Point-set topology notes

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These notes are mainly my own rendition of roughly how I would teach or think about point-set topology, written in a way to give appreciation to its basic (and usually perceived to be dry) notions.

Contents

1 Motivation

We describe some of the shortcomings with abstract metric spaces, as well as how the notion is usually adapted while still keeping some of the important underlying structure.

1.1 Lack of universal morphisms

In abstract metric spaces, (from a more algebraic perspective) there isn't a universally accepted notion of a morphism or "structure preserving map" $f : X \to Y$ between two metric spaces X, Y.

We have *isometries* which preserve the exact distance (i.e. by setting $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$). This notion is "too strong", as the map f is then necessarily injective (forcing all such maps to necessarily be inclusions), and the image $f(X) \subseteq Y$ can be viewed equivalently as an exact "copy" (from a metric space perspective) of X in Y.

Weakening slightly, we have *Lipschitz continuous* maps, which are such that $d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$ for some fixed constant C. This notion of a "structure preserving map" is less restrictive and allows for non-injective maps, and so is slightly more "sensible" than an isometry from this perspective. The associated "isomorphisms" would then be the maps for which the metrics are comparable or equivalent, and so these maps are likely the most sensible morphism notion. Lipschitz continuous maps preserve most notions, but as with the remaining structure preserving maps, they do not preserve the exact distances (which as above is too strong for such a notion).

We have uniform and normal continuous maps as notions of a structure preserving map. Here, *continuity* of a map $f : X \to Y$ is usually phrased as saying that "for any $\varepsilon > 0$ and any x in X, we can find $\delta > 0$ with $d_Y(f(x), f(y)) < \varepsilon$ for any y with $d_X(x, y) < \delta$", and *uniform continuity* is the same statement where the same value of δ works for all pairs of x and y whose images under f are at most ε apart.

Continuity and uniform continuity are inherently local properties, as the ε-δ definitions are inherently skewed towards smaller values, in the sense that if $0 < \varepsilon_1 < \varepsilon_2$, then a value of δ satisfying the (uniform) continuity constraint for ε_1 will also work for ε_2. In particular, these are local properties, and often fail to capture the geometry induced by the metric structure. For example, the identity function is a uniform equivalence (and hence homeomorphism) between $\mathbb{Z}$ equipped with the discrete and Euclidean metrics respectively. However, the distances associated to $\mathbb{Z}$ with the discrete metric would make it the analogue of a triangle with countably infinitely points, while the Euclidean metric would make it a line.

1.2 Metric spaces can only "be so big"

In a metric space, every pair of points necessarily have only a finite distance between them (so in principle you could walk from any point to another). Explicitly, a metric space structure on a set X involves a function $d : X \times X \to \mathbb{R}_{\geq 0}$, explicitly making a metric space "locally as big as $\mathbb{R}$". This prevents metric spaces from being too pathological, but also forces any metric space with interesting structure to be relatively "small". We describe some sensible spaces without suitable metrics for the associated geometry.

Example 1 (Non-metric spaces). *1. (Functions) Let $(a, b) \subseteq \mathbb{R}$, $k \in \mathbb{N}_0$ and consider $C^k((a, b), \mathbb{R})$. For any $[a', b'] \subseteq (a, b)$, $C^k([a, b], \mathbb{R})$ is a metric space with the standard ℓ^p $(p \geq 1)$ norm*

$$\|f\|_p = \left(\int_{a'}^{b'} |f(x)|^p \, dx \right)^{1/p}$$

as such functions are necessarily bounded since $[a', b']$ is compact in $\mathbb{R}$. This metric cannot be extended naturally to $C^k((a,b),\mathbb{R})$ since the boundedness of a function $f:(a,b)\to\mathbb{R}$ is not guaranteed, and in fact we can have functions which grow arbitrarily fast near either a or b, such as $f(x)=(x-a)^{-1/p}$.

Since every C^k function on (a,b) restricts to a C^k function on any $[a',b']\subseteq(a,b)$, $C^k((a,b),\mathbb{R})$ can be viewed in some part as the "limit" of the spaces $C^k([a',b'],\mathbb{R})$, under which this failure to yield a metric space can be viewed as metric spaces not being closed under these limiting procedures.

