CHAPTER

# RELATIVE MOTION AND RENDEZVOUS

7

## 7.1 INTRODUCTION

Up to now we have mostly referenced the motion of orbiting objects to a nonrotating coordinate system fixed to the center of attraction (e.g., the center of the earth). This platform served as an inertial frame of reference, in which Newton's second law can be written as

$$\mathbf{F}_{\text{net}} = m\mathbf{a}_{\text{absolute}}$$

An exception to this rule was the discussion of the restricted three-body problem at the end of Chapter 2, in which we made use of the relative motion equations developed in Chapter 1. In a rendezvous maneuver two orbiting vehicles observe one another from each of their own free-falling, rotating, clearly noninertial frames of reference. To base impulsive maneuvers on observations made from a moving platform requires transforming relative velocity and acceleration measurements into an inertial frame. Otherwise, the true thrusting forces cannot be sorted out from the fictitious "inertial forces" that appear in Newton's law when it is written incorrectly as

$$\mathbf{F}_{\text{net}} = m\mathbf{a}_{\text{rel}}$$

The purpose of this chapter is to use relative motion analysis to gain some familiarity with the problem of maneuvering one spacecraft relative to another, especially when they are in close proximity.

## 7.2 RELATIVE MOTION IN ORBIT

A rendezvous maneuver usually involves a target vehicle $A$, which is passive and nonmaneuvering, and a chase vehicle $B$, which is active and performs the maneuvers required to bring itself alongside the target. An obvious example was the Space Shuttle, the chaser, rendezvousing with the International Space Station, the target. The position vector of target $A$ in the geocentric equatorial frame is $\mathbf{r}_A$. This radial is sometimes called the "$r$-bar." The moving frame of reference has its origin at the target, as illustrated in Fig. 7.1. The $x$ axis is directed along the outward radial $\mathbf{r}_A$ to the target. Therefore, the unit vector $\hat{\mathbf{i}}$ along the moving $x$ axis is

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} \tag{7.1}$$

Orbital Mechanics for Engineering Students. https://doi.org/10.1016/B978-0-08-102133-0.00007-6

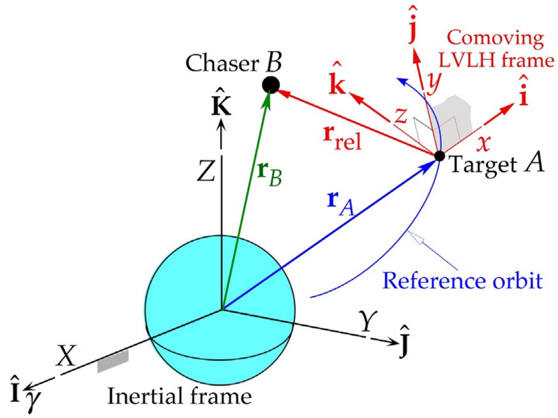



**FIG. 7.1**

Comoving reference frame attached to $A$, from which body $B$ is observed.

The $z$ axis is normal to the orbital plane of the target spacecraft and therefore lies in the direction of $A$'s angular momentum vector. It follows that the unit vector along the $z$ axis of the moving frame is given by

$$\hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} \tag{7.2}$$

The $y$ axis is perpendicular to both $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ and points in the direction of the target satellite's local horizon. Therefore, both the $x$ and $y$ axes lie in the target's orbital plane, with the $y$ unit vector completing a right triad; that is,

$$\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} \tag{7.3}$$

We may refer to the comoving $xyz$ frame defined here as a local vertical/local horizontal (LVLH) frame.

The position, velocity, and acceleration of $B$ relative to $A$, measured in the comoving frame, are given by

$$\mathbf{r}_{\text{rel}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{7.4a}$$

$$\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{7.4b}$$

$$\mathbf{a}_{\text{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{7.4c}$$

The angular velocity vector $\boldsymbol{\Omega}$ of the $xyz$ axes attached to the target is just the angular velocity of the target's position vector. It is obtained with the aid of Eqs. (2.31) and (2.46) from the fact that

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = (r_A v_{A_\perp})\hat{\mathbf{k}} = \left(r_A{}^2 \Omega\right)\hat{\mathbf{k}} = r_A{}^2 \boldsymbol{\Omega}$$

from which we obtain

$$\boldsymbol{\Omega} = \frac{\mathbf{h}_A}{r_A{}^2} = \frac{\mathbf{r}_A \times \mathbf{v}_A}{r_A{}^2} \tag{7.5}$$

To find the angular acceleration vector $\dot{\mathbf{\Omega}}$ of the $xyz$ frame we take the time derivative of $\mathbf{\Omega}$ in Eq. (7.5) and use the fact that the angular momentum $\mathbf{h}_A$ of the passive target is constant,

$$\dot{\mathbf{\Omega}} = \mathbf{h}_A \frac{d}{dt}\frac{1}{r_A{}^2} = -2\frac{\mathbf{h}_A}{r_A{}^3}\dot{r}_A$$

Recall from Eq. (2.35a) that $\dot{r}_A = \mathbf{v}_A \cdot \mathbf{r}_A / r_A$, so this may be written as

$$\dot{\mathbf{\Omega}} = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A{}^4}\mathbf{h}_A = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A{}^2}\mathbf{\Omega} \tag{7.6}$$

After first calculating

$$\mathbf{r}_{\text{rel}} = \mathbf{r}_B - \mathbf{r}_A \tag{7.7}$$

we use Eqs. (7.5) and (7.6) to determine the angular velocity and angular acceleration of the comoving frame, both of which are required in the relative velocity and acceleration formulas (Eqs. 1.66 and 1.70),

$$\mathbf{v}_{\text{rel}} = \mathbf{v}_B - \mathbf{v}_A - \mathbf{\Omega} \times \mathbf{r}_{\text{rel}} \tag{7.8}$$

$$\mathbf{a}_{\text{rel}} = \mathbf{a}_B - \mathbf{a}_A - \dot{\mathbf{\Omega}} \times \mathbf{r}_{\text{rel}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{\text{rel}}) - 2\mathbf{\Omega} \times \mathbf{v}_{\text{rel}} \tag{7.9}$$

The vectors in Eqs. (7.7)–(7.9) are all referred to the inertial $XYZ$ frame in Fig. 7.1. To find their components in the accelerating $xyz$ frame at any instant we must first form the orthogonal direction cosine matrix $[\mathbf{Q}]_{Xx}$, as discussed in Section 4.5. The rows of this matrix comprise the direction cosines of each of the $xyz$ axes with respect to the $XYZ$ axes. That is, from Eqs. (7.1)–(7.3) we find

$$\begin{aligned}\hat{\mathbf{i}} &= l_x\hat{\mathbf{I}} + m_x\hat{\mathbf{J}} + n_x\hat{\mathbf{K}}\\ \hat{\mathbf{j}} &= l_y\hat{\mathbf{I}} + m_y\hat{\mathbf{J}} + n_y\hat{\mathbf{K}}\\ \hat{\mathbf{k}} &= l_z\hat{\mathbf{I}} + m_z\hat{\mathbf{J}} + n_z\hat{\mathbf{K}}\end{aligned} \tag{7.10}$$

where the $l$s, $m$s, and $n$s are the direction cosines. Then,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} \begin{array}{l} \leftarrow \text{components of } \hat{\mathbf{i}} \\ \leftarrow \text{components of } \hat{\mathbf{j}} \\ \leftarrow \text{components of } \hat{\mathbf{k}} \end{array} \tag{7.11}$$

