MATH3354 (Representation Theory) Summary Notes

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Contents

1 Algebras, representations and basic definitions

1.1 Algebras, representations, ideals and quotients

Definition (Algebra): Let $\mathbb{K}$ be a field. An *algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space with a bilinear map (or multiplication)

$$\begin{aligned} \mu : A \times A &\to A \\ (a, b) &\mapsto ab \end{aligned}$$

An algebra is *associative* if the bilinear map satisfies $\mu(\mu(a,b),c) = \mu(a,\mu(b,c))$, or $(ab)c = a(bc)$, *commutative* if $ab = ba$, and *unital* if it has an element 1 with $1a = a1 = a$ for all $a \in A$.

The bilinearity of the multiplication on A can be thought of as left and right distributivity. For the remainder of these notes, the algebra A will be assumed to be associative, and the field $\mathbb{K}$ assumed to be algebraically closed (and usually $\mathbb{C}$). As usual, the unit 1 in any algebra is unique.

(Important) Examples:

- $A = \mathbb{K}\langle x_i \mid i \in I\rangle$, the (non-commutative) free algebra over the set $\{x_i \mid i \in I\}$, with multiplication given by concatenation.
- $A = \mathrm{End}(\mathcal{V})$, the space of linear maps $\mathcal{V} \to \mathcal{V}$ with multiplication given by composition.

Definition (Algebra homomorphism, isomorphism): Let $\mathbb{K}$ be a field and A and B be $\mathbb{K}$-algebras. An *algebra homomorphism* $\varphi : A \to B$ is a linear map such that for any $a_1, a_2 \in A$, $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$.

An algebra homomorphism $\varphi : A \to B$ is an isomorphism if there exists an algebra homomorphism $\psi : B \to A$ such that $\psi \circ \varphi = \mathrm{id}_A$ and $\varphi \circ \psi = \mathrm{id}_B$.

As is standard with homomorphisms of algebraic objects, a homomorphism $\varphi : A \to B$ is an isomorphism if and only if it is bijective; and if and only if it has a set-theoretic inverse.

Here the set of linear maps $\mathcal{V} \to \mathcal{V}$ forms an algebra under composition, and we denote this as $\mathrm{End}(\mathcal{V})$. The key point in our study will be to look at how an algebra A "acts" on vector spaces $\mathcal{V}$, by assigning each element in A a linear map in a compatible manner. In this sense we are studying linear algebra over objects which are not necessarily fields.

Definition (Representation): A representation of A is a vector space $\mathcal{V}$ with an algebra homomorphism $\rho : A \to \mathrm{End}(\mathcal{V})$.

The map $\rho(a)$ is usually just denoted a, and we write av to mean $[\rho(a)](v)$.

(Important) Examples:

- A representation of $A = \mathbb{K}$ is just a $\mathbb{K}$-vector space.
- The regular representation, with $\mathcal{V} = A$ and

$$\begin{aligned} \rho : A &\to \mathrm{End}(\mathcal{V}) \\ a &\mapsto (b \mapsto ab) \end{aligned}$$

- If $\{\mathcal{V}_i \mid i \in I\}$ are representations of A, then

$$\mathcal{V} = \bigoplus_{i \in I} \mathcal{V}_i$$

is also a representation of $\mathcal{V}$ in a canonical manner.

Definition (Subrepresentation): A subspace $\mathcal{W} \subseteq \mathcal{V}$ of an A-representation $\mathcal{V}$ is a subspace such that $A\mathcal{W} \subseteq \mathcal{W}$, or that for any $a \in A$, $w \in \mathcal{W}$, $aw \in \mathcal{W}$.

Definition (Representation homomorphism): Let $\mathcal{V}_1, \mathcal{V}_2$ be A-representations. A homomorphism of representations is a linear map $f : \mathcal{V}_1 \to \mathcal{V}_2$ which commutes with the action of A. That is, $f(av) = af(v)$ for any $a \in A$ and $v \in \mathcal{V}$.

This definition is summarised by the following commutative diagram.

$$\begin{tikzcd}
\mathcal{V}_1 \arrow[r, "f"] \arrow[d, "\rho_{\mathcal{V}_1}(a)"] & \mathcal{V}_2 \arrow[d, "\rho_{\mathcal{V}_2}(a)"] \\
\mathcal{V}_1 \arrow[r, "f"] & \mathcal{V}_2
\end{tikzcd}$$

Definition (Representation isomorphism): A homomorphism $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ of representations is an isomorphism if there exists a homomorphism $\psi : \mathcal{V}_2 \to \mathcal{V}_1$ such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{V}_1}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{V}_2}$.

The same remarks for algebra isomorphisms hold in this case. We abbreviate "algebra homomorphism" or "representation homomorphism" in each case to just "homomorphism", as the type of homomorphism should be clear from context.

Definition (Irreducible / simple; indecomposable): We say that a non-zero A-representation $\mathcal{V}$ is

- Simple or irreducible if its only subrepresentations are 0 and $\mathcal{V}$
- Indecomposable if whenever $\mathcal{V} \cong \mathcal{V}_1 \oplus \mathcal{V}_2$, we have $\mathcal{V}_1 = 0$ or $\mathcal{V}_2 = 0$.

Lemma (Schur's Lemma): Let $f : \mathcal{V}_1 \to \mathcal{V}_2$ be a non-zero homomorphism of A-representations. Then

1. If $\mathcal{V}_1$ is irreducible, then $\mathcal{V}_2$ is injective.
2. If $\mathcal{V}_2$ is irreducible, then $\mathcal{V}_2$ is surjective.

Corollary 1: Let $\mathcal{V}_1, \mathcal{V}_2$ be irreducible A-representations. Then $f : \mathcal{V} to \mathcal{V}$ is either the zero map or an isomorphism.

Corollary 2: Let $\mathbb{K}$ be an algebraically closed field, A be a $\mathbb{K}$-algebra and $\mathcal{V}$ be an irreducible A-representation. Then the only homomorphisms $\varphi : \mathcal{V} \to \mathcal{V}$ are scalar multiplication $\varphi = \lambda\, \mathrm{id}_{\mathcal{V}}$, for $\lambda \in \mathbb{K}$.

Definition (Centre of an algebra): Let A be an algebra. Define the *centre* of A to be

$$Z(A) := \{a \in A \mid ba = ab \text{ for all } b \in A\}$$

Proposition: If $\mathcal{V}$ is an A-representation and $a \in Z(A)$, then $\rho(a) : \mathcal{V} \to \mathcal{V}$ is a homomorphism of representations.