2. *(Sequences in $\mathbb{R}$) The space $\mathbb{R}^\infty$ of real-valued sequences (viewed as infinite vectors indexed by $\mathbb{N}$) naturally contains a copy of $\mathbb{R}^n$ for each $n\in\mathbb{N}$, by taking the first n coordinates of each sequence. Each $\mathbb{R}^n$ is usually equipped with the Euclidean metric, and so $\mathbb{R}^\infty$ with the Euclidean metric (so that the inclusions $\mathbb{R}^n\hookrightarrow\mathbb{R}^\infty$ are isometries) is not an unreasonable notion to consider.*

   We can actually achieve this, but in a rather unsatisfying fashion involving the axiom of choice: beginning with the standard basis vectors $E=\{e_i\mid i\in\mathbb{N}\}$, by Zorn's lemma (equivalent to AC) we can extend E to a basis $\mathcal{E}$ of $\mathbb{R}^\infty$. Define an inner product on $\mathbb{R}^\infty$ by declaring $\mathcal{E}$ to be orthonormal. This induces a norm and hence a metric, and the inclusions are indeed isometries. This construction is however effectively useless from a metric space perspective - the use of the axiom of choice means that we cannot write down $\mathcal{E}$, and actually computing distances between most vectors[1] is not feasible. The closest commonly used construction is ℓ^2, which is the set of $x=(x_n)_{n\in\mathbb{N}}\in\mathbb{R}^\infty$ for which the "norm"

$$\|x\|_{\ell^2}=\left(\sum_{k=1}^{\infty}x_n^2\right)^{1/2}$$

   makes sense.

3. *(The long line) One way to view $\mathbb{R}$ is as the disjoint union*

$$\mathbb{R}=\coprod_{n\in\mathbb{Z}}[n,n+1)$$

   From this perspective $\mathbb{R}$ can be viewed as $|\mathbb{Z}|$ copies of the interval $[0,1)$ patched together at the end-points, i.e. "$\mathbb{R}\cong\mathbb{Z}\times[0,1)$". Replacing $\mathbb{Z}$ with a well-ordered set of bigger cardinality[2], we can find longer line-like spaces, of which a metric cannot capture the true geometry.

1.3 Infinitude of products

The failure of $\mathbb{R}^\infty$ to be a metric space in spite of $\mathbb{R}^n$ (for $n\geq 1$) is an example of a more general failure of metric spaces to generalise to infinite products. For a finite product

$$X=\prod_{i=1}^{k}X_i$$

of metric spaces X_i with associated metrics d_i, we have standard ℓ^p product metrics of the form

$$d^p((x_1,\ldots,x_k),(y_1,\ldots,y_k))=\left(\sum_{i=1}^{k}d_i(x_i,x_j)^p\right)^{1/p}$$

for which the canonical inclusions $X_i\hookrightarrow X$ are isometries, but as we saw in the above example, even when the index set Λ is just countable,

$$X=\prod_{\alpha\in\Lambda}X_\alpha$$

may not be a metric space, let alone for even larger index sets. In this sense, metric spaces are not closed under arbitrary products. To allow for a suitable notion which is closed under products, we need a more general notion independent of $\mathbb{R}$.

[1] i.e. those with infinitely non-zero coefficients when expressed as an arbitrary formal linear combination of e_i

[2] This step explicitly requires the well-ordering principle, equivalent to the axiom of choice

1.4 Quotients

Let X be a metric space, and $A \subseteq X$ be a subspace. Consider the (set-theoretic) quotient X/A given by the relation $x \sim y$ if $x = y$ or $x, y \in A$ (i.e. collapsing A to a point). Ideally, we should be able to put a metric space structure on X/A so that the quotient map $\pi : X \twoheadrightarrow X/A$ is continuous. This is possible if A is closed by taking $d_{X/A}([x], A) = d_X(x, A)$ and $d_{X/A}([x], [y]) = d_{X/A}(x, y)$ for $x, y \notin A$, but this definition of $d_{X/A}$ fails in general as it is possible that $d_{X/A}([x], A) = 0$ for $x \notin A$ (though all other metric space conditions are satisfied). It turns out that A being closed is exactly equivalent to X/A having a metric space structure, with the metric as given above, which makes π 1-Lipschitz.

More generally, given an equivalence relation $\sim$ on a metric space X, $X/\sim$ does not always carry a metric space structure, though under certain conditions such a structure exists.

