MATH2320 Summary

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Sets, functions and cardinality

Some basic set constructions

Naively, a set is a collection of distinct elements. This gives us Russell's paradox as a problem, namely the "set"

$$S = \{T \mid T \notin T\}$$

With more formal and careful axiomatisation, this is not considered a set, though we will not worry much about this.

From sets, we can build more sets. Namely, we define the intersection, union and cartesian product of two sets. These are given below.

Let A, B be sets. Then

$$A \cup B := \{x \mid x \in A \vee x \in B\}$$
$$A \cap B := \{x \mid x \in A \wedge x \in B\}$$
$$A \setminus B := \{x \mid x \in A \wedge x \notin B\}$$

We also define more general unions and intersections.

Let $\{A_i\}_{i \in I}$ be sets. Then the following are also sets

$$\bigcup_{i \in I} A_i := \{x \mid x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i \in I} A_i := \{x \mid x \in A \text{ for all } i \in I\}$$
$$A \times B := \{(a, b) \mid a \in A, b \in B\}$$

Given these definitions, we also have the following results.

Within a fixed set X, we can also define the complement X^c, as

$$A^c := X \setminus A$$

Proposition(Basic set properties): Let A, B, C be sets and $\{B_i\}_{i\in I}$ a family of sets.

Commutativity	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associativity	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributivity	$A \cap \left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in A} (A \cap B_i)$	$A \cup \left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in A} (A \cup B_i)$
De Morgan's Laws	$\left(\bigcup_{i\in I} B_i\right)^c = \bigcap_{i\in I} B_i^c$	$\left(\bigcap_{i\in I} B_i\right)^c = \bigcup_{i\in I} B_i^c$

The cartesian product of a family of sets $X_1, \ldots, X_n$ is then defined as follows

$$X_1 \times \ldots \times X_n := \{(x_1, \ldots, x_n) \mid x_i \in X_i\}$$

Functions

Let X, Y be sets. A function $f : X \to Y$ is a rule which assigns each $x \in X$ a unique $y \in Y$.

The graph of a function $f : X \to Y$ is the subset of $X \times Y$ defined by

$$G(f) := \{(x, f(x)) \mid x \in X\}$$

Given a function $f : X \to Y$, and subsets $A \subseteq X$, $B \subseteq Y$, we define the image and inverse image of A as

$$f(A) := \{f(x) \in Y \mid x \in A\}$$
$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}$$

Let $f : X \to Y$ be a function. It is

1. *Injective* if whenever $a \neq b$, $f(a) = f(b)$.
2. *Surjective* if for every $y \in Y$, there is $x \in X$ with $f(x) = y$
3. *Bijective* if it is injective and surjective.

Proposition(Basic image properties): With these definitions, the following hold:

1. For all $A \subseteq X$, $A \subseteq f^{-1}(f(A))$. This is an equality iff f is injective.
2. For all $B \subseteq Y$, $f(f^{-1}(B)) \subseteq B$. This is an equality iff f is surjective.
3. For a family of sets $\{A_i\}_{i\in I}$,

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i) \qquad f\left(\bigcup_{i\in I} A_i\right) \supseteq \bigcup_{i\in I} f(A_i)$$

$$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i) \qquad f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i)$$

Given two functions $f : X \to Y$, $g : Y \to Z$, their composition is the function $g \circ f : X \to Z$, with $(g \circ f)(x) = g(f(x))$. This composition is associative. That is, $(f \circ g) \circ h = f \circ (g \circ h)$.

A function $f : X \to Y$ is invertible if there exists a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In this case, write $g = f^{-1}$.

A function is invertible if and only if it is a bijection. Further, the composition of two bijections is a bijection.

Cardinality

Definition(Equinumerous): Let X, Y be sets. We say that X and Y are equinumerous if there exists a bijection $f : X \to Y$. We write $X \sim Y$ in this case.

The equinumerous relation $\sim$ is an equivalence relation between sets.

Note that $\mathbb{N} \sim \mathbb{Z}$ and $(0, 1) \sim \mathbb{R}$.

Definition(Finite, infinite, denumerable): We say that a set is **finite** if it is equinumerous to $\{1, \ldots, n\}$ for some n. We call a set **infinite** if it is not finite. A set is **denumerable** if it is equinumerous to $\mathbb{N}$.

Definition(Countable, uncountable): Putting these together, we have that a set is **countable** if it is either finite or denumerable. A set is **uncountable** if it is not countable.

Definition(Cardinality): Let A be a set. The cardinality $\overline{\overline{A}}$ is defined by

1. $\overline{\overline{\emptyset}} = 0$
2. $\overline{\overline{\{1, \ldots, n\}}} = n$
3. $\mathbb{N} = d$
4. If $A \sim B$, then $\overline{\overline{A}} = \overline{\overline{B}}$
5. If A is equinumerous to a subset of B, then $\overline{\overline{A}} \leq \overline{\overline{B}}$ or $\overline{\overline{B}} \geq \overline{\overline{A}}$

This last condition is equivalent to saying that there is an injective map $A \hookrightarrow B$, or a surjective map $B \twoheadrightarrow A$.

Proposition(Uncountability of $\mathbb{R}$): There does not exist a bijection $\mathbb{N} \to \mathbb{R}$.

We refer to $\overline{\overline{\mathbb{R}}}$ as c.

Lemma(Smallest infinite set): Every infinite set has a denumerable subset.

Proposition(Basic properties):

1. $0 < 1 < 2 < \ldots < d < c$
2. If A is infinite $\overline{\overline{A}} \geq d$
3. $\overline{\overline{A}} \leq \overline{\overline{B}} \wedge \overline{\overline{B}} \leq \overline{\overline{C}} \implies \overline{\overline{A}} \leq \overline{\overline{C}}$

Definition(Disjoint, mutually disjoint): Two sets A, B are called disjoint if $A \cap B = \emptyset$.

A family of sets $\{A_i\}_{i \in I}$ is mutually disjoint if for $i \neq j$, $A_i \cap A_j = \emptyset$.