The components of the relative position, velocity, and acceleration are computed as follows:

$$\{\mathbf{r}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{r}_{\text{rel}}\}_X \tag{7.12a}$$

$$\{\mathbf{v}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{v}_{\text{rel}}\}_X \tag{7.12b}$$

$$\{\mathbf{a}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{a}_{\text{rel}}\}_X \tag{7.12c}$$

in which

$$\{\mathbf{r}_{\text{rel}}\}_X = \begin{Bmatrix} X_B - X_A \\ Y_B - Y_A \\ Z_B - Z_A \end{Bmatrix} \tag{7.13a}$$

$$\{\mathbf{v}_{\text{rel}}\}_X = \begin{Bmatrix} \dot{X}_B - \dot{X}_A + \Omega_Z(Y_B - Y_A) - \Omega_Y(Z_B - Z_A) \\ \dot{Y}_B - \dot{Y}_A - \Omega_Z(X_B - X_A) + \Omega_X(Z_B - Z_A) \\ \dot{Z}_B - \dot{Z}_A + \Omega_Y(X_B - X_A) - \Omega_X(Y_B - Y_A) \end{Bmatrix} \tag{7.13b}$$

$$\{\mathbf{a}_{\text{rel}}\}_X = \begin{Bmatrix} \ddot{X}_B - \ddot{X}_A + 2\Omega_Z\left(\dot{Y}_B - \dot{Y}_A\right) - 2\Omega_Y\left(\dot{Z}_B - \dot{Z}_A\right)\cdots \\ -\left({\Omega_Y}^2 + {\Omega_Z}^2\right)(X_B - X_A) + (\Omega_X\Omega_Y + a\Omega_Z)(Y_B - Y_A) + (\Omega_X\Omega_Z - a\Omega_Y)(Z_B - Z_A) \\ \ddot{Y}_B - \ddot{Y}_A - 2\Omega_Z\left(\dot{X}_B - \dot{X}_A\right) + 2\Omega_X\left(\dot{Z}_B - \dot{Z}_A\right)\cdots \\ +(\Omega_X\Omega_Y - a\Omega_Z)(X_B - X_A) - \left({\Omega_X}^2 + {\Omega_Z}^2\right)(Y_B - Y_A) + (\Omega_Y\Omega_Z + a\Omega_X)(Z_B - Z_A) \\ \ddot{Z}_B - \ddot{Z}_A + 2\Omega_Y\left(\dot{X}_B - \dot{X}_A\right) - 2\Omega_X\left(\dot{Y}_B - \dot{Y}_A\right)\cdots \\ +(\Omega_X\Omega_Z + a\Omega_Y)(X_B - X_A) + (\Omega_Y\Omega_Z - a\Omega_X)(Y_B - Y_A) - \left({\Omega_X}^2 + {\Omega_Y}^2\right)(Z_B - Z_A) \end{Bmatrix} \tag{7.13c}$$

The components of $\mathbf{\Omega}$ are obtained from Eq. (7.5), and $\dot{\mathbf{\Omega}} = a\mathbf{\Omega}$, where, according to Eq. (7.6), $a = -2\mathbf{v}_A \cdot \mathbf{r}_A/{r_A}^2$.

**ALGORITHM 7.1**

Given the state vectors $(\mathbf{r}_A, \mathbf{v}_A)$ of target spacecraft $A$ and $(\mathbf{r}_B, \mathbf{v}_B)$ of chaser spacecraft $B$, find the position $\{\mathbf{r}_{\text{rel}}\}_x$, velocity $\{\mathbf{v}_{\text{rel}}\}_x$, and acceleration $\{\mathbf{a}_{\text{rel}}\}_x$ of $B$ relative to $A$ along the LVLH axes attached to $A$. See Appendix D.31 for an implementation of this procedure in MATLAB.

1. Calculate the angular momentum of $A$, $\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A$.
2. Calculate the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ of the comoving frame by means of Eqs. (7.1)–(7.3).
3. Calculate the orthogonal direction cosine matrix $[\mathbf{Q}]_{Xx}$ using Eq. (7.11).
4. Calculate $\mathbf{\Omega}$ and $\dot{\mathbf{\Omega}}$ from Eqs. (7.5) and (7.6).
5. Calculate the absolute accelerations of $A$ and $B$ using Eq. (2.22).

$$\mathbf{a}_A = -\frac{\mu}{{r_A}^3}\mathbf{r}_A \qquad \mathbf{a}_B = -\frac{\mu}{{r_B}^3}\mathbf{r}_B$$

6. Calculate $\mathbf{r}_{\text{rel}}$ using Eq. (7.7).
7. Calculate $\mathbf{v}_{\text{rel}}$ using Eq. (7.8).
8. Calculate $\mathbf{a}_{\text{rel}}$ using Eq. (7.9).
9. Calculate $\{\mathbf{r}_{\text{rel}}\}_x$, $\{\mathbf{v}_{\text{rel}}\}_x$, and $\{\mathbf{a}_{\text{rel}}\}_x$ using Eqs. (7.12).

## EXAMPLE 7.1

In Fig. 7.2, spacecraft A is in an elliptical earth orbit having the following parameters:

$$h = 52{,}059\ \text{km}^2/\text{s} \quad e = 0.025724 \quad i = 60^\circ \quad \Omega = 40^\circ \quad \omega = 30^\circ \quad \theta = 40^\circ \tag{a}$$

Spacecraft $B$ is likewise in an earth orbit with these parameters:

$$h = 52{,}362\ \text{km}^2/\text{s} \quad e = 0.0072696 \quad i = 50^\circ \quad \Omega = 40^\circ \quad \omega = 120^\circ \quad \theta = 40^\circ \tag{b}$$

Calculate the position $\mathbf{r}_{\text{rel}})_x$, velocity $\mathbf{v}_{\text{rel}})_x$, and acceleration $\mathbf{a}_{\text{rel}})_x$ of spacecraft $B$ relative to spacecraft $A$, measured along the $xyz$ axes of the comoving coordinate system of spacecraft $A$, as defined in Fig. 7.1.

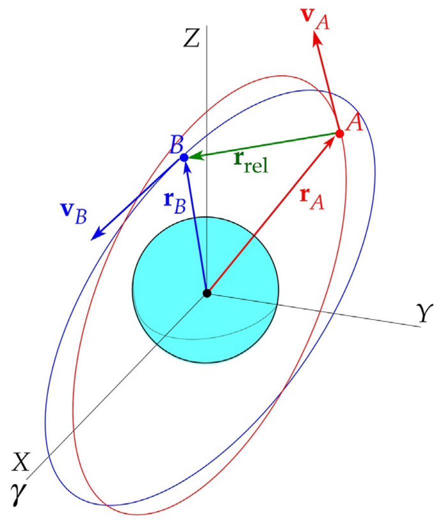



**FIG. 7.2**

Spacecraft *A* and *B* in slightly different orbits.

