# THE SPECTRAL THEOREM FOR GENERAL SELF-ADJOINT OPERATORS

STANLEY LI

## 1. BACKGROUND

For a compact, self-adjoint operator $T$ on a Hilbert space $H$, we have a decomposition into orthogonal projections $T = \sum_{n \in \mathbb{N}} \lambda_n P_{E_n}$ for mutually orthogonal finite-dimensional subspaces $E_n$, and in this essay we describe a generalisation to a much larger class of operators, and the theory behind its proof. The class of operators will be partial linear maps with domain a subspace of $H$, with an associated notion of being self-adjoint.

Unlike in the compact case, we cannot guarantee existence of eigenvalues, or that such an operator splits into countably many orthogonal projections up to scale, and so in the generalisation we replace this countable sum with an integral against a suitable notion of a projection-valued measure $E$ on a subset of $\mathbb{C}$, so that $T$ has a "diagonal decomposition" $\int \lambda dE$. The proof relies heavily on a correspondence between general self-adjoint operators and a certain class of unitary operators. We combine this with a way to convert measurable functions into operators related to $T$, alongside the spectral theorem for normal, bounded operators.

We will deal mostly with the complex case as we will have more tools to work with in this case, and briefly discuss how the real version of this theorem follows from this.

## 2. UNBOUNDED OPERATORS

We give some basic definitions and properties of unbounded operators. Most of the proofs in this section are elementary, and so will either be omitted or given a brief sketch, and the full proofs can be found in [1].

An **unbounded operator** $T$ on a Hilbert space $H$ is a linear map defined on a linear subspace $D(T)$ of $H$, and taking values in $H$. In this case, $D(T)$ is the **domain** of $T$, and we write $N(T) := \{x \in D(T) \mid Tx = 0\}$, $R(T) := \{Tx \mid x \in D(T)\}$ and $G(T) := \{(x, Tx) \in H \times H \mid x \in D(T)\}$ for the **null space**, **range** and **graph of** $T$ respectively. We say that $T$ **extends** $S$ if $D(S) \subseteq D(T)$ and $T|_{D(S)} = S$, and write $S \subset T$ for this. We say $T$ is **densely defined** if $\overline{D(T)} = H$.

Throughout the remaining sections, unless otherwise specified, $H$ is understood to be a *complex* Hilbert space, and operators understood to be unbounded and on $H$.

**Lemma 1.** *Let $T$ be a densely defined operator. Then*
  (1) *$D(T^*) = \{y \mid x \mapsto (\langle Tx, y \rangle) \in D(T)^*\} \subseteq H$ is a subspace; and*
  (2) *For each $y \in D(T^*)$ there is unique $T^*y \in H$ with $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in D(T)$, and $T^*$ is linear.*

*Proof.* For each $y \in D(T^*)$, by Hahn-Banach and density, the functional $x \mapsto \langle Tx, y \rangle$ extends uniquely to a functional on $H$, and the Riesz representation theorem gives unique $T^*y \in H$ with the desired property. $\square$

**Definition 1** (Adjoint). *The **adjoint** of a densely defined operator $T$ is the operator $T^*$ as defined above.*

It is also clear from the above definitions that if $S \subset T$, then $T^* \subset S^*$. We will be interested in the following class of operators, which will correspond to having closed spectrum.

**Definition 2.** *An operator $T$ is **closed** if $G(T) \subseteq H \times H$ is closed.*

This relates nicely with the adjoint of an operator:

**Proposition 1** (19.5 in [1])**.** *If $T$ be a densely defined operator, then $T^*$ is closed, and $N(T^*) = R(T)^\perp$.*

We now extend sums, compositions and inverses to unbounded operators, being careful of domains in doing so.

**Definition 3.** *Let $S$ and $T$ be operators. Then $S+T$ and $ST$ are the operators with domains $D(S+T) = D(S) \cap D(T)$ and $D(ST) = \{x \in D(T) \mid Tx \in D(S)\}$, and $(S+T)(x) := S(x) + T(x)$, $(ST)(x) = S(T(x))$. If $T$ is injective, its inverse $T^{-1}$ has $D(T^{-1}) = R(T)$, and sends $Tx$ to $x$.*

These operators have the following properties with regards to closedness and adjoints.

**Lemma 2** (19.8 in [1])**.** *Let $S$ and $T$ be operators. Then*
  (1) *if $S$ is closed and $T \in B(H)$, then $S+T$ and $S^{-1}$ are closed;*
  (2) *if $S+T$ is densely defined, then $S^* + T^* \subset (S+T)^*$, with equality if $T \in B(H)$.*

For spectra of unbounded operators, the resolvent will also require having image $H$, on top of being bijective.