Lemma(Mutually disjoint unions): Let I be an indexing set, and $\{A_i\}_{i \in I}, \{B_i\}_{i \in I}$ be families of sets with $A_i \sim B_i$ for each $i \in I$. Then

$$\bigcup_{i \in I} A_i \sim \bigcup_{i \in I} B_i$$

Theorem(Schröder-Bernstein Theorem): If $\overline{\overline{A}} \leq \overline{\overline{B}}$ and $\overline{\overline{B}} \leq \overline{\overline{A}}$, then $\overline{\overline{A}} = \overline{\overline{B}}$.

Theorem(Countable subsets): Any subset of a countable set is countable.

Proposition(Countable unions of countable sets): If I is countable and each A_i in the family of sets $\{A_i\}_{i \in I}$ is countable, then $\bigcup_{i \in I} A_i$ is countable.

Proposition(Uncountable unions of uncountable sets): If $\overline{\overline{I}} \leq c$ and $\{A_i\}_{i\in I}$ have cardinality c, then $\bigcup_{i\in I} A_i$ has cardinality c.

Proposition(Countability of $\mathbb{Q}$): $\mathbb{Q}$ is countable, with $\overline{\overline{\mathbb{Q}}} = d$.

Proposition(Infinite unions with denumerable sets): If A is infinite and B is denumerable, then $\overline{\overline{A \cup B}} = \overline{\overline{A}}$.

Corollary(Uncountability of $\mathbb{R} \setminus \mathbb{Q}$): $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

Proposition(Finite products of countable sets): Let $\{A_i\}_{i=1}^n$ each have cardinality d. Then the product $A_1 \times \ldots \times A_n$ has cardinality d.

Theorem(Cardinality of $\mathbb{R}^\infty$): The set of real-valued sequences $\mathbb{R}^\infty$ has $\overline{\overline{R}} = c$.

Proposition(Finite products of uncountable sets): Let $\{A_i\}_{i=1}^n$ each have cardinality at most c. If at least one A_i has cardinality c, then the product $A_1 \times \ldots \times A_n$ has cardinality c.

Proposition(Infinite cardinalities): Let A be a set. Then $\overline{\overline{A}} < \overline{\overline{\mathcal{P}(A)}}$

The **Continuum Hypothesis** hypothesises that there are no sets A with $d < \overline{\overline{A}} < c$. This is independent of the standard ZFC axioms of mathematics.

Metric spaces

Definition(Metric, metric space): Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

1. (Positivity) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$.
3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$

We call the set X along with the metric d a *metric space*, and write (X, d) for this.

Examples(Some important metric space examples):

1. The discrete metric, with $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$.
2. The p-adic metric (p prime) on $\mathbb{Z}$, defined by writing $x - y = p^k n$ with $p \nmid n$, and $d(x, y) = \frac{1}{k+1}$ when $x \neq y$.

Definition(Vector space): A vector space on $\mathbb{R}$ is a set X with two operations $+ : X \times X \to X$, $\cdot : \mathbb{R} \times X \to X$, such that $(X, +)$ is an abelian group, $\cdot$ and $+$ distribute over one another, and $\cdot$ follows group action axioms.

Lemma(Basic properties on a vector space)

1. The zero element is unique.
2. For any $x \in X$, $-x$ is unique.

3. $0x = 0, a0 = 0$, and $-x = (-1)x$.

Definition(Norm, normed vector spaces): A norm on a vector space X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $a \in \mathbb{R}$,

1. (Positivity) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.
2. (Homogeneity) $\|ax\| = |a| \, \|x\|$
3. (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$

We call a vector space X equipped with a norm a *normed vector space.*

Proposition(Induced metric): Let $\|\cdot\|$ be a norm. Then the function $d : X \times X \to \mathbb{R}$, $d(x, y) = \|x - y\|$ is a metric.

Lemma(Equivalent condition for an induced norm): A metric d on a vector space is induced by a norm if and only if $d(ax, ay) = |a|d(x, y)$.

Definition(Inner product, inner product space): An inner product on a vector space over $\mathbb{R}$ is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

1. (Positivity) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.
2. (Symmetry) $\langle x, y \rangle = \langle y, x \rangle$.
3. (Linearity) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

Proposition(Induced norm): Let $\langle \cdot, \cdot \rangle$ be an inner product. Then the function $\|\cdot\| : X \times X \to \mathbb{R}$ given by $\|x\| = \langle x, x \rangle$ satisfies $| \langle x, y \rangle | \leq \|x\| \, \|y\|$. and induces a norm on X.

Definition(Open ball): Let (X, d) be a metric space, and $x \in X$. The open ball at x of radius $r \geq 0$ is defined to be

$$B_r(x) := \{y \in X \mid d(y, x) < r\}$$

Definition(Neighbourhood): A set $U \subseteq X$ is called a neighbourhood of a point x if there is $r > 0$ such that $B_r(x) \subseteq U$.

Definition(Interior, exterior, boundary point): Let (X, d) be a metric space, and let $A \subseteq X$ and $x \in X$.

1. x is called an interior point if there is $r > 0$ such that $B_r(x) \subseteq A$. The set of interior points of A is denoted $\text{int}(A)$.
2. x is called an exterior point if there is $r > 0$ such that $B_r(x) \subseteq A^c$. The set of exterior points of A is denoted $\text{ext}(A)$.
3. x is called a boundary point if for every $r > 0$, the open ball $B_r(x)$ contains points in A and A^c.

Lemma(Some simple properties):

1. $\text{int}(A) \subseteq A$, and $\text{ext}(A) \subseteq A^c$
2. $\text{ext}(A) = \text{int}(A^c)$, and $\text{int}(A) = \text{ext}(A^c)$
3. $X = \text{int}(A) \coprod \text{ext}(A) \coprod \partial A$

4. If $A \subseteq B \subseteq X$, then $\text{int}(A) \subseteq \text{int}(B)$, and $\text{ext}(B) \subseteq \text{ext}(A)$.

Lemma(Interior, exterior and boundary of an open ball):

1. $\text{int}(B_r(x)) = B_r(x)$
2. $\text{ext}(B_r(x)) \supseteq \{y \in X \mid d(y, x) > r\}$
3. $\partial B_r(x) \subseteq \{y \in X \mid d(y, x) = r\}$

In a normed metric space, the above inclusions are all equalities.

