Some good-to-know results for functional analysis

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Uniform equicontinuity and the Arzela-Ascoli theorem

Recall that for metric spaces X and Y, a set $\mathcal{F}$ of functions $f : X \to Y$ is *uniformly equicontinuous* if for every $\varepsilon > 0$, there is $\delta > 0$ so that if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$ for all $f \in \mathcal{F}$.

In full generality, we have the following remarkable statement.

Theorem 1 (Arzela-Ascoli)**.** *Let X be a compact metric space. Then a subset $\mathcal{F} \subseteq C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded and uniformly equicontinuous.*

We also have the following weaker equivalence as a result, by considering closures. We say that a subset S of a metric space X is *pre-compact* (or relatively compact) if every sequence in S has a convergent subsequence (with limit in X), or equivalently if it has compact closure.

Corollary 1. *If X is a complete metric space, then a subset $\mathcal{F} \subseteq C(X, \mathbb{R}^n)$ is bounded and uniformly equicontinuous if and only if it is pre-compact.*

Some measure theory inequalities

We summarise some key inequalities in abstract measure theory; the background details of abstract measure theory are given in the section below.

On the space of measurable functions $X \to [-\infty, \infty]$ (or alternatively to $\mathbb{C} \cup \{\infty\}$), for each $p \in [1, \infty]$ we have a norm[1] given by

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} \tag{1}$$

for $p < \infty$, and $\|f\|_\infty = \sup_{x \in X} |f(x)|$ is the usual sup-norm; these yield the spaces $L^p(X, \mu)$ (defined in the natural way). We then have Hölder's inequality:

Theorem 2 (Holder's inequality)**.** *Let (X, Σ, μ) be a measure space, and f, g be measurable real or complex-valued functions, and $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

We can apply this to see that (1) defines a norm:

[1] Though knowledge that this formula defines a norm only follows from Minkowski's inequality

Theorem 3 (Minkowski's inequality)**.** *Let (X, Σ, μ) be a measure space, and f, g be measurable real or complex-valued functions, and $p \in [1, \infty]$. Then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Abstract measure theory background

We first define some of the basic notions of abstract measure theory, before describing some important inequalities. For a set X and Σ a σ-algebra on X, we say a function $\mu : \Sigma \to [0, \infty]$ is a *measure* if

1. (Non-negativity) $\mu(\emptyset) = 0$; and
2. (Countable additivity) If $\{E_k\}_{k=1}^{\infty}$ is a collection of disjoint sets in Σ, then

$$\mu\left(\coprod_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

We say that the triple (X, Σ, μ) is a *measure space*, and that Σ is the set of *measurable sets*. Given a measure space (X, Σ, μ), we say a function $f : X \to [-\infty, \infty]$ is *measurable* if $f^{-1}[-\infty, a)$ is measurable for every $a \in \mathbb{R}$. The space of measurable functions $X \to [-\infty, \infty]$ has the same closure properties as in the Euclidean case: it is closed under pointwise addition and multiplication, suprema, infima and hence also limsup and liminfs.

We can then define the integral for measurable functions in stages as in the Euclidean case:

1. If $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ is a simple function, then

$$\int_X f d\mu := \sum_{i=1}^{n} a_i \mu(E_i)$$

2. If f is bounded and supported on a measurable set of finite measure, then

$$\int_X f d\mu := \lim_{n \to \infty} \int_X \varphi_n d\mu$$

   for any sequence $\varphi_n \to f$ pointwise almost everywhere;
3. If f is non-negative, then

$$\int_X f d\mu := \sup_{\substack{0 \leq g \leq f \\ g \text{ as in (2)}}} \int_X g d\mu$$

4. If f is measurable, letting $f^{\pm} = \max(0, \pm f)$ (so that $f = f^+ - f^-$),

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu$$

In a similar vein, a complex-valued function $f = u + iv$ is measurable if the components u and v are measurable, and its integral is

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu$$