A comparison theorem on de Rham cohomology for regular algebraic vector bundles

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For my parents.

Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

We give a complete presentation of André-Baldassarri's proof of Deligne's classical comparison theorem between algebraic and complex-analytic de Rham cohomology. Their proof differs mainly in the development and usage of algebraic techniques to deduce the theorem, avoiding the use of Hironaka's resolution of singularities altogether.

We begin in the algebraic setting, by introducing $\mathcal{O}_X$-modules and algebraic vector bundles with connection. We then introduce the inverse image and relative de Rham cohomology functors, and prove basic results on flat base change and vanishing. We then prove a key lemma on dévissage, from which we deduce results on coherence, arbitrary base change and regularity for regular algebraic vector bundles with connection.

We then turn to the analytic setting, introducing the same objects over a complex-analytic variety. We use the theory of complex-local systems and the Riemann-Hilbert correspondence to deduce the same results on coherence and arbitrary base change for analytic vector bundles with connection. We finish with the proof of the comparison theorem, which uses dévissage-style arguments to the (absolute) de Rham cohomology of a curve, which is handled easily using formal comparison criteria.

Contents

Notation and terminology

Notation

$\mathcal{O}_X$	The structure sheaf of a ringed space C
$X \times_S Y$	The fibre product of ringed spaces X and Y over S
$\mathbb{A}^n$	The affine space $\mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n]$ over $\mathbb{C}$
$\mathbb{P}^n$	The projective space $\mathrm{Proj}(\mathbb{C}[x_0, \dots, x_n])$ over $\mathbb{C}$
$\mathbb{A}^n_S$	The affine space $\mathbb{A}^n \times S$ of dimension n over a scheme S.
$\mathbb{P}^n_S$	The projective space $\mathbb{P}^n \times S$ of dimension n over a scheme S.
$X \times Y$	The product of algebraic (resp. analytic) varieties as ringed spaces over $\mathrm{Spec}(\mathbb{C})$ (resp. the point).
$\mathrm{Mod}(\mathcal{R})$	The category of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings.
$\mathcal{H}om_{\mathcal{R}}(-, -)$	The internal Hom of modules over a sheaf of rings $\mathcal{R}$.
$\Omega^n_{X/S}$	The n^{th} wedge power of the relative Kähler differentials $\Omega^1_{X/S}$, where $X \to S$ is a morphism of schemes.
Ω^n_X	The n^{th} wedge power of the absolute Kähler differentials Ω^1_X, relative to $X \to \mathrm{Spec}(\mathbb{C})$, where X is a scheme.
Der_{R_2/R_1}	The ring of R_1-linear derivations $R_2 \to R_2$, for an R_1-algebra R_2.
$\mathcal{D}er_{\mathcal{R}_1}(\mathcal{R}_2, \mathcal{F})$	The $\mathcal{R}_1$-linear derivations of a sheaf of algebras $\mathcal{R}_2$, takin values in an $\mathcal{R}_1$-module $\mathcal{F}$
$\mathcal{D}er_{\mathcal{R}_1}(\mathcal{R}_2)$	The derivations $\mathcal{D}er_{\mathcal{R}_1}(\mathcal{R}_2, \mathcal{R}_2)$

$d\omega$	The exterior derivative (induced by the universal (relative) derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$ of the n-form ω (a local section of $\Omega^n_{X/S}$)
$\mathcal{T}_{X/S}$	The dual $(\Omega^1_{X/S})^\vee = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$ of $\Omega^1_{X/S}$, identified with $\mathrm{Der}_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ by precomposition with d.
f^{-1}	The sheaf-theoretic inverse image by f.
f^*	The module-theoretic inverse image by f.
$R^q f_*$	The sheaf-theoretic higher direct image by f.
$j_!$	The extension by zero functor, where j is an open immersion.
$C^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$	The Čech bicomplex associated to the open cover $\mathcal{U}$ and complex $\mathcal{F}^\bullet$.
$X_{S'}$	The base change $X \times_S S'$ of a ringed space X over another ringed space S, by a morphism $S' \to S$ of ringed spaces.
$f_{S'}$	The base change $X_{S'} \to S'$ of a morphism $f : X \to S$ by a morphism $S' \to S$.
$\mathcal{F}\vert_U$	The restriction $i^{-1}\mathcal{F}$ of the sheaf $\mathcal{F}$ to U.
X_s	The fibre $X \times_S \{s\}$ of a map $f : X \to S$, above $s \in S$.
$H^n(X, \mathcal{F})$	The n^{th} cohomology of a sheaf $\mathcal{F}$ on X.
$\Omega^n_{\mathcal{X}}$	The sheaf of holomorphic n-forms on a complex manifold $\mathcal{X}$. (Note that this is sometimes written $\Omega^{n,0}_{\mathcal{X}}$ in the literature.)
$\mathcal{T}_{\mathcal{X}}$	The holomorphic tangent sheaf on a complex manifold $\mathcal{X}$.

Terminology

complex-algebraic variety	A reduced, separated scheme of finite type over $\mathbb{C}$.
complex-analytic variety	A reduced, Hausdorff, complex-analytic space.
complex manifold	A Hausdorff, possibly non-second countable topological space, with an atlas of charts to some $\mathbb{C}^n$, whose transition maps are holomorphic.

topologically trivial (map)	A continuous map $f : X \to S$ where $X \cong S \times F$ for some F, and f corresponds to pr_1 under this homeomorphism.
étale covering	A surjective étale map, in particular possibly non-finite.
Galois covering	An étale covering $X \to S$ so that $\mathrm{Aut}_S(X)$ acts transitively on any geometric fibre $X_{\overline{s}}$
étale cover (of S)	A collection of étale morphisms $\{\varphi_i : V_i \to S\}_{i \in I}$ so that $S = \bigcup_{i \in I} \varphi_i(V_i)$.

Introduction

This thesis studies de Rham cohomology on algebraic and analytic complex varieties, leading up to a proof of a comparison theorem between the two variants. A comparison theorem on cohomology generally states that two cohomology theories agree on a large class of objects, usually by a natural isomorphism. The most classical example is de Rham's theorem (see [41, 5.36]), which relates the de Rham cohomology of a smooth manifold to its singular cohomology with real coefficients.

The comparison theorems we will consider take place at the interface between de Rham cohomology on algebraic and complex-analytic varieties, due to a process known as *analytification*. This assigns, to each algebraic variety X an analytic variety X^{an}, in a functorial manner. In the simplest setting, the *de Rham cohomology* $H^q_{DR}(X)$ of an (algebraic or analytic) complex variety X is the hypercohomology $\mathbb{H}^q(\Gamma(X, \Omega^\bullet_X))$ (see [42, 5.7]) of the complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \dots,$$

with differentials given by the universal derivation in the algebraic case, and exterior derivative in the analytic case. In this setting, we have the following classical theorem of Grothendieck [19, Theorem 1′]

Theorem (Grothendieck, 1966)**.** For any smooth variety X, there is an isomorphism

$$H^\bullet_{DR}(X) \xrightarrow{\cong} H^\bullet_{DR}(X^{an}),$$

which is natural in X.

The setting described above has two notable generalisations. The first is to take *non-constant coefficients*, by replacing $\mathcal{O}_X$ with a vector bundle $\mathcal{O}_X$-module $\mathcal{E}$, and d with a $\mathbb{C}$-linear integrable connection $\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$. Analytification carries over to the case of vector bundles, and we may make sense of $(\mathcal{E}^{an}, \nabla^{an})$ for an algebraic vector bundle $(\mathcal{E}, \nabla)$. The *de Rham cohomology* $H^q_{DR}(X, (\mathcal{E}, \nabla))$

with coefficients in $(\mathcal{E},\nabla)$ is then the hypercohomology $\mathbb{H}^q(\Gamma(X,\Omega_X^\bullet\otimes_{\mathcal{O}_X}\mathcal{E}))$ of the complex

$$0\to\mathcal{E}\xrightarrow{\nabla}\Omega_X^1\otimes_{\mathcal{O}_X}\mathcal{E}\xrightarrow{\nabla^1}\Omega_X^2\otimes_{\mathcal{O}_X}\mathcal{E}\to\dots.$$

The above theorem generalises in the following way, due to Deligne in [10, II.6.2].

Theorem (Deligne, 1970)**.** Let X be a smooth variety, and $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module, with integrable connection $\nabla:\mathcal{E}\to\Omega_X^1\otimes_{\mathcal{O}_X}\mathcal{E}$. We have an isomorphism

$$H_{DR}^\bullet(X,(\mathcal{E},\nabla))\xrightarrow{\cong}H_{DR}^\bullet(X^{an},(\mathcal{E}^{an},\nabla^{an})).$$

This isomorphism is natural in X and the pair $(\mathcal{E},\nabla)$.

Since $\Gamma(X,-)$ is the pushforward by the map $X\to\mathrm{Spec}(\mathbb{C})$, we may then consider the *relative setting*, of a smooth morphism of varieties $f:X\to S$. To give the definition in the relative setting, we replace Γ with f_*, and the absolute differentials $\Omega_X^\bullet$ with the relative differentials $\Omega_{X/S}^\bullet$, and a *relative* $f^{-1}\mathcal{O}_S$ connection $\nabla:\mathcal{E}\to\Omega_{X/S}^1\otimes_{\mathcal{O}_X}\mathcal{E}$ (which is $f^{-1}\mathcal{O}_S$-linear). This yields the complex

$$0\to\mathcal{E}\xrightarrow{\nabla}\Omega_{X/S}^1\otimes_{\mathcal{O}_X}\mathcal{E}\xrightarrow{\nabla^1}\Omega_{X/S}^2\otimes_{\mathcal{O}_X}\mathcal{E}\to\dots,$$

and we set the *relative de Rham cohomology of a vector bundle* $(\mathcal{E},\nabla)$ to be $R_{DR}^qf_*(\mathcal{E},\nabla):=\mathbb{R}^qf_*(\Omega_{X/S}^\bullet\otimes_{\mathcal{O}_X}\mathcal{E})$. The same definition applies in the analytic setting, substituting the appropriate notion of smoothness, and relative differentials. Unlike the first two definitions, the functor $R_{DR}^qf_*$ in the relative setting takes values in $\mathcal{O}_S$-modules with integrable connection, hence the change in notation. The comparison theorem in this setting also requires *regularity* of the connection, which broadly says that the singularities of the connection at infinity are poles, as opposed to essential. Deligne's original proof can be found as [10, II.6.13].

Theorem (Deligne, 1970)**.** Let X and S be smooth algebraic varieties over $\mathbb{C}$, and $f:X\to S$ be a smooth morphism. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module, with a regular integrable connection $\nabla:\mathcal{E}\to\Omega_X^1\otimes_{\mathcal{O}_X}\mathcal{E}$. Then for each $q\geq 0$, there is a dense open subset U_q of the image of f, independent of $\mathcal{E}$, so that we have an isomorphism

$$\varphi^q:(R_{DR}^qf_*(\mathcal{E},\nabla)|_{U_q})^{an}\xrightarrow{\cong}R_{DR}^qf_*^{an}(\mathcal{E}^{an},\nabla^{an})|_{U_q^{an}}\qquad(*)$$

of $\mathcal{O}_{S^{an}}$-modules with integrable connection.

The explicit statement of this theorem is (7.2.1). There are also analogues of these theorems for smooth varieties over $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$, and one may ask whether there is a proof general enough to apply in both of these settings. This led André-Baldassarri to develop an algebraic program as in [1], which not only provides an affirmative answer to this question, but also uses techniques which are interesting in their own right. In the case of just Deligne's theorem ($*$), their proof is sufficiently algebraic as to avoid any use of Hironaka's resolution of singularities, on which Grothendieck's and Deligne's original proofs rely heavily on.

The most notable technique, and most essential to our proof of ($*$), is their lemma on dévissage* (see (3.2.4)). The rough statement of this lemma is that to prove that $\mathcal{P}$ holds generically on the image of† f for the cohomology sheaves $R^q_{DR}f_*$ of any smooth morphism, it suffices to prove it in the case where $X = \mathbb{A}^1_S - \coprod_{i=1}^r \sigma_i(S)$ is the complement of finitely many disjoint sections $\sigma_i : S \to \mathbb{A}^1_S$ in $\mathbb{A}^1_S$, and f is the fibration $\mathrm{pr}_1|_X : \mathbb{A}^1_S - \coprod_{i=1}^r \sigma_i(S) \to S$. In fact, if f is locally a fibration in the sense of (3.1.11), then $\mathcal{P}$ will hold globally for the $\mathcal{O}_S$-modules given by $R^q_{DR}f_*$.

Outline of the thesis

This thesis looks to give a complete account of André-Baldassarri's proof, developing the theory of vector bundles on algebraic and analytic spaces, and proving intermediate results on dévissage, regularity, preservation of coherence, and base change. We rely on topological arguments in the analytic case, which greatly simplifies the exposition, at the cost of not being able to provide the most general statement as above. The Chapters 1 – 5 are purely algebraic, Chapter 6 is purely analytic, while Chapter 7 is a mix of both. We will work almost entirely in the relative setting as described above.

In Chapter 1, we provide background on differential algebra and hypercohomology, and define $\mathbf{MIC}(X)$, the category of $\mathcal{O}_X$-modules with integrable connection, which will be our central objects of study.

In Chapter 2, we define inverse and direct images of $\mathcal{O}_X$-modules. We then give an explicit description of the higher direct images $R^q_{DR}f_*$, and prove results

*This is the French word for *unscrewing*, and refers to reducing statements down to simpler cases.

†That is, on a dense open subset of the image of f.

on flat base change in (2.5.3), and vanishing for higher direct images in (2.6.1) and (2.6.2).

In Chapter 3, we define elementary fibrations, and in particular what it means for such a map to be rational or coordinatised. We then turn to proving the dévissage lemma (3.2.4), giving an explicit description of the dense open subsets described by the above genericness statement.

In Chapter 4, we define and discuss regularity of connections, equipped with the language of Chapter 2.

In Chapter 5, we prove our main results on coherence, base change, and regularity for regular algebraic vector bundles, as theorems (5.4.2), (5.4.3) and (5.4.4). The proofs of coherence and regularity are a nice application of dévissage, where the bulk of the proof goes into dealing with the case of a rational elementary fibration in (5.1.1). For base change, we reduce to the case of a coordinatised elementary fibration using spectral sequences, and deal with this case in (5.2.1).

In Chapter 6, we delve into the analytic setting. The first section two sections are the analogous of Chapters 1 and 2 respectively. In the third section, we recall various GAGA-style results on the analytification functor. The final section is dedicated to proving the coherence (6.4.10) and base change (6.4.12) theorems, which are analogues of the results in Chapter 5. For the proofs of these results, we rely on topological arguments and the theory of local systems.

In Chapter 7, we complete the proof of the comparison theorem (7.2.3) on de Rham cohomology. We start by proving some abstract comparison criteria in the case of $\mathbb{A}^1 - \{\theta_1, \dots, \theta_r\} \to \mathrm{Spec}(\mathbb{C})$, and then combine all our previous results to reduce to this case.

The content of this thesis mostly follows Chapters 3 and 4 of the book [1]. The primary deviations to this are in Chapters 5 for the theorem on coherence, and the étale and smooth base change results of Chapter 6. We will also refer at times to [2], which despite nominally being a second edition of [1], is substantially different.

Chapter 1

Background

We assume the reader is familiar with schemes, general locally ringed spaces, sheaves and sheaf cohomology, up to the level of [22]. We also assume the reader has a working understanding of the basic concepts of category theory and abelian categories, as in Chapters I – IV and VIII in [30]. We also assume basic familiarity with derived functors and basic knowledge of spectral sequences, in the setting of an arbitrary abelian category, as in Chapters 1 – 4, §5.1 and §5.2 of [42].

In this chapter, we define our main objects of interest. We will phrase most of our statements in terms of $\mathcal{O}_X$-modules with connection, rather than $\mathcal{D}$-modules. In the first 5 chapters, we will refer to algebraic varieties as just varieties, since we only begin considering the analytic setting in Chapter 6. In this chapter, X and Y denote smooth $\mathbb{C}$-varieties over another smooth $\mathbb{C}$-variety S.

1.1 Differential algebra

We define various objects which will be central for our later discussion. In the affine case, we have the following.

Definition 1.1.1.

1. A *differential algebra over* $\mathbb{C}$ is a pair (R, ∂), where R is a $\mathbb{C}$-algebra, and $\partial \in \mathrm{Der}_{R/\mathbb{C}}$.

2. A *differential subalgebra* $(R', \partial') \subseteq (R, \partial)$ (over $\mathbb{C}$) consists of a sub-$\mathbb{C}$-algebra $R' \subseteq R$ such that $\partial|_{R'} = \partial'$;

3. A *differential module* (M, ∇_∂) over a differential ring (R, ∂) is a projective R-module M of finite rank, with a $\mathbb{C}$-linear endomorphism ∇_∂ of M, such that for any $f \in R$ and $m \in M$, we have

$$\nabla_\partial(fm) = \partial(f)m + f\nabla_\partial(m).$$

This will be the local case of the analogous definition 1.2.1(3) in the following section, for a fixed tangent vector ∂. There are other definitions we could make in the affine case, but we will not need them, and we will only refer specifically to this definition in the final proof of the comparison theorem.

Definition 1.1.2. Let $\Delta : X \to X \times_S X$ be the diagonal map, and $\mathcal{I}$ be the ideal sheaf of $\Delta(X)$. The *n^{th}-order jet sheaf $\mathcal{P}^n_{X/S}$ on X* is $\Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}^{n+1})$.

Letting $p_1, p_2 : X \times_S X \to X$ be the projections, we note that $\mathcal{P}^n_{X/S}$ has "left" and "right" $\mathcal{O}_X$-module structures given by

$$\mathcal{O}_X \to p_{i*}(\mathcal{O}_{X\times_S X}) \to p_{i*}\Delta_*\Delta^{-1}(\mathcal{O}_{X\times_S X}) \to \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}^{n+1}) = \mathcal{P}^n_{X/S}. \quad (1.1)$$

We will view $\mathcal{P}^n_{X/S}$ as an $\mathcal{O}_X$-module through its left $\mathcal{O}_X$-module structure.

Definition 1.1.3. The *n^{th} jet map* $j^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$ is induced by p_2 in (1.1).

Note that these maps are injective, with left inverse given by

$$\mathcal{P}^n_{X/S} = \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}^{n+1}) \twoheadrightarrow \Delta^{-1}(\mathcal{O}_{X\times_S X}/\mathcal{I}) \cong \mathcal{O}_X. \quad (1.2)$$

We will mostly make use of $\mathcal{P}^1_{X/S}$ combined with the following result, mainly to get around well-definedness issues with non-linearity when taking tensor products.

Proposition 1.1.4 (The sequence of principal parts)**.** *For any $\mathcal{O}_X$-module $\mathcal{E}$, there is a split short exact sequence of right $\mathcal{O}_X$-modules,*

$$0 \to \Omega^1_{X/S} \xrightarrow{\alpha_{X/S}} \mathcal{P}^1_{X/S} \xrightarrow{\beta_{X/S}} \mathcal{O}_X \to 0, \quad \text{(SPP)}$$

with splitting given by $j^1_{X/S} : \mathcal{O}_X \to \mathcal{P}^1_{X/S}$.

For $f : X \to Y$, the splittings as in the above proposition, combined with the map $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ and the equality $f^*\mathcal{O}_Y = \mathcal{O}_X$ induces a map on jets:

$$f^*\mathcal{P}^1_{Y/S} = f^*\mathcal{O}_Y \oplus f^*\Omega^1_{Y/S} \to \mathcal{O}_X \oplus \Omega^1_{X/S} = \mathcal{P}^1_{X/S}. \quad (1.3)$$

This is functorial in the same ways as for differentials. Note that $\mathcal{E}nd_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X)$ is a sheaf of (non-commutative) rings, so pointwise subsets will generate a subsheaf.

Definition 1.1.5. The sheaf of *differential operators $\mathcal{D}iff_{X/S}$ on X relative to S* is the subsheaf of $\mathcal{E}nd_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X)$ generated by $\mathcal{D}er_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathcal{O}_X$ (viewed as multiplication operators). We say $\partial \in \mathcal{D}iff_{X/S}$ has *order at most n* if it factors as $\partial = \overline{\partial} \circ j^n_{X/S}$, for some $\mathcal{O}_X$-linear $\overline{\partial} : \mathcal{P}^n_{X/S} \to \mathcal{O}_X$.

When $S = \mathrm{Spec}(\mathbb{C})$, we set $\mathcal{D}_X := \mathcal{D}iff_{X/\mathbb{C}}$.

The above definition generalises to arbitrary $\mathcal{O}_X$-modules in the following way.

Definition 1.1.6. Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{O}_X$-modules. The *sheaf of differential operators on $\mathcal{E}$ taking values in $\mathcal{E}'$, relative to S* of rank at most n is

$$\mathcal{D}iff^n_{X/S}(\mathcal{E},\mathcal{E}') := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X}, \mathcal{E}'),$$

viewed as a subset of $\mathcal{E}nd_{f^{-1}\mathcal{O}_S}(\mathcal{E},\mathcal{E}')$ by precomposing with $\mathrm{id}_{\mathcal{E}} \otimes j^n_{X/S}$. We set

$$\mathcal{D}iff_{X/S}(\mathcal{E},\mathcal{E}') := \bigcup_{n\geq 0} \mathcal{D}iff^n_{X/S}(\mathcal{E},\mathcal{E}')$$

These definitions also give a notion of the *order* of a differential operator ∂, by taking the minimal such n for which ∂ admits such a factorisation.

1.2 Connections

Definition 1.2.1. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. An *X/S-connection on $\mathcal{E}$* is one of the following equivalent pieces of data.

1. An $f^{-1}\mathcal{O}_S$-linear map $\nabla : \mathcal{E} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ satisfying the Leibniz rule

$$\nabla(fe) = df \otimes e + f\nabla(e)$$

   on local sections f of $\mathcal{O}_X$ and e of $\mathcal{E}$.

2. An $\mathcal{O}_X$-linear section J_∇ of the projection $p : \mathcal{P}^1_{X/S} \to \mathcal{O}_X$ of 1.2; and if X is smooth over S, these are equivalent to

3. An $\mathcal{O}_X$-linear map $\nabla : \mathcal{T}_{X/S} \to \mathcal{E}nd_{f^{-1}\mathcal{O}_S}(\mathcal{E})$, satisfying the Leibniz rule

$$\nabla_\partial(fe) = \partial(f)e + f\nabla_\partial(e)$$

   on local sections ∂ of $\mathcal{T}_{X/S}$, f of $\mathcal{O}_X$, and e of $\mathcal{E}$.

An *$\mathcal{O}_X$-module with X/S-connection* is a pair $(\mathcal{E},\nabla)$ where ∇ is an X/S-connection on an $\mathcal{O}_X$-module $\mathcal{E}$.

The equivalence between (1) and (3) uses the $\otimes$-$\mathcal{H}om$ adjunction, and that $\mathcal{T}_{X/S}$ is locally free of finite rank. (1) and (2) are equivalent by setting $\alpha_{X/S} \circ \nabla = J_\nabla - j^1_{X/S} \otimes \mathrm{id}_{\mathcal{E}}$ (with $\alpha_{X/S}$ as in (SPP)). The details can be found in [2, 4.2.6].

We will mostly deal with connections in the absolute case, where $S = \mathrm{Spec}(\mathbb{C})$. If S is clear from context, we will refer to such a map ∇ as just a *connection.*

The *horizontal sections of* ∇ are $\mathcal{E}^{\nabla} := \ker(\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E})$. If $f\colon X \to Y$ is a morphism over S and ∇ is an X/S-connection on an $\mathcal{O}_X$-module $\mathcal{E}$, the associated *relative X/Y-connection* is the map $\mathcal{E} \xrightarrow{\nabla} \Omega^1_{X/S} \otimes \mathcal{E} \to \Omega^1_{X/Y} \otimes \mathcal{E}$ (using (1)). If all maps are smooth, this can be formulated in terms of (3), as $\nabla_{X/Y} := \nabla|_{\mathcal{T}_{X/Y}}$.

In the next chapter will define inverse and (higher) direct image functors on modules with connections, so we need an appropriate notion of a morphism.

Definition 1.2.2. A *morphism* $\varphi : (\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$ *of $\mathcal{O}_X$-modules with X/S-connection* is a map of $\mathcal{O}_X$-modules $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$, satisfying $(\mathrm{id}_{\Omega^1_{X/S}} \otimes \varphi) \circ \nabla_1 = \nabla_2 \circ \varphi$ The *category of quasi-coherent $\mathcal{O}_X$-modules with connection relative to S* will be denoted $\mathbf{MC}(X/S)$.

The quasi-coherence condition is standard when considering $\mathcal{D}$-modules, and will be crucial for our later arguments involving the dévissage lemma 3.2.4. Note that a connection "propagates" to higher connections by taking

$$\begin{aligned} \nabla^n : \Omega^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} &\to \Omega^{n+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \\ \omega \otimes e &\mapsto d\omega \otimes e + (-1)^n \omega \wedge \nabla(e) \end{aligned} \tag{1.4}$$

for $n \geq 1$, where the $\wedge$ is taken with the $\Omega^1_{X/S}$ component of $\nabla(e)$. The composition $K_\nabla = \nabla^1 \circ \nabla$ is $\mathcal{O}_X$-linear, as for local sections f of $\mathcal{O}_X$ and e of $\mathcal{E}$,

$$K_\nabla(fe) = \nabla^1(df \otimes e + f\nabla(e)) = -df \wedge \nabla(e) + df \wedge \nabla(e) + f\nabla^1 \circ \nabla(e) = fK_\nabla(e)$$

Definition 1.2.3. The *curvature* K_∇ of an $\mathcal{O}_X$-module $\mathcal{E}$ with S-connection ∇ is the map

$$K_\nabla := \nabla^1 \circ \nabla \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$$

We say ∇ is *integrable* if $K_\nabla = 0$.

In terms of tangent vectors, the curvature is given by (see [2, 4.3.6])

$$[\nabla_\partial, \nabla_\delta] - \nabla_{[\partial,\delta]} = (\partial \wedge \delta) K_\nabla.$$

Thus ∇ being integrable is equivalent to saying that for local sections ∂, δ of $\mathcal{T}_{X/S}$,

$$[\nabla_\partial, \nabla_\delta] = \nabla_{[\partial,\nabla]} \tag{1.5}$$

with the commutator Lie brackets (on identifying $\mathcal{T}_{X/S} = \mathcal{D}er_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X)$).

Definition 1.2.4. The *category of $\mathcal{O}_X$-modules with integrable connections relative to S* is denoted $\mathbf{MIC}(X/S)$, and is the full subcategory of $\mathbf{MC}(X/S)$ with objects as $\mathcal{O}_X$-modules with integrable connections.

In the absolute setting, we denote $\mathbf{MIC}(X) := \mathbf{MIC}(X/\operatorname{Spec}(\mathbb{C}))$.

When ∇ is integrable, we have $\nabla^{n+1} \circ \nabla^n = 0$, by the formula

$$\nabla^{n+1} \circ \nabla^n(\omega \otimes e) = \omega \wedge \nabla^1 \circ \nabla(e)$$

for sections ω of $\Omega^n_{X/S}$ and e of $\mathcal{E}$ (see [2, 4.3.4]).

Definition 1.2.5 (De Rham complex)**.** Let $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}(X/S)$. Its associated *De Rham complex* $\mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$ is the cochain complex

$$\mathrm{DR}_{X/S}(\mathcal{E}, \nabla) := \left[\mathcal{E} \xrightarrow{\nabla} \Omega^1_{X/S} \otimes \mathcal{E} \xrightarrow{\nabla^1} \Omega^2_{X/S} \otimes \mathcal{E} \to \cdots \to \Omega^n_{X/S} \otimes \mathcal{E}\right]$$

where n is the maximum rank of the $\mathcal{O}_{X,x}$-modules $\Omega^1_{X/S,x}$ across points x of X.

Definition 1.2.6. Let $\mathcal{P}$ be one of the properties *coherence* or *(local) freeness* of $\mathcal{O}_X$-modules. We say that an object $(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(X)$ *has property* $\mathcal{P}$ if $\mathcal{E}$ has property $\mathcal{P}$. When $\mathcal{E}$ is locally free of finite rank, we call $(\mathcal{E}, \nabla)$ an *(algebraic) vector bundle with connection.*

Theorem 1.2.7. *An object $(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(X)$ is coherent if and only if it is locally free of finite rank.*

We refer to [27, Proposition 8.8] or [4, Lecture 2, 1a] for the proof.

Theorem 1.2.8. *$\boldsymbol{MIC}(X)$ is equivalent to the category of left $\mathcal{D}_X$-modules, and is hence a symmetric monoidal abelian category with enough injectives and an internal Hom.*

This result is [5, Theorem 2.15]. Explicit formulas for the tensor product and internal Hom can be found in [2, 2.7.3, 4.3.8], though we will not need these.

1.3 Hypercohomology

We briefly review some important facts we will use about hypercohomology, which is essentially a notion of higher derived functors for cochain complexes. In this section, $\mathcal{A}$ and $\mathcal{B}$ denote abelian categories, where $\mathcal{A}$ has enough injectives.

Definition 1.3.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. Let $K^\bullet$ be a bounded-below complex in $\mathcal{A}$, let $\mathcal{I}^\bullet$ is a complex of injectives in $\mathcal{A}$, and $\Phi : K^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism. The i^{th} *hypercohomology of* $K^\bullet$ is $\mathbb{R}^i F(K^\bullet) := H^i(F(\mathcal{I}^\bullet))$.

Hypercohomology is a special case of right derived functors on derived categories (see [24, 05RU]), though we will not discuss this further.

Proposition 1.3.2. *Suppose $\mathcal{A}$ has enough injectives, and $K^\bullet$ be a bounded-below complex in $\mathcal{A}$. Then $\mathbb{R}^i F(K^\bullet)$ is well-defined, i.e. independent of the complex of injectives $\mathcal{I}^\bullet$, and quasi-isomorphism $K^\bullet \to \mathcal{I}^\bullet$.*

For $\mathcal{O}_X$-modules, we denote $\mathbb{H}^i(X, -) := \mathbb{R}^i\Gamma(-)$, as in the introduction.

Proposition 1.3.3. *Let $f : X \to S$ be a morphism. Then for any bounded-below complex of $\mathcal{O}_X$-modules $K^\bullet$, $\mathbb{R}^i f_*(K^\bullet)$ is the sheaf associated to the presheaf*

$$U \mapsto \mathbb{H}^i(f^{-1}(U), K^\bullet|_{f^{-1}(U)}),$$

and hence its formation commutes with taking open subsets.

This statement and its proof can be found as [18, 0.12.4.3]. As for higher direct images, we have a projection formula, which is [24, 01E8].

Proposition 1.3.4. *If K is a bounded-below complex of $\mathcal{O}_X$-modules, and $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module of finite rank, then we have an isomorphism*

$$\mathcal{F} \otimes_{\mathcal{O}_Y} \mathbb{R}^q f_*(K^\bullet) \cong \mathbb{R}^q f_*(\mathcal{F} \otimes_{\mathcal{O}_X} K^\bullet).$$

which is functorial in $K^\bullet$.

We also specify some spectral sequences which we will make use of later.

Theorem 1.3.5 (Spectral sequences on hypercohomology)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor of abelian categories, where $\mathcal{A}$ has enough injectives, and let $K^\bullet$ be a bounded-below complex in $\mathcal{A}$.*

1. *There is a spectral sequence*

$$ {}_I E_1^{p,q} = R^q F(K^p) \Rightarrow \mathbb{R}^{p+q} F(K^\bullet). \tag{1.6}$$

   with E_2-page given by ${}_I E_2^{p,q} = H^p(R^q F(K^\bullet))$

2. *Let F be a decreasing filtration of subcomplexes on $K^\bullet$ compatible with the differential. Suppose that that for any n, $F^m K^n = 0$ for all sufficiently large m. Then there is a spectral sequence*

$$ {}_{FF} E_1^{p,q} = \mathbb{R}^{p+q} F(\mathrm{gr}^p) \Rightarrow \mathbb{R}^{p+q} F(K^\bullet) \tag{1.7}$$

   Further, $d_1^{0,q}$ is the coboundary map in the long-exact sequence of $\mathbb{R}^q F$ induced by

$$0 \to \mathrm{gr}^1 \to F^0/F^2 \to \mathrm{gr}^0 \to 0 \tag{1.8}$$

We refer to [18, 12.4.1] and [35, Theorem 1.1] for the statements and their proofs.

Chapter 2

Images by smooth morphisms

In this chapter, we set up the theory for inverse images and higher direct images under smooth morphisms, expanding on [1, III.2]. Inverse images will primarily be used for base changes and localisations, to reduce to simpler cases. The higher direct images will be the central objects of study, being the generalisation of the de Rham cohomology groups to the relative setting; these are what were referred to as the *relative de Rham cohomology* in the abstract, though we will not use this terminology going forward.

We will begin by defining the image functors. We then give an alternate explicit description of the higher direct images, and some related spectral sequences. We finish the chapter off with results on flat base change and vanishing.

For notation, in the first two sections, we fix $\mathbb{C}$-varieties X and Y which are smooth over another smooth $\mathbb{C}$-variety S, and a morphism $f : X \to Y$. We then look to define the inverse image $f^* : \mathbf{MIC}(Y/S) \to \mathbf{MIC}(X/S)$, and, in the case where f is smooth, the direct images $R^q_{DR} f_* : \mathbf{MIC}(X/S) \to \mathbf{MIC}(Y/S)$, for $q \geq 0$. We denote the exact sequence of differentials associated to $X \xrightarrow{f} Y \to S$ by

$$f^*\Omega^1_{Y/S} \xrightarrow{\iota} \Omega^1_{X/S} \xrightarrow{\pi} \Omega^1_{X/Y} \to 0.$$

2.1 Inverse images

As in §1.1, we let $\alpha_\bullet : \Omega^1_{\bullet/S} \to \mathcal{P}^1_{\bullet/S}$ be the canonical map of (1.1.4).

Given an object $(\mathcal{E}, \nabla)$ of $\mathbf{MC}(Y/S)$, we avoid the non-$\mathcal{O}_Y$-linearity of ∇ using definition 1.2.1(2). Writing $(\alpha_{Y/S} \otimes \mathrm{id}_\mathcal{E}) \circ \nabla = j^1_{X/S} \otimes \mathrm{id}_\mathcal{E} - J_\nabla$ for an $\mathcal{O}_Y$-linear section J_∇ of $\mathcal{P}^1_{Y/S} \otimes \mathcal{E} \to \mathcal{E}$, applying f^* gives $f^* J_\nabla : f^*\mathcal{E} \to f^*\mathcal{P}^1_{Y/S} \otimes_{\mathcal{O}_X} f^*\mathcal{E}$, which is an $\mathcal{O}_X$-linear section of $f^*\mathcal{P}^1_{Y/S} \otimes_{\mathcal{O}_X} f^*\mathcal{E} \to f^*\mathcal{E}$. We then get a section of $\mathcal{P}^1_{X/S} \otimes_{\mathcal{O}_X} f^*\mathcal{E} \to f^*\mathcal{E}$ by composing with the map $f^*\mathcal{P}^1_{Y/S} \otimes f^*\mathcal{E} \to \mathcal{P}^1_{X/S} \otimes f^*\mathcal{E}$ on jets. Together, this defines our connection $f^*\nabla$ by the formula

$$(\alpha_{X/S} \otimes \mathrm{id}_{f^*\mathcal{E}}) \circ f^*\nabla - j^1_{X/S} \otimes \mathrm{id}_{f^*\mathcal{E}} = f^*J_\nabla = f^*((\alpha_{Y/S} \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla - j^1_{Y/S} \circ \mathrm{id}_{\mathcal{E}}). \tag{2.1}$$

For functoriality, we note that for a morphism $\varphi : (\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$, we have

$$(j^1_{Y/S} \otimes \mathrm{id}_{\mathcal{E}_2}) \circ \varphi = j^1_{Y/S} \circ \varphi = (\mathrm{id}_{\mathcal{P}^1_{X/S}} \otimes \varphi) \circ (j^1_{Y/S} \otimes \mathrm{id}_{\mathcal{E}_1}).$$

Taking the difference of this with the equation

$$(\alpha_{Y/S} \otimes \mathrm{id}_{\mathcal{E}_2}) \circ \nabla_2 \circ \varphi = (\alpha_{Y/S} \otimes \mathrm{id}_{\mathcal{E}_2}) \circ (\mathrm{id}_{\Omega^1_{Y/S}} \otimes \varphi) \circ \nabla_1$$

from φ being a morphism, and writing J_{∇_i} for the corresponding $\mathcal{O}_Y$-linear sections yields that the diagram of $\mathcal{O}_Y$-linear maps

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\varphi"] \arrow[d, "J_{\nabla_1}"] & \mathcal{E}_2 \arrow[d, "J_{\nabla_2}"] \\
\mathcal{P}^1_{X/S} \otimes \mathcal{E}_1 \arrow[r, "\mathrm{id} \otimes \varphi"] & \mathcal{E}_2
\end{tikzcd}$$

commutes. Hence by applying f^* (on $\mathcal{O}_Y$-modules) and adding

$$(j^1_{X/S} \otimes \mathrm{id}_{f^*\mathcal{E}_2}) \circ f^*\varphi j^1_{X/S} \otimes f^*\varphi = (\mathrm{id}_{\mathcal{P}^1_{X/S}} \otimes f^*\varphi) \circ j^1_{X/S}$$

to both sides gives $\alpha_{X/S} \otimes \mathrm{id}_{f^*\mathcal{E}} \circ (f^*\nabla_2 \circ \varphi) = \alpha_{X/S} \otimes \mathrm{id}_{f^*\mathcal{E}} \circ (\mathrm{id}_{\Omega^1_{X/S}} \otimes \varphi) \circ f^*\nabla_1$. The identity and composition properties then follow as the maps are just given by f^* on the underlying sheaves.