Definition(Limit point, isolated point, closure): Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$.

1. x is a *limit point* of A if for every $r > 0$,
$$A \cap (B_r(x) \setminus \{x\}) \neq \emptyset$$
2. x is an *isolated point* of A if there is $r > 0$ such that
$$A \cap B_r(x) = \{x\}$$
3. The *closure* of A is the set
$$\overline{A} := A \cup \{x \in X \mid x \text{ is a limit point of } A\}$$

Proposition(Equivalent formulations): Let X be a metric space, and $A \subseteq X$. Then

1. x is a limit point of A if and only if every ball $B_r(x)$ contains infinitely many points in A (for $r > 0$)
2. $x \in \overline{A}$ if and only if $B_r(x)$ contains a point in A for every $r > 0$.

Theorem(Equivalent closure formulations): Let X be a metric space, and $A \subseteq X$. Then
$$\overline{A} = \text{ext}(A)^c = \text{int}(A) \cup \partial A = A \cup \partial A$$

Definition(Open, closed): Let X be a metric space, and $A \subseteq X$. We say that A is *open* if $\text{int}(A) \subseteq A$. We say that A is *closed* if $A^c = X \setminus A$ is open.

Proposition(Simple equivalences): Let X be a metric space, and $A \subseteq X$. Then

1. A is open if and only if A^c is closed.
2. A is open if and only if $A = \text{int}(A)$
3. A is open if and only if for every $x \in A$, there is $r > 0$ such that $B_r(x) \subseteq A$

Theorem(Some open sets): Let X be a metric space, and $A \subseteq X$. Then $\text{int}(A)$ and $\text{ext}(A)$ are open sets.

Theorem(Openness and intersection / union): A finite intersection of open sets is open, and an infinite union of open sets is open.

Theorem(Closure and closedness): Let X be a metric space and $A \subseteq X$. Then A is closed if and only if $A = \overline{A}$.

Theorem(Some closed sets): Let X be a metric space and $A \subseteq X$. Then $\overline{A}$ and ∂A are closed sets.

Theorem(Closedness and intersection / union): An arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.

Theorem(Smallest closed, largest open set): Let X be a metric space and $A \subseteq X$. If $A \subseteq B$ is a closed set, then $\overline{A} \subseteq X$. If $C \subseteq A$ is an open set, then $C \subseteq \text{int}(A)$

Theorem(Structure of open sets in $\mathbb{R}$)

Definition(Metric subspace): Let (X, d) be a metric space. A metric subspace of X is a set S with metric $d_S = d\big|_S$, or

$$d_S : S \times S \to \mathbb{R}$$
$$d_S(x, y) = d(x, y)$$

The open ball in S centred at $x \in S$ of radius r is written $B_r^S(x)$.

Lemma(Open balls in a subspace): For every $r > 0$ and $x \in S$, $B_r^S(x) = S \cap B_r(x)$.

Theorem(Open sets in a subspace): A set $A \subseteq S$ is open if and only if it can be written as $A = S \cap O$ where $O \subseteq X$ is open.

Theorem(Closed sets in a subspace): A set $A \subseteq S$ is closed if and only if it can be written as $A = S \cap C$ where $C \subseteq X$ is closed.

metric subspaces, open sets in a subspace

Definition(Sequence): Let X be a metric space. A sequence in X is a function $f : \mathbb{N} \to X$. We write x_n for $f(n)$, and write the sequence as $(x_n) = (x_n)_{n=1}^{\infty} = (x_n)_{n \in \mathbb{N}}$.

Definition(Subsequence): Let (x_n) be a sequence. A subsequence of (x_n) is a sequence (x_{n_k}) where (n_k) is a strictly increasing sequence of natural numbers.

Definition(Convergence): A sequence (x_n) in X is convergent to a point $x \in X$ if as $n \to \infty$, $x_n \to x$. Equivalently, $x_n \to x$ if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > N$.

Proposition(Unique limits): Let (x_n) be a sequence. If $x_n \to x$ and $x_n \to y$, then $x = y$.

Definition(Bounded set): Let X be a metric space and $A \subseteq X$. We say that A is bounded if there is $x \in X$ and $r > 0$ such that $A \subseteq B(x, r)$.

Definition(Bounded sequence): Let X be a metric space and $(x_n) \in X$ be a sequence. We say that (x_n) is bounded if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded.

Theorem(Convergent implies bounded): Let X be a metric space. Every convergent sequence (x_n) in X is bounded.

Theorem(Distance between distances): Let X be a metric space, and $(x_n), (y_n)$ be convergent sequences in X with $x_n \to x$ and $y_n \to y$. Then $d(x_n, y_n) \to d(x, y)$.

Theorem(Norms and convergence): Let X be a normed vector space, $(x_n), (y_n)$ be sequences in X, and (α_n) be a sequence in $\mathbb{R}$, with $x_n \to x$, $y_n \to y$ and $\alpha_n \to \alpha$. Then

1. $x_n + y_n \to x + y$
2. $\alpha_n x_n \to \alpha x$

Theorem(Inner products and convergence): Let X be an inner product space, and $(x_n), (y_n)$ be sequences in X with $x_n \to x$, $y_n \to y$. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

Theorem(Equivalent closure formulation): Let X be a metric space, and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a sequence (x_n) in A such that $x_n \to x$.

Corollary(Equivalent closed formulation): Let X be a metric space and $A \subseteq X$. Then A is closed if and only if every convergent sequence in A has a limit in A.

Definition(Increasing, decreasing): Let (x_n) be a sequence in $\mathbb{R}$. It is

1. *Increasing* if $x_{n+1} \geq x_n$
2. *Strictly increasing* if $x_{n+1} > x_n$
3. *Decreasing* if $x_{n+1} \leq x_n$
4. *Strictly decreasing* if $x_{n+1} < x_n$

for all $n \in \mathbb{N}$

Definition(Monotonicity): A sequence (x_n) is monotone if it is either increasing or decreasing.