**Definition 4.** *Let $H$ be a Hilbert space over $F = \mathbb{R}$ or $\mathbb{C}$, and $T$ be an operator on $H$. The **resolvent set** of $T$ is $\rho(T) := \{z \in F \mid N(zI - T) = 0, R(zI - T) = H\}$, and the **spectrum** of $T$ is $\sigma(T) := F \setminus \rho(T)$.*

Although we have given the general definition, we will almost always work over complex Hilbert spaces, except for a brief discussion of the spectral theorem over a real Hilbert space. The spectrum of a closed operator has the following properties.

**Proposition 2.** *Let $T$ be a closed operator. Then $(zI - T)^{-1} \in B(H)$ for every $z \in \rho(T)$; $\sigma(T)$ is closed.*

*Proof.* Each $zI - T$ is closed by above, and for $z \in \rho(T)$, $(zI - T)^{-1}$ has domain $H$ and closed graph, so is continuous by the closed graph theorem. If $z_0 \in \rho(T)$, for $|z - z_0| \, \|(z_0 I - T)^{-1}\| < 1$, $(z - z_0)(z_0 I - T)^{-1} + I \in B(H)$ is invertible, and so $zI - T = ((z - z_0)(z_0 I - T)^{-1} + I)(z_0 I - T)$ is injective and with range $H$. $\qquad\square$

We give the generalisation of a self-adjoint operator on $H$, which we will end up proving a spectral theorem for.

**Definition 5.** *A densely defined operator $T$ is **self-adjoint** if $T = T^*$.*

Note that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ only implies $T \subset T^*$, so being self-adjoint also requires $D(T^*) \subseteq D(T)$. By proposition 1, self-adjoint operators are closed. A slightly stronger statement holds for spectra of such operators:

**Lemma 3** (20.3 in [1])**.** *If $T$ is self-adjoint, $\sigma(T)$ is a non-empty subset of $\mathbb{R}$.*

In proving the spectral theorem for unbounded self-adjoint operators, we will use the following transform to reduce to the bounded, normal case. The above result shows that $\pm iI - T$ is injective with domain $D(T)$ and range $H$ when $T$ is self-adjoint, so the operator defined below is well-defined.

**Definition 6.** *Let $T$ be self-adjoint. The operator $U = \mathbf{U}(T) := (iI - T)(-iI - T)^{-1} \in B(H)$ is the **Cayley transform** of $T$.*

We are interested in the Cayley transform as it yields the following correspondence between unbounded self-adjoint operators and unitary operators on $H$, which will be essential to our proof of the spectral theorem.

**Proposition 3.** *There is a one-to-one correspondence between unbounded self-adjoint operators $T$ and unitary operators $U \in B(H)$ with $I - U$ injective, sending a self-adjoint operator $T$ to its Cayley transform $\mathbf{U} = \mathbf{U}(T) := (iI - T)(-iI - T)^{-1}$, and such a unitary operator $U$ to $\mathbf{T} = \mathbf{T}(U) := i(I + U)(I - U)^{-1}$, with $D(\mathbf{T}) = R(I - U)$.*

*Proof.* The Cayley transform of an operator $T$ is unitary as $\|(iI - T)(x)\|^2 = \|T(x)\|^2 + \|x\|^2 = \|(-iI - T)(x)\|^2$. For $T$ self-adjoint, $I - \mathbf{U}(T)$ is injective as for $h = (iI - T)^{-1}(x)$ we have $(iI - T)(h) = U(x) = x = (-iI - T)(h)$, so $h = 0$ and thus $x = 0$. To show $\mathbf{T} := \mathbf{T}(U)$ is self-adjoint, $\mathbf{T}$ is densely defined as $R(I - U)^{\perp} = N((I - U)^*) = N(I - U^{-1}) = N(I - U) = \{0\}$. We can show $\langle \mathbf{T}(x), y \rangle = \langle x, \mathbf{T}(y) \rangle$ by direct computation, and for $y \in D(\mathbf{T}^*)$, $z = \mathbf{T}^*y$, we can show that $U(z + iy) = z - iy$, and thus $y = (I - U)\left(\frac{1}{2i}(z + iy)\right) \in R(I - U) = D(\mathbf{T})$. Checking that $\mathbf{U}$ and $\mathbf{T}$ are mutual inverses is similar to showing $I - \mathbf{U}(T)$ is injective. $\qquad\square$

In lieu of this correspondence, we state a result on the spectrum of a unitary operator, which we will need later.

**Proposition 4** (12.26 in [2])**.** *Let $U \in B(H)$ be unitary. Then $\sigma(U) \subseteq S^1$.*

## 3. The functional calculus

A **$C^*$-algebra** $X$ is an associative $\mathbb{C}$-algebra with unit $e$ and a Banach space, such that $\|xy\| \leq \|x\| \, \|y\|$ for all $x$ and $y$, with a $\mathbb{C}$-antilinear map $* : X \to X$ such that $x^{**} = x$, $(xy)^* = y^*x^*$, and $\|x^*x\| = \|x^*\| \, \|x\|$. If $X$ and $Y$ are $C^*$-algebras, $\varphi \in B(X, Y)$ is a **$*$-homomorphism** if $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ for all $x$ and $y$. If $X \subseteq \mathbb{C}$ is compact, we say a map $\Psi : \mathcal{M}_\infty(X) \to B(H)$ is **weakly-continuous** if for any $x, y \in H$ and $(f_n)_{n \in \mathbb{N}}$ uniformly bounded and pointwise convergent to $f$, we have $\langle \Psi(f_n)x, y \rangle \to \langle \Psi(f)x, y \rangle$.

