Global function fields, zeta functions for curves and the analogies to number fields

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Abstract

We give an exposition of global function fields with an emphasis on the analogies to number fields. We introduce the underlying adelic theory and adelic zeta functions, with the goal of describing several non-trivial properties of the analogue of a Dedekind zeta function for a curve. While doing this, we highlight the extent and limitation of the analogy between number fields and function fields.

Chapter 1

Introduction and motivation

In the grand scheme of mathematics, we often find that two concepts or objects which are seemingly unrelated in definition are in fact deeply connected. When such a connection arises, it is also often the case that the arguments made in one case can be translated naturally into one for the other. One of these surprising, deep and rich connections is between the rational numbers $\mathbb{Q}$ and the rational function field $\mathbb{F}_q(t)$ over a finite field $\mathbb{F}_q$. This connection extends to finite field extensions $K/\mathbb{Q}$ and $K/\mathbb{F}_q(t)$, which we refer to as number fields and global function fields respectively. Within this analogy, the global function field case generally has more tools to work with and is easier to reason about, but the number field case is the more important of the two.

We focus mainly on global function fields, and this analogy with number fields. One of the nice tools for a global function field K is that we can view these as rational functions over an algebraic curve (which consists of zero loci of polynomials defined over some affine space $\mathbb{F}_q^n$) over $\mathbb{F}_q$. On its own, this may seem like a set of scattered points, but a better way to view this is as a restriction of an algebraic curve on $\overline{\mathbb{F}}_q^n$ (or equivalently over $\overline{\mathbb{F}}_q$), which we can think of as a connected space where our usual continuous intuition applies.

At the heart of this report, we look the analogy between the associated zeta functions of a number field and a global function field. These are meromorphic (analytic except for finitely many poles) functions defined as generating functions corresponding to primes. We will often identify the global function field K with a nice curve $\mathcal{C}$ for which it is the field of rational functions. The essential object of study will be the Hasse-Weil zeta function, defined $Z_{\mathcal{C}}(q^{-s})$ for the generating function

$$Z_{\mathcal{C}}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_{\mathcal{C}}(n)}{n} t^n\right)$$

where the integer $N_{\mathcal{C}}(n)$ is the number of points of $\mathcal{C}$ with coordinates in $\mathbb{F}_{q^n}$. We will prove several highly non-trivial properties of these zeta functions, including an analogue of the Riemann hypothesis.

We develop the theory on global function fields from the perspective of norms, and use much more general notions than those outlined above in order to establish these properties and draw analogies between the number field and global function field cases.

Chapter 2

Background

In this section we briefly discuss some field theory and the valuations of global function fields. We give a recap of various relevant definitions, a complete description of the valuations on global function fields, and various other important results we will use.

To make explicit what is meant by a global function field, we have the following:

Definition 1. *Let L/K be a field extension. We say that $S \subseteq L$ is* algebraically independent *over K if every $s \in S$ is transcendental over $K(S \setminus \{s\})$. The* transcendence degree *of L/K is the largest cardinality of an algebraically independent subset of L over K.*

A global function field is then defined as follows.

Definition 2. *Let $\mathbb{F}_q$ be the finite field on q elements. A* global function field *is an extension $K/\mathbb{F}_q$ of transcendence degree 1.*

We can thus view every global function field as a finite extension of the function field $\mathbb{F}_q(x)$. As $K/\mathbb{F}_q(t)$ can be written as $K = \mathbb{F}_q(t)[x_1, \ldots, x_j]/(m_1, \ldots, m_k)$ in general, we can view the field K as the rational functions acting on the zero locus $\mathcal{C}$ of $m_1, \ldots, m_k$ in $\overline{\mathbb{F}}_q^{j+1}$. In general we tend to think of the curves $\mathcal{C}$ as including a "point at infinity", and we refer to such curves as projective. For simplicity, we also assume that the associated projective curve $\mathcal{C}$ is non-singular, which can intuitively be thought of as saying that the curve has a well-defined tangent at each point. This usually corresponds to the curve having no self-overlaps or cusps.

For simplicity, we let K denote a global function field with field of constants $\mathbb{F}_q$. That is, we take $\mathbb{F}_q$ is algebraically closed in K, i.e. $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$, or equivalently that every $t \in K \setminus \mathbb{F}_q$ is transcendental over $\mathbb{F}_q$. The extension $K/\mathbb{F}_q(t)$ depends on the choice of t, but by the following result, we can assume without loss of generality that this extension is separable.

Lemma 1. *Let $K/\mathbb{F}_q$ be an extension of transcendence degree 1. Then there is $w \in K$ such that $K/\mathbb{F}_q(w)$ is a finite, separable extension.*

We also have the following lemma, which says that the extension $K/\mathbb{F}_q(y)$ remains finite regardless of the choice of $y \in K \setminus \mathbb{F}_q$.

Lemma 2. *Let K be an extension of $\mathbb{F}_q$ of transcendence degree 1. Then for any $x \in K \setminus \mathbb{F}_q$, $K/\mathbb{F}_q(x)$ is a finite extension.*

2.1 Valuations

For convenience, we first give a definition for the contexts we will usually be working over.

Definition 3. *A field is a* global field *if it is either a number field or a global function field.*

This definition corresponds exactly to finite extensions of either $\mathbb{Q}$ or $\mathbb{F}_q(x)$. As we will see, these two notions are highly analogous, and so it will make sense to consider both simultaneously when making definitions. To each function field, we define the ring of regular functions to be the integral closure of $\mathbb{F}_q[x]$, and to each number field, we set the ring of integers to be the integral closure of $\mathbb{Z}$. In both cases, we denote this ring by $\mathcal{O}_K$. This is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$ in the number field case, and a free $\mathbb{F}_q[t]$-module of rank $[K : \mathbb{F}_q(t)]$ when $K/\mathbb{F}_q(t)$ is separable.

To study the norms $|\cdot|$ on a field, we often study the additive maps $v(x) = -\log|x|$ (for a suitable base) instead. These maps and their corresponding properties are defined below.

Definition 4. *Let K be a field. A* valuation v on K *is a map $v : K \to \mathbb{R} \cup \{\infty\}$ such that*

1. $v(xy) = v(x) + v(y)$
2. *There is $c \geq 0$ such that $v(x+y) \geq -c + \min(v(x), v(y))$*
3. *$v(x) = \infty$ if and only if $x = 0$.*

We say that a valuation is non-archimedean *if $c = 0$ in (2); and discrete if the image of $v : K^* \to \mathbb{R}$ is discrete.*

A non-archimedean valuation corresponds to the ultrametric triangle inequality $|x+y| \leq \max(|x|, |y|)$ for the corresponding norm. For such a norm, we have the property that $v(x+y) = \min(v(x), v(y))$ when $v(x) \neq v(y)$. This is as if $v(x) < v(y)$, $v(x+y) \geq v(x) \geq \min(v(x+y), v(y))$, and so as $v(x) < v(y)$, we must have $v(x+y) \leq v(x)$ and thus $v(x+y) = v(x)$. We can view this as saying that every triangle is isosceles.

When we refer to valuations, we consider valuations which are scalar multiples of one another to be the same. This is as they correspond to equivalent norms, and thus also equivalent topologies.

To each non-trivial valuation v on a global field K, we may consider the completion K_v of the field with respect to $|\cdot|_v$. We refer to this as the local field at v, and for valuations on $\mathbb{F}_q(t)$ this is explicitly a ring of formal Laurent series in every case. A general local field is defined as follows.

Definition 5. *A* local field *is a locally compact topological field with respect to a non-discrete topology.*

The local fields corresponding to valuations are each local fields in this sense as every bounded set has compact closure. We may refer to the local field K_v as a local factor or term corresponding to v.

For non-archimedean valuations, we also have the following objects of importance:

1. The closed unit disk $\mathcal{O}_v = \{x \in K_v \mid |x|_v \leq 1\}$ is the maximal closed, bounded subring of K_v, which will be referred to as the regular functions (for $K/\mathbb{F}_q(x)$) or the ring of integers at v (for $K/\mathbb{Q}$).
2. The open unit disk $\mathfrak{p}_v = \{x \in K_v \mid |x|_v < 1\}$ is the unique prime ideal in $\mathcal{O}_v$, and thus $\mathcal{O}_v$ is a local ring.

3. The residue class field $K(v) = \mathcal{O}_v/\mathfrak{p}_v$. This is a finite field when K is a global field, and we denote its cardinality by q_v.

We will soon describe the valuations on global fields, from which it will follow that $\mathcal{O}_v$ is always compact. Further, when v is discrete, $\mathfrak{p}_v$ is principal and generated by an element of minimal positive valuation. In these cases we write π_v for a generator of $\mathfrak{p}_v$, and call such a generator a *uniformizer* for v. The rings can thus be viewed as formal power or Laurent series in the respective uniformizers. For a valuation on a global function field, we set its degree to be $\deg(v) = [K(v) : \mathbb{F}_q]$, so that $q_v = q^{\deg(v)}$.

By potentially rescaling, we assume where relevant that we are working with the valuation which is "normalised", that is, $v(\pi_v) = 1$. For extensions of local fields, we have the following:

Proposition 1. *If L/K is an algebraic extension and K is a local field with respect to a valuation v, then there is a unique valuation w on L with $w|_K = v$, given by $w(\alpha) = v(N_{K(\alpha)/K}(\alpha))/[K(\alpha) : K]$ for any $\alpha \in K$.*

This can be viewed as a "locality" of local fields, in that the extension of a valuation is determined entirely in the local setting. We denote the extension of v on K to w on L by w/v.

We also define the following objects and notions for a finite local extension.

Definition 6. *Let L/K be an extension of global fields, and w/v be an extension of valuations.*

1. *The* local different *is the different of local fields $\mathfrak{D}_{w/v} = \mathfrak{D}_{L_w/K_v}$. If π_w is a uniformizer for L_w, we define the differential exponent $d(w/v)$ to be such that $\mathfrak{D}_{w/v} = \pi_w^{d(w/v)}\mathcal{O}_w$. We say the different is trivial if $\mathfrak{D}_{w/v} = \mathcal{O}_w$.*
2. *The local ramification index is the positive integer $e(w/v)$ with $\pi_v\mathcal{O}_w = \pi_w^{e(w/v)}\mathcal{O}_w$. We say that w is unramified over v if $e(w/v) = 1$.*
3. *The residue class degree is the degree $f(w/v) = [L(w) : K(v)]$.*

The residue class degree is well-defined as $\mathcal{O}_v \cap \mathfrak{p}_w = \mathfrak{p}_v$, and so $L(w)$ is indeed an extension of $K(v)$.

As in the number field case, the different is related to the ramification of the corresponding extension in the following way.

Theorem 1. *Let L/K be an extension, and w/v be an associated extension of valuations. Then w is unramified over v if and only if the different $\mathfrak{D}_{w/v}$ is trivial.*

Thus we view the different as a measure of the ramification in the associated extension. The different of an extension of global fields is related to the local differents by the following result.

Theorem 2. *If L/K is an extension of global fields, then the different*

$$\mathfrak{D}_{L/K} = \{x \in L \mid x \in \mathfrak{D}_{w/v} \text{ for all } w \mid v\} = \bigcap_{w|v} \mathfrak{D}_{w/v}$$

generates the ideal $\mathfrak{D}_{w/v} \subseteq \mathcal{O}_w \subseteq L_w$ at each local factor w.

For a finite extension of global fields, it will be useful to know how large the local field extensions are. The following result describes exactly how "large" this local extension is.

Lemma 3. *Let L/K be a finite extension with associated valuations w/v. Then*

$$[L_w : K_v] = e(w/v)f(w/v)$$

and $\mathcal{O}_w$ is a free $\mathcal{O}_v$-module of rank $e(w/v)f(w/v)$.

It is also true as a separate result for global fields that $\sum_{w|v} e(w/v)f(w/v) = [L : K]$, so local extensions are indeed smaller than global ones.

To describe the valuations on a general global field, we first describe the valuations in the basic cases of $K = \mathbb{Q}, \mathbb{F}_q(x)$. A well-known theorem of Ostrowski says that, up to a non-zero scalar, the valuations on $\mathbb{Q}$ fall into 3 categories:

1. The trivial valuation $v|_{\mathbb{Q}^*} = 0$,
2. The p-adic valuations $v_p(ap^k/b) = k$ for $p \in \mathbb{Z}$ prime and $(a, p) = (b, p) = 1$, or
3. The standard "infinite" (or archimedean) valuation $v_\infty(x) = -\log_e |x|$.

We have the following analogue for $\mathbb{F}_q(x)$:

Theorem 3. *Let v be a valuation of $\mathbb{F}_q(x)$. Then, up to a scalar multiple, v is exactly one of:*

1. *The trivial valuation $v|_{\mathbb{F}_q(x)^*} = 0$,*
2. *The P-adic valuation $v_P(AP^k/B) = k$ for $P \in \mathbb{F}_q[x]$ irreducible and $(A, P) = (B, P) = 1$, or*
3. *The infinite valuation $v_\infty(P/Q) = \deg(Q) - \deg(P)$.*

We refer to the P-adic valuations and their extensions as the finite primes (or finite valuations), and the infinite valuation and its extensions as the infinite primes. From these classifications, we see that, aside from the archimedean valuation, all valuations on $\mathbb{Q}$ and $\mathbb{F}_q(x)$ are discrete and non-archimedean.

In terms of the norms induced, the non-archimedean valuations are exponentiated with base corresponding to the size of the residue field, with corresponding norm $|x|_v = |K(v)|^{-v(x)}$, while the usual archimedean norm has $|x| = e^{-v_\infty(x)}$.

While the analogy between the trivial valuations and the p (or P)-adic valuations on the two fields is somewhat clear, the infinite valuations is not immediately clear. Indeed, they are analogous in that they are both induced by the ideal norms: we have $|n| = |\mathbb{Z}/n\mathbb{Z}|$ for $n \in \mathbb{Z}$ and $|f| = |\mathbb{F}_q[t]/f\mathbb{F}_q[t]|$ for $f \in \mathbb{F}_q[t]$.

Alongside having the perspective that v_∞ for a function field corresponds to the ideal norm, the infinite valuation can also be viewed as a $1/t$-adic valuation. This is as $v_\infty(f) = -\deg(f)$ corresponds exactly to writing $f(t) = (1/t)^n p(1/t)$ for $p \in \mathbb{F}_q[t]$, and so unlike in the number field case, v_∞ is very similar to the P-adic valuations in the global function field case. The difference in the resulting infinite norms can thus be viewed as coming from this, as the infinite norm in the number field case is far from $\mathfrak{p}$-adic.

As mentioned previously, the completions can be viewed as Laurent series. Namely, we can think of $\mathbb{F}_q(t)_\infty$ as the Laurent series $\mathbb{F}_q((1/t))$, and $\mathbb{F}_q(t)_P$ as Laurent series in P with coefficients as polynomials in t of degree less than $\deg(P)$. As an additive group, $\mathbb{F}_q(t)_P$ can be seen as $\mathfrak{q}(P)((P))$.

We can also view valuations geometrically in the following way. Valuations of degree 1 have residue field $K(v) = \mathbb{F}_q$, and so correspond to points defined on $\mathcal{C}$ defined over $\mathbb{F}_q$. A general valuation of degree $\deg(v) \geq 1$ correspond points defined over $K(v)$, and splits as an orbit of $\deg(v)$ points on $\mathcal{C}(K(v))$ under the Frobenius map ϕ_q. This is clearest for a polynomial $P \in \mathbb{F}_q(t)$, corresponding to the projective line $\mathbb{P}^1(\mathbb{F}_q)$, which can be thought of as a line including the point at infinity, and is explicitly given by $\mathbb{P}^1(\mathbb{F}_q) = (\mathbb{F}_q^2 \setminus \{0\})/\sim$ where $a \sim \lambda a$ for $\lambda \in \mathbb{F}_q^*$.

For a general global field, we have the following lemma.

Lemma 4. *If $L = K(\alpha)/K$ is a separable extension where α has minimal polynomial $m \in K[t]$, then there is a bijection*

$$\left\{\begin{matrix} \textit{irreducible factors} \\ \textit{of } m \textit{ over } K_v \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{valuations of} \\ L \textit{ extending } v \end{matrix}\right\}$$

This bijection sends each irreducible factor m_i to the valuation induced by $K[t]/(m) \hookrightarrow K_v[t]/(m_i)$. It follows from this that the finite valuations correspond to the prime ideals in $\mathcal{O}_K$: given a prime ideal $\mathfrak{p}$, we can define $v_{\mathfrak{p}}(a)$ for $a \neq 0$ by taking the order of $\mathfrak{p}$ in the prime ideal factorisation of the principal ideal (a). Conversely, given a finite valuation v of K, $\pi_v \mathcal{O}_v \cap \mathcal{O}_K$ is a prime ideal in $\mathcal{O}_K$, and $v = v_{\pi_v \mathcal{O}_v \cap \mathcal{O}_K}$. From this we see that the valuations on both number fields and global function fields are attained in the same way, with the infinite valuations coming from embeddings of K into $\mathbb{C}$ and $\overline{\mathbb{F}_q(t)_\infty}$ respectively.

Similar to as in just the number field case, there are only finitely many primes which ramify in a separable extension of global function fields, and each non-zero element in a global function field vanishes at all but finitely many valuations. This former follows by a similar line of reasoning to the number field case, where we write $K = \mathbb{F}_q(t)[x]/(m(x))$ (for $m \in \mathbb{F}_q[t][x]$ monic in x). We can apply a variant of the Kummer-Dedekind theorem for global function fields, and consider the irreducible factors in $\mathbb{F}_q(t)$ dividing the discriminant of m. The results required to make this explicit are given in Appendix 1. The latter statement follows from the following proposition, which describes which valuations vanish on each $x \in K^*$.

Proposition 2. *Let $K/\mathbb{F}_q$ be a global function field and $x \in K^*$. If w is a valuation on K with $w(x) \neq 0$, then $w|_{\mathbb{F}_q(x)}$ is either v_x or v_∞*

The condition that $x \notin \mathbb{F}_q$ can be viewed exactly as saying that x is non-constant. The valuations lying above v_x correspond to the zeroes of x in K, and the valuations lying above v_∞ correspond to the poles of x.

Once we have developed the theory locally, we extend to the global case by looking at a subset of the products $\prod_v K_v$ (taken over all valuations v), and a corresponding norm on these products by taking the norms pointwise. The following lemma states that each element in K^* has norm 1 under this interpretation.

Lemma 5. *Suppose that $K/\mathbb{F}_q$ is a global function field and $y \in K$ be non-constant. If v_y and v_∞ are the valuations in $\mathbb{F}_q(y)$ as above, then*

$$\sum_{w/v_y} e(w/v_y) f(w/v_y) = \sum_{w/v_\infty} e(w/v_\infty) f(w/v_\infty) = [K : \mathbb{F}_q(y)]$$

From the perspective of zeroes and poles, when y is expressed as a rational function, each w/v_y corresponds to the zeroes of an irreducible polynomial factor of y, with $f(w/v_y)$ corresponding to the degree and $e(w/v_y)$ the multiplicity of this factor. A similar correspondence holds for the poles of y.

Chapter 3

Local Fourier theory

We assume from now that K is a fixed global function field with associated non-singular projective curve $\mathcal{C}$, and $t \in K$ is such that $K/\mathbb{F}_q(t)$ is finite and separable, unless otherwise stated.

In the following two sections, we describe Fourier theory for function fields. The goal of this is to use adelic zeta functions (to be defined) to deduce various important properties of the usual Weil zeta functions of curves. We develop the Fourier theory locally first.

