Analytical Solution for Electromagnetic Ray Trajectories in a Quadratic Plasma Profile

Plasma Profile Definition

The following profile will be described in cylindrical coordinates with variation only along the r axis. The plasma profile is designed to have a background electron density of n_o. The minima of the quadratic electron density distribution is located at r_o. The density well has a width of W. At a distance of $0.5W$ from the minima, the density will reach the critical value causing the electromagnetic rays to reverse their trajectory.

$$n_e(r) = \frac{4(n_c - n_o)}{W^2}(r - r_o)^2 + n_o$$

Initial Conditions

The relevant initial conditions for an electromagnetic ray are its initial position and wavevector. The rays will be launched with no θ component within the wavevector. The initial position will be (z_i, r_i) and the initial wavevector will be (k_{zi}, k_{ri}).

$$x_i = (z_i, r_i)$$

$$k_i = (k_{zi}, k_{ri})$$

Dispersion Equation and Hamilton's Ray Equation

In SI units, the dispersion relation can be expressed as the following:

$$\omega^2 = \omega_{pe}^2 + c^2k^2$$

Expanding the plasma frequency results in:

$$\omega^2 = \alpha n_e(r) + c^2k^2$$

$$\alpha = \frac{e^2}{m_e \epsilon_0}$$

The Hamilton's ray equations are described below

$$V_{g,z} = \frac{dz}{dt} = \left(\frac{\partial \omega}{\partial k_z}\right)_{x,t}$$

$$V_{r,z} = \frac{dr}{dt} = \left(\frac{\partial \omega}{\partial k_r}\right)_{x,t}$$

$$\frac{dk_z}{dt} = -\left(\frac{\partial \omega}{\partial z}\right)_{k,t}$$

$$\frac{dk_r}{dt} = -\left(\frac{\partial \omega}{\partial r}\right)_{k,t}$$

Substituting in the dispersion relation for electromagnetic waves in plasma:

$$V_{g,z} = \frac{dz}{dt} = \frac{c^2 k_z}{\omega}$$

$$V_{r,z} = \frac{dr}{dt} = \frac{c^2 k_r}{\omega}$$

$$\frac{dk_z}{dt} = 0$$

$$\frac{dk_r}{dt} = -\beta(r - r_o)$$

$$\beta = \frac{4(n_c - n_o)}{\omega W^2}\alpha = \frac{4(n_c - n_o)e^2}{\omega W^2 m_e \epsilon_0}$$

Solving for trajectory in the z-dimension

As there is no change in k_z over time, it retains the value of the initial condition.

$$k_z(t) = \left(\int_0^t \frac{dk_z}{d\tau} d\tau\right)_{k_z(0)=k_{zi}} = \left(\int_0^t 0\, d\tau\right)_{k_z(0)=k_{zi}} = k_{zi}$$

With an analytical definition of $k_z(t)$, it can be substituted into the expression for $z(t)$

$$z(t) = \left(\int_0^t \frac{d\omega}{dk_z} d\tau\right)_{z(0)=z_i} = \left(\int_0^t \frac{c^2 k_z(t)}{\omega} d\tau\right)_{z(0)=z_i}$$

The trajectory in the z dimesion is defined analytically as:

$$z(t) = \frac{c^2 k_{zi}}{\omega} + z_i$$

Solving for trajectory in the r-dimension

The coupled set of equations is stated below:

$$\frac{dr}{dt} = \frac{c^2 k_r}{\omega}$$

$$\frac{dk_r}{dt} = -\beta(r - r_o)$$

Differentiating the expression for $V_{g,r}$ and substitution results in a second order ODE:

$$\frac{d^2 r}{dt^2} = \frac{c^2}{\omega}\frac{dk_r}{dt} = -\frac{c^2\beta}{\omega}(r - r_o)$$

The ODE requires finding both the general and specific solution before fitting the complete solution to the initial conditions.

$$r(t) = r_s(t) + r_g(t)$$

General Solution

Determining the general solution requires finding the determinant

$$r_s'' + \frac{c^2\beta}{\omega} r_s = 0$$

$$D = B^2 - 4AC = 0 - 4(1)\frac{c^2\beta}{\omega} = -\frac{4c^2\beta}{\omega}$$

$$D < 0$$

As all constants are positive, the determinant is negative. The general solution will be in the form of trigonometric functions with frequency γ. As $B = 0$, the roots of the ODE will have no real component, therefore there will be no exponential factor in the general solution

$$\gamma^2 = \frac{c^2\beta}{\omega}$$

$$r_g(t) = A_1 \cos(\gamma t) + A_2 \sin(\gamma t)$$

Specific Solution

As the driving function is constant, the specific solution will also be constant

$$r_s'' + \frac{c^2\beta}{\omega} r_s = f(t)$$

$$f(t) = \frac{c^2\beta r_o}{\omega}$$

$$r_s(t) = r_o$$

Fitting to Initial Conditions

The general solution and its first derivative is stated below:

$$r(t) = A_1 \cos(\gamma t) + A_2 \sin(\gamma t) + r_o$$

$$r'(t) = -A_1\gamma \sin(\gamma t) + A_2\gamma \cos(\gamma t)$$

At $t = 0$, the general solution and its derivative must satisfy the initial conditions.

$$r(0) = A_1 + r_o = r_i$$

$$A_1 = r_i - r_o$$

$$r'(0) = A_2\gamma = \left(\frac{dr}{dt}\right)_{t=0} = \frac{c^2 k_{ri}}{\omega}$$

$$A_2 = \frac{c^2 k_{ri}}{\omega\gamma}$$

Complete Solution

The complete solution stated below:

$$z(t) = \frac{c^2 k_{zi}}{\omega} + z_i$$

$$r(t) = (r_i - r_o)\cos(\gamma t) + \frac{c^2 k_{ri}}{\omega\gamma}\sin(\gamma t) + r_o$$

$$\gamma = \sqrt{\frac{c^2\beta}{\omega}}$$

$$\beta = \frac{4(n_c - n_o)e^2}{\omega W^2 m_e \epsilon_0}$$