We now check that this restricts to a functor $\mathbf{MIC}(Y/S) \to \mathbf{MIC}(X/S)$. In terms of the splitting $\mathcal{P}^1_{\bullet/S} = \Omega^1_{\bullet/S} \oplus \mathcal{O}_\bullet$ as right $\mathcal{O}_\bullet$-modules (SPP), the maps J_∇ and $J_{f^*\nabla}$ are given by $J_\nabla(e) = (\nabla(e), e)$ and $J_{f^*\nabla}(f^*\nabla(x \otimes e), x \otimes e)$. In particular, these are such that the diagram

$$\begin{tikzcd}
f^{-1}\mathcal{E} \arrow[r, "f^{-1}\nabla"] \arrow[d, "1_{\mathcal{O}_X} \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{id}_{f^{-1}\mathcal{E}}"] & f^{-1}\Omega^1_{Y/S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E} \arrow[d] \\
f^*\mathcal{E} \arrow[r, "f^*\nabla"] & \Omega^1_{X/S} \otimes_{\mathcal{O}_X} f^*\mathcal{E},
\end{tikzcd} \tag{2.2}$$

where the right map is the composition of $1_{\mathcal{O}_X} \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{id}$ (which has codomain $f^*\Omega^1_{Y/S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E} = f^*\Omega^1_{Y/S} \otimes_{\mathcal{O}_X} f^*\mathcal{E}$), followed by the map $f^*\Omega^1_{Y/S} \otimes f^*\mathcal{E} \to \Omega^1_{X/S} \otimes f^*\mathcal{E}$. Hence we may explicitly write $f^*\nabla(x \otimes e) = dx \otimes e + x\nabla'(e)$, where ∇'

is the diagonal arrow in the above diagram, given by $\nabla'(e) = 1_{\mathcal{O}_X} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\nabla(e)$. In particular, we compute that for a local section $x \otimes e$ of $f^*\mathcal{E}$, noting the sign swaps due to the wedge product in the line (2.3)

$$\begin{aligned}(f^*\nabla)^1 \circ f^*\nabla(x \otimes e) &= (f^*\nabla)^1(dx \otimes e + x \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\nabla(e)) \\ &= -dx \wedge f^{-1}\nabla(e) + dx \wedge f^{-1}\nabla(e) \\ &\quad + x \wedge (f^*\nabla)^1(1 \otimes_{f^{-1}\mathcal{O}_Y} \nabla(e)) \qquad (2.3) \\ &= xf^{-1}(\nabla^1 \circ \nabla)(e) = xf^{-1}K_\nabla(e).\end{aligned}$$

Hence we find that $K_{f^*\nabla} = f^*K_\nabla$, and so $f^*K_\nabla = 0$ implies $K_{f^*\nabla} = 0$, so we get the desired restriction to the subcategories $\mathbf{MIC}(-)$.

Definition 2.1.1. The *inverse image by f* is the functor

$$\begin{aligned} f^* : \mathbf{MIC}(Y/S) &\to \mathbf{MIC}(X/S) \\ (\mathcal{E}, \nabla) &\mapsto (f^*\mathcal{E}, f^*\nabla)\end{aligned}$$

as described above.

We note that for $Z \xrightarrow{g} X \xrightarrow{f} Y \to S$, the formula (2.1) yields $(g \circ f)^* = f^* \circ g^*$.

Notation 2.1.2. If $i : U \to X$ is an étale morphism with i clear from context (such as in an étale cover, or an open immersion), and $(\mathcal{E}, \nabla)$ is an object of $\mathbf{MIC}(X)$, we will write $(\mathcal{E}, \nabla)|_U$ for $i^*(\mathcal{E}, \nabla)$.

2.2 Direct images

Now suppose that f is smooth, so that we have a short exact sequence of differentials

$$0 \to f^*\Omega^1_{Y/S} \xrightarrow{\iota} \Omega^1_{X/S} \xrightarrow{\pi} \Omega^1_{X/Y} \to 0.$$

As f is smooth, $\Omega^1_{X/Y}$ is locally free of finite rank, so the associated long-exact sequence on taking $(-)^\vee = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ degenerates (see [22, III.6.3, III.6.7]), giving a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{T}_{X/Y} \xrightarrow{\pi^*} \mathcal{T}_{X/S} \xrightarrow{\iota^*} f^*\mathcal{T}_{Y/S} \to 0. \qquad (2.4)$$

We will first define $R^0_{DR}f_*$, and the higher direct images will then be its right derived functors in $\mathbf{MIC}(-)$. The notation is intended to suggest that the underlying $\mathcal{O}_X$-module of $R^0_{DR}f_*(\mathcal{E}, \nabla)$ will not be $f_*\mathcal{E}$ in general, and this will correspond to difficulty in defining an $\mathcal{T}_{Y/S}$-action on the underlying module.

We note that for each U, the modules of tangent sheaves on U are Lie algebras, where each of $\mathcal{T}_{X/Y}$, $\mathcal{T}_{X/S}$ and $f^*\mathcal{T}_{Y/S}$ are given the commutator Lie brackets from the identifications $\mathcal{T}_\bullet \cong \mathcal{D}er_\bullet \subseteq \mathcal{E}nd(\mathcal{O}_\bullet)$. Since $d_X = d_{X/S} \circ \pi$ and $f^*d_S = d_X \circ \iota$, it follows that the maps π^* and ι^* are homomorphisms of sheaves of Lie algebras, so we get short-exact sequences of Lie algebras on local sections.

Now given an object $(\mathcal{E}, \nabla)$ of $\mathbf{MIC}(X/S)$, the sheaf of relative horizontal sections $\mathcal{E}^{\nabla_{X/Y}}$ of $\nabla_{X/Y} : \mathcal{T}_{X/Y} \to \mathcal{E}nd(\mathcal{E})$ admits an action of $\mathcal{T}_{X/S}$: for local sections e of $\mathcal{E}^{\nabla_{Y/S}}$, ∂' of $\mathcal{T}_{X/S}$, and ∂ of $\mathcal{T}_{X/Y}$, $[\partial, \partial']$ is a local section of $\mathcal{T}_{X/Y}$ as $\mathcal{T}_{X/Y}$ is a subsheaf of Lie algebra ideals. Using the description (1.5) of the curvature gives

$$\nabla_\partial(\nabla_{\partial'}(e)) = \nabla_{\partial'}(\nabla_\partial(e)) + \nabla_{[\partial,\partial']}(e) = \nabla_{\partial'}(0) + 0 = 0,$$

and so $\nabla_{\partial'}(e)$ is also a local section of $\mathcal{E}^{\nabla_{X/Y}}$. Since this action is trivial on $\mathcal{T}_{X/Y}$, it descends to an action of $f^*\mathcal{T}_{Y/S} \cong \mathcal{T}_{X/S}/\mathcal{T}_{X/Y}$ on $\mathcal{E}$ by the short-exact sequence (2.4), so we get a map $f^*\mathcal{T}_{Y/S} \otimes \mathcal{E} \to \mathcal{E}$, and applying f_* and since $\mathcal{T}_{Y/S}$ is locally free of finite rank, the projection formula yields

$$\mathcal{T}_{Y/S} \otimes_{\mathcal{O}_Y} f_*\mathcal{E}^{\nabla_{X/Y}} \cong f_*(f^*\mathcal{T}_{Y/S} \otimes_{\mathcal{O}_X} \mathcal{E}^{\nabla_{X/Y}}) \to f_*\mathcal{E}^{\nabla_{X/Y}}. \tag{2.5}$$

This satisfies the Leibniz rule as a section of $R^0_{DR}f_*(\mathcal{E}, \nabla)$ over U is a section of $\mathcal{E}^{\nabla_{X/Y}}$ over $f^{-1}(U)$, and so defines a connection $R^0_{DR}f_*\nabla : \mathcal{T}_{Y/S} \to \mathcal{E}nd(f_*\mathcal{E}^{\nabla_{X/Y}})$.

This is functorial as for any morphism $\varphi : (\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$ in $\mathbf{MC}(X)$, on the level of sheaves, $R^0_{DR}f_*(\varphi)$ on an open set $U \subseteq Y$ is given by restricting $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ to the subsheaves $\mathcal{E}_i^{\nabla|_{\mathcal{T}_{X/S}}}$, on the open set $f^{-1}(U) \subseteq X$. Further, if ∇ is integrable, then so is $R^0_{DR}f_*\nabla$, since for a given $U \subseteq S$, the value of $R^0_{DR}f_*\nabla$ on U is given by the value of ∇ on $f^{-1}(U)$, restricted to the subsheaf $\mathcal{E}^{\nabla|_{\mathcal{T}_{X/S}}}$.

Definition 2.2.1. The *(zeroth) direct image by f* is the functor

$$\begin{aligned} R^0_{DR}f_* : \mathbf{MIC}(X/S) &\to \mathbf{MIC}(Y/S) \\ (\mathcal{E}, \nabla) &\mapsto (f_*\mathcal{E}^{\nabla_{X/Y}}, R^0_{DR}f_*\nabla) \end{aligned}$$

To take higher direct images, we need the following property of $R^0_{DR}f_*$.

Lemma 2.2.2. *$R^0_{DR}f_*$ is left-exact.*

Proof. Let $(\mathcal{E}_i, \nabla_i)$ be objects of $\mathbf{MIC}(X)$ for $i = 1, 2, 3$, and suppose

$$0 \to \mathcal{E}_1 \xrightarrow{i} \mathcal{E}_2 \xrightarrow{j} \mathcal{E}_3$$

is exact. As f_* is left-exact, it suffices to check that

$$0 \to \mathcal{E}_1^{\nabla_1} \to \mathcal{E}_2^{\nabla_2} \to \mathcal{E}_3^{\nabla_3}$$

is exact.

For $U \subseteq X$ open, note that $\mathcal{E}_1^{\nabla_1}(U) \to \mathcal{E}_2^{\nabla_2}(U)$ is injective as $\mathcal{E}_1(U) \to \mathcal{E}_2(U)$ is. For exactness at $\mathcal{E}_2^{\nabla_2}$, suppose $e \in \mathcal{E}_2^{\nabla_2}(U)$ has $j_U(e) = 0$. By exactness of $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3$, there is $y \in \mathcal{E}_1(U)$ with $i_U(e') = e$, and $0 = \nabla_2(e) = (i_U \otimes \mathrm{id})(\nabla_1(e'))$. As f is smooth, $\Omega^1_{X/S}$ is locally free (in particular flat), so $i_U \otimes \mathrm{id}$ is injective and $\nabla_1(e') = 0$, so $e' \in \mathcal{E}_1^{\nabla_1}(U)$. □

Definition 2.2.3. 1. The q^{th} *(higher) direct image by* f is

$$R^q_{DR} f_*(\mathcal{E}, \nabla) := R^q(R^0_{DR} f_*(\mathcal{E}, \nabla));$$

2. In the case $S = \mathrm{Spec}(\mathbb{C})$, the q^{th} *de Rham cohomology group of* X is

$$H^q_{DR}(X) := R^q_{DR} f_*(\mathcal{O}_X, d_X)$$

This definition will be used primarily for the validity of the Leray spectral sequence of 2.12, and is given first as it requires much less work to properly define.

Remark 2.2.4. Though we will not use this fact, it is worth noting that this description also describes the underlying $\mathcal{O}_X$-module of the higher direct images: since $\mathcal{D}_X$ is flat over $\mathcal{O}_X$, the underlying $\mathcal{O}_X$-module of any injective left $\mathcal{D}_X$-module is injective as an $\mathcal{O}_X$-module (see [29, 3.5]). Hence the underlying $\mathcal{O}_X$-module of $R^q_{DR} f_*(\mathcal{E}, \nabla)$ is exactly $R^q f_*(\mathcal{E}^{\nabla_{X/S}})$.

Lemma 2.2.5. *If f is étale, $R^0_{DR} f_*(\mathcal{E}, \nabla)$ has underlying $\mathcal{O}_X$-module $f_*\mathcal{E}$, and its connection is $f_*(\nabla : \mathcal{E} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$.*

Proof. Since $\mathcal{T}_{X/Y} = 0$ for f étale, we have $\mathcal{E}^{\nabla_{X/Y}} = \mathcal{E}$. The second claim follows by replacing $\mathcal{E}^{\nabla_{X/Y}}$ with $\mathcal{E}$ in the equation (2.5). □

In lieu of this lemma, we will just write f_* for the direct image functor in the case where f is étale.

2.3 Higher direct images

The goal of this section is to give alternate definitions (and hence more explicit descriptions) for the higher direct images of §2.2, from which we will deduce flat base change, and vanishing of higher cohomology. We do this by first describing a connection ∇^q_{GM} on the hypercohomology $\mathbb{R}^q f_* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$, and then showing that this agrees with the higher direct images as described above.

We first describe the construction of the Gauss-Manin connection on the hypercohomology sheaves $\mathbb{R}^q f_* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$, following [28]. The connection will arise from a d_1 differential in a spectral sequence, and hence can also be viewed as a connecting map of a hypercohomology short-exact sequence.

We build a spectral sequence by filtering the absolute de Rham complex $\mathrm{DR}_{X/\mathbb{C}}(\mathcal{E}, \nabla)$ by taking the subcomplexes

$$F^p = \mathrm{im}\left(f^*(\Omega^p_S) \otimes_{\mathcal{O}_X} \Omega^{\bullet - p}_X \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{((-)\wedge(-))\otimes \mathrm{id}} \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{E} \right) \tag{2.6}$$

By smoothness, all sheaves of differentials are locally free (of finite rank), and the exact sequence

$$0 \to f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0$$

is locally split. Thus we have an isomorphism

$$\bigoplus_{i=0}^{p} f^*\Omega^i_S \otimes \Omega^{p-i}_{X/S} \cong \bigwedge^p \Omega^1_X = \Omega^p_X,$$

and thus the i^{th} term of the complex F^p is

$$F^{p,i} = \bigoplus_{j \geq i} f^*\Omega^j_S \otimes \Omega^{p-j}_{X/S} \otimes \mathcal{E}.$$

Hence the filtration quotients are given by

$$\mathrm{gr}^p = F^p/F^{p+1} = f^*\Omega^p_S \otimes_{\mathcal{O}_X} \Omega^{\bullet - p}_{X/S} \otimes \mathcal{E} = f^*\Omega^p_S \otimes_{\mathcal{O}_X} \mathrm{DR}^{\bullet - p}_{X/S}(\mathcal{E}, \nabla)$$

Since Ω^p_S is a locally free $\mathcal{O}_S$-module of finite rank, applying the projection formula for hypercohomology of Proposition 1.3.4 yields

$$\mathbb{R}^{p+q} f_*(\mathrm{gr}^p) \cong \Omega^p_S \otimes_{\mathcal{O}_S} \mathbb{R}^q f_* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$$

Using the spectral sequence of a finitely filtered complex (1.7), there is a spectral sequence whose E_1 terms are

$$_{FF}E_1^{p,q} = \mathbb{R}^{p+q} f_*(\mathrm{gr}^p) \cong \Omega^p_S \otimes_{\mathcal{O}_S} \mathbb{R}^q f_* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla) \tag{2.7}$$

Theorem 2.3.1. *The maps ∇^q_{GM} are induced by a connection $\widetilde{\aleph}$ on the bicomplex $C^\bullet(\{U_\alpha\}, h^q(f, \mathrm{DR}_{X/S}(\mathcal{E}, \nabla))$, where $h^q(f, \mathrm{DR}_{X/S}(\mathcal{E}, \nabla))$ is the presheaf $U \mapsto R^q f|_{U*}(K^j|_U)$ on S.*

We expand on and give the proof of this theorem in Appendix B, as Theorem B.2.3. This shows, in conjunction with the identity $d_1^{1,q} \circ d_1^{0,q} = 0$, that the maps ∇^q_{GM} are in fact integrable connections.

Definition 2.3.2. Let $q \geq 0$. The *Gauss-Manin connection*

$$\nabla^q_{GM} : \mathbb{R}^q f_* \, \mathrm{DR}_{X/S}(\mathcal{E}, \nabla) \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathbb{R}^q f_* \, \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$$

is given by the differential $d_1^{0,q} : E_1^{0,q} \to E_1^{1,q}$ of the spectral sequence in (2.7). The de Rham complex of $(\mathbb{R}^q f_* \, \mathrm{DR}_{X/S}(\mathcal{E}, \nabla), \nabla^q_{GM})$ is the column $E_1^{\bullet,q}$ of the spectral sequence (2.7).

The latter part of Theorem 1.3.5(2) yields the following.

Proposition 2.3.3. *The Gauss-Manin connection ∇^q_{GM} is the coboundary map in the long-exact sequence on $\mathbb{R}^q f_*$ induced by the short-exact sequence*

$$0 \to \mathrm{gr}^1 \to F^0/F^2 \to \mathrm{gr}^0 \to 0$$

in the filtration (2.6).

We will use this formulation in showing the stability of regularity under higher direct images.

We now turn to showing that the higher direct images coincide with the hypercohomology sheaves with the Gauss-Manin connection. To do this, we first introduce a (homological) chain complex "dual" to the de Rham complex. For this we will use Theorem 1.2.8 to describe objects of $\mathbf{MIC}(X)$ as $\mathcal{D}_X$-modules.

Definition 2.3.4. The *Spencer complex* $\mathrm{Sp}^{X/S}_\bullet(\mathcal{D}_X)$ is the (chain) complex

$$\mathrm{Sp}^{X/S}_\bullet(\mathcal{D}_X) = \left[\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \mathcal{T}_{X/S} \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{X/S} \to \mathcal{D}_{X/S} \right]$$

with differentials given by

$$\begin{aligned} d_k : \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \mathcal{T}_{X/S} &\to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k-1} \mathcal{T}_{X/S} \\ \partial \otimes (\xi_1 \wedge \ldots \wedge \xi_k) &\mapsto \sum_{i=1}^k (-1)^{i-1} \partial \xi_i \otimes (\xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k) + \\ &\quad \sum_{i<j} (-1)^{i+j} \partial \otimes ([\xi_i, \xi_j] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \xi_k) \end{aligned} \tag{2.8}$$

The main property we will need is the following.

Lemma 2.3.5. $\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X)$ *is a resolution of* $f^*\mathcal{D}_S$ *by locally free left* $\mathcal{D}_X$*-modules.*

Proof. We prove that $\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X)$ is exact at each $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^i \mathcal{T}_{X/S}$ for $i > 0$ in Appendix A. Thus $\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X)$ is a resolution of $\mathcal{D}_X/\mathcal{D}_X\mathcal{T}_{X/S}$. As $\mathcal{D}_X$ is locally free, the short exact sequence on tangent sheaves gives rise to a sequence of maps on differential operators

$$0 \to \mathcal{D}_X\mathcal{T}_{X/S} \to \mathcal{D}_X \to f^*\mathcal{D}_S \to 0,$$

given by extending multiplicatively, and well-defined as the commutation relations are preserved. We claim this is short-exact. First, this is exact at $\mathcal{D}_X\mathcal{T}_{X/S}$ and $f^*\mathcal{D}_S$ as the associated maps remain injective and surjective. At $\mathcal{D}_X$, we note that the short exact sequence (2.4) of tangent vectors is locally split. By localising on X, we may assume f factors through $\mathbb{A}_S^n$ by an étale map, we may choose a splitting such that elements of $\mathcal{D}_X\mathcal{T}_{X/S}$ and $f^*\mathcal{D}_S$ commute. Using the commutation rule $[\partial, f] = \partial(f)$, we may write any $D \in \mathcal{D}_{X,x}$ in the form

$$\sum_{i=1}^{n} a_i \partial_{i,1} \dots \partial_{i,r_i}, \tag{2.9}$$

where a_i are local sections of $\mathcal{O}_X$, and each ∂_{i_j} is a local section of either $\mathcal{D}_X\mathcal{T}_{X/S}$ or $f^*\mathcal{D}_S$. By the above discussion we may move all local sections of $\mathcal{D}_X\mathcal{T}_{X/S}$ to the right in each term in the sum (2.9). Then if such an element descends to 0 in $f^*\mathcal{D}_S$, each term must have a factor in $\mathcal{T}_{X/S}$, and so lie in $\mathcal{D}_X\mathcal{T}_{X/S}$, and this shows exactness at $\mathcal{D}_X$.

Hence $f^*\mathcal{D}_S \cong \mathcal{D}_X/\mathcal{D}_X\mathcal{T}_{X/S}$, and we finish by noting that each of the modules in the Spencer complex is a locally free left $\mathcal{D}_X$-module. □

Remark 2.3.6.

1. We have $\mathcal{E}^{\nabla_{X/S}} \cong \mathcal{H}om_{\mathcal{D}_X}(f^*\mathcal{D}_S, \mathcal{E})$, since a section of the latter sheaf on is equivalent to a section of $\mathcal{E}$ vanishing under the action of $\mathcal{D}_X\mathcal{T}_{X/S}$, hence a section of $\mathcal{E}^{\nabla_{X/S}}$.

2. For any left $\mathcal{D}_X$-module $\mathcal{E}$, we have

$$\mathrm{DR}_{X/S}^\bullet(\mathcal{E}, \nabla) \cong \mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X), \mathcal{E})$$

We are now in a position to prove that this coincides with the higher direct images of §2.2.

Proposition 2.3.7. *For any object $(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(X)$, and any $i \geq 0$, we have*

$$R^i_{DR} f_*(-) \cong (\mathbb{R}^i f_* \, \mathrm{DR}_{X/S}(-), \nabla^i_{GM})$$

as functors $\boldsymbol{MIC}(X) \to \boldsymbol{MIC}(S)$.

This will follow from the following lemma.

Lemma 2.3.8. *If $(\mathcal{I}, \nabla)$ is an injective object of $\boldsymbol{MIC}(X)$, then $\mathrm{DR}_{X/S}(\mathcal{I}, \nabla)$ is an injective resolution of $\mathcal{I}^{\nabla_{X/S}}$, and $\mathcal{I}^{\nabla_{X/S}}$ is flabby as an $\mathcal{O}_X$-module.*

Proof of Lemma 2.3.8. Since $\mathcal{I}$ is injective, $\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{I})$ is exact. Applying this to the Spencer resolution

$$\mathrm{Sp}^{X/S}_{\bullet}(\mathcal{D}_X) \to f^* \mathcal{D}_S \to 0$$

of Lemma 2.3.5 and noting the isomorphisms of Remark 2.3.6 yields the locally-free resolution

$$0 \to \mathcal{I}^{\nabla_{X/S}} \to \mathrm{DR}^{\bullet}_{X/S}(\mathcal{I}, \nabla)$$

of $\mathcal{I}^{\nabla_{X/S}}$. For the termwise injectivity of the modules of $\mathrm{DR}^{\bullet}_{X/S}(\mathcal{I}, \nabla)$, we note that as $\mathrm{Sp}^p_{X/S}(\mathcal{D}_X)$ is locally free over $\mathcal{D}_X$ and $\mathcal{I}$ is injective, the functor

$$\mathcal{H}om_{\mathcal{D}_X}(-, \mathrm{DR}^p_{X/S}(\mathcal{I}, \nabla)) \cong \mathcal{H}om_{\mathcal{D}_X}((-) \otimes_{\mathcal{D}_X} \mathrm{Sp}^p_{X/S}(\mathcal{D}_X), \mathcal{I})$$

is exact.

Finally, to show that $\mathcal{I}^{\nabla_{X/S}}$ is flabby, by taking $\mathcal{G} = f^* \mathcal{D}_S$, it suffices to show that $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})$ is flabby for any $\mathcal{G}$. Let $\mathcal{G}_U = \mathcal{G} \otimes_{\mathcal{O}_X} j_! \mathcal{O}_U$, and note that

$$\begin{aligned} \Gamma(U, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})) &= \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, j^* \mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})) \\ &= \mathrm{Hom}_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})) \\ &= \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{G} \otimes_{\mathcal{O}_X} j_! \mathcal{O}_U, \mathcal{I}) \\ &= \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{G}_U, \mathcal{I}) \end{aligned} \tag{2.10}$$

As $\mathcal{I}$ is injective, the inclusion $\mathcal{G}_U \hookrightarrow \mathcal{G}$ induces a surjection

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{G}, I) \twoheadrightarrow \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{G}_U, \mathcal{I})$$

and in lieu of (2.10), this shows the surjectivity of

$$\Gamma(X, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})) \twoheadrightarrow \Gamma(U, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})),$$

and hence $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{G}, \mathcal{I})$ is flabby. □

Proof of Proposition 2.3.7. We show that

$$\mathbb{R}^0 f_* \,\mathrm{DR}_{X/S}(\mathcal{E},\nabla) \cong R^0_{DR} f_*(\mathcal{E},\nabla) \tag{2.11}$$

for all objects $(\mathcal{E},\nabla)$ of $\mathbf{MIC}(X)$, and that $(\mathbb{R}^i f_* \,\mathrm{DR}_{X/S}(-))_{i\geq 0}$ is the universal such δ-functor.

First, for any object $(\mathcal{E},\nabla)$ of ob $\mathbf{MIC}(X)$, we note that H^0 of the resolutions defining $\mathbb{R}^\bullet f_* \,\mathrm{DR}_{X/S}(\mathcal{E},\nabla)$ and $R^\bullet_{DR} f_*(\mathcal{E},\nabla)$ are both just $\mathcal{E}^{\nabla_{X/S}} = \ker(\nabla_{X/S})$, and their connections are induced by $f_*(\nabla)$ in the same way, so (2.11) follows.

If $\mathcal{I}$ is injective, then $\mathcal{I}^{\nabla_{X/S}}$ is f_*-acyclic as it is flabby. Hence for $i > 0$, we have

$$\mathbb{R}^i f_* \,\mathrm{DR}_{X/S}(\mathcal{I},\nabla) = H^i(f_* \,\mathrm{DR}_{X/S}(\mathcal{I},\nabla)) = 0,$$

so injectives are acyclic for $\mathbb{R}^i f_* \,\mathrm{DR}_{X/S}(-)$. Thus as $\mathbf{MIC}(X)$ has enough injectives, $\mathbb{R}^i f_* \,\mathrm{DR}_{X/S}(-)$ is an effaceable δ-functor for each $i > 0$. We conclude that $(\mathbb{R}^i f_* \,\mathrm{DR}_{X/S}(-))_{i\geq 0}$ is a universal δ-functor by [22, III.1.3A], and the desired result by [22, III.1.4]. $\square$

The definition with hypercohomology will underpin most of our results from here. From the proof of Proposition 2.3.7, and the explicit description of hypercohomology from Proposition 1.3.3, we obtain the following.

Corollary 2.3.9.
1. *Any injective object $(\mathcal{I},\nabla)$ of* ***MIC***(X) *is acyclic for $R^0_{DR} f_*$;*
2. *For any object $(\mathcal{E},\nabla)$ of* ***MIC***(X)*, $R^i_{DR} f_*(\mathcal{E},\nabla)$ has underlying sheaf associated to the presheaf*

$$U \mapsto \mathbb{H}^i(f^{-1}(U), \mathrm{DR}_{f^{-1}(U)/S}(\mathcal{E},\nabla));$$

3. *For any open $U \subseteq S$,*

$$(R^i_{DR} f_*(\mathcal{E},\nabla))|_U = R^i_{DR} f_*((\mathcal{E},\nabla)|_{f^{-1}(U)})$$

2.4 A few spectral sequences

We describe 3 classes of spectral sequences on higher direct images, which we will use to reduce our arguments to simpler cases, and rely on our description in §2.3 of $R^q_{DR} f_*$ in terms of hypercohomology. The arguments behind these spectral sequences can be mostly found in [2, 23.3]. The first is the *Leray spectral sequence*, which concerns the de Rham higher direct images of a composition of smooth morphisms, whose validity relies on Corollary 2.3.9 (1), and our description in §2.2 of $R^q_{DR} f_*$ as the right-derived functors of $R^0_{DR} f_*$.

Proposition 2.4.1 (Leray spectral sequence)**.** *Let $g : X \to Y$ and $f : Y \to S$ be smooth morphisms of smooth $\mathbb{C}$-varieties, and $(\mathcal{E}, \nabla)$ be an $\mathcal{O}_X$-module with integrable connection. Then there is a spectral sequence of $\mathcal{O}_S$-modules, given by*

$$_LE_2^{p,q} = R^p_{DR} f_* \circ R^q_{DR} g_* (\mathcal{E}, \nabla) \Rightarrow R^{p+q}_{DR} (f \circ g)_* (\mathcal{E}, \nabla) \tag{2.12}$$

Note that this is a (very) special case of the Grothendieck spectral sequence of [24, 015N].

Proposition 2.4.2 (Zariski spectral sequence)**.** *Let $f : X \to S$ be a smooth morphism of smooth $\mathbb{C}$-varieties. There is a spectral sequence of $\mathcal{O}_S$-modules*

$$_ZE_1^{p,q} = \bigoplus_{\alpha_0 < \alpha_1 < \ldots < \alpha_p} R^q_{DR} f_{\underline{\alpha}*} (\mathcal{E}, \nabla)|_{U_{\underline{\alpha}}} \Rightarrow R^{p+q}_{DR} f_* (\mathcal{E}, \nabla).$$

This comes from a spectral sequence on hypercohomology, using the description as in Proposition 2.3.7.

The following *Čech spectral sequence* relates de Rham direct images to the usual sheaf direct image. This is stated for a general complex of $f^{-1}\mathcal{O}_S$-modules, as we will also apply it to a logarithmic de Rham complex in the proof of the regularity theorem 5.3.1. In the follow proposition, we use $h^q(f, K^j)$ to denote the presheaf $U \mapsto R^q f|_{U*}(K^j|_U)$, as in the statement of Theorem 2.3.1. The last part of the following statement will only be used in Appendix B

Proposition 2.4.3 (Čech spectral sequence)**.** *Let $K^\bullet$ be a (cochain) complex of $f^{-1}\mathcal{O}_S$-modules on X. Then there is a spectral sequence*

$$_{\check{C}}E_1^{p,q} = \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} R^q f_{\underline{\alpha}*} \left(K^j|_{U_{\underline{\alpha}}} \right) \Rightarrow \mathbb{R}^{p+q} f_*(K^\bullet).$$

The values on the E^2 page $_CE_2^{p,q}$ are given by the total cohomology in degree p of the bicomplex $C^\bullet(\{U_\alpha\}, h^q(f, K^\bullet))$.

Applying this for $K^\bullet = \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$, this spectral sequence becomes

$$\bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} R^q f_{\underline{\alpha}*} \left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E} \right) \Rightarrow R^{p+q}_{DR}(\mathcal{E}, \nabla) \tag{2.13}$$

This spectral sequence specialises to the *Hodge-de Rham spectral sequence*, which is the special case $\mathcal{U} = \{X\}$, and takes the form

$$_{HDR}E_1^{p,q} = R^q f_*(\Omega^p_{X/S} \otimes \mathcal{E}) \Rightarrow R^{p+q}_{DR} f_*(\mathcal{E}, \nabla) \tag{2.14}$$

2.5 Flat base change

In this section, we prove flat base change for de Rham higher direct images in Proposition 2.5.3. Let $f : X \to S$ be a smooth morphism, and $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}(X)$. Suppose that $u : S' \to S$ is a morphism of smooth $\mathbb{C}$-varieties, and let

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "u"] & S \end{tikzcd} \tag{2.15}$$

be a fibred square.

We will first want an appropriate definition of the complex $u'^* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla)$. This complex should have modules $u'^*\Omega^\bullet_{X/S} \cong \Omega^\bullet_{X'/S'}$, except the differentials $\nabla^n : \Omega^n_{X/S} \otimes \mathcal{E} \to \Omega^{n+1}_{X/S} \otimes \mathcal{E}$ are only $f^{-1}\mathcal{O}_S$-linear, so we cannot directly make sense of $u'^*(\nabla^n)$.

We get around this using the jet sheaf $\mathcal{P}^1_{X/S}$, in a similar way to defining the inverse image functor. Note that $\nabla^n : \Omega^n_{X/S} \otimes \mathcal{E} \to \Omega^{n+1}_{X/S} \otimes \mathcal{E}$ is a differential operator of order 1, so factors as

$$\Omega^n_{X/S} \otimes \mathcal{E} \xrightarrow{d_{univ}} \mathcal{P}^1_{X/S} \otimes \Omega^n_{X/S} \otimes \mathcal{E} \xrightarrow{\overline{\nabla^n}} \Omega^{n+1}_{X/S} \otimes \mathcal{E} \tag{2.16}$$

where $d_{univ} = j^1_{X/S} \otimes \mathrm{id}_{\Omega^n_{X/S} \otimes \mathcal{E}}$ is the universal differential operator of order one, and $\overline{\nabla}^n$ is $\mathcal{O}_X$-linear. We may now make sense of $u'^*(\overline{\nabla^n})$, which is a map

$$u'^*(\overline{\nabla^n}) : u'^*\mathcal{P}^1_{X/S} \otimes (\Omega^n_{X'/S'} \otimes u'^*\mathcal{E}) \to \Omega^{n+1}_{X'/S'} \otimes u'^*\mathcal{E}$$

However, since $u'^*\Omega^1_{X/S} \cong \Omega^1_{X'/S'}$, the induced map (1.3) is an isomorphism $u'^*\mathcal{P}^1_{X/S} \cong \mathcal{P}^1_{X'/S'}$. We then take $u'^*(\nabla^n)$ to be the composition

$$u'^*(\nabla^n) : \Omega^n_{X'/S'} \otimes u'^*\mathcal{E} \xrightarrow{d'_{univ}} \mathcal{P}^1_{X'/S'} \otimes \Omega^n_{X'/S'} \otimes u'^*\mathcal{E} \xrightarrow{u'^*(\overline{\nabla}^n)} \Omega^{n+1}_{X'/S'} \otimes u'^*\mathcal{E} \tag{2.17}$$

where $d'_{univ} = j^1_{X'/S'} \otimes \mathrm{id}_{\Omega^n_{X'/S'} \otimes u'^*\mathcal{E}}$.

Definition 2.5.1. In the setting of (2.15), the complex $u'^* \mathrm{DR}^\bullet_{X/S}(\mathcal{E}, \nabla)$ of objects in $\mathbf{MIC}(X'/S')$ has

1. n^{th} underlying module $u'^* \mathrm{DR}^n_{X/S}(\mathcal{E}, \nabla) = \Omega^n_{X'/S'} \otimes u'^*\mathcal{E}$; and
2. n^{th} differential $u'^*(\nabla^n) : \Omega^n_{X'/S'} \otimes u'^*\mathcal{E} \to \Omega^{n+1}_{X'/S'} \otimes u'^*\mathcal{E}$ as in (2.17).

Lemma 2.5.2. *There is a canonical isomorphism*

$$u'^* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla) \cong \mathrm{DR}_{X'/S'}(u'^*(\mathcal{E}, \nabla)) \tag{2.18}$$

induced by the pointwise identity maps.

Proof. It suffices to check that the constructions of the differentials agree. Let $u'^*\nabla$ denote the connection on the right-hand side. For the 0^{th} differentials, similarly to (2.1) we have the formula

$$\alpha_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}} \circ u'^*\nabla - j^1_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}} = u'^*((\alpha_{X/S} \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla - j^1_{X/S} \otimes \mathrm{id}_{\mathcal{E}}),$$

by instead pulling back from X/S to X'/S'. We note that $u'^* j^1_{X/S} = j^1_{X'/S'}$ and $u'^*\alpha_{X/S} = \alpha_{X'/S'}$, as $u'^*\mathcal{P}^1_{X/S} = \mathcal{P}^1_{X'/S'}$. Writing $\nabla = \overline{\nabla} \circ (j^1_{X/S} \otimes \mathrm{id}_{\mathcal{E}})$, on distributing the functor u'^* on the right-hand side and cancelling the latter terms, we have

$$(\alpha_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}}) \circ u'^*\nabla = (\alpha_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}}) \circ u'^*\overline{\nabla} \circ (j^1_{X'/S'} \circ \mathrm{id}_{u'^*\mathcal{E}}).$$

Since $\alpha_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}}$ is injective by (1.1.4), we get $u'^*\nabla = u'^*\overline{\nabla} \circ (j^1_{X'/S'} \circ \mathrm{id}_{u'^*\mathcal{E}})$, so this coincides with the connection on the left-hand side.

