$$\int_{\partial D} f(z)dz = 0$$

Theorem (Cauchy's integral formula for Jordan domains): Let D be a Jordan domain, and $f \in H(\overline{D})$. Then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

6 Sequences and series

Definition (Normal convergence): Let $\mathcal{U}$ be an open set, and $f_n, f : \mathcal{U} \to \mathbb{C}$. We say that sequence of functions f_n converges normally to f on D if either

1. For every compact subset $K \subseteq \mathcal{U}$, $f_n|_K \to f|_K$ uniformly on K.
2. For every $z \in D$ there is a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ such that $f_n|_{\mathcal{U}'} \to f|_{\mathcal{U}'}$ uniformly.

Proposition (Equivalence of definitions): The above definitions for normal convergence are equivalent.

Proposition (Normal convergence preserves continuity): Let $f_n \in C(\mathcal{U})$ and $f_n \to f$ normally. Then $f \in C(\mathcal{U})$.

Proposition (Normal convergence preserves integrals): Let γ be a piecewise C^1 path in $\mathcal{U}$, and $f_n, f \in C(\gamma[0,1])$. If $f_n \to f$ uniformly on $\gamma([0,1])$, then

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$$

In particular, as $\gamma([0,1])$ is compact, so if $f_n \to f$ normally on D and γ is a path in D, then we have integral convergence.

Theorem (Normal convergence and holomorphic functions): Let D be a domain, $f_n \in H(D)$ and $f_n \to f$ normally on D. Then $f \in H(D)$, and

$$f_n^{(k)} \to f^{(k)}$$

normally for $k \in \mathbb{Z}_{\geq 0}$.

Definition (Series convergence): A series $\sum_{k=0}^\infty z_k$ converges absolutely if

$$\sum_{k=0}^\infty |z_k| < \infty$$

A series $\sum_{k=0}^\infty f_k$ converges *uniformly* / *pointwise* / *normally* if the series

$$F_n = \sum_{k=0}^n f_k$$

does. It converges *absolutely* if for every $z \in D$

$$\sum_{k=0}^\infty f_k(z)$$

converges absolutely.

Theorem (Series manipulations):

1. If $\sum_{k=0}^\infty z_k$ converges absolutely, then any rearrangement converges to the same limit.
2. If $\sum_{k,l=0}^\infty |z_{k,l}| < \infty$, then

$$\sum_{k=0}^\infty \sum_{l=0}^\infty z_{k,l} = \sum_{l=0}^\infty \sum_{k=0}^\infty z_{k,l} = \sum_{k,l=0}^\infty z_{k,l}$$

3. If $\sum_{k=0}^{\infty} z_k$ and $\sum_{k=0}^{\infty} w_k$ converge absolutely, then

$$\sum_{k,l=0}^{\infty} z_k w_l = \left(\sum_{k=0}^{\infty} z_k\right)\left(\sum_{k=0}^{\infty} w_l\right)$$

Proposition (M-test for numbers): Suppose that z_k and a_k are sequences with $|z_k| < a_k$ and $\sum_{k=0}^{\infty} a_k < \infty$. Then $\sum_{k=0}^{\infty} z_k$ converges absolutely.

Theorem (M-test for functions): Suppose that $S \subseteq \mathbb{C}$, $f_k : S \to \mathbb{C}$ and a_k are sequences with $|f_k(z)| < a_k$ for all $z \in S$ and $\sum_{k=0}^{\infty} a_k < \infty$. Then $\sum_{k=0}^{\infty} f_k$ converges uniformly and absolutely.

Definition (Power series): A power series centred at a is an expression of the form

$$\sum_{k=0}^{\infty} c_k (z-a)^k$$

Proposition (Abel's theorem): Let $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ be a power series. If $\sum_{k=0}^{\infty} c_k(z_0-a)^k$ converges, then $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges normally and absolutely on $\Delta(a, |z_0-a|)$.

Corollary (Radius of convergence): For any power series $\sum_{k=0}^{\infty} c_k(z-a)^k$, there is a number $R \in [0,\infty]$ such that the power series converges normally and absolutely on $\Delta(a,R)$ and diverges on $\mathbb{C} \setminus \overline{\Delta(a,R)}$.