2 Topological spaces

2.1 Notation

Unless otherwise stated, X, Y and Z are topological spaces, and $\mathcal{T}_S$ (or sometimes simply $\mathcal{T}$) is the set of open sets of S. If $\mathbb{S}$ is a collection of sets, $\bigcup \mathbb{S}$ denotes $\bigcup_{S \in \mathbb{S}} S = \{s \in S \mid S \in \mathbb{S}\}$, and similar for $\bigcap \mathbb{S}$.

2.2 Basic definitions

The basic definitions of topology are "very loose", and it is not difficult to find counterintuitive examples. In any case, we try to paint a certain picture with each definition and avoid counterintuitive examples, to follow a roughly intuitive and sensible progression.

Leaving only the notion of an open set, which we can think of in the same way as a metric space (i.e. a set in which every point is still locally "far enough" from its edge). These open sets give us a qualitative notion of "closeness", without being strictly quantified by a metric. We abstract the properties of open sets from metric spaces, yielding the following notion.

Definition 1 (Topological space)**.** *Let X be a set. A topology $\mathcal{T}$ on X is a set of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying*

1. $\emptyset, X \in \mathcal{T}$*;*
2. *if $\{\mathcal{U}_\alpha\}_{\alpha \in \Lambda}$ is a collection of sets in $\mathcal{T}$, then $\bigcup \{\mathcal{U}_\alpha \mid \alpha \in \Lambda\} \in \mathcal{T}$; and*
3. *if $\{\mathcal{U}_i\}_{i=1}^n$ is a (finite) collection of sets in $\mathcal{T}$, then $\bigcap_{i=1}^n \mathcal{U}_i \in \mathcal{T}$.*

The sets in $\mathcal{T}$ are referred to as open sets*, and we say that a space X equipped with a topology $\mathcal{T}$ is a* topological space*.*

The notion of a topological space then naturally generalises the notion of a metric space, in the sense that any metric space is a topological space with its usual open sets. Since topological spaces are a vast generalisation of metric spaces, it is usually not hard to find cathartic or pathological examples of such spaces.

Example 2 (Topological spaces)**.**

1. *If X is a metric space, the open sets (in the metric sense) form a topology, and this is called the* metric topology *on X. In particular, any inner product space or normed vector space has an associated topology, which we call the* norm topology*.*
2. *If X is a set, we have two "simple topologies":*

(a) The discrete topology is given by $\mathcal{T} = \mathcal{P}(X)$. This corresponds to the metric topology where X is given the discrete metric.
(b) The trivial topology is given by $\mathcal{T} = \{\emptyset, X\}$. This can be thought of as treating all points in X as "close enough to be inseparable", and when X is non-empty, this is equivalent to a single-point topological space in some sense.

It will be easier to construct some more interesting topological spaces with some more tools, and we will return to this in example **??**.

In a metric space, to talk about the space near a given point x, we usually refer to the open balls $B_r(x)$. Since topological spaces are no longer tied to $\mathbb{R}$ (and hence have no notion of distance), we instead have the following analogue.

Definition 2 (Neighbourhood). *Let $x \in X$. A* neighbourhood *of x is an open set $\mathcal{U}$ containing x.*

The statement that two points are in the same open set can then be viewed as saying the points are "somewhat close", with the notion of "somewhat" being slightly ambiguous (since the space X is a neighbourhood of every point), though the smaller the neighbourhood, the closer two points in that neighbourhood can be thought of. Continuing with the motivation from metric spaces:

Definition 3 (Closed set). *We say that $C \subseteq X$ is* closed *if $X \setminus C = C^c$ is open.*

The above definition is relatively opaque, and appealing to our metric space intuition, we would expect a closed set to "contain its boundary points":

Definition 4 (Limit point; isolated point). *Let $A \subseteq X$. We say that a point $x \in X$ is*

1. *a* limit point of A *if every neighbourhood $\mathcal{U} \ni x$ has non-empty intersection with $A \setminus \{x\}$, that is*

$$(A \setminus \{x\}) \cap \mathcal{U} \neq \emptyset$$

2. *an* isolated point of A *and there is a neighbourhood $\mathcal{U} \ni x$ which does not meet $A \setminus \{x\}$.*

The limit points of a set are thus those which are "close to the set", captured by the notion that every open set "nearby" also touches the set. We can also think of these as the points which are "approximated well" by other points in the set. The constraint $(A \setminus \{x\}) \cap \mathcal{U} \neq \emptyset$ can instead be written $A \cap \mathcal{U} \supsetneq \{x\}$. The isolated points are then in some sense an opposite, in that they are points in the set "far from other points" or "isolated from the rest of the set", or points not "well-approximated by other points in the set".