For the higher connections, we note that

$$\overline{\nabla^n} \circ (j^1_{X/S} \otimes \mathrm{id}_{\Omega^n_{X/S} \otimes \mathcal{E}}) = \nabla^n = d^n_{X/S} \otimes \mathrm{id}_{\mathcal{E}} + (-1)^n \mathrm{id}_{\Omega^n_{X/S}} \wedge (\overline{\nabla} \circ (j^1_{X/S} \otimes \mathrm{id}_{\mathcal{E}})),$$

and hence we get

$$\begin{aligned} u'^*(\overline{\nabla^n}) \circ (j^1_{X'/S'} \otimes \mathrm{id}_{\Omega^n_{X'/S'} \otimes u'^*\mathcal{E}}) &= d^n_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}} \\ &\quad + (-1)^n \mathrm{id}_{\Omega^n_{X'/S'}} \wedge (u'^*\overline{\nabla} \circ (j^1_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}})) \\ &= d^n_{X'/S'} \otimes \mathrm{id}_{u'^*\mathcal{E}} + (-1)^n \mathrm{id}_{\Omega^n_{X'/S'}} \\ &\quad + (-1)^n \mathrm{id}_{\Omega^n_{X'/S'}} \wedge u'^*\nabla \\ &= (u'^*\nabla)^n \end{aligned}$$

and so the two constructions agree. □

The above lemma also shows that the above complex is a cochain complex. Armed with a well-defined notion of a pullback of de Rham complexes, we now deduce flat base change for higher direct images.

Proposition 2.5.3 (Flat base change)**.** *Let $f : X \to S$ be a smooth morphism, $u : S' \to S$ be a flat morphism, and*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{u} & S \end{array} \tag{2.19}$$

be a fibred square. Let $(\mathcal{E}, \nabla)$ be an object of $\boldsymbol{MIC}(X)$. Then for any $q \geq 0$, we have an isomorphism

$$u^* R^q_{DR} f_*(\mathcal{E}, \nabla) \cong R^q_{DR} f'_* u'^*(\mathcal{E}, \nabla) \tag{2.20}$$

Proof. Since the question is local on S', we may assume that S' and S are affine, and hence so is u. Choosing a finite affine open cover $\{U_a\}_{\alpha=1}^n$ of X, applying (2.4.3) to $(\mathcal{E}, \nabla)$ gives

$${}_{\check{C},1}E_1^{p,q} = \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} R^q f_{\underline{\alpha}*} \left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_{\underline{\alpha}}} \right) \Rightarrow R^{p+q}_{DR} f_*(\mathcal{E}, \nabla).$$

Since $\mathcal{E}$ is quasi-coherent, so is $\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_{\underline{\alpha}}}$, and hence ${}_{\check{C},1}E_1^{p,q} = 0$ as quasi-coherent sheaves are acyclic on affines. Hence this spectral sequence degenerates, and for each $p > 0$ we have

$$R^p_{DR} f_*(\mathcal{E}, \nabla) =_{\check{C},1} E_1^{p,0} = \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} f_{\underline{\alpha}*} \left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_{\underline{\alpha}}} \right). \tag{2.21}$$

Since u is affine, $\{u'^{-1}(U_\alpha)\}_{i=1}^n$ is an affine open cover of X', and applying (2.4.3) to $u'^*(\mathcal{E}, \nabla)$ gives

$${}_{\check{C},2}E_1^{p,q} = \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} R^q f'_{\underline{\alpha}*} \left(\Omega^j_{u'^{-1}(U_{\underline{\alpha}})/S'} \otimes u'^* \mathcal{E}|_{u'^{-1}(U_{\underline{\alpha}})} \right) \Rightarrow R^{p+q}_{DR} f'_*(u^*(\mathcal{E}, \nabla))$$

As $u'^*\mathcal{E}$ is quasi-coherent, we get ${}_{\check{C},2}E_1^{p,q} = 0$ for $p > 0$, As $(u'^*\mathcal{E})|_{u'^{-1}(U_{\underline{\alpha}})} = u'^*(\mathcal{E}|_{U_{\underline{\alpha}}})$, we find that

$$\begin{aligned} R^p_{DR} f_*(u^*(\mathcal{E}, \nabla)) &= \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} f'_{\underline{\alpha}*} \left(\Omega^j_{u'^{-1}(U_{\underline{\alpha}})/S'} \otimes u'^* \mathcal{E}|_{u'^{-1}(U_{\underline{\alpha}})} \right) \\ &= \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} f'_{\underline{\alpha}*} u'^* \left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_{\underline{\alpha}}} \right) \end{aligned}$$

Hence applying u^* to (2.21), and flat base change ([24, 02KH]) to the direct summands yields

$$\begin{aligned} u^* R^p_{DR} f_*(\mathcal{E}, \nabla) &= \bigoplus_{\substack{i+j=p \\ \alpha_0<\ldots<\alpha_i}} u^* f_{\underline{\alpha}*}\left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_{\underline{\alpha}}}\right) \\ &\xrightarrow{\cong} \bigoplus_{\substack{i+j=p \\ \alpha_0<\ldots<\alpha_i}} f'_{\underline{\alpha}*} u'^* \left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_{\underline{\alpha}}}\right) \\ &= R^p_{DR} f_*(u^*(\mathcal{E}, \nabla)). \end{aligned}$$

□

Remark 2.5.4. From the proof of the above proposition, we see that the base change isomorphism is induced by the natural isomorphisms $u^* f_* \Rightarrow f'_* u'^*$ on $\mathcal{O}$-modules over affines. Hence, the base change isomorphism commutes in the same ways as the base change morphism on sheaves, and in particular with differentials of the Zariski spectral sequence (2.4.2).

2.6 Vanishing of higher direct images

We use various theorems on the vanishing of higher direct images to deduce bounds on where the de Rham higher direct images are non-zero. Note that $R^j_{DR} f_* = 0$ for all $j < 0$. We will be interested mainly in the case of an affine map with constant relative dimension 1.

Proposition 2.6.1. *Let $f : X \to S$ be a smooth morphism of smooth varieties over $\mathbb{C}$, and d be the maximum dimension of the fibres. Suppose $(\mathcal{E}, \nabla)$ is an object of* ***MIC***(X)*. Then*

1. $R^j_{DR} f_*(\mathcal{E}, \nabla) = 0$ *for* $j > d + \dim(X)$*; and*
2. *if f is also affine, then* $R^j_{DR} f_*(\mathcal{E}, \nabla) \cong \mathcal{H}^j(f_* \mathrm{DR}_{X/S}(\mathcal{E}, \nabla))$*, where* $\mathcal{H}^q$ *denotes cohomology of complexes of objects of* ***MIC***(X)*.*

Proof. We have the Hodge-de Rham spectral sequence

$${}_{HDR}E_1^{p,q} = R^q f_*(\Omega^p_{X/S} \otimes \mathcal{E}) \Rightarrow R^{p+q}_{DR} f_*(\mathcal{E}, \nabla),$$

of 2.14 and since $\Omega^n_{X/S} \otimes \mathcal{E}$ is quasi-coherent (see definition 1.2.2), by Grothendieck's vanishing theorem [22, III.2.7]*, $R^q f_*(\Omega^n_{X/S} \otimes \mathcal{E}) = 0$ for $q > \dim(X)$. Since

*Applied in conjunction with [22, III.8.5], using an affine open cover.

$\Omega^p_{X/S} \otimes \mathcal{E} = 0$ for $p > d$, the E_1-page of this spectral sequence is concentrated in $0 \leq p \leq d$, $0 \leq q \leq \dim(X)$, and so $R^j_{DR} f_*(\mathcal{E}, \nabla)$ must be 0 for $j > d + \dim(X)$.

When f is affine, the E_2-page of 1.6 combined with the isomorphism of Proposition 2.3.7 yields

$$_I\mathcal{E}_2^{p,q} = \mathcal{H}^p(R^q f_* \operatorname{DR}_{X/S}(\mathcal{E}, \nabla)) \Rightarrow R^{p+q}_{DR} f_*(\mathcal{E}, \nabla)$$

Since f is affine, we have $R^q f_* \operatorname{DR}_{X/S}(\mathcal{E}, \nabla) = 0$, by applying [22, III.3.5] on $\mathbf{MIC}(S)$, and extending to complexes by [24, 07K7 4(c)]. Hence this spectral sequence degenerates at the E_2-page, yielding the isomorphism in (2). □

We will be particularly interested in the cases where $d \leq 1$.

Corollary 2.6.2. *In the setting of Proposition 2.6.1,*

1. *If $\pi : X' \to X$ is finite étale, then*

$$R^q_{DR}(f \circ \pi)_* = R^q_{DR} f_* \circ \pi_* \tag{2.22}$$

   as functors $\boldsymbol{MIC}(X') \to \boldsymbol{MIC}(S)$.

2. *If f is also affine, $R^q_{DR} f_*(\mathcal{E}, \nabla) = 0$ for $q > d$. Furthermore, if $d = 1$, we have*

$$R^0_{DR} f_*(\mathcal{E}, \nabla) \cong \ker(f_*(\nabla_{X/S})), \tag{2.23}$$

$$R^1_{DR} f_*(\mathcal{E}, \nabla) \cong \operatorname{coker}(f_*(\nabla_{X/S})), \tag{2.24}$$

   and there is an exact sequence of $\mathcal{O}_S$-modules

$$0 \to \ker(f_*(\nabla_{X/S})) \to f_*\mathcal{E} \xrightarrow{f_*(\nabla_{X/S})} f_*(\Omega^1_{X/S} \otimes \mathcal{E}) \to \operatorname{coker}(f_*(\nabla_{X/S})) \to 0 \tag{2.25}$$

Proof. We first prove (2), then use the first statement of (2) to prove (1). For (2), we note that $f_* \operatorname{DR}_{X/S}(\mathcal{E}, \nabla)$ is concentrated in degrees $0 \leq j \leq d$. The rest of (2) follows as the higher direct images are computed as the cohomology sheaves of the complex

$$f_*\mathcal{E} \xrightarrow{f_*(\nabla_{X/S})} f_*(\Omega^1_{X/S} \otimes \mathcal{E}).$$

For (1), the Leray spectral sequence (2.12) of $f \circ \pi$ and $(\mathcal{E}, \nabla) \in \mathbf{MIC}(X)$ has $_LE_1^{p,0} = R^p_{DR} f_* \circ \pi_*(\mathcal{E}, \nabla)$ since π is étale, and $_LE_1^{p,q} = 0$ for $q > 0$ as π is finite (in particular affine). Hence the spectral sequence degenerates at the E_1-page, yielding (2.22). □

Chapter 3

Elementary fibrations and dévissage

The goal of this chapter is to prove the main lemma on dévissage (3.2.4), following and expanding on [1, III.3]. This will allow us to reduce our proofs of theorems to a simple "base" case, using an inductive argument. This is the key technique underlying the algebraic proofs of the theorems on coherence, arbitrary base change and regularity in Chapter 5. In doing so, we will define and develop the theory of (rational) elementary fibrations, which will serve as the aforementioned base case. In this chapter, X, Y, S' and S denote smooth varieties over $\mathbb{C}$.

3.1 Elementary fibrations

Definition 3.1.1. A morphism $f : X \to S$ of smooth varieties is an *elementary fibration* if it fits into a diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Z \arrow[l, hook'] \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{3.1}$$

where

(i) Z is a closed reduced subscheme of $\overline{X}$ and j is the complementary open immersion;

(ii) $\overline{f}$ is projective and smooth with geometric fibres irreducible of dimension 1; and

(iii) g is a finite étale covering.*

Lemma 3.1.2. *Any elementary fibration $f : X \to S$ is a surjective, affine morphism.*

Proof. Surjectivity follows from 2.1.(ii), as f has geometric fibres of dimension 1. Since f being affine is local on the base for the étale topology, we may assume that Z is a disjoint union of divisors, each isomorphic to S via g. Let s be a point in S. Then the fibre $Z_s = g^{-1}(s)$ is a positive, effective divisor on the projective curve $\overline{X}_s/\kappa(s)$, hence ample by [22, IV.3.3]. By [18, 4.7.1], s has has a neighbourhood U so that the restriction of the associated invertible sheaf $\mathcal{L}(Z)|_{f^{-1}(U)}$ is relatively ample for f_U, and so Z is a relatively ample divisor in the relative projective curve $\overline{X}/S$. Choosing a section $\ell \in \Gamma(\overline{X}, \mathcal{L}(Z))$ with $X_\ell = \overline{X} \setminus Z = X$, it follows by [17, 5.5.7] that $f = \overline{f}|_X : X \to S$ is affine. □

We now give the two special cases of elementary fibrations which will be the base cases in our arguments.

Definition 3.1.3. Let $f : X \to S$ be an elementary fibration as in 3.1. We say f is *rational* if $\overline{X} = \mathbb{P}^1_S$, $\overline{f} = \mathrm{pr}_S : \mathbb{P}^1_S \to S$ is the projection, and Z is a disjoint union of images of sections σ_i, for $i = 1, \ldots, r, \infty$ of $\overline{f}$, where σ_∞ is the section $\infty \times \mathrm{id} : S \to \{\infty\} \times S$. This gives rise to a diagram of the form

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{A}^1_S \arrow[d, "\mathrm{pr}_S"] & \coprod_{i=1}^{r} \sigma_i(S) \arrow[l, hook'] \arrow[dl] \\
& S &
\end{tikzcd} \tag{3.2}$$

Related, but slightly weaker, we have the following notion:

Definition 3.1.4. An elementary fibration $f : X \to S$ is *coordinatised* if it fits in a diagram of the form

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, "\pi"] & \overline{X} \arrow[d, "\overline{\pi}"] & Z \arrow[l, hook'] \arrow[d, "\pi'"] \\
Y \arrow[r, hook, "j'"] \arrow[dr, "f'"'] & \mathbb{P}^1_S \arrow[d, "\mathrm{pr}_S"] & Z' \arrow[l, hook'] \arrow[dl, "g'"] \\
& S &
\end{tikzcd} \tag{3.3}$$

*In [1, III.3.1.1], the term étale covering seems to mean multiple things in different contexts, which we have distinguished between by taking étale covering to mean a *surjective, étale morphism.*

where

1. $\overline{\pi}$ is finite, and π is finite étale;
2. $f = f' \circ \pi$, $\overline{f} = \mathrm{pr}_S \circ \overline{\pi}$, and $g = g' \circ \pi'$; and
3. The lower part of the diagram is an elementary fibration with $j'(Y) \subseteq \mathbb{A}^1_S$.

Remark 3.1.5. We note that in the analytic setting, a coordinatised elementary fibration is topologically locally trivial in the analytic topology. This is because the finite étale maps π and g' are analytically (possibly non-surjective) finite-sheeted covering space maps, and pr_S is topologically trivial. In the analytic topology, the fibres are a finite disjoint union of sets of the form $\mathbb{C} - \{\theta_1, \ldots, \theta_r\}$.[†]

We prove some important, basic properties of elementary fibrations, which we will need later.

Lemma 3.1.6. *Let $f : X \to S$ be an elementary fibration.*

1. *If $u : S' \to S$ is a morphism, then the base change $f' : X' \to S'$ is also an elementary fibration. If f is also rational (respectively coordinatised), then so is f'.*
2. *If f is coordinatised as in (3.3), there is a finite étale covering $\varepsilon : S' \to S$ so that the lower part of the base change to S' is a rational elementary fibration.*

Proof. The first part of (1) follows as the properties described in an elementary fibration (closed immersion, open immersion, projective, etc.) are stable under base change (see [24, 02WE]). For coordinatised and rational f, we note in addition that the base change of $\mathbb{A}^1_S$ (resp. $\mathbb{P}^1_S$) is $\mathbb{A}^1_{S'}$ (resp. $\mathbb{P}^1_{S'}$), and that sections of pr_S base change to sections of $\mathrm{pr}_{S'}$.

For the second claim, since $j'(Y) \subseteq \mathbb{A}^1_S$, $\sigma_\infty := \infty \times \mathrm{id} : S \to \mathbb{P}^1_S$ is in fact a section of S', so $Z' = \sigma_\infty(S) \sqcup Z''$ for some $Z'' \subseteq Z'$. Since g' is a finite étale covering, there is $h : Z_1 \to Z$ so that $g' \circ h$ is a Galois covering with group G (see [14, 2.2.5]), and a finite étale covering $\varepsilon : S' \to S$ so that the base change $(Z_1)_{S'} = S' \times G$ (see [14, 2.2.1]). For each $\gamma \in G$, the section $s_\gamma = \mathrm{id} \times \gamma : S' \to S' \times G$ then gives the section $h_{S'} \circ s_\gamma : S' \to Z_{S'}$ of $g'_{S'}$, whose images cover $Z_{S'}$. The base change of (the inverse of) $(\sigma_\infty)_{S'}$ is then (the inverse of) the section $\infty \times \mathrm{id}_{S'} : S' \to \mathbb{P}^1_{S'}$ at infinity. □

[†]Though we will not use this, it is also worth noting that a rational elementary fibration is (globally) topologically trivial in the analytic topology.

Definition 3.1.7. A smooth morphism $f : X \to S$ of smooth $\mathbb{C}$-varieties is a *tower of elementary fibrations of height* $d \geq 0$ if it factors as

$$X = X_d \xrightarrow{f_{d-1}} X_{d-1} \xrightarrow{f_{d-2}} \dots \xrightarrow{f_0} X_0 = S$$

where each f_i is an elementary fibration.

A tower of height 0 is then an isomorphism $X \cong S$.

A key result in applying dévissage will be that a smooth morphism is generically a tower of coordinatized elementary fibrations (see 3.1.9), up to Zariski-localisation on the source and étale-localisation on the base. We will give a sketch of its proof, up to application of the following lemma.

Lemma 3.1.8. *If $f : X \to S$ is an elementary fibration and $\xi \in X$ is closed, then there are neighbourhoods $\xi \in U$ and $f(\xi) \in T$ so that $f|_U : U \to T$ is a coordinatised elementary fibration.*

This lemma is proved by applying properties of Lefschetz pencils, which we will not discuss here, and can be found as [1, III.1.5].

Proposition 3.1.9. *Let $f : X \to S$ be a smooth morphism of smooth $\mathbb{C}$-varieties, with S connected. Then there is an étale dominant morphism $\varepsilon : S' \to S$ so that $X' = X \times_S S'$ admits a finite, open affine cover $\{U_\alpha\}$, so that each $U_\alpha \to S'$ is a tower of coordinatised elementary fibrations.*

Proof sketch. Letting $K = K(S)^{alg}$, we reduce to when S is a point by replacing S with $\operatorname{Spec} K$, using descent as in [24, 01ZM]. Suppose we have an open cover $\{U_\alpha\}$ of $X_K = X \times_S \operatorname{Spec} K$ as in the proposition, and let $\{S_i\}_{i \in I}$ be the inverse system of schemes affine and étale over S, and note that $\varprojlim_{i \in I} S_i = \operatorname{Spec} K$.

We descend $X_K \to \operatorname{Spec} K$ to some index $i \in I$, then the open sets $U_j \subseteq X_K$ (by descending the diagram $U_1 \cap U_2 \rightrightarrows U_1 \coprod U_2 \to \operatorname{Spec} K$) to some $i' \geq i$. We do the same for $\overline{X_K}$ and Z (using $\overline{X_K} \to \mathbb{P}^n_K \to \operatorname{Spec} K$ to get a projective morphism $\overline{X_i} \to S_i$), and then descending the remaining properties required in Definition 3.1.1, using [24, 081C].

In the case $S = \operatorname{Spec} K$, f has pure relative dimension $d \geq 0$. The case $d = 0$ is immediate, while $d = 1$ follows by taking an open cover of X by subsets isomorphic to $\mathbb{A}^1_K$ with finitely many points removed.

For $d \geq 2$, let $\xi \in X$ be a point in X. By localising, we may take X to be affine, and we let $\overline{X}$ be a normal projective closure. We embed $\overline{X}$ into some $\mathbb{P}^N$ so that X has degree at least 2, and use this (see [3, XI, 2.1.ii]) to find $L \subseteq \mathbb{P}^N$

of codimension $d-1$ passing through ξ, cutting $\overline{X}$ and $\overline{X} \setminus X$ transversally, and avoiding the singular locus of $\overline{X}$.

Choosing another hyperplane $H \subseteq \mathbb{P}^N$ avoiding ξ and $L \cap (\overline{X} \setminus X)$, we blow up $\mathbb{P}^N$ along $L \cap H$, and let $p : \tilde{\mathbb{P}}^N \to \mathbb{P}^{d-1}$ be the morphism induced by the linear projection with centre $L \cap H$. We then set $\tilde{\overline{X}} \subseteq \tilde{\mathbb{P}}^N$ to be the strict transform of $\overline{X}$, and Z to be the disjoint union of $\overline{X} \setminus X$ and the exceptional divisor from blowing up. The desired elementary fibration comes from pulling back the following diagram over an appropriate affine neighbourhood of $p(\xi) \in \mathbb{P}^{d-1}$, to ensure that conditions (ii) and (iii) of an elementary fibration hold:

$$\begin{tikzcd}
\tilde{\overline{X}} \setminus Z \arrow[r, hook] \arrow[dr] & \tilde{\overline{X}} \arrow[d, "p|_{\tilde{\overline{X}}}"] & Z \arrow[l, hook'] \arrow[dl] \\
& \mathbb{P}^{d-1} &
\end{tikzcd}$$

We then coordinatise using Lemma 3.1.8. $\square$

The full argument for producing a tower of elementary fibrations can be found in [3, XI, 3.3]. We now come to the main definition underlying dévissage. We first fix some notational short-hands for intersections of a (finite) open cover.

Notation 3.1.10. Let $f : X \to S$ be a smooth morphism of smooth $\mathbb{C}$-varieties. We denote by d the maximum of the dimensions of the fibres. For a (finite) open cover $\{U_\alpha\}_{\alpha=0}^r$ and $\underline{\alpha} = (\alpha_0, \ldots, \alpha_p)$ where $\alpha_0 < \ldots < \alpha_p$, we denote $|\underline{\alpha}| := p+1$, $U_{\underline{\alpha}} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$, and $f_{\underline{\alpha}} = f|_{U_{\underline{\alpha}}}$.

Definition 3.1.11. The *Artin set* $A_i(f)$ *of* f *of level* $i \geq 0$ is the union of images $\varepsilon(S') \subseteq S$ of étale morphisms $\varepsilon : S' \to S$, finite over their image, so that

1. $X_{S'}$ admits a finite open cover $\{U_\alpha\}_\alpha$ so that for each α, the restriction of $f_{S'}$ to each U_α is a tower of coordinatised elementary fibrations, such that $|\underline{\alpha}| - 1 \leq i$;

2. For each $|\underline{\alpha}| > 1$, $U_{\underline{\alpha}}$ admits a finite open cover $\{U_{\underline{\alpha},\beta}\}_\beta$ so that the restriction of $f_{S'}$ to each $U_{\underline{\alpha},\beta}$ is a tower of coordinatised elementary fibrations, such that
$$(|\underline{\alpha}| - 1) + (|\beta| - 1) \leq i$$
and so on, for all $(\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \ldots)$ such that $(|\underline{\alpha}| - 1) + (|\underline{\beta}| - 1) + (|\underline{\gamma}| - 1) + \ldots \leq i$.

The compatibility with intersections of (2) will be used in conjunction with taking Čech complexes. Applying Proposition 3.1.9 to the connected components of S yields the following statement on Artin sets.

Corollary 3.1.12. *For any $i \geq 0$ and smooth morphism $f : X \to S$ of smooth varieties, $A_i(f)$ is a dense open subset of the image of f in S.*

Some basic facts we will use on Artin sets are as follows.

Lemma 3.1.13. *1. For any i, $A_i(f) \supseteq A_{i+1}(f)$, and $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) \supseteq A_i(f)$ for any $U_{\underline{\alpha}}$ as in the above definition;*

2. If $f : X \to S$ is a tower of coordinatised elementary fibrations, then for all $i \geq 0$, $A_i(f) = S$;

3. If f is étale, then for all $i \geq 0$, $A_i(f)$ is the largest open subset $U_{fét}$ of the image of f with $f_{U_{fét}} : f^{-1}(U_{fét}) \to U_{fét}$ a finite étale covering.

Proof. (1) The condition for $\varepsilon(S') \subseteq A_{i+1}(f)$ encompasses that of $A_i(f)$, so any base change of satisfying the conditions for $A_{i+1}(f)$ also satisfies those for $A_i(f)$. The second part of this claim follows as the conditions for $A_{i-|\alpha|+1}(f_{\underline{\alpha}})$ start from (2) in the definition of $A_i(f)$, and as $(|\underline{\alpha}| - 1) + (|\underline{\beta}| - 1) + \ldots \leq i$ if and only if $(|\underline{\beta}| - 1) + \ldots \leq i - |\alpha| + 1$. (2) follows by taking the open cover $\{X\}$ of X.

For (3), we first have $U_{fét} \subseteq A_i(f)$ by taking $\varepsilon : U_{fét} \hookrightarrow S$, and noting that $\varepsilon_{U_{fét}} = \mathrm{id}_{U_{fét}} : U_{fét} \times_S U_{fét} = U_{fét} \to U_{fét}$.

To show $A_i(f) \subseteq U_{fét}$, note that the condition for $\varepsilon : S' \to S$ in Definition 3.1.11 simplifies to $X_{S'}$ admitting a finite open cover $\{U_\alpha\}$ where $f_{S'}|_{U_\alpha} : U_\alpha \xrightarrow{\cong} S'$. Hence $X_{S'}$ is a union of the sections $S' \xrightarrow{f_{S'}|_{U_\alpha}^{-1}} U_\alpha \hookrightarrow X_{S'}$. As $f_{S'}$ is étale, by [24, 024T(3)], $X_{S'}$ splits as a disjoint union of copies of S' (indexed by α). Thus $X_{S'} \cong \coprod_\alpha S' \to S' \to \varepsilon(S')$ is finite étale. Hence by [24, 01W6 (2)] applied to $X_{S'} \to f^{-1}(\varepsilon(S')) \to \varepsilon(S')$, the maps $f^{-1}(\varepsilon(S')) \to \varepsilon(S')$ are finite étale, thus so is $f^{-1}(A_i(f)) \to A_i(f)$. $\square$

The final statement of the previous lemma will be important in handling the height 0 cases of towers of elementary fibrations.

3.2 Dévissage

We use abstract theory to reduce the arguments for higher direct images of modules with integrable connections to a much simpler case. For X a smooth $\mathbb{C}$-variety, we consider *properties $\mathcal{P}$ of $\mathcal{O}_X$-modules with integrable connection*, which we will just refer to as a *property (of **MIC**(X)* where unambiguous. For any object $(\mathcal{E}, \nabla)$ of $\mathbf{MIC}(X)$, we write $\mathcal{P}((\mathcal{E}, \nabla))$ as a shorthand for "$(\mathcal{E}, \nabla)$ satisfies the property $\mathcal{P}$".

Definition 3.2.1. For a property $\mathcal{P}$ as above, we say that $\mathcal{P}$ is

1. *local for the étale topology* if for any X, any object $(\mathcal{E}, \nabla)$ of $\mathbf{MIC}(X)$ and étale cover $\{\varphi_i : V_i \to X\}_{i \in I}$, we have

$$\mathcal{P}((\mathcal{E}, \nabla)) \iff \forall i(\mathcal{P}(\varphi_i^*(\mathcal{E}, \nabla)));$$

2. *stable under finite étale direct image* if for any X, any object $(\mathcal{E}, \nabla)$ of $\mathbf{MIC}(X)$ and finite étale $\pi : X \to X'$, we have

$$\mathcal{P}((\mathcal{E}, \nabla)) \implies \mathcal{P}(\pi_*(\mathcal{E}, \nabla));$$

3. *strongly exact* if $\mathcal{P}(0)$ holds, and for any exact sequence

$$(\mathcal{E}_1, \nabla_1) \to (\mathcal{E}, \nabla) \to (\mathcal{E}_2, \nabla_2)$$

in $\mathbf{MIC}(X)$,

$$\mathcal{P}((\mathcal{E}_1, \nabla_1)) \wedge \mathcal{P}((\mathcal{E}_2, \nabla_2)) \implies \mathcal{P}((\mathcal{E}, \nabla)).$$

We note strong exactness is equivalent to $\mathcal{P}$ being closed under subobjects, extensions, and quotients. The main use of strong exactness is to reduce to simpler cases using spectral sequences: given a convergent spectral sequence $E_r^{p,q} \Rightarrow H^{p+q}$ in $\mathbf{MIC}(X)$, to show $\mathcal{P}(H^{p+q})$, we can instead show $\mathcal{P}(E_r^{p,q})$ for all p and q.

Remark 3.2.2. 1. Since any variety is noetherian and the underlying modules of objects of $\mathbf{MIC}(X)$ are quasi-coherent, we have that coherence is strongly exact as a property on $\mathbf{MIC}(X)$.

2. Any property $\mathcal{P}$ local under the étale topology is stable under pullback by an étale map φ, since by extending to an étale cover, it follows that $\mathcal{P}((\mathcal{E}, \nabla)) \implies \mathcal{P}(\varphi^*(\mathcal{E}, \nabla))$.

We adopt the notation of 2.1.2, writing $(\mathcal{E}, \nabla)|_U$ where the étale map $U \to X$ is clear from context. The above properties are related in the following way.

Lemma 3.2.3. *Let $\mathcal{P}$ be a property of $\boldsymbol{MIC}(X)$, for X a smooth $\mathbb{C}$-variety, which is strongly exact and local for the étale topology. Then $\mathcal{P}$ is also stable under finite étale direct image.*

The bulk of the argument goes by lifting a finite étale map to an appropriate Galois covering.

Proof of Lemma 3.2.3. Let $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}(X)$, and $\pi : X \to Z$ be finite étale. Since $\mathcal{P}$ is local for the étale topology, we may assume Z is connected by taking its connected components, and that $X \neq \emptyset$ as the only sheaf on $\emptyset$ is 0. We may also assume that X is connected, since if $X = X_1 \coprod X_2$, we may instead consider $\pi_i : X_i \hookrightarrow X \to Z$, since $\pi_*(\mathcal{E}, \nabla) = \pi_{1*}((\mathcal{E}, \nabla)|_{X_1}) \oplus \pi_{2*}((\mathcal{E}, \nabla)|_{X_2})$.

Thus π is then a finite étale covering, and by [14, 2.2.5] there is scheme X' with a map $\tau : X' \to X$ so that $\pi \circ \tau$ is a Galois covering with Galois group G, and by [21, V.2.6(ii bis)], there is a finite étale covering $\sigma : Z' \to Z$ so that $X'_{Z'} \cong Z' \times G$. Then τ is also finite étale by [24, 02GW], since $X \to Z$ is separated and étale, and this yields the diagram of morphisms

$$\begin{tikzcd} Z' \times G \arrow[r, "\tau'"] \arrow[d, "\sigma''"] & X_{Z'} \arrow[r, "\pi'"] \arrow[d, "\sigma'"] & Z' \arrow[d, "\sigma"] \\ X' \arrow[r, "\tau"] & X \arrow[r, "\pi"] & Z \end{tikzcd} \tag{3.4}$$

with $\pi' \circ \tau' = \mathrm{pr}_1$. For $\gamma \in G$, let $i_\gamma = \mathrm{id}_{Z'} \times \gamma : Z' \hookrightarrow Z' \times G$ be the inclusion with second component γ. For each $\gamma \in G$, and any object $(\mathcal{M}, \nabla)$ of $\mathbf{MIC}(Z' \times G)$, we have

$$\mathrm{pr}_1^* \mathrm{pr}_{1*}(\mathcal{M}, \nabla)_\gamma := i_\gamma^*(\mathrm{pr}_1^* \mathrm{pr}_{1*}(\mathcal{M}, \nabla)) \cong \bigoplus_{\delta \in G} (\mathcal{M}, \nabla)$$

as objects of $\mathbf{MIC}(Z' \times \{\gamma\}) = \mathbf{MIC}(Z')$. Hence $\mathcal{P}$ being strongly exact and local for the étale topology, and pr_1 being an étale covering yields that

$$\mathcal{P}((\mathcal{M}, \nabla)) \iff \mathcal{P}(\mathrm{pr}_1^* \mathrm{pr}_{1*}(\mathcal{M}, \nabla)) \iff \mathcal{P}(\mathrm{pr}_{1*}(\mathcal{M}, \nabla))$$

Applying this to $(\mathcal{M}, \nabla) = \sigma''^* \tau^*(\mathcal{E}, \nabla)$ yields $\mathcal{P}(\mathrm{pr}_{1*} \sigma''^* \tau^*(\mathcal{E}, \nabla))$. Applying flat base change (2.5.3) to the two squares in (3.4), as functors $\mathbf{MIC}(X) \to \mathbf{MIC}(Z')$ we have

$$\mathrm{pr}_{1*} \sigma''^* \tau^* = \pi'_*(\tau'_* \sigma''^*)\tau^* = (\pi'_* \sigma'^*)\tau_* \tau^* = \sigma^* \pi_* \tau_* \tau^*,$$

and so we get $\mathcal{P}(\pi_* \tau_* \tau^*(\mathcal{E}, \nabla))$ since $\mathcal{P}$ is étale-local. As τ is faithfully flat (being an étale covering), the functor τ^* is faithful, hence it follows that the sheaf unit map $(\mathcal{E}, \nabla) \to \tau_* \tau^*(\mathcal{E}, \nabla)$ is a monomorphism by the argument of [24, 07RB]. Hence as π_* is left-exact, $\pi_*(\mathcal{E}, \nabla) \to \pi_* \tau_* \tau^*(\mathcal{E}, \nabla)$ is also a monomorphism. Thus by strong exactness, we get $\mathcal{P}(\pi_*(\mathcal{E}, \nabla))$. $\square$

The following lemma allows us to prove results about direct images of general smooth morphisms of smooth varieties, by instead arguing about an (a priori) much more specific case. Its proof acts as a stand-in for when we wish to prove results in the general setting by arguing about that specific case.

Lemma 3.2.4. *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth $\mathbb{C}$-varieties, which is strongly exact and local for the étale topology. Suppose that*

For any rational elementary fibration $f' : X' \to S'$ with S' affine, and for any object $(\mathcal{E}', \nabla')$ of $\boldsymbol{MIC}(X')$ and $j = 0, 1$,

$$\mathcal{P}((\mathcal{E}', \nabla')) \implies \mathcal{P}(R^j_{DR} f_*(\mathcal{E}', \nabla')).$$

Then for any $i \geq 0$, any smooth morphism $f : X \to S$ of smooth $\mathbb{C}$-varieties and any $(\mathcal{E}, \nabla)$ in $\boldsymbol{MIC}(X)$, we have

$$\mathcal{P}((\mathcal{E}, \nabla)) \implies \mathcal{P}((R^i_{DR} f_*(\mathcal{E}, \nabla)|_{A_i(f)}).$$

Proof. We first reduce to the case where f is of pure relative dimension $d \geq 1$, and X and S are connected. Since $\mathcal{P}$ is local for the étale topology, by taking connected components, we may assume that S is connected. We may also assume that X is connected, as if $X = X_1 \coprod X_2$, then $f = f_1 \coprod f_2 : X_1 \coprod X_2 \to S$. Writing $\iota_j : X_j \hookrightarrow X$, the $\mathcal{O}_X$-module $(\mathcal{E}, \nabla)$ then splits as

$$(\mathcal{E}, \nabla) = \iota_{1*}((\mathcal{E}, \nabla)|_{X_1}) \oplus \iota_{2*}((\mathcal{E}, \nabla)|_{X_2}).$$

Thus $R^0_{DR} f_*(\mathcal{E}, \nabla) = R^0_{DR} f_{1*}((\mathcal{E}, \nabla)|_{X_1}) \oplus R^0_{DR} f_{2*}((\mathcal{E}, \nabla)|_{X_2})$ by Corollary 2.6.2, as ι_j are finite étale (being open immersions as connected components). Hence, taking higher direct images gives

$$R^i_{DR} f_*(\mathcal{E}, \nabla) = R^i_{DR} f_{1*}((\mathcal{E}, \nabla)|_{X_1}) \oplus R^i_{DR} f_{2*}((\mathcal{E}, \nabla)|_{X_2})$$

Thus by strong exactness, we may assume that X is connected, hence that f is of pure relative dimension d. When $d = 0$, f is étale, so $f_{A_i(f)}$ is finite étale by Lemma 3.1.13 (3), and hence $\mathcal{P}(f_*(\mathcal{E}, \nabla)|_{A_i(f)})$ follows by Lemma 3.2.3, and $R^i_{DR} f_*((\mathcal{E}, \nabla)|_{A_i(f)}) = 0$ for $i > 0$ by Corollary 2.6.2. Since $R^j_{DR} f_*(\mathcal{E}, \nabla) = 0$ for $j > d + \dim(X)$, we may take $0 \leq j \leq d + \dim(X)$.

We now reduce to when f is a coordinatised elementary fibration. To simplify notation, without loss of generality we may take $S = A_i(f)$. By flat base change, $R^j_{DR} f_*$ is commutes with étale localisation $\mathcal{E}$. Thus as $\mathcal{P}$ is local for the étale topology, we may replace S with an affine, connected (by the same argument as above) étale neighbourhood S', so that $X_{S'}$ has a finite open cover $\{U_\alpha\}$ as in the definition of $A_i(f)$. Note that as $A_i(f) = S \supseteq A_{i-|\alpha|+1}(f_{\underline{\alpha}})$, we have $A_i(f) = A_{i-|\alpha|+1}(f_{\underline{\alpha}})$ from Lemma 3.1.13 (2).