Corollary: If A is a commutative algebra, then the irreducible representations of A have dimension 1.

Proposition: Every non-zero finite-dimensional representation $\mathcal{V}$ has an irreducible subrepresentation $\mathcal{W}$.

Definition (Ideal): Let A be an algebra. A subset $I \subseteq A$ is

- A left ideal if $AI \subseteq I$
- A right ideal if $IA \subseteq I$
- A two-sided ideal if it is both a left and right ideal.

Example(s):

- For any algebra homomorphism $f : A \to B$, $\ker(f)$ is a 2-sided ideal of A.

Definition (Quotient): Let $I \subseteq A$ be a 2-sided ideal, and $A/I := \{a + I \mid a \in A\}$. Then A/I is an algebra[1] with the multiplication $(a+I)(b+I) = ab + I$.

Algebras by generators and relations: Given a subset $S \subseteq A$, we can consider the 2-sided ideal

$$\langle S \rangle = \left\{ \sum_{i=1}^{n} a_i s_i b_i \middle| a_i, b_i \in A, s_i \in S \right\}$$

and the corresponding quotient $A / \langle S \rangle$. When $A = \mathbb{K}\langle x_1, \dots, x_n \rangle$ and $S = \{f_1, \dots, f_k\}$, we may form the quotient $\mathbb{K}\langle x_1, \dots, x_n \rangle / \langle f_1, \dots, f_k \rangle$.

Definition (The Weyl algebra): The Weyl algebra is the algebra $W = \mathbb{C}\langle x, y \rangle / \langle yx - xy - 1 \rangle$.

Proposition (Properties of the Weyl algebra):

1. The set $\left\{x^i y^j \mid i, j \in \mathbb{Z}_{\geq 0}\right\}$ is a basis for W.
2. The algebra W is simple; that is, the only 2-sided ideals in W are 0 and W.

Definition (Faithful representation): An A-representation $\mathcal{V}$ is faithful if the map $\rho : A \to \mathrm{End}(\mathcal{V})$ is injective.

1.2 Quivers

Definition (Quiver): A *quiver* Q is a directed multigraph.

The edge set of a quiver is usually denoted H, and the vertex set denoted I. Given an edge h, the vertex it maps from is denoted h', and the vertex it maps to is denoted h''.

Definition (Quiver representation): A *representation of a quiver* Q assigns

[1] and a vector space as a quotient space

1. Each vertex $i \in I$ a vector space $\mathcal{V}_i$.
2. Each edge $h \in H$ a linear map $x_h : \mathcal{V}_{h'} \to \mathcal{V}_{h''}$

Definition (Path algebra): Let Q be a quiver. The *path algebra* $P(Q)$ of Q has $\mathbb{K}$-basis $\{p_i \mid i \in I\}$, and relations

- $p_i^2 - p_i$
- $p_i p_j$ for $i \neq j$
- $p_i a_h - a_h$ when $i = h''$ and $p_i a_h$ otherwise
- $a_h p_i - a_h$ when $i = h'$, and $a_h p_i$ otherwise

Proposition: A quiver representation of Q is equivalent to an algebra representation of $P(Q)$.

Lie algebras

Definition (Lie algebra): Let $\mathfrak{g}$ be a vector space, and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a skew-symmetric bilinear map, that is, for any $a, b \in \mathfrak{g}$

1. $[a, b] = -[b, a]$
2. $[a, a] = 0$

The pair $(\mathfrak{g}, [\cdot, \cdot])$ is called a Lie algebra if it satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

In this case, the function $[\cdot, \cdot]$ is called the Lie bracket.

Examples:

1. $(\mathbb{R}^3, \times)$
2. $\mathfrak{sl}_n = \{M \in M_{n \times n}(\mathbb{K}) \mid \mathrm{Tr}(M) = 0\}$
3. Every associative algebra A is naturally a Lie algebra with the Lie bracket $[a, b] := ab - ba$

Definition (Lie algebra homomorphism): A homomorphism of Lie algebras $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map such that for all $a, b \in \mathfrak{g}_1$,

$$\varphi[a, b] = [\varphi(a), \varphi(b)]$$

Definition (Lie algebra representation): A representation of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is a vector space $\mathcal{V}$ and a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathrm{End}(\mathcal{V})$.

Definition (Basis dependent version of enveloping algebra): Let $\mathfrak{g}$ be a Lie algebra, and $\{x_i\}$ be a basis of $\mathfrak{g}$. Write $[x_i, x_j] = \sum_l c_{ij}^l x_l$. The *enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is given by

$$\mathcal{U}(\mathfrak{g}) := \mathbb{K} \langle x_i \rangle \,/\, \left\langle x_i x_j - x_j x_i - \sum_l c_{ij}^l x_l \right\rangle$$

Proposition: A Lie algebra representation $\mathfrak{g}$ is equivalent to a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$.

1.3 Tensor products

Definition (Tensor product): Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and Z be a vector space with the basis $\{(v, w) \mid v \in \mathcal{V}, w \in \mathcal{W}\}$. Then the tensor product $\mathcal{V} \otimes \mathcal{W}$ is given by

$$\mathcal{V} \otimes \mathcal{W} = Z / \sim$$

1. $(v_1, w) + (v_2, w) \sim (v_1 + v_2, w)$
2. $(v, w_1) + (v, w_2) \sim (v, w_1 + w_2)$
3. $\lambda(v, w) \sim (\lambda v, w)$
4. $\lambda(v, w) \sim (v, \lambda w)$

In this case, we denote the tuple (v, w) as $v \otimes w$.

Proposition (Tensor products): Let $\mathcal{V}, \mathcal{W}$ be vector spaces. Then the following hold.