Theorem(Monotone Convergence): Every bounded, monotone sequence in $\mathbb{R}$ is convergent.

Proposition(Monotone subsequences): Every sequence in $\mathbb{R}$ has a monotone subsequence.

Theorem(Bolzano-Weierstrass in $\mathbb{R}$): Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem(Bolzano-Weierstrass in $\mathbb{R}^n$): Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Summary of important, applicable results, viewpoints and definitions

Sets and functions

For a function $f : X \to Y$,

1. [[**Left-invertibility**]] $A \subseteq f^{-1}(f(A))$, and $f^{-1}(f(A)) = A$ for all $A \subseteq X$ if and only if f is injective.
2. [[**Right-invertibility**]] $f(f^{-1}(B)) \subseteq B$, and $f(f^{-1}(B)) = B$ for all $B \subseteq Y$ if and only if f is surjective.
3. [[**Intersections and unions with image**]] $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$, $f\left(\bigcap_{i\in I} A_i\right) \supseteq \bigcap_{i\in I} f(A_i)$

Cardinality

$X \sim Y \iff \exists f : X \to Y$ bijective $\iff \overline{\overline{X}} = \overline{\overline{Y}}$

$\overline{\overline{X}} \leq \overline{\overline{Y}} \iff \exists f : X \to Y$ injective $\iff \exists g : Y \to X$ surjective

The Schröder-Bernstein Theorem: If there is $f : A \to B$ and $g : B \to A$ injective, then there is a bijection $h : A \to B$.

We can equivalently write this as $\overline{\overline{A}} \leq \overline{\overline{B}} \wedge \overline{\overline{B}} \leq \overline{\overline{A}} \implies \overline{\overline{A}} = \overline{\overline{B}}$.

The following sets are **countable**:

1. A countable union of countable sets
2. A finite product of countable sets
3. $\mathbb{N}^n, \mathbb{Z}^n, \mathbb{Q}^n$

If B is denumerable and A is infinite, then $\overline{\overline{A \cup B}} = \overline{\overline{A}}$

The following sets have **cardinality c**:

1. $\mathbb{R}^n, \mathbb{R}^\infty, \mathbb{R} \setminus \mathbb{Q}, \mathcal{P}(\mathbb{N})$
2. A union over an index set $1 \leq \overline{\overline{I}} \leq c$ with sets of cardinality c.
3. A countable union of cardinality c sets

Metric spaces

Basic objects: Metrics, metric spaces, metric subspaces, vector spaces, norms, normed vector spaces, inner products, inner product spaces.

Induced metric: $d(x, y) = \|x - y\|$. Induced if and only if $d(ax, ay) = |a|d(x, y)$.

Induced norm: $\|x\| = \sqrt{\langle x, x\rangle}$. Induced if and only if $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Equivalent formulations of open, closed, interior, exterior, closure, subspaces

Interior, exterior and boundary

Definitions in quantifiers $(\mathrm{int}(A), \partial A, \mathrm{ext}(A))$

$$\begin{aligned}
x \in \mathrm{int}(A) &\iff (\exists r > 0)(B_r(x) \subseteq A) \iff (\exists r > 0)(\forall y \in X)(d(y,x) < r \implies y \in A) \\
x \in \mathrm{ext}(A) &\iff (\exists r > 0)(B_r(x) \subseteq A^c) \iff (\exists r > 0)(\forall y \in X)(d(y,x) < r \implies y \in A^c) \\
x \in \partial A &\iff (\forall r > 0)((B_r(x) \cap A \neq \emptyset) \wedge (B_r(x) \cap A^c \neq \emptyset))
\end{aligned}$$

General properties:

1. $\mathrm{int}(A) \subseteq A$ and $\mathrm{ext}(A) \subseteq A^c$
2. $\mathrm{int}(A) = \mathrm{ext}(A^c)$ and $\mathrm{ext}(A) = \mathrm{int}(A^c)$
3. $X = \mathrm{int}(A) \coprod \partial A \coprod \mathrm{ext}(A)$
4. If $A \subseteq B \subseteq X$, $\mathrm{int}(A) \subseteq \mathrm{int}(B)$ and $\mathrm{ext}(A) \supseteq \mathrm{ext}(B)$.

Closure
Let A' be the set of limit points of A. Then

$$\overline{A} = A \cup A' = \mathrm{ext}(A)^c = \mathrm{int}(A) \cup \partial A = A \cup \partial A$$

For set membership, we also have

$$x \in \overline{A} \iff (\exists (x_n) \in X^\infty)(x_n \to x)$$

The closure can also be seen as the smallest closed set containing A. That is, if B is closed and $A \subseteq B$, then $\overline{A} \subseteq B$.

Openness

$$\begin{aligned}
A \text{ open} &\iff A \subseteq \mathrm{int}(A) \iff A = \mathrm{int}(A) \iff (\forall x \in A)(\exists r > 0)(B_r(x) \subseteq A) \\
&\iff (\forall x \in A)(\exists r > 0)(\forall y \in X)(d(y,x) < r \implies y \in A) \\
&\iff A = \bigcup_{B_r(x) \subseteq A} B_r(x)
\end{aligned}$$

For every set $A \subseteq X$, $\mathrm{int}(A), \mathrm{ext}(A)$ are open sets. A finite intersection and arbitrary union of open sets is also open.

The interior of a set A can be seen as the largest open set contained in A. That is, if B is open and $B \subseteq A$, $B \subseteq \mathrm{int}(A)$.

Closedness

$$A \text{ closed} \iff A^c \text{ open} \iff A = \overline{A} \iff \partial A \subseteq A$$

For every set $B \subseteq X$, the sets ∂A and $\overline{A}$ are closed.

Open and closed sets in subspaces

A set A in a metric subspace $(S, d_S) \subseteq (X, d)$ is open if and only if $A = S \cap O$ for some open set $O \subseteq X$.

A set B in a metric subspace $(S, d_S) \subseteq (X, d)$ is closed if and only if $A = S \cap C$ for some closed set $C \subseteq X$.

Sequences

Some basic definitions: Sequence, subsequence, convergence, bounded,

These results hold somewhat generally.