Proposition 1. *Let $A \subseteq X$, and A' denote the limit points of A. Then $\overline{A} = A \cup A'$, and hence A is closed if and only if it contains all its limit points.*

We can generally think of the open sets as relatively "large", as the sets which are "unbounded" when viewed from internally. On the flip side, most closed sets can generally be viewed as "small", or for which we could "reach a boundary". This way of thinking of closed sets however breaks down when talking about sets without boundaries, such as $\emptyset$ or the whole space X.

Extending from metric spaces, we further have:

Definition 5 (Interior, exterior, closure, boundary). *Let $A \subseteq X$.*

1. *The* interior $\operatorname{int}(A)$ of A *is the unique open set satisfying*
   (a) $\operatorname{int}(A) \subseteq A$*; and*
   (b) if $O \subseteq A$ is open, $O \subseteq \operatorname{int}(A)$.

2. *The* exterior $\mathrm{ext}(A)$ of A *is the interior of* A^c.
3. *The* closure $\overline{A}$ of A *is the unique closed set satisfying*
   (a) $A \subseteq \overline{A}$*; and*
   (b) *if* $C \supseteq A$ *is closed, then* $C \supseteq \overline{A}$.
4. *The* boundary ∂A of A *is* $\partial A := \overline{A} \cap \overline{A^c}$.

The definitions of the interior and closure of A correspond to saying that $\mathrm{int}(A)$ is the largest open set contained in A, and that $\overline{A}$ is the smallest closed set containing A. The following proposition makes this precise, alongside giving a way in which the interior and closure of a set are "dual".

Proposition 2. *Let* $A \subseteq X$. *Then*

1. $$\mathrm{int}(A) = \bigcup \{O \subseteq A \mid O \in \mathcal{T}\}$$
2. $$\overline{A} = \bigcap \{C \supseteq A \mid X \setminus C \in \mathcal{T}\}$$
3. $\mathrm{int}(A) = (\overline{A^c})^c$, *or equivalently* $\overline{A} = \mathrm{int}(A^c)^c = \mathrm{ext}(A)^c$
4. $\partial A = \partial A^c$, *and* $X = \mathrm{int}(A) \coprod \partial A \coprod \mathrm{ext}(A)$.

The following proposition sharpens the usual metric space intuition for closed sets.

Proposition 3. *Let* $A \subseteq X$*, and let* A' *denote the set of limit points of* A*. Then* $\overline{A} = A \cup A'$*, and hence* A *is closed if and only if* $A' \subseteq A$.

As in metric spaces, it will sometimes be easier (cardinality-wise or otherwise) to consider *dense* subsets, that is, those which can "approximate" all points in the space. The notion is the same as in metric spaces: a subset where every point in the ambient space is a limit point.

Definition 6 (Dense subset)**.** *A subset* $E \subseteq X$ *is dense if* $\overline{E} = X$.

Equivalently, $E \subseteq X$ is dense if for each $x \in X$, every neighbourhood $U \ni X$ contains a point in E, i.e. $E \cap U \neq \emptyset$.

In some cases we may endow a space X with more than one topology, and we use the following descriptions in this case:

Definition 7. *Let* X *be a set, and* $\mathcal{T}_1, \mathcal{T}_2$ *be topologies on* X*. If* $\mathcal{T}_1 \subseteq \mathcal{T}_2$*, we say* $\mathcal{T}_1$ *is* coarser *or* weaker *than* $\mathcal{T}_2$*, and* $\mathcal{T}_2$ *is* finer *or* stronger *than* $\mathcal{T}_1$.

2.3 Bases

In other (algebraic) objects such as vector spaces and ideals in rings, it is usually equivalent and easier to consider a generating subset. In these cases we tend to prefer minimal generating subsets (such as a basis for a vector space), as a smaller generating subset tends to be more illuminating to the structure of the associated object. We have the following analogous notion in a topological space.

