Completions and formal power series

Stanley Li

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For any metric space X, we have an associated "larger" metric space which is *complete*, that is, where every Cauchy sequence converges. This space also has an associated universal property:

Theorem 1. *Let X be a metric space. Then there is a complete metric space $\overline{X}$ and an isometry $i : X \hookrightarrow \overline{X}$. Further, if Y is a complete metric space and $j : X \to Y$ is a uniformly continuous injective map, then there is a unique uniformly continuous injective map $\overline{j} : \overline{X} \to Y$ such that $\overline{j} \circ i = j$.*

The last condition can be described in the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, hook, "i"] & Y \\
\tilde{X} \arrow[ur, hook, dashed, "\exists ! \tilde{j}"'] &
\end{tikzcd}$$

and in this sense $\overline{X}$ is the smallest complete metric space containing X, unique up to isometry. We call this $\overline{X}$ the *completion* of X.

To construct this space, we note that the space

$$\mathcal{C}_X = \left\{ (x_n)_{n\in\mathbb{N}} \mid d(x_n, x_m) \xrightarrow{n,m\to\infty} 0 \right\} \subseteq X^{\mathbb{N}}$$

of Cauchy sequences in X has a *pseudometric* (that is, a metric except potentially $\tilde{d}(x, y) = 0$ for $x \neq y$) given by $\tilde{d}((x_n), (y_n)) = \lim_{n\to\infty}(x_n - y_n)$. We then get a (genuine) metric space $\tilde{X}$ by taking the quotient

$$\tilde{X} = \frac{\mathcal{C}_X}{d((x_n), (y_n)) = 0}$$

with metric $d([(x_n)], [(y_n)]) = \tilde{d}((x_n), (y_n))$, which is independent of representative. This space is complete as any sequence of $\mathbf{x}_k = [(x_{n,k})_{n\in\mathbb{N}}]$ converges to the diagonal sequence $\mathbf{x} = [(x_{n,n})]$. The isometry i is given by sending $x \in X$ to the class of the constant sequence $[(x)_{n\in\mathbb{N}}]$.

Identifying X with the dense subset $i(X) \subseteq \overline{X}$, for any $j : X \to Y$ injective and uniformly continuous with Y complete, the map $\overline{j} : \overline{X} \to Y$ is necessarily given by $\overline{j}(\overline{x}) = \lim_{n\to\infty} j(x_n)$ for any sequence (x_n) in X with $i(x_n) \to \overline{x}$, and this is well-defined and uniformly continuous as j is uniformly continuous. The injectivity of $\overline{j}$ follows from that of j.

To compute the completion of a space, we have the following useful property.

Corollary 1. *Let X be a metric space. If $\tilde{X}$ is a complete metric space and $i : X \hookrightarrow \tilde{X}$ is an isometry with dense image, then $\tilde{X}$ is the completion of X.*

Since closed subspaces of complete metric spaces are themselves complete metric spaces, we have the following.

Corollary 2. *The following are equivalent.*

1. *X is complete;*
2. *$X = \overline{X}$;*
3. *X is closed in $\overline{X}$.*

Concretely, the metric space X is constructed by "adjoining" the limits of Cauchy sequences, which are demonstrated in the following.

Example 1.
1. *Let $X = \mathbb{Q}$ with its usual Euclidean metric $d(x, y) = |x - y|$. Then $\overline{X} = \mathbb{R}$;*
2. *If Y is a complete metric space and $X \subseteq Y$, then the completion of X is its closure $\overline{X}$;*
3. *(Formal power series) Let R be an integral domain (usually $R = \mathbb{R}$ or $\mathbb{C}$), and $R(x)$ be the associated ring of rational functions. For $c > 1$, define a metric d on $R(x)$ by defining $\|\cdot\| : R(x) \to \mathbb{R}_{\geq 0}$ by $\|0\| = 0$ and $\|(f/g)x^k\| = c^{-k}$, where $x \nmid f, g$, and $d(p, q) = \|p - q\|$. In this sense larger powers of x become "smaller", and the associated completion $R((x))$ is the set of formal rational functions:*

   $$R((x)) = \left\{ \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} \;\middle|\; a_n, b_n \in R \right\} = \left\{ \sum_{n=-N}^{\infty} a_n x^n \;\middle|\; N \in \mathbb{Z}, a_n \in R \right\}$$

   And in particular we can use this metric space to justify the usual formal series manipulations and identities without worrying about "genuine" convergence. Note that the metric on R as a subspace here is always the discrete metric, and not the usual metric we would associate to R (as in the cases of $\mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$);
4. *p-adics*

Formal power series

p-adics

m-adic completions in an arbitrary ring