Assignment 3

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1 Question 1

1.1 a

$$\begin{aligned}
Y_t &= \theta X_{t-1} + Z_t + W_t \\
&= \theta^2 X_{t-2} + Z_t + \theta^1 Z_{t-1} + W_t \\
&= \theta^3 X_{t-3} + Z_t + \theta^1 Z_{t-1} + \theta^2 Z_{t-2} + W_t \\
&= \theta^t X_0 + \sum_{i=0}^{t-1} \theta^i Z_{t-i} + W_t
\end{aligned}$$

$$\begin{aligned}
E[Y_t] &= E[\theta^t X_0 + \sum_{i=0}^{t-1} \theta^i Z_{t-i} + W_t] \\
&= E[\theta^t X_0] + E[\sum_{i=0}^{t-1} \theta^i Z_{t-i}] + E[W_t] \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

for h ¿ 0

$$\gamma_y(t+h,t) = E[(\theta^t X_0 + \sum_{i=0}^{t-1}\theta^i Z_{t-i} + W_t - 0) * (\theta^{t+h}X_0 + \sum_{i=0}^{t+h-1}\theta^i Z_{t+h-i} + W_{t+h} - 0)]$$

$$= \begin{array}{r} E[(\theta^{2t+h}X_0^2 + \theta^t X_0 * \sum_{i=0}^{t+h-1}\theta^i Z_{t+h-i} + \theta^{t+h}X_0 * \sum_{i=0}^{t-1}\theta^i Z_{t-i} + \sum_{i=0}^{t+h-1}\theta^i Z_{t+h-i} * \sum_{i=0}^{t-1}\theta^i Z_{t-i} \\ +\theta^t X_0 W_{t+h} + \theta^{t+h}X_0 W_t + \sum_{i=0}^{t-1}\theta^i Z_{t-i} * W_{t+h} + \sum_{i=0}^{t+h-1}\theta^i Z_{t+h-i} * W_t] \end{array}$$

$$= E[\sum_{i=0}^{t+h-1}\theta^i Z_{t+h-i} * \sum_{j=0}^{t-1}\theta^i Z_{t-j}]$$

$$= \sum_{i=h}^{t-1}\theta^i\theta^{i-h}\sigma_z^2$$

$$= \theta^h\sigma_z^2\sum_{i=0}^{t-1}\theta^{2i}\sigma_z^2$$

$$= \theta^h\sigma_z^2/(1-\theta^2)$$

for h = 0

$$\gamma_y(0) = \sigma_z^2/(1-\theta^2) + \sigma_w^2$$

So its stationary.

1.2 b

(b) Show that the process $U_t = Y_t - \phi Y_{t-1}$ is 1-correlated.

Recall that

$$Y_t = X_t + W_t,$$

where

$$X_t = \phi X_{t-1} + Z_t,$$

with $Z_t \sim WN(0,\sigma_z^2)$ and $W_t \sim WN(0,\sigma_w^2)$.

Define

$$U_t = Y_t - \phi Y_{t-1}.$$

Substituting Y_t, we have

$$U_t = (X_t + W_t) - \phi(X_{t-1} + W_{t-1}) = (X_t - \phi X_{t-1}) + (W_t - \phi W_{t-1}).$$

From the AR(1) equation for X_t,

$$X_t - \phi X_{t-1} = Z_t,$$

so

$$U_t = Z_t + W_t - \phi W_{t-1}.$$

Since Z_t and W_t are independent white noise sequences, define

$$\xi_t = Z_t + W_t,$$

which is white noise with variance

$$\sigma_\xi^2 = \sigma_z^2 + \sigma_w^2.$$

Then

$$U_t = \xi_t - \phi W_{t-1}.$$

Because W_{t-1} is a lagged noise term and related to ξ_{t-1}, U_t can be represented as

$$U_t = \xi_t + \theta \xi_{t-1}$$

for some θ.

Thus, $\{U_t\}$ has zero autocovariance for lags $|h| > 1$, making it a *1-correlated* process.

1.3 c

(c) Show that $\{Y_t\}$ follows an ARMA(1,1) model and find its parameters for $\phi = 0.5$, $\sigma_z^2 = \sigma_w^2 = 1$.

From part (b), we have

$$U_t = Y_t - \phi Y_{t-1} = \xi_t + \theta \xi_{t-1},$$

where $\{\xi_t\}$ is white noise with variance σ_ξ^2, and U_t is an MA(1) process.