The Fourier transform we seek to define will be attained in a similar way to the usual $\mathbb{C}$ Fourier transform, by integrating our original function against functions analogous to $\exp(2\pi i\xi x)$. We first describe these functions explicitly, along with the notion of integration that we use.

3.1 Additive character theory and measure

In this section we focus on the character group of the additive group $(K_v, +)$ (which we will just denote K_v in this section), and an additive Haar measure. For the purposes of having a Fourier inversion formula and determining a suitable measure, we seek to find an identification of K_v with its character group.

To describe this identification, we first fix a topology on the character group of K_v. The standard topology for this is the compact-open topology, which has a subbasis given by the sets $V(K_v, \mathcal{U}) = \left\{f \in C(K_v, S^1) \mid f(K) \subseteq \mathcal{U}\right\}$ for $K \subseteq K_v$ compact and $\mathcal{U} \subseteq S^1$ open. As in Tate's thesis, we have the following lemma.

Lemma 6. *If $\chi_v : K_v \to S^1$ is a non-trivial character, then $y \leftrightarrow (x \mapsto \chi_v(yx))$ is a topological and algebraic isomorphism between K_v and its character group $\hat{K}_v$.*

Thus to fix an identification of K_v with its character group $\hat{K}_v$, it suffices to find a single non-trivial character of K_v. We do this first for $\mathfrak{q}$, and then extend to a general global function field K. The proof of the above lemma only uses the fact that K_v is a ring with a locally compact abelian additive group, and so we can also extend such an identification to the global setting.

3.2 Additive characters of $\mathfrak{q}$

As described previously, the non-trivial valuations on $\mathfrak{q}$ are exactly the P-adic valuations and the infinite valuation corresponding to $1/t$. We describe non-trivial additive characters for each of these completions. We note that there is a natural identification of $\mathbb{Z}/p\mathbb{Z}$ with the field $\mathbb{F}_p$ on p-elements, and so field trace $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p$ for $a \in \mathbb{F}_q$ gives a well-defined element in $\mathbb{Q}/\mathbb{Z}$. Following van Frankenhuysen, we define these characters on monomials as follows.

Case 1. v infinite, $\mathfrak{q}_v = \mathbb{F}_q((1/t))$ the formal Laurent series in $1/t$.

$$\chi_\infty(aT^n) = \begin{cases} \exp\left(-\frac{2\pi i}{p}\,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)\right) & n = -1 \\ 1 & \text{Otherwise} \end{cases}$$

Case 2. v P-adic, $\mathfrak{q}_v = [\mathfrak{q}(P)]((P))$ the formal Laurent series in P with coefficients in the residue field $\mathfrak{q}(P)$.

$$\chi_P(aT^nP^k) = \begin{cases} \exp\left(\frac{2\pi i}{p}\,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)\right) & n = \deg(P) - 1, k = -1 \\ 1 & \text{Otherwise} \end{cases}$$

To complete this identification, we check that these do indeed give non-trivial additive characters.

Lemma 7. *The maps χ_v for $v = v_P, v_\infty$ are non-trivial, continuous additive maps of K_v into S^1.*

Proof. It suffices to show this for the map $K_v \to \mathbb{R}/\mathbb{Z}$ given by $\psi_v = \frac{1}{2\pi i}\log \circ \chi_v$, as $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups by $x \mapsto \exp(2\pi i x)$. These maps are additive as the coefficients of the sum are the sum of the coefficients for each monomial, and the field trace $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ is additive.

To show continuity, by additivity we have $\psi_v(x) - \psi_v(y) = \psi_v(x-y)$, and so continuity around any $x \in K_v$ reduces to continuity around 0. It thus suffices to show continuity around 0 for each valuation. For finite $v = v_P$, when $|x|_P < q$, we have $v_P(x) \geq 0$, and so $x \in \mathcal{O}_P = [\mathfrak{q}(P)][[P]]$. Thus the coefficient of $T^{\deg(P)-1}P^{-1}$ is zero, and $|\psi_P(x)|_P = 0$. Similarly, for $v = v_\infty$, when $|x|_\infty < q^{-1}$ we have $v_\infty(x) \geq 2$, and so $x \in T^{-2}\mathcal{O}_\infty = T^{-2}\mathbb{F}_q[[T^{-1}]]$. Thus the coefficient of T^{-1} is zero, and $|\psi_\infty(x)|_\infty = 0$. □

Unlike Tate's thesis, the finite characters do not have the property that $\psi_P(x) - x$ is P-adic, nor do they have $\psi_\infty(x) = -x \mod 1$. Instead, these characters can be viewed as closely tied to the residues of the respective formal power series. This connection is covered in more detail in van Frankenhuysen's exposition [van Frankenhuysen].

3.3 Additive characters of K

We can extend our above result to any K_v using the field trace, giving $\chi_v := \chi_P \circ \mathrm{Tr}_{K_v/\mathfrak{q}_P}$ as a non-trivial character of K_v. By Lemma 6, this proves the following result.

Theorem 4. *The local field K_v is naturally its own character group $\hat{K}_v$ under the identification $y \leftrightarrow (x \mapsto \chi_v(yx))$.*

Before defining the local Fourier transform, we discuss which characters act trivially on the functions $\mathcal{O}_v$ regular at v. In our proof of continuity we have used the fact that χ_P is trivial (i.e. always 1) on $\mathcal{O}_P$ for P finite, and trivial on $T^{-2}\mathcal{O}_\infty$ for the infinite valuation. In a similar vein, we have the following lemma.

Lemma 8. *Let v be a valuation of K.*

- *If v is finite, then $x \mapsto \chi_v(yx)$ is trivial on $\mathcal{O}_v$ if and only if $y \in \mathfrak{D}_{v/P}^{-1}$.*
- *If v is infinite (i.e. $v(T) < 0$), then $x \mapsto \chi_v(yx)$ is trivial on $\mathcal{O}_v$ if and only if $y \in T^{-2}\mathfrak{D}_{v/\infty}^{-1}$.*

Proof. We give the proof in the infinite case, and describe how to adapt the proof to the finite case. Suppose first that $y \in T^{-2}\mathfrak{D}_{v/\infty}^{-1}$. Then for any $x \in \mathcal{O}_v$, we have $T^2 \operatorname{Tr}_{v/\infty}(yx) = \operatorname{Tr}_{v/\infty}(T^2yx) \in \mathcal{O}_\infty$. Thus $\operatorname{Tr}_{v/\infty}(yx) \in T^{-2}\mathcal{O}_\infty$, and so $\chi_v(yx) = 1$ for all $x \in \mathcal{O}_v$.

Conversely, if $y \notin T^{-2}\mathfrak{D}_{v/\infty}^{-1}$, then there is $x \in \mathcal{O}_v$ with $\operatorname{Tr}_{v/\infty}(T^2yx) \notin \mathcal{O}_\infty$. Let

$$\operatorname{Tr}_{v/\infty}(yx) = \sum_{n=N}^{\infty} a_n T^{-n}$$

and suppose that $a_N \neq 0$. Then as $T^2 \operatorname{Tr}_{v/\infty}(yx) \notin \mathcal{O}_\infty$, the highest power (T^{-N+2}) of $T^2 \operatorname{Tr}_{v/\infty}(yx)$ with non-zero coefficient must be more than 0. Thus $-N + 2 > 0$, or equivalently $N \leq 1$. Hence $T^{-N+1}x \in \mathcal{O}_v$ as $v(T^{-N+1}x) = (1 - N)v(T) + v(x) \geq 0$. Choosing $b \in \mathbb{F}_q$ so that $\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(ba_N) \neq 0$, for $bT^{-N+1}x \in \mathcal{O}_v$ we have

$$\chi_v(y(bT^{-N+1}x)) = \exp\left(-\frac{2\pi i}{p} \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(ba_N)\right) \neq 1$$

Hence the character associated to y is not trivial on $\mathcal{O}_v$.

To adapt this proof to the finite case, we replace $T^{-2}\mathfrak{D}_{v/\infty}^{-1}$ with $\mathfrak{D}_{v/P}$, and multiply instead by a monomial T^nP^k which makes the coefficient of $T^{\deg(P)-1}P^{-1}$ have non-zero image under $\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. $\square$

For each valuation v, this shows that the set of elements whose associated character acts trivially on $\mathcal{O}_v$ is a fractional ideal. This also motivates the following definition.

Definition 7. *Let K be a global function field, and v a valuation on K. The* canonical exponent *of v is the integer k_v such that $y \in \pi_v^{-k_v}\mathcal{O}_v$ if and only if its associated character is trivial on $\mathcal{O}_v$.*

We note that T^{-1} is a uniformizer for v_∞, and so this can be written explicitly as

$$k_v = \begin{cases} d(v/P) & v(T) \geq 0 \\ d(v/\infty) - 2e(v/\infty) & v(T) < 0 \end{cases}$$

In particular, we see that for finite valuations v, the local different $\mathfrak{D}_{v/P}$ has ideal norm $q_v^{k_v}$. We can restate our above result in terms of canonical exponents, and also extend our result to any fractional ideal in K.

Theorem 5. *If v is a valuation of K, then $x \mapsto \chi_v(yx)$ is trivial on $\pi_v^n\mathcal{O}_v$ if and only if $y \in \pi_v^{-n-k_v}\mathcal{O}_v$.*

Before we introduce our choice of measure, we give some background from abstract point-set topology, group theory and measure theory.

Definition 8. *1. We say that $\Sigma \subseteq \mathcal{P}(X)$ is a σ-*algebra *on a set X if it is non-empty and closed under complement, countable unions and countable intersections.*

2. *Let X be a topological space.*
   (a) *The* Borel algebra of X *is the σ-algebra generated by the open sets in X.*
   (b) *Let $x \in X$. We say that a* compact neighbourhood *of x is a neighbourhood $\mathcal{U}$ contained in a compact set K.*
   (c) *We say that X is* locally compact *if every $x \in X$ has a compact neighbourhood.*
3. *A* topological group *G is a group with a topology where multiplication and inversion are continuous maps, and a* topological ring *is a ring where addition and multiplication are continuous.*

We thus state the following definition, which we can think of as generalising the properties of the Lebesgue measure on Euclidean space.

Definition 9. *Let G be a locally compact abelian topological group. A* Haar measure *on G is a measure $\mu : G \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ on the Borel algebra of G such that:*

1. *For any measureable $E \subseteq G$ and $x \in G$, we have $\mu(xE) = \mu(E)$.*
2. *If $K \subseteq G$ is compact, then $\mu(K) < \infty$.*
3. *If $\mathcal{U} \subseteq G$ is open, then $\mu(\mathcal{U}) > 0$.*

These 3 constraints can be viewed as being invariant under translation, compact sets being "small" and open sets being "large". Notably, if we add the constraint that the topology on G is Hausdorff, we have the following result.

Theorem 6. *Let G be a locally compact Hausdorff abelian topological group. Then there is a unique Haar measure on G up to multiplication by a non-zero scalar.*

In our case, we have the following result for scaling our Haar measure.

Proposition 3. *The measure $\mu_a(M) = \mu(aM)$ for $a \in K_v^*$ and M measureable is a Haar measure, and we have $\mu(aM) = |a|_v \, \mu(M)$.*

A proof of this result can be found in Tate's thesis (Lemmas 2.2.4. and 2.2.5.). In terms of the corresponding integrals, this means that $d\mu(ax) = |a|_v \, d\mu(x)$.

3.4 Fourier transform

As above, our choice of Haar measure on G is decided up to a scalar. We will define the local Fourier transform $\mathcal{F}_v f$ on $f \in L^1(K_v)$ by

$$\mathcal{F}_v f(y) := \int_{K_v} f(x)\chi_v(yx) d_v x$$

with respect to a suitable measure $d_v x$. A result of abstract Fourier analysis says that once we identify K_v with its $\hat{K}_v$ as above, for continuous functions $f \in L^1(K_v)$ with $\mathcal{F}_v f \in L^1(\hat{K}_v)$ we will always have an inversion formula of the form $\mathcal{F}_v \mathcal{F}_v f(x) = cf(-x)$ for some scalar $c > 0$. We thus choose our Haar measure $d_v x$ as the one which gives $\mu(\mathcal{O}_v) = q_v^{-k_v/2}$, which we claim is the measure for which $c = 1$ in this inversion formula.

To check this, it suffices to compute this constant for a specific choice of function. Conveniently, in showing various properties about the adelic zeta functions, we will also need to compute the Fourier transforms of the characteristic functions ρ_v^n of the sets $\pi_v^n \mathcal{O}_v$. Before giving this computation, we note that this measure gives the fractional ideals $\pi_v^n \mathcal{O}_v$ measure $\mu(\pi_v^n \mathcal{O}_v) = |\pi_v|^n \mu(\mathcal{O}_v) = q_v^{-n-k_v/2}$.

Lemma 9. *Under the Fourier transform as defined above, we have*

$$\mathcal{F}_v \rho_v^n = q_v^{-n-k_v/2} \rho_v^{-n-k_v}$$

Proof. We need to check that $\mathcal{F}_v \rho_v^n(y)$ is $q_v^{-n-k_v/2}$ if $y \in \pi_v^{-n-k_v} \mathcal{O}_v$, and 0 otherwise. Suppose that $y \in \pi_v^{-n-k_v} \mathcal{O}_v$. Then the character $x \mapsto \chi_v(yx)$ is trivial on $\pi_v^n \mathcal{O}_v$, and so

$$\mathcal{F}_v \rho_v^n(y) = \int_{K_v} \rho_v^n(x) \chi_v(yx) d_v x = \int_{\pi_v^n \mathcal{O}_v} d_v x = \mu(\pi_v^n \mathcal{O}_v) = q_v^{-n-k_v/2}$$

Now suppose that $y \notin \pi_v^{-n-k_v \mathcal{O}_v}$. Let $M = y\pi_v^n \mathcal{O}_v$. We note that the image of χ_v is contained in the roots of unity μ_p by definition. For $k \in \mathbb{Z}/p\mathbb{Z}$, let $S_k = \chi_v^{-1}\left(\zeta_p^k\right) \cap M$. We have that $M = \coprod_{k \in \mathbb{Z}/p\mathbb{Z}} S_k$, and we claim now that $\mu(S_k)$ is independent of k. The measures $\mu(S_k)$ are finite as M is a fractional ideal of $\mathcal{O}_v$, and so has finite measure.

Let $z \in M$ have non-trivial character $e^{2\pi i k/p}$, and consider $z' = z/k \in M$ - this element has character ζ_p. The map $S_k \to S_l$ given by $x \mapsto x + (l-k)z$ is thus well-defined and has inverse $x \mapsto x - (l-k)z$, so the sets S_k are additive translations of one another and have the same measure $\mu(S_0)$. We thus have

$$\mathcal{F}_v \rho_v^n(y) = \int_{\pi_v^n \mathcal{O}_v} \chi_v(yx) d_v x = \frac{1}{|y|_v} \int_{y\pi_v^n \mathcal{O}_v} \chi_v(x) d_v x = \frac{1}{|y|_v} \sum_{k \in \mathbb{Z}/p\mathbb{Z}} \int_{S_k} \chi_v(x) d_v x = \frac{\mu(S_0)}{|y|_v} \sum_{k \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^k = 0$$

□

This also proves that the Fourier inversion formula holds for $c = 1$, as for the characteristic function ρ_v^0 we have $\mathcal{F}_v \mathcal{F}_v \rho_v^0 = \mathcal{F}_v(q_v^{-k_v/2} \rho_v^{-k_v}) = q_v^{-k_v/2}(q_v^{k_v/2} \rho_v^0) = \rho_v^0$. While we have used the Fourier inversion formula as motivation for our choice of measure, we will only make use of this in Appendix 2.

We also see that the characteristic functions ρ_v^n are continuous, as if $v(x-y) \geq n$. then if $x \in \pi_v^n \mathcal{O}_v$ we have $v(y) \geq \min(v(x), v(x-y)) \geq n$, while if $x \notin \pi_v^n$ we have $v(y) = \min(v(x), v(x-y)) = v(x) < n$.

3.5 Multiplicative character theory and measure

To introduce the integral form of the zeta function and the local zeta factors, we briefly describe the multiplicative characters of K_v^*, and define a multiplicative Haar measure on the unit group K_v^* of the local field K_v.

We refer to a multiplicative character $c : K_v^* \to \mathbb{C}$ as a quasi-character. The quintessential example of a quasi-character is the norm map $|\cdot|_v$, which has image in $\langle q_v \rangle \subseteq \mathbb{Q}^*$ and whose kernel is exactly the unit group $\mathcal{O}_v^*$. The theory of zeta functions can be developed in more generality by looking at all quasi-characters, but we will focus on the quasi-characters which act trivially on $\mathcal{O}_v^*$. We call these unramified quasi-characters, and as in Tate's thesis [Lemma 2.3.1], we have the following classification.

Lemma 10. *The unramified quasi-characters c of K_v are exactly the maps*

$$c(a) = |a|_v^s$$

where s is well-defined modulo $2\pi i/\log(q_v)$.

We seek to define a multiplicative measure d_v^* on K_v^* which is closely related to the additive measure d_v on K_v. We note that as $d_v(ax)/\left|ax\right|_v = d_vx/\left|x\right|_v$ for any $a, x \in K_v^*$, the measure $\mu_0^*(x) = \mu(x)/\left|x\right|_v$ on K_v^* is invariant under multiplicative translation, and inherits the other desired properties of a Haar measure from the additive Haar measure. Thus the Haar measures on K_v^* are scalar multiples of the measure $d_0^*x = d_vx/\left|x\right|_v$. For a specific choice of measure, we choose the measure which gives the group $\mathcal{O}_v^*$ unit measure.

Lemma 11. *The Haar measure*

$$d_v^*x := \frac{q_v^{k_v/2}}{1-q_v^{-1}}\frac{d_vx}{\left|x\right|_v}$$

gives the unit group $\mathcal{O}_v^*$ unit measure.

Proof. As $\mathcal{O}_v$ is a local ring, $\mathcal{O}_v = \mathcal{O}_v^* \sqcup \pi_v\mathcal{O}_v$ and that $\left|x\right|_v = 1$ for $x \in \mathcal{O}_v^*$, thus

$$\mu^*(\mathcal{O}_v^*) = \int_{\mathcal{O}_v^*}\frac{q_v^{k_v/2}}{1-q_v^{-1}}\frac{d_vx}{\left|x\right|_v} = \frac{q_v^{k_v/2}}{1-q_v^{-1}}\left(\int_{\mathcal{O}_v} d_vx - \int_{\pi_v\mathcal{O}_v} d_vx\right) = \frac{q_v^{k_v/2}}{1-q_v^{-1}}(q_v^{k_v/2}(1-q_v^{-1})) = 1$$

□

By the invariance of our Haar measure under multiplication, this also shows that $\mu^*(\alpha\mathcal{O}_v^*) = 1$ for every $\alpha \in K_v^*$, and in particular for $\alpha = \pi_v^n$ where $n \in \mathbb{Z}$. For the integral, this measure can equivalently be written as the formula

$$\int_{K_v^*} f(x)d_v^*x = \frac{q_v^{k_v/2}}{1-q_v^{-1}}\int_{K_v\setminus\{0\}}\frac{f(x)}{\left|x\right|_v}d_vx$$

3.6 Local zeta function

The general local zeta function is defined in terms of an integral over the multiplicative group K_v^*. We will use this to describe the local factors of the zeta function. As in Tate's thesis, suppose for convergence that

1. $f, \mathcal{F}f \in L^1(K_v)$ are continuous.
2. $f(a)\left|a\right|^\sigma, \mathcal{F}f(a)\left|a\right|^\sigma \in L^1(K_v^*)$ for $\sigma > 0$.