This notion of weak continuity is similar to weak properties we have seen, and when $H = \mathbb{C}(= B(H))$ this reduces to the usual notion of continuity in the weak topology. We look to define the functional calculus of an operator, which associates measurable functions to operators on $H$ in a way compatible with $T$. In the remaining sections, for $X \subseteq \mathbb{C}$, Borel$(X)$ denotes the Borel $\sigma$-algebra on $X$, $\mathcal{M}_\infty(X)$ denotes the set of bounded measurable functions $f : X \to \mathbb{C}$, while $\mathcal{M}(X)$ denotes the set of all measurable functions $f : X \to \mathbb{C}$.

For $X \subseteq \mathbb{C}$ compact, $\mathcal{M}_\infty(X)$ is a commutative $C^*$-algebra, by taking pointwise operations, the sup-norm, and $f^* = \overline{f}$ the pointwise conjugate. $B(H)$ is also a $C^*$-algebra, with the usual operator norm and composition, and $*$ sending an operator to its adjoint.

**Proposition 5** (18.3, 18.4 in [1])**.** *If $T \in B(H)$ is normal, then there is a unique weakly-continuous $*$-homomorphism $\Psi : \mathcal{M}_\infty(\sigma(T)) \to B(H)$ with $\Psi(z) = T$. Further,*

(1) $\Psi$ *is continuous with* $\|\Psi\| = 1$;
(2) $R(\Psi)$ *consists of normal operators which commute with one another, and* $\Psi(f)$ *is self-adjoint for* $f$ *real-valued;*
(3) $\Psi(\chi_M)$ *is orthogonal projection for each* $M \in \mathrm{Borel}(\sigma(T))$.

**Definition 7.** *The* **functional calculus** *of a normal operator* $T$ *is the operator in the proposition above.*

## 4. Spectral Measures

**Definition 8.** *Let* $F = \mathbb{R}$ *or* $\mathbb{C}$, $Y \subseteq F$, *and* $H$ *be a Hilbert space over* $F$. *A* **spectral measure** *on* $Y$ *is a map* $E : \mathrm{Borel}(Y) \to B(H)$ *such that*

(1) $E(M)$ *is an orthogonal projection for each* $M \in \mathrm{Borel}(Y)$; $E(\emptyset) = 0$ *and* $E(Y) = I$;
(2) $E(M_1 \cap M_2) = E(M_1)E(M_2)$ *for all* $M_i \in B(Y)$;
(3) $E(M_1 \sqcup M_2) = E(M_1) + E(M_2)$ *for all disjoint* $M_i \in B(Y)$; *and*
(4) *For every* $x \in H$, $E_{x,x} : M \mapsto \langle E(M)x, x \rangle$ *is a Radon measure in* $Y$.

Throughout the remaining results, $Y \subseteq \mathbb{C}$ is a fixed subset, and $E$ is a fixed spectral measure on $Y$. To define integrals of measurable functions with respect to spectral measures, we want to assign to each $f \in \mathcal{M}(Y)$ a densely defined operator on $H$ which corresponds to "integrating" the function with respect to the spectral measure. We first establish this for bounded measurable functions.

**Lemma 4.** *Let* $E$ *be a spectral measure on* $Y$. *Then for each* $f \in \mathcal{M}_\infty(Y)$ *there is a unique operator* $\int f dE \in B(H)$ *such that for each* $x \in H$:

(1) $\left\langle \int f dE x, x \right\rangle = \int f dE_{x,x}$; *and*
(2) $\left\| \int f dE x \right\|^2 = \int |f|^2 dE_{x,x}$

*Further, the map* $\psi_0 : f \mapsto \int f dE$ *is a* $*$-*homomorphism* $\psi_0 : \mathcal{M}_\infty(Y) \to B(H)$.