1. Given a vector space $\mathcal{U}$, there is a natural bijection between bilinear maps $\mathcal{V} \times \mathcal{W} \to \mathcal{U}$ and linear maps $\mathcal{V} \otimes \mathcal{W} \to \mathcal{U}$.
2. If $\mathcal{B}_{\mathcal{V}}$ is a basis of $\mathcal{V}$ and $\mathcal{B}_{\mathcal{W}}$ is a basis of $\mathcal{W}$, then

$$\mathcal{B}_{\mathcal{V} \otimes \mathcal{W}} := \{v \otimes w \mid v \in \mathcal{B}_{\mathcal{V}}, w \in \mathcal{B}_{\mathcal{W}}\}$$

   is a basis for $\mathcal{V} \otimes \mathcal{W}$.
3. If $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional, then $\dim(\mathcal{V} \otimes \mathcal{W}) = \dim(\mathcal{V}) \dim(\mathcal{W})$.
4. If $\mathcal{V}$ is finite-dimensional, the map

$$\begin{aligned} f : \mathcal{V}^* \otimes \mathcal{W} &\to \mathrm{Hom}(\mathcal{V}, \mathcal{W}) \\ \phi \otimes w &\mapsto (v \mapsto \phi(v) w) \end{aligned}$$

   is a natural isomorphism, i.e. $\mathcal{V}^* \otimes \mathcal{W} \cong \mathrm{Hom}(\mathcal{V}, \mathcal{W})$.
5. Suppose that $\mathcal{V}$ is finite-dimensional. Define the symmetric and exterior squares by

$$\begin{aligned} \mathrm{Sym}^n(\mathcal{V}) &= \mathcal{V}^{\otimes n} / \{T - \sigma(T)\} \\ \wedge^n(\mathcal{V}) &= \mathcal{V}^{\otimes n} / \{T \mid \sigma(T) = T\} \end{aligned}$$

   Then
   (a) $\dim(\mathrm{Sym}^n(\mathcal{V})) = \binom{\dim(\mathcal{V})+n-1}{n}$ and $\dim(\mathrm{Sym}^n(\mathcal{V})) = \binom{\dim(\mathcal{V})}{n}$.
   (b) $\mathcal{V} \otimes \mathcal{V} \cong \mathrm{Sym}^2(\mathcal{V}) \oplus \wedge^2(\mathcal{V})$

Definition (The tensor algebra) Let $\mathcal{V}$ be a vector space. The tensor algebra $T(\mathcal{V})$ is defined by

$$T(v) = \bigoplus_{n=0}^{\infty} \mathcal{V}^{\otimes n} = \mathbb{K} \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2} \oplus \dots$$

with multiplication given by concatenation of tensors.

Definition (Some further algebras):

1. The symmetric algebra $\mathrm{Sym}(\mathcal{V}) = \mathbb{S}(\mathcal{V}) = T(\mathcal{V}) / \langle v \otimes w - w \otimes v \rangle$
2. The exterior algebra $\wedge(\mathcal{V}) = T(\mathcal{V}) / \langle v \otimes v \rangle$
3. If $\mathfrak{g}$ is a Lie algebra, the enveloping algebra $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / \langle v \otimes w - w \otimes v - [v, w] \rangle$

2 Further representation theory

Theorem (Finite-dimensional representations of $\mathfrak{sl}_2$): There is a unique irreducible dimension N representation of $\mathfrak{sl}_2$ up to isomorphism for each $N \in \mathbb{N}$, and every finite-dimensional representation is a direct sum of these irreducible representations[2].

Definition (Semi-simple): A representation $\mathcal{V}$ is semi-simple if $\mathcal{V} \cong \bigoplus_i \mathcal{W}_i$ where each $\mathcal{W}_i$ is simple.

Lemma: Let $\mathcal{V}$ be a finite-dimensional simple representation. Then $\mathrm{End}(\mathcal{V}) \cong \mathcal{V}^{\oplus \dim(\mathcal{V})}$.

Proposition (Semi-simple representations): Let $\{\mathcal{V}_i\}$ be the irreducible representations of A, and $\mathcal{W} \cong \bigoplus_i n_i \mathcal{V}_i$ be a semisimple representation of A, where each $n_i \geq 0$. If $\mathcal{U} \subseteq \mathcal{W}$ is a subrepresentation, then

1. $\mathcal{U} \cong \bigoplus_i r_i \mathcal{V}_i$ for $r_i \leq n_i$
2. The inclusion $\mathcal{U} \hookrightarrow \mathcal{W}$ is a direct sum of inclusions $f_i : r_i \mathcal{V}_i \to n_i \mathcal{V}_i$, where each f_i is given by an injective linear map $\mathbb{K}^r \to \mathbb{K}^n$, or equivalently an $r_i \times n_i$ matrix.

Corollary: Let $\mathcal{V}$ be a finite-dimensional simple representation, and suppose that $v_1, \dots, v_n \in \mathcal{V}$ be linearly independent. For any n vectors $w_1, \dots, w_n \in \mathcal{V}$, there is $a \in A$ with $a(v_i) = w_i$.

Theorem (The Density Theorem):

1. If $\rho : A \to \mathrm{End}(\mathcal{V})$ is irreducible, then ρ is surjective.
2. Let $\mathcal{V} \cong \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_r$ where the representations $\mathcal{V}_i$ are irreducible and pairwise non-isomorphic. Then $\rho : A \to \bigoplus_{i=1}^r \mathrm{End}(\mathcal{V}_i)$ is surjective

Definition (Dual representation): Let $\mathcal{V}$ be a representation of A. The dual representation is the representation of A^{op} given by $(f \cdot a)(v) = af(v)$.

Theorem (Finite-dimensional representations of matrix algebras): Let $A = \bigoplus_{i=1}^n M_{d_i \times d_i}(\mathbb{K})$. Then the only irreducible representation of $M_{d_i}(\mathbb{K})$ up to isomorphism is $\mathbb{K}^{d_i}$, and every finite dimensional representation of A is a direct sum of these.

Definition (Filtration): Let A be an algebra and $\mathcal{V}$ be an A-representation. A finite filtration of $\mathcal{V}$ is a sequence of subrepresentations

$$0 = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \ldots \subseteq \mathcal{V}_k = \mathcal{V}$$

Lemma (Existence of Jordan-Hölder filtrations): Let $\mathcal{W}$ be a finite-dimensional representation of A. Then $\mathcal{W}$ has a finite filtration

$$0 = \mathcal{W}_0 \subseteq \ldots \subseteq \mathcal{W}_k = \mathcal{W}$$

such that each $\mathcal{V}_i = \mathcal{W}_i / \mathcal{W}_{i-1}$ is irreducible.

Such a filtration is called a *Jordan-Hölder* filtration.

[2]More can be said here, but I have no motivation to add that now.

Definition (Radical of an algebra): Let A be an algebra. The *radical* or *Jacobson radical* of A is the set of elements $a \in A$ which act by 0 in all finite-dimensional irreducible representations $\rho : A \twoheadrightarrow \mathrm{End}(\mathcal{V})$ of A, and is denoted $\mathrm{Rad}(A)$ or $J(A)$.

All results proved about radicals from here onwards are working with the underlying assumption that every representation is finite-dimensional.