## Solution

From the orbital elements in Eqs. (a) and (b) we can use Algorithm 4.5 to find the position and the velocity of both spacecraft relative to the geocentric equatorial reference frame. Omitting those familiar calculations here, the reader can verify that for spacecraft *A*

$$\mathbf{r}_A = -266.77\hat{\mathbf{I}} + 3865.8\hat{\mathbf{J}} + 5426.2\hat{\mathbf{K}}(\text{km}) \quad (r_A = 6667.8\,\text{km}) \tag{c}$$

$$\mathbf{v}_A = -6.4836\hat{\mathbf{I}} - 3.6198\hat{\mathbf{J}} + 2.4156\hat{\mathbf{K}}(\text{km/s}) \quad (v_A = 7.8087\,\text{km/s}) \tag{d}$$

and for spacecraft *B*

$$\mathbf{r}_B = -5890.7\hat{\mathbf{I}} - 2979.8\hat{\mathbf{J}} + 1792.2\hat{\mathbf{K}}(\text{km}) \quad (r_B = 6840.4\,\text{km}) \tag{e}$$

$$\mathbf{v}_B = 0.93583\hat{\mathbf{I}} - 5.2403\hat{\mathbf{J}} - 5.5009\hat{\mathbf{K}}\,(\text{km/s}) \quad (v_B = 7.6548\,\text{km/s}) \tag{f}$$

Having found the state vectors we can proceed with Algorithm 7.1.
Step 1:

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -266.77 & 3865.8 & 5426.2 \\ -6.4836 & -3.6198 & 2.4156 \end{vmatrix}$$
$$= 28{,}980\hat{\mathbf{I}} - 34{,}537\hat{\mathbf{J}} + 26{,}029\hat{\mathbf{K}}\left(\text{km}^2/\text{s}\right)$$
$$\left(h_A = 52{,}059\,\text{km}^2/\text{s}\right)$$

Step 2:

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} = -0.040009\hat{\mathbf{I}} + 0.57977\hat{\mathbf{J}} + 0.81380\hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} = 0.55667\hat{\mathbf{I}} - 0.66341\hat{\mathbf{J}} + 0.5000\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.55667 & -0.66341 & 0.5000 \\ -0.040008 & 0.57977 & 0.81380 \end{vmatrix} = -0.82977\hat{\mathbf{I}} - 0.47302\hat{\mathbf{J}} + 0.29620\hat{\mathbf{K}}$$

Step 3:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -0.040009 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}$$

Step 4:

$$\boldsymbol{\Omega} = \frac{\mathbf{h}_A}{{r_A}^2} = 0.00065183\hat{\mathbf{I}} - 0.00077682\hat{\mathbf{J}} + 0.00058547\hat{\mathbf{K}}(\text{rad/s})$$

$$\dot{\boldsymbol{\Omega}} = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{{r_A}^2}\boldsymbol{\Omega} = -2.47533\left(10^{-8}\right)\hat{\mathbf{I}} + 2.9500\left(10^{-8}\right)\hat{\mathbf{J}} - 2.2233\left(10^{-8}\right)\hat{\mathbf{K}}\left(\text{rad/s}^2\right)$$

Step 5:

$$\mathbf{a}_A = -\mu\frac{\mathbf{r}_A}{{r_A}^3} = 0.00035870\hat{\mathbf{I}} - 0.00051980\hat{\mathbf{J}} - 0.0072962\hat{\mathbf{K}}\left(\text{km/s}^2\right)$$

$$\mathbf{a}_B = -\mu\frac{\mathbf{r}_B}{{r_B}^3} = 0.0073359\hat{\mathbf{I}} - 0.0037108\hat{\mathbf{J}} - 0.0022319\hat{\mathbf{K}}\left(\text{km/s}^2\right)$$

Step 6:

$$\mathbf{r}_{\text{rel}} = \mathbf{r}_B - \mathbf{r}_A = -5623.9\hat{\mathbf{I}} - 6845.5\hat{\mathbf{J}} - 3634.0\hat{\mathbf{K}}(\text{km})$$

Step 7:

$$\begin{aligned}
\mathbf{v}_{\text{rel}} &= \mathbf{v}_B - \mathbf{v}_A - \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} \\
&= \left(0.93583\hat{\mathbf{I}} - 5.2403\hat{\mathbf{J}} - 5.5009\hat{\mathbf{K}}\right) - \left(-6.4836\hat{\mathbf{I}} - 3.6198\hat{\mathbf{J}} + 2.4156\hat{\mathbf{K}}\right) \\
&\quad - \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix} \\
\mathbf{v}_{\text{rel}} &= 0.58855\hat{\mathbf{I}} - 0.69663\hat{\mathbf{J}} + 0.91436\hat{\mathbf{K}}(\text{km/s})
\end{aligned}$$

Step 8:

$$\begin{aligned}
\mathbf{a}_{\text{rel}} &= \mathbf{a}_B - \mathbf{a}_A - \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) - 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} \\
&= \left(0.0073359\hat{\mathbf{I}} + 0.0037108\hat{\mathbf{J}} - 0.0022319\hat{\mathbf{K}}\right) - \left(0.00035870\hat{\mathbf{I}} + 0.0051980\hat{\mathbf{J}} - 0.0072962\mathbf{K}\right) \\
&\quad - \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -2.4753\left(10^{-8}\right) & 2.9500\left(10^{-8}\right) & -2.2233\left(10^{-8}\right) \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix} \\
&\quad - \left(0.00065183\hat{\mathbf{I}} - 0.00077682\hat{\mathbf{J}} + 0.0005854\hat{\mathbf{K}}\right) \times \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix} \\
&\quad - 2\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ 0.58855 & -0.69663 & 0.91436 \end{vmatrix} \\
\mathbf{a}_{\text{rel}} &= 0.00044050\hat{\mathbf{I}} - 0.00037900\hat{\mathbf{J}} + 0.00001858\hat{\mathbf{K}}\ \left(\text{km/s}^2\right)
\end{aligned}$$

Step 9:

$$\mathbf{r}_{\text{rel}})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix} \begin{Bmatrix} -5623.9 \\ -6845.5 \\ -3634.0 \end{Bmatrix}$$

$$\boxed{\mathbf{r}_{\text{rel}})_x = \begin{Bmatrix} -6701.2 \\ 6828.3 \\ -406.26 \end{Bmatrix} \text{(km)}}$$

$$\mathbf{v}_{\text{rel}})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix} \begin{Bmatrix} 0.58855 \\ -0.69663 \\ 0.91436 \end{Bmatrix}$$

$$\boxed{\mathbf{v}_{\text{rel}})_x = \begin{Bmatrix} 0.31667 \\ 0.11199 \\ 1.2470 \end{Bmatrix} \text{(km/ s)}}$$

$$\mathbf{a}_{\text{rel}})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix} \begin{Bmatrix} 0.00044050 \\ -0.00037900 \\ 0.000018581 \end{Bmatrix}$$

$$\boxed{\mathbf{a}_{\text{rel}})_x = \begin{Bmatrix} -0.00022222 \\ -0.00018074 \\ 0.00050593 \end{Bmatrix} \text{(km/ s}^2\text{)}}$$

See Appendix D.31 for the MATLAB solution to this problem.

---

The motion of one spacecraft relative to another in orbit may be hard to visualize at first. Fig. 7.3 is offered as an assist. Orbit 1 is circular, and orbit 2 is elliptical with an eccentricity of 0.125. Both coplanar orbits were chosen to have the same semimajor axis length, so they both have the same period. A comoving frame is shown attached to the observers $A$ in circular orbit 1. At the initial time $I$ the spacecraft $B$ in elliptical orbit 2 is directly below the observers. In other words, $A$ must draw an arrow in the negative local $x$ direction to determine the position vector of $B$ in the lower orbit. The figure shows eight different instants ($I$, $II$, $III$, …, $VIII$), equally spaced around the circular orbit, at which observers $A$ construct the position vector pointing from them toward $B$ in the elliptical orbit. Of course, $A$'s frame is rotating, because its $x$ axis must always be directed away from the earth. Observers $A$ cannot sense this rotation and record the set of observations in their (to them) fixed $xy$ coordinate system, as shown at the right in the figure. Coasting at a uniform speed along this circular orbit, observers $A$ see the other vehicle orbiting them clockwise in a sort of bean-shaped path. The distance between the two spacecraft in this case never becomes so great that the earth intervenes.