We now show that, by arguing that $\mathcal{P}(R^i_{DR} f_*(\mathcal{E}, \nabla))$ by induction on i, we can reduce to the case of when f is a coordinatised elementary fibration. The covering $\{U_\alpha\}$ has associated Zariski spectral sequence

$$_Z E_1^{p,i-p} = \bigoplus_{\alpha_0 < \alpha_1 < \ldots < \alpha_p} R^{i-p}_{DR} f_{\underline{\alpha}*}(\mathcal{E}, \nabla)|_{U_{\underline{\alpha}}} \Rightarrow R^i_{DR} f_*(\mathcal{E}, \nabla)$$

and by induction and strong exactness, to show $\mathcal{P}(R^i_{DR} f_*(\mathcal{E}, \nabla))$ it suffices to show $\mathcal{P}(R^{i-p}_{DR} f_{\underline{\alpha}*}(\mathcal{E}, \nabla)|_{U_{\underline{\alpha}}})$ for all such $\underline{\alpha}$. But when $i - |\underline{\alpha}| + 1 < i$ or equivalently $|\underline{\alpha}| = p + 1 > 1$, we have $\mathcal{P}(R^{i-p}_{DR} f_{\underline{\alpha}*}(\mathcal{E}, \nabla))$ by induction. Thus we are left to show $R^i_{DR} f_{\alpha*}(\mathcal{E}, \nabla)|_{U_\alpha}$ for a single open set U_α. By replacing X with U_α, we are reduced to the case where f is a tower

$$X = X_d \xrightarrow{f_{d-1}} X_{d-1} \xrightarrow{f_{d-2}} \ldots \xrightarrow{f_0} X_0 = S$$

of coordinatised elementary fibrations. By repeated applications of Leray spectral sequences of the form

$$_{L,k} E_2^{p,i-p} = R^p_{DR}(f_0 \circ \cdots \circ f_{k-1})_* \circ R^{i-p}_{DR} f_{k*}(\mathcal{E}, \nabla) \Rightarrow R^i_{DR}(f_0 \circ \cdots \circ f_k)_*(\mathcal{E}, \nabla)$$

we are reduced to proving $\mathcal{P}(R^i_{DR} f_*(\mathcal{E}, \nabla))$ for f a single coordinatised elementary fibration.

When f is a coordinatised elementary fibration, since f is affine of relative dimension 1 by Lemma 3.1.2, $R^i_{DR} f_*(\mathcal{E}, \nabla)$ can only be non-zero when $i = 0, 1$ by Corollary 2.6.2. In this case, we have a diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "\pi"] & \overline{X} \arrow[d] & Z \arrow[l, hook'] \arrow[d] \\
Y \arrow[r, hook] \arrow[dr, "f'"] & \mathbb{P}^1_S \arrow[d] & Z' \arrow[l, hook'] \arrow[dl] \\
 & S &
\end{tikzcd}$$

where π is étale, $f = f' \circ \pi$, and if necessary, we replace S by an étale covering so that f' becomes a rational elementary fibration. By Corollary 2.6.2, we have $R^i_{DR} f'_*(\pi_*(\mathcal{E}, \nabla)) = R^i_{DR} f_*(\mathcal{E}, \nabla)$. We hence get $\mathcal{P}(\pi_*(\mathcal{E}, \nabla))$, and thus as f' is now a rational elementary fibration and $i = 0, 1$, we get $\mathcal{P}(R^i_{DR} f'_*(\pi_*(\mathcal{E}, \nabla))$, that is, $\mathcal{P}(R^i_{DR} f_*(\mathcal{E}, \nabla))$, as desired. $\square$

Remark 3.2.5. By replacing S with the dense open subset $A_{d+\dim(X)}(f)$ of the image, we have a version of the dévissage lemma 3.2.4 which holds globally on S, for all $i \geq 0$. This is due to the vanishing of $R^i_{DR} f_*$ for $i > d + \dim(X)$ as in Proposition 2.6.1(1).

Chapter 4

Regularity of connections

We take an interlude to discuss the regularity of connections, drawing mostly from [2, Chapter III]. These will be connections whose singularities behave nicely, in the sense that they are *meromorphic at infinity*, in a way that we will formalise shortly. Although the content of this chapter runs roughly parallel to Chapter 1, we have deferred its content as we will mostly make use of this in the next chapter (Chapter 5), with a brief mention in Chapter 7.

We will first define an appropriate notion of a meromorphic connection, and state a few key results on regularity which we will require in the next chapter. In this chapter X denotes a smooth $\mathbb{C}$-variety, and Z denotes a strict normal crossing divisor* in X, with closed immersion $j : Z \hookrightarrow X$. In this chapter we will only consider the absolute setting $X/\mathbb{C}$.

4.1 Logarithmic singularities

We describe Kähler differentials and connections with logarithmic singularities along strict normal crossing divisors.

Definition 4.1.1. 1. The sheaf $\Omega^1_X(\log Z)$ of *Kähler differentials logarithmic along* Z is the subsheaf of $j_*\Omega^1_{X\setminus Z}$ generated by Ω^1_X and the local sections df/f, for any local section f of $j_*\mathcal{O}_{X\setminus Z}$;

2. We set $\Omega^h_X(\log Z) := \bigwedge^h \Omega^1_X(\log Z)$;

*i.e. a normal, effective Cartier divisor whose irreducible components are smooth, and so that each point z of Z has a neighbourhood U and an étale map $\varphi : U \to \mathbb{A}^n$, so that $Z \cap U \subseteq U$ is the preimage of $V(x_1 \ldots x_m) \subseteq \mathbb{A}^n$ for some $m \leq n$; see [24, 0BI9]

The smoothness of X and local structure of Z imply the following about $\Omega^1_X(\log Z)$.

Lemma 4.1.2. *The $\mathcal{O}_X$-module $\Omega^1_X(\log Z)$ is locally free of finite rank.*

We refer to [10, II, 3.2, 3.3.1] for the proof (this is given in analytic context, but can be translated directly into the étale topology).

Definition 4.1.3. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. A *connection* ∇ *with logarithmic poles along* Z is one of the following equivalent pieces of data:

1. A $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \Omega^1_X(\log Z) \otimes_{\mathcal{O}_X} \mathcal{E}$;

2. An $\mathcal{O}_X$-linear map $\nabla_\bullet : \mathcal{T}_X(\log Z) \to \mathcal{E}nd(\mathcal{E})$.

An *$\mathcal{O}_X$-module with a connection logarithmic along Z is a pair* $(\mathcal{E}, \nabla)$ is a pair where $\mathcal{E}$ is a $\mathcal{O}_X$-module, and ∇ is a connection logarithmic along Z.

A *morphism* of pairs $(\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$ is a map $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ so that $(\mathrm{id}_{\Omega^1_X(\log Z)} \otimes \varphi) \circ \nabla_1 = \nabla_2 \circ \varphi$.

Remark 4.1.4. As in the case without poles, a connection with logarithmic poles propogates to higher connections $\nabla^q : \Omega^q_X(\log Z) \otimes \mathcal{E} \to \Omega^{q+1}_X(\log Z) \otimes \mathcal{E}$ by the same formula as in (1.4). As in the discussion proceeding (1.4), this gives rise to a curvature morphism

$$K_\nabla := \nabla^1 \circ \nabla \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^2_X(\log Z) \otimes \mathcal{E}).$$

Definition 4.1.5. A connection ∇ as in Definition 4.1.3 is *integrable* if $K_\nabla = 0$.

Definition 4.1.6. The category $\mathbf{MIC}(X(\log Z))$ has

1. Objects as $\mathcal{O}_X$-modules with integrable connections $(\mathcal{E}, \nabla)$, logarithmic along Z; and

2. Morphisms as in Definition 4.1.3.

Remark 4.1.7 (Pullbacks for logarithmic connections)**.** Let $i : X \hookrightarrow \overline{X}$ be an open immersion, and suppose that the complementary reduced, closed subvariety $Z \subseteq \overline{X}$ is a strict normal crossing divisor. Then $i^* = i^{-1}$ as functors $\mathrm{Mod}(\mathcal{O}_{\overline{X}}) \to \mathrm{Mod}(\mathcal{O}_X)$, and we may directly pull back any connection using $i^{-1} = (-)|_X$ to give

$$i^*\nabla = i^{-1}\nabla : i^*\mathcal{E} = \mathcal{E}|_X \to \Omega^1_{\overline{X}}(\log Z)|_X \otimes_{\mathcal{O}_{\overline{X}}|_X} \mathcal{E}|_X = \Omega^1_X \otimes_{\mathcal{O}_X} i^*\mathcal{E}$$

4.2 Regularity

We look to define regularity for coherent $\mathcal{O}_X$-modules with integrable connection. There is a notion of regularity which makes sense for non-coherent $\mathcal{O}_X$-modules, but this will not be necessary for our purposes. We use normal compactification to refer to an normal, complete $\mathbb{C}$-variety $\overline{X}$ with an open immersion $i : X \hookrightarrow \overline{X}$. Note that in this case, $Z := \overline{X} \setminus X$ is a strict normal crossing divisor.

Definition 4.2.1. Let $(\mathcal{E}, \nabla)$ be a coherent object of $\mathbf{MIC}(X)$. We say ∇ is *regular* if for any normal compactification $i : X \hookrightarrow \overline{X}$, there is a coherent, locally free sheaf $\tilde{\mathcal{E}}$ and an (integrable) connection $\widetilde{\nabla}$ on $\widetilde{\mathcal{E}}$ with logarithmic poles along Z, such that $(\widetilde{\mathcal{E}}, \widetilde{\nabla})|_X \cong (\mathcal{E}, \nabla)$.

Remark 4.2.2. The definition given in [2, 11.2.2, 13.1.5(b)] requires that $\widetilde{\mathcal{E}}$ is a subsheaf of $j_*\mathcal{E}$, but this will be guaranteed (up to identifying by an isomorphism) by $\widetilde{\mathcal{E}}$ being locally free, and the isomorphism $\varphi : i^*(\tilde{\mathcal{E}}, \tilde{\nabla}) \cong (\mathcal{E}, \nabla)$. Indeed, on the level of sheaves, we have

$$\widetilde{\mathcal{E}} \hookrightarrow i_* i^* \widetilde{\mathcal{E}} \xhookrightarrow{i_*\varphi} i_*\mathcal{E},$$

where the first arrow is injective as it is given by restrictions, and $\widetilde{\mathcal{E}}$ is locally free (in particular torsion-free); and the second arrow is injective as i_* is left-exact.

Note that unlike in the setting of §1.2, we also need to assert that the sheaf $\widetilde{\mathcal{E}}$ is regular, as the connection may have poles.

Remark 4.2.3. This definition is not vacuous, as a normal compactification always exists. Indeed, given a smooth variety X, we may take a Nagata compactification X' of X [24, 0F41], and $X \hookrightarrow X'$ factors through the normalisation $\overline{X}$ of X', and $\overline{X}$ is complete as X' is complete and $\overline{X} \to X'$ is integral [24, 035Q].

To actually check whether a connection is regular, we will use the following criterion.

Theorem 4.2.4. *Let $(\mathcal{E}, \nabla)$ be a coherent object of $\boldsymbol{MIC}(X)$. Then ∇ is regular if and only if for any locally closed immersion of a smooth curve $i : C \hookrightarrow X$, $i^*(\nabla)$ is regular.*

Hence we can test for regularity by cutting X by curves, and we refer to [1, I.5.7] for the proof. In lieu of this, it is worth describing the case of regularity for a curve in a bit more detail.

Remark 4.2.5. Let C be a curve, and $(\mathcal{E}, \nabla)$ be a coherent object of $\mathbf{MIC}(C)$. A compactification $\overline{C}$ is a complete curve, and hence projective [Sta, 0A26; 0B45]; the complement $\overline{C} \backslash C$ is then a finite set of points. Checking regularity for ∇ then amounts to checking that *for any projective curve $\overline{C}$ where C is the complement of a finite set of points Σ in $\overline{C}$, there is an extension as in Definition 4.2.1.*

Lemma 4.2.6. *Regularity is strongly exact, and local for the étale topology.*

This will be necessary to establish that regularity is preserved under direct images in the next chapter, and we refer to [2, 8.3.9, 10.3.2] for the proof.

4.3 Residues and τ-extensions

We now consider the setting where

1. $i : X \hookrightarrow \overline{X}$ is an open immersion of smooth varieties of dimension d, (in particular with completeness no assumptions on $\overline{X}$);

2. $Z = \overline{X} \setminus X = \bigcup_{j=1}^{n} Z_j$ is a strict normal crossing divisor, and $\iota_j : Z_j \hookrightarrow X$ are the associated closed immersions; and

3. $(\mathcal{E}, \nabla)$ is a coherent object of $\mathbf{MIC}(X)$ with a regular connection.

In the proof of Theorem 5.1.1, we will want slightly more control over extending $(\mathcal{E}, \nabla)$ to an object of $\mathbf{MIC}(X(\log Z))$.

Remark 4.3.1. Since Z is a strict normal crossing divisor, around each point of Z we have étale coordinates $x_1, \ldots, x_d$, so that $Z_j = V(x_j)$ for $1 \leq j \leq r$, and for some $r \leq d$ (see [24, 0CBN]).

Proposition 4.3.2. *Let $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ be an object of $\boldsymbol{MIC}(\overline{X}(\log Z))$. Then for each $1 \leq j \leq n$, there is a well-defined $\mathcal{O}_{Z_j}$-linear endomorphism*

$$\operatorname{Res}_{Z_j} : \iota_j^* \mathcal{M} \to \iota_j^* \mathcal{M}$$

such that

1. Res_{Z_j} *locally induced by the action of $\tilde{\nabla}(x_j \partial_{x_j})$ in étale coordinates as in Remark 4.3.1;*

2. *The characteristic polynomial $\phi_j(t)$ of* Res_{Z_j} *lies in* $\mathbb{C}[t]$.

We refer to [10, II.3.5, II.3.6, II.3.7] for the construction of Res_{Z_j} and the proof of (1), and [10, II.3.10] for the proof of (2).

Since the characteristic polynomials are in fact complex polynomials, their roots will be in $\mathbb{C}$. We will find it useful later when these eigenvalues are *non-resonant*, i.e. when that these eigenvalues do not differ by an integer, or take on a non-zero integer value. This is equivalent to asserting that they are in the image of some section τ of the quotient $\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathbb{Z}$ (of abelian groups) with $\tau(0) = 0$.

Definition 4.3.3. Let $(\mathcal{E}, \nabla)$ be a coherent object of $\mathbf{MIC}(X)$, and τ be a section of $\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathbb{Z}$ with $\tau(0) = 0$. A *τ-extension of* $(\mathcal{E}, \nabla)$ *on* $\overline{X}$ is an object $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ of $\mathbf{MIC}(X(\log Z))$ so that

1. $\widetilde{\mathcal{E}}$ is coherent and locally free;
2. $(\widetilde{\mathcal{E}}, \widetilde{\nabla})|_X \cong (\mathcal{E}, \nabla)$; and
3. the roots of each characteristic polynomial $\phi_j(t)$ of Res_{Z_j} are in the image of τ.

Due to the *non-resonance* of the eigenvalues, we may extend any coherent $(\mathcal{E}, \nabla)$ on X to $\overline{X}$.

Theorem 4.3.4. *Let $(\mathcal{E}, \nabla)$ be a coherent object of **MIC**(X). Then for any τ as in Definition 4.3.3, there is a τ-extension $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ on $\overline{X}$, unique up to unique isomorphism.*

We refer to [1, I.4.9] for the statement and its proof.

Chapter 5

Algebraic vector bundles with regular connection

We prove the main structure theorems required for the proof of the comparison theorem, mostly following [1, III.5, III.6, III.7], with an alternate argument to. By Theorem 1.2.7, for an object $(\mathcal{E}, \nabla)$ in $\mathbf{MIC}(X)$, we reformulate the statement $(\mathcal{E}, \nabla)$ *is an algebraic vector bundle* as just the coherence of $(\mathcal{E}, \nabla)$ within the proofs. In this chapter we will first prove each statement for either a coordinatised or rational elementary fibration, and use dévissage for the general case.

5.1 Coherence of higher direct images

We look to show the following.

Theorem 5.1.1. *Let $f : X \to S$ be a coordinatised elementary fibration. Let $(\mathcal{E}, \nabla)$ be a locally free $\mathcal{O}_X$-module of finite rank with regular, integrable connection ∇. Then for $i = 0, 1$, the $\mathcal{O}_X$-module with integrable connection $R^i_{DR} f_*(\mathcal{E}, \nabla)$ is locally free of finite rank.*

We will make use of regularity by taking a τ-extension of $(\mathcal{E}, \nabla)$ to $\mathbb{P}^1 \times S$.

Remark 5.1.2. Since coherence is étale-local and the formation of $R^1_{DR} f_*$ commutes with étale localisation (by the flat base change proposition 2.5.3), we may assume that S is affine and connected in the proof of Theorem 5.1.1.

The following will allow us to make an important reduction to an even simpler case in the proof of Theorem 5.1.1. The main idea behind the following lemma is due to Anand Deopurkar.

Lemma 5.1.3. *Let S be a smooth, connected, affine variety over $\mathbb{C}$. Then for any locally free $\mathcal{O}_{\mathbb{P}^1\times S}$-module $\mathcal{F}$ of finite rank, there is an open cover $\{V_i\}$ of S such that for each i, the sheaf $\mathcal{F}|_{\mathbb{A}^1\times V_i}$ is free.*

Proof. We show this by induction on $d = \dim(\mathcal{F})$. Note that when $d = 1$, since S is a smooth, connected variety, $\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(S \times \mathbb{A}^1)$ is an isomorphism by [22, II.6.6, II.6.16], so $\mathcal{F}|_{\mathbb{A}^1_S} = \pi^*\mathcal{L}$ for some line bundle on S, and choosing a trivialising cover $\{V_i\}$ for S yields the result.

For $d > 1$, let $s \in S$ and $\mathcal{F}_s = \mathcal{F}|_{\mathbb{P}^1\times\{s\}}$, and $\mathcal{O}_{\mathbb{P}^1}(1)$ be the twisting sheaf of $\mathbb{P}^1$. Since S is affine, by Serre's theorems on vanishing and global generation [22, III.5.2, II.5.17], for $k > 0$ and n sufficiently large we have $H^k(\mathcal{F}_s(n)) = 0$, and $\Gamma(\mathbb{P}^1, \mathcal{F}_s(n)) \neq 0$. Since $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-n), V_s) \cong \Gamma(\mathbb{P}^1, V_s(n))$ (by the tensor-Hom adjunction), we may choose $\varphi : \mathcal{O}_{\mathbb{P}^1}(-n) \to V_s$ nowhere vanishing (and hence injective).

By cohomology and base change [40, 24.8.1] and [22, III.12.9, III.12.11(b)], it follows that $\Gamma(\mathbb{P}^1, \mathcal{F}_s) \cong \mathrm{pr}_{1*}(\mathcal{F}) \otimes \kappa(s)$, so the section φ admits an extension $\widetilde{\varphi}$ to a Zariski neighbourhood U of s. As the zero set Z of $\widetilde{\varphi}$ is closed and disjoint from $\mathbb{P}^1 \times \{s\}$, by replacing S with $U - Z$ we may assume that $\widetilde{\varphi}$ is globally defined and nowhere vanishing. This gives a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^1\times S}(-n) \xrightarrow{\widetilde{\varphi}} \mathcal{F} \to Q \to 0,$$

where $Q|_{\mathbb{P}^1\times S}$ is locally free exactly as $\widetilde{\varphi}$ is nowhere vanishing. On restricting to $\mathbb{A}^1 \times S$, $\mathcal{O}_{\mathbb{P}^1\times S}(-n)$ becomes free, and by induction we may further shrink S so that Q is free. Since S is affine, there are no non-trivial extensions of free modules, so $\mathcal{F}|_{\mathbb{A}^1\times S}$ must also be free. $\square$

Remark 5.1.4. Andre-Baldassari rely on a special case of the Bass-Quillen conjecture, though the line of reasoning provided in their text [1, III.6.3.1] seemed dubious and unclear.

Notation 5.1.5. In the rational elementary fibration

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{A}^1 \times S \arrow[d, "\mathrm{pr}_{\mathbb{A}^1}"] & \coprod_{i=1}^{r} \theta_i(S) \arrow[l, hook'] \arrow[dl] \\ & S & \end{tikzcd}$$

of Theorem 5.1.1, let

1. x denote the affine coordinate on $\mathbb{A}^1 \times S$, and $\mathbb{A}^1_\infty := \mathbb{P}^1 - \{0\}$ ($\cong \mathbb{A}^1$);

2. For $i = 1, \dots, r, \infty$, $Z_i := \theta_i(S)$, and $\iota_i : Z_i \hookrightarrow \mathbb{P}^1 \times S$ be the associated inclusion;

3. $Z = \coprod_{i=1,\dots,r,\infty} Z_i$; and

4. $\Theta(x) := \prod_{i=1}^r (x - \theta_i) \in \mathcal{O}(S)[x]$; and $\Theta_\infty(x) := \frac{1}{x} \prod_{i=2}^r \left(\frac{1}{x} - \frac{1}{\theta_i}\right) \in K(S)\left[\frac{1}{x}\right]$, when $\theta_2, \dots, \theta_r$ are nowhere vanishing.

For convenience, we also recall what a τ-extension is, in the setting $X \subseteq \mathbb{P}^1 \times S$. For a given $(\mathcal{E}, \nabla) \in \operatorname{ob} \mathbf{MIC}(X)$ and a section τ of the quotient $\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathbb{Z}$ with $\tau(0) = 0$, a τ-extension is a pair $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ where

1. $\widetilde{\mathcal{E}}$ is a locally free $\mathcal{O}_{\mathbb{P}^1 \times S}$-module of finite rank such that $\widetilde{\mathcal{E}}|_X = \mathcal{E}$;

2. $\widetilde{\nabla} : \mathcal{T}_{\mathbb{P}^1_S}(\log Z) \to \mathcal{E}nd_{\mathbb{C}} \widetilde{\mathcal{E}}$ is an integrable connection with logarithmic poles along Z such that $\widetilde{\nabla}|_{\mathcal{T}_X} = \nabla$; and

3. The endomorphisms $\operatorname{Res}_{\theta_i} \widetilde{\nabla} : \iota_i^* \widetilde{\mathcal{E}} \to \iota_i^* \widetilde{\mathcal{E}}$ locally induced by $\widetilde{\nabla}_{(x-\theta_i)\partial_x}$ have eigenvalues in the image of τ.

We will show coherence by showing the following proposition, which will also aid us in showing the stability of regularity under direct image.

Proposition 5.1.6. *Suppose that $f : X \to S$ is a coordinatisd elementary fibration, $(\mathcal{E}, \nabla)$ is a locally free $\mathcal{O}_X$-module of finite rank with integrable connection, and $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ is a τ-extension of $(\mathcal{E}, \nabla)$ to $\mathbb{P}^1 \times S$. Then the morphism of $\mathcal{O}_S$-modules*

$$\mathbb{R}^i \overline{f}_*(\Omega^\bullet_{(\mathbb{P}^1 \times S)/S}(\log Z) \otimes \widetilde{\mathcal{E}}) \to \mathbb{R}^i \overline{f}_*(j_*(\Omega^\bullet_{X/S} \otimes \mathcal{E})) \cong R^i_{DR} f_*(\mathcal{E}, \nabla)$$

induced by the open immersion $j : X \to \mathbb{P}^1 \times S$ is an isomorphism for $i = 0, 1$.

Proof of Theorem 5.1.1 from Proposition 5.1.6. We first consider the case where f is a rational elementary fibration. By the spectral sequence (1.6) on hypercohomology, $\mathbb{R}^{p+q} \overline{f}_*(\Omega^\bullet_{(\mathbb{P}^1 \times S)/S}(\log Z) \otimes \widetilde{\mathcal{E}})$ is the abutment of a spectral sequence with terms ${}_I E_1^{p,q} = R^q \overline{f}_*(\Omega^p_{(\mathbb{P}^1 \times S)/S}(\log Z) \otimes \widetilde{\mathcal{E}})$, which are coherent by the stability of coherence under projective higher direct image. Hence as coherence is strongly exact, $R^i_{DR} f_*(\mathcal{E}, \nabla)$ is coherent.

This argument extends to the case of a coordinatised elementary fibration by the same argument as in the end of the proof of the dévissage lemma 3.2.4. $\square$

Proof of Proposition 5.1.6. By Remark 5.1.2 we may take S to be affine and connected. By applying the translation $x \mapsto x - \theta_1$ on the affine coordinate, we may assume that $\theta_1 = 0$. Let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module with integrable connection, and $d = \dim(\mathcal{E})$, which is constant as S is connected. Fixing a section τ of $\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathbb{Z}$ with $\tau(0) = 0$, by Theorem 4.3.4 we have a τ-extension of $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ of $(\mathcal{E}, \nabla)$ on $\mathbb{P}^1 \times S$.

By Lemma 5.1.3 applied to $\mathbb{A}^1, \mathbb{A}^1_\infty \subseteq \mathbb{P}^1$, as rational elementary fibrations are stable under base change due by Lemma 3.1.6, by shrinking S further, we may assume in addition that $\widetilde{\mathcal{E}}|_{\mathbb{A}^1 \times S}$ is a free $\mathcal{O}_{\mathbb{A}^1 \times S}$-module, and $\widetilde{\mathcal{E}}|_{\mathbb{A}^1_\infty \times S}$ is a free $\mathcal{O}_{\mathbb{A}^1_\infty \times S}$-module. Hence for $M = \mathcal{O}(S)^{\oplus d}$, we may write $\widetilde{\mathcal{E}}(\mathbb{A}^1 \times S) = M[x]$ and $\widetilde{\mathcal{E}}(\mathbb{A}^1_\infty \times S) = M[x^{-1}]$.

Since f is affine, the higher direct images of $\mathcal{E}$ are the kernel and cokernel of $\nabla_{\partial_x} : M[x, 1/\Theta(x)] \to M[x, 1/\Theta(x)]$ respectively, by (2.25). Since we have $(1/\Theta(x))\widetilde{\nabla}_{\Theta(x)\partial_x} = \widetilde{\nabla}_{\partial_x}$, multiplication by $1/\Theta(x)$ induces a well-defined map on cokernels:

$$\frac{M[x]}{\widetilde{\nabla}_{\Theta(x)\partial_x}(M[x])} \to \frac{M[x, 1/\Theta(x)]}{\nabla_{\partial_x}(M[x, 1/\Theta(x)])}. \tag{5.1}$$

We will show that this map is an isomorphism, and use the Čech spectral sequence of Proposition 2.4.3 to compute $R^i_{DR} f_*(\mathcal{E}, \nabla)$. We first establish surjectivity of (5.1).

Claim 5.1.7. We have

$$M\left[x, \frac{1}{\Theta(x)}\right] = \widetilde{\nabla}_{\Theta(x)\partial_x}\left(M\left[x, \frac{1}{\Theta(x)}\right]\right) + M[x]$$

Proof of Claim 5.1.7. We first compute the leading term of $\widetilde{\nabla}_{\Theta(x)\partial_x}(m(x-\theta_i)^{-n})$, for $m \in M$ and $n \geq 0$. We claim that

$$\widetilde{\nabla}_{(x-\theta_i)\partial_x} m - (\operatorname{Res}_{\theta_i} \widetilde{\nabla})(m) \in (x - \theta_i) M\left[x, \frac{1}{\Theta(x)}\right].$$

This is as restriction to Z_i corresponds to the map $M[x] \to M$ given by evaluation at θ_i, and that $\widetilde{\nabla}_{(x-\theta_i)\partial_x} m$ is sent to $(\operatorname{Res}_{\theta_i} \widetilde{\nabla})m$. Writing $\Theta_i(x) = \prod_{k \neq i}(x - \theta_k)$, we find that

$$\begin{aligned}\widetilde{\nabla}_{\Theta(x)\partial_x}(m(x - \theta_i)^{-n}) \in \Theta_i(x)\Big((\operatorname{Res}_{\theta_i} \widetilde{\nabla})(m) - nm\Big)&(x - \theta_i)^{-n} \\ &+ (x - \theta_i)^{-n+1} M[x].\end{aligned}$$

We further note that for $\gamma_i = \prod_{k\neq i}(\theta_i - \theta_k) \in \mathcal{O}(S)^*$, $\Theta_i(x) \in \gamma_i + (x-\theta_i)M[x]$, and hence we find the leading term:

$$\widetilde{\nabla}_{\Theta(x)\partial_x}(m(x-\theta_i)^{-n}) \in \gamma_i(\operatorname{Res}_{\theta_i}\widetilde{\nabla} - n\,\mathrm{id}_M)(m)(x-\theta_i)^{-n} + (x-\theta_i)^{-n+1}M[x]. \tag{5.2}$$

We claim now that for $n > 0$, $\operatorname{Res}_{\theta_i}\widetilde{\nabla} - n\,\mathrm{id}_M$ is an automorphism of $M = \mathcal{O}(S)^d$. Indeed, this gives an automorphism of $M \otimes_{\mathcal{O}(S)} K(S) = K(S)^d$ as n is not an eigenvalue of $\operatorname{Res}_{\theta_i}\widetilde{\nabla}$. By the Cayley-Hamilton theorem, its inverse on $K(S)^d$ is a polynomial in $\operatorname{Res}_{\theta_i}\widetilde{\nabla}$, and hence also defines an endomorphism of $\mathcal{O}(S)^d$, and this is a two-sided inverse to $\operatorname{Res}_{\theta_i}\widetilde{\nabla} - n\,\mathrm{id}_M$ as an endomorphism of M.

Replacing m with $(\operatorname{Res}_{\theta_i}\widetilde{\nabla} - n\,\mathrm{id}_M)^{-1}(\gamma_i^{-1}m)$ in (5.2) yields

$$m(x-\theta_i)^{-n} \in \widetilde{\nabla}_{\Theta(x)\partial_x}(M(x-\theta_i)^{-n}) + (x-\theta_i)^{-n+1}M[x].$$

Hence

$$M(x-\theta_i)^{-n} \subseteq \widetilde{\nabla}_{\Theta(x)\partial_x}(M(x-\theta_i)^{-n}) + (x-\theta_i)^{-n+1}M[x].$$

Combining this with the Mittag-Leffler (partial fraction) decomposition

$$M\left[x, \frac{1}{\Theta(x)}\right] = M[x] \oplus \bigoplus_{k=1}^{r} \frac{1}{x-\theta_k} M\left[\frac{1}{x-\theta_k}\right],$$

this yields

$$M\left[x, \frac{1}{\Theta(x)}\right] = \widetilde{\nabla}_{\Theta(x)\partial_x} M\left[x, \frac{1}{\Theta(x)}\right] + M[x]$$

as claimed. $\square$

We now turn to injectivity of (5.1). Note that the class of $f \in M[x]$ is zero under the map in (5.1) if and only if $f/\Theta(x) \in \widetilde{\nabla}_{\partial_x}(M[x, 1/\Theta(x)])$, or equivalently $f \in M[x] \cap \nabla_{\Theta(x)\partial_x}(M[x, 1/\Theta(x)])$. This thus requires us to show

$$M[x] \cap \widetilde{\nabla}_{\Theta(x)\partial_x}\left(M\left[x, \frac{1}{\Theta(x)}\right]\right) = \widetilde{\nabla}_{\Theta(x)\partial_x}(M[x]).$$

We will actually show the following.

Claim 5.1.8. If $f \in M[x, 1/\Theta(x)]$ has $\widetilde{\nabla}_{\Theta(x)\partial_x} f \in M[x]$, then $f \in M[x]$.

This will also show that

$$\ker_{M[x]} \widetilde{\nabla}_{\Theta(x)\partial_x} = \ker_{M[x,1/\Theta(x)]} \nabla_{\partial_x},$$

as the forward inclusion is clear, and for the reverse inclusion, if $\widetilde{\nabla}_{\partial_x} f = 0$, then $\widetilde{\nabla}_{\Theta(x)\partial_x} f = 0 \in M[x]$, so $f \in M[x]$ by the claim.

Proof of Claim 5.1.8. If f has a pole of order $n \geq 0$ at θ_i, then by taking the lowest non-zero term $m(x-\theta_i)^{-n}$, either $n = 0$, or by the formula (5.2), we have $(\operatorname{Res}_{\theta_i} \widetilde{\nabla} - n\,\mathrm{id}_M)(m) = 0$. But then in the latter case, $\ker(\operatorname{Res}_{\theta_i} \widetilde{\nabla} - n\,\mathrm{id}_M$ has non-trivial kernel, so n is an eigenvalue of $\operatorname{Res}_{\theta_i} \widetilde{\nabla}$, and $n \in \operatorname{im}(\tau) \cap \mathbb{Z} = \{0\}$. $\square$

Combining this, we get isomorphisms

$$(\mathrm{co})\ker_{M[x]} \widetilde{\nabla}_{\Theta(x)\partial_x} \cong (\mathrm{co})\ker_{M[x,1/\Theta(x)]} \widetilde{\nabla}_{\partial_x}$$

The same arguments applied to the inclusion $M[x] \hookrightarrow M[x,1/x]$ also show

$$(\mathrm{co})\ker_{M[x]} \nabla_{\Theta(x)\partial_x} \cong (\mathrm{co})\ker_{M[x,1/x]} \widetilde{\nabla}_{\Theta(x)\partial_x}$$

Similarly, by the symmetry $x \mapsto x^{-1}$, for $M[x^{-1}]$, recalling that $\theta_1 = 0$, in the notation of 5.1.5 we find

$$(\mathrm{co})\ker_{M[1/x]} \widetilde{\nabla}_{\Theta_\infty(x)\partial_{1/x}} \cong (\mathrm{co})\ker_{M[x,1/\Theta(x)]} \nabla_{\partial_x}$$

and by the multiplication map by $(-1)^r x^{r-1} \prod_{i=2}^r \theta_i = (-1)^r \gamma_1 x^{r-1}$,

$$(\mathrm{co})\ker_{M[1/x]} \widetilde{\nabla}_{\Theta_\infty(x)\partial_{1/x}} \cong (\mathrm{co})\ker_{M[x,1/x]} \nabla_{\Theta(x)\partial_x},$$

noting that $\partial_{1/x} = x^2\partial_x$. This together yields a diagram of isomorphisms

$$\begin{tikzcd}
(\mathrm{co})\ker_{M[x]} \widetilde{\nabla}_{\Theta(x)\partial_x} \arrow[d] & \\
(\mathrm{co})\ker_{M[x,1/x]} \widetilde{\nabla}_{\Theta(x)\partial_x} \arrow[r, "\cdot(1/\Theta(x))"] & (\mathrm{co})\ker_{M[x,1/\Theta(x)]} \nabla_{\partial_x} \\
(\mathrm{co})\ker_{M[1/x]} \widetilde{\nabla}_{\Theta_\infty(x)\partial_{1/x}} \arrow[u, "\cdot(-1)^r\gamma_1 x^r"'] &
\end{tikzcd}$$

We now re-express this in terms of the associated (cohomology) sheaves $\mathcal{H}^i(K^\bullet) = H^i(\widetilde{K^\bullet(S)})$ on S, and the open cover

$$\mathcal{U} = \left\{U_0 := \mathbb{A}^1 \times S, U_1 = \mathbb{A}^1_\infty \times S\right\}$$

with $U_{01} = U_0 \cap U_1$ as usual. Noting that $X \subseteq U_{01} = \mathbb{P}^1_S - (\sigma_1(S) \sqcup \sigma_\infty(S))$, for

$i = 0, 1$ we hence have the isomorphisms of $\mathcal{O}_S$-modules

$$\begin{array}{ccc}
\mathcal{H}^i(\overline{f}_{U_0*}(\Omega^\bullet_{U_0/S}(\log(U_0 \cap Z)) \otimes \widetilde{\mathcal{E}}|_{U_0}) & & \\
\downarrow & & \\
\mathcal{H}^i(\overline{f}|_{U_{01}*}(\Omega^\bullet_{U_{01}/S}(\log(U_{01} \cap Z)) \otimes \widetilde{\mathcal{E}}|_{U_{01}}) & \longrightarrow & \mathcal{H}^i f_*(\Omega^\bullet_{X/S} \otimes \mathcal{E}) \cong R^i_{DR} f_*(\mathcal{E}, \nabla) \\
\uparrow & & \\
\mathcal{H}^i(\overline{f}_*(\Omega^\bullet_{U_1/S}(\log(U_1 \cap Z)) \otimes \widetilde{\mathcal{E}}|_{U_1}) & &
\end{array} \tag{5.3}$$

We now use the Čech bicomplex $K = C^\bullet(\mathcal{U}, \Omega^\bullet_{(\mathbb{P}^1 \times S)/S}(\log Z) \otimes \widetilde{\mathcal{E}})$ to compute the hypercohomology $\mathbb{R}^i \overline{f}_*(\Omega^\bullet_{(\mathbb{P}^1 \times S)/S}(\log(Z)) \otimes \widetilde{\mathcal{E}})$. Writing $Z^{\underline{\alpha}} = U_{\underline{\alpha}} \cap Z$ for $\underline{\alpha} = 0, 1, 01$, this takes the form

$$\begin{array}{ccc}
(\Omega^1_{U_0/S}(\log(Z^0)) \otimes \widetilde{\mathcal{E}}|_{U_0}) \oplus (\Omega^1_{U_1/S}(\log Z^1) \otimes \widetilde{\mathcal{E}}|_{U_1}) & \xrightarrow{(-j_0^*, j_1^*)} & \Omega^1_{U_{01}/S}(\log(Z^{01})) \otimes \widetilde{\mathcal{E}}|_{U_{01}} \\
\Big\uparrow{\scriptstyle (\widetilde{\nabla}|_{U_0}, \widetilde{\nabla}|_{U_1})} & & \Big\uparrow{\scriptstyle \widetilde{\nabla}|_{U_{01}}} \\
\widetilde{\mathcal{E}}|_{U_0} \oplus \widetilde{\mathcal{E}}|_{U_1} & \xrightarrow{(j_0^*, j_1^*)} & \widetilde{\mathcal{E}}|_{U_{01}}
\end{array}$$

In this setting, applying the Čech spectral sequence to this open cover, we have that

$$_{\check{C}}E_1^{p,q} = \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} R^q f_{\underline{\alpha}*} \left(\Omega^j_{U_{\underline{\alpha}}/S}(\log Z^{\underline{\alpha}}) \otimes \widetilde{\mathcal{E}}|_{U_{\underline{\alpha}}} \right) \Rightarrow \mathbb{R}^{p+q} \overline{f}_*(\Omega^\bullet_{(\mathbb{P}^1 \times S)/S}(\log Z) \otimes \widetilde{\mathcal{E}}),$$

and the E_2-page can be computed by then taking cohomology with respect to the horizontal differentials. The E_1-page of this spectral sequence is

$$R^1_{DR} f_*(\mathcal{E}, \nabla)^{\oplus 2} \xrightarrow{(-1,1)} R^1_{DR} f_*(\mathcal{E}, \nabla)$$

$$R^0_{DR} f_*(\mathcal{E}, \nabla)^{\oplus 2} \xrightarrow{(1,1)} R^0_{DR} f_*(\mathcal{E}, \nabla)$$

Now, taking cohomology of the horizontal differentials yields the E_2-page

$$\begin{array}{cc}
R^1_{DR} f_*(\mathcal{E}, \nabla) & 0 \\
& \\
R^0_{DR} f_*(\mathcal{E}, \nabla) & 0
\end{array}$$

Thus this sequence degenerates at the E_2 page, and its convergence yields isomorphisms

$$\mathbb{R}^i \overline{f}_*(\Omega^1_{(\mathbb{P}^1 \times S)/S}(\log(Z)) \otimes \widetilde{\mathcal{E}}) \cong R^i_{DR} f_*(\mathcal{E}, \nabla)$$

induced by the open immersion $j : X \hookrightarrow \mathbb{P}^1 \times S$, from the diagram (5.3). $\square$

5.2 Base change for smooth morphisms

We establish that for $\mathcal{O}_X$-modules with regular integrable connections, the higher direct images of coordinatised elementary fibrations commute with arbitrary base change. This contrasts with flat base change, since the conditions we impose are on the morphism subject to the base change, rather than the base change morphism.