The autocovariance function (ACVF) of an MA(1) process is

$$\gamma_U(0) = (1+\theta^2)\sigma_\xi^2, \quad \gamma_U(1) = \theta \sigma_\xi^2, \quad \gamma_U(h) = 0 \text{ for } |h| > 1.$$

On the other hand, from the expression

$$U_t = Z_t + W_t - \phi W_{t-1},$$

and using independence and variances, the autocovariances are

$$\gamma_U(0) = \mathrm{Var}(Z_t) + (1+\phi^2)\mathrm{Var}(W_t) = \sigma_z^2 + (1+\phi^2)\sigma_w^2,$$

$$\gamma_U(1) = \mathrm{Cov}(U_t, U_{t-1}) = -\phi \sigma_w^2,$$

$$\gamma_U(h) = 0 \quad \text{for } |h| > 1.$$

Plugging in the given values $\phi = 0.5$, $\sigma_z^2 = \sigma_w^2 = 1$, we get

$$\gamma_U(0) = 1 + (1 + 0.5^2) \cdot 1 = 1 + 1 + 0.25 = 2.25,$$

$$\gamma_U(1) = -0.5 \cdot 1 = -0.5.$$

Equate these with the MA(1) autocovariances:

$$(1+\theta^2)\sigma_\xi^2 = 2.25,$$

$$\theta\sigma_\xi^2 = -0.5.$$

From the second equation,

$$\sigma_\xi^2 = \frac{-0.5}{\theta}.$$

Substitute into the first:

$$(1+\theta^2)\frac{-0.5}{\theta} = 2.25,$$

which simplifies to

$$-(1+\theta^2) = 4.5\theta,$$

or equivalently,

$$\theta^2 + 4.5\theta + 1 = 0.$$

Solving this quadratic equation:

$$\theta = \frac{-4.5 \pm \sqrt{4.5^2 - 4\cdot 1\cdot 1}}{2} = \frac{-4.5 \pm \sqrt{20.25-4}}{2} = \frac{-4.5 \pm \sqrt{16.25}}{2}.$$

Approximating,

$$\sqrt{16.25} \approx 4.03,$$

so

$$\theta_1 \approx \frac{-4.5+4.03}{2} = -0.235, \quad \theta_2 \approx \frac{-4.5-4.03}{2} = -4.265.$$

The invertibility condition requires $|\theta| < 1$, so we choose

$$\theta \approx -0.235.$$

Finally, from

$$\sigma_\xi^2 = \frac{-0.5}{\theta} \approx \frac{-0.5}{-0.235} \approx 2.13.$$

Conclusion: The ARMA(1,1) representation of $\{Y_t\}$ is

$$Y_t = \phi Y_{t-1} + \xi_t + \theta\xi_{t-1},$$

with parameters

$$\boxed{\phi = 0.5, \quad \theta \approx -0.235, \quad \sigma_\xi^2 \approx 2.13.}$$

2 Question 2

(a) Multiply equation (1) by X_{t-h}, for $h = 0, 1, 2$, and take expectations to find equations for $\gamma_X(0), \gamma_X(1), \gamma_X(2)$.

The ARMA(2,1) model is

$$X_t = \phi X_{t-2} + Z_t + \theta Z_{t-1},$$

where $\{Z_t\}$ is white noise with mean zero and variance σ^2.

Multiply both sides by X_{t-h} and take expectations:

$$\mathrm{E}[X_t X_{t-h}] = \phi \mathrm{E}[X_{t-2} X_{t-h}] + \mathrm{E}[Z_t X_{t-h}] + \theta \mathrm{E}[Z_{t-1} X_{t-h}].$$

By stationarity, define the autocovariance function $\gamma_X(h) = \mathrm{E}[X_t X_{t-h}]$. Also, note that

$$\mathrm{E}[Z_t X_{t-h}] = \begin{cases} \sigma^2, & h = 0, \\ 0, & h > 0, \end{cases} \quad \text{and} \quad \mathrm{E}[Z_{t-1} X_{t-h}] = \begin{cases} \sigma^2, & h = 1, \\ 0, & h \neq 1. \end{cases}$$

Therefore, for $h = 0, 1, 2$, we get the system:

$$\begin{cases} \gamma_X(0) = \phi \gamma_X(2) + \sigma^2, \\ \gamma_X(1) = \phi \gamma_X(1) + \theta \sigma^2, \\ \gamma_X(2) = \phi \gamma_X(0). \end{cases}$$

2.1 b

(b) Solve the system to express $\gamma_X(0)$, $\gamma_X(1)$, $\gamma_X(2)$ in terms of ϕ, θ, σ^2.

From part (a), we have the system:

$$\begin{cases} \gamma_X(0) = \phi \gamma_X(2) + \sigma^2, \\ \gamma_X(1) = \phi \gamma_X(1) + \theta \sigma^2, \\ \gamma_X(2) = \phi \gamma_X(0). \end{cases}$$

Using the third equation in the first:

$$\gamma_X(0) = \phi(\phi \gamma_X(0)) + \sigma^2 = \phi^2 \gamma_X(0) + \sigma^2,$$

which implies

$$\gamma_X(0)(1 - \phi^2) = \sigma^2 \quad \Rightarrow \quad \boxed{\gamma_X(0) = \frac{\sigma^2}{1 - \phi^2}}.$$

Then, from the third equation,

$$\gamma_X(2) = \phi \gamma_X(0) = \frac{\phi \sigma^2}{1 - \phi^2}.$$

From the second equation,

$$\gamma_X(1) - \phi\gamma_X(1) = \theta\sigma^2 \quad \Rightarrow \quad \gamma_X(1)(1-\phi) = \theta\sigma^2,$$

so

$$\boxed{\gamma_X(1) = \frac{\theta\sigma^2}{1-\phi}.}$$

(c) Find the probability limit of the estimator $\hat{\phi}$ when fitting an AR(1) to the ARMA(2,1) process.

