Compactness of operators

Stanley Li

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Let H_1, H_2 be Hilbert spaces, and recall that an operator $T : H_1 \to H_2$ is *compact* if one (and hence all) of the following hold:

1. $\overline{T(B(0,1))}$ is compact;
2. For any bounded sequence (f_n) in H_1, (Tf_n) has a convergent subsequence in H_2;
3. The image of any bounded set is pre-compact.

Note that if T is continuous (bounded) we have $T(\overline{B(0,1)}) \subseteq \overline{T(B(0,1))}$, and so $\overline{T(B(0,1))} = \overline{T(\overline{B(0,1)})}$.

If T has finite rank (i.e. $\dim(T(H_1)) < \infty$) then T is automatically compact. In complete generality, we have the following characterisation.

Theorem 1. *Let $T : H_1 \to H_2$ be a operator. Then*

1. *T is compact if and only if there is a sequence $(T_n : H_1 \to H_2)_{n\in\mathbb{N}}$ of finite rank operators with $T_n \to T$ in operator norm; and*
2. *T is compact if and only if T^* is compact.*

In theory these should then give a way to show that any compact operator is indeed compact, but this may end up being unwieldy.

The Arzela-Ascoli theorem

Theorem 2 (Arzela-Ascoli)**.** *Let X be a compact metric space. Then a subset $\mathcal{F} \subseteq C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded and uniformly equicontinuous.*

Note that by the Arzela-Ascoli theorem, we can deduce compactness (an extrinsic property) from uniform continuity (intrinsic property), which is a statement about "compatibility" of the functions in the space $C(X, \mathbb{R}^n)$.

Let $H_1 \subseteq C(X, \mathbb{R}^n), H_2 \subseteq C(Y, \mathbb{R}^m)$ be function spaces, and $T : H_1 \to H_2$ be an operator – usually we will have $H_i = L^p(\Omega), C^k(\Omega)$ or a Sobolev space. To show that T is compact, that is, $\overline{T(B(0,1))}$ is compact, we have that the set $\overline{T(B(0,1))}$ is

1. bounded *iff* T is bounded, i.e. $\|T\| < \infty$;
2. always closed; and

3. uniformly equicontinuous *iff* $T(B(0,1))$ is uniformly equicontinuous.

Where the last of these follows from the more general fact that $\overline{A}$ is uniformly equicontinuous whenever A is. Thus it suffices to just show that

1. T is bounded, i.e. there is M with $\|Tx\| \leq M \|x\|$; and
2. $T(B(0,1))$ is uniformly equicontinuous, i.e. for any $\varepsilon > 0$, there is $\delta > 0$ so that for any $f \in B(0,1) \subseteq H_1$ and $y_1, y_2 \in Y$ with $d(y_1, y_2) < \delta$, we have

$$\|Tf(y_1) - Tf(y_2)\| < \varepsilon$$