Definition 8 (Basis)**.** *A* basis (or base) for X *is a subset* $\mathcal{B} \subseteq \mathcal{T}$ *with the property that for every* $U \in \mathcal{T}$*, there is* $\mathcal{B}_U \subseteq \mathcal{B}$ *such that*

$$U = \bigcup \mathcal{B}_U$$

In this sense a basis $\mathcal{B}$ is a "spanning set" for $\mathcal{T}$ (in the sense that every $U \in \mathcal{T}$ can be reached as a union of $B \in \mathcal{B}$). In comparison to vector spaces we don't have a sensible notion of "independence" for open sets, nor is there a sensible notion of a minimal subset of $\mathcal{T}$ with the above properties in general.

The following examples illustrate the notion of a basis.

Example 3. *1. $\mathcal{T}$ is a basis for X, but this does not give any additional insight into the structure of the topological space.*

2. For $X = \mathbb{R}^n$ with the usual (metric) topology, we have a countable basis given by balls of rational radius with rational centre. That is,

$$\mathcal{B}_{\mathbb{R}^n} = \{B_r(x) \mid r \in \mathbb{Q}, x \in \mathbb{Q}^n\}$$

In fact, for any dense $E \subseteq \mathbb{R}$, the collection of $B_r(x)$ for $r \in E$ and $x \in E^n$ is a basis for $\mathbb{R}^n$ as a topological space.

We should expect that a basis for a topological space cannot be "much smaller" than the space itself. The following result gives an explicit bound on the size of a basis relative to the size of the ambient space, and shows that the above bases for $\mathbb{R}^n$ are of minimal cardinality.

Proposition 4. *Let $\mathcal{B}$ be a basis for X. Then $|\mathcal{T}| \leq |\mathcal{P}(\mathcal{B})|$.*

We have an equivalent and sometimes more useful formulation of what it means to be a basis. In essence, the idea of this statement comes from the fact that $U = \bigcup_{i \in I} U_i$ if and only if every $x \in U$ has $x \in U_i \subseteq U$.

Lemma 1. *A subset $\mathcal{B} \subseteq \mathcal{T}$ is a basis for X if and only if for every $U \in \mathcal{T}$ and $x \in U$, there is $B_{x,U} \in \mathcal{B}$ with $x \in B_{x,U} \subseteq U$.*

If X is a set, we may also ask what collections of subsets of X form bases for topologies, and this is answered exactly in the following result.

Proposition 5. *A subset $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for a (unique) topology on X if and only if*

1. *$\bigcup_{B \in \mathcal{B}} B = X$; and*
2. *For all $B_1, B_2 \in \mathcal{T}$ and $x \in B_1 \cap B_2$, there is B_3 with $x \in B_3 \subseteq B_1 \cap B_2$.*

It thus suffices to exhibit a basis to define a topology. This gives the following example, which is essential in functional analysis:

Example 4 (The weak topology on a normed vector space)**.** *Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with norm $\|\cdot\|$, and $X^* = C(X, \mathbb{F}) \subseteq \mathrm{Hom}(X, \mathbb{F})$ be its dual. Then we have an associated topology $\mathcal{T}_w$ on X which is minimal and makes every $x^* \in X^*$ continuous, which we can think of as being associated to "testing elements $y \in X$ against functionals". This is explicitly given as follows: for $n \in \mathbb{N}$, $\varepsilon > 0$, $x \in X$ and $x_1^*, \ldots, x_n^* \in X^*$, let*

$$V_{\varepsilon,x,x_1^*,\ldots,x_n^*} = \{y \in X \mid |x^*(x-y)| < \varepsilon\}$$

*Then by **Proposition 5**, the collection of all such $V_{\varepsilon,x,x_1^*,\ldots,x_n^*}$ forms a basis for a topology $\mathcal{T}_w$, which we call the* weak topology*. Since every $x^* \in X^* = C(X, \mathbb{F})$ is continuous with respect to the norm topology by definition, the norm topology on X is stronger than the weak topology. In general, the norm topology may be strictly stronger than the weak topology.*

The only issue with directly building a topology from declaring an arbitrary subset $S \subseteq X$ to be a basis is that it may not be closed under finite intersections, which the second point of the above proposition deals with. We can build a topology by specifying even less data, by allowing finite intersections.