1. Every convergent sequence is bounded.
2. Every sequence has a monotone subsequence.
3. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
4. x is a limit point of A if there is a sequence (x_n) in A with $x_n \to x$.

Let $(x_n), (y_n)$ be sequences in X with $x_n \to x$ and $y_n \to y$, and (α_n) be a sequence in $\mathbb{R}$ with $\alpha_n \to \alpha$. Then

1. $d(x_n, y_n) \to d(x, y)$
2. If X is a normed vector space, then $x_n + y_n \to x + y$ and $\alpha_n x_n \to \alpha x$
3. If X is an inner product space, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

Further metric space theory: convergence; compactness; continuity; ODEs

Completeness

Definition(Cauchy sequence): A sequence (x_n) in a metric space (X, d) is called Cauchy if for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$m, n > N \implies d(x_m, x_n) < \varepsilon$$

Theorem(Convergence implies Cauchy): Every convergent sequence is Cauchy.

Theorem(Cauchy implies bounded): Every Cauchy sequence is bounded.

Theorem(Cauchy with convergent subsequence): If (x_n) is Cauchy and has a convergent subsequence, then (x_n) is convergent.

Theorem(Cauchy in $\mathbb{R}^k$): Every Cauchy sequence in $\mathbb{R}^k$ is convergent.

Definition(Completeness): A metric space (X, d) is complete if every Cauchy sequence is convergent.

Theorem(Complete subspaces): Let (X, d) be a complete metric space, and $S \subseteq X$. Then (S, d_S) is complete if and only if S is closed in X.

Theorem(Completion): *Let (X, d) be a metric space. There exists a complete metric space $(\tilde{X}, \tilde{d})$ and an injective function $f : X \to \tilde{X}$ such that $d(x, y) = \tilde{d}(f(x), f(y))$.*

Banach's contraction mapping theorem

Definition(Contraction mapping): Let (X, d) be a metric space. A function $f : X \to X$ is called a contraction mapping if there is some $0 \leq \lambda < 1$ with $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

Theorem(Banach's contraction mapping theorem): Let (X, d) be a complete metric space, and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point.

Introduction to compactness

Definition(Covering; open covering): Let (X, d) be a metric space and $S \subseteq X$. A covering of S is a set $\{U_i\}_{i\in I}$ such that $S \subseteq \bigcup_{i\in I} U_i$. It is an open covering if each U_i is open.

Definition(Compact): Let (X, d) be a metric space, and $S \subseteq X$. Then S is compact if for any open covering $\{U_i\}_{i\in I}$ of S, there is a finite subcovering $\{U_j\}_{j=1}^{n}$ of S.

Definition(Sequentially compact): Let (X, d) be a metric space, and $S \subseteq X$ is sequentially compact if every sequence (x_n) in S has a subsequence which converges to some $x \in S$.

Theorem(Closed and bounded): Every sequentially compact set is closed and bounded.

Definition(Continuous): Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous at a point x if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$d(x, y) < \delta \to \rho(f(x), f(y)) < \varepsilon$$

Equivalently, $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. We say that the function f is continuous if it is continuous at every point $x \in X$.

Definition(Neighbourhood): Let (X, d) be a metric space, and $x \in X$. A set $U \subseteq X$ is called a neighbourhood of x if there is some $r > 0$ such that $B_r(x) \subseteq U$.

Theorem(Pointwise continuity): Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at x.
2. For every sequence $x_n \to x$, $f(x_n) \to f(x)$.
3. For every neighbourhood U of $f(x)$, $f^{-1}(U)$ is a neighbourhood of x.

Theorem(Global continuity): Let (X, d), (Y, ρ) be metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. f is a continuous function.
2. For every open set $O \subseteq Y$, $f^{-1}(O) \subseteq X$ is open.
3. For every closed set $O \subseteq Y$, $f^{-1}(O) \subseteq X$ is closed.

Theorem(Continuity and compactness): Let $f : (X, d) \to (Y, \rho)$ be a continuous function, and $K \subseteq X$ be compact. Then $f(K)$ is compact.

Theorem(Extreme Value Theorem on $\mathbb{R}$): Let (X, d) be a metric space, $f : X \to \mathbb{R}$ and $K \subseteq X$ be compact. Then f is bounded on K, and achieves its maximum and minimum values.

Definition(Uniform continuity): Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ be a function. f is *uniformly continuous* if for each $\varepsilon > 0$, there is some $\delta > 0$ such that for any $x, x' \in X$,

$$d(x, x') < \delta \implies \rho(f(x), f(x')) < \varepsilon$$

Assignment definitions(Isometry, isometric): Let (X, d) and (Y, ρ) be metric spaces. An *isometry* is a function $f : X \to Y$ such that $\rho(f(x), f(x')) = d(x, x')$ for any $x, x' \in X$. Two metric spaces are said to be *isometric* if there exists a bijective isometry between them.

Assignment definitions(Lipschitz continuous, comparable): A function $f : (X, d) \to (Y, \rho)$ is Lipschitz continuous if there is $M \in \mathbb{R}$ such that $\rho(f(x), f(x')) \le Md(x, x')$ for all $x, x' \in X$. Two metric spaces (X, d) and (Y, ρ) are comparable if there exists a bijection $f : X \to Y$ such that f and f^{-1} are Lipschitz continuous.

Assignment definitions(Uniformly equivalent, homeomorphism, homeomorphic): Two metric spaces (X, d) and (Y, ρ) are *uniformly equivalent* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are uniformly continuous. A homeomorphism is a bijection $f : X \to Y$ such that f and f^{-1} are continuous, and in this case we say that X and Y are homeomorphic.

The notion of a homeomorphism only requires a notion of continuity, which can be expressed in terms of open sets (and so can be adapted to topological spaces).

Theorem(Continuity and compactness cont.): Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and $K \subseteq X$ is compact, then f is uniformly continuous on K.

Function convergence

Definition(Pointwise convergence): Let S be a set and (Y, ρ) a metric space. A sequence (f_n) of functions $f_n : S \to Y$ is convergent to $f : S \to Y$ pointwise if for each $x \in S$ and for every $\varepsilon > 0$, there is some $n \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

Pointwise convergence is very weak, and instead motivates us to define a stronger notion of convergence.