*Proof.* For a simple function $f = \sum_{j=1}^n \lambda_j \chi_{M_j}$, we have $\int f dE_{x,x} = \sum_{j=1}^n \lambda_j \langle E(M_j)x, x \rangle = \left\langle \sum_{j=1}^n \lambda_j E(M_j)x, x \right\rangle$. Since an operator $T$ on a complex Hilbert space is determined by the values of $\langle T(x), x \rangle$, $\sum_{j=1}^n \lambda_j E(M_j)$ is the unique operator satisfying (1), so the map $\psi_0$ on simple functions sending $\sum_{j=1}^n \lambda_j \chi_{M_j} \mapsto \sum_{j=1}^n \lambda_j E(M_j)$ is well-defined. We can check the $*$-homomorphism properties on simple functions, from which it follows that $\|\psi_0(f)x\|^2 = \langle \psi_0(f)^* \psi_0(f)x, x \rangle = \int |f|^2 dE_{x,x}$. $\psi_0$ is then continuous as $\|\psi_0(f)x\|^2 \leq \|f\|_\infty^2 E_{x,x}(Y) = \|f\|_\infty^2 \|x\|^2$, so extends continuously to a map on $\mathcal{M}_\infty(Y)$, which has the desired properties by taking limits with weak-continuity. $\square$

We extend this to all unbounded operators and measurable functions $f$ by taking sequences in $\mathcal{M}_\infty(Y)$ which are $L^2(Y, E_{x,x})$ and pointwise convergent, and for fixed $f \in \mathcal{M}(Y)$ we write $Y_{f,n} = \{ y \in Y \mid |f(y)| \leq n \}$.

**Proposition 6.** *Let* $E$ *be a spectral measure on* $Y$. *Then for each* $f \in \mathcal{M}(Y)$, *there is a unique densely defined operator* $\psi(f)$ *in* $H$ *with domain*

$$D(\psi(f)) = \left\{ x \in H \ \middle| \ \int_Y |f|^2 dE_{x,x} < \infty \right\}$$

*so that if* $x \in D(\psi(f))$ *and* $(f_n)_{n \in \mathbb{N}}$ *in* $\mathcal{M}_\infty(Y)$ *converges in* $L^2(Y, E_{x,x})$ *to* $f$, $\lim_{n \to \infty} \psi_0(f_n)x = \psi(f)x$, *and*

(1) $\langle \psi(f)x, x \rangle = \int f dE_{x,x}$; *and*
(2) $\left\| \int f dE x \right\|^2 = \int |f|^2 dE_{x,x}$.

*Proof.* For each $f \in \mathcal{M}(Y)$, $D(\psi(f))$ is a subspace as $E_{x+y,x+y}(M) \leq 2(E_{x,x}(M) + E_{y,y}(M))$ by Cauchy-Schwarz, and $E_{\lambda x, \lambda x}(M) = |\lambda|^2 E_{x,x}$; and dense as for $x \in H$, $R(E(Y_{f,n})) \subseteq D(\psi(f))$, and $\|x - E(Y_{f,n})x\|^2 = E_{x,x}(Y \setminus Y_{f,n}) \to 0$.

To show $\psi$ is well-defined as above, such a sequence exists as the sequence $f_n := f\chi_{Y_{f,n}}$ converges pointwise to $f$. Since $f_n, f$ are integrable with respect to $E_{x,x}$, the dominated convergence theorem gives that $f_n \to f$ in $L^2(Y, E_{x,x})$. We have $\|\psi_0(f_n)x - \psi_0(f_m)x\| = \|f_n - f_m\|_{L^2(Y, E_{x,x})}$, so has a limit for $f_n \to f$ in $L^2(Y, E_{x,x})$. This is independent of sequence as for $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ converging to $f$ in $L^2(Y, E_{x,x})$, we can take $(h_n)$ with $h_{2n-1} = f_n$ and $h_{2n} = g_n$.

We can thus compute $\|\psi(f)x\|^2 = \int |f|^2 dE_{x,x}$ and $\langle \psi(f)x, x \rangle = \int f dE_{x,x}$ by taking limits with $f_n \to f$, noting that $f \in L^1(Y, E_{x,x})$ since $E_{x,x}$ is a finite-valued measure. $\square$

As in lemma 4, we will write $\int f dE$ for $\psi(f)$ as above. This notion of an integral has the following properties:

**Proposition 7.** *Let* $E$ *be a spectral measure on* $Y$, $f \in \mathcal{M}(Y)$ *and* $\psi(f) = \int f dE$ *be as above. Then for* $f, g \in \mathcal{M}(Y)$,

(1) $\psi(f) + \psi(g) \subset \psi(f + g)$;
(2) $\psi(f)\psi(g) \subset \psi(fg)$, *and* $D(\psi(f)\psi(g)) = D(\psi(g)) \cap D(\psi(fg))$. *In particular,* $D(\psi(f)\psi(g)) = D(\psi(fg))$ *whenever* $D(\psi(g)) = H$; *and*
(3) $\psi(f)^* = \psi(\bar{f})$, *and* $\psi(f)\psi(f^*) = \psi(|f|^2) = \psi(f^*)\psi(f)$, *and hence* $\psi(f)$ *is closed.*