Theorem 5.2.1. *Let $f : X \to S$ be a coordinatised elementary fibration, and $u : S' \to S$ be a morphism of smooth varieties over $\mathbb{C}$. Let*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{u} & S \end{array}$$

be a fibred square. Then for any coherent object $(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(X)$ with regular connection, and $i = 0, 1$, we have

$$u^* R^i_{DR} f_*(\mathcal{E}, \nabla) \cong R^i_{DR} f'_*(u'^*(\mathcal{E}, \nabla))$$

Proof. Since f is affine by Lemma 3.1.2, by base change for affine maps [24, 30.5.1] we have

$$u^* f_* \mathcal{E} \cong f'_* u'^* \mathcal{E} \tag{5.4}$$

$$u^* f_*(\Omega^1_{X/S} \otimes \mathcal{E}) \cong f'_* u'^*(\Omega^1_{X/S} \otimes \mathcal{E}), \tag{5.5}$$

Since f also has fibres of dimension 1, by (2.25) we have an exact sequence

$$0 \to \ker(f_*(\nabla_{X/S})) \to f_*\mathcal{E} \xrightarrow{f_*(\nabla_{X/S})} f_*(\Omega^1_{X/S} \otimes \mathcal{E}) \to \operatorname{coker}(f_*(\nabla_{X/S})) \to 0. \tag{5.6}$$

We first argue that each of these $\mathcal{O}_S$-modules are flat. The flatness of $f_*\mathcal{E}$ and $f_*(\mathcal{E} \otimes \Omega^1_{X/S})$ follows from the flatness of f and the local freeness of $\mathcal{E}$ and $\Omega^1_{X/S} \otimes \mathcal{E}$. By equations (2.23) and (2.24) we have

$$R^0_{DR} f_*(\mathcal{E}, \nabla) \cong \ker(f_*(\nabla_{X/S})), \tag{5.7}$$

$$R^1_{DR} f_*(\mathcal{E}, \nabla) \cong \operatorname{coker}(f_*(\nabla_{X/S})). \tag{5.8}$$

By Theorem 5.1.1 these are locally free, and hence flat $\mathcal{O}_S$-modules. The exact sequence (5.6) breaks into 2 short-exact sequences

$$0 \to \ker(f_*(\nabla_{X/S})) \to f_*\mathcal{E} \to \operatorname{im}(f_*(\nabla_{X/S})) \to 0 \tag{5.9}$$

$$0 \to \operatorname{im}(f_*(\nabla_{X/S})) \to f_*(\Omega^1_{X/S} \otimes \mathcal{E}) \to \operatorname{coker}(f_*(\nabla_{X/S})) \to 0 \tag{5.10}$$

Since $f_*(\Omega^1_{X/S}\otimes\mathcal{E})$ and $\operatorname{coker}(f_*(\nabla_{X/S}))$ are flat, so is $\operatorname{im}(f_*(\nabla_{X/S}))$, by the exact sequence (5.10). Hence by the flatness of $\operatorname{im}(f_*(\nabla_{X/S}))$ and $\operatorname{coker}(f_*(\nabla_{X/S}))$, we have

$$\operatorname{Tor}_1^{u^{-1}\mathcal{O}_S}(u^{-1}\operatorname{im}(f_*(\nabla_{X/S})),\mathcal{O}_X)=\operatorname{Tor}_1^{u^{-1}\mathcal{O}_S}(u^{-1}\operatorname{coker}(f_*(\nabla_{X/S})),\mathcal{O}_X)=0.$$

Applying u^* to (5.9), it follows (see [42, 3.2.1]) that the respective long-exact sequences on $\operatorname{Tor}_\bullet^{u^{-1}\mathcal{O}_S}(-,\mathcal{O}_X)$ associated to (5.9) and (5.10) degenerate to give short-exact sequences

$$0\to u^*\ker(f_*(\nabla_{X/S}))\to u^*f_*\mathcal{E}\to u^*\operatorname{im}(f_*(\nabla_{X/S}))\to 0 \tag{5.11}$$

$$0\to u^*\operatorname{im}(f_*(\nabla_{X/S}))\to u^*f_*(\Omega^1_{X/S}\otimes\mathcal{E})\to u^*\operatorname{coker}(f_*(\nabla_{X/S}))\to 0 \tag{5.12}$$

Applying the same reasoning to the elementary fibration $f':X'\to S'$ and the coherent object $u'^*(\mathcal{E},\nabla)$ of $\mathbf{MIC}(X')$ (noting that coherence is preserved under inverse image by morphisms of locally Noetherian schemes) yields similar short-exact sequences to (5.9) and (5.10) (the bottom rows of (5.13) and (5.14)), and the isomorphisms (5.4) and (5.5) give rise to a map $u^*(\operatorname{im}(f_*(\nabla_{X/S}))\to\operatorname{im}(f'_*u'^*(\nabla_{X/S}))$, as in the diagrams

$$\begin{tikzcd}
0 \arrow[r] & u^*\ker(f_*(\nabla_{X/S})) \arrow[r] & u^*f_*\mathcal{E} \arrow[r] \arrow[d, "\cong"] & u^*\operatorname{im}(f_*(\nabla_{X/S})) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(f'_*u'^*(\nabla_{X/S})) \arrow[r] & f'_*u'^*\mathcal{E} \arrow[r] & \operatorname{im}(f'_*u'^*(\nabla_{X/S})) \arrow[r] & 0
\end{tikzcd} \tag{5.13}$$

$$\begin{tikzcd}
0 \arrow[r] & u^*\operatorname{im}(f_*(\nabla_{X/S})) \arrow[r] \arrow[d] & u^*f_*(\Omega^1_{X/S}\otimes\mathcal{E}) \arrow[r] \arrow[d, "\cong"] & u^*\operatorname{coker}(f_*(\nabla_{X/S})) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{im}(f'_*(u'^*\nabla_{X/S})) \arrow[r] & f'_*u'^*(\Omega^1_{X/S}\otimes\mathcal{E}) \arrow[r] & \operatorname{coker}(f'_*u'^*(\nabla_{X/S})) \arrow[r] & 0
\end{tikzcd} \tag{5.14}$$

This map on images is well-defined and surjective by a diagram chase and a four-lemma on (5.13), and injective by the other four-lemma on (5.14). Thus it is an isomorphism, and the maps that this induces on kernels and cokernels are exactly the base change maps. Another set of diagram chases show that $u^*\ker(f_*(\nabla_{X/S}))\cong\ker(f'_*u'^*(\nabla_{X/S}))$ and $u^*\operatorname{coker}(f_*(\nabla_{X/S}))\cong\operatorname{coker}(f'_*u'^*(\nabla_{X/S}))$ through these maps. By two applications of (2.23), we have

$$u^*R^0_{DR}f_*(\mathcal{E},\nabla)\cong u^*\ker(f_*(\nabla_{X/S}))\cong\ker(f'_*u'^*(\nabla)_{X'/S'})\cong R^0_{DR}f'_*(u'^*(\mathcal{E},\nabla))$$

and similarly for R^1_{DR}, using (2.24) instead. □

5.3 Regularity of higher direct images

We establish that regularity is stable under higher direct image by rational elementary fibrations.

Theorem 5.3.1. *Let $f : X \to S$ be a rational elementary fibration. Then for any coherent object $(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(X)$ with regular connection, the Gauss-Manin connection on $R^i_{DR} f_*(\mathcal{E}, \nabla)$ is also regular for $i = 0, 1$.*

Proof. By the base change result of Theorem 5.2.1 combined with the criterion Theorem 4.2.4, in lieu of the discussion in Remark 4.2.5, we are reduced to considering the case where S is a curve embedded in a smooth projective curve $\overline{S}$, and $\Sigma := \overline{S} \setminus S$ is a finite set of points, and hence X is a surface embedded in $\overline{X} = \mathbb{P}^1 \times S$.

We look to embed the variety $\overline{X}$ given by the coordinatised elementary fibration into a smooth, projective surface $\overline{\overline{X}}$, so that $\overline{Z} := \overline{\overline{X}} \setminus X$ is a strict normal crossing divisor, and extend the morphism $\overline{f} : \overline{X} \to S$ to a projective morphism $\overline{\overline{f}} : \overline{\overline{X}} \to \overline{S}$, so that we have a fibred square

$$\begin{array}{ccc} \overline{X} & \hookrightarrow & \overline{\overline{X}} \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{\overline{f}}} \\ S & \hookrightarrow & \overline{S} \end{array}$$

Since $\overline{X} = \mathbb{P}^1 \times S$, the closure X' of $\overline{X}$ in $\mathbb{P}^1 \times S$ is $\mathbb{P}^1 \times \overline{S}$ itself. We note that $\overline{Z} = \overline{\overline{X}} \setminus X$ is a union of the smooth curves Z and $\mathbb{P}^1 \times \Sigma$, and so by applying an embedded resolution of curves as in [24, 0BIC] to $\overline{Z}$, so that we get a smooth projective surface $\overline{\overline{X}}$, and the resulting divisor $\overline{Z}$ is in fact a strict normal crossing divisor.

We now construct an extension of $R^i_{DR} f_*(\mathcal{E}, \nabla)$ to $\overline{S}$ with logarithmic poles along Σ, in a similar way to as in Proposition 2.3.3. We define $\mathcal{O}_{\overline{\overline{X}}}$-modules $\Omega^p_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma)$ by setting

$$\Omega^1_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) = \frac{\Omega^1_{\overline{\overline{X}}}(\log \overline{Z})}{\overline{\overline{f}}^* \Omega^1_{\overline{S}}(\log \Sigma)},$$

and $\Omega^p_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) = \bigwedge^p \Omega^1_{\overline{\overline{X}}/\overline{S}}$. Since $(\mathcal{E}, \nabla)$ is regular, there is an extension $(\widetilde{\mathcal{E}}, \widetilde{\nabla})$ of $(\mathcal{E}, \nabla)$ to $\overline{\overline{X}}$, with logarithmic poles along $\overline{Z}$. We claim then that

$$R^i_{DR} \overline{\overline{f}}_*(\widetilde{\mathcal{E}}, \widetilde{\nabla}) := \frac{\mathbb{R}^i_{DR} \overline{\overline{f}}_* \left(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes \widetilde{\mathcal{E}} \right)}{\mathbb{R}^i_{DR} \overline{\overline{f}}_* \left(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma) \otimes \widetilde{\mathcal{E}} \right)^{\mathcal{O}_{\overline{S}}-tor}}$$

yields the desired extension of $R^i_{DR}f_*(\mathcal{E},\nabla)$ to $\overline{S}$, when endowed with an appropriate connection (where the denominator indicates $\mathcal{O}_{\overline{S}}$ torsion). (This notation is meant to be suggestive, noting that $R^i_{DR}\overline{\overline{f}}_*$ is not actually defined for connections with poles.)

Indeed, since $\overline{\overline{f}}$ is projective, $\mathbb{R}^i_{DR}\overline{\overline{f}}_*\left(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma)\otimes\widetilde{\mathcal{E}}\right)$ is coherent, and since $\overline{S}$ is a smooth curve, this $\mathcal{O}_{\overline{S}}$-module splits as the direct sum of a locally free sheaf and a torsion sheaf (see [38]). Hence $R^i_{DR}\overline{\overline{f}}_*(\widetilde{\mathcal{E}},\widetilde{\nabla})$ is locally free of finite rank, and on the level of sheaves, we have

$$\begin{aligned}
R^i_{DR}\overline{\overline{f}}_*(\widetilde{\mathcal{E}},\widetilde{\nabla})|_S &= \mathbb{R}^i\overline{\overline{f}}_*\left(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log \overline{Z}/\Sigma)\otimes_{\mathcal{O}_{\overline{\overline{X}}}}\widetilde{\mathcal{E}}\right)\Big|_S \\
&\cong \mathbb{R}^i\overline{f}_*\left(\Omega^\bullet_{\overline{X}/\overline{S}}(\log Z)\otimes_{\mathcal{O}_{\overline{X}}}\widetilde{\mathcal{E}}|_{\overline{X}}\right) && \text{by Proposition 1.3.3} \\
&\cong R^i_{DR}f_*(\mathcal{E},\nabla) && \text{by Proposition 5.1.6}
\end{aligned}$$

Thus it remains to construct an appropriate connection on $R^i_{DR}\overline{\overline{f}}_*(\widetilde{\mathcal{E}},\widetilde{\nabla})$. We filter the complex $\Omega^\bullet_{\overline{\overline{X}}}(\log\overline{Z})\otimes_{\mathcal{O}_{\overline{\overline{X}}}}\widetilde{\mathcal{E}}$ of $\mathcal{O}_{\overline{\overline{X}}}$-modules by the subcomplexes

$$F^p = \operatorname{im}\left(\overline{\overline{f}}^*\Omega^p_{\overline{S}}(\log\Sigma)\otimes\Omega^{\bullet-p}_{\overline{\overline{X}}}(\log Z)\otimes\widetilde{\mathcal{E}}\xrightarrow{((-)\wedge(-))\otimes\mathrm{id}}\Omega^\bullet_{\overline{\overline{X}}}(\log Z)\otimes\widetilde{\mathcal{E}}\right).$$

As with (2.6), these have associated filtration quotients

$$\mathrm{gr}^p = \overline{\overline{f}}^*\Omega^p_{\overline{S}}(\log\Sigma)\otimes\Omega^{\bullet-p}_{\overline{\overline{X}}/\overline{S}}(\log\overline{Z}/\Sigma)\otimes\widetilde{\mathcal{E}}$$

With respect to this filtration, the short-exact sequence

$$0\to\mathrm{gr}^1\to F^0/F^2\to\mathrm{gr}^0\to 0 \tag{5.15}$$

induces a long exact sequence on hypercohomology $\mathbb{R}^\bullet\overline{\overline{f}}_*$, and by the projection formula, we have that

$$\mathbb{R}^{i+1}\overline{\overline{f}}_*(\mathrm{gr}^1) = \Omega^1_{\overline{S}}(\log\Sigma)\otimes_{\mathcal{O}_{\overline{S}}}\mathbb{R}^i\overline{\overline{f}}_*\left(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log\overline{Z}/\Sigma)\otimes\widetilde{\mathcal{E}}\right)$$

Hence the coboundary map of the long exact sequence associated to (5.15) is an extension of the Gauss-Manin connection. It satisfies the Leibniz rule for the same reason as the usual Gauss-Manin connection (see Appendix B), instead using the logarithmic de Rham complex $\mathbb{R}^i\overline{\overline{f}}_*\,\mathrm{DR}_{\overline{\overline{X}}/\overline{S}}(\Omega^\bullet_{\overline{\overline{X}}/\overline{S}}(\log\overline{Z}/\Sigma)\otimes\widetilde{\mathcal{E}})$. This coboundary map factors through $\mathcal{O}_{\overline{S}}$-torsion as $\Omega^1_{\overline{S}}(\log\Sigma)$ is locally free, and so gives an extension of the Gauss-Manin connection on $R^i_{DR}\overline{\overline{f}}_*(\widetilde{\mathcal{E}},\widetilde{\nabla})$. □

Remark 5.3.2. The result of Theorem 5.3.1 generalises immediately to the case of a coordinatised elementary fibration, which is the case we will use later. This is as since regularity is preserved under finite étale direct image by Lemmas 4.2.6 and 3.2.3, and we may use the same argument as in the proof of Theorem 5.1.1. This will however be superceded by the results of the next section.

5.4 The main theorems in the regular case

We finish by using dévissage to deduce coherence, regularity and base change for arbitrary smooth morphisms. This section is not strictly necessary for the proof of the comparison theorem, but its results are substantial in their own right, and this provides a nice application of Lemma 3.2.4.

Notation 5.4.1. Let $i \geq 0$, and

1. X, S be smooth varieties over $\mathbb{C}$, and $f : X \to S$ be smooth, with $S = A_i(f)$;
2. $(\mathcal{E}, \nabla)$ be a coherent, regular object of $\mathbf{MIC}(X)$.

These theorems also have variants when $S \neq A_i(f)$. For an arbitrary smooth f, the results of this section will hold on base change to the dense open subset $A_i(f)$ of the image of f (Lemma 3.1.12) (and for all i simultaneously if one chooses $A_{d+\dim(X)}(f)$, by Proposition 2.6.1(1)).

Theorem 5.4.2 (Coherence)**.** $R^i_{DR} f_*(\mathcal{E}, \nabla)$ *is coherent.*

Theorem 5.4.3 (Regularity)**.** $R^i_{DR} f_*(\mathcal{E}, \nabla)$ *is regular.*

Theorem 5.4.4 (Base change)**.** *For any smooth $\mathbb{C}$-variety S' and morphism $u : S' \to S$, and associated fibred square*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{u} & S \end{array}$$

the base change map is an isomorphism

$$u^* R^i_{DR} f_*(\mathcal{E}, \nabla) \xrightarrow{\cong} R^i_{DR} f'_* u'^*(\mathcal{E}, \nabla)$$

Proof of Theorems 5.4.2, 5.4.3, 5.4.4. To see that $R^i_{DR} f_*(\mathcal{E}, \nabla)$ is coherent and regular, by Remark 3.2.2 and 4.2.6, we may applying the dévissage lemma 3.2.4.

This reduces us to the case where f is a rational elementary fibration and $i = 0, 1$, which was settled in Theorems 5.1.1 and 5.3.1.

For base change, we reduce to where f is a coordinatised elementary fibration. We first choose a finite open cover $\{U_\alpha\}$ as in Definition 3.1.11. By Remark 2.5.4, the base change maps on pullback to $U_{\underline{\alpha}}$ now give rise to a morphism of Zariski spectral sequences as in Proposition 2.4.2

$$\begin{tikzcd}
\bigoplus_{\alpha_0<\ldots<\alpha_p} u^* R^{i-p}_{DR} f_{\underline{\alpha}*}(\mathcal{E},\nabla)|_{U_{\underline{\alpha}}} \arrow[r, Rightarrow] \arrow[d, "\varphi_{i-p}"] & u^* R^i_{DR} f_*(\mathcal{E},\nabla) \arrow[d, "\varphi_i"] \\
\bigoplus_{\alpha_0<\ldots<\alpha_p} R^{i-p}_{DR} f'_{\underline{\alpha}*} u'^*(\mathcal{E},\nabla)|_{u'^{-1}(U_{\underline{\alpha}})} \arrow[r, Rightarrow] & R^i_{DR} f'_* u'^*(\mathcal{E},\nabla)
\end{tikzcd}$$

as the base change maps commute with the associated differentials. We have $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) = A_i(f) = S$ in this case (by Claim 3.1.13 (1)), and by inducting on i, we may assume that φ_{i-p} is an isomorphism for $p > 0$. This reduces us to considering the case of each $U_\alpha \to S$, which is a tower of coordinatised elementary fibrations.

In the case where $f = f_{d-1} \circ \ldots \circ f_0$ for f_j coordinatised elementary fibrations, letting

$$\begin{tikzcd}
 & X' \arrow[r, "u'=u_d"] \arrow[d, "f'_{d-1}"] & X \arrow[d, "f_{d-1}"] & \\
 & X'_{d-1} \arrow[r, "u_{d-1}"] \arrow[d, "f'_{d-2}"] & X_{d-1} \arrow[d, "f_{d-2}"] & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
S' \arrow[r, equal] & X'_0 \arrow[r, "u=u_0"] & X_0 \arrow[r, equal] & S
\end{tikzcd}$$

and inducting on j in the composition $f_j \circ (f_{j-1} \circ \ldots \circ f_0)$, these maps give rise to morphisms of Leray spectral sequences

$$\begin{tikzcd}
u^* R^p_{DR}(f_0 \circ \ldots \circ f_{j-1})_* \circ R^{i-p}_{DR} f_{j*}(\mathcal{E},\nabla) \arrow[r, Rightarrow] \arrow[d, "\cong"] & u^* R^i_{DR}(f_0 \circ \ldots \circ f_j)_*(\mathcal{E},\nabla) \arrow[dd] \\
R^{i-p}_{DR}(f_0 \circ \ldots \circ f_{j-1})_* \circ u^*_{j-1} R^p_{DR} f_{j*}(\mathcal{E},\nabla) \arrow[d] & \\
R^{i-p}_{DR}(f'_0 \circ \ldots \circ f'_{j-1})_* \circ R^p_{DR} f_{j*} u^*_j(\mathcal{E},\nabla) \arrow[r, Rightarrow] & R^i_{DR}(f'_0 \circ \ldots \circ f'_j)_* u'^*(\mathcal{E},\nabla)
\end{tikzcd}$$

For $0 < p < i$, the left maps are isomorphisms by induction, and so it suffices to consider a single f_j, i.e. the case of a single coordinatised elementary fibration, which follows from Theorem 5.2.1. $\square$

Chapter 6

The analytic setting

In this chapter, we expand on [1, IV.1], and describe connections on modules over complex-analytic varieties, whose setting is very similar to that of complex-algebraic varieties. We then describe analytification for modules with integrable connection, which is a functorial way to associate a complex-analytic vector bundle to any algebraic vector bundle. The full details of these results will be essentially the same as for in the algebraic case, so we will sketch most of the details. We finish this chapter by proving suitable analogues of the results of Chapter 5, which will be of use in the next chapter. The proofs of these will be relatively different to the sketches provided in [1, IV.1].

In the remaining chapters, we assume basic familiarity with complex-analytic spaces and Stein spaces. The required definitions pertaining to complex-analytic varieties, their morphisms (in particular smoothness, étale, finiteness and properness), and Stein spaces can be found in Appendix C.

6.1 Vector bundles on analytic varieties

We first recall the basic definitions of jets and differentials in the complex-analytic setting. We will refer to complex-analytic varieties as just analytic varieties, since we will always be working over $\mathbb{C}$.

Since the spaces underlying smooth analytic varieties are complex manifolds, the notion of smoothness for manifolds carries over to maps of smooth analytic varieties, giving a notion of smooth morphisms of smooth analytic varieties. To define differentials, we note that the category of analytic spaces has all fibre products (see [13, 0.32]).

Definition 6.1.1. Let $\mathcal{X}, \mathcal{S}$ be smooth analytic varieties, $f : \mathcal{X} \to \mathcal{S}$ be a smooth morphism, $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ be the associated diagonal morphism, with projections $p_1, p_2 : \mathcal{X} \times_{\mathcal{S}} \mathcal{X} \to \mathcal{X}$, and $\mathcal{I}$ be the ideal sheaf of $\Delta(\mathcal{X})$. We define

1. the *relative analytic differentials* $\Omega^1_{\mathcal{X}/\mathcal{S}} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ and *relative tangent vectors* $\mathcal{T}_{\mathcal{X}/\mathcal{S}} := \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\Omega^1_{\mathcal{X}/\mathcal{S}}, \mathcal{O}_{\mathcal{X}})$;

2. the n^{th} *order analytic jets* $\mathcal{P}^n_{\mathcal{X}/\mathcal{S}} := \Delta^{-1}(\mathcal{O}_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}}/\mathcal{I}^{n+1})$, viewed as an $\mathcal{O}_{\mathcal{X}}$-module by the left structure map

$$\mathcal{O}_{\mathcal{X}} \to p_{1*}\mathcal{O}_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}} \to p_{1*}\Delta_*\Delta^{-1}\mathcal{O}_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}} = \Delta^{-1}\mathcal{O}_{\mathcal{X}\times_{\mathcal{S}}\mathcal{X}} \to \mathcal{P}^n_{\mathcal{X}/\mathcal{S}},$$

   with *jet map* $j^n_{\mathcal{X}/\mathcal{S}} : \mathcal{O}_{\mathcal{X}} \to \mathcal{P}^n_{\mathcal{X}/\mathcal{S}}$ induced by the right $\mathcal{O}_{\mathcal{X}}$-linear structure from p_{2*}.

3. the *sheaf of differential operators* $\mathcal{D}_{\mathcal{X}}$ to be the subsheaf of $\mathcal{E}nd_{\underline{\mathbb{C}}}(\mathcal{O}_{\mathcal{X}})$ generated by $\mathcal{O}_{\mathcal{X}}$ and $\mathcal{D}er_{\underline{\mathbb{C}}}(\mathcal{O}_{\mathcal{X}})$.

Note that when $\mathcal{S} = *$ is the point, we have $\Omega^1_{\mathcal{X}/*} = \Omega^1_{\mathcal{X}}$, the sheaf of holomorphic differential 1-forms on the complex manifold $\mathcal{X}$, and that the dual of $\Omega^1_{\mathcal{X}}$ is exactly the holomorphic tangent sheaf $\mathcal{T}_{\mathcal{X}}$ of $\mathcal{X}$.

Proposition 6.1.2. *Let $\mathcal{X}, \mathcal{Y}$ be analytic varieties smooth over a smooth analytic variety $\mathcal{S}$, and $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of analytic spaces over $\mathcal{S}$. Then*

1. $\Omega^1_{\mathcal{X}/\mathcal{S}}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of finite rank, and we have an exact sequence

$$f^*\Omega^1_{\mathcal{Y}/\mathcal{S}} \to \Omega^1_{\mathcal{X}/\mathcal{S}} \to \Omega^1_{\mathcal{X}/\mathcal{Y}} \to 0; \tag{6.1}$$

which is short-exact and locally split if f is smooth.

2. We have an split short-exact sequence of right $\mathcal{O}_{\mathcal{X}}$-modules

$$0 \to \Omega^1_{\mathcal{X}/\mathcal{S}} \xrightarrow{\alpha_{\mathcal{X}/\mathcal{S}}} \mathcal{P}^1_{\mathcal{X}/\mathcal{S}} \xrightarrow{\beta_{\mathcal{X}/\mathcal{S}}} \mathcal{O}_{\mathcal{X}} \to 0, \tag{6.2}$$

with a canonical splitting given by $j^1_{\mathcal{X}/\mathcal{S}} : \mathcal{O}_{\mathcal{X}} \to \mathcal{P}^1_{\mathcal{X}/\mathcal{S}}$.

We refer to [20, 4.5] and [12, p. 569] for the details.

The maps on differentials also induce maps $f^*\mathcal{P}^1_{\mathcal{Y}/\mathcal{S}} \to \mathcal{P}^1_{\mathcal{X}/\mathcal{S}}$ and $\mathcal{P}^1_{\mathcal{X}/\mathcal{S}} \to \mathcal{P}^1_{\mathcal{X}/\mathcal{Y}}$ on jets, using the splitting given in (2).

As in Chapter 1, we may define the sheaves of differential operators, and analytic connections.

Definition 6.1.3. Let $f : \mathcal{X} \to \mathcal{S}$ be a smooth morphism, and $\mathcal{E}, \mathcal{F}$ be sheaves over $\mathcal{X}$.

1. An *$\mathcal{X}/\mathcal{S}$-connection on $\mathcal{E}$ relative to $\mathcal{S}$* is an $f^{-1}\mathcal{O}_\mathcal{S}$-linear map $\nabla : \mathcal{E} \to \Omega^1_{\mathcal{X}/\mathcal{S}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}$ satisfying
$$\nabla(fe) = df \otimes e + f\nabla(e)$$
for any local sections f of $\mathcal{O}_\mathcal{X}$ and e of $\mathcal{E}$.

2. An *$\mathcal{O}_\mathcal{X}$-module with $\mathcal{X}/\mathcal{S}$-connection* $(\mathcal{E}, \nabla)$ is a pair where ∇ is an $\mathcal{X}/\mathcal{S}$-connection on an $\mathcal{O}_\mathcal{X}$-module $\mathcal{E}$.

3. A *morphism* $(\mathcal{E}_1, \nabla_1) \to (\mathcal{E}_2, \nabla_2)$ *of $\mathcal{O}_X$-modules with $\mathcal{S}$-connection* is an $\mathcal{O}_\mathcal{X}$-linear map $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ so that $\nabla_2 \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla_1$

The analogous equivalent definitions in 1.2.1 also carry over, by essentially the same reasoning. The same discussion as for (1.4) also applies, yielding higher connections
$$\nabla^n : \Omega^n_{\mathcal{X}/\mathcal{S}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E} \to \Omega^{n+1}_{\mathcal{X}/\mathcal{S}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}$$
$$\omega \otimes e \mapsto d\omega \otimes e + \omega \wedge (-1)^n \nabla(e),$$
and an $\mathcal{O}_X$-linear map $K_\nabla := \nabla^1 \circ \nabla$, with the higher connections satisfying
$$\nabla^{n+1} \circ \nabla^n(\omega \otimes e) = \omega \wedge K_\nabla(e).$$

Definition 6.1.4. The *curvature of the connection* ∇ is
$$K_\nabla := K_\nabla := \nabla^1 \circ \nabla \in \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{E}, \Omega^2_{\mathcal{X}/\mathcal{S}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}).$$
We say ∇ is *integrable* if $K_\nabla = 0$.

As in Definition 1.2.4, we get a category $\mathbf{MIC}(\mathcal{X}/\mathcal{S})$ of $\mathcal{O}_\mathcal{X}$-modules with integrable $\mathcal{S}$-connections, and we set $\mathbf{MIC}(\mathcal{X}) := \mathbf{MIC}(\mathcal{X}/*)$. As in Chapter 1, the *de Rham complex* $\mathrm{DR}_{\mathcal{X}/\mathcal{S}}(\mathcal{E}, \nabla)$ of an object $(\mathcal{E}, \nabla)$ of $\mathbf{MIC}(\mathcal{X}/\mathcal{S})$ is
$$\mathrm{DR}_{\mathcal{X}/\mathcal{S}}(\mathcal{E}, \nabla) := \left[\mathcal{E} \xrightarrow{\nabla} \Omega^1_{\mathcal{X}/\mathcal{S}} \otimes \mathcal{E} \xrightarrow{\nabla^1} \Omega^2_{\mathcal{X}/\mathcal{S}} \otimes \mathcal{E} \to \cdots \to \Omega^n_{\mathcal{X}/\mathcal{S}} \otimes \mathcal{E}\right]$$
with $n = \max_{x \in X} \dim_{\mathcal{O}_{X,x}}(\Omega^1_{X/S,x})$. If $\mathcal{P}$ is a property of $\mathcal{O}_\mathcal{X}$-modules, we say that $(\mathcal{E}, \nabla)$ *has property* $\mathcal{P}$ if $\mathcal{E}$ does. The analogues of Theorems 1.2.7 and 1.2.8 hold in the analytic setting, and can be found as [6, 1.2.12] and [23, 4.1.1, p.100].

Theorem 6.1.5. *If $(\mathcal{E}, \nabla)$ is a coherent object of $\boldsymbol{MIC}(\mathcal{X})$, then $\mathcal{E}$ is locally free.*

Theorem 6.1.6. *$\boldsymbol{MIC}(\mathcal{X})$ is equivalent to the category of left $\mathcal{D}_\mathcal{X}$-modules.*

6.2 Analytic images by morphisms

Let $\mathcal{X}, \mathcal{Y}$ be analytic varieties smooth over a smooth analytic variety $\mathcal{S}$, and $f : \mathcal{X} \to \mathcal{Y}$ be a morphism over $\mathcal{S}$. As in §2.1 and §2.2, we have an inverse image functor $f^* : \mathbf{MIC}(\mathcal{Y}/\mathcal{S}) \to \mathbf{MIC}(\mathcal{X}/\mathcal{S})$, and when f is smooth, direct image functors $R^q_{DR} f_* : \mathbf{MIC}(\mathcal{X}/\mathcal{S}) \to \mathbf{MIC}(\mathcal{Y}/\mathcal{S})$. These are constructed in the same way as in §2.1 and §2.2, and we briefly recall their constructions, with the full details given in those sections. For f^*, given $(\mathcal{E}, \nabla)$ over $\mathcal{Y}$, ∇ determines an $\mathcal{O}_\mathcal{Y}$-linear section $J_\nabla = j^1_{\mathcal{Y}/\mathcal{S}} \otimes \mathrm{id}_\mathcal{E} - (\alpha_{\mathcal{Y}/\mathcal{S}} \otimes \mathrm{id}_\mathcal{E}) \circ \nabla$. As in §2.1, we in turn get the pullback connection by the formula

$$\alpha_{\mathcal{X}/\mathcal{S}} \otimes \mathrm{id}_{f^*\mathcal{E}} \circ f^*\nabla - j^1_{\mathcal{X}/\mathcal{S}} \otimes \mathrm{id}_{f^*\mathcal{E}} = f^*((\alpha_{\mathcal{Y}/\mathcal{S}} \otimes \mathrm{id}_\mathcal{E}) \circ \nabla - j^1_{\mathcal{Y}/\mathcal{S}} \circ \mathrm{id}_\mathcal{E}).$$

This is functorial and descends to $\mathbf{MIC}(-)$ for the same reasons as in §2.1. In particular, as in (2.2), we have the diagram

$$\begin{tikzcd}
f^{-1}\mathcal{E} \arrow[r, "f^{-1}\nabla"] \arrow[d, "\mathrm{id}\otimes 1"] & f^{-1}\Omega^1_{\mathcal{Y}/\mathcal{S}} \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} f^{-1}\mathcal{E} \arrow[d] \\
f^*\mathcal{E} \arrow[r, "f^*\nabla"] & \Omega^1_{\mathcal{X}/\mathcal{S}} \otimes_{\mathcal{O}_\mathcal{X}} f^*\mathcal{E}
\end{tikzcd} \tag{6.3}$$

Now suppose f is smooth. To define $R^0_{DR} f_*$, given $(\mathcal{E}, \nabla)$ on $\mathcal{X}$, we recall that a connection is equivalent to an $\mathcal{O}_\mathcal{X}$-linear map $\nabla : \mathcal{T}_{\mathcal{X}/\mathcal{S}} \to \mathcal{E}nd(\mathcal{E})$ satisfying the Leibniz rule. This restricts to $\mathcal{T}_{\mathcal{X}/\mathcal{S}} \to \mathcal{E}nd(\mathcal{E}^{\nabla_{\mathcal{X}/\mathcal{Y}}})$ as in §2.2, so descends to $f^*\mathcal{T}_{\mathcal{Y}/\mathcal{S}} \to \mathcal{E}nd(\mathcal{E}^{\nabla_{\mathcal{X}/\mathcal{Y}}})$ using the dual of (6.1). The $f^* \dashv f_*$ adjunction on modules yields $R^0_{DR} f_* \nabla : \mathcal{T}_{\mathcal{Y}/\mathcal{S}} \to \mathcal{E}nd(f_*\mathcal{E}^{\nabla_{\mathcal{X}/\mathcal{Y}}})$. The Leibniz rule for $R^0_{DR} f_* \nabla$ follows from that of ∇ on $\mathcal{E}^{\nabla_{\mathcal{X}/\mathcal{Y}}}(f^{-1}(U)) = f_*\mathcal{E}^{\nabla_{\mathcal{X}/\mathcal{Y}}}(U)$.