Definition (Disk of convergence): Let $\sum_{k=0}^{\infty} c_k(z-a)^k$ be a power series, and R be its radius of convergence. Its *disk of convergence* is $\Delta(a,R)$.

Lemma (Convergence and limsups): Let (a_n) be a sequence in $\mathbb{C}$.

1. If $\limsup_{n\to\infty} |a_n|^{1/n} > 1$, then $a_n \not\to 0$.
2. If $\limsup_{n\to\infty} |a_n|^{1/n} < 1$, then $\sum_{k=0}^{\infty} a_k$ converges absolutely.

Proposition (Formula for radius of convergence): Let $\sum_{k=0}^{\infty} c_k(z-a)^k$ be a power series. Then the radius of convergence is

$$R = \left(\limsup_{n\to\infty} |c_n|^{1/n}\right)^{-1}$$

Proposition (Existence of a Taylor series): Let $f \in H(\Delta(a,r))$. Then for $z \in \Delta(a,r)$, we have

$$f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k$$

where $c_k = \frac{f^{(k)}(z)}{k!}$.

Theorem (Uniqueness of Taylor series): Every $f \in H(\Delta(a,r))$ is given by a unique Taylor series centred at a on $\Delta(a,r)$.

Theorem (Uniqueness): Let D be a domain and $f, g : D \to \mathbb{C}$. If there is a convergent sequence (z_n) in D with limit z in D and all z_n, z are distinct, then $f = g$.

Theorem (Stronger Liouville)[1]**:** Let f be an entire function. If there are $A, B \in \mathbb{R}$ with $|f(z)| \leq A|z|^n + B$, then f is a polynomial of degree at most n.

[1] Credit: Assignment 2 question 6

7 Laurent series and singularities

Definition (Laurent series): A *Laurent series* is a formal series of the form $\sum_{n=-\infty}^{\infty} c_n z^n$, where $c_n \in \mathbb{C}$ for each $n \in \mathbb{Z}$.

Definition (Annulus): Let $a \in \mathbb{C}$ and $r_1, r_2 \in [0, \infty]$. Then

$$\Delta(a, r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z - a| < r_2\}$$

Proposition (Convergence of Laurent series): Every Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ converges on an annulus of the form $\Delta(a, r_1, r_2)$.

Theorem (Existence and uniqueness of Laurent series): Let $f \in \Delta(a, r_1, r_2)$. Then f is given by the unique Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$$

with

$$c_k = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(\zeta)}{(\zeta - a)^{k+1}} d\zeta$$

Aside (Computing Laurent series): To compute Laurent series we usually use geometric series expansions or standard Taylor series for functions (such as e^z or $\sin(z)$), rather than using this integral formula.

Definition (Isolated singularity): Let D be a domain and $f \in H(D)$. We say that a is an isolated singularity of f if $a \notin D$, and $\Delta(a, 0, r) \subseteq D$ for some $r > 0$.

Definition (Types of isolated singularities): Let a be an isolated singularity of $f \in H(D)$. Let $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$ be the Laurent expansion of f around a. We say that a is

1. Removable if $c_k = 0$ for all $k < 0$.
2. A pole if $c_k = 0$ for all but finitely many $k < 0$.
3. Essential if $c_k \neq 0$ for infinitely many $k < 0$.

Theorem (Classification of removable singularities): Let a be an isolated singularity of $f \in H(D)$. The following are equivalent

1. a is a removable singularity
2. $\lim_{z \to a} f(z)$ exists in $\mathbb{C}$
3. There is a holomorphic extension of f to $D \cup \{a\}$
4. There is some $\varepsilon > 0$ such that f is bounded on $\Delta(0, 0, \varepsilon)$

Theorem (Classification of poles): Let a be an isolated singularity of $f \in H(D)$. The following are equivalent

1. a is a pole
2. $\lim_{z \to a} |f(z)| = \infty$ in $\hat{\mathbb{C}}$

Proposition (Factoring out zeroes from poles): Suppose that $f \in H(D)$ has a pole at a. Then there are unique $d \in \mathbb{Z}$ and $p \in H(D \cup \{a\})$ with $p(a) \neq 0$ such that

$$f(z) = (z-a)^d p(z)$$

Definition (Orders): Let D be a domain and $f \in H(D)$. If $f \not\equiv 0$ and $f(z) = (z-a)^d p(z)$ where $p(a) \neq 0$. If $d \geq 0$, we say that f has a zero of order d at a, and if $d \leq 0$, we say that f has a pole of order $-d$ at a.