Example: In a matrix algebra, the radical corresponds to the set of all *strictly* upper triangular matrices.

Proposition (Radicals): The following hold.

- $\mathrm{Rad}(A)$ is a 2-sided ideal in A.
- If $I \subseteq A$ is a nilpotent 2-sided ideal, then $I \subseteq \mathrm{Rad}(A)$.
- $\mathrm{Rad}(A)$ is nilpotent.

Hence $\mathrm{Rad}(A)$ is the largest nilpotent 2-sided ideal of A with respect to inclusion.

Theorem (Finite dimensional algebras): Let A be a finite-dimensional algebra. Then

1. There are finitely many irreducible representations of A up to isomorphism.
2. Every irreducible representation of A is finite-dimensional.
3. if $\mathcal{V}_1, \ldots, \mathcal{V}_k$ are the irreducible representations of A, then
$$A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^{k} \mathrm{End}(\mathcal{V}_i) \cong \bigoplus_{i=1}^{n} M_{d_i \times d_i}(\mathbb{K})$$
where $d_i = \dim(\mathcal{V}_i)$ for each $1 \le i \le k$.

Corollary (Sums of squares): Let A be a finite-dimensional algebra. If $\mathcal{V}_1, \ldots, \mathcal{V}_k$ are the irreducible representations of A, then

$$\sum_{i=1}^{k} \dim(\mathcal{V}_i)^2 \le \dim(A)$$

Definition (Semisimple algebras): Let A be a finite-dimensional algebra. Then A is *semisimple* if $\mathrm{Rad}(A) = 0$.

Theorem (Semisimplicity): Let A be a finite-dimensional algebra, and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be its irreducible representations with dimensions $d_1, \ldots, d_k$. Then the following are equivalent:

1. A is semisimple.
2. $\sum_{i=1}^{k} \dim(\mathcal{V}_i)^2 = \dim(A)$.
3. $A \cong \bigoplus_{i=1}^{k} M_{d_i \times d_i}(\mathbb{K})$
4. Any finite-dimensional representation of A is *completely reducible*, i.e. isomorphic to a direct sum of irreducible representations.
5. The regular representation of A is completely reducible.

2.1 Characters

Definition (Character of a representation): Let $\rho : A \to \mathrm{End}(\mathcal{V})$ be a finite-dimensional representation of A. The character $\chi_{\mathcal{V}}$ of $\mathcal{V}$ (or technically of ρ) is the map

$$\begin{aligned} \chi_{\mathcal{V}} : A &\to \mathbb{K} \\ a &\mapsto \mathrm{Tr}(\rho(a)) \end{aligned}$$

This is linear and satisfies $\chi_{\mathcal{V}}(ab) = \chi_{\mathcal{V}}(ba)$, so $\chi_{\mathcal{V}}([A, A]) = 0$, and so the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "\chi_{\mathcal{V}}"] \arrow[dr, two heads, "\pi"'] & & \mathbb{K} \\ & A/[A,A] \arrow[ur, "``\chi_{\mathcal{V}}''"'] & \end{tikzcd}$$

Where we treat the maps $\chi_{\mathcal{V}} : A \to \mathbb{K}$ and the corresponding quotient map $\chi_{\mathcal{V}} : A/[A, A] \to \mathbb{K}$ as the same.

Lemma (Splitting of characters): If $\mathcal{V}$ is a representation and $\mathcal{W} \subseteq \mathcal{V}$ a subrepresentation, then

$$\chi_{\mathcal{V}} = \chi_{\mathcal{W}} + \chi_{\mathcal{V}/\mathcal{W}}$$

In particular, $\chi_{\mathcal{V}\oplus\mathcal{W}} = \chi_{\mathcal{V}} + \chi_{\mathcal{W}}$

Theorem (Characters): The following hold.

- If $\mathcal{V}$ is finite-dimensional, $\chi_{\mathcal{V}} \in \mathrm{span}\,\{\chi_{\mathcal{U}} \mid \mathcal{U}$ is irreducible$\}$.
- The characters of distinct finite-dimensional irreducible representations of A are linearly independent.
- If A is semisimple and finite-dimensional, then $\{\chi_{\mathcal{U}} \mid \mathcal{U}$ irreducible$\}$ is a basis for $(A/[A, A])^*$

This proof uses the fact that $[M_d(\mathbb{K}), M_d(\mathbb{K})] = \mathfrak{sl}_d = \{X \in M_d(\mathbb{K}) \mid \mathrm{Tr}(X) = 0\}$, which has dimension $d^2 - 1$.

Theorem (Jordan-Hölder theorem): The irreducible representations (with multiplicity) that show up in a Jordan-Hölder filtration of a finite-dimensional representation $\mathcal{V}$ are independent of the filtration.

Theorem (Krull-Schmidt property): Let $\mathcal{V}$ be a finite-dimensional representation. The indecomposable representations with multiplicity that occur in the decomposition

$$\mathcal{V} \cong \bigoplus_i \mathcal{W}_i$$

are independent of the decomposition.

Proposition (Characters and isomorphism): Let A be a semisimple algebra. Then $\mathcal{V} \cong \mathcal{W}$ if and only if $\chi_{\mathcal{V}} = \chi_{\mathcal{W}}$.

3 Representation theory of finite groups

3.1 Foundations

Note that a representation $\mathcal{V}$ of $\mathbb{K}[G]$ as an algebra is equivalent to a group representation $\rho : G \to \mathrm{GL}(\mathcal{V})$.

In this section, the basis vectors of $\mathbb{K}[G]$ (corresponding to the group elements $g \in G$) are also denoted as $g \in \mathbb{K}[G]$.

Definition (G-invariant inner product): Let $\mathcal{V}$ be a representation of $\mathbb{K}[G]$. We say that an inner product $\langle -, - \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ is *G-invariant* if for every $g \in G$ and $u, v \in \mathcal{V}$,

$$\langle gu, gv \rangle = \langle u, v \rangle$$

Lemma (G-invariance and subrepresentations): Let $\mathcal{V}$ be a representation of $\mathbb{K}[G]$ and $\langle -, - \rangle$ be a G-invariant inner product. If $\mathcal{W} \subseteq \mathcal{V}$ is a subrepresentation, then so is $\mathcal{W}^{\perp}$.

Theorem (Maschke's Theorem): If $\mathrm{char}(\mathbb{K}) \nmid |G|$, then $\mathbb{K}[G]$ is semisimple.