Definition(Uniform convergence): Let S be a set and (Y, ρ) be a metric space. A sequence (f_n) of functions $f_n : S \to Y$ is convergent to $f : S \to Y$ uniformly if for every $\varepsilon > 0$, there is some $n \in \mathbb{N}$ such that for every $x \in S$,

$$d(f_n(x), f(x)) < \varepsilon$$

Theorem(Uniform convergence on $\mathbb{R}$): Let (X, d) be a metric space and (f_n) be an increasing sequence of real-valued functions on a set $S \subseteq X$. If S is compact, $f_n \to f$ pointwise and f is continuous, then $f_n \to f$ uniformly.

Theorem(Preservation of continuity): Let (X, d) and (Y, ρ) be metric spaces, and $f_n : X \to Y$ be a sequence of continuous functions such that $f_n \to f$ uniformly. Then $f : X \to Y$ is continuous on X.

Theorem(Preservation of integrals): Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n \to f$ uniformly on $[a, b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Definition(Bounded, $C(X, Y)$): Let (X, d) and (Y, ρ) be metric spaces.

1. A function $f : X \to Y$ is bounded if there is $y \in Y$ and $r > 0$ such that $f(X) \subseteq B_r(y)$. That is, $f(X)$ is bounded in Y.

2. The set of continuous functions from X to Y is denoted $C(X,Y)$, and $BC(X,Y)$ denotes the set of bounded continuous functions.

Definition(Uniform metric): Let (X,d) and (Y,ρ) be metric spaces. Then

$$d_u(f,g) := \sup_{x\in X} \rho(f(x), g(x))$$

is finite and a metric on $BC(X,Y)$.

Theorem(Completeness with uniform metric): Let (X,d) and (Y,ρ) be metric spaces, with (Y,ρ) complete. Then $(BC(X,Y), d_u)$ is a complete metric space.

Corollary(More general completeness): Let (X,d) be compact, and (Y,ρ) be complete. Then $(C(X,Y), d_u)$ is a complete metric space.

Ordinary differential equations

Definition(ODE): An ordinary differential equation relates a function $x(t)$ of a single variable $t \in I$ with its derivatives, where I is an interval.

The highest order derivative of the ODE is called the order of the differential equation.

Definition(IVP): An initial value problem takes the form

$$x^{(n)}(t) = G(t, x(t), x'(t), \ldots, x^{(n-1)}(t))$$

for $t \in I$, and $x(t_0) = u_0$, $x'(t_0) = u_1$, …, $x^{(n-1)}(t_0) = u_{n-1}$.

Any initial value problem can be reduced to one of the form $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, where $x \in \mathbb{R}^n$.

Lemma(Equivalent integral equation): Let f be continuous. Then a function $x \in C^1(I, \mathbb{R}^n)$ satisfies the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

if and only if it is a solution of

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

Theorem(Existence of [unique] solutions to ODEs): Consider the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

Suppose that U is an open set, and that

1. there are $h_0, \kappa_0 > 0$ such that

$$R_{h_0,\kappa_0}(t_0, x_0) = \{(t,x) \mid |t - t_0| \le h_0, \|x - x_0\| \le \kappa_0\} \subseteq U$$

2. $f(t,x)$ is bounded on $(t,x) \in R_{h_0,\kappa_0}(t_0, x_0)$

3. There is $L > 0$ such that $|f(t,x) - f(t,y)| \leq L\,|x-y|$ for all $(t,x),(t,y) \in R_{h_0,\kappa_0}(t_0,x_0)$.

Then there is $0 < h \leq h_0$ such that the initial value problem has a unique solution defined on $[t_0 - h, t_0 + h]$.

Theorem(Extension theorem): Assuming the above conditions, the initial value problem has a unique solution on a maximal time interval $[t_0, T)$, where either $T = \infty$, or $T < \infty$, and for any compact $K \subseteq U$, we have $(t, x(t)) \notin K$ for all t sufficiently close to T.

Definition(Total boundedness): A set A in a metric space (X,d) is totally bounded if for each $\varepsilon > 0$, there are points $x_1, \ldots, x_n \in X$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

Theorem(Compactness and completeness): A metric space (X,d) is complete and totally bounded if and only if it is compact.

Definition(Uniformly equicontinuous): Let (X,d) and (Y,ρ) be metric spaces. A set F of functions $f : X \to Y$ is uniformly equicontinuous if for each $\varepsilon > 0$, there is some $\delta > 0$ such that for any $f \in F$ and $x, x' \in X$,

$$\rho(f(x), f(x')) < \varepsilon$$

Theorem(Arzela-Ascoli): Let (X,d) be a complete metric space. If $F \subseteq C(X, \mathbb{R}^n)$ is closed, bounded and uniformly equicontinuous, then F is compact in $C(X, \mathbb{R}^n)$.

Theorem(Peano): Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^n$ be a continuous function, and $(t_0, x_0) \in U$. Then the initial value problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

has a solution defined on an interval $[t_0 - h, t_0 + h]$ for some $h > 0$.

Topology

Foundations

Definition(Topology): Let X be a set. A topology on X is a set $\tau \subseteq \mathcal{P}(X)$ such that

1. $\emptyset, X \in \tau$
2. $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau \implies \bigcup_\alpha U_\alpha \in \tau$
3. $\{U_i\}_{i=1}^n \subseteq \tau \implies \bigcap_{i=1}^n U_i \in \tau$

The sets in τ are called open sets. A set X alongside a topology τ is called a topological space, denoted (X,τ).

(General) topology examples: Discrete, trivial, subspace, product

Definition(Coarser/weaker; comparable): Let X be a set, and τ_1, τ_2 be topologies on X. We say τ_1 is coarser/weaker than τ_2 (or τ_2 is finer/stronger than τ_1) if $\tau_1 \subseteq \tau_2$. The two topologies are comparable if either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$.