This is left-exact by the argument of Lemma 2.2.2, and we set $R^q_{DR} f_*$ to be the q^{th} derived functor of $R^0_{DR} f_*$.

6.3 Analytifying algebraic vector bundles

We describe the setup for converting algebraic spaces (respectively vector bundles) into complex-analytic spaces (respectively vector bundles). Most properties have the same definitions, though separated analytic spaces have Hausdorff underlying topological space, and the properties of smoothness, properness, étale and finiteness are defined slightly differently (see Definition C.1.2).

We denote the category of schemes of locally finite type over $\mathbb{C}$ by $\mathrm{Sch}_\mathbb{C}$, and the category of $\mathbb{C}$-analytic spaces by $\mathrm{AnSp}_\mathbb{C}$. We recall the explicit form of the analytification functors, as we will make use of these in the remaining sections.

Theorem 6.3.1 (GAGA for spaces)**.** *Let X be a scheme over $\mathbb{C}$.*

1. *There is a locally ringed space $(X^{an}, \mathcal{O}_{X^{an}})$, with points*

$$X(\mathbb{C}) = \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbb{C}), X);$$

and with topology and sheaf so that for any affine open

$$U = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)) \subseteq X,$$

$U^{an} \subseteq X^{an}$ is isomorphic to the local model space $V(f_1, \dots, f_m) \subseteq \mathbb{C}^n$, so that the assignment

$$\begin{aligned} (-)^{an} : \mathrm{Sch}_{\mathbb{C}} &\to \mathrm{AnSp}_{\mathbb{C}} \\ (X, \mathcal{O}_X) &\mapsto (X^{an}, \mathcal{O}_{X^{an}}) \end{aligned}$$

is functorial.

2. *The inclusion $\lambda_X : X^{an} \to X$ is continuous, and $\lambda_{(-)} : (-)^{an} \to \mathrm{id}_{\mathrm{Sch}_{\mathbb{C}}}$ is a natural transformation.*

3. *Let $\mathcal{P}$ be one of the properties reduced, separated, irreducible, connected or of dimension n. Then X has $\mathcal{P}$ if and only if X^{an} has $\mathcal{P}$. Further, $(X \times_S Y)^{an}$ is naturally isomorphic to $X^{an} \times_{S^{an}} Y^{an}$, and if X is affine, then X^{an} is a Stein space.*

4. *Let $f : X \to Y$ be a morphism of algebraic varieties, and let $\mathcal{P}$ be one of the properties étale, smooth, proper, finite. Then f has property $\mathcal{P}$ if and only if f^{an} has property $\mathcal{P}$.*

This result can be found as [21, XII.1.1, XII.1.2, XII.2.1, XII.3.1, XII.3.2]. Hence the analytification of an algebraic variety is also an analytic variety. Let $\mathrm{Mod}(-)$ denote the category of $\mathcal{O}$-modules on a locally ringed space, and $\mathrm{Coh}(-)$ denote the full subcategory of coherent sheaves.

Theorem 6.3.2 (Analytification for sheaves)**.** *Let X be a scheme of locally finite type over $\mathbb{C}$. Then there is a functor*

$$\begin{aligned} (-)^{an} = \lambda_X^* : \mathrm{Mod}(X) &\to \mathrm{Mod}(X^{an}) \\ \mathcal{F} &\mapsto \mathcal{F}^{an} := \lambda_X^* \mathcal{F} = \lambda_X^{-1} \mathcal{F} \otimes_{\lambda_X^{-1} \mathcal{O}_X} \mathcal{O}_{X^{an}} \end{aligned}$$

such that

1. *$(-)^{an}$ restricts to a functor* $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{an})$*;*

2. *$(-)^{an}$ is exact and faithful, commutes with arbitrary limits, internal Homs, tensor products, and pullbacks.*

This result is can be found as [21, XII.1.3]. We note however that analytification does not commute with direct images, and we refer to [8, 2.6] for an example of this. To analytify connections, we also note the following compatibility of analytification with jets and differentials.

Proposition 6.3.3. *Let $f : X \to S$ be a smooth morphism of smooth algebraic varieties. Then $(\Omega^1_{X/S})^{an} \cong \Omega^1_{X^{an}/S^{an}}$, $(\mathcal{P}^1_{X/S})^{an} \cong \mathcal{P}^1_{X^{an}/S^{an}}$, and we have $(j^1_{X/S})^{an} = j^1_{X^{an}/S^{an}}$ as maps $\mathcal{O}_{X^{an}} \to \mathcal{P}^1_{X^{an}/S^{an}}$. These isomorphisms are compatible with the exact sequences in Proposition 6.1.2.*

The first part is shown using that f factors locally on X as $X \xrightarrow{\pi} \mathbb{A}^n \times S$ in the algebraic setting (for π étale, see [24, 054L]), and using this to explicitly define maps on a basis of local sections. The second part follows from the first, as the constructions of the jet maps coincide.

These together yield an analytification functor

$$\begin{aligned}(-)^{an} = \lambda_X^* : \mathbf{MIC}(X) &\to \mathbf{MIC}(X^{an}) \\ (\mathcal{E}, \nabla) &\mapsto (\mathcal{E}^{an}, \nabla^{an})\end{aligned}$$

with ∇^{an} defined in the same way as for a pullback: writing $J_\nabla = \alpha_X \circ \nabla - j^1_X \otimes \mathrm{id}_{\mathcal{E}}$, we define ∇^{an} by

$$\alpha_{X^{an}} \circ \nabla^{an} - j^1_{X^{an}} \circ \mathrm{id}_{\mathcal{E}^{an}} = J^{an}_\nabla = (\alpha_X \circ \nabla - j^1_X \circ \mathrm{id}_{\mathcal{E}})^{an},$$

using the right linearity of the jet maps.

We now turn to describing the analytification comparison map on de Rham cohomology. For any morphism $f : X \to S$ of algebraic $\mathbb{C}$-varieties and any sheaf $\mathcal{F}$ on X, we have an analytification comparison map $(R^p f_* \mathcal{F})^{an} \to R^p f^{an}_* \mathcal{F}^{an}$. We can describe this using the Leray spectral sequence (see [24, 01F6]) of $f \circ \lambda_X = \lambda_S \circ f^{an}$ on regular higher direct images, as the composition

$$\begin{aligned}\theta_p : \lambda_S^* R^p f_* \mathcal{F} \to \lambda_S^* R^p f_* \lambda_{X*} \mathcal{F}^{an} &\xrightarrow{i} \lambda_S^* R^p (f \circ \lambda_X)_* \mathcal{F}^{an} = \lambda_S^* R^p (\lambda_S \circ f^{an})_* \mathcal{F}^{an} \\ &\xrightarrow{j} \lambda_S^* \lambda_{S*} R^p f^{an} \mathcal{F}^{an} \to R^p f^{an} \mathcal{F}^{an} \end{aligned} \tag{6.4}$$

where the first and last maps are given by the unit and counit of the adjunction $\lambda_\bullet^* \dashv \lambda_{\bullet *}$, and i, j are given by the $E_2^{p,0} \to H^p$ and $H^p \to E_2^{0,p}$ maps of the Leray spectral sequences of $f \circ \lambda_X$ and $\lambda_S \circ f^{an}$.

Theorem 6.3.4 (Serre's GAGA)**.** *If $f : X \to S$ is proper and $\mathcal{F}$ is coherent, then θ_p is an isomorphism for all $p \geq 0$.*

We refer to [21, XII.4.2] for the proof. To define the analytification comparison map on de Rham higher direct images, we note that λ_{X*} also defines a functor

$$\lambda_{X*} : \mathbf{MIC}(X^{an}) \to \mathbf{MIC}(X)$$

by sending a connection $\nabla : \mathcal{E} \to \Omega^1_{X^{an}} \otimes \mathcal{E}$ to

$$\lambda_{X*}\nabla : \lambda_{X*}\mathcal{E} \to \lambda_{X*}(\lambda_X^*\Omega^1_X \otimes \mathcal{E}) \cong \Omega^1_X \otimes \lambda_{X*}\mathcal{E},$$

where we have used the projection formula [24, 01E8] for Ω^1_X. For f smooth and each $p \geq 0$, we define a comparison map

$$\varphi_p : (R^p_{DR}f_*(\mathcal{E}, \nabla))^{an} \to R^p_{DR}f^{an}_*(\mathcal{E}^{an}, \nabla^{an}) \tag{6.5}$$

in the same way as (6.4). We use a variant of the spectral sequence (2.12): the functors $R^0_{DR}f_*$, λ_{X*} and $R^0_{DR}f^{an}_*$ all* send injectives to acyclic objects. Thus by applying the Grothendieck spectral sequence [24, 015N], for any coherent $(\mathcal{E}, \nabla)$ in $\mathbf{MIC}(X^{an})$, we get 2 spectral sequences converging to

$$R^\bullet_{DR}(f \circ \lambda_X)_*(\mathcal{E}, \nabla) = R^\bullet_{DR}(\lambda_X \circ f^{an})_*(\mathcal{E}, \nabla),$$

with $E_2^{p,q}$-terms $R^p_{DR}f_* \circ R^q\lambda_{X*}(\mathcal{E}, \nabla)$ and $R^p_{DR}\lambda_{X*} \circ R^q f^{an}_*(\mathcal{E}, \nabla)$.

We note that we still have an adjunction $\lambda^*_\bullet \dashv \lambda_{\bullet*}$ induced by the maps on the underlying modules†, and so the composition

$$\begin{aligned}\varphi_p : \lambda_S^* R^p_{DR}f_*(\mathcal{E}, \nabla) &\to \lambda_S^* R^p_{DR}f_*\lambda_{X*}(\mathcal{E}^{an}, \nabla^{an}) \xrightarrow{i} \lambda_S^* R^p_{DR}(f \circ \lambda_X)_*(\mathcal{E}^{an}, \nabla^{an}) \\ &= \lambda_S^* R^p_{DR}(\lambda_S \circ f^{an})_*(\mathcal{E}^{an}, \nabla^{an}) \xrightarrow{j} \lambda_S^*\lambda_{S*}R^p_{DR}f^{an}(\mathcal{E}^{an}, \nabla^{an}) \\ &\to R^p_{DR}f^{an}(\mathcal{E}^{an}, \nabla^{an})\end{aligned}$$

with the maps essentially the same as in (6.4), but with i and j given by the $E_2^{p,0} \to H^p$ and $H^p \to E_2^{0,p}$ maps of the Grothendieck spectral sequences associated to $R^0_{DR}f_* \circ \lambda_{X*}$ and $\lambda_{S*} \circ R^0_{DR}f_*$ instead.

Remark 6.3.5. From this construction, we see that the maps φ_p (and their restrictions) commute with the differentials on the Leray and Zariski spectral sequences (6.6) and (6.7).

*This result is Proposition 6.4.2 of the next section, but follows from identical reasoning to Proposition 2.3.9.

†For general f (in the algebraic or analytic settings), we have a natural transformation $f^*R^0_{DR}f_* \to \mathrm{id}$ induced by the usual module counit through an inclusion $\mathcal{E}^{\nabla_{X/S}} \to \mathcal{E}$, but we do not get a well-defined *unit*, since taking $R^0_{DR}f_*$ involves passing to horizontal sections.

6.4 The main analytic theorems

In this section, we prove analogues to Theorems 2.5.3 and 5.2.1 on flat and smooth base change in Theorems 6.4.7) and 6.4.12; Corollary 2.6.2 on vanishing in Theorem 6.4.5; and Theorem 5.4.2 on coherence for analytic de Rham higher direct images in Theorem 6.4.10.

We note that our arguments for étale and smooth base change and coherence are very different to the sketches provided in [1, IV.1.1.5, IV.1.1.7, IV.1.1.9] (resp. [2, 29.1.5, 29.1.7, 29.1.9]), which we elaborate on in the following remark.

Remark 6.4.1. The sketch of étale base change provided in [1, IV.1.1.5] is to show base change for finite flat morphisms (and open immersions), and use Zariski's main theorem. It seems however that étale morphisms are much simpler in (local) structure than finite flat morphisms in the analytic topology.

The sketches of the coherence and smooth base change theorems in [1, IV.1.1.7, IV.1.1.9] refer to using the theory of relative local systems[‡] applied to maps which are topologically trivial locally on the base. However, it seems that such maps only have compatible results with complex local systems (see Definition C.2.1), rather than relative local systems.

The arguments of §2.3 carry over in the analytic setting, and yield connections ∇^q_{GM} on $\mathbb{R}^q f_* \operatorname{DR}_{X/S}(\mathcal{E}, \nabla)$ for each $q > 0$, and the following analogue of Corollary 2.3.9.

Proposition 6.4.2. *Let $f : \mathcal{X} \to \mathcal{S}$ be a smooth morphism of analytic varieties.*

1. *Any injective object $(\mathcal{I}, \nabla)$ of $\boldsymbol{MIC}(\mathcal{X})$ is acyclic for $R^0_{DR} f_*$;*

2. *For any object $(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(\mathcal{X})$, the object $R^q_{DR} f_*(\mathcal{E}, \nabla)$ of $\boldsymbol{MIC}(\mathcal{S})$ is isomorphic to $(\mathbb{R}^q f_* \operatorname{DR}_{\mathcal{X}/\mathcal{S}}(\mathcal{E}, \nabla), \nabla^q_{GM})$, and hence has underlying sheaf associated to the presheaf*

$$U \mapsto \mathbb{H}^i(f^{-1}(U), \operatorname{DR}_{f^{-1}(U)/\mathcal{S}}(\mathcal{E}, \nabla));$$

3. *For any $U \subseteq \mathcal{S}$,*

$$(R^i_{DR} f_*(\mathcal{E}, \nabla))|_U = R^i_{DR} f_*((\mathcal{E}, \nabla)|_{f^{-1}(U)}).$$

[‡]A *relative local system* in this context refers to an $f^{-1}\mathcal{O}_S$-modules on X which is locally of the form $f^{-1}\mathcal{F}$ for some coherent sheaf $\mathcal{F}$ on S, see [10, I.2.22]).

As in Propositions 2.4.1, 2.4.2 and 2.4.3, we have the following spectral sequences in the analytic setting.

Proposition 6.4.3. *Let $g : \mathcal{X} \to \mathcal{Y}$ and $f : \mathcal{Y} \to \mathcal{S}$ be smooth morphisms of analytic varieties, and $(\mathcal{E}, \nabla)$ be an object of $\boldsymbol{MIC}(\mathcal{X})$.*

1. *(Leray spectral sequence) We have a spectral sequence of $\mathcal{O}_{\mathcal{S}}$-modules*

$$ {}_LE_2^{p,q} = R^p_{DR} f_* \circ R^q_{DR} g_*(\mathcal{E}, \nabla) \Rightarrow R^{p+q}_{DR}(f \circ g)_*(\mathcal{E}, \nabla); \tag{6.6} $$

2. *Let $\{U_\alpha\}$ be an open cover of $\mathcal{X}$. Then we have the following spectral sequences of $\mathcal{O}_{\mathcal{S}}$-modules.*

   (a) *(Zariski spectral sequence)*

$$ {}_ZE_1^{p,q} = \bigoplus_{\alpha_0 < \alpha_1 < \ldots < \alpha_p} R^q_{DR} f_{\underline{\alpha}*}((\mathcal{E}, \nabla)|_{U_{\underline{\alpha}}}) \Rightarrow R^{p+q}_{DR} f_*(\mathcal{E}, \nabla); \tag{6.7} $$

   (b) *(Čech-to-de Rham spectral sequence)*

$$ {}_{\check{C}}E_1^{p,q} = \bigoplus_{\substack{i+j=p \\ \alpha_0 < \ldots < \alpha_i}} R^q f_{\underline{\alpha}*} \left(\Omega^j_{U_{\underline{\alpha}}/\mathcal{S}} \otimes \mathcal{E}|_{U_{\underline{\alpha}}} \right) \Rightarrow R^{p+q}_{DR} f_*(\mathcal{E}, \nabla); \tag{6.8} $$

We will now restrict to the case of analytic varieties which come from algebraic varieties, i.e. X^{an}, S^{an} for smooth algebraic varieties X, S. We deduce base change for finite étale morphisms using (6.8), by first showing the analogous statement for analytic higher direct images.

Proposition 6.4.4. *Let $f : X \to S$ be a smooth morphism of smooth algebraic varieties, $u : S' \to S$ be an étale morphism, and*

$$ \begin{array}{ccc} X' & \xrightarrow{u'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{u} & S \end{array} \tag{6.9} $$

be a fibred square. Then for any coherent sheaf $\mathcal{E}$ on X^{an} and $q \geq 0$, the base change map

$$ u^{an*} R^q f^{an}_* \mathcal{E} \xrightarrow{\cong} R^q f'^{an}_* u'^{an*} \mathcal{E} $$

is an isomorphism.

Proof. The square (6.9) remains fibred on analytification by Theorem 6.3.1(4). Since the question is local on S'^{an} (see [22, III.8.2]), we may shrink S'^{an} and S^{an}. We may thus assume u^{an} is an isomorphism of complex-analytic varieties by Theorem 6.3.1(5), in which case the base change property follows immediately. □

We also have the following acyclicity result, in analogy with quasi-coherent sheaves under direct image by affine morphisms.

Proposition 6.4.5. *Let $f : X \to S$ be a smooth morphism of smooth, affine algebraic varieties. Then for any coherent sheaf $\mathcal{E}$ on X^{an}, we have*

$$R^q f^{an}_* \mathcal{E} = 0 \qquad \text{for } q > 0.$$

Proof. We note that $R^q f^{an}_* \mathcal{E}$ is the sheaf associated to the presheaf

$$U \mapsto H^q(f^{an-1}(U), \mathcal{E}).$$

Since X and S are affine, we may embed $X \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ and $S \hookrightarrow \mathbb{A}^m_{\mathbb{C}}$ as closed subvarieties. Since S^{an} is a smooth submanifold of $\mathbb{C}^n$, it has a basis $\{U_\alpha\}$ of open sets given by intersections of polydiscs in $\mathbb{C}^n$ with S^{an}. Since f is a polynomial map in these coordinates, it follows from Proposition C.1.5 that each $f^{an-1}(U_\alpha)$ is a Stein manifold, so

$$H^q(f^{an-1}(U_\alpha), \mathcal{E}) = 0$$

for $q > 0$. Upon sheafification, we see that $R^q f^{an}_* \mathcal{E} = 0$ for $q > 0$. □

The arguments of Proposition 2.6.1 and Corollary 2.6.2 carry over, replacing the acyclicity of quasi-coherent sheaves on affines (see [22, III.3.5]) with the above proposition, and yield the following vanishing results.

Corollary 6.4.6. *Let $f : X \to S$ be a smooth morphism of smooth algebraic varieties over $\mathbb{C}$, and d be the maximum dimension of the fibres of f. Let $(\mathcal{E}, \nabla)$ be a coherent object of $\boldsymbol{MIC}(X^{an})$. Then*

1. *If π is finite étale, then*

$$R^q_{DR}(f^{an} \circ \pi^{an})_*(\mathcal{E}, \nabla) = R^q_{DR} f^{an}_* \circ \pi^{an}_*(\mathcal{E}, \nabla);$$

2. *If f is also affine, then for any $q \geq 0$,*

$$R^q_{DR} f^{an}_*(\mathcal{E}, \nabla) \cong \mathcal{H}^q(f^{an}_* \, \mathrm{DR}_{X^{an}/S^{an}}(\mathcal{E}, \nabla)),$$

where $\mathcal{H}^q$ denotes cohomology of complexes of objects of $\boldsymbol{MIC}(X^{an})$. In particular, we have $R^q_{DR} f^{an}_*(\mathcal{E}, \nabla) = 0$ for $q > d$.

In particular, if f is a rational elementary fibration, then $R^q_{DR}f^{an}_*(\mathcal{E},\nabla) = 0$ for $q > 1$.

Proposition 6.4.7. *In the setting of Proposition 6.4.4, suppose $(\mathcal{E},\nabla)$ is a coherent object of $\boldsymbol{MIC}(X^{an})$. Then for any $q \geq 0$, the base change map*

$$u^{an*}R^q_{DR}f^{an}_*(\mathcal{E},\nabla) \xrightarrow{\cong} R^q_{DR}f'^{an}_*u'^{an*}(\mathcal{E},\nabla)$$

is an isomorphism.

The argument is identical to that of Proposition 2.5.3, replacing all spaces with analytifications, substituting the appropriate Čech spectral sequences, and using Proposition 6.4.5 to show that the spectral sequences degenerate at the E_1 page.

Remark 6.4.8. From the proof of 6.4.7 (see Proposition 2.5.3), the base change isomorphism is induced by the natural isomorphisms $u^{an*}f^{an}_* \Rightarrow f'^{an}_*u'^{an*}$, and so commutes with the differentials of the Zariski spectral sequence (6.7), as in Remark 2.5.4.

We now turn to the coherence of higher direct images, and we begin with an analogue of Lemma 3.2.4 on dévissage.

Lemma 6.4.9. *Let $\mathcal{P}$ be a property of modules with integrable connection on smooth analytic $\mathbb{C}$-varieties, which is strongly exact and local for the (analytic) étale topology. Suppose that*

For any rational elementary fibration $f' : X' \to S'$ with S' affine, and for any coherent object $(\mathcal{E}',\nabla')$ of $\boldsymbol{MIC}(X'^{an})$ and $j = 0, 1$,

$$\mathcal{P}((\mathcal{E}',\nabla')) \implies \mathcal{P}(R^j_{DR}f^{an}_*(\mathcal{E}',\nabla'))$$

Then for any $i \geq 0$, any smooth morphism $f : X \to S$ of smooth algebraic varieties and any coherent object $(\mathcal{E},\nabla)$ of $\boldsymbol{MIC}(X^{an})$, we have

$$\mathcal{P}((\mathcal{E},\nabla)) \implies \mathcal{P}\left((R^i_{DR}f^{an}_*(\mathcal{E},\nabla)|_{A_i(f)^{an}}\right).$$

For our theorems on coherence and base change, we will rely on various results on complex local systems, which can be found in Appendix C.

Theorem 6.4.10. *Let $f : X \to S$ be a smooth morphism of smooth algebraic varieties, and $(\mathcal{E},\nabla)$ be a coherent object of $\boldsymbol{MIC}(X^{an})$. Then for any $i \geq 0$, $R^i_{DR}f^{an}_*(\mathcal{E},\nabla)|_{A_i(f)^{an}}$ is coherent.*

Proof. We note that coherence on analytic varieties is strongly exact as it has the 2-out-of-3 property for short exact sequences, combined with [6, 1.2.6] for subobjects; and local for the étale topology as it is local for the analytic topology (see [15, A.3.3]). Thus applying Lemma 6.4.9 reduces us to the case where f is a rational elementary fibration, and $i = 0, 1$.

When f is a rational elementary fibration, for each point s of S^{an}, applying Proposition C.2.5 (noting that $\mathbb{A}^{1an} - \{\theta_1, \ldots, \theta_r\}$ is homotopy equivalent to a wedge sum of r circles S^1) yields that $H^j(X_s^{an}, \mathcal{E}^\nabla|_{X_s^{an}})$ is a finite-dimensional $\mathbb{C}$-vector space for $j = 0, 1$. Since f^{an} is topologically trivial, applying Theorem C.2.4 yields

$$R^q f_*^{an}(\mathcal{E}, \nabla) \cong R^j f_*^{an} \mathcal{E}^\nabla \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}. \tag{6.10}$$

Again by the topological triviality of f^{an}, it follows by Theorem C.2.7 that $R^q f_*^{an} \mathcal{E}^\nabla$ is a complex local system on S^{an}, and hence the identity (6.10) yields that $R^q f_*^{an}(\mathcal{E}, \nabla)$ is a coherent $\mathcal{O}_{S^{an}}$-module. □

We note that Theorem C.2.4 holds for topologically locally trivial maps, with fibrewise finite-dimensional cohomology. Thus by Remark 3.1.5, the proof of the above theorem yields the following corollary, which we will use in the next theorem on arbitrary base change.

Corollary 6.4.11. *If $f : X \to S$ is a coordinatised elementary fibration, then*

$$R^i_{DR} f_*^{an}(\mathcal{E}, \nabla) \cong R^i f_*^{an} \mathcal{E}^\nabla \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}.$$

Theorem 6.4.12. *Let $f : X \to S$ be a smooth morphism of smooth algebraic varieties, and $u' : S' \to S$ be an arbitrary morphism of smooth algebraic varieties. Suppose we have a fibred square as in (6.9). Then for any $i \geq 0$ and coherent $(\mathcal{E}, \nabla)$ in $\boldsymbol{MIC}(X^{an})$, the restriction of*

$$u^{an*} R^i_{DR} f_*^{an}(\mathcal{E}, \nabla) \to R^i_{DR} f_*'^{an} u'^{an*}(\mathcal{E}, \nabla)$$

to $(u^{-1} A_i(f))^{an}$ is an isomorphism.

Proof. We start by reducing to the case where f is a coordinatised elementary fibration. Since the question is local on S'^{an}, by shrinking S, we may assume that $S = A_i(f)$. Hence, by Definition 3.1.11 and replacing S by étale neighbourhoods using Theorem 6.4.7, we may assume X has an a finite open cover $\{U_i\}$ so that each $U_i \to S$ is a tower of coordinatised elementary fibrations.

As in the argument of Theorem 5.4.4, using Remark 6.4.8, we use a morphism of Čech spectral sequences (as in (6.8)) to reduce to the case of each $U_i \to S$, followed by repeatedly using the Leray spectral sequence of (6.6) to reduce to the case of a single coordinatised elementary fibration.

We note that on the level of underlying modules, the previous corollary yields

$$R^q_{DR} f'^{an}_* u'^{an*}(\mathcal{E}, \nabla) = R^q f'^{an}_* (u'^{an*}\mathcal{E})^{u'^{an*}\nabla} \otimes_{\mathbb{C}} \mathcal{O}_{S'^{an}},$$

and

$$\begin{aligned} u^{an*} R^q_{DR} f^{an}_*(\mathcal{E}, \nabla) &= u^{an*}\left(R^q f^{an}_* \mathcal{E}^{\nabla} \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}\right) \\ &= u^{an-1} R^q f^{an}_* \mathcal{E}^{\nabla} \otimes_{\mathbb{C}} \mathcal{O}_{S'^{an}}, \end{aligned}$$

where we treat $\otimes_{\mathbb{C}}$ as a tensor over the respective constant sheaves $\underline{\mathbb{C}}$. We thus see that it suffices to prove that the base change is an isomorphism on the first factors, and we achieve this with the following proposition. □

Proposition 6.4.13. *Let $f : \mathcal{X} \to \mathcal{S}$ be smooth morphism of smooth complex-analytic varieties, topologically trivial locally on $\mathcal{S}$. Let $u : \mathcal{S}' \to \mathcal{S}$ be an arbitrary morphism of complex-analytic varieties and*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "u'"] \arrow[d, "f'"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{S}' \arrow[r, "u"] & \mathcal{S} \end{tikzcd}$$

be a fibred square. Then for any coherent $(\mathcal{E}, \nabla)$ in $\boldsymbol{MIC}(\mathcal{X})$,

1. *For any $i \geq 0$, the base change map*

$$u^{-1} R^q f_* \mathcal{E}^{\nabla} \to R^q f'_* u'^{-1}(\mathcal{E}^{\nabla})$$

   is an isomorphism; and

2. *The map $u^{-1}\mathcal{E} \to u^*\mathcal{E}$ induces an isomorphism $u^{-1}(\mathcal{E}^{\nabla}) \cong (u^*\mathcal{E})^{u^*\nabla}$.*

Proof. Let $\mathcal{X}_0$ be the fibre of f above any s in $\mathcal{S}$. By localising on $\mathcal{S}'$ and $\mathcal{S}$, we may assume that f' and f are trivial. For (1), by Theorem C.2.7, for any contractible $U \subseteq \mathcal{S}'$, the sheaf $R^q f'_* u'^{-1}(\mathcal{E}^{\nabla})$ restricted to U is the constant sheaf with value

$$H^q(\{x'\} \times \mathcal{X}_0, u'^{-1}\mathcal{E}^{\nabla}|_{\{x'\}\times\mathcal{X}_0}) = H^q(\{u'(x')\} \times \mathcal{X}_0, \mathcal{E}^{\nabla}_{\{u'(x')\}\times\mathcal{X}_0})$$

for any $x' \in U$. Similarly, for contractible $V \subseteq \mathcal{S}$, $R^q f_* \mathcal{E}^\nabla|_V$ is the constant sheaf with value

$$H^q(\{x\} \times \mathcal{X}_0, \mathcal{E}^\nabla|_{\{x\}\times\mathcal{X}_0})$$

for any $x \in V$. Now for any contractible $U \subseteq \mathcal{S}'$ with $u(U) \subseteq V$ for some contractible $V \subseteq \mathcal{S}$, the value of $u^{-1}R^q f_* \mathcal{E}^\nabla$ is also $H^q(\{x'\} \times \mathcal{X}_0, \mathcal{E}^\nabla|_{\{u(x')\}\times\mathcal{X}_0})$, and on these open sets, the base change map is just the identity, hence an isomorphism on this open set. As these form a fundamental system of open sets, we get the desired isomorphism.

For (2), since u'^{-1} is exact, by (6.3), we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & u'^{-1}\mathcal{E}^\nabla \arrow[r] & u'^{-1}\mathcal{E} \arrow[r, "u'^{-1}\nabla"] \arrow[d, "\mathrm{id}\otimes 1"] & u'^{-1}\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} u'^{-1}\Omega^1_\mathcal{X} \arrow[d] \\
0 \arrow[r] & (u'^*\mathcal{E})^{u'^*\nabla} \arrow[r] & u'^*\mathcal{E} \arrow[r, "u'^*\nabla"] & u'^*\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}'}} \Omega^1_{\mathcal{X}'}
\end{tikzcd} \tag{6.11}$$

with exact rows, which yields a map of kernels $\varphi : u'^{-1}\mathcal{E}^\nabla \to (u'^*\mathcal{E})^{u'^*\nabla}$. For $x' \in \mathcal{X}'$, taking the stalk at x' gives a map

$$(u'^*\mathcal{E})_x = \mathcal{E}_{u'(x')} \otimes_{\mathcal{O}_{\mathcal{X}',u'(x')}} \mathcal{O}_{\mathcal{X}',x'} \to \mathcal{E}_{u'(x')} \otimes_{\mathcal{O}_{X',u'(x')}} \kappa(x')$$

given by the quotient map $\mathcal{O}_{\mathcal{X}',x'} \to \kappa(x') \cong \mathbb{C}$. Taking stalks at x' in the diagram (6.11) yields the diagram

$$\begin{tikzcd}
\mathcal{E}^\nabla_{u'(x')} \arrow[rr] \arrow[d, "\varphi"] \arrow[ddrr, dashed] & & \mathcal{E}_{u'(x')} \arrow[d] \\
(u'^*\mathcal{E})^{u'^*\nabla}_{x'} \arrow[rr] & & \mathcal{E}_{u'(x')} \otimes_{\mathcal{O}_{\mathcal{X},u'(x')}} \mathcal{O}_{\mathcal{X}',x'} \arrow[d] \\
& & \mathcal{E}_{u'(x')} \otimes_{\mathcal{O}_{\mathcal{X},u'(x')}} \kappa(x')
\end{tikzcd} \tag{6.12}$$

where $\mathcal{E}^\nabla_{u'(x')}$, $(u'^*\mathcal{E})^{u'^*\nabla}_x$ and $\mathcal{E}_{u'(x')} \otimes_{\mathcal{O}_{\mathcal{X},u'(x')}} \kappa(x')$ are $\mathbb{C}$-vector spaces of dimension $\mathrm{rank}_{\mathcal{O}_\mathcal{X}} \mathcal{E} = \mathrm{rank}_{\mathcal{O}_{\mathcal{X}'}} u'^*\mathcal{E}$. Applying the Riemann-Hilbert correspondence (C.2.2) to the right column of this diagram, the dashed arrow factors (through $\mathcal{E}_{u'(x')}$) as

$$\mathcal{E}^\nabla_{u'(x')} \xrightarrow{\mathrm{id}\otimes 1} \mathcal{E}^\nabla_{u'(x')} \otimes_\mathbb{C} \mathcal{O}_{\mathcal{X},u'(x')} \xrightarrow{\mathrm{id}\times u^\#_{x'}} \mathcal{E}^\nabla_{u'(x')} \otimes_\mathbb{C} \mathcal{O}_{\mathcal{X}',x'} \to \mathcal{E}^\nabla_{u'(x')} \otimes_\mathbb{C} \kappa(x')$$

where the map $\mathbb{C} \to \mathcal{O}_{\mathcal{X},u'(x')}$ identifies $\mathbb{C}$ with the constants, so descends to give an isomorphism $\mathbb{C} \to \kappa(x')$. We hence conclude that the dashed arrow is an isomorphism, so that $\varphi_{x'}$ is injective, hence an isomorphism as the domain and codomain have the same dimension as $\mathbb{C}$-vector spaces. Thus we conclude that φ itself is an isomorphism. $\square$

Chapter 7

The comparison theorem on de Rham cohomology

The goal of this chapter is to prove comparison theorem described in the introduction (see $(*)$). We deduce this by reducing to the case of the affine line, and applying some comparison criteria to deduce that the comparison maps

$$\varphi_i : (R^i_{DR} f_*(\mathcal{E}, \nabla))^{an} \to R^i_{DR} f_*^{an}(\mathcal{E}^{an}, \nabla^{an})$$

of (6.5) are isomorphisms.

7.1 Comparison criteria

In this section we will work purely in the affine case. We first introduce some notation and standing assumptions.

Notation 7.1.1. Let (C, ∂) be a differential algebra over $\mathbb{C}$, and A, B be differential subalgebras of C so that

1. $A^\partial = C^\partial$;
2. $A^\partial \to A$ is faithfully flat; and
3. $\partial|_A : A \to A$ is surjective.

We also let (E, ∇_∂) be a differential module over $(A \cap B, \partial)^*$. Unless otherwise indicated, tensor products will be taken over $A \cap B$.

*Note that the restriction $\partial|_{A\cap B} : A \cap B \to A \cap B$ is well-defined as each of A and B are sub-differential algebras of C.

In the above notation, the comparison map will correspond to the map

$$\mathrm{id} \otimes 1 : E \to E \otimes B.$$

where we should think of B as the regular functions over an appropriate analytic space.

Definition 7.1.2. We say that E is *solvable in A* if the canonical morphism

$$(E \otimes A)^{\partial} \otimes_{A^{\partial}} A \dot{\to} E \otimes A \tag{7.1}$$

is an isomorphism.

We will see how solvability relates to regularity, though we will first demonstrate why solvability is important.

Proposition 7.1.3. *Suppose that E is solvable in A. Then*

1. $\varphi_0 : \ker_E \partial \to \ker_{E \otimes B} \partial$ *is an isomorphism; and*

2. $\varphi_1 : \mathrm{coker}_E \partial \to \mathrm{coker}_{E \otimes B} \partial$ *is injective.*

Proof. We note that the injectivity of φ_0 will follow from that of $E \to E \otimes B$. Since E is flat over $A \cap B$, applying the functor $E \otimes (-)$ to the sequence $0 \to A \cap B \to B$ yields that $E \to E \otimes B$ is injective.

We have the following diagram of inclusions

$$\begin{array}{ccc} E & \longrightarrow & E \otimes A \\ \downarrow & & \downarrow \\ E \otimes B & \longrightarrow & E \otimes C \end{array}$$

where we note that as E is locally free over $A \cap B$, we have

$$(E \otimes A) \cap (E \otimes B) = E \otimes (A \cap B) = E \tag{7.2}$$

We require the following statements for the surjectivity of φ_0 and injectivity of φ_1.

Claim 7.1.4. 1. $(E \otimes A)^{\partial} = (E \otimes C)^{\partial}$, and hence $(E \otimes C)^{\partial} \subseteq E \otimes A$;

2. $\partial : E \otimes A \to E \otimes A$ is surjective.

Proof of Claim 7.1.4. By the isomorphism given by the solvability of E in A, applying faithfully flat descent for projectivity of modules [24, 058S] yields that $(E \otimes A)^\partial \subseteq (E \otimes C)^\partial$ is projective of finite rank μ as an A^∂-module, so we in fact get equality as in (1).

For (2), we note that $E \otimes A \cong A^\mu$ as a differential module over A, locally in the Zariski topology on $\mathrm{Spec}(A^\partial)$. Hence as $\partial : A \to A$ is surjective, $\partial : E \otimes A \to E \otimes A$ is surjective Zariski-locally, hence globally surjective. □

The surjectivity of φ_0 is now an easy consequence: if $e \in E \otimes B$ has $\partial e = 0$, then $e \in (E \otimes C)^\partial \subseteq E \otimes A$, so $e \in E$ by (7.2).