The field in the above theorem is not required to be algebraically closed. The converse of this theorem also holds, and the idea of its proof is that we can find $\mathrm{Triv} \subsetneq \mathbb{K}[G]$, but $\mathbb{K}[G] \ncong \mathrm{Triv} \oplus \mathcal{W}$ for any $\mathcal{W} \subseteq \mathbb{K}[G]$.

Note now that $\chi_{\mathcal{V}} : \mathbb{K}[G] \to \mathbb{K}$ is constant on conjugacy classes, that is, $\chi_{\mathcal{V}}(h) = \chi_{\mathcal{V}}(ghg^{-1})$ for any $g, h \in G$.

Definition (Class function): A function $f : G \to \mathbb{K}$ is called a *class function* if it is constant on conjugacy classes. That is, for $g, h \in G$, $f(h) = f(ghg^{-1})$. We then denote the space of all class functions

$$\mathcal{C}\ell(G) := \{f : G \to \mathbb{K} \mid f \text{ is a class function}\}$$

Note that as vector spaces, we have a canonical isomorphism $(A/[A, A])^* \cong \mathcal{C}\ell(G)$

Lemma (Dimension of $\mathcal{C}\ell(G)$): The dimension of $\mathcal{C}\ell(G)$ is exactly the number of conjugacy classes in G. Namely, the maps

$$\begin{aligned} \xi_{\mathcal{C}} : G &\to \mathbb{K} \\ g &\mapsto \begin{cases} 1 & \text{if } g \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for the conjugacy classes $\mathcal{C} \subseteq G$ are a basis for $\mathcal{C}\ell(G)$.

Lemma (Another expression for $\dim(\mathcal{C}\ell(G))$**)** Let $\mathbb{K}[G]$ be semisimple. Then $\dim(\mathcal{C}\ell(G))$ is equal to the number of irreducible representations of $\mathcal{V}$.

Theorem (Counting irreducible representations): Suppose that $\mathrm{char}(\mathbb{K}) \nmid |G|$. Then the number of irreducible representations of $\mathbb{K}[G]$ is equal to the number of conjugacy classes of G.

3.2 Characters

Definition (Character table): The *character table* of G is the change of basis matrix between the basis $\{\xi_\mathcal{C} \mid \mathcal{C} \text{ is a conjugacy class of } G\}$ and $\{\chi_{\mathcal{V}_i} \mid \mathcal{V}_i \text{ irreducible}\}$.

Proposition (Characters): When $\mathbb{K}[G]$ is semisimple, the character $\chi_\mathcal{V}$ of a representation $\mathcal{V}$ determines $\mathcal{V}$ up to isomorphism. That is, $\chi_\mathcal{V} = \chi_\mathcal{W}$ if and only if $\mathcal{V} \cong \mathcal{W}$.

Lemma (Eigenvalues of a tensor product): Let $f : \mathcal{V} \to \mathcal{V}$ and $g : \mathcal{W} \to \mathcal{W}$ be linear maps with eigenvalues $\{\lambda_i\}_i$ and $\{\mu_j\}_j$. Then $f \otimes g : \mathcal{V} \otimes \mathcal{W} \to \mathcal{V} \otimes \mathcal{W}$ has eigenvalues $\{\lambda_i \mu_j\}_{i,j}$.

Proposition (Building representations): Let $\mathcal{V}, \mathcal{W}$ be representations of $\mathbb{K}[G]$. Then

1. The dual space $\mathcal{V}^* = \mathrm{Hom}(\mathcal{V}, \mathbb{K})$ of $\mathcal{V}$ is a representation with $\rho_{\mathcal{V}^*}(g) := \rho(g^{-1})^*$, and $\chi_{\mathcal{V}^*}(g) = \overline{\chi_\mathcal{V}(g)}$.
2. The tensor product $\mathcal{V} \otimes \mathcal{W}$ is a representation with $\rho_{\mathcal{V}\otimes\mathcal{W}}(g) = \rho_\mathcal{V}(g) \otimes \rho_\mathcal{W}(g)$, and

$$\chi_{\mathcal{V}\otimes\mathcal{W}}(g) = \chi_\mathcal{V}(g)\chi_\mathcal{W}(g)$$

Proposition (Tensors and G-invariance): Let $\mathcal{V}, \mathcal{W}$ be representations with $\mathcal{W}$ be finite-dimensional, Under the natural isomorphism $\mathrm{Hom}(\mathcal{W}, \mathcal{V}) \cong \mathcal{W}^* \otimes \mathcal{V}$, restricting gives an isomorphism $\mathrm{Hom}_G(\mathcal{W}, \mathcal{V}) \cong [\mathcal{W}^* \otimes \mathcal{V}]^G$ between the representation homomorphisms $\mathcal{W} \to \mathcal{V}$ and the G-invariant vectors in $\mathcal{W}^* \otimes \mathcal{V}$.

Lemma (Hermitian inner product on class functions): Let G be a finite group. Then the bilinear form

$$\begin{aligned} \langle -, - \rangle : \mathcal{C}\ell(G) \times \mathcal{C}\ell(G) &\to \mathbb{C} \\ (f_1, f_2) &\mapsto \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} \end{aligned}$$

is a Hermitian inner product.

Theorem (Properties of the inner product): Let $\mathcal{V}$ and $\mathcal{W}$ be representations of a group G. Then

$$\langle \chi_\mathcal{V}, \chi_\mathcal{W} \rangle = \dim(\mathrm{Hom}_G(\mathcal{W}, \mathcal{V}))$$

Further, if $\mathcal{V}$ and $\mathcal{W}$ are irreducible, then

$$\langle \chi_\mathcal{V}, \chi_\mathcal{W} \rangle = \begin{cases} 1 & \text{if } \mathcal{V} \cong \mathcal{W} \\ 0 & \text{otherwise} \end{cases}$$

That is, the characters of irreducible representations form an orthonormal basis with respect to this inner product.

Corollary (Decomposing representations): Let G be a group, $\mathcal{V}$ be a representation, and $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be its irreducible representations.

1. If $\mathcal{V} \cong \bigoplus_{i=1}^n n_i \mathcal{V}_i$, then $\langle \chi_\mathcal{V}, \chi_\mathcal{V} \rangle = \sum_{i=1}^n n_i^2$, and $n_i = \langle \mathcal{V}, \mathcal{V}_i \rangle$.
2. If $\langle \chi_\mathcal{V}, \chi_\mathcal{V} \rangle = 1$, then $\mathcal{V}$ is irreducible.