Definition(Basis): Let (X, τ) be a topological space. A basis $\mathcal{B} \subseteq \tau$ is such that for every $U \in \tau$, there is $\mathcal{B}' \subseteq \mathcal{B}$ such that

$$U = \bigcup_{B \in \mathcal{B}'} B$$

Theorem(Equivalent basis formulation): Let (X, τ) be a topological space. A subset $\mathcal{B} \subseteq \tau$ is a basis if and only if for every open set U and $x \in U$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B} \subseteq U$.

Theorem(Unique topology): Let X be a set, and $\mathcal{B}$ be a subset of $\mathcal{P}(X)$. Then if

1. $$\bigcup_{B \in \mathcal{B}} B = X$$
2. For every $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Then there is a unique topology on X with $\mathcal{B}$ as a basis.

Theorem(Equivalent formulation for coarseness): Let X be a set and $\mathcal{B}_1, \mathcal{B}_2$ generate topologies τ_1, τ_2 on X. Then $\tau_1 \subseteq \tau_2$ if and only if $\mathcal{B}_1 \subseteq \mathcal{B}_2$.

Theorem(Basis for the product topology): The product topology $X \times Y$ has basis

$$\tau_{X \times Y} = \big\{ U \times V \bigm| U \in \tau_X, V \in \tau_Y \big\}$$

Sequences

Definition(Closed set): Let (X, τ) be a topological space. A set $S \subseteq X$ is closed if $S^c \in \tau$. That is, S^c is open.

Definition(Closure, interior): Let (X, τ) be a topological space and $A \subseteq X$. The closure $\overline{A}$ of A is the smallest closed set containing A. The interior $\operatorname{int}(A)$ of A is the largest open set contained in A. That is,

$$\overline{A} = \bigcap_{\substack{A \subseteq C \\ C^c \in \tau}} B$$

$$\operatorname{int}(A) = \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O$$

It is clear that $\operatorname{int}(A^c)^c = \overline{A}$ and $\operatorname{int}(A) = \left(\overline{A^c}\right)^c$

Definition(Neighbourhood): Let (X, τ) be a topological space, and $x \in X$. A neighbourhood of X is a set $x \in U \subseteq X$ such that there is some $O \in \tau$ with $x \in O \subseteq U$. An open neighbourhood is a neighbourhood which is itself open.

Theorem(Equivalent formulations of interior/closure): Let $A \subseteq X$. Then

1. $x \in \text{int}(A)$ if and only if there is some neighbourhood U of x contained in A.
2. $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.

Definition(Limit point): Let (X, τ) be a topological space, and $A \subseteq X$. A point $x \in X$ is a limit point of A if for every neighbourhood U of x,

$$A \cap U \setminus \{x\} \neq \emptyset$$

Theorem(Closure in terms of limit points): Let (X, τ) be a topological space and $A \subseteq X$. Then $\overline{A} = A \cup \{\text{limit points of } A\}$.

Definition(Boundary): Let (X, τ) be a topological space, and $A \subseteq X$. The boundary ∂A of A is the set $\partial A = \overline{A} \cap \overline{A^c}$.

Theorem(Open/closed in terms of boundaries): Let (X, τ) be a topological space and $A \subseteq X$. Then

1. A is open if and only if $A \cap \partial A = \emptyset$.
2. A is closed if and only if $\partial A \subseteq A$.

We also trivially have that $A = \text{int}(A)$ if and only if A is open.

Definition(Sequence convergence): Let (X, τ) be a topological space and (x_n) be a sequence in X. We say that x_n converges to x if for every neighbourhood U of x, there is $N \in \mathbb{N}$ such that $x_n \in U$ for every $n > N$. In this case we write $x_n \to x$.

This definition is very weak and allows for multiple limits (such as any set X with the trivial topology). To constrain this, we have the following property of topological spaces.

Definition(Hausdorff space): We say that a topological space (X, τ) is Hausdorff if for every $x, y \in X$, there is $U, V \in \tau$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

The notion of a Hausdorff space formalises what it means to be "separable" within a space. That is, from the pure perspective of neighbourhoods, we can still tell apart any two points.

Theorem(Uniqueness of sequence limits): Let (X, τ) be a Hausdorff space, and let (x_n) be a sequence in X. Then if (x_n) is convergent, its limit is unique.

Theorem(Finite sets in a Hausdorff space): Let (X, τ) be a Hausdorff space. Then every finite subset $A \subseteq X$ is closed.

Theorem(Hausdorff limit points): Let (X, τ) be a Hausdorff metric space, and $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if every neighbourhood of x contains infinitely many points in A.

Functions and compactness

Definition(Continuity): Let (X, τ_1) and (Y, τ_2) be topological spaces, and $f : (X, \tau_1) \to (Y, \tau_2)$ be a function. We say that

1. f is continuous at $x \in X$ if for every neighbourhood U of $f(x)$, $f^{-1}(U)$ is a neighbourhood of x.
2. f is continuous as a function if for every open set $O \in \tau_2$, $f^{-1}(O) \in \tau_1$.

Theorem(Equivalent formulations of continuity): Let (X, τ_1) and (Y, τ_2) be topological spaces, and $f : (X, \tau_1) \to (Y, \tau_2)$ be a function. The following are equivalent.

1. f is continuous.
2. f is continuous at every point $x \in X$.
3. For any closed set $C \subseteq Y$, $f^{-1}(C)$ is closed in X.
4. For any basis $\mathcal{B}$ of τ_2 and $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X.
5. For any subbasis $\mathbb{S}$ of τ and $S \in \mathbb{S}$, $f^{-1}(S)$ is open in X.
6. $f(\overline{A}) \subseteq \overline{f(A)}$ for any set $A \subseteq X$.
7. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$.
8. $f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B))$ for any $B \subseteq Y$.

(Where the last 3 formulations are from Workshop 10.)

Theorem(Continuity and subsets): Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function, and $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of X. Then f is continuous if and only if $f\big|_{U_\alpha}$ is continuous for each U_α.

This result says that we can deduce the continuity of a function by looking at its continuity on individual (open) parts of the set.

Definition(Compact, sequentially compact): Let (X, τ) be a topological space and $K \subseteq X$. We say that K is *compact* if for every open covering $\{\mathcal{U}_\alpha\}_{\alpha \in \Lambda}$, there is some finite subset $\{\mathcal{U}_i\}_{1 \leq i \leq n}$ which is also a covering of K.