Poles and zeroes of functions add and subtract in the usual way when functions are multiplied together and divided by one another.

Theorem (Casorati-Weierstrass): Let a be an essential singularity of $f \in H(D)$. Then for any $w \in \mathbb{C}$, there is a sequence (z_n) with $z_n \to a$ and $f(z_n) \to w$.

8 Residue theory

Definition: We say that $f \in H(\overline{D} \setminus \{a_1, \dots, a_n\})$ if f has a holomorphic extension to an open set $\overline{D} \setminus \{a_1, \dots, a_n\} \subseteq \mathcal{U}$.

Definition (Residue): Let $f \in H(\overline{D} \setminus \{a_1, \dots, a_n\})$, and $\Delta(a_i, \rho) \subseteq D$. The residue of f at a_j is

$$\operatorname{Res}_{a_j} f = \frac{1}{2\pi i} \int_{|z-a_j|=\rho} f(z) dz$$

Equivalently, $\operatorname{Res}_{a_i} f = c_{-1}$ in the Laurent expansion of f around a_i.

Theorem (Integrals and residues): Let D be a bounded Jordan domain & $f \in H(\overline{D} \setminus \{a_1, \dots, a_n\})$. Then

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{a_j} f$$

A special case of residues is $\operatorname{Res}_a \frac{f(z)}{g(z)} = \frac{f(a)}{g'(a)}$ if $f(a) \neq 0$ and g has a zero of order 1.

8.1 The argument principle

Lemma (Counting zeroes): Let $f(z) = (z-a)^d p(z)$ in a neighbourhood of a for $p(a) \neq 0$, then

$$\operatorname{Res}_a \frac{f'(z)}{f(z)} = d$$

Theorem (Argument principle): Let D be a bounded Jordan domain and $f \in H(\overline{D} \setminus \{a_1, \dots, a_n\})$. Suppose that f has zeroes of orders $K_1, \dots, K_m$ at $b_1, \dots, b_m$ and poles of orders $N_1, \dots, N_n$ at $a_1, \dots, a_n$. Then for $K = \sum_{j=1}^{m} K_j$ and $N = \sum_{j=1}^{n} N_j$,

$$\int_{\partial D} \frac{f'(z)}{f(z)} dz = 2\pi i (K - N)$$

Definition (Winding number): Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a path. The winding number of γ is

$$\operatorname{ind}_0 \gamma := \frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz$$

Theorem: Every path γ in $\mathbb{C} \setminus \{0\}$ is homotopic to some $\gamma_n(t) = e^{2\pi i k t}$. We thus have $\operatorname{ind}_0 \gamma \in \mathbb{Z}$ for all paths γ in $\mathbb{C} \setminus \{0\}$.

Proposition (Winding number and argument principle): Let γ be a path in $\mathbb{C}$, and $f \in H(D)$ be such that $f(\gamma(t)) \neq 0$ for all t. Then

$$\operatorname{ind}_0 f \circ \gamma = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

Definition (Change along the boundary): Let D be a Jordan domain, and f be holomorphic and non-zero in a neighbourhood of ∂D. Then

$$\Delta_{\partial D} f = 2\pi \operatorname{ind}_0 (f \circ \partial D)$$

Where the latter term is taken to mean a sum across boundary components.

Theorem (Argument principle, second version): Let D be a bounded Jordan domain and $f \in H(\overline{D} \setminus \{a_1, \dots, a_n\})$. Suppose that f has zeroes of orders $K_1, \dots, K_m$ at $b_1, \dots, b_m$ and poles of orders $N_1, \dots, N_n$ at $a_1, \dots, a_n$. Then for $K = \sum_{j=1}^{m} K_j$ and $N = \sum_{j=1}^{n} N_j$. Then

$$\Delta_{\partial D} f = 2\pi(K - N)$$

We denote the number of roots of a function h on a domain D as $K_{h,D}$ in the below theorem, and assume f, g have no poles.