Definition (Idempotents): Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be the irreducible representations of a group g, and $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \nmid |G|$. The idempotent ψ_i is the element

$$\psi_i := \frac{\dim(\mathcal{V}_i)}{|G|} \sum_{g \in G} \chi_{\mathcal{V}_i}(g) g^{-1} \in \mathbb{K}[G]$$

Proposition (Idempotents as projections): The map $\rho_j(\psi_i) : \mathcal{V}_j \to \mathcal{V}_j$ is the identity map if $i = j$, and the null map otherwise. We also have $\psi_i^2 = \psi_i$, and $\psi_i \psi_j = 0$ for $i \neq j$.

Definition (Unitary representation): Let G be a group, and $\mathcal{V}$ be a representation of G. We say that $\mathcal{V}$ is unitary if it is endowed with a G-invariant Hermitian inner product.

Theorem (Characters and centralisers): Let G be a group, $g \in G$, and $Z_g = \{h \in G \mid gh = hg\}$ be the stabiliser of g. Then

$$\sum_{\mathcal{V}_i \text{ irreducible}} \chi_{\mathcal{V}_i}(g)\overline{\chi_{\mathcal{V}_i}(h)} = \begin{cases} |Z_g| & \text{if } g, h \text{ conjugate} \\ 0 & \text{otherwise} \end{cases}$$

Proposition (Induced representations by quotients): Let $N \triangleleft G$, and $\rho : G/N \to \mathrm{GL}(\mathcal{V})$ be a representation of G/N. Then $\tilde{\rho} = \rho \circ \pi : G \to \mathrm{GL}(\mathcal{V})$ is a representation of G, with $\chi_{\mathcal{V}}^G(g) = \chi_{\mathcal{V}}^{G/N}(gN)$. If ρ is irreducible, then so is $\tilde{\rho}$.

Definition (Division algebra): An algebra A is a *division algebra* if each $a \neq 0$ is invertible. That is, there exists $b \in A$ with $ab = ba = 1$.

Theorem (Division algebras over $\mathbb{R}$): The division algebras over $\mathbb{R}$ are exactly

1. The complex numbers $\mathbb{C}$
2. The real numbers $\mathbb{R}$
3. The quaternions $\mathbb{H}$, generated by $1, i, j, k$ with $i^2 = j^2 = k^2 = ijk = -1$.

Definition (Representation types): Let G be a group, and $\mathcal{V}$ be an irreducible representation of G over $\mathbb{C}$. We say that G is of

1. complex type if $\mathcal{V} \ncong \mathcal{V}^*$
2. real type if it admits a non-degenerate symmetric bilinear form.
3. quaternionic type if it admits a non-degenerate skew-symmetric bilinear form.

These are mutually exclusive, although this is not immediately obvious.

Lemma: If $\mathcal{V}$ admits a non-degenerate bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$, then

$$\begin{aligned} \tilde{B} : \mathcal{V} &\to \mathcal{V}^* \\ v &\mapsto (w \mapsto B(v, w)) \end{aligned}$$

is an isomorphism.

Definition (Frobenius-Schur indicator): Let $\mathcal{V}$ be a representation of a finite group G. The *Frobenius-Schur indicator* of $\mathcal{V}$ is

$$\mathrm{FS}(\mathcal{V}) := \frac{1}{|G|} \sum_{g \in G} \chi_{\mathcal{V}}(g^2)$$

Theorem: Let $\mathcal{V}$ be a representation of a finite group G. Then

$$\mathrm{FS}(\mathcal{V}) = \begin{cases} 1 & \text{if } \mathcal{V} \text{ is of complex type} \\ 0 & \text{if } \mathcal{V} \text{ is of real type} \\ -1 & \text{if } \mathcal{V} \text{ is of quaternionic type} \end{cases}$$

Lemma: The number of involutions in G is given by

$$\left|\{g \in G \mid g^2 = e\}\right| = \frac{1}{|G|} \sum_{\mathcal{V}_i \text{ irreducible}} \dim(\mathcal{V}) \chi_{\mathcal{V}} \left(\sum_{g \in G} g^2 \right)$$

Corollary: The number of involutions in G is

$$\sum_{\mathcal{V} \text{ irreducible}} \mathrm{FS}(\mathcal{V}) \dim(\mathcal{V}) = \sum_{\mathcal{V} \text{ irreducible, } \mathbb{R}} \dim(\mathcal{V}) - \sum_{\mathcal{V} \text{ irreducible, } \mathbb{H}} \dim(\mathcal{V})$$

3.3 Algebraic numbers and integers

Definition (Algebraic number, integer): An algebraic number is a root of a monic polynomial $f \in \mathbb{Q}[x]$, and an algebraic integer is a root of a monic polynomial $f \in \mathbb{Z}[x]$.

Definition (Algebraic number, integer): An algebraic number is an eigenvalue of a matrix M with $\mathbb{Q}$-coefficients, and an algebraic integer is an eigenvalue of a matrix M with $\mathbb{Z}$-coefficients.

Proposition: The above definitions are equivalent.

The set of algebraic integers is denoted $\mathbb{A}$, and the set of algebraic numbers $\overline{\mathbb{Q}}$.

Proposition: $\mathbb{A}$ is a ring, and $\overline{\mathbb{Q}}$ is a field.

Proposition: $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.

3.4 Further representation theory of finite groups

Proposition: Let G be a finite group, $\mathcal{V}$ be an irreducible representation of $\mathbb{C}[G]$, and $C_1, \ldots, C_n$ be the irreducible representations of G. Then

$$\lambda_i = \chi_{\mathcal{V}}(g_{C_i}) \frac{|C_i|}{\dim(\mathcal{V}_i)} \in \mathbb{A}$$

for each $1 \leq i \leq n$.

Theorem (Frobienius divisibility): Let G be a group, and $\mathcal{V}$ be an irreducible representation of $\mathbb{C}[G]$. Then $\dim(\mathcal{V}) \mid |G|$.

Theorem: Let G be a finite group, $\mathcal{V}$ be an irreducible representation, and C be a conjugacy class with $\gcd(\dim(\mathcal{V}), |C|) = 1$. Then either $\chi_{\mathcal{V}}(g_C) = 0$ for every $g_C \in C$, or every $g_C \in C$ acts by a scalar.

Lemma: Let $\varepsilon_1, \ldots, \varepsilon_n$ be roots of unity, and suppose that $\frac{\varepsilon_1 + \ldots + \varepsilon_n}{n}$ is an algebraic integer. Then either $\varepsilon_1 + \ldots + \varepsilon_n = 0$, or $\varepsilon_1 = \ldots = \varepsilon_n$.

