Notes on Category Theory

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Abstract

This will be a collection of random category theory junk that I find useful.

1 Foundations

Informally, view a category as a transitively closed directed graph.

Definition 1 (Universal arrow)**.** *Let $\mathcal{F} : C \to D$ be a functor, and d be an object of D. A* universal arrow from d to $\mathcal{F}$ *is a pair (r, u) where*

1. *r is an object in C;*
2. *$u : d \to \mathcal{F}r$ is an arrow in D; and*
3. *For any pair (c, f) with c an object in C and $f : d \to \mathcal{F}c$ an arrow of D, there is a unique arrow $f' : r \to c$ of C with $\mathcal{F}f' \circ u = f$.*

2 Limits and colimits

2.1 Limits in algebraic objects

We assume for the following definitions (until otherwise stated) that we are considering some collection of algebraic objects.

Definition 2 (Direct system)**.** *A* direct system *is a pair $\langle A_i, f_{ij} \rangle$ where $\{A_i\}_{i \in I}$ be a family of objects indexed by a poset $\langle I, \leq \rangle$ be a poset, and $f_{ij} : A_i \to A_j$ are homomorphisms for $i \leq j$ with*

1. *$f_{ii} = 1_{A_i}$; and*
2. *$f_{ik} = f_{jk} \circ f_{ij}$ whenever $i \leq j \leq k$.*

Definition 3. *Let $\langle A_i, f_{ij} \rangle$ be a directed system. Define an equivalence relation $\sim$ on $\bigsqcup A_i$ by $x_i \sim x_j$ iff there is $i, j \leq k$ with $f_{ik}(x_i) = f_{jk}(x_j)$. The* direct limit *or* colimit *$\varinjlim A_i$ is then*

$$\varinjlim A_i := \bigsqcup A_i \;/\sim$$

We can think of points in the direct limit as the result of following the arrows f_{ij} "to infinity".