For such functions, we then define their zeta functions as below.

Definition 10. *Let $f : K_v \to \mathbb{C}$ be as above. The corresponding* local zeta function $\zeta_v(f, -) : \mathbb{C} \to \mathbb{C}$ of f at v *is*

$$\zeta_v(f, s) := \int_{K_v^*} f(x)\left|x\right|_v^s d_v^*x$$

As $\left|x\right|_v$ is always an integer power of q_v, the local zeta function $\zeta_v(f, s)$ is periodic with period $\frac{2\pi i}{\log(q_v)}$, and hence also with period $\frac{2\pi i}{\log(q)}$. A more general definition can be made in terms of quasicharacters c, by replacing $\left|x\right|_v^s$ with c.

We aim to define the local zeta function at v by choosing a function which is closely tied to the valuation, and for which the function converges. The clear choice for this would be the characteristic function ρ_v^0 associated to $\mathcal{O}_v$. This is as from the perspective of the valuation v, this is the most natural non-trivial subring of K_v, being the unit disk and the maximal compact subring of K_v.

Definition 11. *The local zeta function at v is*

$$\zeta_v(s) := \zeta_v(\rho_v^0, s)$$

We can readily compute a formula for the zeta functions of ρ_v^n (and hence ζ_v) by noting that $K_v^* \cap \pi_v^n \mathcal{O}_v = \pi_v^n \mathcal{O}_v \setminus \{0\} = \coprod_{k \geq n} \pi_v^k \mathcal{O}_v^*$ and that $|x|_v = q_v^{-ks}$ on $\pi_v^k \mathcal{O}_v^*$, and hence

$$\zeta_v(\rho_v^n, s) = \int_{\pi_v^n \mathcal{O}_v \setminus \{0\}} |x|_v^s \, d_v^* x = \sum_{k \geq n} \int_{\pi_v^k \mathcal{O}_v^*} |x|_v^s \, d_v^* x = \sum_{k \geq n} \mu^*(\pi_v^n \mathcal{O}_v^*) q_v^{-ks} = \frac{q_v^{-ns}}{1 - q_v^{-s}}$$

Since the Fourier transform of ρ_v^n is $q_v^{-n-k_v/2} \rho_v^{-n-k_v}$, this also shows that the zeta function of the Fourier transform is

$$\zeta_v(\mathcal{F}_v \rho_v^n, s) = q_v^{-n-k_v/2} \frac{q_v^{(n+k_v)s}}{1 - q_v^{-s}}$$

We will use these Fourier transforms to derive a functional equation for the global zeta function, which we can view as the zeta function we arrive at by “patching together” the local information at each valuation v.

Chapter 4

Global theory and zeta functions

Having described the local components of the additive, multiplicative and zeta function theory, we move to the richer global setting, where we consider the valuations all at once, rather than once at a time. We begin by describing the global analogues for the various local objects we have considered so far.

Definition 12. *Let K be a global function field.*

1. *(Adeles) The* ring of adeles $\mathbb{A}_K$ of K *is*

$$\mathbb{A}_K := \left\{ (x_v)_v \in \prod_v K_v \;\middle|\; x_v \in \mathcal{O}_v \textit{ for all but finitely many } v \right\} \subseteq \prod_v K_v$$

with pointwise operations, and the subring of integral adeles is $\mathcal{O} := \prod_v \mathcal{O}_v$.

2. *(Ideles) The* group of ideles $\mathbb{A}_K^*$ of K *is*

$$\mathbb{A}_K^* := \left\{ (x_v)_v \in \prod_v K_v^* \;\middle|\; x_v \in \mathcal{O}_v^* \textit{ for all but finitely many } v \right\} \subseteq \prod_v K_v^*$$

with operations taken pointwise, and the subgroup of integral ideles is $\mathcal{O}^* := \prod_v \mathcal{O}_v^*$.

3. *(Measures) The* additive and multiplicative measures dx on $\mathbb{A}_K$ and d^*x on $\mathbb{A}_K^*$ *are*

$$dx := \prod_v d_v x_v \qquad\qquad d^*x := \prod_v d_v^* x_v$$

4. *(Norms) The* norm of an idele $x \in \mathbb{A}_K^*$ *is the product of the local norms*

$$|x| := \prod_v |x|_v$$

The adele ring is a topological ring with the product topology, and the idele group is a topological group under the topology induced by the embedding $x \mapsto (x, x^{-1})$ of $\mathbb{A}_K^*$ into $\mathbb{A}_K \times \mathbb{A}_K$. Notice also that the idele group is indeed the unit group of the adele ring, as for an arbitrary element $x \in \prod_v K_v^*$, $v(x_v^{-1}) = -v(x_v)$, and so for all but finitely many coordinates of both x and x^{-1} to be in the respective $\mathcal{O}_v$, we require $v(x_v) = 0$ (or equivalently $x_v \in \mathcal{O}_v^*$) for all but finitely many v.

The norm is well-defined as the local norms $|x_v|_v$ of an idele $x = (x_v)_v$ vanish at all but finitely many valuations v. We can see that the measures are inherited from the local cases, so the global measures satisfy $d(ax) = |a|\,dx$ and $d^*x = dx/|x|$. We can also view the adeles $\mathbb{A}_K$ as being analogous to the local fields K_v, the ideles $\mathbb{A}_K^*$ as analogous to the units K_v^*, and $\mathcal{O}, \mathcal{O}^*$ as being analogous to $\mathcal{O}_v$ and $\mathcal{O}_v^*$ respectively.

We refer to a function $f : \mathbb{A}_K \to \mathbb{C}$ on the adeles as an adelic function.

We have a non-trivial global character χ of K is of the form $\chi(x) = \prod_v \chi_v(x_v)$ for the local characters we defined above. This global character is well-defined as the characters $\chi_v(x_v)$ are trivial for all but finitely many v. This is as there are all but finitely many v for which either $\mathfrak{D}_{v/P} = \mathcal{O}_v$ or $x_v \in \mathcal{O}_v$, and there are only finitely many valuations above infinity. By the lemma described in the local section on identifying additive groups with their character groups, we have the following result.

Theorem 7. *$\mathbb{A}_K$ is its own character group under the identification $y \longleftrightarrow (x \mapsto \chi(yx))$.*

We can thus define the Fourier transform in a way analogous to the local setting.

Definition 13. *Let $f \in L^1(\mathbb{A}_K)$ be an adelic function. The* global Fourier transform *$\mathcal{F}f$ is the adelic function*

$$\mathcal{F}f(y) = \int_{\mathbb{A}_K} f(x)\chi(yx)dx$$

From the local Fourier inversion formula, we also see that the global Fourier inversion formula $\mathcal{F}\mathcal{F}f(x) = f(-x)$ holds. We can view this as reassurance that our choice of measure is the most reasonable such choice.

In the local case, we could describe the fractional ideals of $\mathcal{O}_v \subseteq K_v$ (that is, those generated by elements of K_v^*) as powers of the unique prime ideal $\pi_v\mathcal{O}_v$. We seek to do the same for the fractional ideals of $\mathcal{O} \subseteq \mathbb{A}_K$ generated by elements of $\mathbb{A}_K^*$. These ideals generated by elements of $\mathbb{A}_K^*$ are trivial in all but finitely many valuations, and so we can describe them with formal $\mathbb{Z}$-linear combinations of valuations.

Definition 14. *A* divisor *is a finite formal $\mathbb{Z}$-linear combination $D = \sum_v D_v v$ of valuations v.*

Hence the divisors correspond naturally to the fractional ideals of $\mathcal{O}$ as described above under the map $D \mapsto \prod_v \pi_v^{D_v} K_v$, and this map is an isomorphism between the group of divisors under addition and the group of such fractional ideals under ideal multiplication, with inverse $\mathfrak{x}\mathbb{A}_K \mapsto \sum_v v(\mathfrak{x}_v)v$. We denote the group of divisors on K by $\mathrm{Div}(K)$, but we could equivalently think of this as a property of the curve $\mathcal{C}$. For a given divisor $D = \sum_v D_v v$, we denote the associated ideal $\prod_v \pi_v^{D_v} K_v$ as $\pi^D\mathbb{A}_K$, and its characteristic function as $\rho^D = \prod_v \rho_v^{D_v}$.

Corresponding to the canonical exponents k_v for the valuations v, we have the following.

Definition 15. *The* canonical divisor *is the divisor $\mathcal{K} = \sum_v k_v v$. The* genus *of the curve $\mathcal{C}$ is $g := 1 + \deg(\mathcal{K})/2$.*

This definition of the canonical divisor leans much more on the number theoretic side of $K/\mathbb{F}_q(t)$. It is also special in that we can write this divisor down explicitly, unlike in more general cases where the ability to do so is far but guaranteed. In line with other, more orthodox definitions of the genus and canonical divisor, we have $\deg(\mathcal{K}) = 2g - 2$. For $\mathbb{P}^1(\mathbb{F}_q)$ with function field $\mathbb{F}_q(t)$, we see that $g = 0$.

From this, we see that the Fourier transform of $\rho^{\mathcal{D}}$ is

$$\mathcal{F}\rho^{\mathcal{D}} = \prod_v \mathcal{F}_v\rho_v^{\mathcal{D}_v} = \prod_v q_v^{-\mathcal{D}_v - k_v/2}\rho_v^{-\mathcal{D}_v - k_v} = q^{-\deg(\mathcal{D}) - \deg(\mathcal{K})/2}\rho^{-\mathcal{K}-\mathcal{D}} = q^{1-g-\deg(D)}\rho^{-\mathcal{K}-\mathcal{D}}$$

For general divisors, we have the following associated definitions.

Definition 16. *Let $\mathcal{D} = \sum_v \mathcal{D}_v v$ be a divisor. We say that $\mathcal{D}$ is* positive *if $\mathcal{D}_v \geq 0$ for all valuations v. The* degree of $\mathcal{D}$ *is the integer* $\deg(\mathcal{D}) := \sum_v \mathcal{D}_v \deg(v)$.

We denote the set of divisors of degree n on K by $\mathrm{Div}^n(K)$, and note that $\mathrm{Div}^0(K)$ is a subgroup of $\mathrm{Div}(K)$. The degree of an idele and its norm satisfy $|\mathfrak{x}| = q^{-\deg(\alpha)}$, as

$$|\mathfrak{x}| = \prod_v |\mathfrak{x}_v|_v = \prod_v q_v^{-v(\mathfrak{x}_v)} = q^{-\deg(\mathfrak{x})}$$

The global function field K embeds into the adele ring via the the diagonal embedding $\alpha \mapsto (\alpha)_v$, and to each $\alpha \in K^*$ we have an associated *principal divisor* $(\alpha) = \sum_v v(\alpha)v$. This principal divisor can be seen as a list of the orders of zeroes of α at each point or prime ideal v, and the degrees $\deg(v)$ of individual valuations as the number of zeroes of a uniformizer π_v in an algebraic closure. From this perspective, the degree $\deg(\alpha)$ counts the difference between the number of zeroes and number of poles of α, which by Lemma 5.

The set of principal divisors $\mathcal{P}(K)$ is thus a subgroup of $\mathrm{Div}^0(K)$, as the map sending any $\alpha \in K^*$ to its principal divisor is a group homomorphism, with kernel $\mathbb{F}_q^*$ (the set of elements with empty factorisation or no zeroes).

For any $\mathrm{Div}^n(K)$, we can define the equivalence relation $\mathcal{D} \sim \mathcal{D}'$ if and only if $\mathcal{D} - \mathcal{D}' = (\alpha)$ for some $\alpha \in K$. We say that two divisors are *linearly equivalent* if they are related in this way, and set $\mathscr{C}^n(K)$ to be the set of equivalence classes of $\mathrm{Div}^n(K)$ under this relation. When $n = 0$ this corresponds to the quotient group $\mathscr{C}(K) := \mathscr{C}^0(K) = \mathrm{Div}^0(K)/\mathcal{P}(K)$, which we refer to as the *divisor class group*. This is the analogous object to the ideal class group of a number field, except that the divisor class group also accounts for the “infinite primes”.

If there is a divisor $\mathcal{D}_n$ of degree n, then the map $\mathcal{D} \mapsto \mathcal{D} + \mathcal{D}_n$ from $\mathscr{C}^0(K)$ to $\mathscr{C}^n(K)$ is a bijection, and so $|\mathscr{C}^n(K)|$ is either 0 or $\left|\mathscr{C}^0(K)\right|$ for each n.

To each divisor $\mathcal{D}$, we have an associated vector space over $\mathbb{F}_q$ of functions $\alpha \in K^*$ with pole divisor bounded by the coefficients of $\mathcal{D}$.

Definition 17. *Let $\mathcal{D}$ be a divisor. The* set of functions with pole divisor bounded by $\mathcal{D}$ *is*

$$L(\mathcal{D}) := \{x \in K \mid \mathcal{D} + (a) \geq 0\} \cup \{0\}$$

This is indeed a vector space over $\mathbb{F}_q$. It is finite dimensional for any given $\mathcal{D}$ as $(a) = (b)$ if and only if $a/b \in \mathbb{F}_q$, and there are only finitely many valuations of degree at most N for any $N \in \mathbb{N}$, and so finitely many possibilities for (a) in $L(\mathcal{D}) \setminus \{0\}$. We denote its dimension by $\dim_{\mathbb{F}_q}(L(\mathcal{D})) = l(\mathcal{D})$.

4.1 The Riemann-Roch theorem

Similar to in Tate's thesis, we have a corresponding Poisson summation formula, which we use to derive the Riemann-Roch theorem. This involves an additive fundamental domain $D \subseteq \mathbb{A}_K$ for $\mathcal{O}_K$, whose analogue in the function field case is defined below.

Definition 18. *Let $\{\omega_1, \ldots, \omega_n\}$ be a $\mathbb{F}_q[t]$-basis for $\mathcal{O}_K$. The* additive fundamental domain $D \subseteq \mathbb{A}_K$ *is the set*

$$D = \prod_{v \nmid \infty} \mathcal{O}_v \times D^\infty$$

where $D^\infty := \left\{\sum_{i=1}^n f_i\omega_i \in \prod_{v|\infty} K_v \;\middle|\; \deg(f_i) < 0\right\}$ *is a fundamental domain for the image of* $\mathcal{O}_K$ *in* $\prod_{v|\infty} K_v$ *under the diagonal embedding.*

We note that for any $y = (y_v)_v \in D$, the finite valuations have $v(y_v) \geq 0$, and the infinite valuations are bounded below by some $m_D \in \mathbb{Z}_{\leq 0}$. This is as the valuations $v(\omega_i)$ for $1 \leq i \leq n$ and $v \mid \infty$ are bounded below by their minimum, and a linear combination $\sum_{i=1}^n f_i\omega_i$ has

$$v\left(\sum_{i=1}^n f_i\omega_i\right) \geq \min_{\substack{1\leq i\leq n \\ v|\infty}} v\left(f_i\omega_i\right) \geq \min_{\substack{1\leq i\leq n \\ v|\infty}} v\left(\omega_i\right)$$

With this definition, we can state the Poisson summation formula.

Theorem 8. *If* $f : \mathbb{A}_K \to \mathbb{C}$ *is such that*

1. $f \in L^1(\mathbb{A}_K)$ *and is continuous.*
2. *The map* $\varphi(y) = \sum_{x\in K} f(y + x)$ *is uniformly convergent for* $y \in D$.
3. $\sum_{x\in K} |\mathcal{F}f(x)|$ *is convergent.*

then

$$\sum_{x\in K} f(x) = \sum_{x\in K} \mathcal{F}f(x)$$

The proof of this result and the corresponding definitions are looked at in more detail in appendix 2.

From this, we attain the number theoretic statement of the Riemann-Roch theorem, which is the above theorem applied to $g(x) = f(ax)$ for f continuous.

Theorem 9. *If* $f : \mathbb{A}_K \to \mathbb{C}$ *and* $a \in \mathbb{A}_K^*$ *is such that*

1. $f \in L^1(\mathbb{A}_K)$ *and is continuous.*
2. *The map* $\varphi(y) = \sum_{x\in K} f(a(y + x))$ *is uniformly convergent for* $x \in D$.
3. $\sum_{x\in K} |\mathcal{F}f(x/a)|$ *is convergent.*

then

$$\sum_{x\in K} f(ax) = \frac{1}{|a|}\sum_{x\in K} \mathcal{F}f(x/a)$$

Proof. We see that $g(x) = f(ax)$ is the composition $f \circ (x \mapsto ax)$, and so is continuous and in $L^1(\mathbb{A}_K)$. We have

$$\mathcal{F}g(y) = \int_{\mathbb{A}_K} f(ax)\chi(yx)dx = \frac{1}{|a|}\int_{\mathbb{A}_K} f(x)\chi\left((y/a)x\right) = \frac{1}{|a|}\mathcal{F}f(y/a)$$

and so the claim follows by the Poisson summation formula. □

Unlike the number theoretic case, there is also a reasonably clear geometric view on global function fields. Corresponding to this, we can recover the Riemann-Roch theorem as usually stated in the geometric case.

For any fixed divisor $\mathcal{D}$, let the idele $\pi^{\mathcal{D}}$ be a generator of the corresponding ideal. The map $\rho^0 = \prod_v \rho_v^0$ is continuous as each of the components are, and we find that for $x \in K$:

$$\rho^0(\pi^{\mathcal{D}} x) = \prod_v \rho_v^0(\pi_v^{v(x)+\mathcal{D}_v}) = \begin{cases} 1 & \text{if } (x) + \mathcal{D} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus we have $\sum_{x \in K} \rho^0(\pi^{\mathcal{D}}(x)) = |L(\mathcal{D})| = q^{l(\mathcal{D})}$. To see that the map $\varphi(y) = \sum_{x \in K} \rho^0(\pi^{\mathcal{D}}(y+x))$ is also uniformly convergent in general, we show that there are only finitely many $x \in K$ for which there are any $y = (y_v)_v \in D$ with $\rho^0(\pi^{\mathcal{D}}(y+x)) = \prod_v \rho^0(\pi_v^{\mathcal{D}_v}(y_v + x)) \neq 0$.

For fixed $y \in D$, at any valuations v with $\mathcal{D}_v + v(y_v) < 0$, if we have $v(x) \neq v(y)$, then we find that $\mathcal{D}_v + v(y_v + x) = \mathcal{D}_v + \min(v(y_v), v(x)) \leq \mathcal{D}_v + v(y_v) < 0$. We thus require $v(x) = v(y) \geq m_D$ at these valuations. For valuations where $D_v + v(y_v) \geq 0$, if we have $\mathcal{D}_v + v(x) < 0$, we have $v(x) < -D_v \leq v(y_v)$ and hence $\mathcal{D}_v + v(y_v + x_v) < 0$. Hence we must have $\mathcal{D}'_v + v(x) \geq 0$ for $\mathcal{D}'_v = \mathcal{D}_v$ when $v(y_v) \geq 0$ and $-m_D$ when $v(y_v) \leq 0$, and so the set of x with $\rho^0(\pi^{\mathcal{D}}(y+x)) \neq 0$ is alwayd contained in $L(\mathcal{D}')$ for any $y \in D$. This shows that $\varphi(y)$ is uniformly convergent as it is ultimately equal to a partial sum sum at every point.