Theorem: Let G be a finite group, and C be a conjugacy class of order p^k. Then G has a proper normal subgroup.

Lemma: With the above setup, there is a non-trivial subrepresentation $\mathcal{V}$ of G with $p \nmid \dim(\mathcal{V})$ and $\chi_{\mathcal{V}}(g) \neq 0$ for $g \in C$.

Definition (Solvable group): Let G be a finite group. We say that G is solvable if there exists a chain $\{0\} = G_0 \triangleleft \ldots \triangleleft G_n = G$ such that each quotient G_{i+1}/G_i is abelian.

Theorem (Burnside's theorem): Let p, q be distinct primes, and $|G| = p^a q^b$ for $a, b \geq 1$. Then G is solvable.

Corollary (Class functions as characters): Let $\{\mathcal{V}_i\}$ be the set of irreducible representation of a group G, and $f \in \mathcal{C}\ell(G)$ be of the form $f = \sum_i n_i \mathcal{V}_i$. Then f is the character of an irreducible representation if and only if $\langle f, f \rangle = 1$ and $f(1) > 0$.

3.5 Restriction and Induction

Definition (Restriction and induction) Let $H \subseteq G$ be an inclusion of groups. If $\mathcal{V}$ is a representation of G, then the restriction of G to H is the representation $\rho_{\mathrm{Res}_G^H(\mathcal{V})} = \rho_{\mathcal{V}}|_H : H \to \mathrm{GL}(\mathcal{V})$. If $\mathcal{W}$ is a representation of H, then the induced representation on G by H is the representation with underlying vector space

$$\mathrm{Ind}_H^G(\mathcal{V}) := \{f : G \to \mathcal{V} \mid f(hx) = \rho_{\mathcal{V}}(h)f(x) \text{ for all } h \in H, g \in G\}$$

and corresponding G-action

$$\begin{aligned} \rho_{\mathrm{Ind}_H^G(\mathcal{V})} : G &\to \mathrm{GL}(\mathrm{Ind}_H^G(\mathcal{V})) \\ g &\mapsto (f \mapsto (x \mapsto f(xg))) \end{aligned}$$

That is, $[[\rho_{\mathrm{Ind}_H^G(\mathcal{V})}(g)](f)](x) = f(xg)$.

Lemma (Properties of the induced representation): If $H \subseteq G$ is an inclusion of groups, then the induced representation is a well-defined representation of G. If $\mathcal{V}$ is finite dimensional and G is a finite group, then $\dim(\mathrm{Ind}_H^G(\mathcal{V})) = \dim(\mathcal{V})[G : H]$.

Theorem (Frobenius reciprocity): Let $H \subseteq G$ be an inclusion of groups, $\mathcal{V}$ be a finite-dimensional representation of G and $\mathcal{W}$ be a finite-dimensional representation of H. Then there is a canonical isomorphism

$$\mathrm{Hom}_H(\mathrm{Res}_G^H(\mathcal{V}), \mathcal{W}) \cong \mathrm{Hom}_G(\mathcal{V}, \mathrm{Ind}_H^G(\mathcal{W}))$$

In particular, we have

$$\left\langle \mathrm{Res}_G^H(\mathcal{V}), \mathcal{W} \right\rangle = \left\langle \mathcal{V}, \mathrm{Ind}_H^G(\mathcal{W}) \right\rangle$$

4 More on quiver representations

4.1 Dynkin diagrams

Definition (Adjacency matrix, Cartan matrix, Dynkin diagram): Let Γ be a finite (undirected) graph, and $v_1, \ldots, v_n$ be its vertices. Its adjacency matrix is the matrix

$$(R_\Gamma)_{i,j}(\#\text{ edges between } i \ \& \ j)$$

Its Cartan matrix is then $A_\Gamma = 2I - R_\Gamma$.

A graph Γ is a *Dynkin diagram* if the associated quadratic form $x \mapsto x^T A_\Gamma x$ is positive definite.

Theorem (Classification of Dynkin diagrams): A graph Γ is Dynkin if and only if it is one of A_N, D_N, E_6, E_7 or E_8.

4.2 Roots of Dynkin diagrams

Definition (Finite type): A quiver has *finite type* if it has finitely many indecomposable representations up to isomorphism.

Definition (Lattice): A lattice is a free abelian group with a symmetric bilinear form.

Definition (Root lattice): Let Γ be a Dynkin diagram. A root lattice of type Γ is a lattice $(\mathbb{Z}^n, B)$ where B agrees with $(u, v)_\Gamma = u^T A_\Gamma v$ with respect to some basis.

Lemma (Properties of the bilinear form): Let Γ be a Dynkin diagram, and B be the associated bilinear form. Then

1. A_Γ is positive definite.
2. $B(x, x) \in 2\mathbb{Z}$ for all $x \in \mathbb{Z}^n$.

Corollary (Minimal norms): If $x \neq 0$, then $B(x, x) \geq 2$.

Definition (Simple root): Let Γ be a Dynkin diagram and i be a vertex. The corresponding element α_i the root lattice $(\mathbb{Z}^n, B)$ is called a simple root.

The simple roots are trivially a basis for $(\mathbb{Z}^n, B)$.

Lemma (Positive and negative): Let $\alpha = \sum_{k=1}^n k_i \alpha_i$ be a root. Then either all $k_i \geq 0$ or all $k_i \leq 0$.

Definition (Positive and negative roots): Let $\{\alpha_i\}$ be the simple roots of a Dynkin diagram Γ and $\alpha = \sum_{k=1}^n k_i \alpha_i$ be a root. We say that α is positive if $k_i \geq 0$ for all i, and negative if $k_i \leq 0$ for all i.

Proposition (Roots of A_N): The roots of A_N are of the form $\alpha_i + \ldots + \alpha_j$ for $i < j$, and there are exactly $\frac{N(N+1)}{2}$ roots.

Definition (Weyl group): For a Dynkin diagram Γ and each root α of Γ, the reflection in the hyperplane orthogonal to α is the map

$$\begin{aligned} s_\alpha : \mathbb{Z}^n &\to \mathbb{Z}^n \\ v &\mapsto v - (v, \alpha)_\Gamma \alpha \end{aligned}$$

The Weyl group W_Γ is the subgroup of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^n)$ generated by these maps. We write the map corresponding to α_i as s_i.