9 Large final results

9.1 Counting zeroes

Theorem (Rouche): Let D be a bounded Jordan domain and $f, g \in H(\overline{D})$. If $|f(z)| > |g(z)|$ on ∂D, then $K_{f,D} = K_{f+g,D}$.

9.2 Constraints on holomorphic functions

Theorem (Open mapping theorem): Let D be a domain and $f \in H(D)$ be non-constant. Then $f(D)$ is open.

Theorem (Injectivity of holomorphic functions): Let $f \in H(D)$ be injective. Then $f'(z) \neq 0$ for all $z \in D$.

Theorem (Biholomorphic functions): Let $f \in H(D)$ be injective. Then f is a conformal equivalence between D and $f(D)$.

Theorem (Strong maximum modulus principle): If D is a domain and $f \in H(D)$ achieves its maximum, then f is constant.

Corollary (Weak maximum modulus principle): If $f \in H(\overline{D})$, then

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|$$

Theorem (The Schwarz Lemma): Let $f \in H(\Delta)$ be such that $|f(z)| \leq 1$ for all $z \in \Delta$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \Delta$, and if $|f(z)| = |z|$ for $z \in \Delta \setminus \{0\}$, then $f(z) = \beta z$ for some $|\beta| = 1$.

9.3 Conformal maps and the unit disk

We denote the conformal maps from the unit disk to itself as $\mathrm{Conf}(\Delta)$

Proposition: Every function of the form $f(z) = \beta \frac{z-w}{\overline{w}z+1}$ for $|\beta| = 1$ and $|w| < 1$ is in $\mathrm{Conf}(\Delta)$.

Lemma: If $f \in \mathrm{Conf}(\Delta)$ is such that $f(0) = 0$, then $f(z) = \beta z$ for some $|\beta| = 1$.

Theorem (Classification of maps in Conf(Δ)): Every map in $\mathrm{Conf}(\Delta)$ is of the form

$$f(z) = \beta \frac{z - w}{\overline{w}z + 1}$$

for some $|\beta| = 1$ and $|w| < 1$.

Lemma (Square roots on simply connected domains): Let $D \subseteq \mathbb{C}$ be simply connected and $0 \notin D$. Then there is injective $k \in H(D)$ with $k(z)^2 = z$ for all $z \in D$, and at most one of $w, -w$ is in $k(D)$ for each $w \in \mathbb{C}$.

Proposition (Conformal equivalences on simply connected domains): Let D be a simply connected domain. Then D is conformally equivalent to a bounded domain.

Lemma (Contrapositive of the Schwarz Lemma): Let $D' \subsetneq \Delta$ be a simply connected domain with $0 \notin D'$. Then there exists an injective map $h \in H(D')$ with $h(0) = 0$, $h'(0) > 1$ and $h(D') \subseteq \Delta$.

Theorem (Hurwitz): Suppose that $f_n, f \in H(D)$, $f_n \to f$ normally and $f(z_0) = 0$. Then for all $r > 0$, there is some $N \in \mathbb{N}$ such that

$$K_{f_n,\Delta(a,r)} \geq 1$$

for all $n > N$

Lemma (Buildup to Montel's theorem): Let D be a domain, and $\mathcal{F} \subseteq H(D)$ be such that $|f(z)| \leq M$ for all $z \in D$ and $f \in \mathcal{F}$. Then for any compact subset $K \subseteq D$, every sequence $(f_n|_K)$ has a uniformly convergent subsequence.

Theorem (Montel): Let D be a domain, $\mathcal{F} \subseteq H(D)$ be such that for every compact subset $K \subseteq D$, $|f(z)| \leq M$ for all $z \in K$ and $f \in \mathcal{F}$. Then every sequence (f_n) has a normally convergent subsequence.

Theorem (The Riemann Mapping Theorem): Let D be a simply connected domain and $z_0 \in D$. Then there is a conformal equivalence $f : D \to \Delta$ with $f(z_0) = 0$.