For its Fourier transform, we have

$$\mathcal{F}\rho^0\left(\frac{x}{\pi^{\mathcal{D}}}\right) = q^{1-g}\rho^{\mathcal{K}}\left(\frac{x}{\pi^{\mathcal{D}}}\right) = \begin{cases} q^{1-g} & \text{if } (x) + (\mathcal{K} - \mathcal{D}) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence $\sum_{x \in K} \left|\mathcal{F}\rho^0(x/\pi^{\mathcal{D}})\right|$ is convergent with value $q^{1-g}q^{l(\mathcal{K}-\mathcal{D})}$. We have $\left|\pi^{\mathcal{D}}\right| = \prod_v \left|q_v^{\mathcal{D}_v}\right| = q^{-\deg(\mathcal{D})}$, and so applying the number theoretic Riemann-Roch theorem to our situation yields

$$q^{l(\mathcal{D})} = \sum_{x \in K} \rho^0(\pi^{\mathcal{D}} x) = \frac{1}{|\pi^{\mathcal{D}}|} \sum_{x \in K} \mathcal{F}\rho^0\left(\frac{x}{\pi^{\mathcal{D}}}\right) = q^{\deg(\mathcal{D}) + 1 - g + l(\mathcal{K} - \mathcal{D})}$$

Taking the corresponding exponents, we have proved

Theorem 10. *For any divisor $\mathcal{D}$ of $\mathcal{C}$, we have*

$$l(\mathcal{D}) = \deg(\mathcal{D}) + 1 - g + l(\mathcal{K} - \mathcal{D})$$

This justifies our somewhat unorthodox definition of $\mathcal{K}$ and hence the genus g, and says that the mostly number-theoretic definition we gave matches the usual geometric definition. For divisors $\mathcal{D}$ with $\deg(\mathcal{D}) \geq g$, we see that $l(\mathcal{D}) = 1 + l(\mathcal{K} - \mathcal{D}) \geq 1$, and so there is a function f with $(f) + \mathcal{D} \geq 0$, and so every divisor class of degree $n \geq g$ is represented by a positive divisor.

As usual, it follows from this that $g \geq 0$: Taking $\mathcal{D} = 0$, we have $L(0) = \mathbb{F}_q$ and thus $1 = 1 - g + l(\mathcal{K})$, so $g = l(\mathcal{K}) \geq 0$.

4.2 Some classical finiteness results

A classical result of algebraic number theory is that the ideal class group of a number field is finite. It follows from this formulation of the Riemann-Roch theorem that we have the analogous statement for global function fields.

Theorem 11. *The divisor class group $\mathscr{C}(K)$ of a global function field is finite.*

Proof. Let $\mathcal{D}$ be a divisor with $n = \deg(\mathcal{D}) \geq g$ (we can do this by picking a valuation v and taking a sufficiently large multiple), so that $|\mathscr{C}(K)| = |\mathscr{C}^n(K)|$. By our above argument, the divisor classes of degree n are represented by positive divisors. There are only finitely many valuations of degree at most n, so only finitely many possible positive divisors of degree n. Thus $\mathscr{C}^n(K)$ is finite, and hence so is $\mathscr{C}(K)$. $\square$

In lieu of this result and parallel to the number field case, we give the following definition.

Definition 19. *The* class number h *of the curve* $\mathcal{C}$ *over* $\mathbb{F}_q$ *is* $h := |\mathscr{C}(K)|$.

The divisor class group is related to the ideal class group $\mathcal{I}\mathscr{C}(K)$ in that the ideal class group can be identified with a quotient of $\mathscr{C}(K)$. Considering the map $\psi : \sum_v \mathcal{D}_v v \mapsto \prod_{v\nmid\infty} \mathfrak{p}_v^{\mathcal{D}_v}$ from the fractional ideals $\mathrm{Div}^0(K)$ to $\mathcal{I}(\mathcal{O}_K)$, by unique prime ideal factorisation we see that this map is surjective, and that the image of any principal divisor (α) is the corresponding ideal $\alpha\mathcal{O}_K$. The above theorem thus also tells us that the ideal class group of a global function field is also finite.

The other notable classical result of algebraic number theory is that the unit group $\mathcal{O}_K^* \subseteq K^*$ of a number field $K/\mathbb{Q}$ is $\mathcal{O}_K^* \cong \mu_K \times \mathbb{Z}^{r_1+r_2-1}$, where μ_K is the set of roots of unity in K, r_1 is the number of real embeddings, and r_2 is the number of conjugate pairs of complex embeddings. As in a paper by Cassels (Theorem 18.3 of [Cassels]), we have the following generalisation, which in particular applies to the unit groups of function fields.

Theorem 12. *Let K be a global field, and suppose that S is a finite, non-empty set of valuations, containing all archimedean valuations. Let*

$$H_S := \{x \in K \mid v(x) = 0, v \notin S\}$$

be the group of S-units *in K. Then H_S is the direct sum of the roots of unity in K and a free abelian group of rank $s-1$.*

Applying this to a global function field $K/\mathbb{F}_q(t)$ and letting S_∞ be the set of infinite valuations (of cardinality s_∞), we see that unit group $\mathcal{O}_K^* \cong \mathbb{F}_q^* \times \mathbb{Z}^{s_\infty-1}$, in complete analogy with the number field case.

4.3 The adelic zeta function

Similar to the local case, we define the global zeta function of an adelic function f by an integral with respect to the multiplicative measure d^*x. For such an integral to make sense, we want our function f and its Fourier transform $\mathcal{F}f$ to decay quickly as $|a| \to \infty$. As in Tate's thesis, we mean that

1. $f, \mathcal{F}f \in L^1(\mathbb{A}_K)$ are continuous.
2. The sums $\sum_{x\in K} f(a(x+x)), \sum_{x\in K} \mathcal{F}f(a(x+x))$ are convergent for any $a \in \mathbb{A}_K^*$ and $x \in \mathbb{A}_K$, with uniform convergence in (a, x) if $x \in D$ and a ranging over compact subsets of $\mathbb{A}_K^*$.
3. $f(a)\,|a|^\sigma, \mathcal{F}f(a)\,|a|^\sigma \in L^1(\mathbb{A}_K^*)$ for $\sigma > 1$.

Definition 20. *Let $f : \mathbb{A}_K \to \mathbb{C}$ be as above. The corresponding zeta function $\zeta_{\mathcal{C}}(f, -) : \mathbb{C} \to \mathbb{C}$ is defined for $Re(s) > 1$ by*

$$\zeta_{\mathcal{C}}(f, s) := \int_{\mathbb{A}_K^*} f(a)\,|a|^s\,d^*a$$

If $f = \prod_v f_v$ is given by local functions $f_v : K_v \to \mathbb{C}$ at each coordinate, we have the Euler product $\zeta_{\mathcal{C}}(f,s) = \prod_v \zeta_v(f,s)$. It will be convenient to view the zeta function as an integral over a fundamental domain $\mathbb{A}_K^*/K^*$, and we can restrict a function f on $\mathbb{A}_K^*$ to one on $\mathbb{A}_K^*/K^*$ in the following natural way, by averaging over the values of the function in αK^*.

Definition 21. *Let $f : \mathbb{A}_K \to \mathbb{C}$. The* restriction of f to $\mathbb{A}_K^*/K^*$ *is the map*

$$Rf(a) := \sqrt{|a|} \sum_{\alpha \in K^*} f(\alpha a)$$

As in Tate's thesis, we can choose a fundamental domain E of K^* in $\mathbb{A}_K^*$ contained in a compact subset, and identify this with the quotient $\mathbb{A}_K^*/K^*$. As $|\alpha| = 1$ for $\alpha \in K^*$, we find that

$$\zeta_{\mathcal{C}}(f,s) = \sum_{\alpha \in K^*} \int_E f(\alpha a) \, |\alpha a|^s \, d^*a = \int_{\mathbb{A}_K^*/K^*} \sum_{\alpha \in K^*} f(\alpha a) \, |a|^s \, d^*a = \int_{\mathbb{A}_K^*/K^*} Rf(a) \, |a|^{s-1/2} \, d^*a$$

We seek to derive an analytic continuation and a functional equation for these adelic zeta functions. Writing $\mathbb{A}_n^*$ for the ideles of degree n, as $K^* \subseteq \ker |\cdot| = \mathbb{A}_0^*$, we have $\mathbb{A}_K^* = \coprod_{n \in \mathbb{Z}} \mathbb{A}_n^*$, and we can write

$$\zeta_{\mathcal{C}}(f,s) = \int_{\mathbb{A}^*/K^*} Rf(a) \, |a|^{s-1/2} \, d^*a = \sum_{n \in \mathbb{Z}} \int_{\mathbb{A}_n^*/K^*} Rf(a) \, |a|^{s-1/2} \, d^*a$$

To compute this, we note that if m is the minimal positive degree of a divisor, then by applying division with remainder on the degrees of ideles, we see that $\mathbb{A}_n^*/K^*$ will be non-empty if and only if n is a multiple of m. We split the above sum into 2 parts: $n < 0$ and $n \geq 0$.

The sum over negative degrees is

$$\sum_{k=1}^{\infty} \int_{\mathbb{A}_{-mk}^*/K^*} Rf(a) \, |a|^{s-1/2} \, d^*a$$

and this is holomorphic everywhere as we have assumed that f is of fast decay as in (3). For the non-negative degrees, we see that by the number-theoretic Riemann-Roch theorem, we have $Rf(a) + f(0) \, |a|^{1/2} = R(\mathcal{F}f)(a^{-1}) + \mathcal{F}f(0) \, |a|^{-1/2}$. Thus the sum across non-negative degrees is

$$\sum_{k=0}^{\infty} \int_{\mathbb{A}_{mk}^*/K^*} Rf(a) \, |a|^{s-1/2} \, d^*a = \sum_{k=0}^{\infty} \int_{\mathbb{A}_{mk}^*/K^*} \left(R\mathcal{F}f(a^{-1}) \, |a|^{s-1/2} + \mathcal{F}f(0) \, |a|^{s-1} - f(0) \, |a|^s \right) d^*a$$

The sum of the first terms is holomorphic as we have again assumed that $\mathcal{F}f$ is of fast decay. To compute the sums of the other terms, we require the volume of the sets $\mathbb{A}_{mk}^*/K^*$. These each have the same volume as $\ker |\cdot| / K^* = \mathbb{A}_0^*/K^*$ as they are each multiplicative translations of one another. For $\mathrm{vol}(\ker |\cdot| / K^*)$, we have the decomposition

$$\ker |\cdot| = \coprod_{c \in \mathscr{C}(K)} \{c\} \times \mathcal{O}^* \times K^*/\mathbb{F}_q^*$$

where $c \in \mathscr{C}(K)$ ranges over a fixed set of representatives. Indeed, for a given $\alpha \in \ker |\cdot|$, we can divide by a representative of its class in $\mathscr{C}(K)$ to give the image of a function in K^*. The result of dividing by that function (determined up to a multiple in $\mathbb{F}_q^*$) is a unit everywhere, and so in $\mathcal{O}^*$. This shows the above decomposition, and since $\mathcal{O}^* \cong \mathcal{O}^*/\mathbb{F}_q^* \times \mathbb{F}_q^*$ and $K^* \cong K^*/\mathbb{F}_q^* \times \mathbb{F}_q^*$, we can equivalently write $\ker |\cdot| = \coprod_{c \in \mathscr{C}(K)} \{c\} \times \mathcal{O}^*/\mathbb{F}_q^* \times K^*$. Thus the quotient $\ker |\cdot| / K^*$ is h copies of $\mathcal{O}^*/\mathbb{F}_q^*$, and so $\mathrm{vol}(\ker \cdot / K^*) = h \, \mathrm{vol}(\mathcal{O}^*) / \left|\mathbb{F}_q^*\right| = h/(q-1)$. For the last 2 terms we thus have

$$\sum_{k=0}^{\infty} \int_{\mathbb{A}_{mk}^*} \left(\mathcal{F}f(0) \, |a|^{s-1} - f(0) \, |a|^s \right) d^*a = \frac{h}{q-1} \sum_{k=0}^{\infty} \left(\mathcal{F}f(0) q^{mk(1-s)} - f(0) q^{-mks} \right)$$

$$= \frac{h}{q-1}\left(\frac{\mathcal{F}f(0)}{1-q^{m(1-s)}} - \frac{f(0)}{1-q^{-ms}}\right)$$

This yields the analytic continuation of $\zeta_c(f,s)$ to $\mathbb{C}\setminus\{0,1\}$ in general up to the value of m, with the zeta function also being defined at 0 if $f(0)=0$ and at 1 and $\mathcal{F}f(0)=0$. To determine the value of m, we see that the residues at $s=0,1$ (and all translations by $2k\pi i/m\log(q)$) depend on m. Indeed, as $1-q^{-ms}$, $1-q^{m(1-s)}$ have simple zeroes at $s=0,1$ respectively and the other terms are holomorphic, we readily compute

$$\operatorname{Res}_0 \zeta_{\mathcal{C}}(f,s) = \operatorname{Res}_0 \frac{-hf(0)}{(q-1)1-q^{-ms}} = \left[\frac{-hf(0)}{(q-1)mq^{-ms\log(q)}}\right]_{s=0} = -\frac{hf(0)}{m(q-1)\log(q)}$$

and

$$\operatorname{Res}_1 \zeta_{\mathcal{C}}(f,s) = \operatorname{Res}_1 \frac{h\mathcal{F}f(0)}{(q-1)1-q^{m(1-s)}} = \left[\frac{h\mathcal{F}f(0)}{(q-1)mq^{-ms\log(q)}}\right]_{s=1} = \frac{h\mathcal{F}f(0)}{m(q-1)\log(q)}$$

To compute the value of m, we take advantage of the fact that this residue depends on m, and look at how the value of this residue changes when we instead view $\mathcal{C}$ over an extension of the field $\mathbb{F}_q$.

4.4 Zeta functions over constant field extensions

The goal of this section is to compute the value of m. For this, we compare the residue of $\zeta_{\mathcal{C}/\mathbb{F}_q}$ to $\zeta_{\mathcal{C}/\mathbb{F}_{q^n}}$, corresponding to an extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. The important fact we will use is that this pole has order at most 1, independent of the choice of constant field.

We have already computed the local factors of $\rho^0 = \prod_v \rho_v^0$ as $\rho_v^0(s) = \frac{1}{1-q_v^{-s}}$ and we will make use of these soon, so we choose $f=\rho^0$. Of the fast decay constraints we outlined above, (1) and (2) follow from our discussion while deriving the geometric Riemann-Roch theorem (swapping ρ^0 for $\rho^{-\mathcal{K}}$ for $\mathcal{F}\rho^0$), while (3) follows as a geometric series sum.

In this section we denote $\zeta_{\mathcal{C}/F}(s) := \zeta_{\mathcal{C}/F}(\rho_F^0, s)$ where ρ_F^0 is the characteristic function of the the integral adeles $\mathcal{O}\subseteq\mathbb{A}_L$ of the function field L of $\mathcal{C}/F$. This always has a simple pole at $s=1$ as $\rho_F^0(0)=1$. Viewing $\mathcal{C}$ over $\mathbb{F}_q$, we have the Euler product

$$\zeta_{\mathcal{C}/\mathbb{F}_q}(s) = \prod_v \frac{1}{1-q_v^{-s}}$$

Writing $K=\mathbb{F}_q(t)(\alpha)$, we set $K_n = \mathbb{F}_{q^n}(t)(\alpha) = K[x]/(m)$ for some irreducible $m\in\mathbb{F}_q[x]$ of degree n. Viewed over $\mathbb{F}_{q^n}$, $\mathcal{C}$ has function field K_n, and the corresponding zeta function to the characteristic function Ψ_n^0 of $\mathcal{O}_n\subseteq\mathbb{A}_{K_n}$ is

$$\zeta_{\mathcal{C}/\mathbb{F}_{q^n}}(s) = \prod_w \frac{1}{1-q_w^{-s}}$$

where w ranges over the valuations of K_n. As we have seen previously, the extensions of a given valuation v of K correspond to the irreducible factors of m in K_v. We thus have the following lemma relating $\zeta_{\mathcal{C}/\mathbb{F}_q}(s)$ to $\zeta_{\mathcal{C}/\mathbb{F}_{q^n}}(s)$.

Lemma 12. *We have*

$$\zeta_{\mathcal{C}/\mathbb{F}_{q^n}}(s) = \prod_{k=1}^{n} \zeta_{\mathcal{C}/\mathbb{F}_q}\left(s + k\frac{2\pi i}{n\log(q)}\right)$$

Proof. For a given valuation w of K_n, the residue field $K_n(w)$ is the smallest field containing both $K(v)$ and $\mathbb{F}_{q^n}$, and so we find that $[K_n(w) : \mathbb{F}_q] = n\deg(v)/d$ for $d = \gcd(\deg(v), n)$. Thus the degree $[K_n(w) : K(v)] = n/d$, and so m factors as d irreducible factors of degree n/d, and there are d valuations of K_n above v with degree $\deg(w) = n\deg(v)/d$. The local factor corresponding to v in the Euler product of $\zeta_{\mathcal{C}/\mathbb{F}_{q^n}}(s)$ is

$$\prod_{w|v} \frac{1}{1 - q_w^{-s}} = (1 - q_v^{-ns/d})^{-d}$$

Since $\gcd(\deg(v)/d, n/d) = 1$, $e^{2\pi i \deg(v)/n}$ is a primitive n/d-th root of unity, and so we see that

$$(1 - x^{n/d})^d = \prod_{k=1}^{n} (1 - e^{2\pi i k \deg(v)/n} x)$$

as these polynomials have the same roots, with same multiplicities and agree at $x = 0$. Setting $x = q_v^{-s}$, we find that

$$\zeta_{\mathcal{C}/\mathbb{F}_{q^n}}(s) = \prod_w \frac{1}{1 - q_w^{-s}} = \prod_v \frac{1}{\left(1 - q_v^{-ns/d}\right)^d} = \prod_{k=1}^{n} \prod_v \frac{1}{1 - e^{2\pi i k \deg(v)/n} q_v^{-s}} = \prod_{k=1}^{n} \zeta_{\mathcal{C}/\mathbb{F}_q}\left(s + \frac{2\pi i k}{n\log(q)}\right)$$

□

We can thus determine the value of m as above, and this is stated as the following theorem.

Theorem 13. *There is a divisor on $\mathcal{C}$ of degree 1.*

Proof. For m as above, we saw that $\zeta_{\mathcal{C}/\mathbb{F}_q}(s) = \zeta_{\mathcal{C}/\mathbb{F}_q}(s + 2\pi i/m\log(q))$ for any s. Thus by the above lemma, as $\zeta_{\mathcal{C}/\mathbb{F}_q}$ has a pole of order 1 at $s = 1 + 2k\pi i/m\log(q)$, $\zeta_{\mathcal{C}/\mathbb{F}_{q^m}}$ has a pole of order m at $s = 1$. This is a simple pole, so we conclude $m = 1$. □

4.5 The functional equation of an adelic zeta function

We derive the following functional equation stated in the theorem below.