*Proof.* We use the fact that $L^2(Y, E_{x,x})$ is closed under addition for (1). Letting $g \in \mathcal{M}(Y)$, for $f \in \mathcal{M}_\infty(Y)$, we can show (2) by taking $(g_n)$ in $\mathcal{M}_\infty(Y)$ converging to $g$ in $L^2(Y, E_{x,x})$ and taking limits. In this case we also have $\int |f|^2 \, dE_{\psi(g)x,\psi(g)x} = \|\psi(f)\psi(g)x\|^2 = \|\psi(fg)x\|^2 = \int |fg|^2 \, dE_{x,x}$, and so the same equation holds for all $f \in \mathcal{M}(Y)$ by the monotone convergence theorem. This equation then shows that $x \in D(\psi(fg))$ is equivalent to $\psi(g)x \in D(\psi(f))$, so the claimed set equality holds. The first statement of (2) follows as $\|\psi(fg)x - \psi(f_n g)x\| = \int |f - f_n|^2 \, dE_{\psi(g)x,\psi(g)x}$ and thus for $f_n \to f$ in $L^2(Y, E_{x,x})$ we have $\psi(f)\psi(g)x = \lim_{n\to\infty} \psi(f_n g)x = \psi(fg)x$. For (3), $\psi(\overline{f}) \subset \psi(f)^*$ follows by again taking $\mathcal{M}_\infty(Y) \ni f_n \to f$, and the reverse inclusion follows as for $y \in D(\psi(f)^*)$ and $f_n := f\chi_{Y_{f,n}}$, we have $E(Y_n)\psi(f)^* = \psi\left(\overline{f_n}\right)$, and thus $\int_{Y_{f,n}} |f|^2 \, dE_{y,y} = \left\|\psi\left(\overline{f_n}\right)y\right\|^2 \leq \|\psi(f)^*y\|^2$, so $y \in D(\psi(\overline{f}))$. The remaining equality follows as $D(\psi(|f|^2)) \subseteq D(\psi(f))$ by Cauchy-Schwarz, and $\psi(f) = \psi(\overline{f})^*$ is closed. $\qquad\square$

It will be convenient to characterise the spectrum of $\int f dE$ in terms of $f$, and our characterisation will use the following notion:

**Definition 9.** *Let $E$ be a spectral measure on $Y$ and $f \in \mathcal{M}(Y)$. The **essential range** $\mathrm{ess.\,im}(f)$ (with respect to $E$) is such that $\mathrm{ess.\,im}_E(f)^c \subseteq \mathbb{C}$ is the largest open subset whose preimage under $f$ has measure zero.*

**Proposition 8.** *Let $E$ be a spectral measure. Then for any $f \in \mathcal{M}(Y)$,*

$$\sigma\left(\int f dE\right) = \mathrm{ess.\,im}_E(f)$$

*Proof.* First, $M := f^{-1}(\mathrm{ess.\,im}_E(f)) \in \mathrm{Borel}(Y)$ has $I = E(M) + E(\mathbb{C} \setminus M) = \psi(\chi_M)$ by definition of the essential range. For $\lambda \notin \mathrm{ess.\,im}_E(f)$, let $g : Y \to \mathbb{C}$ be the map $z \mapsto \frac{\chi_M(z)}{\lambda - f(z)}$; we have $g \cdot (\lambda - f) = (\lambda - f) \cdot g = \chi_M$, so by (2) of the previous proposition, as $\psi(g) \in B(H)$ and $\psi(\lambda - f) = \lambda I - \int f dE$, we have $\psi(g)\left(\lambda I - \int f dE\right) \subset \left(\lambda I - \int f dE\right)\psi(g) = \psi(\chi_M) = I$ and thus $\lambda \in \rho\left(\int f dE\right)$.

Letting $Y_\lambda = f^{-1}(\{\lambda\})$, any $x \in R(E(Y_\lambda))$ has $\psi(f)x = \lambda x$ as $f\chi_{Y_\lambda} = \lambda \chi_{Y_\lambda}$. Thus for $\lambda \in \mathrm{ess.\,im}_E(f)$, if $E(Y_\lambda) \neq 0$ we are done, and if $E(Y_\lambda) = 0$, letting $M_n = f^{-1}(B(\lambda, 1/n)) = \{y \in Y \mid |\lambda - f(y)| < 1/n\}$, we must have $E(M_n) \neq 0$. Choosing $x_n \in R(E(M_n))$ with $\|x_n\| = 1$, we get $\left\|\lambda x_n - \int f dE x_n\right\| = \|\psi(\lambda - f)\|\|x_n\| \leq \|\lambda - f\|_\infty \leq 1/n$, so $\left(\lambda I - \int f dE\right)x_n \to 0$. Now if $\lambda \in \rho\left(\int f dE\right)$, then since $\lambda I - \int f dE$ is closed by proposition 7, $\left(\lambda I - \int f dE\right)^{-1} \in B(H)$ by lemma 2, and we would have $x_n = \left(\lambda I - \int f dE\right)^{-1}\left(\lambda I - \int f dE\right)x_n \to 0$. Since $\|x_n\| = 1$ this is not possible, so $\lambda \in \sigma\left(\int f dE\right)$. $\qquad\square$

To use the correspondence in proposition 3, we use the following result to reduce to the Cayley transform.