If observers $A$ declared theirs to be an inertial frame of reference, they would be faced with the task of explaining the physical origin of the force holding $B$ in its bean-shaped orbit. Of course, there is no such force. The apparent path is due to the actual, combined motion of both spacecraft in their free fall around the earth. When $B$ is below $A$ (having a negative $x$ coordinate), conservation of angular momentum demands that $B$ move faster than $A$, thereby speeding up in $A$'s positive $y$ direction until the orbits cross ($x = 0$) between $III$ and $IV$. When $B$'s $x$ coordinate becomes positive (i.e., $B$ is above $A$) the laws of momentum dictate that $B$ slow down, which it does, progressing in $A$'s negative $y$ direction until the

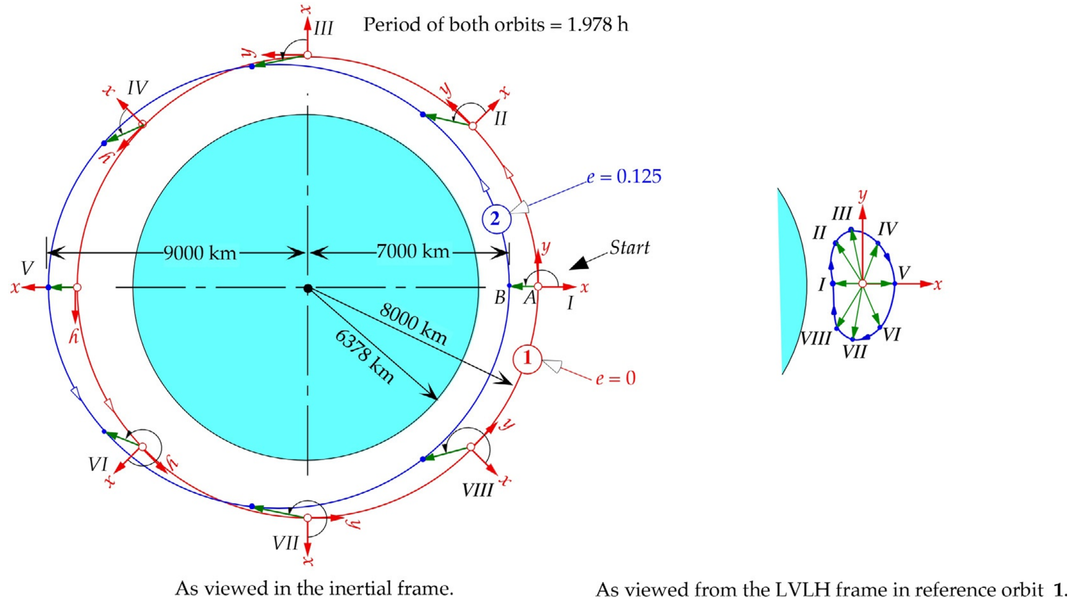



**FIG. 7.3**

Spacecraft *B* in elliptical orbit 2 appears to orbit the observer *A* in circular orbit 1.

next crossing of the orbits between *VI* and *VII*. *B* then falls below *A* and begins to pick up speed. The process repeats over and over again. From inertial space the process is the motion of two satellites on intersecting orbits, appearing not at all like the orbiting motion seen by the moving observers *A*.

## EXAMPLE 7.2

Plot the motion of spacecraft *B* relative to spacecraft *A* in Example 7.1.

### Solution

In Example 7.1 we found $\mathbf{r}_{\rm rel})_x$ at a single time. To plot the path of *B* relative to *A* we must find $\mathbf{r}_{\rm rel})_x$ at a large number of times, so that when we "connect the dots" in three-dimensional space a smooth curve results. Let us outline an algorithm and implement it in MATLAB.

1. Given the orbital elements of spacecraft *A* and *B*, calculate their state vectors $(\mathbf{r}_{A_0}, \mathbf{v}_{A_0})$ and $(\mathbf{r}_{B_0}, \mathbf{v}_{B_0})$ at the initial time $t_0$ using Algorithm 4.5 (as we did in Example 7.1).
2. Calculate the period $T_A$ of *A*'s orbit from Eq. (2.82). (For the data of Example 7.1, $T_A = 5585\,\text{s}$.)
3. Let the final time $t_f$ for the plot be $t_0 + mT_A$, where $m$ is an arbitrary integer.
4. Let $n$ be the number of points to be plotted, so that the time step is $\Delta t = (t_f - t)/n$.
5. At time $t \geq t_0$:
   a. Calculate the state vectors $(\mathbf{r}_A, \mathbf{v}_A)$ and $(\mathbf{r}_B, \mathbf{v}_B)$ using Algorithm 3.4.
   b. Calculate $\mathbf{r}_{\rm rel})_x$ using Algorithm 7.1.
   c. Plot the point $(x_{\rm rel}, y_{\rm rel}, z_{\rm rel})$.
6. Let $t \leftarrow t + \Delta t$ and repeat Step 5 until $t = t_f$.

This algorithm is implemented in the MATLAB script *Example_7_02.m* listed in Appendix D.32. The resulting plot of the relative motion for a time interval of 60 periods of spacecraft *A* is shown in Fig. 7.4. The arrow drawn from *A* to *B* is the initial position vector $\mathbf{r}_{\rm rel})_x$ found in Example 7.1. As can be seen the trajectory of *B* is a looping, clockwise motion around a circular path about 14,000 km in diameter. The closest approach of *B* to *A* is 105.5 km at an elapsed time of 25.75 h.

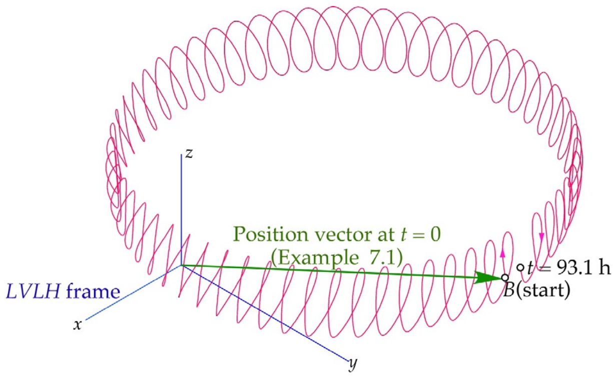



**FIG. 7.4**

Trajectory of spacecraft $B$ relative to spacecraft $A$ for the data in Example 7.1. The total time is 60 periods of $A$'s orbit (93.1 h).

## 7.3 LINEARIZATION OF THE EQUATIONS OF RELATIVE MOTION IN ORBIT

Fig. 7.5, similar to Fig. 7.1, shows two spacecraft in earth orbit. Let the inertial position vector of the target vehicle $A$ be denoted $\mathbf{R}$ and that of the chase vehicle $B$ be denoted $\mathbf{r}$. The position vector of the chase vehicle relative to the target is $\delta\mathbf{r}$, so that

$$\mathbf{r} = \mathbf{R} + \delta\mathbf{r} \tag{7.14}$$

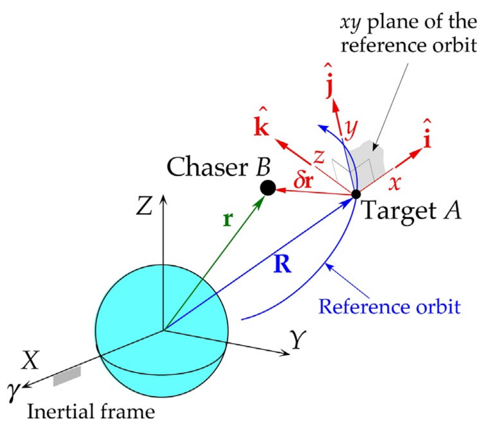



**FIG. 7.5**

Position of chaser $B$ relative to the target $A$.