Theorem 14. *Let $f : \mathbb{A}_K \to \mathbb{C}$ be an adelic function of fast decay. Then*

$$\zeta_{\mathcal{C}}(f, s) = \zeta_{\mathcal{C}}(\mathcal{F}f, 1 - s)$$

Proof. We find an expression invariant under $(f, s) \longleftrightarrow (\mathcal{F}f, 1 - s)$. Continuing with our previous computation with the knowledge that $m = 1$, we have

$$\begin{aligned}\zeta_{\mathcal{C}}(f, s) = &\sum_{k=1}^{\infty} \int_{\mathbb{A}^*_{-k}/K^*} Rf(a)\, |a|^{s-1/2}\, d^*a + \sum_{k=0}^{\infty} \int_{\mathbb{A}^*_k/K^*} R(\mathcal{F}f)(a^{-1})\, |a|^{s-1/2}\, d^*a \\ &+ \frac{h}{q-1}\left(\frac{\mathcal{F}f(0)}{1 - q^{(1-s)}} - \frac{f(0)}{1 - q^{-s}}\right)\end{aligned}$$

On substituting $a \to a^{-1}$ into the second integral, this becomes

$$\sum_{k=1}^{\infty} \int_{\mathbb{A}^*_{-k}/K^*} \left(Rf(a)\, |a|^{s-1/2}\, d^*a + R(\mathcal{F}f)(a)\, |a|^{1/2-s}\right) d^*a + \int_{\mathbb{A}^*_0/K^*} R(\mathcal{F}f)(a) d^*a$$

$$+\frac{h}{q-1}\left(\frac{\mathcal{F}f(0)}{1-q^{(1-s)}}-\frac{f(0)}{1-q^{-s}}\right)$$

The sum is invariant under $(f,s)\longleftrightarrow(\mathcal{F}f,1-s)$, so we consider the latter terms. We recall that $Rf(a)+f(0)\,|a|^{1/2}=R(\mathcal{F}f)(a^{-1})+\mathcal{F}f(0)\,|a|^{-1/2}$ by the Riemann-Roch theorem, and applying this to the latter terms when $|a|=1$ gives

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{A}_0^*/K^*}(R(\mathcal{F}f)(a)+Rf(a^{-1})-f(0)+\mathcal{F}f(0))d^*a+\frac{h}{q-1}\left(\frac{\mathcal{F}f(0)}{1-q^{1-s}}-\frac{f(0)}{1-q^{-s}}\right)\\
=&\frac{1}{2}\left(\int_{\mathbb{A}_0^*/K^*}R(\mathcal{F}f)(a)d^*a+\int_{\mathbb{A}_0^*/K^*}Rf(a^{-1})d^*a\right)\\
&\quad+\frac{h}{2(q-1)}\left(\mathcal{F}f(0)\left(\frac{2}{1-q^{1-s}}-1\right)-f(0)\left(\frac{2}{1-q^{-s}}-1\right)\right)\\
=&\frac{1}{2}\int_{\mathbb{A}_0^*/K^*}(R(\mathcal{F}f)(a)+Rf(a))d^*a+\frac{h}{2(q-1)}\left(\mathcal{F}f(0)\frac{1+q^{1-s}}{1-q^{1-s}}+f(0)\frac{1+q^{s}}{1-q^{s}}\right)
\end{aligned}$$

Where we have substituted $a\to a^{-1}$ in the latter integral. This expression is also invariant in $(f,s)\longleftrightarrow(\mathcal{F}f,1-s)$, so we conclude the functional equation $\zeta_{\mathcal{C}}(f,s)=\zeta_{\mathcal{C}}(\mathcal{F}f,1-s)$. □

4.6 The Hasse-Weil zeta function of a curve

We restate the definition of the Hasse-Weil zeta function of a curve $\mathcal{C}$ in terms of adeles.

Definition 22. *Let $\mathcal{C}$ be a curve over $\mathbb{F}_q$. The* Hasse-Weil zeta function $\zeta_{\mathcal{C}}:\mathbb{C}\to\mathbb{C}$ of $\mathcal{C}$ *is the zeta function $\zeta_{\mathcal{C}}(s):=\zeta_{\mathcal{C}}(\rho^0,s)$ of the characteristic function ρ^0 of $\mathcal{O}$.*

In this regard, the zeta function and its local components correspond exactly to the elements integral at each valuation v. When referring to the zeta function of the curve $\mathcal{C}$ (or just zeta function) from now, we mean the Hasse-Weil zeta function. The definition of any zeta function is in general ambiguous up to a finite number of factors, and this corresponds to looking at the underlying object from different viewpoints, such as number theoretic or geometric. We will refer to all such functions as Hasse-Weil zeta functions, and we will use whichever form is most appropriate for the context.

Surprisingly, this definition exactly matches the point-counting definition given in the introduction. From the computations in section 4.2, we have the Euler product

$$\zeta_{\mathcal{C}}(s)=\prod_v\frac{1}{1-q_v^{-s}}$$

for $\mathrm{Re}(s)>1$. We also have a series representation analogous to that of a Dedekind zeta function, summing over the norms $\left|q^{-\deg(\mathcal{D})}\right|$ of the integral ideals $\pi^{\mathcal{D}}\mathcal{O}\subseteq\mathcal{O}$ corresponding to divisors $\mathcal{D}$:

$$\zeta_{\mathcal{C}}(s)=\prod_v\left(\sum_{n=0}^{\infty}q^{-s\deg(v)}\right)=\sum_{\mathcal{D}\geq 0}q^{-s\deg(\mathcal{D})}=\sum_{n=0}^{\infty}|\mathrm{PDiv}^n(K)|\,q^{-ns}$$

where $\mathrm{PDiv}^n(L)$ denotes the set of positive divisors of $\mathcal{C}$ of degree n.

Recalling the point-counting definition, the zeta function of the curve was given as $Z_{\mathcal{C}}(q^{-s})$ where $\log(Z_{\mathcal{C}}(t))=\sum_{n=1}^{\infty}N_{\mathcal{C}}(n)t^n/n$ was a generating function with coefficients depending on the number of points on $\mathcal{C}$ defined over $\mathbb{F}_{q^n}$. We can split this sum by the corresponding valuations, and we see that

$$\log(Z_{\mathcal{C}}(q^{-s})) = \sum_{n=1}^{\infty} \frac{N_{\mathcal{C}}(n)}{n} q^{-sn} = \sum_{v} \sum_{n=1}^{\infty} \frac{1}{n} q^{-s\deg(v)n} = -\sum_{v} \log(1 - q_v^{-s})$$

and exponentiating both sides gives the desired equality between our two definitions. The purpose of describing adelic zeta functions for global function fields is that we have been able to extend the number field-function field analogy. We have also been able to prove some non-trivial properties about adelic zeta functions in full generality, which we will soon apply to the Hasse-Weil zeta function.

4.6.1 Rationality of the zeta function of a curve

In this section we show the following surprising property of the zeta function, which is the 1-dimensional case of the (now proven) rationality Weil conjecture.

Theorem 15. *Let $\mathcal{C}$ be a curve over $\mathbb{F}_q$ with genus g. The zeta function $\zeta_{\mathcal{C}}(s)$ is a rational function in $t = q^{-s}$ of the form*

$$\frac{L_{\mathcal{C}}(t)}{(1-qt)(1-t)}$$

where $L_{\mathcal{C}} \in \mathbb{Z}[x]$ has degree $2g$.

The proof of this theorem is just a computation using the formula

$$\zeta_{\mathcal{C}}(s) = \sum_{n=0}^{\infty} |\mathrm{PDiv}^n(K)|\, q^{-ns}$$

We will carry out this computation first for the projective line $\mathbb{P}^1(\mathbb{F}_q)$, and then for a general curve $\mathcal{C}$ over $\mathbb{F}_q$.

For $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{F}_q)$, given a fixed $n \geq 0$, we note that the positive divisors $\mathcal{D}$ of degree n with $v_\infty(\mathcal{D}) = 0$ are in bijection with monic polynomials, and so there are q^n divisors of degree n with this constraint. The positive divisors of degree n with $v_\infty(\mathcal{D}) = k$ are in bijection with those of degree $n-k$ (by adding or subtracting $k[v_\infty]$), so there are q^{n-k} such polynomials. Summing up across $0 \leq v_\infty(\mathcal{D}) \leq k$, we see that

$$|\mathrm{PDiv}(K)| = 1 + q + \ldots + q^n = \frac{q^{n+1}-1}{q-1}$$

We can thus compute the zeta function in this case as

$$\zeta_{\mathbb{P}^1}(s) = \sum_{n=0}^{\infty} q^{-ns}\frac{q^{n+1}-1}{q-1} = \sum_{n=0}^{\infty} \frac{q^{n(1-s)} - q^{-ns}}{q-1} = \frac{1}{q-1}\left(\frac{q}{1-q^{1-s}} - \frac{1}{1-q^{-s}}\right) = \frac{1}{(1-q^{1-s})(1-q^{-s})}$$

This verifies the result for $\mathbb{P}^1$ of genus 0.

In the general case, we count the number of positive divisors in each class of the divisor classes $|\mathscr{C}^n(K)|$. For each class, every $\alpha \in L(D) \setminus \{0\}$ yields a positive divisor $D + (\alpha)$, and two elements give rise to the same divisor if and only if they are $\mathbb{F}_q^*$ multiples of one another. Thus if $\mathcal{D}$ is a positive divisor, the number of positive divisors in its class is $|L(\mathcal{D}) \setminus \{0\}| \,/\, |\mathbb{F}_q^*| = (q^{l(\mathcal{D})} - 1)/(q-1)$. We thus have the explicit formula

$$\zeta_{\mathcal{C}}(s) = \sum_{n=0}^{\infty} q^{-ns} \sum_{\mathcal{D} \in \mathscr{C}^n(K)} \frac{q^{l(\mathcal{D})}-1}{q-1}$$

When $\deg(\mathcal{D}) > 2g-2 = \deg(\mathcal{K})$ we have $L(\mathcal{K}-\mathcal{D}) = 0$, and so $l(D) = n+1-g$ in these cases, and so $l(\mathcal{D}) \geq \max(0, n+1-g)$ with equality for $n > 2g-2$. We thus have

$$\begin{aligned}
\zeta_{\mathcal{C}}(s) &= \sum_{n=0}^{\infty} q^{-ns} \sum_{\mathcal{D}\in\mathscr{C}^n(K)} \frac{q^{l(\mathcal{D})}-1}{q-1} \\
&= \sum_{n=0}^{2g-2} q^{-ns} \sum_{\mathcal{D}\in\mathscr{C}^n(K)} \frac{q^{l(\mathcal{D})}-q^{\max(0,n+1-g)}}{q-1} + \sum_{n=0}^{\infty} q^{-ns} \sum_{\mathcal{D}\in\mathscr{C}^n(K)} \frac{q^{\max(0,n+1-g)}-1}{q-1} \\
&= \sum_{n=0}^{2g-2} q^{-ns} \sum_{\mathcal{D}\in\mathscr{C}^n(K)} \frac{q^{l(\mathcal{D})}-q^{\max(0,n+1-g)}}{q-1} + q^{-gs} \sum_{n=g}^{\infty} q^{-(n-g)s} h \frac{q^{(n-g)+1}-1}{q-1} \\
&= \zeta_{\mathbb{P}^1}(s) \left((1-q^{1-s})(1-q^{-s}) \sum_{n=0}^{2g-2} q^{-ns} + hq^{-gs} \right)
\end{aligned}$$

Since $\zeta_{\mathbb{P}^1}(s)$ is exactly the denominator as stated in the theorem, this shows that the theorem holds for the polynomial

$$L_{\mathcal{C}}(t) = (1-qt)(1-t) \sum_{n=0}^{2g-2} \frac{q^{l(\mathcal{D})}-q^{\max(0,n+1-g)}}{q-1} t^n + ht^g$$

as $q-1 \mid q^{l(\mathcal{D})} - q^{\max(0,n+1-g)}$ since $l(\mathcal{D}) \geq \max(0, n+1-g)$, so these coefficients are indeed integral. From the expression $\zeta_{\mathcal{C}}(s) = \frac{L_{\mathcal{C}}(q^{-s})}{(1-q^{1-s})(1-q^{-s})}$, writing the roots of $L_{\mathcal{C}}$ as $\omega_1, \ldots, \omega_{2g}$, we see that the Riemann Hypothesis (the statement that the zeroes of $\zeta_{\mathcal{C}}$ lie on $\mathrm{Re}(s) = 1/2$) is equivalent to saying all zeroes ω of $L_{\mathcal{C}}$ have $|\omega| = q^{-1/2}$, as $|q^{-s}| = q^{-\mathrm{Re}(s)}$.

We can also view the fact that all zeta functions $\zeta_{\mathcal{C}}$ are a polynomial multiple of $\zeta_{\mathbb{P}^1}$ as saying that every curve only differs "finitely much" from the base case (projective line). The numerator $L_{\mathcal{C}}(t)$ of this rational expression carries a significant amount of information about the curve, such as the class number in its value at 1, and twice the genus in its degree. There is likely no such analogue or form in the number field case, but if it did it would certainly be deep and insightful.

4.6.2 Analytic continuation

From the perspective that the zeta function of a curve is a rational function in q^{-s}, most statements in the following theorem are somewhat clear, and could be derived without knowledge of adeles. In any case, its proof is independent of the simple form of the zeta function, and using only the theory developed for an arbitrary adelic zeta function, we have the following theorem.

Theorem 16. *Let $\mathcal{C}$ be a curve defined over $\mathbb{F}_q$. Its corresponding zeta function $\zeta_{\mathcal{C}}$ has a meromorphic continuation with simple poles at $q^s = 1, q$, with residues $-h/(q-1)\log(q)$ and $hq^{1-g}/(q-1)\log(q)$, and the formula*

$$q^{(g-1)s}\zeta_{\mathcal{C}}(s)$$

is invariant under the transformation $s \mapsto 1-s$.

Proof. The meromorphic continuation and their residues follow from the general adelic case as ρ^0 and $\mathcal{F}\rho^0 = q^{1-g}\rho^{-\mathcal{K}}$ are 1 and q^{1-g} respectively at 0. To check the functional equation, we see that as $\deg(\mathcal{K}) = 2g-2$,

$$\zeta_{\mathcal{C}}(s) = \zeta_{\mathcal{C}}(q^{1-g}\rho^{-\mathcal{K}}, 1-s) = q^{1-g}\prod_v \zeta_v(\rho_v^{-k_v}, s) = q^{1-g}\prod_v \frac{q_v^{-k_v s}}{1-q_v^{-s}} = q^{1-g+(2-2g)s}\prod_v \frac{1}{1-q_v^{-s}}$$

and multiplication by $q^{(g-1)s}$ gives the desired formula. ∎

4.7 Values of zeta functions

To find analogies between the number field and global function field cases, we compare values of their corresponding zeta functions.

4.7.1 Analytic class number formula

For $L/\mathbb{Q}$ with r_1 embeddings $L \to \mathbb{R}$, r_2 embeddings $L \to \mathbb{C}$, class number h, regulator $R = \mathrm{Reg}(L)$ (given exactly by $\mathrm{Reg}(L) = \left|\det(n_i \log|\sigma_i(\eta_j)|)_{i,j=1}^{r+s-1}\right|$ where $\eta_1, \ldots, \eta_{r+s-1}$ generate the free part of $\mathcal{O}_L^*$, and viewed as a measure of how "dense" the units are in L), discriminant Δ_L and w the number of roots of unity in L, the corresponding zeta function ζ_L has a zero of order $r_1 + r_2 - 1$ at $s = 0$ and a pole at $s = 1$, with values

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\zeta_L(s) = -\frac{hR}{w} \qquad \lim_{s\to 1}(s-1)\zeta_L(s) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|\Delta_L|}}$$

The residue at $s = 1$ as above is usually referred to the analytic class number formula, and carries many pieces of important data about L. For a curve $\mathcal{C}$ over $\mathbb{F}_q$ with rational function field K and Hasse-Weil zeta function $\zeta_{\mathcal{C}}$, we have found that

$$\lim_{s\to 0} s\zeta_{\mathcal{C}}(s) = -\frac{h}{(q-1)\log(q)} \qquad \lim_{s\to 1}(s-1)\zeta_{\mathcal{C}}(s) = \frac{hq^{1-g}}{(q-1)\log(q)}$$

We see that these pairs of values are highly analogous, and we immediately see that the class numbers correspond to one another, and w to $q - 1 = \left|\mathbb{F}_q^*\right|$. Since we have $g = 1 + \deg(\mathcal{K})/2$, we see that $q^{1-g} = \prod_v q_v^{-k_v/2}$. Recalling that $k_v = d(v/P)$ for finite v and so $N(\mathfrak{D}_{v/P}) = q_v^{k_v}$, we see that

$$\prod_{v\nmid\infty} q_v^{-k_v/2} = \prod_{v\nmid\infty} N(\mathfrak{D}_{v/P})^{-1/2} = \frac{1}{\sqrt{N(\mathfrak{D}_{K/\mathbb{F}_q(t)})}}$$

corresponds to the $1/\sqrt{|\Delta_K|}$ term, and that the remaining product over infinite valuations corresponds to $2^{r_1}(2\pi)^{r_2}$. The q^{1-g} term corresponds to the product of all prime terms $2^{r_1}(2\pi)^{r_2}/\sqrt{|\Delta_L|}$, and we can also see this as these are the factors omitted in going from the value at $s = 1$ to that at $s = 0$. This thus gives a correspondence between the Euler characteristic $2 - 2g$ and $\log_q(2^{r_1}(2\pi)^{r_2}/\sqrt{|\Delta_L|})$. This analogy however has the flaw that the genus of $\mathcal{C}$ then corresponds to something dependent on the constant field $\mathbb{F}_q$, but the genus of the curve $\mathcal{C}$ stays the same regardless of the choice of constant field.

This leaves the non-sensical correspondence between the regulator R and $1/\log(q)$, as the density of units $\mathcal{O}_K^*$ in $K/\mathbb{F}_q(t)$ is not only dependent on the base field $\mathbb{F}_q$, but also on the valuations above ∞ in K. However, another issue with the correspondences outlined is that the orders of the zeroes at 0 do not match up. This comes as the Dedekind zeta function only considers the finite information corresponding to prime ideals $\mathfrak{p} \subseteq \mathcal{O}_L$, and for a better analogy (and analytic class number formula) we can also restrict the Hasse-Weil zeta function to the finite primes, giving

$$\zeta_K(s) := \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \zeta_{\mathcal{C}}(s)\prod_{v|\infty}(1 - q_v^{-s}) = \frac{L_{\mathcal{C}}(q^{-s})}{(1-q^{1-s})(1-q^{-s})}\prod_{v|\infty}(1 - q_v^{-s})$$

where the first 2 formulae hold for $\mathrm{Re}(s) > 1$ and the last globally. This then instead has a zero of order $s_\infty - 1$ (where s_∞ is the number of infinite valuations on K), in direct analogy to ζ_L. For the corresponding values, we find that

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \mathrm{Res}_1\,\zeta_K = \left[\frac{L_{\mathcal{C}}(q^{-s})}{q^{1-s}\log(q)(1-q^{-s})}\prod_{v\mid\infty}(1-q_v^{-s})\right]_{s=1} = \frac{hq^{1-g}\prod_{v\mid\infty}(1-q_v^{-1})}{(q-1)\log(q)}$$

From this residue we have the more reasonable correspondence between R and $[\prod_{v\mid\infty}(1-q_v^{-1})]/\log(q)$. This however does not carry over to the value at 0 as in the number field case, as we instead have

$$\lim_{s\to 0} s^{-(s_\infty-1)}\zeta_K(s) = \frac{h}{1-q}\lim_{s\to 0}\frac{1}{(1-q^{-s})/s}\prod_{v\mid\infty}\frac{1-q_v^{-s}}{s} = -\frac{h\prod_{v\mid\infty}\frac{d}{ds}(-q_v^{-s})|_{s=0}}{(q-1)\frac{d}{ds}(-q^{-s})|_{s=0}} = -\frac{h\prod_{v\mid\infty}\log(q_v)}{(q-1)\log(q)}$$

A minor difference which arises from the direct comparison between ζ_L and $\zeta_{\mathcal{C}}$ is that the analytic class number formulas of the various zeta functions. On an adelic level, for an adelic zeta function $\zeta(f,s)$, its residues in the number field case are $-\kappa f(0)$ at $s=0$ and $\kappa\mathcal{F}f(0)$, where κ is the analytic class number formula as above, which also corresponds to the volume of a multiplicative fundamental domain of $K^* \subseteq \ker|\cdot|$.