**Lemma 5.** *Let $Y, Z \subseteq \mathbb{C}$, $\varphi : Y \to Z$ be a homeomorphism, and $F : \mathrm{Borel}(Z) \to B(H)$ be a spectral measure on $Z$. Then $E = F \circ \varphi : \mathrm{Borel}(Y) \to B(H)$ given by $E(M) = F(\varphi(M))$ is a spectral measure on $Y$, satisfying*

$$\int_Y f dE = \int_Z f \circ \varphi^{-1} dF$$

*Proof.* The spectral measure properties hold as $\varphi$ is a homeomorphism, so images under $\varphi$ respect unions and intersections. Since $\int_Y \chi_M dE = E(M) = F(\varphi(M)) = \int_Z \chi_{\varphi(M)} dF = \int_Z \chi_M \circ \varphi^{-1} dF$, the claimed equality holds for all simple functions, and hence all $f \in \mathcal{M}_\infty(Y)$ by density. To show this for all $f \in \mathcal{M}(Y)$ we must show that $D(\int_Y f dE) = D(\int_Z f \circ \varphi^{-1} dF)$, and for this we have $\int_Y |f|^2 \, dE_{x,x} = \int_Z \left|f \circ \varphi^{-1}\right|^2 \, dF_{x,x}$ for any $f \in \mathcal{M}_\infty(Y)$, and thus for any $f \in \mathcal{M}(Y)$ by the monotone convergence theorem. $\qquad\square$

## 5. The Spectral Theorem

We are almost ready to prove the spectral theorem. We sketch the theorem in the case of normal bounded operators, since it will be insightful to the underlying structure.

**Theorem 1.** *Let $T \in B(H)$ be normal, and $\Psi : \mathcal{M}_\infty(\sigma(T)) \to B(H)$ be its functional calculus. Then $E : \mathrm{Borel}(\sigma(T)) \to B(H)$ given by $E(M) = \Psi(\chi_M)$ is the unique spectral measure on $\sigma(T)$ with $T = \int z dE$.*

*Proof.* To show each $E_{x,x}$ defines a Radon measure, for each $x \in H$, define $\hat{\mu}_{x,x} \in \mathcal{M}_\infty(\sigma(T))^*$ by $f \mapsto \langle \Psi(f)x, x \rangle$. This is a functional as $\|\Psi\| = 1$, and positive as for $f \in C(\sigma(A))$ and $f \geq 0$, $\sqrt{f} \in C(\sigma(A))$ and $\Psi(\sqrt{f})$ is self-adjoint, so $\hat{\mu}_{x,x}(f) = \left\|\Psi(\sqrt{f})x\right\|^2 \geq 0$, and this extends to $\mathcal{M}_\infty(\sigma(T))$ by density. $\hat{\mu}_{x,x}$ thus defines a Radon measure by taking $\mu_{x,x}(M) = \hat{\mu}_{x,x}(\chi_M)$, and is weakly-continuous by Proposition 5, so $\langle \Psi(f)x, x \rangle = \int f d\mu_{x,x}$, and thus $E_{x,x}(M) = \langle \Psi(\chi_M)x, x \rangle = \int \chi_M d\mu_{x,x} = \mu_{x,x}(M)$, so $E_{x,x}$ is a Radon measure.

For uniqueness, if $F$ is a spectral measure with $T = \int z dF$, then $f \mapsto \int f dF$ is a $*$-homomorphism, so agrees with $\Psi$ on polynomials in $z$ and $\overline{z}$. This extends to $C(\sigma(T))$ by Stone-Weierstrass, as $\sigma(T)$ is compact, and this in turn extends to $\chi_O$ for $O \subseteq \sigma(T)$ open, by taking $(g_n)$ in $C(\sigma(T))$ bounded with $g_n \to \chi_O$ pointwise and applying the dominated convergence theorem. Thus the Radon measures $F_{x,x}$ coincide with $E_{x,x}$ for all $x \in H$, thus $F = E$. $\qquad\square$

We also give a lemma relating the eigenvalue decomposition of a bounded normal operator to its spectral measure.