The symbol $\delta$ is used here to represent the fact that the relative position vector has a magnitude that is very small compared with the magnitude of $\mathbf{R}$ (and $\mathbf{r}$); that is,

$$\frac{\delta r}{R} \ll 1 \tag{7.15}$$

where $\delta r = \|\delta\mathbf{r}\|$ and $R = \|\mathbf{R}\|$. This is true if the two vehicles are in close proximity to each other, as is the case in a rendezvous maneuver or close formation flight. Our purpose in this section is to seek the equations of motion of the chase vehicle relative to the target when they are close together. Since the relative motion is seen from the target vehicle, its orbit is also called the reference orbit.

The equation of motion of the chase vehicle $B$ relative to the inertial geocentric equatorial frame is Eq. (2.22),

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \tag{7.16}$$

where $r = \|\mathbf{r}\|$. Substituting Eq. (7.14) into Eq. (7.16) and writing $\delta\ddot{\mathbf{r}} = \left(\mathrm{d}^2/\mathrm{d}t^2\right)\delta\mathbf{r}$ yields the equation of motion of the chaser relative to the target,

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu\frac{\mathbf{R}+\delta\mathbf{r}}{r^3} \quad (\text{where } r = \|\mathbf{R}+\delta\mathbf{r}\|) \tag{7.17}$$

We will simplify this equation by making use of the fact that $\|\delta\mathbf{r}\|$ is very small, as expressed in Eq. (7.15).

First, note that

$$r^2 = \mathbf{r}\cdot\mathbf{r} = (\mathbf{R}+\delta\mathbf{r})\cdot(\mathbf{R}+\delta\mathbf{r}) = \mathbf{R}\cdot\mathbf{R} + 2\mathbf{R}\cdot\delta\mathbf{r} + \delta\mathbf{r}\cdot\delta\mathbf{r}$$

Since $\mathbf{R}\cdot\mathbf{R} = R^2$ and $\delta\mathbf{r}\cdot\delta\mathbf{r} = \delta r^2$, we can factor out $R^2$ on the right to obtain

$$r^2 = R^2\left[1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2} + \left(\frac{\delta r}{R}\right)^2\right]$$

By virtue of Eq. (7.15) we can neglect the last term in the brackets, so that

$$r^2 = R^2\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right) \tag{7.18}$$

In fact, we will neglect all powers of $\delta r/R$ greater than unity wherever they appear. Since $r^{-3} = (r^2)^{-3/2}$ it follows from Eq. (7.18) that

$$r^{-3} = R^{-3}\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/2} \tag{7.19}$$

Using the binomial theorem (Eq. 5.44) and neglecting terms of higher order than 1 in $\delta r/R$, we obtain

$$\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/2} = 1 + \left(-\frac{3}{2}\right)\left(\frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)$$

Therefore, to our level of approximation, Eq. (7.19) becomes

$$r^{-3} = R^{-3}\left(1 - \frac{3}{R^2}\mathbf{R}\cdot\delta\mathbf{r}\right)$$

which can be written as

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\delta\mathbf{r} \tag{7.20}$$

Substituting Eq. (7.20) into Eq. (7.17) (the equation of motion), we get

$$\begin{aligned}\delta\ddot{\mathbf{r}} &= -\ddot{\mathbf{R}} - \mu\left(\frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\delta\mathbf{r}\right)(\mathbf{R}+\delta\mathbf{r}) \\ &= -\ddot{\mathbf{R}} - \mu\left[\frac{\mathbf{R}+\delta\mathbf{r}}{R^3} - \frac{3}{R^5}(\mathbf{R}\cdot\delta\mathbf{r})(\mathbf{R}+\delta\mathbf{r})\right] \\ &= -\ddot{\mathbf{R}} - \mu\left[\frac{\mathbf{R}}{R^3} + \frac{\delta\mathbf{r}}{R^3} - \frac{3}{R^5}(\mathbf{R}\cdot\delta\mathbf{r})\mathbf{R} + \overbrace{\text{terms of higher order than 1 in } \delta\mathbf{r}}^{\text{neglect}}\right]\end{aligned}$$

That is, to our degree of approximation,

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu\frac{\mathbf{R}}{R^3} - \frac{\mu}{R^3}\left[\delta\mathbf{r} - \frac{3}{R^2}(\mathbf{R}\cdot\delta r)\mathbf{R}\right] \tag{7.21}$$

But the equation of motion of the reference orbit is

$$\ddot{\mathbf{R}} = -\mu\frac{\mathbf{R}}{R^3} \tag{7.22}$$

Substituting this into Eq. (7.21) finally yields

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left[\delta\mathbf{r} - \frac{3}{R^2}(\mathbf{R}\cdot\delta\mathbf{r})\mathbf{R}\right] \tag{7.23}$$

This is the linearized version of Eq. (7.17), the equation that governs the motion of the chaser with respect to the target. The expression is linear because the unknown $\delta\mathbf{r}$ appears only in the numerator and only to the first power throughout. We achieved this by dropping a lot of terms that are insignificant when Eq. (7.15) is valid. Eq. (7.23) is nonlinear in $\mathbf{R}$, which is not an unknown because it is determined independently by solving Eq. (7.22).

In the comoving frame of Fig. 7.5 the $x$ axis lies along the radial $\mathbf{R}$, so that

$$\mathbf{R} = R\hat{\mathbf{i}} \tag{7.24}$$

In terms of its components in the comoving frame the relative position vector $\delta\mathbf{r}$ in Fig. 7.5 is (cf. Eq. 7.4a)

$$\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \tag{7.25}$$

Substituting Eqs. (7.24) and (7.25) into Eq. (7.23) yields

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left[\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right) - \frac{3}{R^2}\left[\left(R\hat{\mathbf{i}}\right)\cdot\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right)\right]\left(R\hat{\mathbf{i}}\right)\right]$$

After expanding the dot product on the right and collecting terms, we find that the linearized equation of relative motion takes a rather simple form when the components of $\mathbf{R}$ and $\delta\mathbf{r}$ are given in the comoving frame,

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left(-2\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right) \tag{7.26}$$

Recall that $\delta\ddot{\mathbf{r}}$ is the acceleration of chaser $B$ relative to target $A$ as measured in the inertial frame. That is,

$$\delta\ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta\mathbf{r} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{r}_B - \mathbf{r}_A) = \ddot{\mathbf{r}}_B - \ddot{\mathbf{r}}_A = \mathbf{a}_B - \mathbf{a}_A$$

$\delta\ddot{\mathbf{r}}$ is not to be confused with $\delta\mathbf{a}_{\text{rel}}$, which is the relative acceleration measured in the comoving frame. These two quantities are related by Eq. (7.9),

$$\delta\mathbf{a}_{\text{rel}} = \delta\ddot{\mathbf{r}} - \dot{\boldsymbol{\Omega}} \times \delta\mathbf{r} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \delta\mathbf{r}) - 2\boldsymbol{\Omega} \times \delta\mathbf{v}_{\text{rel}} \tag{7.27}$$

Since we arrived at an expression for $\delta\ddot{\mathbf{r}}$ in Eq. (7.26), let us proceed to evaluate each of the three terms on the right that involve $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$. First, recall that the angular momentum of $A$ $\left(\mathbf{h} = \mathbf{R} \times \dot{\mathbf{R}}\right)$ is normal to $A$'s orbital plane, and so is the $z$ axis of the comoving frame. Therefore, $\mathbf{h} = h\hat{\mathbf{k}}$. It follows that Eqs. (7.5) and (7.6) may be written as

$$\boldsymbol{\Omega} = \frac{h}{R^2}\hat{\mathbf{k}} \tag{7.28}$$

and

$$\dot{\boldsymbol{\Omega}} = -\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\hat{\mathbf{k}} \tag{7.29}$$

where $\mathbf{V} = \dot{\mathbf{R}}$.