The Yule-Walker estimator for an AR(1) model is approximately

$$\hat{\phi} \approx \frac{\gamma_X(1)}{\gamma_X(0)}.$$

Using the results from (b), we have

$$\frac{\gamma_X(1)}{\gamma_X(0)} = \frac{\frac{\theta\sigma^2}{1-\phi}}{\frac{\sigma^2}{1-\phi^2}} = \theta\frac{1-\phi^2}{1-\phi}.$$

Simplifying the fraction:

$$\frac{1-\phi^2}{1-\phi} = \frac{(1-\phi)(1+\phi)}{1-\phi} = 1+\phi.$$

Therefore,

$$\boxed{\lim_{n\to\infty} \hat{\phi} = (1+\phi)\theta.}$$

3 Question 3

Question 3. Consider the invertible MA(1) model

$$X_t = Z_t + \theta Z_{t-1},$$

where $\{Z_t\}$ is a Gaussian white noise process with variance σ^2 and $|\theta| < 1$. Given a sample $\mathbf{X}_T = [X_T, \ldots, X_1]^\top$, the conditional log-likelihood function is

$$\ln \tilde{L}(\theta, \sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T} Z_t^2,$$

where the Z_t depend on θ and the data.

(a) Verify that the joint conditional density of $\mathbf{X}_T$ given Z_0 can be written as

$$f_{\mathbf{X}_T|Z_0}(\mathbf{x}_T|z_0) = \prod_{t=1}^{T} f_{X_t|Z_{t-1}}(x_t|z_{t-1}).$$

Explanation: Since the model can be written recursively as

$$X_t = Z_t + \theta Z_{t-1},$$

and Z_t are i.i.d. Gaussian, the sequence $\{X_t\}$ given $Z_0 = z_0$ forms a Markov chain where each X_t depends only on Z_{t-1}. Therefore, the joint conditional density factorizes as the product of the conditional densities:

$$f_{\mathbf{X}_T|Z_0}(\mathbf{x}_T|z_0) = \prod_{t=1}^{T} f_{X_t|Z_{t-1}}(x_t|z_{t-1}).$$

(b) Find the value of σ^2 that maximizes $\ln \tilde{L}(\theta, \sigma^2)$ for fixed θ, say $\hat{\sigma}^2(\theta)$, and show that the concentrated log-likelihood function is

$$\ln \tilde{L}(\theta) = \ln \tilde{L}(\theta, \hat{\sigma}^2(\theta)) = C_T - \frac{T}{2} \ln \left(\frac{1}{T} \sum_{t=1}^{T} Z_t^2 \right),$$

where C_T is a constant depending only on T.

Solution:
Starting with the conditional log-likelihood:

$$\ln \tilde{L}(\theta, \sigma^2) = -\frac{T}{2} \ln(2\pi) - \frac{T}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{t=1}^{T} Z_t^2.$$

For fixed θ, maximize w.r.t. σ^2 by setting the derivative to zero:

$$\frac{\partial}{\partial \sigma^2} \ln \tilde{L}(\theta, \sigma^2) = -\frac{T}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{t=1}^{T} Z_t^2 = 0.$$

Multiply both sides by $2(\sigma^2)^2$:

$$-T\sigma^2 + \sum_{t=1}^{T} Z_t^2 = 0 \quad \Rightarrow \quad \hat{\sigma}^2(\theta) = \frac{1}{T} \sum_{t=1}^{T} Z_t^2.$$

Substitute $\hat{\sigma}^2(\theta)$ back into the log-likelihood:

$$\ln \tilde{L}(\theta, \hat{\sigma}^2(\theta)) = -\frac{T}{2} \ln(2\pi) - \frac{T}{2} \ln \hat{\sigma}^2(\theta) - \frac{1}{2\hat{\sigma}^2(\theta)} \sum_{t=1}^{T} Z_t^2.$$

Using the expression for $\hat{\sigma}^2(\theta)$,

$$\frac{1}{2\hat{\sigma}^2(\theta)} \sum_{t=1}^{T} Z_t^2 = \frac{1}{2\hat{\sigma}^2(\theta)} \times T\hat{\sigma}^2(\theta) = \frac{T}{2}.$$

Hence,

$$\ln \tilde{L}(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln\hat{\sigma}^2(\theta) - \frac{T}{2} = C_T - \frac{T}{2}\ln\hat{\sigma}^2(\theta),$$

where

$$C_T = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}$$

is a constant depending only on T.

Thus,

$$\boxed{\ln \tilde{L}(\theta) = C_T - \frac{T}{2}\ln\left(\frac{1}{T}\sum_{t=1}^{T} Z_t^2\right).}$$

3.1 c