**Lemma 6.** *Let $T \in B(H)$ be normal, and $E$ be the spectral measure of $T$. Then $E(\{\lambda\})$ is orthogonal projection onto the eigenspace $H_\lambda := N(\lambda I - T)$, and in particular $\lambda \in \sigma(T)$ is an eigenvalue if and only if $E(\{\lambda\}) \neq 0$.*

*Proof.* Let $\Psi$ be the functional calculus of $T$. It suffices to show $R(E(\{\lambda\})) = H_\lambda$ since $E(\{\lambda\})$ is an orthogonal projection. For $x \in R(E(\{\lambda\}))$, we have $Tx = TE(\{\lambda\})x = \Psi(z\chi_{\{\lambda\}}) = \psi(\lambda\chi_{\{\lambda\}}) = \lambda E(\{\lambda\})x = \lambda x$. Conversely, for $x \in H_\lambda$, letting $M_n = \{z \in \sigma(T) \mid |z - \lambda| \geq 1/n\}$, the functions $f_n : \sigma(T) \to \mathbb{C}$ given by $z \mapsto \frac{\chi_{M_n}(z)}{\lambda - z}$ are bounded and measurable, and so $E(M_n)x = \Psi(f_n)\Psi(\lambda - z)x = \Psi(f_n)(\lambda I - T)x = 0$. Since $E_{x,x}$ is a measure and $M_n \nearrow \sigma(T) \setminus \{\lambda\}$, we have $\|E(\sigma(T) \setminus \{\lambda\})x\|^2 = \lim_{n\to\infty} \langle E(M_n)x, x \rangle = 0$, so $x = E(\{\lambda\})x + E(\sigma(T) \setminus \{\lambda\})x = E(\{\lambda\})x \in R(E(\{\lambda\}))$. $\square$

We are now equipped to prove the spectral theorem, which we state explicitly.

**Theorem 2** (The spectral theorem for unbounded self-adjoint operators). *Let $T$ be self-adjoint. Then there is a unique spectral measure $E$ on $\sigma(T)$ with $T = \int z dE$. Explicitly, $E(M) = \Psi'\left(\chi_{\varphi(M)}\right)$, for $\varphi : \sigma(T) \to \sigma(\mathbf{U}(T)) \setminus \{1\}$ given by $z \mapsto \frac{i-z}{-i-z}$, and $\Psi'$ the functional calculus of the Cayley transform $\mathbf{U}(T)$ of $T$.*

*Proof.* First, $U := \mathbf{U}(T)$ admits a spectral measure $F$ by theorem 1. As $I - U$ is injective by proposition 3, $F(\{1\}) = 0$ by the previous lemma, so $F$ restricts to a spectral measure on $Z := \sigma(U) \setminus \{1\}$, with $\int_{\sigma(U)} g dF = \int_Z g dF$ for all $g \in \mathcal{M}(Z)$. The map $f : S^1 \setminus \{1\} \to \mathbb{R}$, $f(z) = i\frac{1+z}{1-z}$ is a homeomorphism and real-valued, so $B := \Psi'(f) = \int_Z f dF$ is self-adjoint. Applying $\Psi'$ to the equation $f(z)(1 - z) = i(1 + z)$ with proposition 7 we get $B(I - U) = i(I + U)$. Thus $A = i(I + U)(I - U)^{-1} \subset B$, and since $A$ and $B$ are self-adjoint we get $A = A^* \supset B^* = B$.

The same formula given for $\varphi$ defines a homeomorphism $\mathbb{R} \to S^1 \setminus \{1\}$ with inverse $f$, and for $\varphi$ to be well-defined we must show that $f(Z) = \sigma(T)$. We note $G = \mathbb{C} \setminus f(Z)$ is open and $F(f|_Z^{-1}(G)) = F(\emptyset) = 0$, while if $G_0 \supsetneq G$ is open, then $G_0 \cap Y \neq \emptyset$, so $f|_Z^{-1}(G_0) \subseteq Z$ is non-empty and open in $Z$, and so has $F(f|_Z^{-1}(G_0)) \neq 0$. We thus have $f(Z) = \text{ess.im}(f|_Z) = \sigma(T)$, so $f^{-1}$ has a well-defined restriction $\varphi : \sigma(T) \to Z = \sigma(\mathbf{U}(T)) \setminus \{1\}$. Defining $E = F \circ \varphi$ as above, we have that $T = \int_Z f dF = \int_Z \varphi^{-1} dF = \int_{\sigma(T)} z dF$.

For uniqueness, if $E$ is any spectral measure on $\sigma(T)$ with $T = \int z dE$, for $g = f|_{\sigma(T)}^{-1} : \sigma(T) \to \mathbb{C}$, we have $g(t)(-i - t) = i - t$, and applying $\psi$ yields $\left(\int g dE\right)(-iI - T) = iI - T$, so $\int g dE$ is the Cayley transform of $T$, and its spectral measure is uniquely determined, hence so is the spectral measure of $T$. $\square$

As in the spectral theorem for compact operators, we will reason about the eigenvalue decomposition of $T$.

**Definition 10.** *Let $T$ be a self-adjoint operator, and $E$ its spectral measure, and let $E_\lambda := E((-\infty, \lambda] \cap \sigma(T))$. Then $(E_\lambda)_{\lambda \in \mathbb{R}}$ is the **spectral resolution** of $E$.*

Since each $E_{x,x}$ is a finite measure on $\sigma(T)$, the spectral resolution determines the spectral measure $E$. Using the spectral resolution, we obtain the following description of the spectrum and eigenvalues of $T$.