K is *sequentially compact* if every sequence (x_n) in K has a subsequence which converges to some $x \in K$.

These notions are not equivalent, and there is no reason to believe that they are without a notion of a distance.

Theorem(Compactness of closed sets; ambient compact): Let (X, τ) be a compact topological space, and $K \subseteq X$ be closed. Then K is also compact.

Theorem(Closedness of compact sets; Hausdorff): Let (X, τ) be a Hausdorff space, and $K \subseteq X$ be a compact set. Then K is closed.

Lemma(Open neighbourhoods in compact products): Let (X, τ_1) and (Y, τ_2) be topological spaces, and Y be compact. For $p \in X$, if there is an open set $N \subseteq X \times Y$ such that $\{p\} \times Y \subseteq N$, then there is an open set $U \subseteq X$ such that $U \times Y \subseteq N$.

Theorem(Compactness of products): If X and Y are compact, then so is $X \times Y$.

Theorem(Compactness and continuity): Let $(X, \tau_1), (Y, \tau_2)$ be topological spaces, and $f : X \to Y$ be a continuous function. If $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is also compact.

Theorem(Continuity and products): Let A, X, Y be topological spaces, and $f : A \to X \times Y$ be a function. Then f is continuous if and only if the projections $\pi_1 \circ f : A \to X$ and $\pi_2 \circ f : A \to Y$ are continuous.

Corollary(Coarsest topology for projection continuity): The product topology is the coarsest topology for which the projections π_1 and π_2 are continuous.

Definition(Arbitrary product): Let $\{X_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of sets. The product $\prod_{\alpha \in \mathscr{A}}$ is the set of all functions $x : \mathscr{A} \to \bigcup_{\alpha \in \mathscr{A}} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for each $\alpha \in \mathscr{A}$. We write this function as $(x_\alpha)_{\alpha \in \mathscr{A}}$ (which can be thought of as a tuple indexed over the set $\mathscr{A}$).

[This definition *explicitly* uses the axiom of choice in most infinite cases, to assert that this product is non-empty.]

On $\prod_{\alpha \in \mathscr{A}} X_\alpha$, we have two "natural" product topologies. The first is generated by the basis

$$\mathcal{B} = \left\{ \prod_{\alpha \in \mathscr{A}} U_\alpha \;\middle|\; U_\alpha \subseteq X_\alpha \text{ is open} \right\}$$

This is called the box topology, and is not the standard product topology on an arbitrary product. The standard product topology on an arbitrary product is generated by the subbasis

$$\mathbb{S} = \left\{ \prod_{\alpha \in \mathscr{A}} U_\alpha \;\middle|\; U_\alpha \subseteq X_\alpha, U_\gamma \neq X_\gamma \text{ for at most one } \gamma \in A \right\}$$

Theorem(Products and Hausdorff): Let $\{X_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of Hausdorff topological spaces. Then $\prod_{\alpha \in \mathscr{A}} X_\alpha$ is Hausdorff.

Theorem(Products and compact): Let $\{X_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of compact topological spaces. Then $\prod_{\alpha \in \mathscr{A}} X_\alpha$ is compact.

[[Insert tangent about ***ZOING NEMMER*** here]]

Theorem(Alexander subbase): Let (X, τ) be a topological space, and $\mathbb{S}$ be a subbasis. If for every open covering of X from sets in $\mathbb{S}$, there is a finite subcovering, then X is compact.

Connectedness

Definition(Connected): Let (X, τ) be a topological space. We say that X is *connected* if there do not exist disjoint, non-empty, open sets $U, V \in \tau$ such that $X = U \cup V$.

This definition is phrased in terms of a negation, and so proofs involving connectedness are easier to approach with contrapositives.

We can define connected subsets $K \subseteq X$ by taking the subspace topology (note that this gives open sets of the form $K \cap U$ and $K \cap V$)

(The non-negated version says that for any two disjoint open sets $U, V \in \tau$ with $X = U \cup V$, either $U = \emptyset$ or $V = \emptyset$, which is harder to work with.)

Definition(Interval): Let $I \subseteq \mathbb{R}$. We say that I is an *interval* if for any $a < b \in I$, every x with $a < x < b$ is in I.

Theorem(Connected sets in $\mathbb{R}$): Let $\mathbb{R}$ have its standard topology. $I \subseteq \mathbb{R}$. Then I is connected if and only if I is an interval.

Theorem(Continuity and connectedness): Let $f : X \to Y$ be a continuous function. If X is connected, then so is $f(X)$.

This also holds for any $A \subseteq X$, by taking $f\big|_A : A \to Y$ and noting that this is also continuous.

Corollary(An intermediate value theorem): Let X be a connected topological space, and $f : X \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). If $a < b \in f(X)$, then for any $a < x < b$, $x \in f(X)$.

Definition(Path; path connectedness): Let X be a topological space, and $x, y \in X$. A *path* from x to y is a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

A set $S \subseteq X$ is *path connected* if for every $x, y \in S$, there exists a path from x to y.

Theorem(Path connectedness implies connectedness): Every path connected set is connected.

Theorem(Equivalence in $\mathbb{R}^n$): In $\mathbb{R}^n$ with the standard topology, a set S is connected if and only if it is path connected.

Assignment definitions: *Riemann integrable, homeomorphism, uniform equivalence, Lipschitz equivalent, isometric*

Properties in a metric space and where they relate (on which level they act on):

1. **POINTWISE**
   (a) Continuity, uniform continuity
   (b) Pointwise convergence
2. **SETWISE/SPACEWISE**
   (a) Open, closed, bounded, compact
   (b) Complete, connected, path-connected
   (c) Totally bounded, subspace
3. **SEQUENCEWISE**
   (a) Cauchy, convergence, sequential compactness
4. **FUNCTIONWISE**
   (a) Bounded ($f(X) \subseteq B_r(x)$, some x)
   (b) Uniform equicontinuity, uniform convergence

Other useful formulations: The sets from connectedness can also be viewed as closed