For the injectivity of φ_1, suppose that $e \in \partial(E \otimes B)$. Writing $e = \partial e'$ for $e' \in E \otimes B$, we also have that $e = \partial e''$ for some $e'' \in E \otimes A$ by Claim 7.1.4 (2). But then $\partial(e' - e'') = 0$, so $e' - e'' \in (E \otimes C)^\partial \subseteq E \otimes A$, and $e' \in E$ by (7.2). □

The surjectivity of φ_1 is much harder, and requires more constraints on the structure of A and C.

Proposition 7.1.5. *Suppose in addition to Notation 7.1.1 and Proposition 7.1.3 that $C = \bigoplus_{i=0}^r C_i$ and $A = \bigoplus_{i=0}^r A_i$, where $A_i \subseteq C_i$ are differential $\mathbb{C}$-subalgebras of A and C respectively, and suppose that we have*

1. *differential subalgebras $C_i' \subseteq C_i$ and $A_i' \subseteq A_i$, with $A_i' = A_i \cap C_i'$, and a differential A_i'-subalgebra T_i' of C_i;*

2. *a multiple $\partial_i := u_i \partial$, for some $u_i \in A_i'^*$; and*

3. *a sequence $(\partial_i^{-j} 1)_{i \geq 0}$ of elements of A_i'.*

so that

1. *$\partial : C_i \to C_i$ is surjective, and C_i is faithfully flat over C_i^∂;*

2. *The projection $B \to C \xrightarrow{\mathrm{pr}_i} C_i$ factors through an injection $B \hookrightarrow C_i$;*

3. *Letting*
$$C_i^{\log} := \bigoplus_{j \geq 0} C_i' \partial_i^{-j} 1,$$
we have that
$$C_i = T_i \otimes_{A_i'} C_i^{\log}$$

4. *We have $C'_0 \subseteq A'_0 + \mathrm{pr}_0(B)$, and for $i > 0$,*

$$C'_i \subseteq A'_i + \bigcap_{j<i} \mathrm{pr}_i(B \cap A_j)$$

where the latter intersection is taken in C_j.

We have just introduced a lot of objects in the statement of this proposition, so it is worth giving some idea for what these objects should be. Roughly speaking, C'_i and A_i should be thought of as meromorphic functions and formal function germs around a point θ_i respectively, and C_i as the algebra generated by their (formal) products. The existence of a sequence $(\partial_i^{-j}1)_{j\geq 0}$ should be viewed as saying that A_i "contains all powers of logarithms". The objects and their dependencies are summarised in the following diagram:

$$\begin{tikzcd}
A_i^\partial \arrow[rrr, "\cong"] \arrow[ddd, hook] & & & C_i^\partial \arrow[ddd, hook] \\
& A'_i \arrow[r, hook] \arrow[d, hook] \arrow[ddl, hook] & C'_i \arrow[d, hook] \arrow[ddr, hook] & \\
& T'_i \arrow[r, hook] \arrow[drr, hook] & C_i^{\log} \arrow[dr, hook] & \\
A_i \arrow[rrr, hook] \arrow[dd, hook] & & & C_i \arrow[dd, hook] \\
& & B \arrow[ur, hook] \arrow[dr, hook] & \\
A \arrow[rrr, hook] & & & C
\end{tikzcd}$$

Proof of Proposition 7.1.5. We first by establish a few facts from our conditions.

Claim 7.1.6. For each i, we have

a. $(E \otimes A_i)^\partial \cong (E \otimes C_i)^\partial$; and

b. ∂ is surjective as a map

   i. $E \otimes A_i \to E \otimes A_i$; and

   ii. $E \otimes C_i \to E \otimes C_i$

Proof of Claim 7.1.6. We note the (b.ii) follows by applying Claim 7.1.4 to C_i (using (1)). We show the rest by applying Claim 7.1.4, where E, C, A, B are replaced by $E \otimes (A_i \cap T_i)$, C_i, A_i and T_i respectively, where $A_i \cap T_i$ is viewed as an $A \cap B$-module through the projection pr_i. This requires us to check the conditions of Notation 7.1.1 and Proposition 7.1.3.

The faithful flatness of A over A^∂ also yields that is A_i over A_i^∂ is faithfully flat, and $A_i^\partial = C_i^\partial$ and $\partial|_{A_i} : A_i \to A_i$ is surjective by restricting $\partial : A \to A$ to the appropriate component.

Now, $E \otimes T_i$ is a differential module over T_i, and since

$$(E \otimes (A_i \cap T_i)) \otimes_{A_i \cap T_i} A_i \cong E \otimes A_i,$$

the solvability of $E \otimes (A_i \cap T_i)$ in $A_i \cap T_i$ reduces to that of E in A_i. But this follows from the diagram of isomorphisms

$$\begin{tikzcd}
(E \otimes A)^\partial \otimes_{A^\partial} A \arrow[r, "\cong"] & E \otimes A \\
\bigoplus_{i=0}^{r} (E \otimes A_i)^\partial \otimes_{A_i^\partial} A_i \arrow[u, "\cong"] \arrow[r, "\cong"] & \bigoplus_{i=0}^{r} E \otimes A_i \arrow[u, "\cong"']
\end{tikzcd}$$

where the lower arrow is an isomorphism coordinatewise, and hence we conclude (a) and (b.i). $\square$

By intersecting the domain and codomain of the map $B \hookrightarrow C_i'$ of condition (2) with A, we get an injection $A \cap B \hookrightarrow A_i'$, making A_i' an $(A \cap B)$-algebra. Treating B as a subset of C_i' via the inclusion also yields $B \cap A_i = B \cap A_i'$. Hence we may make sense of $E \otimes A_i'$, and by the local freeness of E we have $(E \otimes A_i) \cap (E \otimes C_i') = E \otimes A_i'$ as a submodule of $E \otimes C_i$.

In the following statement, since $\mathrm{pr}_i|_B$ is an injection, we identify $B \cap A_j$ with its image under pr_i for each i. Under this identification, we have that $\bigcap_{j=0}^r B \cap A_j = B \cap A$.

Claim 7.1.7. For any $e \in E \otimes B$, there is a sequence $e_i' \in E \otimes \left(\bigcap_{j<i} B \cap A_j\right)$ (intersection taken in C_i') for $-1 \le i \le r$, so that $e = e_{-1}$ and

$$e_i := e_{i-1} - \partial e_i' \in E \otimes \left(\bigcap_{j \le i} B \cap A_j\right)$$

To see that the proposition follows from the claim, for any $e \in E \otimes B$, taking $e' = \sum_{j=0}^r e_i'$ and $f = e_r \in E \otimes (B \cap A) = E$ yields that $f = e - \partial e'$, so that the class $[f]$ maps to the class $[e]$ under the map on cokernels.

Proof of Claim 7.1.7. We show this inductively. For each i, we have $e_{i-1} \in E \otimes \left(\bigcap_{j<i} B \cap A_j\right)$ (note that $B \subseteq C'_i$ for $i = 0$). By Claim 7.1.6 (b.i), there is $f_i \in E \otimes C_i = E \otimes (T_i \otimes_{A'_i} C_i^{\log})$ (by condition (3)) so that $\partial f_i = e_{i-1}$. Thus we may write

$$f_i = \sum_{j=0}^{N} f_{i,j} \partial_i^{-j} 1$$

for some $f_{i,j} \in (E \otimes T_i) \otimes_{A'_i} C'_i$. Equating the coefficients of $\partial_i^{-j} 1$ in the equation

$$e_i = \partial f_i = \sum_{j=0}^{N} (\partial f_{i,j} \partial_i^{-j} 1 + u_i^{-1} f_{i,j} \partial_i^{-j+1} 1)$$

yields that

$$\begin{aligned} \partial f_{i,N} &= 0 && (\textit{coefficient of } \partial_i^{-N} 1) \\ \partial f_{i,j} &= -u_i^{-1} f_{i,j+1} && (\textit{coefficient of } \partial_i^{-j}, 1 \leq j \leq N-1) \\ \partial f_{i,0} &= -u_i^{-1} f_{i,1} + e_{i-1} && (\textit{coefficient of } 1) \end{aligned}$$

We thus deduce that $f_{i,N} \in ((E \otimes A_i) \otimes_{A'_i} C'_i)^{\partial} \subseteq (E \otimes C_i)^{\partial} = (E \otimes A_i)^{\partial}$, by identifying $A_i \otimes_{A'_i} C'_i$ compositum of A_i and C'_i in C_i. By repeatedly applying (a) and (b.i) of Claim 7.1.6 with decreasing indices, we find that $f_{i,j} \in E \otimes A_i$ for each j, and in particular $f_{i,1} \in E \otimes A_i$.

Finally, by condition (4), we may write $f_{i,0} = f'_i + e'_i$ where $f'_i \in E \otimes A'_i$ and $e'_i \in E \otimes \bigcap_{j<i} B \cap A_j$, and our last equation above yields

$$e_i := e_{i-1} - \partial e'_i = (\partial f_{i,0} + u_i^{-1} f_{i,1}) - (\partial f_{i,0} - \partial f'_i) = u_i^{-1} f_{i,1} + \partial f'_i \in E \otimes A'_i$$

and hence we conclude that $e_i \in E \otimes \left(\bigcap_{j \leq i} B \cap A_j\right)$, so by induction the claim (and hence the proposition) follows. $\square$

$\square$

7.2 The main theorem

We are now in a position to prove the comparison theorem for de Rham cohomology.

Theorem 7.2.1. *Let $i \geq 0$, and $f : X \to S$ be a smooth morphism of smooth varieties with $S = A_i(f)$. Let $(\mathcal{E}, \nabla)$ be a coherent $\mathcal{O}_X$-module with a regular, integrable connection. Then the comparison map*

$$\varphi_i : (R^i_{DR} f_*(\mathcal{E}, \nabla))^{an} \to R^i_{DR} f^{an}_*(\mathcal{E}^{an}, \nabla^{an})$$

is an isomorphism.

By taking $S = \mathrm{Spec}(\mathbb{C})$ and $(\mathcal{E}, \nabla) = (\mathcal{O}_X, d_X)$, we recover Grothendieck's original comparison theorem.

Corollary 7.2.2. *For any smooth algebraic $\mathbb{C}$-variety X and $i \geq 0$, we have*

$$H^i_{DR}(X) \cong H^i_{DR}(X^{an})$$

Proof of Theorem 7.2.1. We first reduce to when S is a point, and $X = \mathbb{A}^1 - \{\theta_1, \ldots, \theta_r\}$. Since the question is local on S, we may assume that S is affine and connected, and that f has constant relative dimension d. If $d = 0$, the morphism f is finite étale by Lemma 3.1.13 (3), and in this case we have $(f_*\mathcal{E})^{an} \cong f_*^{an}\mathcal{E}^{an}$ by Serre's GAGA (6.3.4). Thus it remains to consider when $d \geq 1$.

We note that the statement is local on S^{an} for the étale topology, and that algebraic and analytic de Rham cohomology commutes with étale localisation by Propositions 2.5.3 and 6.4.7. Hence by the definition of the Artin set, we may replace S with an affine connected étale neighbourhood so that X admits an open cover $\{U_\alpha\}$ with the properties of Definition 3.1.11.

By Remark 6.3.5, the comparison maps for $U_{\underline{\alpha}}$ commute with the associated differentials, so give rise to a morphism of Čech spectral sequences as in Proposition 2.4.3

$$\begin{array}{ccc}
\displaystyle\bigoplus_{\alpha_0 < \ldots < \alpha_p} (R^{i-p}_{DR} f_{\underline{\alpha}*}(\mathcal{E}, \nabla)|_{U_{\underline{\alpha}}})^{an} & \Longrightarrow & (R^i_{DR} f_*(\mathcal{E}, \nabla))^{an} \\
\Big\downarrow{\scriptstyle \varphi_{i-p}} & & \Big\downarrow{\scriptstyle \varphi_i} \\
\displaystyle\bigoplus_{\alpha_0 < \ldots < \alpha_p} R^{i-p}_{DR} f^{an}_{\underline{\alpha}*}(\mathcal{E}^{an}, \nabla^{an})|_{U^{an}_{\underline{\alpha}}} & \Longrightarrow & R^i_{DR} f^{an}_*(\mathcal{E}^{an}, \nabla^{an})
\end{array}$$

We note that $A_{i-|\underline{\alpha}|+1}(f_{\underline{\alpha}}) = A_i(f) = S$ in this case (by Claim 3.1.13 (1)). Hence by inducting on i, we may assume that φ_{i-p} is an isomorphism for $p > 0$, and we are reduced to considering the case of each $U_\alpha \to S$, which is a tower of coordinatised elementary fibrations.

In the case where $f = f_0 \circ \ldots \circ f_{d-1}$ for f_i coordinatised elementary fibrations, again by Remark 6.3.5, the comparison maps give rise to morphisms of the Leray spectral sequences associated to $(f_0 \circ \ldots \circ f_{j-1}) \circ f_j$:

$$\begin{array}{ccc}
(R^{i-p}_{DR}(f_0 \circ \ldots \circ f_{j-1})_* \circ R^p_{DR} f_{j*}(\mathcal{E}, \nabla))^{an} & \Longrightarrow & (R^i_{DR}(f_0 \circ \ldots f_j)_*(\mathcal{E}, \nabla))^{an} \\
\Big\downarrow & & \Big\downarrow \\
R^{i-p}(f_0 \circ \ldots f_{j-1})^{an}_* \circ R^p_{DR} f^{an}_{j*}(\mathcal{E}^{an}, \nabla^{an}) & \Longrightarrow & R^i_{DR}(f_0 \circ \ldots \circ f_j)^{an}_*(\mathcal{E}^{an}, \nabla^{an})
\end{array}$$

For $0 < p < i$, the left maps are isomorphisms by induction. Since regularity is preserved under higher direct images by coordinatised elementary fibrations Theorem 5.3.1, we are reduced to the case of a single coordinatised elementary fibration.

We may now write $f = f' \circ \pi$ for π finite étale, and by a further étale localisation, we may assume by Lemma 3.1.6 (2) that f' is rational. Now by the vanishing results of Corollaries 2.6.2 and 6.4.6, we have $R^i_{DR} f_*(\mathcal{E}, \nabla) = R^i_{DR} f'_* \circ \pi_*(\mathcal{E}, \nabla)$ and $R^i_{DR} f^{an}_*(\mathcal{E}^{an}, \nabla^{an}) = R^i_{DR} f'^{an}_* \circ \pi^{an}_*(\mathcal{E}^{an}, \nabla^{an})$, and analytification commutes with direct image by π (by Serre's GAGA (6.3.4)), and so we are reduced to proving the theorem for when f is a rational elementary fibration, and where $i = 0, 1$.

In this case, by the coherence theorems 5.1.1 and 6.4.10, the source and target of φ_i are locally free $\mathcal{O}_{S^{an}}$-modules of finite rank. By local freeness, it thus suffices to check that our map is an isomorphism fibre-wise. Applying the smooth base change theorems 5.2.1 and 6.4.12, to a point $u : \mathrm{Spec}(\mathbb{C}) \to S$, we may assume that $S = \mathrm{Spec}(\mathbb{C})$, and $X = \mathbb{A}^1 - \{\theta_1, \ldots, \theta_r\}$ as f is a rational elementary fibration. In this case, $\mathcal{E}$ is locally free of finite rank on the affine variety X, so given by $\widetilde{E}$ for some finitely-generated projective $O(X)$-module E.

We note that for $X = \mathbb{A}^1 - \{\theta_1, \ldots, \theta_r\}$, we have $\Omega^1_X(X) = \mathcal{O}(X)dx$ and hence $\mathcal{T}_X(X) = \mathcal{O}(X)\partial_x$, so ∇ is determined by ∇_{∂_x}. Using the descriptions of the higher direct images in Corollaries 2.6.2(2) and 6.4.6(2) and that analytification is given globally by $\mathcal{E}^{an}(X^{an}) = \mathcal{E}(X) \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{an})$ (by Theorem 6.3.2†), the theorem is now reduced to the following special case. $\square$

Proposition 7.2.3. *Let $X = \mathrm{Spec}(\mathbb{C}[x, 1/\prod_{i=1}^r (x - \theta_i)])$, and (E, ∇_{∂_x}) be a differential module over $(\mathcal{O}(X), \partial_x)$ of rank μ. Suppose that the associated algebraic vector bundle $(\widetilde{E}, \nabla)$ is regular. Then*

$$\varphi_0 : \ker_E \nabla_{\partial_x} \to \ker_{E \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{an})} \nabla_{\partial_x}$$

and

$$\varphi_1 : \mathrm{coker}_E \nabla_{\partial_x} \to \mathrm{coker}_{E \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{an})} \nabla_{\partial_x}$$

are isomorphisms.

We now look to apply the comparison criteria of §7.1. We first specify notation for our case, in line with that of Proposition 7.1.5.

†Noting also that the only open subset U of X containing $\lambda_X(X^{an})$ is X itself.

Notation 7.2.4. In the setting of Proposition 7.2.3, we set

1. $X = \mathrm{Spec}(\mathbb{C}[x, 1/\prod_{i=1}^r (x - \theta_i)])$;

2. $B := \mathcal{O}(X^{an})$; and

3. For each $1 \le i \le r$,

   (a) $C_i' := \bigcup_{\varepsilon>0} \mathcal{O}_{\mathbb{C}}(D(\theta_i, \varepsilon) - \{\theta_i\})$, functions with an isolated singularity at θ_i, identified using the restriction maps;

   (b) $\mathcal{M}_{\theta_i} :=$ the field of function germs meromorphic at θ_i;

   (c) $A_i := \mathcal{M}_{\theta_i}[\log(x - \theta_i), (x - \theta_i)^\alpha]_{\alpha \in \mathbb{C}}$, with multiplication

   $$(x - \theta_i)^\alpha (x - \theta_i)^\beta = (x - \theta_i)^{\alpha+\beta};$$

   (d) $C_i = A_i C_i'$

4. For $i = 0$, we replace θ_i with ∞, which amounts to replacing $D(\theta_i, \varepsilon) - \{\theta_i\}$ with $\mathbb{C} - \overline{D(0, 1/\varepsilon)}$ in (a), and $x - \theta_i$ with $1/x$ in (c).

5. We set

   $$C = \bigoplus_{i=0}^r C_i; \qquad A = \bigoplus_{i=0}^r A_i; \qquad \text{and} \qquad \partial = \partial_x$$

   with B embedded diagonally in C.

In this notation, we have

1. $A \cap B = (\prod_{i=0}^r \mathcal{M}_{\theta_i}) \cap B = \mathcal{O}(X) = \mathbb{C}[x, 1/\prod_{i=1}^r (x - \theta_i)]$, diagonally embedded in A;

2. $A_i^\partial = C_i^\partial = \mathbb{C}$, hence $A^\partial = C^\partial = \mathbb{C}^{r+1}$;

3. $A_i' := A_i \cap C_i' = \mathcal{M}_{\theta_i}$.

To apply the abstract comparison criteria of §7.1, we require the solvability of (E, ∇_{∂_x}). This is however guaranteed by ∇ being a regular connection.

Lemma 7.2.5. *In the setting of Notation 7.2.4, E is solvable in A.*

The proof of this is purely formal, and can be found in Appendix D.

Proof of Proposition 7.2.3. In lieu of the previous lemma, we are reduced to checking the conditions of Notation 7.1.1 and Proposition 7.1.5. First, A and C are faithfully flat over $A^\partial = C^\partial = \mathbb{C}^{r+1}$. Since

$$\partial\left(x^\alpha \log^j(x)\right) = x^{\alpha-1}\log^{j-1}(x)(\alpha\log(x)+j),$$

for $\alpha \neq 0$ we have

$$x^{\alpha-1}\frac{\log^k(x)}{k!} = \partial\left(\frac{x^\alpha}{\alpha}\sum_{i+j=k}\left(-\frac{1}{\alpha}\right)^i\frac{\log^j(x)}{j!}\right),$$

and for $\alpha = 0$ we get

$$x^{-1}\frac{\log^k(x)}{k!} = \partial\left(\frac{\log^{k+1}(x)}{(k+1)!}\right).$$

and hence ∂ acts surjectively on both A and C. This shows the requirements for Proposition 7.1.3, and (1) of Proposition 7.1.5. For the remaining requirements of Proposition 7.1.5, (2) amounts to the injectivity of the restrictions, which follows by analytic continuation. To show (3), for $i \geq 1$, setting

$$u_i = x - \theta_i; \qquad \partial_i^{-j}1 = \frac{\log^j(x-\theta_i)}{j!}; \qquad \text{and} \qquad T_i = \mathcal{M}_{\theta_i}[(x-\theta_i)^\alpha]_{\alpha\in\mathbb{C}}$$

yields the desired sequence $(\partial_i^{-j}1)_{j\geq 0}$, differential algebra T_i, and their desired properties. For $i = 0$, we replace $x - \theta_i$ with $1/x$ in each of the above. Finally, to see that $C_i' \subseteq A_i' + \bigcap_{j<i}\mathrm{pr}_i(B\cap A_j)$ for all (with this intersection treated as $\mathrm{pr}_0(B)$ for $i = 0$), given $f \in C_i'$, we may separate out the negative Laurent series terms at θ_i and all other θ_j (for $j \neq i$) to write $f = f_1 + f_2$, for $f_1 \in A_i'$ and $f_2 \in \bigcap_{j\neq i}\mathrm{pr}_i(B\cap A_j)$. Hence the conditions of §7.1 are satisfied, and we conclude the proposition. □

Appendix A

The Spencer resolution

In this appendix, we explain why the Spencer complex $\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X)$, as defined in Definition 2.3.4, is an acyclic chain complex. For the acyclicity, we require the following definition and lemma.

Definition A.0.1. Let R be a commutative ring, and $s : R^r \to R$ be an R-linear map. The *Koszul complex of* s is the (chain) complex

$$K(s) = \left[\bigwedge^r A^r \to \bigwedge^{r-1} \to \ldots \to \bigwedge^1 A^r \to \bigwedge^0 A^r \cong A \right],$$

with maps

$$\alpha_1 \wedge \ldots \wedge \alpha_k \mapsto \sum_{i=1}^{k} (-1)^{i+1} s(\alpha_i) \alpha_1 \wedge \ldots \wedge \hat{\alpha}_i \wedge \ldots \wedge \alpha_k.$$

The particular case we are interested in is when $R = \mathbb{C}[x_1, \ldots, x_n]$, and

$$s = (x_1, \ldots, x_n) : \mathbb{C}[x_1, \ldots, x_n]^{\oplus n} \to \mathbb{C}[x_1, \ldots, x_n]$$

is given componentwise by multiplication by x_i.

Lemma A.0.2. *The Koszul complex $K(x_1, \ldots, x_n)$ is a resolution of $\mathbb{C}$ by free $\mathbb{C}[x_1, \ldots, x_n]$-modules.*

This is a consequence of [33, 16.5(i)]. We recall the Spencer complex, as in Chapter 2.

Notation A.0.3. 1. Let $f : X \to S$ be a smooth morphism of smooth varieties. The Spencer complex $\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X)$ is the (chain) complex

$$\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X) = \left[\mathcal{D}_x \otimes_{\mathcal{O}_X} \bigwedge^n \mathcal{T}_{X/S} \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{X/S} \to \mathcal{D}_{X/S} \right]$$

with differentials given by

$$\begin{aligned} d_k : \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \mathcal{T}_{X/S} &\to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k-1} \mathcal{T}_{X/S} \\ \partial \otimes (\xi_1 \wedge \ldots \wedge \xi_k) &\mapsto \sum_{i=1}^{k} (-1)^{i-1} \partial \xi_i \otimes (\xi_1 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_k) + \\ & \sum_{i<j} (-1)^{i+j} \partial \otimes ([\xi_i, \xi_j] \wedge \xi_1 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \\ & \wedge \widehat{\xi_j} \wedge \ldots \wedge \xi_k) \end{aligned} \tag{A.1}$$

2. Let
$$\mathcal{D}_X = \bigoplus_{n \geq 0} \mathcal{D}_n$$
denote the grading on $\mathcal{D}_X$ by order, and $\overline{\mathcal{D}}_n = \mathcal{D}_n / \mathcal{D}_{n-1}$.

The following proof is based on [32, III.3].

Proposition A.0.4. *The Spencer complex is an acyclic chain complex.*

Proof. Checking that $d_{k+1} \circ d_k = 0$ is a direct (albeit long-winded) computation. Since the differentials respect the total grading on $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \mathcal{T}_{X/S}$, we have grading-wise sequences

$$0 \to \mathcal{D}_{p-n} \otimes \bigwedge^n \mathcal{T}_{X/S} \to \ldots \to \mathcal{D}_p \to \mathcal{T}_{X/S} \to 0. \tag{A.2; p}$$

The quotient of (A.2; p) by (A.2; $p-1$) is

$$0 \to \overline{\mathcal{D}}_{p-n} \otimes \bigwedge^n \mathcal{T}_{X/S} \to \ldots \to \mathcal{D}_p \to 0, \tag{A.2$'$; p}$$

with differential

$$\begin{aligned} d_k : \overline{\mathcal{D}}_{p-k} \otimes \bigwedge^k \mathcal{T}_{X/S} &\to \overline{\mathcal{D}}_{p-k+1} \otimes \bigwedge^{k-1} \mathcal{T}_{X/S} \\ [a] \otimes (\xi_1 \wedge \ldots \wedge \xi_k) &\to \sum_{i=1}^{k} (-1)^{i-1} \partial \xi_i \otimes (\xi_1 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_k) \end{aligned}$$

The acyclicity of all (A.2; p) follows from that of all (A.2$'$; p), and so it suffices to prove that (A.2$'$; p) is acyclic. For this, since X is smooth, given $x \in X$, we have that $(\overline{\mathcal{D}}_q)_x$ is the space of homogeneous polynomials of degree q in the ring

$$\mathbb{C}\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right],$$

where $(x_1, \ldots, x_n)$ is a system of local étale coordinates for X in a neighbourhood of x. Hence (A.2′; p) is the degree p part of the Koszul complex of a polynomial ring, and so acyclic (i.e. exact at each point). Hence (A.2; p) is also acyclic, and we finish by noting that $\mathrm{Sp}_\bullet^{X/S}(\mathcal{D}_X)$ is the colimit of the complexes in (A.2; p), hence also acyclic. $\square$

Appendix B

A Čech description of the Gauss-Manin connection

We elaborate on the statement of Theorem 2.3.1 in this appendix, introducing a relative interior product and Lie derivative, and explicitly defining a connection on the Čech complex

$$C_f^{p,q}(\mathcal{E},\nabla) := C_f^{\bullet}(\{U_\alpha\}, \Omega^{\bullet}_{X/S} \otimes \mathcal{E}) = \bigoplus_{\alpha_0 < \ldots < \alpha_i} f_{\underline{\alpha}*}\left(\Omega^j_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_\alpha}\right). \tag{B.1}$$

for some open cover $\{U_\alpha\}$ of X. We mainly follow and give the omitted computations from [1, 2.5.2], and [28, 3] in some parts.

B.1 The setup and preliminary results

Notation B.1.1. We let

1. $f : X \to S$ be a smooth morphism of smooth complex varieties, where since the question is local on S, we assume S is affine;

2. $\mathcal{U} = \{U_\alpha\}$ be a finite affine open cover of X, so that each U_α is étale over $\mathbb{A}^n_S$ for some $n \geq 0$. This yields a splitting

   $$\Omega^1_{U_\alpha} = \Omega^1_{U_\alpha/S} \oplus f_\alpha^*\Omega^1_S.$$

   where $f_\alpha = f|_{U_\alpha}$, and so that each of these modules is free, and $\Omega^1_{U_\alpha/S}$ has basis $\{dx_1^\alpha, \ldots, dx_d^\alpha\}$ as a free $\mathcal{O}_{U_\alpha}$-module. As usual, for $\underline{\alpha} = (\alpha_0, \ldots, \alpha_p)$, we write $U_{\underline{\alpha}} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$.

3. For a vector field $\partial \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_S, \mathcal{O}_S)$, we let $\partial_\alpha \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{U_\alpha}, \mathcal{O}_{U_\alpha})$ be its unique extension sending $dx_1^\alpha, \ldots, dx_d^\alpha$ to 0.

Definition B.1.2. Let ∂ be a vector field on S. For each α, we define the associated *interior product on* U_α by

$$\begin{aligned} \iota_{\partial_\alpha} : \Omega^p_{U_\alpha/S} &\to \Omega^{p-1}_{U_\alpha/S} \\ h dg_1 \wedge \ldots dg_q &\mapsto h \sum_{i=1}^{q} (-1)^i \partial_\alpha(g_i) dg_1 \wedge \ldots \wedge \widehat{dg_i} \wedge \ldots \wedge dg_q \end{aligned}$$

for local sections $h, g_1, \ldots, g_q$ of $\mathcal{O}_{U_\alpha}$.

The Lie derivative is given, as usual, by *Cartan's magic formula* (see [31, 18.13].

Definition B.1.3. Let ∂ be a vector field on S. The associated *Lie derivative on* U_α is

$$\mathcal{L}_{\partial_\alpha} := d\iota_{\partial_\alpha} + \iota_{\partial_\alpha} d : \Omega^p_{U_\alpha/S} \to \Omega^p_{U_\alpha/S}$$

We note the following identities for these operators.

Lemma B.1.4. *Let ∂ be a vector field on S. Then for each α, and any local sections f of $\mathcal{O}_{U_\alpha}$, β of $\Omega^1_{U_\alpha/S}$, ω of $\Omega^j_{U_\alpha/S}$ and η of $\Omega^k_{U_\alpha/S}$, we have*

1. $\iota_{\partial_\alpha}(\beta) = \alpha(\beta)$;
2. $\iota_{\partial_\alpha}(\omega \wedge \eta) = \iota_{\partial_\alpha}(\omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge \iota_{\partial_\alpha}(\eta)$;
3. $\mathcal{L}_{\partial_\alpha}(f) = \partial_a(f)$;
4. $\mathcal{L}_{\partial_\alpha}(\omega \wedge \eta) = \mathcal{L}_{\partial_\alpha}\omega \wedge \eta + \omega \wedge \mathcal{L}_{\partial_\alpha}\eta$;
5. $\mathcal{L}_{\partial_\alpha} \circ d = d \circ \mathcal{L}_{\partial_\alpha}$, *or equivalently*

$$[\mathcal{L}_{\partial_\alpha}, d] = 0$$

as maps $\bigoplus_{n\geq 0} \Omega^n_{U_\alpha/S} \to \bigoplus_{n \geq 0} \Omega^n_{U_\alpha/S}$.

The proof of these statements can be found as [31, 13.11, 18.9, 18.11, 18.13] (this is given for smooth manifolds, but works formally in the algebraic setting).

Definition B.1.5. Let $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}(X)$, and ∂ be a vector field on S.

1. The *interior product on* $(\mathcal{E}, \nabla)$ *with* ∂_α is

$$\iota_{\partial_\alpha} \otimes \mathrm{id}_{\mathcal{E}|_{U_\alpha}} : \Omega^n_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha} \to \Omega^{n-1}_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha},$$

and by abuse of notation, we will denote this by ι_{∂_α} when $\mathcal{E}$ is clear from context.

2. The *Lie derivative on* $(\mathcal{E}, \nabla)$ *in direction* ∂_α is

$$\begin{aligned} \mathcal{L}_{\partial_\alpha} : \Omega^n_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha} &\to \Omega^n_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha} \\ \omega \otimes e &\mapsto \mathcal{L}_{\partial_\alpha}\omega \otimes e + \omega \otimes \nabla_{\partial_\alpha}(e). \end{aligned}$$

Lemma B.1.6. *We have* $\iota_{\partial_\alpha} \circ \nabla = \nabla_{\partial_\alpha}$ *as maps* $\mathcal{E}|_{U_\alpha} \to \mathcal{E}|_{U_\alpha}$.

This follows directly from the definitions. We write

$$d_{DR} : \bigoplus_{n\geq 0} \Omega^n_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha} \to \bigoplus_{n\geq 0} \Omega^n_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha}$$

for the differential given by ∇^n in the n^{th} component.

We will deduce the main theorem of this section using the following formulae.

Lemma B.1.7. *1. For any* α_0, *we have*

$$[\mathcal{L}_\partial, d_{DR}] = 0$$

as endomorphisms of $\bigoplus_{n\geq 0} \Omega^n_{U_\alpha/S} \otimes \mathcal{E}|_{U_\alpha}$;

2. For any α_0, α_1, *we have*

$$\mathcal{L}_{\partial_{\alpha_0}} - \mathcal{L}_{\partial_{\alpha_1}} = d_{DR}\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} + \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} d_{DR}$$

on $U_{\alpha_0\alpha_1}$;

3. For any $\alpha_0, \alpha_1, \alpha_2$, *we have*

$$\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} + \iota_{\partial_{\alpha_1}-\partial_{\alpha_2}} + \iota_{\partial_{\alpha_2}-\partial_{\alpha_0}} = 0$$

on $U_{\alpha_0\alpha_1\alpha_2}$.

Proof. Let $\omega \otimes e$ be a local section of $\Omega^n_{U_{\underline{\alpha}}} \otimes \mathcal{E}|_{U_{\underline{\alpha}}}$, with $\underline{\alpha}$ interpreted appropriately in each case. For (1), $\mathcal{L}_{\partial_{\alpha_0}} d_{DR}$ applied to $\omega \otimes e$ yields

$$\begin{aligned} \mathcal{L}_{\partial_{\alpha_0}} d_{DR}(\omega \otimes e) &= \mathcal{L}_{\partial_{\alpha_0}}(d\omega \otimes e + (-1)^n \omega \wedge \nabla(e)) \\ &= d\iota_{\partial_{\alpha_0}} d\omega \otimes e + d\omega \otimes \nabla_{\partial_{\alpha_0}}(e) + (-1)^n(\iota_{\partial_{\alpha_0}} d\omega \wedge \nabla(e) \\ &\quad + d\iota_{\partial_{\alpha_0}}\omega \wedge \nabla(e) + \omega \wedge \mathcal{L}_{\partial_{\alpha_0}}\nabla(e)), \end{aligned}$$

while $d_{DR}\mathcal{L}_{\partial_{\alpha_0}}$ applied to $\omega \otimes e$ yields

$$\begin{aligned} d_{DR}\mathcal{L}_{\partial_{\alpha_0}}(\omega \otimes e) &= d_{DR}\left(d\iota_{\partial_{\alpha_0}} e + \iota_{\partial_{\alpha_0}} d\omega \otimes e + (-1)^n \omega \otimes \nabla_{\partial_{\alpha_0}}(e)\right) \\ &= (-1)^n d\iota_{\partial_{\alpha_0}}\omega \wedge \nabla(e) + d\iota_{\nabla_{\alpha_0}} d\omega \otimes e + (-1)^n \iota_{\partial_{\alpha_0}} d\omega \wedge \nabla(e) \\ &\quad + d\omega \otimes \nabla_{\partial_{\alpha_0}}(e) + (-1)^n \omega \wedge \nabla(\nabla_{\partial_{\alpha_0}}(e)). \end{aligned}$$

On comparing terms, we see that it suffices to show that

$$(-1)^n \omega \wedge \mathcal{L}_{\partial_{\alpha_0}} \nabla(e) = (-1)^n \omega \wedge \nabla(\nabla_{\partial_{\alpha_0}}(e)),$$

and so it in turn suffices to show

$$\mathcal{L}_{\partial_{\alpha_0}} \nabla(e) = \nabla(\nabla_{\partial_{\alpha_0}}(e)).$$

Writing $\nabla(e) = \sum_i \omega_i \otimes e_i$, computing the left-hand side yields

$$\begin{aligned} \mathcal{L}_{\partial_{\alpha_0}} \nabla(e) &= \sum_i \mathcal{L}_{\partial_{\alpha_0}}(\omega_i \otimes e_i) \\ &= \sum_i \left(d\iota_{\partial_{\alpha_0}}\omega_i \otimes e_i + \iota_{\partial_{\alpha_0}} d\omega_i \otimes e_i + \omega_i \otimes \nabla_{\partial_{\alpha_0}}(e_i)\right), \end{aligned}$$

while the right-hand side is

$$\begin{aligned} \nabla(\nabla_{\partial_{\alpha_0}}(e)) &= \sum_i \nabla(\iota_{\partial_{\alpha_0}}(\omega_i) e_i) \\ &= \sum_i \left(d\iota_{\partial_{\alpha_0}}\omega_i \otimes e_i + \iota_{\partial_{\alpha_0}}(\omega_i)\nabla(e_i)\right). \end{aligned}$$

Hence, subtracting these expressions, we see that the statement is equivalent to the equality

$$\begin{aligned} 0 &= \sum_i \left(\iota_{\partial_{\alpha_0}} d\omega_i \otimes e_i - \iota_{\partial_{\alpha_0}}\omega_i \wedge \nabla(e_i) + \omega_i \otimes \iota_{\partial_{\alpha_0}} \nabla(e_i)\right) \\ &= \sum_i \left(\iota_{\partial_{\alpha_0}} d\omega_i \otimes e_i - \iota_{\partial_{\alpha_0}}(\omega_i \wedge \nabla(e_i))\right) \qquad \text{By Lemma B.1.4(2)} \\ &= \iota_{\partial_{\alpha_0}}\left(\sum_i d\omega_i \otimes e_i - \omega_i \wedge (e_i)\right) \\ &= \iota\left(\nabla^1\left(\sum_i \omega_i \otimes e_i\right)\right) \\ &= \iota_{\partial_{\alpha_0}}(\nabla^1 \circ \nabla(e)) \end{aligned}$$

which holds by the integrability of ∇.