A similar formula holds for adelic zeta functions of function fields, but with $\kappa = h/(q-1)\log(q)$. This also does not have the same connection with volume, as the multiplicative fundamental domain $\ker|\cdot|/K^*$ instead has volume $h/(q-1)$ in the function field case. From these values, we can also see that the adelic functions f whose adelic zeta functions are the Dedekind or Hasse-Weil zeta functions also differ slightly at these values. This describes the separation between absolute and relative properties which is possible in the function field case, as the differing properties have different values in the function field case.

4.7.2 Zeroes of the zeta functions

By looking at the Euler products and noting that $1/(1-a^{-s}) = 1 + 1/(a^s-1)$, we see that the Dedekind and Hasse-Weil zeta function have no zeroes for $\mathrm{Re}(s) > 1$. By the functional equations for each of these, we also see that aside from a family of trivial zeroes, all zeroes of each zeta function lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

For $L \supsetneq \mathbb{Q}$, the Dedekind zeta function has a set of "trivial zeroes" at the non-positive integers (or non-positive even integers if all embeddings $L \hookrightarrow \mathbb{C}$ have image in $\mathbb{R}$), and this changes to the negative even integers when $L = \mathbb{Q}$. Similarly, for a curve $\mathcal{C}$, the Hasse-Weil zeta function ζ_K over finite primes has trivial zeroes at integer multiples of $s = \frac{2\pi i}{\log(q_v)}$ for each $v \mid \infty$, except for at integer multiples of $\frac{2\pi i}{\log(q)}$ if there is only 1 valuation $v \mid \infty$.

Chapter 5

Geometry over $\mathbb{F}_q$

We outline a geometric perspective to the theory we have developed thus far, and give a proof of the Riemann hypothesis for curves over $\mathbb{F}_q$, due to Bombieri. We assume now that the projective curve $\mathcal{C}$ over $\mathbb{F}_q$ is *geometrically irreducible* (that is, irreducible when viewed over $\overline{\mathbb{F}}_q$) and for simplicity that $\mathcal{C}$ is non-singular.

5.1 A geometric view and point-counting

Recalling that a curve $\mathcal{C}$ in n variables can be viewed as a subset of the affine space $\overline{\mathbb{F}}_q^n$, we can define the Frobenius map $\phi_q : \overline{\mathbb{F}}_q^n \to \overline{\mathbb{F}}_q^n$ by taking q-th powers pointwise. This restricts to a map on $\mathcal{C}$, as for the polynomials $m_i(x_1, \ldots, x_n)$ defining $\mathcal{C}$, we have $m_i(x_1^q, \ldots, x_n^q) = m_i(x_1, \ldots, x_n)^q = 0$, and so $\phi_q(x) \in \mathcal{C}$ for $x = (x_1, \ldots, x_n) \in \mathcal{C}$. We denote the fixed points of ϕ_q^k on $\mathcal{C}$ by $\mathcal{C}(\mathbb{F}_{q^k})$, as these are the points on $\mathcal{C}$ defined over the field $\mathbb{F}_{q^k}$.

We have mostly been viewing valuations of the curve $\mathcal{C}$ algebraically, as corresponding to prime ideals or embeddings of the field K into $\mathbb{F}_q(t)_\infty$. The following theorem allows us to instead interpret these valuations geometrically, in terms of the curve $\mathcal{C}$.

Theorem 17. *Let $\mathcal{C}$ be a curve with rational function field K. Then there is a bijection*

$$\{\textit{valuations on } K\} \leftrightarrow \big\{\textit{orbits of } \phi_q \textit{ on } \mathcal{C}(\overline{\mathbb{F}}_q)\big\}$$

given by sending v to the orbit $\big\{x \in \mathcal{C}(\overline{\mathbb{F}}_q) \mid f(x) = 0 \text{ for all } v(f) > 0\big\}$, and the orbit $\big\{\phi_q^n(x) \mid n \in \mathbb{Z}\big\}$ to the valuation v_x. Under this bijection, we have $\deg(v) = \big|\big\{\phi_q^n(x) \mid n \in \mathbb{Z}\big\}\big| = [K(v) : \mathbb{F}_q]$.

Thus we can instead view each valuation v as an orbit of $\deg(v)$ points on the curve $\mathcal{C}(\overline{\mathbb{F}}_q)$ under the Frobenius map ϕ_q, which for finite valuations can be viewed as the roots of the associated polynomial P. The purpose of having this geometric view is the following surprising connection between the number of points on $\mathcal{C}(\mathbb{F}_{q^n})$ and the roots of the polynomial $L_{\mathcal{C}}(t)$ associated to the zeta function $\zeta_{\mathcal{C}}$.

Proposition 4. *Let $\mathcal{C}$ be a curve with genus g, and set $N_{\mathcal{C}}(n)$ to be the number of points on $\mathcal{C}(\mathbb{F}_q)$. Denoting the roots of $L_{\mathcal{C}}$ by $\omega_1, \ldots, \omega_{2g}$, we have*

$$N_{\mathcal{C}}(n) = q^n + 1 - \sum_{j=1}^{2g} \omega_j^{-n}$$

Proof. We note that $\zeta_{\mathcal{C}}$ can be written as

$$\exp\left(\sum_{n=1}^{\infty}\frac{N_{\mathcal{C}}(n)}{n}q^{-ns}\right)=\zeta_{\mathcal{C}}(s)=\frac{L_{\mathcal{C}}(q^{-s})}{(1-q^{1-s})(1-q^{-s})}$$

Taking logarithmic derivatives and noting that $L_{\mathcal{C}}(t)=k\prod_{j=1}^{2g}(t-\omega_j)$ for some positive $k\in\mathbb{Z}$, we see that

$$\sum_{n=1}^{\infty}N_{\mathcal{C}}(n)q^{-ns}=-\frac{1}{\log(q)}\frac{\zeta_{\mathcal{C}}'}{\zeta_{\mathcal{C}}}(s)=-\sum_{j=1}^{2g}\frac{q^{-s}}{\omega_j-q^{-s}}+\frac{q^{-s}}{1-q^{-s}}+\frac{q^{1-s}}{1-q^{-s}}=\sum_{j=1}^{\infty}\left(q^n+1-\sum_{j=1}^{2g}\omega_j^{-n}\right)q^{-ns}$$

Equating the coefficients of each q^{-ns} term gives the desired result. □

Thus by the above proposition, we can reason about the roots of $L_{\mathcal{C}}$ by point-counting, and this will be essential in proving the Riemann hypothesis for $\mathcal{C}$.

5.2 The Riemann Hypothesis for $\mathcal{C}$

In this section we give a proof of the Riemann hypothesis as formulated for $\mathcal{C}$. Given the analogies we have described previously, it may come as a surprise that the Riemann hypothesis remains open in the number field case. This speaks to the fact that all provided proofs rely heavily on point counting and geometry over $\overline{\mathbb{F}}_q$, something which, as of now, has no good corresponding number field analogy.

The following lemma gives a sufficient condition for the Riemann hypothesis. Recall that the Riemann hypothesis is equivalent to the statement that all roots ω of $L_{\mathcal{C}}$ have $|\omega|=q^{-1/2}$, as the zeroes of $\zeta_{\mathcal{C}}(s)$ come from $L_c(q^{-s})$.

Lemma 13. *If for every $\varepsilon>0$ there is $m\in\mathbb{N}$ and $C>0$ such that*

$$|N_{\mathcal{C}}(nm)-q^{nm}-1|\leq Cq^{nm(1/2+\varepsilon)}$$

for all n, then the Riemann hypothesis holds for $\zeta_{\mathcal{C}}$.

Proof. Let $\varepsilon>0$. We claim that $\mathrm{Re}(\omega_j^{nm})\geq\frac{1}{2}\left|\omega_j^{nm}\right|$ for infinitely many n. To show this, we use a simultaneous version of Dirichlet's approximation theorem: For any real numbers $\alpha_1,\ldots,\alpha_d$ and $N\in\mathbb{N}$, there are $p_1,\ldots,p_d,q\in\mathbb{Z}$ with $1\leq q\leq N$ such that

$$|q\alpha_i-p_i|\leq\frac{1}{N^{1/d}}$$

Setting $d=2g$, $\alpha_i=\arg(\omega_v)/2\pi$ and $N=(6mn)^{2g}$ for each $k\in\mathbb{N}$, we find infinitely many k and corresponding $q_k,p_1,\ldots,p_d$ such that

$$|q_k\arg(\omega_v)-2\pi p_i|<\frac{\pi}{3mk}$$

Exponentiating, we see that this corresponds to $\cos(\arg(\omega_v^{q_kkm}))>\frac{1}{2}$, so $\mathrm{Re}(\omega_v^{q_kmk})>\frac{1}{2}\left|\omega_v^{q_kmk}\right|$, and we get infinitely many such distinct q_kk.

Having established this, we see that for each such n we have

$$Cq^{nm(1/2+\varepsilon)}\geq|N_{\mathcal{C}}(nm)-q^{nm}-1|\geq\mathrm{Re}\left(\sum_{j=1}^{2g}\omega_j^{nm}\right)\geq\frac{1}{2}\sum_{j=1}^{2g}|\omega_j|^{-nm}\geq\frac{1}{2}\max_{1\leq j\leq 2g}|\omega_j|^{nm}$$

Taking $-1/(mn)$ powers and letting $n \to \infty$, we see that $|\omega_j| \leq q^{-1/2-\varepsilon}$. Taking $\varepsilon \to 0$, we see that $|\omega_j| \leq q^{-1/2}$ for all j. By the functional equation for $\zeta_{\mathcal{C}}$, if $|\omega_j| < q^{-1/2}$ for some j, then $|\omega_k| > q^{-1/2}$ for some other k, so every root must have $|\omega_j| = q^{-1/2}$, so the Riemann hypothesis holds. □

5.2.1 The Frobenius action on $\mathcal{C}$

The remainder of the argument used to prove the Riemann hypothesis relies heavily on point-counting and the geometry of $\mathcal{C}$. To study this, for suitable powers $Q = q^n$ we look at two subsets of $\mathcal{C} \times \mathcal{C}$, namely the diagonal

$$\Delta = \{(x, x) \mid x \in \mathcal{C}\}$$

and the graph of the $Q = q^n$-power map $\phi_Q := \phi_q^n$

$$\phi_Q(X) := \{(x, \phi_Q(x)) \mid x \in \mathcal{C}\}$$

The intersection between these graphs gives the points defined over $\mathbb{F}_{q^n}$, and there are $N_{\mathcal{C}}(n)$ of these. Suppose that there is at least 1 point on this intersection, and call this point (∞, ∞), and its associated valuation v_∞.

Consider the space of functions on $\mathcal{C}$ defined over $\mathbb{F}_Q$ with possibly only a pole of order at most m at ∞. We can intuitively think of these as polynomials, and these form an $\mathbb{F}_Q$-vector space, namely $L_m := L(m[v_\infty])$. Letting g be the genus of $\mathcal{C}$, by the Riemann-Roch theorem, we have $l_m = m + 1 - g + l(\mathcal{K} - m[v_\infty])$, and so $l_m \geq m + 1 - g$, with equality where $m > 2g - 2$. We also have $l_m \leq l_{m+1} \leq l_m + 1$, as $L_m \subseteq L_{m+1}$, and for $f, g \in L_{m+1}$ with $f \notin L_m$, we can find $\lambda \in \mathbb{F}_q$ such that $g - \lambda f \in L_m$. Thus we also see that $l_m \leq m + 1$.

Suppose that $s_1, \ldots, s_{l_m}$ is a basis for L_m with increasing pole orders, i.e. $v_\infty(s_{i+1}) < v_\infty(s_i)$. For $k \geq 0$, we can consider the space of functions on $\mathcal{C} \times \mathcal{C}$ defined over $\mathbb{F}_Q$ of the form

$$f(x, y) = \sum_{i=1}^{n} a_i(x) s_i(y)$$

where $a_i \in L_k$.

For each function f in this space, we can get a function on $\phi_Q(X)$ defined over $\mathbb{F}_Q$ by taking the restriction $f|_\phi(x) = f(x, \phi_Q(x))$. The restriction map is $\mathbb{F}_Q$-linear, and we have $s_i(\phi_Q(x)) = s_i(x)^Q$, and hence $f|_\phi \in L_{k+mQ}$. To provide an upper bound for the number of points $N_{\mathcal{C}}(n)$, we will want a function whose restriction to Δ is zero, but its restriction to $\phi_Q(X)$ is non-zero. The below lemma tells us the dimension of this space in the case where $k < Q$.

Lemma 14. *If $k < Q$, the restriction map from functions on $\mathcal{C} \times \mathcal{C}$ defined over $\mathbb{F}_Q$ to functions on the graph of $\phi_Q(X)$ defined over $\mathbb{F}_Q$ is injective, and so an isomorphism onto its image.*

Proof. When $k < Q$, suppose that $f(x, y) = \sum_{i=0}^{l_m} a_i(x) s_i(y)$ satisfies $f|_\phi = 0$. Then $\sum_{i=0}^{l_m} a_i s_i^Q = 0$ in K. Considering the pole at ∞, if $a_i \neq 0$, then for $j < i$ we find that

$$v_\infty(a_i s_i^Q) \leq Q v_\infty(s_i) \leq -Q + Q v_\infty(s_j) < -k + Q v_\infty(s_j) \leq v_\infty(a_j s_j^Q)$$

Thus the highest non-zero term is not cancelled by lower terms, and so must vanish. The only way this is possible is if there is no such term, and so $f = 0$. □

To make the restriction map $f \mapsto f|_\phi$ an isomorphism, we choose values so that $k < Q$. It will be convenient to make the coefficients a_i p^μ-th powers for some $p^\mu \mid q$, so suppose that $a_i = b_i^{p^\mu}$ where $b_i \in L_n$ for some n to be chosen later, subject to the constraint $k = p^\mu n < Q$. The space of functions $f|_\phi$ thus has dimension $l_n l_m$, with a basis given by the pure tensors of p^μ-powers of a basis of L_n with a basis of L_m.

We can also restrict f to the diagonal by taking $f|_\Delta(x) = f(x,x)$, and chaining this with the inverse of $f \mapsto f|_\phi$ gives a linear map $f|_\phi \mapsto f|_\Delta$. A map f with $f|_\phi \neq 0$ but $f|_\Delta = 0$ is then a non-zero element of the kernel of this map, so we choose n, m so that this map has non-zero kernel.

Before choosing values of n and m, we first see where such an function f can get us. We have $N_\mathcal{C}(n)$ intersections between Δ and $\phi_Q(X)$. Since $f|_\phi$ is a p^μ-th power and its only pole is (∞, ∞), this function has at least $p^\mu(N_\mathcal{C}(1) - 1)$ zeroes (counting multiplicity).

We also know that the number of zeroes is the same as the number of poles of $f|_\phi$, and the pole has order at most $p^\mu n + mQ$, so we find that

$$p^\mu(N_\mathcal{C}(1) - 1) \leq \#\text{ zeroes of } f|_\phi \leq p^\mu n + mQ$$

Hence we see that $N_\mathcal{C}(1) \leq 1 + n + mQp^{-\mu}$. Since we require $n < Qp^{-\mu}$ from above, we see that $1 + n \leq qp^{-\mu}$, and so $N_\mathcal{C}(1) \leq Qp^{-\mu}(m+1)$.

For such a function to exist, it suffices to choose n and m so that $l_n l_m \geq l_{p^\mu n + m}$, as every $f|_\Delta$ has possibly only a pole at ∞ of order at most $p^\mu n + m$. As $l_n \geq n + 1 - g$ and similar for m, it thus suffices to choose n and m such that

$$(n + 1 - g)(m + 1 - g) > l_{p^\mu n + m}$$

Letting $n, m \geq g$, we have $p^\mu n + m > 2g - 2$, so $l_{p^\mu n + m} = p^\mu n + m + 1 - g$. This constraint then gives

$$(m + 1 - g - p^\mu)(n - g) = (m + 1 - g)(n + 1 - g) - p^\mu(n - g) - (m + 1 - g) > p^\mu g$$

For the strongest bound we want m as small as possible, and hence $n < Qp^{-\mu}$ as large as possible. We thus set $n = Qp^{-\mu} - 1$, and so rearranging our above equation yields

$$m + 1 > p^\mu + g + \frac{p^\mu g}{Qp^{-\mu} - 1 - g}$$

Multiplication by $Qp^{-\mu}$ yields

$$Qp^{-\mu}(m+1) > q + g\left(\frac{q}{p^\mu} + \frac{p^\mu}{1 - (g+1)/(Qp^{-\mu})}\right)$$

Taking q and μ to be such that $(g+1) < qp^{-\mu}$, this yields the bound

$$Qp^{-\mu}(m+1) > q + g(Qp^{-\mu} + p^\mu) \geq q + 2g\sqrt{q}$$

with equality in the latter inequality when $Qp^{-\mu} = p^\mu$. We thus set $Q = p^{2\mu}$, so we take Q to be a perfect square. Returning to our above lower bound for $m + 1$, we see that

$$m + 1 > \sqrt{Q} + g + \frac{g\sqrt{Q}}{\sqrt{Q} - (g+1)} \geq \sqrt{Q} + g + \frac{g(g+1)}{\sqrt{Q} - (g+1)}$$

where the latter inequality follows as $\sqrt{Q} = n + 1 \geq l_n + 1 \geq g + 1$. From this above equation we see that when $q > (g+1)^4$, $g(g+1) \leq (g+1)((g+1)^3 - 1)$, and so it suffices to take $m = \sqrt{Q} + 2g$. This computation shows the following upper bound on $N_\mathcal{C}(n)$

Theorem 18. *For g the genus of $\mathcal{C}$ and $Q = q^n > (g+1)^4$ a square, we have*

$$N_\mathcal{C}(n) = |\mathcal{C}(\mathbb{F}_Q)| \leq Q + (2g+1)\sqrt{Q}$$

5.3 Galois covers and the Riemann Hypothesis

We still need a lower bound for $N_{\mathcal{C}}(n)$, and to attain this we use a "nice" cover of $\mathcal{C}$ to "flip" the inequality we found above. Before introducing the exact notions we work with, we give some definitions and results for context.

Intuitively, a covering map $f : \mathcal{C}_1 \twoheadrightarrow \mathcal{C}_2$ between curves is a a surjective map which can be described (at least locally) by polynomials, which preserves pointwise data and where each point $x \in \mathcal{C}_2$ has same size fibre $f^{-1}(\{x\})$. The curve $\mathcal{C}_1$ is called a cover of $\mathcal{C}_2$, and such a cover induces a morphism $K(\mathcal{C}_2) \hookrightarrow K(\mathcal{C}_1)$ between the associated function fields by the pullback $g \mapsto g^* = g \circ f$, and so we have an associated field extension $K(\mathcal{C}_1)/K(\mathcal{C}_2)$. We say that this cover is Galois if this associated extension of function fields is Galois.

