Assignment 3

Instructions.

(1) Work in groups of up to three. Select your group in Canvas under People/Groups.
(2) The assignment is due on **16 May, 2025** at **14:15** the latest.
(3) The folder "Assignment 3" has been created. Open the folder and select your group number. After that, submit your assignment to the folder.
(4) The answers can be handwritten although the use of a text processor is encouraged. If the former, please make sure that your handwriting is legible.
(5) You may use any result given in the lecture notes.
(6) When answering the empirical exercises, provide both the MATLAB code you use and the values of the statistics you calculate. You can use screenshots from MATLAB.
(7) If you have any questions or comments regarding this assignment, please send an email to `g.mullin@tilburguniversity.edu`.

Question 1. Let $\{Y_t\}$ be the AR(1) plus noise process defined by

$$Y_t = X_t + W_t,$$

where $W_t \sim \mathrm{WN}(0, \sigma_w^2)$, and $\{X_t\}$ is the AR(1) process, $X_t = \phi X_{t-1} + Z_t$ with $Z_t \sim \mathrm{WN}(0, \sigma_z^2)$ and $|\phi| < 1$. Assume further that $\mathsf{E}[X_t W_s] = 0$ for all $t, s \in \mathbf{Z}$.

Definition. A stationary process $\{\xi_t\}$ is q-correlated if $\gamma_\xi(q) \neq 0$ and $\gamma_\xi(h) = 0$ whenever $|h| > q$. Any such process can be represented as an MA(q) process.

(a) Verify that $\{Y_t\}$ is stationary. Specifically, find its ACVF, $\gamma_Y(\,\cdot\,)$.

(b) Show that the process $\{U_t\}$ defined by $U_t = Y_t - \phi Y_{t-1}$ is 1-correlated.

(c) Conclude from (b) that $\{Y_t\}$ follows the ARMA(1,1) model and find its parameters assuming that $\phi = 0.5$ and $\sigma_z^2 = \sigma_w^2 = 1$.

Hint: write $U_t = \xi_t + \theta\xi_{t-1}$ and find θ and $\mathrm{Var}(\xi_t)$ that match the ACVFs of $\{U_t\}$. (Choose the invertible solution. You may use the `fsolve` function in Matlab to solve a system of nonlinear equations.)

Question 2. Consider a causal and invertible ARMA(2,1) process:

$$X_t = \phi X_{t-2} + Z_t + \theta Z_{t-1}, \tag{1}$$

where $Z_t \sim \mathrm{IID}(0, \sigma^2)$.

(a) Multiply (1) by X_{t-h}, $h = 0, 1$, and 2, to get a system of equations for $\gamma_X(0)$, $\gamma_X(1)$, and $\gamma_X(2)$.

(b) Solve the system of equations in (a) to express $\gamma_X(h)$, $h = 0, 1, 2$, in terms of the model parameters.

(c) Suppose that we fit the AR(1) model, $(1 - \varphi L)X_t = \xi_t$, to (1) using the Yule-Walker equations. Find the probability limit of the estimator $\hat{\varphi}$.

Question 3. Consider the invertible MA(1) model,

$$X_t = Z_t + \theta Z_{t-1},$$

where $\{Z_t\}$ is a Gaussian white noise process with variance σ^2 and $|\theta| < 1$. Recall that given a sample $\boldsymbol{X}_T \equiv [X_T, \dots, X_1]^\top$ from the model, the conditional log-likelihood function is given by

$$\ln \tilde{L}(\theta, \sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln\sigma^2 - \sum_{t=1}^{T} \frac{Z_t^2}{2\sigma^2},$$

where Z_t is computed recursively as $Z_t = X_t - \theta Z_{t-1}$.

(a) Verify that the joint conditional density of $\boldsymbol{X}_T$ given Z_0 can be written as

$$f_{\boldsymbol{X}_T|Z_0}(\boldsymbol{x}_T \mid z_0) = \prod_{t=1}^{T} f_{X_t|Z_{t-1}}(x_t \mid z_{t-1}).$$

(b) Find the value of σ^2 that maximizes $\ln \tilde{L}(\theta, \sigma^2)$ for a fixed value of θ, say $\hat{\sigma}^2(\theta)$, and show that the concentrated log-likelihood function is given by

$$\ln \tilde{L}(\theta) = \ln \tilde{L}(\theta, \hat{\sigma}^2(\theta)) = C_T - \frac{T}{2} \ln\left\{ \frac{1}{T} \sum_{t=1}^{T} Z_t^2 \right\},$$

where C_T is a constant depending only on T.

Remark. This also shows that $\ln \tilde{L}(\hat{\theta}, \hat{\sigma}^2) \propto -T \ln(\hat{\sigma}^2)/2$. After multiplying the RHS by $-2/T$, we get the first term appearing in the information criteria discussed in class.

(c) Use the provided `MA1.mlx` (live script) file to run a simulation study in Matlab.
- Complete the auxiliary functions at the bottom of the file.
- Compare the average estimate of θ with its true value and the sample variance of the estimates with the asymptotic variance of the ML estimator, $1 - \theta^2$ (scaled by the factor of T^{-1}).

Empirical Exercise using Matlab. In this exercise you will fit an ARMA(p, q) model to the time series given in Assignment 2. Work with the subsample of the data ending on August 2022 and use the following Matlab functions:

- `arima` - specifies the chosen model (e.g., `arima(2,0,1)` specifies the ARMA(2,1) model with Gaussian innovations).
- `estimate` - estimates parameters of the partially specified model using maximum likelihood.
- `forecast` - performs linear prediction.

(1) For each $0 \leq p, q \leq 5$ fit the corresponding ARMA(p, q) model and compute the information criteria.

(2) Select the best model according to one of the computed information criteria and check whether the respective theoretical ACFs and PACFs capture the features of the series sufficiently well. (Use the provided `acf` and `pacf` functions.)

(3) For the selected model in (3) report the estimated coefficients and their standard errors. Perform diagnostic checking and report the results.

(4) Compute the forecasts $P_T X_{t+h}$, $h = 1, \ldots, 7$, and their MSPEs.

(5) Report the predicted values obtained in part (4) together with the 95% confidence bands (assuming Gaussian innovations). Compare the predicted values with the actual realizations of the interest rate spread and discuss the quality of the forecasts.