Definition 9 (Subbasis)**.** *Let X be a set, and $S \subseteq X$. Then S is a subbasis for the topology with basis*

$$\mathcal{S} = \left\{ \bigcap_{i=1}^{n} S_i \;\middle|\; S_i \in S, n \in \mathbb{N}_0 \right\}$$

This is indeed a basis (with the convention that the empty intersection is X), and so we have the following result.

Lemma 2. *Let $S \subseteq X$. Then the topology on X with basis $\mathcal{S}$ is the minimal topology on X so that every set in S is open.*

Here we have referred to minimality (with respect to inclusion), which plays an important role in defining topologies: when defining topologies we would ideally have the number of open sets be as small as possible, as this gives nice properties with respect to continuous maps (which we will define soon), which are the structure-preserving maps of topological spaces.

2.4 Continuous maps

To define morphisms between topological spaces, we return to the notion of continuous maps. Recall that continuous maps in metric spaces are defined by the following:

A function $f : X \to Y$ is continuous at a point $x \in X$ *if for every $\varepsilon > 0$, there is $\delta > 0$ so that if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \varepsilon$.*

The statement that $d_X(x, x') < \delta$ can be reformulated as $x' \in B_X(x, \delta)$ and $d(f(x), f(x')) < \varepsilon$ as $f(x') \in B_Y(f(x), \varepsilon)$, so we have the following formulation with open balls:

A function $f : X \to Y$ is continuous at a point $x \in X$ *if for every $\varepsilon > 0$, there is $\delta > 0$ so that $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$.*

Recalling that neighbourhoods are the generalisation of open balls to topological spaces, we have the following definition.

Definition 10 (Continuous)**.** *A function $f : X \to Y$ is* continuous at $x \in X$ *if for every neighbourhood $U' \ni f(x)$, there is a neighbourhood $U \ni x$ with $f(U) \subseteq U'$. The function f is* continuous *if it is continuous at all $x \in X$.*

As a reasonable notion of a morphism, we should expect the following result.

Lemma 3. *If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then so is $g \circ f : X \to Z$.*

We have the following (somewhat trivial) diagram of continuous maps:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

The above definition conveniently shows how continuity comes about from the metric space definition, but isn't immediately useful beyond the following result. To begin with, we have the following reformulations.

Proposition 6. *Let $f : X \to Y$.*

1. *f is continuous at x if and only if for every neighbourhood $U \ni f(x)$, $f^{-1}(U)$ is a neighbourhood of x.*
2. *The following are equivalent.*
   (a) *f is a continuous map.*
   (b) *If $O \subseteq Y$ is open, then so is $f^{-1}(O) \subseteq X$.*
   (c) *If $C \subseteq Y$ is closed, then so is $f^{-1}(C) \subseteq X$.*

Statements 2(b) and 2(c) can be taken as equivalent definitions for continuity of a map f, and are usually the first definitions given. Continuity is ultimately a local property, and the following sharpens this:

Theorem 1. *Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of X, and $f : X \to Y$. Then f is continuous if and only if each $f|_{U_\alpha}$ is.*

There are even more equivalent formulations of continuity, though these are less important than those above.

Proposition 7. *Let $f : X \to Y$. Then the following are equivalent.*

1. *f is continuous;*
2. *f satisfies* ***Prop 6. 2(b)*** *(or* ***(c)****) on a basis $\mathcal{B}$;*
3. *f satisfies* ***Prop 6. 2(b)*** *(or* ***(c)****) on a subbasis $\mathbb{S}$;*
4. *$f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$;*
5. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$;*
6. *$f^{-1}(\mathrm{int}(B)) \subseteq \mathrm{int}(f^{-1}(B))$ for any $B \subseteq Y$.*

We will return to continuous maps in due time, though we will now use these to endow more space with natural topologies.

2.5 Induced topologies

2.5.1 Subspaces

Definition 11. *Let $A \subseteq X$. The* subspace topology $\mathcal{T}_A$ on A *is the minimal topology making the inclusion $i : A \hookrightarrow X$ continuous.*

Given a map $f : X \to Y$, the associated restriction $f|_A = f \circ i$ is always continuous with the subspace topology, and the subspace topology is the unique such topology making i and all such $f|_A$ continuous. We can describe this in the following diagram (where arrows indicate *continuous* maps):

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, dashed, "f|_A"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

The subspace topology is explicitly given by

$$\mathcal{T}_A = \{O \cap A \mid O \in \mathcal{T}_X\}$$

2.5.2 Products

We first deal with finite products.