Over $\overline{\mathbb{F}}_q$, we have a correspondence between extensions of function fields in 1 variable and algebraic curves up to birational equivalence, and finite extensions correspond to Galois coverings. This gives a Galois correspondence between function fields and Galois coverings, where inclusions of function fields correspond to Galois coverings between the associated curves.

Given our curve $\mathcal{C}$, we can thus find a curve $\mathcal{C}'$ whose function field is the Galois closure of $K(\mathcal{C})/\mathbb{F}_q(t)$, and we have associated Galois coverings $\mathcal{C}' \twoheadrightarrow \mathcal{C} \twoheadrightarrow \mathbb{P}^1$. Letting $G = \mathrm{Gal}(K(\mathcal{C}')/\mathbb{P}^1)$, for $\sigma \in G$ we define

$$N_{\mathcal{C}'}(1, \sigma) := \left|\left\{x \in \mathcal{C}'(\overline{\mathbb{F}}_Q)\right\} \mid x \text{ projects to } \mathbb{P}^1(\mathbb{F}_Q), \phi_Q(x) = \sigma(x)\right|$$

As in deriving the upper bound, we have the following theorem.

Theorem 19. *For g' the genus of $\mathcal{C}'$ and $Q = q^n > (g'+1)^4$ a square, we have*

$$N_{\mathcal{C}}(n, \sigma) \leq Q + (2g'+1)\sqrt{Q}$$

Proof. Similarly to above, for f on $\mathcal{C}' \times \mathcal{C}'$ defined over $\mathbb{F}_Q$ we have maps $f \mapsto f|_\phi$ and $f \mapsto f|_\sigma$, where $f|_\sigma(x) = f(x, \sigma(x))$. When $f|_\sigma$ vanishes, $f|_\phi$ vanishes at points counted by $N_{\mathcal{C}'}(1, \sigma)$, and so applying the same argument to $\mathcal{C}'$ and $f|_\phi \mapsto f|_\sigma$ gives the bound. □

We can thus use this to achieve the lower bound, as shown in the following theorem.

Theorem 20. *Let $\mathcal{C}$ be a curve. Then $\mathcal{C}$ satisfies the Riemann Hypothesis: If $\zeta_{\mathcal{C}}(s) = 0$, then $Re(s) = 1/2$.*

Proof. Let $\mathcal{C}'$ be as above. Since $\mathcal{C}'/\mathbb{F}_q$ is Galois, for each $t \in \mathbb{P}^1(\mathbb{F}_Q)$, there are $|G|/e$ valuations which restrict to it, and for each valuation there are e automorphisms which reduce to the Frobenius map on the residue class field. Thus, in the sum $\sum_{\sigma \in G} N_{\mathcal{C}'}(1, \sigma)$, every $t \in \mathbb{P}^1(\mathbb{F}_Q)$ is counted $|G|$ times. We have $\left|\mathbb{P}^1(\mathbb{F}_Q)\right| = Q + 1$, and so

$$\sum_{\sigma \in G} N_{\mathcal{C}'}(n, \sigma) = |G|(Q+1)$$

Letting $H \subseteq G$ be the set of covering transformations which act trivially on $\mathcal{C}$, we have that

$$\sum_{\sigma \in H} N_{\mathcal{C}'}(n, \sigma) = |H| N_{\mathcal{C}}(n)$$

Given $\tau \in G$, applying the bound of the above theorem for $\sigma \neq \tau$ yields

$$N_{\mathcal{C}'}(n,\tau) = |G|\,(Q+1) - \sum_{\sigma\neq\tau} N_{\mathcal{C}'}(n,\sigma) \geq |G|\,(Q+1) - (|G|-1)(Q+(2g'+1)\sqrt{Q}) = |G| + Q + (2g'+1)\sqrt{Q}$$

Hence $N_{\mathcal{C}}(n) \geq |G| + q^n + (2g'+1)q^{n/2}$, and so by Lemma 13, choosing m so that $q^m > (g+1)^4, (g'+1)^4$ is a square, we obtain the Riemann Hypothesis for $\mathcal{C}$. □

An immediate corollary of this is the Hasse-Weil bound $|N_{\mathcal{C}}(n) - q^n - 1| \leq 2gq^{n/2}$, as since we have $|\omega_j| = q^{-1/2}$ for the roots ω_j of $L_{\mathcal{C}}$, we find that

$$|N_{\mathcal{C}}(n) - q^n - 1| = \left|\sum_{j=1}^{2g} \omega_j^{-n}\right| \leq 2gq^{n/2}$$

Unlike many of the analogies we have previously given, much of the geometry we have described for global function fields in this section is not easily translated into the number field case.

Chapter 6

Conclusions

We outline some more unmentioned analogies and comparisons between number field and global function fields, with a tabular summary of these analogies in Appendix 2. We then discuss possible future directions to extend this analogy.

6.1 Comparisons between number fields and global function fields

6.1.1 $\mathbb{Q}$ and $\mathbb{F}_q(t)$

For these “base cases” for the number field and global function field cases, it makes sense to talk about a minimal Dedekind domain, and the usual choices are $\mathbb{Z}$ and $\mathbb{F}_q[t]$ respectively. Both of these natural Dedekind domains are in fact Euclidean rings, and thus unique factorisation domains.

In the number field case, the ring $\mathbb{Z}$ is the unique minimal subring with fraction field $\mathbb{Q}$, and $\mathbb{Q}$ the smallest subfield of itself. To constrast, in the function field case, such a choice is not unique, with isomorphic, non-equal Dedekind domains of the form $\mathbb{F}_q[1/(t+a)]$ for $a \in \mathbb{F}_q$ with same fractional field, and isomorphic, (in general) proper subfields of the form $\mathbb{F}_r[p(t)]$ for $r \mid q$ and $p \in \mathbb{F}_r[t]$ non-constant. This is also reflected in their automorphism groups, with $\mathbb{Q}$ having a unique endomorphism and trivial automorphism group, while an automorphism of $\mathbb{F}_q(t)$ may consist of an automorphism of $\mathbb{F}_q$ and $t \mapsto (at+b)/(ct+d)$ non-constant (any function with a single simple zero and a single simple pole), and with infinitely many endomorphisms which are not automorphisms.

In these regards $\mathbb{F}_q(t)$ is much more free than $\mathbb{Q}$, and although $\mathbb{F}_q(t)$ is the minimal global function field up to isomorphism, no truly minimal example of a global function field exists in the set theoretic sense. The choice of generator for a minimal Dedekind domain also corresponds to the fact that non-constant rational functions cannot be regular everywhere on a projective curve, and so in this regard the choice of $(t+a)^{\pm 1}$ corresponds to choosing a pole in $\mathbb{P}^1(\mathbb{F}_q)$.

6.1.2 General field extensions

As number fields have characteristic zero, all number fields are separable over $\mathbb{Q}$, and in a unique way as every number field $K \neq \mathbb{Q}$ is non-isomorphic. In contrast, $K/\mathbb{F}_q(t)$ is not necessarily separable (for example $\mathbb{F}_q(t)/\mathbb{F}_q(t^p)$), but can be made separable by considering $K/\mathbb{F}_q(u)$ for suitable $u \in K$. The ability to make such a choice describes that $K/\mathbb{F}_q(t)$ is dependent on the choice of $t \in K$, with the exact extension and properties such as its degree, automorphism group, intermediate field extensions, separability and normality vary depending on the exact extension chosen.

In the function field case, we are able to have both constant field and transcendental extensions (with respect to $\mathbb{F}_q$). The field of constants is equivalently the roots of unity and zero, so a constant field extension corresponds to just increasing the number of roots of unity. For comparison, there is no known analogue of a constant field in any $K \supseteq \mathbb{Q}$, and so no notion of a constant field, let alone a transcendental extension with respect to one. The analogous set $\mu_K \cup \{0\}$ is also not a subfield, being only closed multiplicatively.

In the function and number field cases we have the subring $\mathcal{O}_K$ - the regular functions or the ring of integers in K. This is defined initially as the integral closure of $\mathbb{Z}$ or $\mathbb{F}_q[t]$, but can also be viewed the set of $x \in K$ with $v_{\mathfrak{p}}(x) \geq 0$ for all finite $\mathfrak{p}$. This is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$ for $K/\mathbb{Q}$, and a free $\mathbb{F}_q[t]$-module of rank $[K : \mathbb{F}_q(t)]$ for $K/\mathbb{F}_q(t)$ separable. It is Dedekind in both cases, minimal in both the number field and function field cases. However, similar to as outlined above, the choice of minimal Dedekind domain depends on the choice of $t \in K$, and is in general not unique, with the integral closures of $\mathbb{F}_q[1/(t+a)]$ being isomorphic, non-equal minimal Dedekind domains.

For general subrings R of a global field K, we also see that every ideal $I \subseteq R$ has finite index, and the proof of this property reduces to the fact that this holds for the unique factorisation domains $\mathbb{Z}$ and $\mathbb{F}_q[t]$.

The common motifs we can see across these analogies are that global function fields have absolute and relative properties, while no such separation or analogue happens in the number field case. We can see that constant fields are absolute, while properties of the extensions $K/\mathbb{F}_q(t)$ and $\mathcal{O}_K/\mathbb{F}_q[(t+a)^{\pm 1}]$ depend on an initial choice, and so are relative.

6.1.3 Numerics

To each valuation v on $K/\mathbb{F}_q(t)$, we have its associated degree $\deg(v) = [K(v) : \mathbb{F}_q(t)]$. For $P \in \mathbb{F}_q[t]$ irreducible we have $\deg(v_P) = -v_\infty(P)$, and in general we have $\deg(v_{\mathfrak{p}}) = \log_q([\mathcal{O}_K : \mathfrak{p}])$ for $\mathfrak{p} \subseteq \mathcal{O}_K$ prime. From this perspective, the analogue of $\deg(v)$ for the number field case is $\log(N(\mathfrak{p}))$ for a finite prime $\mathfrak{p} \subseteq \mathcal{O}_K$, and this is heuristically supported by the prime number theorem, which says that a number n is prime with asymptotic probability $1/\log(n)$, and so the numbers $\log(p)$ will ultimately appear in roughly even spacings. However, this perspective does not give a clear analogue for the infinite primes $K \hookrightarrow \mathbb{C}$. We can view this as an artifact of the fact that $K/\mathbb{Q}$ has $r_1 + r_2$ archimedean places in general, for which the notions of the local ring of integers and the corresponding unique prime ideal are reduced to only being multiplicative subsets.

In the number field case, the completion $\mathbb{R}$ at the infinite prime is a quadratic extension $(\mathbb{C}/\mathbb{R})$ away from being algebraically closed. This is far from the case for the field $\mathbb{F}_q(t)_\infty = \mathbb{F}_q((1/t))$ which plays a similar role in the function field case, where, for example, adjoining the n-th roots $\sqrt[n]{t}$ give differing, albeit isomorphic extensions.

6.1.4 Geometry

In our above arguments we have used, to a great extent, the view of global function fields $K/\mathbb{F}_q(t)$ as rational functions acting on a curve $\mathcal{C}$. The key fact underpinning this is the fact that each global function field has a natural field of constants, namely $\mathbb{F}_q$. Our proof of the Riemann Hypothesis for geometrically irreducible, non-singular curves relies heavily on the view that the curve $\mathcal{C}$ defined by the polynomials generating $K/\mathbb{F}_q(t)$ can be viewed as restrictions of subsets of affine space $\overline{\mathbb{F}}_q^n$.

In order to get a proof of the Riemann Hypothesis by tweaking the argument for $K/\mathbb{F}_q(t)$, the naive approach would be to first find the analogue of a constant field F for which $\mathbb{Q}/F$ has transcendence degree 1. As in the function field case, we would want all residue fields $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ to be extensions of F, and so in some regard we would have to designate this field as "$F = \mathbb{F}_1$", since 1 is the only positive integer dividing every prime. We may then expect that the extension $\mathbb{Q}$ "$=$" $\mathbb{F}_1(t)$ will depend on $t \in \mathbb{Q}$, and that we will be able to use this perspective to separate the absolute and relative properties of a number field $K/\mathbb{Q}$. In outlining this, it is clear that the analogue $\mathbb{F}_1$ "$\subseteq$" $\mathbb{Q}$ of $\mathbb{F}_q \subseteq \mathbb{F}_q(t)$ will most certainly not be a field, and will likely involve much more advanced tools to construct and define.

With a geometric view on $K/\mathbb{F}_1$, the Riemann-Roch theorem for number fields should then read along the lines of $l(\mathcal{D}) = \deg(\mathcal{D}) + 1 - g + l(\mathcal{K} - \mathcal{D})$. The degree of a divisor $\mathcal{D}$ will likely be transcendental in all cases (with $\deg(v_p) = \log(p)$), and we would like $g = 1 + \deg(\mathcal{K})/2$. To define the canonical divisor in a way inspired by the case of a curve $\mathcal{C}/\mathbb{F}_q$, the differential exponents $d(\mathfrak{p}/p)$ make sense for $\mathfrak{p} \subseteq K$, but neither $d(v/\infty)$ nor $e(v/\infty)$ make sense for the infinite valuations.

6.1.5 Zeta functions

Alongside the comparisons we have made with the values of zeta functions, the associated factors which make up the completed zeta functions

$$\zeta^*_{\mathcal{C}}(s) = q^{(g-1)s} \prod_v \frac{1}{1 - q_v^{-s}}$$
$$\zeta^*_K(s) = (|\Delta_K|^{1/2} \pi^{-r_1/2} (2\pi)^{r_2})^s \left[\Gamma(s/2)^{r_1} [2\Gamma(s)]^{r_2} \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}} \right]$$

can be matched up, with the preceding factors $q^{(g-1)s}$ and $(|\Delta_K|^{1/2} \pi^{-r_1/2} (2\pi)^{r_2})^s$ corresponding to an extra term for each of the valuations.

As shown above, the Hasse-Weil zeta function is a rational function in q^{-s}. This can be viewed in some sense to be from the fact that all norms on $\mathbb{F}_q(t)$ only take values in $\langle q \rangle \cup \{0\}$ (and so the same holds for $K/\mathbb{F}_q(t)$ on rescaling the norms by some power). To contrast, there are no such common factors in the real case, and so no clear choice of what variable the Dedekind zeta function should be a rational function in. While relatively unlikely, the existence of such a rational expression would likely be an important bridge to a proof of the Riemann Hypothesis, and give rise to a point-counting argument similar to that outlined previously.

We can also express the Dedekind zeta function in a way analogous to the point-counting definition of the Hasse-Weil zeta function, with

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \exp\left(-\sum_{\mathfrak{p}} \log(1 - N(\mathfrak{p})^{-s}) \right) = \exp\left(\sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{N(\mathfrak{p})^{-ns}}{n} \right)$$

When expressed in the form $\exp\left(\sum_{n=1}^{\infty} \frac{N_K(n)}{n} n^s\right)$, the coefficients $N_K(n)$ are the number of prime ideal powers $\mathfrak{p}^k$ with norm n if $n = p^l$ is a prime power, and zero otherwise. From this perspective, the geometry of $K/\mathbb{Q}$ is likely much different to that of $K/\mathbb{F}_q(t)$, with this suggesting that all "extensions of $\mathbb{F}_1$" have prime power degree.

6.2 Concluding remarks

We have described adele theory and adelic zeta functions for global function fields to provide emphasis on the number field and global field analogy. These were applied to the Hasse-Weil zeta functions of curves, but the results associated adelic zeta functions are much more general and often times stronger than would be expected from these Hasse-Weil zeta functions alone. We have proved that these Hasse-Weil zeta functions are rational of a specific form, satisfy a functional equation corresponding to "symmetry" about $\mathrm{Re}(s) = 1/2$, and satisfy the analogue of the Riemann Hypothesis. These are cases of the more general, now-proven Weil conjectures, which state that these hold for more general geometric objects defined by polynomials.

The global function field case which we have focused on sits at the crossroad between arithmetic (pertaining to number fields) and geometric (pertaining to curves, surfaces and manifolds). There is an equally good geometric analogy, though its content would be different in flavour. We have been able to make use of the tools from both perspectives to describe properties which may not be known in either the solely arithmetic or solely geometric case. In a broader view, the global function field case acts as a bridge between the arithmetic and geometric worlds, with arithmetic properties being easier to transfer to the global function field case than directly to the geometric case, and vice versa.

The analogy between arithmetic, mixed (global function field) and geometric is a rich and active field of research, with proposals such as in the Langlands program seeking to unify these perspectives. Such a global unified theory would likely speak not of a solely arithmetic or geometric case, but rather treat these all as one and the same.

6.2.1 Future directions

To delve deeper into the analogy between the two types of global field, one could look at open questions in the number field case, and check whether or not such a property holds in the global function field case. An example of this is the Birch-Tate conjecture for number fields, which conjectures the values of $|\zeta_K(-m)|$ for $m \in \mathbb{N}$ in terms of the K-groups of $\mathcal{O}_K$.

It would also be interesting to see a definition of a zeta function in the purely geometric case (say over $\mathbb{C}(t)$) analogous to that of the global field cases, albeit possibly not very insightful or meaningful. The definition of the Riemann-zeta function can be extended to that of a ring-theoretic zeta function of a noetherian commutative ring R by setting

$$\zeta_R(s) := \prod_{I \text{ maximal}} \frac{1}{1 - [R : I]^{-s}}$$

and a similar thing can be done for the Hasse-Weil zeta function and zeta functions of schemes. These definitions only make sense when the residue fields of every maximal $I \subseteq R$ are finite, and so in particular do not extend to the the geometric case where the base field is infinite.

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Chapter 7

References

BOMBIERI, E., (2010), *The Classical Theory of Zeta and L-Functions*, Milan J. Math. vol. 78, 11-59.

CASSELS, J., (1967), *Global Fields, in: Algebraic Number Theory (J.W.S. Cassels, A. Fröhlich, eds.)*, Academic Press, New York, 42 - 84.

GATHMANN, A., (2022), *Algebraic Geometry*, online at *https://www.mathematik.uni-kl.de/ gathmann/class/alggeom-2021/alggeom-2021.pdf*

GLEASON, J., (2010), *Existence and Uniqueness of Haar Measure*, online at *http://www.math.uchicago.edu/˜may/VIGRE/VIGRE2010/REUPapers/Gleason.pdf*

GOLDFELD, D., HUNDLEY, J., (1967), *Automorphic Representations and L-Functions for the General Linear Group*, Cambridge Studies in Advanced Mathematics 129, Cambridge University Press.

LANG, S., (1970), *Algebraic Number Theory*, Graduate Texts in Mathematics 110, Springer.

LICHTENBAUM, S., (1973), *Values of zeta-functions, étale cohomology, and algebraic K-theory*, in: Bass, H. (eds) "Classical" Algebraic K-Theory, and Connections with Arithmetic. Lecture Notes in Mathematics, vol 342. Springer, Berlin, Heidelberg. *https://doi.org/10.1007/BFb0073737*, 487-501.

STEVENHAGEN, P., (2019), *Number rings*, online at *https://websites.math.leidenuniv.nl/algebra/ant.pdf*

SU, D., (2016), *The Fourier Transform for Locally Compact Abelian Groups*, online at *http://math.uchicago.edu/ may/REU2016/REUPapers/Su.pdf*

TAO, T., (2014), *The Bombieri-Stepanov proof of the Hasse-Weil bound*, online at *https://terrytao.wordpress.com/2014/05/02/the-bombieri-stepanov-proof-of-the-hasse-weil-bound/*

TATE, J., (1967), *Fourier analysis in number fields and Hecke's zeta-functions, in: Algebraic Number Theory (J.W.S. Cassels, A. Fröhlich, eds.)*, Academic Press, New York, 305–347.