From Eqs. (7.25)–(7.29), we find

$$\dot{\boldsymbol{\Omega}} \times \delta\mathbf{r} = \left[-\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\hat{\mathbf{k}}\right] \times \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right) = \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\left(\delta y\hat{\mathbf{i}} - \delta x\hat{\mathbf{j}}\right) \tag{7.30}$$

and

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \delta\mathbf{r}) = \frac{h}{R^2}\hat{\mathbf{k}} \times \left[\frac{h}{R^2}\hat{\mathbf{k}} \times \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right)\right] = -\frac{h^2}{R^4}\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right) \tag{7.31}$$

According to Eq. (7.4b), $\delta\mathbf{v}_{\text{rel}} = \delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}}$ where $\delta\dot{x} = (\mathrm{d}/\mathrm{d}t)\delta x$, etc. It follows that

$$2\boldsymbol{\Omega} \times \delta\mathbf{v}_{\text{rel}} = 2\frac{h}{R^2}\hat{\mathbf{k}} \times \left(\delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}}\right) = 2\frac{h}{R^2}\left(\delta\dot{x}\hat{\mathbf{j}} - \delta\dot{y}\hat{\mathbf{k}}\right) \tag{7.32}$$

Substituting Eq. (7.26) along with Eqs. (7.30)–(7.32) into Eq. (7.27) yields

$$\delta\mathbf{a}_{\text{rel}} = \overbrace{-\frac{\mu}{R^3}\left(-2\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right)}^{\delta\ddot{\mathbf{r}}} - \overbrace{\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\left(\delta y\hat{\mathbf{i}} - \delta x\hat{\mathbf{j}}\right)}^{\dot{\boldsymbol{\Omega}}\times\delta\mathbf{r}} - \overbrace{\left[-\frac{h^2}{R^4}\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right)\right]}^{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\delta\mathbf{r})}$$
$$-\overbrace{2\frac{h}{R^2}\left(\delta\dot{x}\hat{\mathbf{j}} + \delta\dot{y}\hat{\mathbf{i}}\right)}^{2\boldsymbol{\Omega}\times\delta\mathbf{v}_{\text{rel}}}$$

Referring to Eq. (7.4c) we set $\delta\mathbf{a}_{\mathrm{rel}} = \delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}}$ (where $\delta\ddot{x} = \left(d^2/dt^2\right)\delta x$, etc.) and collect the terms on the right to obtain

$$\begin{aligned}\delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}} = &\left[\left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)\delta x - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta y + 2\frac{h}{R^2}\delta\dot{y}\right]\hat{\mathbf{i}} \\ &+ \left[\left(\frac{h^2}{R^4} - \frac{\mu}{R^3}\right)\delta y + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta x - 2\frac{h}{R^2}\delta\dot{x}\right]\hat{\mathbf{j}} \\ &- \frac{\mu}{R^3}\delta z\hat{\mathbf{k}}\end{aligned} \tag{7.33}$$

Finally, by equating the coefficients of the three unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, this vector equation yields the three scalar equations,

$$\delta\ddot{x} - \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)\delta x + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta y - 2\frac{h}{R^2}\delta\dot{y} = 0 \tag{7.34a}$$

$$\delta\ddot{y} + \left(\frac{\mu}{R^3} - \frac{h^2}{R^4}\right)\delta y - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta x + 2\frac{h}{R^2}\delta\dot{x} = 0 \tag{7.34b}$$

$$\delta\ddot{z} + \frac{\mu}{R^3}\delta z = 0 \tag{7.34c}$$

This set of linear second-order differential equations must be solved to obtain the relative position coordinates $\delta x$, $\delta y$, and $\delta z$ as a function of time. Eqs. (7.34a) and (7.34b) are coupled since $\delta x$ and $\delta y$ appear in each one of them. $\delta z$ appears by itself in Eq. (7.34c) and nowhere else, which means the relative motion in the $z$ direction is independent of that in the other two directions. If the reference orbit is an ellipse, then $\mathbf{R}$ and $\mathbf{V}$ vary with time (although the angular momentum $h$ of the reference orbit is constant). In that case the coefficients in Eq. (7.34) are time dependent, so there is no easy analytical solution. However, we can solve Eq. (7.34) numerically using the methods in Section 1.8.

To that end we recast Eq. (7.34) as a set of first-order differential equations in the standard form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{7.35}$$

where

$$\mathbf{y} = \begin{Bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{Bmatrix} \quad \dot{\mathbf{y}} = \begin{Bmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \\ \delta\ddot{x} \\ \delta\ddot{y} \\ \delta\ddot{z} \end{Bmatrix} \quad \mathbf{f}(t, \mathbf{y}) = \begin{Bmatrix} y_4 \\ y_5 \\ y_6 \\ \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)y_1 - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}y_2 + 2\frac{h}{R^2}y_5 \\ \left(\frac{h^2}{R^4} - \frac{\mu}{R^3}\right)y_2 + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}y_1 - 2\frac{h}{R^2}y_4 \\ -\frac{\mu}{R^3}y_3 \end{Bmatrix} \tag{7.36}$$

These can be solved by Algorithm 1.1 (Runge-Kutta), Algorithm 1.2 (Heun), or Algorithm 1.3 (Runge-Kutta-Fehlberg). In any case, the state vector of the target orbit must be updated at each time step to provide the current values of **R** and **V**. This is done with the aid of Algorithm 3.4. (Alternatively, Eq. (7.22), the equations of motion of the target, can be integrated along with Eq. (7.36) to provide **R** and **V** as a function of time.)

## EXAMPLE 7.3

At time $t = 0$ the orbital parameters of target vehicle $A$ in an equatorial earth orbit are

$$r_p = 6678\,\text{km} \quad e = 0.1 \quad i = \Omega = \omega = \theta = 0^\circ \tag{a}$$

where $r_p$ is the perigee radius. At that same instant the state vector of the chaser vehicle $B$ relative to $A$ is

$$\delta\mathbf{r}_0 = -1\hat{\mathbf{i}}(\text{km}) \quad \delta\mathbf{v}_{\text{rel}})_0 = 2n\hat{\mathbf{j}}(\text{km/s}) \tag{b}$$

where $n$ is the mean motion of $A$. Plot the path of $B$ relative to $A$ in the comoving frame for five periods of the reference orbit.

### Solution

1. Use Algorithm 4.5 to obtain the initial state vector $(\mathbf{R}_0, \mathbf{V}_0)$ of the target vehicle from the orbital parameters given in Eq. (a).
2. Starting with the initial conditions given in Eq. (b), use Algorithm 1.3 to integrate Eq. (7.36) over the specified time interval. Use Algorithm 3.4 to obtain the reference orbit state vector ($\mathbf{R}$, $\mathbf{V}$) at each time step in order to evaluate the coefficients in Eq. (7.36).
3. Graph the trajectory $\delta y(t)$ vs. $\delta x(t)$.

This procedure is implemented in the MATLAB function *Example_7_03.m* listed in Appendix D.33. The output of the program is shown in Fig. 7.6. Observe that since $\delta z_0 = \delta\dot{z}_0 = 0$, no movement develops in the $z$ direction. The motion of the chaser therefore lies in the plane of the target vehicle's orbit. Fig. 7.6 shows that $B$ rapidly moves away from $A$ along the $y$ direction and that the amplitude of its looping motion about the $x$ axis continuously increases. The accuracy of this solution degrades over time because eventually the criterion in Eq. (7.15) is no longer satisfied.