Definition 12. *Let $X_1, \ldots, X_n$ be topological spaces. The* product topology *on $X_1 \times \ldots \times X_n$ is the minimal topology making each projection $\pi_i : X_1 \times \ldots \times X_n \twoheadrightarrow X_i$ continuous.*

Given a topological space Y and continuous maps $f_i : X_i \to Y$, minimality and the condition that each π_i is continuous translates to the continuity of the map $(f_1, \ldots, f_n) : X_1 \times \ldots \times X_n \to Y$, and similar to the subspace topology, the product topology is the unique such topology making π_i, and all n-tuples of continuous functions $(f_1, \ldots, f_n)$ continuous. This corresponds to the following diagram.

$$\begin{tikzcd}
 & & X_1 \times \ldots \times X_n \arrow[dll, two heads, "\pi_1"'] \arrow[drr, two heads, "\pi_n"] \arrow[dd, dashed] & & \\
X_1 \arrow[drr, "f_1"'] & \cdots \qquad \cdots & & \cdots \qquad \cdots & X_n \arrow[dll, "f_1"] \\
 & & Y & &
\end{tikzcd}$$

The product topology for finitely many sets has a basis given by products of open sets:

$$\mathcal{B}_{X_1 \times \ldots \times X_n} = \{U_1 \times \ldots \times U_n \mid U_i \in \mathcal{T}_{X_i}\}$$

To extend to *infinite* products, we first need to define what we mean set-theoretically by an infinite product. If S_1, S_2 are sets, a function $f : S_1 \to S_2$ can be viewed as a tuple of $|S_1|$ elements in S_2 (fixing some ordering on S_1). The usual product $X_1 \times \ldots \times X_n$ corresponds to the collection of maps $f : \{1, \ldots, n\} \to \bigcup_{i=1}^n X_i$ with $f(i) \in X_i$. Replacing $\{1, \ldots, n\}$ with an arbitrary index set, we get the following.

Definition 13. *Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a collection of sets. Their (set-theoretic)* product *is*

$$\prod_{\alpha \in \Lambda} X_\alpha := \left\{ f : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha \;\middle|\; f(\beta) \in X_\beta \text{ for all } \beta \in \Lambda \right\}$$

For $x_\alpha \in X_\alpha$, we will equivalently write $(x_\alpha)_{\alpha \in \Lambda}$ for the map f sending $\alpha \mapsto x_\alpha$. As with finite products, for each $\beta \in \Lambda$ we have an associated projection map $\pi_\beta : \prod_{\alpha \in \Lambda} X_\alpha \twoheadrightarrow X_\beta$ sending $f \mapsto f(\beta)$ or equivalently $(x_\alpha)_{\alpha \in \Lambda} \mapsto x_\beta$. The above definition is very closely related to the axiom of choice.

Remark 1. *For any set X of sets, a* choice function *is a map $f : X \to \bigcup X$ with each $f(A) \in A$. The* axiom of choice *says that every such non-empty set of sets admits a choice function. This is almost exactly the statement that an arbitrary set-theoretic product is non-empty, up to replacing $\{X_\alpha\}_{\alpha \in \Lambda}$ with the index set Λ.*

We define a topology on an arbitrary product in the same way as for a finite product.

Definition 14. *Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be topological spaces. The* product topology *on $\prod_{\alpha \in \Lambda} X_\alpha$ is the minimal topology making each projection $\pi_\beta : \prod_{\alpha \in \Lambda} X_\alpha \twoheadrightarrow X_\beta$ continuous.*

Before giving an explicit description of this topology, we explore an alternative topology which we might (erroneously) expect to be the product topology. In the finite case, a basis for the product topology was given by pure products of open sets in each space. Considering the topology with basis

$$\mathcal{B}_0 = \left\{ \prod_{\alpha \in \Lambda} U_\alpha \;\middle|\; U_\alpha \in \mathcal{T}_{X_\alpha} \right\}$$

It turns out that this topology is

2.5.3 Quotients

2.6 Separability

2.7 Compactness

2.8 Connectedness

2.9 Locality