Lemma (Weyl group acts on roots): Let α, β be roots of a Dynkin diagram Γ. Then $s_\alpha(\beta)$ is a root.

Corollary: If Γ is a Dynkin diagram, then $|W_\Gamma| < \infty$.

Definition (Dimension vector): Let Q be a quiver and $\mathcal{V}$ be a representation of Q. The *dimension vector* of $\mathcal{V}$ is $\dim(\mathcal{V}) = (\dim(\mathcal{V}_i))_i$.

4.3 Reflection Functors

Definition (Reflection functor): Let Γ be a graph and Q be a quiver with underlying graph Γ. Let i be a vertex in Γ and $\overline{Q_i}$ denote the quiver with orientations at i flipped. If i is a sink in Q, we define the reflection functor $F_i^+ : \mathrm{Rep}(Q) \to \mathrm{Rep}(\overline{Q_i})$ by setting $(F_i^+\mathcal{V})_j = \mathcal{V}_j$ for $j \neq i$, preserving all maps not involving i, and

$$\begin{aligned} (F_i^+\mathcal{V})_i &= \ker\left(\bigoplus_{j\to i} \mathcal{V}_j \to \mathcal{V}_i\right) \\ \varphi_{i\to j}^{F_i^+\mathcal{V}} : (F_i^+\mathcal{V})_i &\to (F_i^+\mathcal{V})_j \\ (v_j)_{j\to i} &\mapsto v_j \end{aligned}$$

Similarly, if i is a source in Q, we define $F_i^- : \mathrm{Rep}(Q) \to \mathrm{Rep}(\overline{Q_i})$ by $(F_i^-\mathcal{V})_j = \mathcal{V}_j$ for $j \neq i$, preserving all maps not involving i, and

$$\begin{aligned} (F_i^+\mathcal{V})_i &= \mathrm{coker}\left(\mathcal{V}_i \to \bigoplus_{j\to i} \mathcal{V}_j\right) \\ \varphi_{j\to i}^{F_i^-\mathcal{V}} : (F_i^-\mathcal{V})_j &\to (F_i^-\mathcal{V})_i \\ v_j &\mapsto \overline{\left(\begin{cases} v_j & \text{if } k = j \\ 0 & \text{else} \end{cases}\right)_{k\to i}} \end{aligned}$$

Proposition (Indecomposable representations): Let Q be a quiver, $\mathcal{V}$ an indecomposable representation of Q.

1. If i is a sink, then either $\dim(\mathcal{V}_i) = 1$ and $\dim(\mathcal{V}_j) = 0$ for all $j \neq i$, or $\varphi : \bigoplus_{j\to i} \mathcal{V}_j \to \mathcal{V}_i$ is surjective.
2. If i is a source, then either $\dim(\mathcal{V}_i) = 1$ and $\dim(\mathcal{V}_j) = 0$ for all $j \neq i$, or $\varphi : \mathcal{V}_i \to \bigoplus_{i\to j} \mathcal{V}_j$ is injective.

Proposition (Reflection functors as inverses): Let Q be a quiver and $\mathcal{V} \in \mathrm{Rep}(Q)$.

1. If i is a sink and $\varphi : \bigoplus_{j\to i} \mathcal{V}_j \to \mathcal{V}_i$ is surjective, then
$$F_i^- F_i^+ \mathcal{V} \cong \mathcal{V}$$
2. If i is a source and $\varphi : \mathcal{V}_i \to \bigoplus_{i\to j} \mathcal{V}_j$ is injective, then
$$F_i^+ F_i^- \mathcal{V} \cong \mathcal{V}$$

Theorem (Indecomposables and reflection functors): Let $\mathcal{V}$ be an indecomposable representation of Q, and i be a sink or source in Q. Then $F_i^{\pm}\mathcal{V}$ is either an indecomposable representation or zero.

Theorem (Reflection functors and the Weyl group): Let Q be a quiver and $\mathcal{V}$ be a representation of Q. If $\mathcal{V}$ is indecomposable and $F_i\mathcal{V} \neq 0$, then $\dim(F_i\mathcal{V}) = s_i(\dim(\mathcal{V}))$.

Definition (Coxeter element): Let Q be a quiver and its vertices be $1, \dots, n$. The *Coxeter element* with respect to this ordering is the element $c = s_1 \dots s_n$.

Lemma (Coxeter element and coefficients): Let c be the Coxeter element of Q with respect to some ordering, and $\beta = \sum_i k_i\alpha_i$. Then there is some $N \in \mathbb{N}$ such that
$$c^N\beta = \sum_i k_i\alpha_i$$
has at least 1 negative coefficient.

Lemma (Ordering vertices): Let Q be a Dynkin quiver (in particular a tree). Then there is an ordering $1, \dots, n$ of the vertices in Γ such that $i \leq j$ if there is a path from i to j.

Theorem (Applying reflection functors): Let $\mathcal{V}$ be an indecomposable representation of a Dynkin quiver Q, and $1, \dots, n$ be an ordering as above. Define a sequence of representations by
$$\begin{aligned}
\mathcal{V}^{(0)} &= \mathcal{V} \\
\mathcal{V}^{(1)} &= F_n^+\mathcal{V} \\
\mathcal{V}^{(2)} &= F_{n-1}^+F_n^+\mathcal{V} \\
&\vdots \\
\mathcal{V}^{(n)} &= F_1^+ \dots F_n^+\mathcal{V} \\
\mathcal{V}^{(n+1)} &= F_n^- F_1^+ \dots F_n^+\mathcal{V} \\
\mathcal{V}^{(n+2)} &= F_{n-1}^- F_n^- F_1^+ \dots F_n^+\mathcal{V} \\
&\vdots
\end{aligned}$$
Then there is some $i \in \mathbb{N}$ and simple root α_p with $\dim(\mathcal{V}^{(i)}) = \alpha_p$.

Corollary (Dimension vectors as roots): Let Q be a Dynkin quiver and $\mathcal{V}$ be an indecomposable representation of Q. Then $\dim(\mathcal{V})$ is a root of Q.

Corollary (Uniqueness of representations by roots): Let Q be a Dynkin quiver and $\mathcal{V}, \mathcal{V}'$ be indecomposable representations with $\dim(\mathcal{V}) = \dim(\mathcal{V}')$. Then $\mathcal{V} \cong \mathcal{V}'$.

Theorem (Existence of a representation for each root): Let Q be a Dynkin quiver and α a positive root of Q. Then there is an indecomposable representation $\mathcal{V}$ with $\dim(\mathcal{V}) = \alpha$.