It is interesting to note that if we change the eccentricity of $A$ to zero, so that the reference orbit is a circle, then Fig. 7.7 results. That is, for the same initial conditions, $B$ orbits the target vehicle instead of drifting away from it.

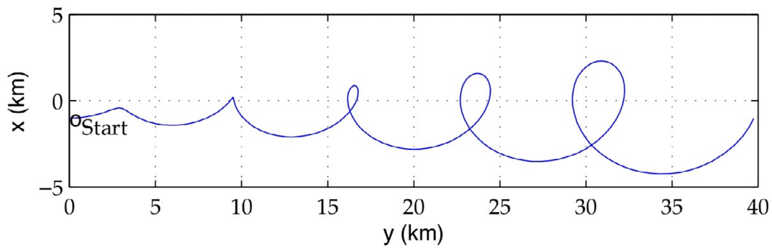



**FIG. 7.6**

Trajectory of $B$ relative to $A$ in the comoving frame during five of the target's orbits. Eccentricity of the target orbit is 0.1.

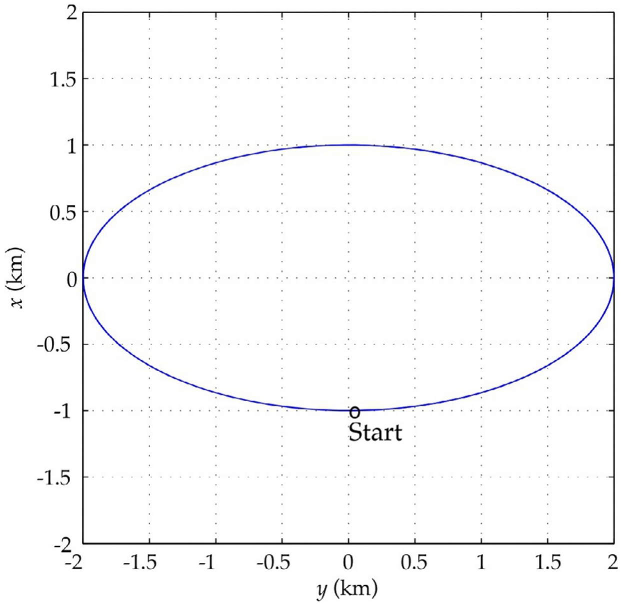



**FIG. 7.7**

Trajectory of $B$ relative to $A$ in the comoving frame during five of the target's orbits. Eccentricity of the target orbit is 0.

## 7.4 CLOHESSY-WILTSHIRE EQUATIONS

If the orbit of the target vehicle $A$ in Fig. 7.5 is a circle, then our LVLH frame is called a Clohessy-Wiltshire (CW) frame (Clohessy) (Clohessy and Wiltshire, 1960). In such a frame Eq. (7.34) simplifies considerably. For a circular target orbit $\mathbf{V} \cdot \mathbf{R} = 0$ and $h = \sqrt{\mu R}$. Substituting these into Eqs. (7.34) yields

$$
\begin{aligned}
\delta\ddot{x} - 3\frac{\mu}{R^3}\delta x - 2\sqrt{\frac{\mu}{R^3}}\delta\dot{y} &= 0 \\
\delta\ddot{y} + 2\sqrt{\frac{\mu}{R^3}}\delta\dot{x} &= 0 \\
\delta\ddot{z} + \frac{\mu}{R^3}\delta z &= 0
\end{aligned}
\tag{7.37}
$$

It is furthermore true for circular orbits that the angular velocity (mean motion) is

$$n = \frac{V}{R} = \frac{\sqrt{\mu/R}}{R} = \sqrt{\frac{\mu}{R^3}}$$

Therefore, Eq. (7.37) may be written as

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = 0 \tag{7.38a}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = 0 \tag{7.38b}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = 0 \tag{7.38c}$$

These are known as the Clohessy-Wiltshire (CW) equations. Unlike Eq. (7.34), where the target orbit is an ellipse, the coefficients in Eq. (7.38) are constant. Therefore, a straightforward analytical solution exists.

We start with the first two equations, which are coupled and define the motion of the chaser in the $xy$ plane of the reference orbit. First, observe that Eq. (7.38b) can be written as $(\mathrm{d}/\mathrm{d}t)(\delta\dot{y}+2n\delta x)=0$, which means that $\delta\dot{y}+2n\delta x=C_1$, where $C_1$ is a constant. Therefore,

$$\delta\dot{y}=C_1-2n\delta x \tag{7.39}$$

Substituting this expression into Eq. (7.38a) yields

$$\delta\ddot{x}+n^2\delta x=2nC_1 \tag{7.40}$$

This familiar differential equation has the following solution, which can be easily verified by substitution:

$$\delta x=\frac{2}{n}C_1+C_2\sin nt+C_3\cos nt \tag{7.41}$$

Differentiating this expression with respect to time gives the $x$ component of the relative velocity,

$$\delta\dot{x}=C_2 n\cos nt-C_3 n\sin nt \tag{7.42}$$

Substituting Eq. (7.41) into Eq. (7.39) yields the $y$ component of the relative velocity

$$\delta\dot{y}=-3C_1-2C_2 n\sin nt-2C_3 n\cos nt \tag{7.43}$$

Integrating this equation with respect to time yields

$$\delta y=-3C_1 t+2C_2\cos nt-2C_3\sin nt+C_4 \tag{7.44}$$

The constants $C_1$ through $C_4$ are found by applying the initial conditions; namely,

$$\text{At } t=0 \quad \delta x=\delta x_0 \quad \delta y=\delta y_0 \quad \delta\dot{x}=\delta\dot{x}_0 \quad \delta\dot{y}=\delta\dot{y}_0$$

Evaluating Eqs. (7.41)–(7.44), respectively, at $t=0$ we get

$$\begin{aligned}\frac{2}{n}C_1+C_3&=\delta x_0\\ C_2 n&=\delta\dot{x}_0\\ -3C_1-2C_3 n&=\delta\dot{y}_0\\ 2C_2+C_4&=\delta y_0\end{aligned}$$

Solving for $C_1$ through $C_4$ yields

$$C_1=2n\delta x_0+\delta\dot{y}_0 \quad C_2=\frac{1}{n}\delta\dot{x}_0 \quad C_3=-3\delta x_0-\frac{2}{n}\delta\dot{y}_0 \quad C_4=-\frac{2}{n}\delta\dot{x}_0+\delta y_0 \tag{7.45}$$

Finally, we turn our attention to Eq. (7.38c), which governs the relative motion normal to the plane of the circular reference orbit. Eq. (7.38c) has the same form as Eq. (7.40) with $C_1=0$. Therefore, its solution is

$$\delta z=C_5\sin nt+C_6\cos nt \tag{7.46}$$

It follows that the relative velocity normal to the reference orbit is

$$\delta\dot{z}=C_5 n\cos nt-C_6 n\sin nt \tag{7.47}$$

The initial conditions are $\delta z = \delta z_0$ and $\delta \dot{z} = \delta \dot{z}_0$ at $t = 0$, which means

$$C_5 = \frac{\delta \dot{z}_0}{n} \qquad C_6 = \delta z_0 \tag{7.48}$$

Substituting Eqs. (7.45) and (7.48) into Eqs. (7.41), (7.44), and (7.46) yields the trajectory of the chaser in the CW frame,

$$\delta x = 4\delta x_0 + \frac{2}{n}\delta \dot{y}_0 + \frac{\delta \dot{x}_0}{n}\sin nt - \left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0\right)\cos nt \tag{7.49a}$$

$$\delta y = \delta y_0 - \frac{2}{n}\delta \dot{x}_0 - 3(2n\delta x_0 + \delta \dot{y}_0)t + 2\left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0\right)\sin nt + \frac{2}{n}\delta \dot{x}_0 \cos nt \tag{7.49b}$$

$$\delta z = \frac{1}{n}\delta \dot{z}_0 \sin nt + \delta z_0 \cos nt \tag{7.49c}$$