VAN FRANKENHUYSEN, M., (2014), *The Riemann Hypothesis for Function Fields*, London Mathematical Society Student Texts 80, Cambridge University Press.

Chapter 8

Appendices

8.1 Appendix 1: The proof that finitely many valuations ramify in $K/\mathbb{F}_q(t)$

It actually suffices to prove that only finitely many *finite* valuations ramify in $K/\mathbb{F}_q(t)$, since there are only finitely many infinite valuations.

The variant of the Kummer-Dedekind theorem we apply is stated as follows. The original statement for number rings can be found in [Stevenhagen].

Theorem 21. *Let $f \in (\mathbb{F}_q[t])[x]$ be a monic irreducible polynomial (in x), $\alpha \in \overline{\mathbb{F}_q(t)}$ be a root of f and $P \in \mathfrak{q} = \mathbb{F}_q[t]$ an irreducible polynomial. Let $R = \mathbb{F}_q[t,\alpha] = \mathbb{F}_q[t][x]/(f(x))$, and suppose that $\overline{g}_i \in \mathfrak{q}(P) = [\mathbb{F}_q[t]/(P(t))]$ are monic, irreducible and pairwise distinct polynomials such that $\overline{f} \in \mathfrak{q}(P)[x]$ factors as*

$$\overline{f} = \prod_{i=1}^{k} \overline{g_i}^{e_i}$$

with $e_i \in \mathbb{N}$. Then

1. *The prime ideals of R are the ideals $\mathfrak{p}_i = PR + g_i(\alpha)R$, and we have an inclusion $\prod_{i=1}^{k} \mathfrak{p}_i^{e_i} \subseteq PR$.*
2. *The equality $pR = \prod_{i=1}^{k} \mathfrak{p}_i^{e_i}$ holds if and only if every prime $\mathfrak{p}_i$ is invertible.*
3. *Writing $r_i \in \mathbb{F}_q[t][x]$ for the remainder of f upon division by g_i, we have*

$$\mathfrak{p}_i \text{ is non-invertible} \iff e_i > 1 \text{ and } P^2 \mid r_i$$

The e_i are exactly the corresponding ramification indices $e(\mathfrak{p}_i/P)$, and so a finite prime $\mathfrak{p}_i$ in $K = \mathbb{F}_q(t)[\alpha] = \mathbb{F}_q(t)[x]/(f)$ ramifies if and only if the corresponding factor $g_i(x)$ has multiplicity ≥ 2 in the factorisation of $f(x)$ modulo P. This factor will divide the partial derivative f_x of f, and so when f is separable, f and f' are coprime in $\mathbb{F}_q(t)$ and so we can write $a(t)f(t,x) + b(t)f_x(t,x) = k(t)$, and so we see that $(f \mod P), (f_x \mod P)$ are not coprime for exactly the (finitely many) irreducibles $P \mid k$.

8.2 Appendix 2: The proof of the Poisson summation formula for a global function field

Our proof in the global function field case roughly follows the structure of Tate's thesis. Fix a global function field $K/\mathbb{F}_q(t)$ with degree $n = [K : \mathbb{F}_q(t)]$, and suppose that this extension is separable. We first establish that the diagonal embedding $\alpha \mapsto (\alpha)_v$ of K into $\mathbb{A}_K$ gives K the discrete topology. For this, we construct an analogue of an additive fundamental domain of K in $\mathbb{A}_K$. Denote by S_∞ the set of valuations $v \mid \infty$, and $(\mathbb{A}_K)_{S_\infty} := \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} \times \prod_{v|\infty} K_v$.

Lemma 15. *Under $K \hookrightarrow \mathbb{A}_K$, we have*

1. $K \cap (\mathbb{A}_K)_{S_\infty} = \mathcal{O}_K$
2. $K + (\mathbb{A}_K)_{s_\infty} = \mathbb{A}_K$

Proof. (1) is just the statement that $x \in K$ is integral if and only if it is integral at each prime $\mathfrak{p} \subseteq K$.

For (2), we want to show that for every $x \in \mathbb{A}_K$, there is $y \in K$ with $y - x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$. For each $\mathfrak{p}$ with $v_{\mathfrak{p}}(x) < 0$, choose $\xi_1 \in \mathcal{O}_K$ with $x_{\mathfrak{p}}\xi_1 \in \mathcal{O}_{\mathfrak{p}}$ and write $\xi_2 \equiv x_{\mathfrak{p}}\xi_1 \mod \mathfrak{p}^a$ where $a = v_{\mathfrak{p}}(\xi_1) > v_{\mathfrak{p}}(\xi_1 x_{\mathfrak{p}})$. Then by the Chinese remainder theorem, we have a solution $\xi_2 \in \mathcal{O}_K$ to these congruences, and so for $y = \xi_2/\xi_1$ has $x_{\mathfrak{p}} - y \in \mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \subseteq \mathcal{O}_K$. □

The set $K^\infty := \prod_{v|\infty} K_v$ is naturally a $\mathbb{F}_q(t)_\infty$-vector space of degree

$$\sum_{v|\infty} \dim_{\mathbb{F}_q(t)_\infty}(K_v) = \sum_{v|\infty} e(v/\infty) f(v/\infty) = n$$

We also have

Lemma 16. *If $\{\omega_1, \ldots, \omega_n\}$ is a basis for $\mathcal{O}_K$ over $\mathbb{F}_q[t]$, then their images under the diagonal embedding $\mathcal{O}_K \hookrightarrow K^\infty$ are a basis for K^∞ over $\mathbb{F}_q(t)_\infty$.*

Proof. The embedding $\mathcal{O}_K \hookrightarrow (\overline{K_v})^n$ sending

$$\alpha \mapsto (\sigma(\alpha))_{\sigma \in \mathrm{Hom}_{\mathbb{F}_q(t)}(K, \overline{K_v})}$$

has image in $(\overline{K})^n$, and factors through K^∞ as $\mathcal{O}_K \hookrightarrow K^\infty \hookrightarrow (\overline{K_v})^n$, where the former map is the diagonal embedding and the latter map is given by sending the coordinate corresponding to v to $(\sigma_v(x))_{\sigma_v \in \mathrm{Hom}_{\mathbb{F}_q(t)_\infty}(K_v, \overline{K_v})}$. The image of each $\omega_i \in \mathcal{O}_K$ under these embeddings have image in $\overline{K}$, and in $(\overline{K_v})^n$ we find that

$$\Delta_K = \Delta_{\mathcal{O}_K/\mathbb{F}_q[t]}(x_1, \ldots, x_n) = \det(\sigma(\omega_j))_{1 \le j \le n}^{\sigma \in \mathrm{Hom}_{\mathbb{F}_q(t)}(K, \overline{K_v})} \neq 0$$

Hence $\omega_1, \ldots, \omega_n$ are linearly independent in $(\overline{K_v})^n$, and so form a basis in K^∞. □

We define the following analogue of a parallelotope in Euclidean space for our above embedding.

Definition 23. *For the basis $\{\omega_1, \ldots, \omega_n\}$ as above, the corresponding* fundamental domain in K^∞ *is*

$$D^\infty := \left\{ \sum_{i=1}^n f_i \omega_i \in K^\infty \;\middle|\; \deg(f_i) < 0 \right\}$$

This choice corresponds to having $0 \le |f_i|_\infty < 1$, similar to the usual parallelotope in Euclidean space with respect to a basis. We extend this to a global additive fundamental domain by taking $\mathcal{O}_\mathfrak{p}$ at the finite valuations.

Definition 24. *The* additive fundamental domain $D \subseteq \mathbb{A}_K$ *is*

$$D := \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p} \times D^\infty$$

Note that D^∞ has a finite measure as a parallelotope in K^∞, and hence $D \subseteq \mathbb{A}_K$ also has finite measure, given by $\mu(D^\infty)\prod_\mathfrak{p} N(\mathfrak{D}_{\mathfrak{p}/p})^{-1/2}$. The following theorem says that calling the subset D an additive fundamental domain is reasonable.

Theorem 22. *We have*

$$\mathbb{A}_K = \coprod_{y \in K} (y + D)$$

Proof. Given $x \in \mathbb{A}_K$, for existence we can add an element $y \in K$ so that $x + y \in (\mathbb{A}_K)_{S_\infty}$. We can then add an element of $\mathcal{O}_K$ so that the infinite components lie in D^∞, by writing the coefficient of ω_i as $f_i = g_i/h_i$, applying division with remainder to get $g_i = q_i h_i + r_i$ and subtracting the quotients $q_i\omega_i$.

For uniqueness, if $x = x_1 + y_1 = x_2 + y_2$ for $x_i \in K$ and $y_i \in D$, then $t = x_1 - x_2 = y_2 - y_1$ is in $D \cap \mathcal{O}_K$. Writing $t = \sum_{i=1}^n f_i\omega_i$, we require $f_i \in \mathcal{O}_K$ and $\deg(f_i) < 0$, so each $f_i = 0$ and $t = 0$. □

We thus have the following theorem.

Theorem 23. *The diagonal embedding $K \hookrightarrow \mathbb{A}_K$ gives K the discrete topology, and $\mathbb{A}_K/K$ is compact.*

Proof. Notice that D is compact and has an interior (namely itself), and so K must be discrete and the quotient $\mathbb{A}_K/K$ compact. □

It follows from this that the Haar measure on K is given by the counting measure, where each point has measure 1.

The remaining results are proved in a way identical to that in Tate's thesis, except that the knowledge that $\mu(D) = 1$ is not known until the Poisson summation formula is proven. Their proofs are thus omitted, and can be found in Tate's thesis as lemmas 4.2.2, 4.2.3 and 4.2.4.

Lemma 17. *If $\varphi(x)$ is continuous and periodic, then $\int_D \varphi(x)dx = \int_{\mathbb{A}_K/K} \varphi(x)dx$.*

The Fourier transform $\mathcal{F}\varphi(y)$ of a continuous function on $\mathbb{A}_K/K$ is then

$$\mathcal{F}\varphi(y) = \frac{1}{\mu(D)} \int_D \varphi(x)\chi(yx)dx$$

Since $K \subseteq \mathbb{A}_K$ is discrete, the Fourier transform and Fourier inversion formula translate to the following lemma.

Lemma 18. *If $\varphi(x)$ is continuous, periodic, and satisfies $\sum_{x\in K} |\mathcal{F}\varphi(x)| < \infty$, then*

$$\varphi(x) = \frac{1}{\mu(D)} \sum_{y \in K} \mathcal{F}\varphi(x)\chi(yx)$$

We then have the following lemma relating a function f on $\mathbb{A}_K$ to its restriction to $\mathbb{A}_K/K$ by summing over K.

Lemma 19. *If $f \in L^1(\mathbb{A}_K)$ is continuous and $\sum_{\eta \in K} f(x+\eta)$ is uniformly convergent for $x \in D$, then for the continuous periodic function $\varphi(x) = \sum_{\eta \in K} f(x+\eta)$ we have $\mathcal{F}\varphi(\eta) = \mathcal{F}f(\eta)$.*

Applying Lemma 19 in conjunction with Lemma 18 with $x = 0$, we obtain the Poisson summation formula:

Theorem 24. *If $f : \mathbb{A}_K \to \mathbb{C}$ is such that*

1. *$f \in L^1(\mathbb{A}_K)$ is continuous*
2. *The map $\varphi(y) = \sum_{x \in K} f(y+x)$ is uniformly convergent for $x \in D$*
3. *$\sum_{x \in K} |\mathcal{F}f(x)|$ is convergent*

then

$$\frac{1}{\mu(D)} \sum_{x \in K} \mathcal{F}f(x) = \sum_{x \in K} f(x)$$

Iteration of this formula using the Fourier inversion formula yields $\mu(D)^2 = 1$, and so we find that $\mu(D) = 1$. Since $\mu(D) = \mu(D^\infty) \prod_{\mathfrak{p}} N(\mathfrak{D}_{\mathfrak{p}/P})^{-1/2}$, it follows then that $D^\infty = \prod_{\mathfrak{p}} N(\mathfrak{D}_{\mathfrak{p}/P})^{1/2}$. In Tate's thesis this volume is calculated to be $\sqrt{|\Delta_K|}$, and this computation tells us that the field discriminant only takes into account the finite primes in both cases. From the numerics we have calculated previously, this is why we match up q^{1-g} to the terms corresponding to all primes, rather than just $1/\sqrt{|\Delta_K|}$.

8.3 Appendix 3: Summary of comparisons between number fields and global function fields

We summarise the connections we have outlined previously, omitting some details where they can be inferred from context.

Property	**Number fields,** $K/\mathbb{Q}$	**Global function fields,** $K/\mathbb{F}_q(t)$
	Field Theory	
	Initial objects F	
Fields F	$\mathbb{Q}$	$\mathbb{F}_q(t)$
Minimality	Yes	$\mathbb{F}_q(t) \supseteq \mathbb{F}_r(P(t))$
Minimal Dedekind domain $\mathcal{O}_F$	$\mathbb{Z}$	$\mathbb{F}_q[t]$
Uniqueness of domain	Yes	$\mathbb{F}_q[(at+b)/(ct+d)]$
	Norms	
Trivial	$\lvert x\rvert_{\text{Triv}} = 1$	$\lvert x\rvert_{\text{Triv}} = 1$
p-adic	$\lvert up^k\rvert_p = \lvert\mathbb{Z}/p\mathbb{Z}\rvert^{-k}$	$\lvert UP^k\rvert = \lvert\mathbb{F}_q[t]/f\mathbb{F}_q[t]\rvert^{-k}$
Infinite	$\lvert n\rvert_\infty = \lvert\mathbb{Z}/n\mathbb{Z}\rvert$	$\lvert f\rvert = \lvert\mathbb{F}_q[t]/f\mathbb{F}_q[t]\rvert$
Other properties	Euclidean algorithm, unique factorisation, principal ideals	
	General objects K/F	
Separabilty of K/F	Yes	Depends on extension
$[K:F]$, $\mathrm{Aut}(K/F)$, $K/E/F$	Fixed	Depends on extension
$\mathcal{O}_K$	Well-defined	Depends on $\mathbb{F}_q[(t+a)^{\pm 1}]$
Isomorphism determines K/F	Yes	No
Non-trivial endomorphisms	No	Yes
Constant field	None	$\mathbb{F}_q$
	Norms	
Trivial	$\lvert x\rvert_{\text{Triv}} = 1$	$\lvert x\rvert_{\text{Triv}} = 1$
$\mathfrak{p}$-adic, $\mathfrak{p} \subseteq \mathcal{O}_K$, $\mathfrak{p}_\mathfrak{p} = \pi_\mathfrak{p}\mathcal{O}_\mathfrak{p}$	$\lvert u\pi_\mathfrak{p}^k\rvert_\mathfrak{p} = \lvert\mathcal{O}_K/\mathfrak{p}\rvert^{-k}$	$\lvert u\pi_\mathfrak{p}^k\rvert_\mathfrak{p} = \lvert\mathcal{O}_K/\mathfrak{p}\rvert^{-k}$
Infinite	Induced by $K \hookrightarrow \mathbb{C}$	Induced by $K \hookrightarrow \overline{\mathbb{F}_q(t)_\infty}$
	Units	
Roots of unity μ_K $\lvert\mu_K\rvert$	$\mu \cap K$ w_K	$\mathbb{F}_q^*$ $\lvert\mathbb{F}_q^*\rvert = q-1$
Unit group $\mathcal{O}_K^*$	$\mu_K \times \mathbb{Z}^{r_1+r_2-1}$	$\mathbb{F}_q^* \times \mathbb{Z}^{s_\infty - 1}$
	Class groups	
Pre-quotient objects	Fractional ideals, $\mathcal{I}(\mathcal{O}_K)$	Divisors, $\mathrm{Div}^0(K)$
Class group $\mathscr{C}(K)$	$\mathcal{I}(\mathcal{O}_K)/\mathcal{P}(\mathcal{O}_K)$	$\mathrm{Div}^0(K)/\mathcal{P}(K)$
Explicit finiteness bound	Minkowski bound on ideal norms, $N(I) \le \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n}\sqrt{\lvert\Delta_K\rvert}$	Number of positive divisors, $\deg(\mathcal{D}) \le g$

Numerics		
Class number	$h = \lvert \mathscr{C}(K) \rvert$	
Degree of (non-arch) valuation	$\log(N(\mathfrak{p}))$	$[K(v) : \mathbb{F}_q]$
Regulator	$\lvert \det(n_i \log \lvert \sigma_i(\eta_j) \rvert)_{i,j=1}^{r+s-1} \rvert$	$[\prod_{v \mid \infty}(1 - q_v^{-1})] / \log(q)$
Norm of $\mathfrak{D}_{K/F}$	$\lvert \Delta_K \rvert$	$\prod_{v \mid \infty} q_v^{-k_v/2}$
Euler characteristic	$\log_q(2^{r_1}(2\pi)^{r_2}/\sqrt{\lvert \Delta_L \rvert})$	$2 - 2g$
Zeta functions		
Standard sum expression	$\sum_{0 \neq I \subseteq \mathcal{O}_K} N(I)^{-s}$	$\sum_{\mathcal{D} \geq 0} q^{-s \deg(\mathcal{D})}$
Euler product with all primes	$\zeta_{K/\mathbb{Q}}(s) = \Gamma(s/2)^{r_1}[2\Gamma(s)]^{r_2} \prod_{\mathfrak{p}} \frac{1}{1-N(\mathfrak{p})^{-s}}$	$\zeta_{\mathcal{C}}(s) = \prod_v \frac{1}{1-q_v^{-s}}$
Analytic class number formula	$\frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{\lvert \Delta_K \rvert}}$	$\frac{hq^{1-g}\prod_v(1-q_v^{-1})}{(q-1)\log(q)}$
Value at zero	$-\frac{hR}{w}$	$-\frac{h\prod_v \log(q_v)}{(q-1)\log(q)}$
Completed zeta functions	$\Lambda_{K/\mathbb{Q}}(s) = (\lvert \Delta_K \rvert^{1/2} \pi^{-r_1}(2\pi)^{-r_2})^s \zeta_{K/\mathbb{Q}}(s)$	$\Lambda_{\mathcal{C}}(s) = q^{(g-1)s}\zeta_{\mathcal{C}}(s)$
Rationality	Unknown	Yes
Functional equation	$\Gamma_{K/\mathbb{Q}}(s) = \Gamma_{K/\mathbb{Q}}(1-s)$	$\Gamma_{\mathcal{C}}(s) = \Gamma_{\mathcal{C}}(1-s)$
Adelic analytic class num. formulas (0; 1)	$-\frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{\lvert \Delta_K \rvert}} f(0)$; $\frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{\lvert \Delta_K \rvert}} \mathcal{F}f(0)$	$-\frac{h}{(q-1)\log(q)} f(0)$; $\frac{h}{(q-1)\log(q)} \mathcal{F}f(0)$
Geometry		
Affine space	Unknown	$\overline{\mathbb{F}}_q^n$
Point counting	Unknown	$N_{\mathcal{C}}(n) = \lvert \mathcal{C}(\mathbb{F}_{q^n}) \rvert$
Valuations	Unknown	Orbit of $\deg(v)$ points on $\mathcal{C}(\overline{\mathbb{F}}_q)$