**Theorem 3.** *Let $T$ be self-adjoint, and $E$ its spectral measure. Then for $\mu \in \mathbb{R}$,*

(1) *$\mu \in \text{EV}(T)$ if and only if $E_\mu \neq E_{\mu^-}$, where $E_{\mu^-}(x) = \lim_{\lambda \to \mu^-} E_\lambda(x)$;*
(2) *If $\mu \in \text{EV}(T)$, then $E(\{\mu\}) = E_\mu - E_{\mu^-}$ is orthogonal projection onto $N(\mu I - T)$; and*
(3) *$\mu \in \rho(T)$ if and only if $\lambda \mapsto E_\lambda$ is constant in some neighbourhood of $\mu$.*

*Proof.* Let $F$ be the spectral measure of $\mathbf{U}(T)$. By the proof of the spectral theorem we have $E(\{\mu\}) = F\left(\left\{\frac{i-\mu}{-i-\mu}\right\}\right)$, and $N(\mu I - T) = N(\frac{i-\mu}{-i-\mu}I - \mathbf{U}(T))$. We have $E_{\mu^-} = E((-\infty, \lambda) \cap \sigma(T))$ and thus $E_\mu - E_{\mu^-} = E(\{\lambda\})$, so (1) and (2) reduce to applying lemma 6 to $\mathbf{U}(T)$. For (3), as $\sigma(T)$ is closed, there is $(\alpha, \beta) \ni \mu$ disjoint from $\sigma(T)$, and $E_\lambda = E_\alpha$ for $\lambda \in (\alpha, \beta)$. Conversely, if $\lambda \mapsto E_\lambda$ is constant on $J = (\alpha, \beta) \ni \mu$, then $E(J) = 0$ for $g \in \mathcal{M}_\infty(\sigma(T))$ given by $t \mapsto \frac{\chi_{\mathbb{R} \setminus J}(t)}{\mu - t}$, we have $g(t)(\lambda - t) = \chi_{\mathbb{R} \setminus J}$, so by proposition 7 we get $\left(\int g dE\right)(\mu I - T) \subset (\mu I - T)\left(\int g dE\right) = E(\mathbb{R} \setminus J) = I$. $\square$

The fact that our results for complex Hilbert spaces still involve real spectra should suggest that an analogous result should hold for real Hilbert spaces. We have also dealt exclusively with real Hilbert spaces, so such a result would tie in well with the results we have already seen.

**Theorem 4.** *Let $H$ be a real Hilbert space, and $T$ be a self-adjoint operator on $H$. Then there is a unique spectral measure $E$ with $T = \int \lambda dE$.*

*Proof.* Let $H_\mathbb{C}$ and $T_\mathbb{C}$ be the complexifications of $H$ and $T$ respectively, with $D(T_\mathbb{C}) = D(T) + iD(T)$. Then $T_\mathbb{C}$ is also self-adjoint, and $\sigma(A) = \sigma(A_\mathbb{C})$ as $\sigma(A_\mathbb{C}) \subseteq \mathbb{R}$. Let $U : H_\mathbb{C} \to H_\mathbb{C}$ be the conjugation map $x + iy \mapsto x - iy$. Then $T_\mathbb{C} = UT_\mathbb{C}U^{-1}$, aetting $E$ be the spectral measure of $T_\mathbb{C}$, we get another spectral measure $F : B(\sigma(A_\mathbb{C})) \to L(H_\mathbb{C})$ with $F(M) = UE(M)U^{-1}$. This gives $\int g dF = U\left(\int g dE\right)U^{-1}$ for real-valued $g \in \mathcal{M}(\sigma(T_\mathbb{C}))$, and in particular we obtain $\int \lambda dF = UT_\mathbb{C}U^{-1} = T_\mathbb{C}$, so by uniqueness we have $F = E$, and so $\xi + i\eta := E(M)(x + i0) = UE(M)U^{-1}(x + i0) = \xi - i\eta$, so $E(M)(x + i0) = \xi + i0$ for any $x \in H$, and so $E_\mathbb{R}(M) = E(M)|_H$ is orthogonal projection in $H$, so $E_\mathbb{R}$ is a spectral measure on $\sigma(A) = \sigma(A_\mathbb{C})$ with values in $L(H)$. Since $E_\mathbb{R}$ is determined by the spectral measure of $A_\mathbb{C}$, $E_\mathbb{R}$ is the unique spectral measure on $\sigma(A)$. $\square$

The eigenvalue decomposition in theorem 3 hence also holds for such operators on real Hilbert spaces.

## 6. References

[1] R. Meise and D. Vogt, (1997), *Introduction to Functional Analysis*, Oxford University Press, Oxford, 198 - 230.

[2] W. Rudin, (1991), *Functional Analysis*, McGraw-Hill, New Jersey.