For (2), we compute directly that

$$\begin{aligned}
&\left(d_{DR} \circ \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} + \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} \circ d_{DR}\right)(\omega \otimes e) \\
&= d_{DR}(\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}(\omega \otimes e)) + \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}(d\omega \otimes e + (-1)^n \omega \wedge \nabla(e)) \\
&= d\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\omega \otimes e + (-1)^{n-1}\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\omega \wedge \nabla(e) + \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} d\omega \otimes e \\
&\quad + (-1)^n \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\omega \wedge \nabla(e) + \omega \otimes \iota_{\partial_{\alpha_0-\partial_{\alpha_1}}}\nabla(e) \\
&= (d\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} + \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}} d)\omega \otimes e + \omega \otimes \nabla_{\partial_{\alpha_0}-\partial_{\alpha_1}}(e) \\
&= \mathcal{L}_{\partial_{\alpha_0}-\partial_{\alpha_1}}(\omega \otimes e) \\
&= \mathcal{L}_{\partial_{\alpha_0}}(\omega \otimes e) - \mathcal{L}_{\partial_{\alpha_1}}(\omega \otimes e).
\end{aligned}$$

Finally, (3) follows from the additivity of the ι in the ∂_α component. □

B.2 A connection on the Čech complex

We now turn to the proof of our desired theorem.

Notation B.2.1. Let

$$\begin{aligned}
C_f^n(\mathcal{E}, \nabla) &:= \bigoplus_{p+q=n} C_f^{p,q}(\mathcal{E}, \nabla) \\
&= \bigoplus_{p+q=n} C_f^p(\mathcal{U}, \Omega^q_{X/S} \otimes \mathcal{E}) \text{by } (B.1) \\
&= \bigoplus_{p+q=n} \bigoplus_{\alpha_0 < \ldots < \alpha_p} \left(\Omega^q_{U_{\underline{\alpha}}/S} \otimes \mathcal{E}|_{U_\alpha}\right)
\end{aligned}$$

be the total complex of our Čech bicomplex. Let $\check{d}$ be the Čech differential of our double complex. The total differential on $C_f^{p,q}(\mathcal{E}, \nabla)$ is then given by

$$\delta = d_{DR} + (-1)^q \check{d}.$$

We note that if $\mathcal{E}$ is quasi-coherent and each U_α is affine (so that $\mathcal{E}|_{U_\alpha}$ is acyclic), $R^j_{DR} f_*(\mathcal{E}, \nabla)$, the description of ${}_C E_2^{p,q}$ yields that

$$H^i(C_f^\bullet(\mathcal{E}, \nabla)) \cong \mathbb{R}^i f_* \operatorname{DR}_{X/S}(\mathcal{E}, \nabla). \tag{B.2}$$

Before stating the main theorem of this appendix, we first extend our notation to maps on the Čech bicomplex. The maps $\mathcal{L}_{\partial_{\alpha_0}}$ and ι_∂ extend to the Čech bicomplex by defining these pointwise. We will however want a variant of the interior product which yields a degree 0 map (i.e. leaves total degree unchanged) on $C_f^\bullet(\mathcal{E}, \nabla)$.*

*André-Baldassarri's sections in [1, 2.5.2] and [2, 23.4.1] use an abuse of notation with $\iota_{\partial_{i_0}-\partial_{i_1}}$, which ends up being highly confusing for explicit computations.

Definition B.2.2. For a vector field ∂ on S, we let $\lambda_\partial : C_f^\bullet(\mathcal{E}, \nabla) \to C_f^\bullet(\mathcal{E}, \nabla)$ be the (degree 0) map defined componentwise by

$$\lambda_\partial : C_f^p(\mathcal{U}, \Omega^q_{X/S} \otimes \mathcal{E}) \to C_f^{p+1}(\mathcal{U}, \Omega^{q-1}_{X/S} \otimes \mathcal{E})$$
$$\lambda_\partial(\beta)_{\alpha_0\ldots\alpha_{p+1}} = (-1)^q \iota_{\partial_{\alpha_0} - \partial_{\alpha_1}}(\beta_{\alpha_1\ldots\alpha_p}) \left(= (-1)^q \iota_{\partial_{\alpha_0} - \partial_{\alpha_1}} \beta\right)_{\alpha_1\ldots\alpha_p}$$

We are now in a position to state the theorem.

Theorem B.2.3. *The maps sending a vector field ∇ on S to $\widetilde{\aleph}_\partial = \mathcal{L}_{\partial_{\alpha_0}} + \lambda_\partial$ defines a connection on $C_f^i(\mathcal{E}, \nabla)$, which descends to the map ∇^i_{GM} of Theorem 2.3.1.*

Proof. We note that $\widetilde{\aleph}$ is functorial as $\mathcal{L}_{\partial_{\alpha_0}}$ and λ_∂ are. It also defines a connection, as $\mathcal{L}_{\partial_{\alpha_0}}$ satisfies a Leibniz rule, and λ_∂ is linear.

We first establish that $[\widetilde{\aleph}_\partial, \delta] = 0$, so that $\widetilde{\aleph}_\partial$ descends to connections $\aleph^i$ on $H^i(C_f^\bullet(\mathcal{E}, \nabla)) \cong \mathbb{R}^i f_* \operatorname{DR}_{X/S}(\mathcal{E}, \nabla)$. Since

$$[\widetilde{\aleph}_\partial, \delta] = \left[\mathcal{L}_{\partial_{\alpha_0}}, d_{DR}\right] + \left(\left[\mathcal{L}_{\partial_{\alpha_0}}, (-1)^q \check{d}\right] + [\lambda_\partial, d_{DR}]\right) + [\lambda_\partial, (-1)^q \check{d}],$$

this will follow from the following claim, in conjunction with (1) of Lemma B.1.7.

Claim B.2.4. We have the identities

1. $\left[\mathcal{L}_{\partial_{\alpha_0}}, (-1)^q \check{d}\right] + [\lambda_\partial, d_{DR}] = 0$; and
2. $[\lambda_\partial, (-1)^q \check{d}] = 0$.

Proof of Claim B.2.4. For $\beta \in C^p(\mathcal{U}, , \Omega^q_{X/S} \otimes \mathcal{E})$, as complex maps, we have

$$\left(\mathcal{L}_{\partial_{\alpha_0}} \check{d}\beta\right)_{\alpha_0\ldots\alpha_{p+1}} = \mathcal{L}_{\partial_{\alpha_1}} \beta_{\alpha_1\ldots\alpha_{p+1}} + \sum_{i=1}^{p+1} \mathcal{L}_{\partial_{\alpha_0}} \beta_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}},$$

while

$$\left(\check{d}\mathcal{L}_{\partial_{\alpha_0}} \beta\right)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \mathcal{L}_{\partial_{\alpha_0}} \beta_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}}.$$

We hence have

$$\begin{aligned}
&\left(\left[\mathcal{L}_{\partial_{\alpha_0}}, (-1)^q \check{d}\right] \beta\right)_{\alpha_0\ldots\alpha_{p+1}} \\
&= (-1)^q \left(\mathcal{L}_{\partial_{\alpha_1}} - \mathcal{L}_{\partial_{\alpha_0}}\right) \beta_{\alpha_1\ldots\alpha_{p+1}} \\
&= (-1)^{q+1} (d_{DR} \iota_{\partial_{\alpha_0} - \partial_{\alpha_1}} + \iota_{\partial_{\alpha_0} - \partial_{\alpha_1}} d_{DR}) \beta_{\alpha_1\ldots\alpha_{p+1}} \qquad \text{by Lemma B.1.7(2)} \\
&= -(-d_{DR}(\lambda\beta)_{\alpha_0\ldots\alpha_{p+1}} + (\lambda(d_{DR}\beta))_{\alpha_0\ldots\alpha_{p+1}} \\
&= -\left([\lambda, d_{DR}]\, \beta\right)_{\alpha_0\ldots\alpha_{p+1}}
\end{aligned}$$

and hence (1) follows. For (2), we have

$$\begin{aligned}\left(\lambda\check{d}\beta\right)_{\alpha_0\ldots\alpha_p} &= \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\left(\check{d}\beta\right)_{\alpha_1\ldots\alpha_p}\\ &= \sum_{i=1}^{p}(-1)^{i-1}\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\beta_{\alpha_1\ldots\widehat{\alpha_i}\ldots\alpha_p}\\ &= \iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\beta_{\alpha_2\ldots\alpha_p}-\sum_{i=2}^{p}(-1)^{i}\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\beta_{\alpha_1\ldots\widehat{\alpha_i}\ldots\alpha_p},\end{aligned}$$

while

$$\begin{aligned}\left(\check{d}\lambda\beta\right)_{\alpha_0\ldots\alpha_p} &= \sum_{i=0}^{p}(-1)^{i}(\lambda\beta)_{\alpha_1\ldots\widehat{\alpha_i}\ldots\alpha_p}\\ &= \iota_{\partial_{\alpha_1}-\partial_{\alpha_2}}\beta_{\alpha_2\ldots\alpha_p}-\iota_{\partial_{\alpha_0}-\partial_{\alpha_2}}\beta_{\alpha_2\ldots\alpha_p}+\sum_{i=2}^{p}(-1)^{i}\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}\beta_{\alpha_1\ldots\widehat{\alpha_i}\ldots\alpha_p}.\end{aligned}$$

We hence get

$$\begin{aligned}&\left([\lambda,(-1)^q\check{d}]\beta\right)_{\alpha_0\ldots\alpha_p}\\ &= \left((-1)^q\lambda\check{d}\beta-(-1)^{q-1}\check{d}\lambda\beta\right)_{\alpha_0\ldots\alpha_p}\\ &= (\iota_{\partial_{\alpha_0}-\partial_{\alpha_1}}+\iota_{\partial_{\alpha_1}-\partial_{\alpha_2}}+\iota_{\partial_{\alpha_2-\alpha_0}})\beta_{\alpha_2\ldots\alpha_p}\\ &= 0 \qquad\qquad \text{by Lemma B.1.7(3).}\end{aligned}$$

□

We now argue by functoriality that $\aleph^i$ agrees with $R^i_{DR}f_*$, up to the exact embedding $\omega : \mathbf{MIC}(S) \to \mathbf{MC}(S)$. Since $\widetilde{\aleph}$ is functorial in $(\mathcal{E}, \nabla)$, this forms (by the snake lemma) a δ-functor. For $i > 0$, this is effaceable by the argument of Proposition 2.3.7, since we may embed into some injective $(\mathcal{I}, \nabla)$, and injectives are acyclic. We thus deduce $\aleph^i = R^i\aleph^0$ by [22, III.1.4].

By the given formulas, we have that $\mathcal{L}_{\partial_{\alpha_0}} = \nabla_{\partial_{\alpha_0}}$ on $\mathcal{E}$, and so the connections defined by $\aleph^0$ and $\omega \circ R^0_{DR}f_*$ on $f_*\mathcal{E}^{\nabla_{X/S}}$ agree, as they are induced by $\mathcal{L}$ and ∇ respectively. Hence $\aleph^0 = \omega \circ R^0_{DR}f_*$, so $\aleph^i = R^i(\omega \circ R^0_{DR}f_*) = \omega \circ R^i_{DR}f_*$ by the exactness of ω, and $\aleph^i$ takes values in $\mathbf{MIC}(S)$. □

Appendix C

Some complex-analytic geometry

In this appendix, we collate, for convenience, the facts we will need about complex-analytic geometry. We start with some background on analytic varieties and Stein spaces, since this thesis is primarily written from the perspective of algebraic geometry. We then give some results on complex local systems on analytic varieties.

C.1 General complex-analytic background

In this section, we define (complex-)analytic varieties and Stein spaces, and state a few results we will need about them.

For the definition of a complex-analytic variety, we note that any open $U \subseteq \mathbb{C}^n$ inherits a sheaf of holomorphic functions $\mathcal{O}_U = \mathcal{O}_{\mathbb{C}^n}|_U$ from $\mathcal{O}_{\mathbb{C}^n}$.

Definition C.1.1.

1. For any $U \subseteq \mathbb{C}^n$ and $f_1, \ldots, f_k \in \mathcal{O}_U(U)$, the associated *local model space* is the locally ringed space whose underlying set is $V(f_1, \ldots, f_k) = \{x \in U \mid f_i(x) = 0\}$, equipped with the subspace topology from U, and the sheaf $\mathcal{O}_U/(f_1, \ldots, f_k)\mathcal{O}_U$.

2. A *complex-analytic space* $\mathcal{X}$ is a locally ringed space which is locally isomorphic to a local model space. A morphism of complex-analytic spaces is a morphism as locally ringed spaces.

3. A complex-analytic space is a *complex-analytic variety* if it is reduced and its underlying space is Hausdorff, and a complex-analytic variety is *smooth* if its underlying space is a complex manifold.

We will just refer to these as analytic varieties, since we will only be working over $\mathbb{C}$. Note also that as analytic varieties are locally ringed spaces, the usual notion of schemes and morphisms over a base scheme translates to a notion of *analytic varieties over a base analytic space* and *morphisms over a base analytic space*. We also have a notion of *smoothness* for complex-analytic spaces: A complex-analytic space is *smooth* if its underlying topological space is a complex manifold.

In terms of morphisms, the notions of smoothness, properness, étale and finiteness are slightly different.

Definition C.1.2. Let $f : \mathcal{X} \to \mathcal{S}$ be a morphism of smooth complex-analytic spaces. We say that f is

1. *smooth* if it is a smooth map of complex manifolds;
2. *proper* if it is universally closed;
3. *étale* if it is locally an isomorphism of complex-analytic spaces; and
4. *finite* if for any $s \in S$, $f^{-1}(s)$ is finite.

We turn to defining Stein spaces, and the main properties of these which we will need.

Definition C.1.3. An analytic space $\mathcal{X}$ is a *Stein space* if for every coherent analytic sheaf $\mathcal{F}$, $H^q(\mathcal{X}, \mathcal{F}) = 0$ for all $q > 0$.

There are more explicit definitions (see [16, IV.4.5]) which are used to prove the below statements, but these are omitted for brevity, and as the above vanishing property is the main one we need. We will need the following examples of Stein spaces.

Proposition C.1.4. *The following are Stein spaces.*

1. $\mathbb{C}^n$*, for any* $n \geq 1$*;*
2. *Any closed subspace of a Stein space;*
3. *Any polydisc* $D = D_1 \times \ldots \times D_n \subseteq \mathbb{C}^n$ *(where* D_i *are discs in* $\mathbb{C}$*).*

We refer to the much more general [16, V.1.1] for the proof of this statement.

Proposition C.1.5. *Let* $f : \mathbb{C}^n \to \mathbb{C}^m$ *be a polynomial map, and* $D \subseteq \mathbb{C}^m$ *be a polydisc. Then* $f^{-1}(D)$ *is a Stein space. In particular, for any closed subspace* $X \subseteq \mathbb{C}^n$*,* $f^{-1}(D)$ *is a Stein space.*

The first part of this result is [34, 5.3.8], and the latter part follows from the previous proposition.

C.2 Complex local systems

We define and state some results on complex local systems, which we use in the proofs of §6.4. The proofs can be found in the provided references.

Definition C.2.1. Let X be a topological space. A sheaf on X is a *complex local system* if it is locally isomorphic to the constant sheaf $\underline{\mathbb{C}^n}$ for some $n \geq 0$.

We denote the full subcategory of $\mathrm{Sh}(\mathcal{X})$ whose objects are complex local systems by $\mathrm{LocSys}(\mathcal{X})$.

The importance of complex local systems is due to the *Riemann-Hilbert correspondence*, which relates complex local systems to $\mathcal{O}_{\mathcal{X}}$-modules with integrable connection.

Theorem C.2.2. *Let $\mathcal{X}$ be a smooth analytic variety. Then* $\mathrm{LocSys}(\mathcal{X})$ *is equivalent to the category $\boldsymbol{MIC}(\mathcal{X})$, with quasi-inverse functors*

$$\begin{aligned} \mathrm{LocSys}(\mathcal{X}) &\to \boldsymbol{MIC}(\mathcal{X}) \\ \mathcal{V} &\mapsto (\mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{C}} \mathcal{V}, d_{\mathcal{X}} \otimes 1) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{MIC}(\mathcal{X}) &\to \mathrm{LocSys}(\mathcal{X}) \\ (\mathcal{E}, \nabla) &\mapsto \mathcal{E}^{\nabla}. \end{aligned}$$

We also have the following *relative Poincaré lemma*.

Theorem C.2.3. *Let $f : \mathcal{X} \to \mathcal{S}$ be a smooth morphism of smooth analytic varieties, and $\mathcal{V}$ be a complex local system on $\mathcal{X}$. Then the de Rham complex of $(\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{V}, d_{X/S} \otimes 1)$ is a resolution of $f^{-1}\mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{V}$.*

These statements are [10, I.2.17] and [10, I.2.23.ii], and their proofs are given in [10, I.2.23].

We now turn to giving the statements necessary for the proof of the coherence theorem 6.4.10. To describe the main result behind the coherence theorem, we give some context.

For notation, we let $f : \mathcal{X} \to \mathcal{S}$ be a smooth morphism of smooth analytic varieties, and let $\mathcal{V}$ is a complex local system on $\mathcal{X}$, with associated vector bundle $(\mathcal{E}, \nabla) = (\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{X}}, 1 \otimes d_{\mathcal{X}})$. This gives rise to the relative vector bundle $(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{X}}, 1 \otimes d_{\mathcal{X}/\mathcal{S}})$.

By Theorem C.2.3, combined with the flatness of $\mathcal{V}$ over $\underline{\mathbb{C}}$, the complex

$$\mathrm{DR}_{\mathcal{X}/\mathcal{S}}(\mathcal{E},\nabla) = \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{X}} \xrightarrow{1\otimes d_{\mathcal{X}/\mathcal{S}}} \mathcal{V} \otimes_{\mathbb{C}} \Omega^1_{\mathcal{X}} \xrightarrow{1\otimes d_{\mathcal{X}}} \dots$$

is a resolution of $\mathcal{V} \otimes_{\mathbb{C}} f^{-1}\mathcal{O}_{\mathcal{S}}$. On applying hypercohomology, for each $i \geq 0$ we get an isomorphism

$$R^q f_*(f^{-1}\mathcal{O}_{\mathcal{S}} \otimes_{\mathbb{C}} \mathcal{V}) \cong \mathbb{R}^q f_* \,\mathrm{DR}_{\mathcal{X}/\mathcal{S}}(\mathcal{E},\nabla) \cong R^q_{DR} f_*(\mathcal{E},\nabla).$$

We also have a map on higher direct images

$$\mathcal{O}_S \otimes_{\mathbb{C}} R^q f_* \mathcal{V} \to R^q f_*(f^{-1}\mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{V})$$

by considering injective resolutions (see [24, 01E8], applied to the morphism of ringed spaces $(X, f^{-1}\mathcal{O}_S) \to (S, \underline{\mathbb{C}})$). Composing these gives

$$\mathcal{O}_{\mathcal{S}} \otimes_{\mathbb{C}} R^q f_* \mathcal{V} \to R^q_{DR} f_*(\mathcal{E},\nabla) = R^q_{DR} f_*(\mathcal{O}_{\mathcal{X}} \otimes_{\mathbb{C}} \mathcal{V}, d_{\mathcal{X}/\mathcal{S}} \otimes 1), \tag{C.1}$$

or equivalently, phrased in terms of $(\mathcal{E},\nabla)$,

$$\mathcal{O}_{\mathcal{S}} \otimes_{\mathbb{C}} R^q f_* \mathcal{E}^{\nabla} \to R^q_{DR} f_*(\mathcal{E},\nabla). \tag{C.2}$$

Theorem C.2.4. *Let $f : \mathcal{X} \to \mathcal{S}$ be a smooth morphism of smooth analytic varieties, and i be an integer, and $\mathcal{V}$ be a complex local system on $\mathcal{X}$. Suppose that*

1. *f is topologically trivial locally on $\mathcal{S}$; and*
2. *the fibres of f satisfy*

$$\dim H^i(f^{-1}(s), \mathcal{V}|_{f^{-1}(s)}) < \infty.$$

Then the map C.1 (hence C.2) is an isomorphism.

This result and its proof can be found in [10, I.2.28]. To apply this result, we further require the following two results, which again rely on topological arguments.

Proposition C.2.5. *Let X be a finite CW-complex of dimension n, and $\mathcal{F}$ be a complex local system on X. Then $H^k(X,\mathcal{F})$ is a finite-dimensional vector space over $\mathbb{C}$ for $0 \leq k \leq n$, and vanishes for $k < 0$ and $k > n$.*

This result is [11, 2.5.4.i]*. We apply this to $\mathbb{A}^{1an} - \{\theta_1, \ldots, \theta_r\}$ using the following homotopy invariance result. In the proof of this theorem, we will need the following homotopy-invariance result on sheaf cohomology.

Theorem C.2.6. *Let $f_0, f_1 : X \rightrightarrows Y$ be homotopic maps, and $\mathcal{F}$ be a locally constant sheaf on Y. Let $f_i^{\#j} : H^j(Y, \mathcal{F}) \to H^j(X, f_i^{-1}\mathcal{F})$ be the induced maps on cohomology, for $i = 0, 1$. Then there are isomorphisms*

$$\theta^j : H^j(X, f_0^{-1}\mathcal{F}) \to H^j(X, f_1^{-1}\mathcal{F})$$

such that $\theta^j \circ f_0^{\#j} = f_1^{\#j}$.

Further, if $f : X \to Y$ is a homotopy equivalence, then

$$f^{\#j} : H^j(Y, \mathcal{F}) \to H^j(X, f^{-1}\mathcal{F})$$

is an isomorphism.

The proof of this theorem can be found in [36, 6.3.4] and [36, 6.3.10]. Hence the conclusion of Theorem C.2.4 holds for any map which is topologically trivial locally on $\mathcal{S}$, and so that the fibres are homotopy equivalent to finite CW complexes, as we may apply Proposition C.2.5.

Our last theorem in this appendix is on the structure of higher direct images of complex local systems, and is essential to the proofs of coherence and smooth base change.

Theorem C.2.7. *Let $f : \mathcal{X} \to \mathcal{S}$ be a complex fibre bundle (i.e. topologically locally trivial on $\mathcal{S}$). Then for any complex local system $\mathcal{V}$ on $\mathcal{X}$ and any $q \geq 0$, $R^q f_* \mathcal{V}$ is a complex local system on $\mathcal{S}$, locally isomorphic to the constant sheaf $\underline{H^q(\mathcal{X}_s, \mathcal{V}|_{\mathcal{X}_s})}$ on any contractible, trivialising open $U \subseteq \mathcal{S}$ for any point s of U, via the homotopy equivalence $\mathcal{X}_s \hookrightarrow f^{-1}(U)$.*

Proof of Theorem C.2.7. By localising on $\mathcal{S}$, we may assume that $\mathcal{X} = \mathcal{S} \times \mathcal{X}_0$ for some fixed fibre $\mathcal{X}_0$, and that $f = \mathrm{pr}_1$. By [9, 2.12], to show that $R^q f_* \mathcal{V}$ is locally constant, it suffices to show that for any contractible open subsets $i : U_1 \hookrightarrow U_2$ of X, the restriction map $R^q f_* \mathcal{V}(U_2) \to R^q f_* \mathcal{V}(U_1)$ is an isomorphism.

On the corresponding presheaf $U \mapsto H^q(f^{-1}(U), \mathcal{V}|_{f^{-1}(U)})$, we note that the restriction map

$$H^q(f^{-1}(U_2), \mathcal{V}|_{f^{-1}(U_2)}) \to H^q(f^{-1}(U_1), \mathcal{V}|_{f^{-1}(U_1)})$$

*This is initially stated for homology, but applies equally as well on cohomology, as specified in the discussion after the proof of [11, 2.5.5].

is induced by the homotopy eqivalence i, hence an isomorphism by Theorem C.2.6. The local description also follows from Theorem C.2.6, as $f^{-1}(U)$ is homotopy equivalent to $\mathcal{X}_s$ for any contractible subset U of X for which $f|_{f^{-1}(U)}$ is trivial. $\square$

Appendix D

Solvability of regular connections on $\mathbb{A}^1$

The goal of this appendix is to prove that any coherent, regular $\mathcal{O}_{\mathbb{A}^1-\{\theta_1,\ldots,\theta_r\}}$-module with connection $(\mathcal{E}, \nabla)$ is solvable. We work exclusively in the affine case described in Definition 1.1.1. We first establish a correspondence between ordinary differential equations, and differential modules generated by a single element. We then define the dual of a differential module with connection, and relate solutions of a differential module to the horizontal sections of the dual.

D.1 Some affine $\mathcal{D}$-module theory

We first introduce some notation for this section, roughly following the first chapter of [2].

Notation D.1.1. Let (R, ∂) be a differential ring, and (E, ∇_∂) be a differential module over (R, ∂), of rank μ. We denote by $R\langle\partial\rangle$ the (non-commutative) subring of $\mathrm{End}_{\mathbb{C}}(E)$ generated by R and ∂.

We note that every element of $R\langle\partial\rangle$ can be expressed uniquely as a polynomial in ∂, since the Leibniz rule $\partial(fg) = \partial(f)g + f\partial(g)$ yields the commutation rule $[\partial, f] = \partial(f)$.

Definition D.1.2. An element $e \in E$ is a *cyclic vector for* (E, ∇_∂) if E is a free R-module, with basis $\left\{e, \nabla_\partial(e), \ldots, \nabla_\partial^{\mu-1}(e)\right\}$. A differential module is *cyclic* if it admits a cyclic vector.

If (E, ∇_∂) is cyclic, we will denote the setting where we have a prescribed cyclic vector e by (E, ∇_∂, e). If (E, ∇_∂, e) is a cyclic differential module, then there are unique coefficients $a_i \in \mathcal{O}(X)$ so that

$$\nabla_\partial^\mu(e) = -a_{\mu-1}\nabla_\partial^{\mu-1}(e) - \ldots - a_i\nabla_\partial(e) - a_0 e. \tag{D.1}$$

Definition D.1.3. Let (E, ∇_∂, e) be a cyclic differential module. Its *associated differential polynomial* is

$$L_{(E,\nabla_\partial,e)} = \partial^\mu + a_{\mu-1}\partial^{\mu-1} + \ldots + a_1\partial + a_0 \in R\langle\partial\rangle\,,$$

with coefficients as in (D.1).

This is hence the minimal polynomial so that $L(\partial)e = 0$, with the action of ∂ on E interpreted through ∇_∂. It turns out that at the cost of possibly localising on $\operatorname{Spec}(R)$, we may always assume that (E, ∇_∂) is cyclic. We first specify what is meant by "localising". In the setting where $(R, \partial) \subseteq (R', \partial')$, we will say that (R', ∂') is an *extension* of (R, ∂), and we will denote this by $(R', \partial')/(R, \partial)$, or just R'/R where the derivations are clear from context.

Definition D.1.4. If (R', ∂) is an extension of (R, ∂), we define the *extension* of an (R, ∂)-differential module (E, ∇_∂) to (R', ∂') to have underlying module $E \otimes_R R'$, and associated derivation

$$\nabla_{\partial'}(e \otimes r') = \nabla_\partial(e) \otimes r' + e \otimes \partial'(r').$$

We will usually write ∂ for both the derivations on R and R'.

Theorem D.1.5. *There is an affine open cover $\{U_\alpha = \operatorname{Spec}(R_\alpha)\}_{\alpha\in\Lambda}$ of $\operatorname{Spec}(R)$, such that for every $\alpha \in \Lambda$, $(E \otimes_R R_\alpha, \nabla_\partial)$ is a cyclic differential module.*

This follows from [26, Theorem 2], and means that the following correspondence can be applied locally to an arbitrary differential module.

Theorem D.1.6. *The assignment $(E, \nabla, e) \mapsto L_{(E,\nabla_\partial,e)}$ gives a one-to-one correspondence between cyclic differential modules (E, ∇_∂, e) with prescribed cyclic vectors, and elements of $R\langle\partial\rangle$. The inverse mapping is given by sending $L \in R\langle\partial\rangle$ to $R\langle\partial\rangle / L(\partial) R\langle\partial\rangle$.*

This essentially follows from [2, 3.2.8].

D.2 The exponential formalism

We define what it means to exponentiate a formal variable x by a $\mu \times \mu$ matrix $A \in M_\mu(\mathbb{C})$, and tie this back to regularity and solving differential equations over A_i. We will work in the case of equations centred at some $x = \theta$.

We begin by describing a logarithm, the powers $(x-\theta)^\alpha$ for $\alpha \in \mathbb{C}$, and an associated extension of $\mathbb{C}((x-\theta))$ (with an appropriate derivation) for which the exponentiation $(x-\theta)^A$ will take values in. Consider the differential ring $(\mathbb{C}((x-\theta)), \partial_x)$, and let $\vartheta_x = (x-\theta)\partial_x$. We then consider the extension (F_θ, ϑ_x) of $(\mathbb{C}((x-\theta)), \vartheta_x)$ given by adjoining

1. A solution $y = \log(x-\theta)$ of the differential equation $\vartheta_x y = 1$; and

2. For each $\alpha \in \mathbb{C}$, a solution $(x-\theta)^\alpha$ of the differential equation $\vartheta_x y = \alpha y$.

Explicitly, we may choose a branch of the logarithm centred at θ over $\mathbb{C}$, and set

$$(x-\theta)^\alpha := \exp(\alpha \log(x-\theta)),$$

from which we also have $(x-\theta)^{\alpha+\beta} = (x-\theta)^\alpha (x-\theta)^\beta$, for $\alpha, \beta \in \mathbb{C}$.

Definition D.2.1. For $A \in M_\mu(\mathbb{C})$, we define $(x-\theta)^A \in \mathrm{GL}_\mu(F_\theta)$ as follows:

1. If $A = \Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_n \end{bmatrix}$ is diagonal, we set

$$(x-\theta)^\Delta = \begin{bmatrix} (x-\theta)^{\Delta_1} & & \\ & \ddots & \\ & & (x-\theta)^{\Delta_n} \end{bmatrix};$$

2. If $A = N$ is nilpotent, we set $(x-\theta)^N$ to be the (finite) sum

$$x^N = \exp\left(N \log(x-\theta)\right) = \sum_{j=0}^{\infty} \frac{N^j}{j!} (\log(x-\theta))^j;$$

3. For a general $A \in M_\mu(\mathbb{C})$, writing $A = P(\Delta + N)P^{-1}$ for its Jordan normal form (with $P \in \mathrm{GL}_\mu(\mathbb{C}) \subseteq \mathrm{GL}_\mu(F_\theta)$), we set

$$(x-\theta)^A := P(x-\theta)^\Delta (x-\theta)^N P^{-1}.$$

We note that a solution $y \in F_\theta$ to the differential equation

$$\vartheta_x^\mu y + a_{\mu-1}\vartheta_x^{\mu-1} y + \ldots + a_1\vartheta_x y + a_0 y = 0 \tag{D.2}$$

is equivalent to a system of the form $\vartheta_x \mathbf{y} = G\mathbf{y}$ with $\mathbf{y} \in F_\theta^\mu$ and $G \in \mathrm{GL}_\mu(F_\theta)$, with

$$\mathbf{y} = \begin{bmatrix} y \\ \theta_x y \\ \vdots \\ \theta_x^{\mu-1} y \end{bmatrix} \qquad\qquad G = \begin{bmatrix} 0 & 0 & \ldots & -a_0 \\ 1 & 0 & \ldots & -a_1 \\ 0 & 1 & \ldots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -a_{\mu-1} \end{bmatrix}$$

In terms of this form, a basis of μ solutions to (D.2) is equivalent to a matrix $Y \in M_\mu(F_\theta)$ such that $\vartheta_x Y = GY$. In the next section, we will use this to describe solutions to systems of the form (D.2) coming from a cyclic vector on a vector bundle with regular connection over $\mathbb{A}^1$.

D.3 The solvability theorem

To set up the theorem, we recall the following notation.

Notation D.3.1. We set

1. $\theta_1, \ldots, \theta_r \in \mathbb{C}$, and $\Theta(x) = \prod_{i=1}^r (x - \theta_i)$;
2. $X = \operatorname{Spec} \mathbb{C}[x.1/\Theta(x)] = \mathbb{A}^1 - \{\theta_1, \ldots, \theta_r\}$;
3. $A_i = \mathcal{M}_{\theta_i}[\log(x-\theta_i), (x-\theta_i)^\alpha]_{\alpha\in\mathbb{C}}$, and $A = \prod_{i=0}^r A_i$;
4. $\partial = \partial_x$ as a derivation on $\mathcal{O}(X) = \mathbb{C}[x, 1/\Theta(x)]$, so that $A_i^\partial = \mathbb{C}$; and
5. (E, ∇_{∂_x}) be a differential module in the sense of Definition 1.1.1(2).

These relate to the discussion of the previous section by the formula

$$F_{\theta_i} = A_i \otimes_{M_{\theta_i}} \mathbb{C}((x - \theta_i)).$$

Note that E equivalently defines a coherent $\mathcal{O}_X$-module with connection, and so we may talk about the regularity of E from this viewpoint. This is as, we have $\Omega_X^1 = \mathcal{O}(X)dx$, and so $\mathcal{T}_X = \mathcal{O}(X)\partial_x$ is free on ∂_x, so ∇_{∂_x} determines ∇ on the associated $\mathcal{O}_X$-module $\widetilde{E}$.

We will need the following formulation of regularity over $(M_{\theta_i}, \partial_x)$, where ∂_x is the usual (formal) derivative on meromorphic function germs. We will require the following reformulation of regularity over M_{θ_i}.

Theorem D.3.2. *Let $(E, \nabla_{\partial_x}, e)$ be a cyclic differential module of finite rank μ over $(M_{\theta_i}, \partial_x)$. Then the following are equivalent.*

1. *∇_{∂_x} defines a regular connection on $\widetilde{E}$;*

2. *If $\Lambda = \vartheta_x^\mu + a_{r-1}\vartheta_x^{r-1} + \ldots + a_1\vartheta_x + a_0$ is the minimal differential operator associated to e (with $a_j \in M_{\theta_i}$), then the associated equation $\theta_x Y = GY$ has a solution of the form*

$$Y = W(x - \theta_i)^A, \tag{D.3}$$

with $W \in \mathrm{GL}_\mu(M_{\theta_i})$ and $A \in M_\mu(\mathbb{C})$.

This statement is obtained as follows. For each i, by extending base up to $\mathbb{C}((x - \theta_i)) \supseteq M_{\theta_i}$, the equivalence of [2, 7.4.1] yields a solution of the form (D.3) with $W \in \mathrm{GL}_\mu(\mathbb{C}((x - \theta_i)))$. We then get that $W \in \mathrm{GL}_\mu(M_{\theta_i})$ by using the regularity criterion [2, 7.5.1] (so that $\mathrm{ord}_{\theta_i}(a_j) \geq 0$ for each j), combined with the associated local convergence result [2, 6.3.5] of Fuchs-Frobenius.

In particular, the previous theorem yields a basis of solutions $y = y_1, \ldots, y_n$ to the equation $\Lambda y = 0$ over A_i.

Theorem D.3.3. *Let $X = \mathrm{Spec}(\mathbb{C}[x, 1/\Theta(x)]) = \mathbb{A}^1 - \{\theta_1, \ldots, \theta_r\}$, and (E, ∇_∂) be a regular differential module over $\mathcal{O}(X)$, projective of finite rank μ over $\mathcal{O}(X)$. Then E is solvable in A, i.e. the map*

$$\left(E \otimes_{\mathcal{O}(X)} A\right)^\partial \otimes_{A^\partial} A \to E \otimes_{\mathcal{O}(X)} A$$

is an isomorphism.

Proof. Since the map splits componentwise as the direct sum of the maps

$$\left(E \otimes_{\mathcal{O}(X)} A_i\right)^\partial \otimes_{A_i^\partial} A_i \to E \otimes_{\mathcal{O}(X)} A_i,$$

we may reduce to the analogous statements with A instead of A_i. Since this question is local at θ_i, it follows from Theorem D.1.5 the modules $E \otimes_{\mathcal{O}(X)} A_i$ are cyclic. We note that it suffices to show that

$$E \otimes_{\mathcal{O}(X)} A_i = \bigoplus_{k=1}^{\mu} A_i y_k$$

for $\alpha_k \in \left(E \otimes_{\mathcal{O}(X)} A_i\right)^\partial$, as the map will then preserve the basis $\{y_1, \ldots, y_\mu\}$. Let $\Lambda \in A_i \langle \partial \rangle$ be the associated differential polynomial to a cyclic vector $e \in E \otimes_{\mathcal{O}(X)} A_i$. Then by Theorem D.3.2, the equation $\Lambda y = 0$ has a basis of solutions $y_1, \ldots, y_\mu$. □

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