Observe that all the three components of $\delta \mathbf{r}$ oscillate with a frequency equal to the frequency of revolution (mean motion $n$) of the CW frame. Only $\delta y$ has a secular term, which grows linearly with time. Therefore, unless $2n\delta x_0 + \delta \dot{y}_0 = 0$, the chaser will drift away from the target and the distance $\delta r$ will increase without bound. The accuracy of Eqs. (7.49) will consequently degrade as the criterion (Eq. 7.15) on which this solution is based eventually ceases to be valid. Fig. 7.8 shows the motion of a particle relative to a CW frame with an orbital radius of 6678 km. The particle started at the origin with a velocity of 0.01 km/s in the negative $y$ direction. This delta-v dropped the particle into a lower energy, a slightly elliptical orbit. The subsequent actual relative motion of the particle in the CW frame is graphed in Fig. 7.8 as is the motion given by Eqs. (7.49), the linearized CW solution. Clearly, the two solutions diverge markedly after one orbit of the reference frame, when the distance of the particle from the origin exceeds 150 km.

Now that we have finished solving the CW equations, let us simplify our notation a bit and denote the $x$, $y$, and $z$ components of relative velocity in the moving frame as $\delta u$, $\delta v$, and $\delta w$, respectively. That is, let

$$\delta u = \delta \dot{x} \qquad \delta v = \delta \dot{y} \qquad \delta w = \delta \dot{z} \tag{7.50a}$$

The initial conditions on the relative velocity components are then written as

$$\delta u_0 = \delta \dot{x}_0 \qquad \delta v_0 = \delta \dot{y}_0 \qquad \delta w_0 = \delta \dot{z}_0 \tag{7.50b}$$

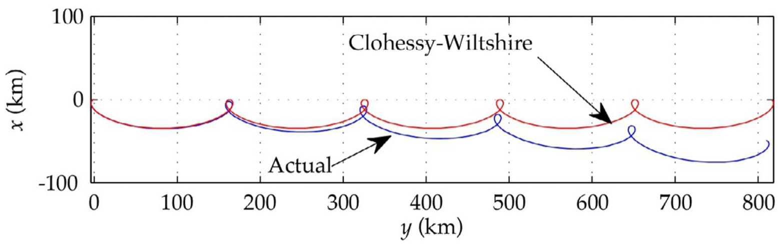



**FIG. 7.8**

Relative motion of a particle and its Clohessy-Wiltshire approximation.

Using this notation in Eq. (7.49) and rearranging the terms we get

$$\begin{aligned}
\delta x &= (4-3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta u_0 + \frac{2}{n}(1-\cos nt)\delta v_0 \\
\delta y &= 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta u_0 + \frac{1}{n}(4\sin nt - 3nt)\delta v_0 \\
\delta z &= \cos nt\delta z_0 + \frac{1}{n}\sin nt\delta w_0
\end{aligned} \tag{7.51a}$$

Differentiating each of these with respect to time and using Eq. (7.50a) yields

$$\begin{aligned}
\delta u &= 3n\sin nt\delta x_0 + \cos nt\delta u_0 + 2\sin nt\delta v_0 \\
\delta v &= 6n(\cos nt - 1)\delta x_0 - 2\sin nt\delta u_0 + (4\cos nt - 3)\delta v_0 \\
\delta w &= -n\sin nt\delta z_0 + \cos nt\delta w_0
\end{aligned} \tag{7.51b}$$

Let us introduce matrix notation to define the relative position and velocity vectors

$$\{\delta\mathbf{r}(t)\} = \begin{Bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{Bmatrix} \qquad \{\delta\mathbf{v}(t)\} = \begin{Bmatrix} \delta u(t) \\ \delta v(t) \\ \delta w(t) \end{Bmatrix}$$

and their initial values (at $t = 0$)

$$\{\delta\mathbf{r}_0\} = \begin{Bmatrix} \delta x_0 \\ \delta y_0 \\ \delta z_0 \end{Bmatrix} \qquad \{\delta\mathbf{v}_0\} = \begin{Bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{Bmatrix}$$

In matrix notation, Eqs. (7.51) appear more compactly as

$$\{\delta\mathbf{r}(t)\} = [\mathbf{\Phi}_{rr}(t)]\{\delta\mathbf{r}_0\} + [\mathbf{\Phi}_{rv}(t)]\{\delta\mathbf{v}_0\} \tag{7.52a}$$

$$\{\delta\mathbf{v}(t)\} = [\mathbf{\Phi}_{vr}(t)]\{\delta\mathbf{r}_0\}[\mathbf{\Phi}_{vv}(t)]\{\delta\mathbf{v}_0\} \tag{7.52b}$$

where the "Clohessy-Wiltshire matrices" comprise the coefficients in Eqs. (7.51):

$$[\mathbf{\Phi}_{rr}(t)] = \left[\begin{array}{cc:c} 4-3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ \hdashline 0 & 0 & \cos nt \end{array}\right] \tag{7.53a}$$

$$[\mathbf{\Phi}_{rv}(t)] = \left[\begin{array}{cc:c} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ \hdashline 0 & 0 & \frac{1}{n}\sin nt \end{array}\right] \tag{7.53b}$$

$$[\mathbf{\Phi}_{vr}(t)] = \left[\begin{array}{cc:c} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ \hdashline 0 & 0 & -n\sin nt \end{array}\right] \tag{7.53c}$$

$$[\boldsymbol{\Phi}_{vv}(t)] = \left[\begin{array}{cc|c} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ \hline 0 & 0 & \cos nt \end{array}\right] \tag{7.53d}$$

The subscripts on $\Phi$ remind us which of the vectors $\delta\mathbf{r}$ and $\delta\mathbf{v}$ is related by that matrix to which of the initial conditions $\delta\mathbf{r}_0$ and $\delta\mathbf{v}_0$. For example, $[\Phi_{rv}]$ relates $\delta\mathbf{r}$ to $\delta\mathbf{v}_0$. The partition lines remind us that motion in the $xy$ plane is independent of that in the $z$ direction normal to the target's orbit. In problems where there is no motion in the $z$ direction ($\delta z_0 = \delta w_0 = 0$), we need only use the upper left 2 by 2 corners of CW matrices. Finally, note also that

$$[\Phi_{vr}(t)] = \frac{d}{dt}[\Phi_{rr}(t)] \quad \text{and} \quad [\Phi_{vv}(t)] = \frac{d}{dt}[\Phi_{rv}(t)]$$

## 7.5 TWO-IMPULSE RENDEZVOUS MANEUVERS

Fig. 7.9 illustrates the rendezvous problem. At time $t = 0^-$ (the instant preceding $t = 0$) the position $\delta\mathbf{r}_0$ and velocity $\delta\mathbf{v}_0^-$ of the chase vehicle $B$ relative to target $A$ are known. At $t = 0$ an impulsive maneuver instantaneously changes the relative velocity to $\delta\mathbf{v}_0^+$ at $t = 0^+$ (the instant after $t = 0$). The components of $\delta\mathbf{v}_0^+$ are shown in Fig. 7.9. We must determine the values of $\delta u_0^+$, $\delta v_0^+$, and $\delta w_0^+$ at the beginning of the

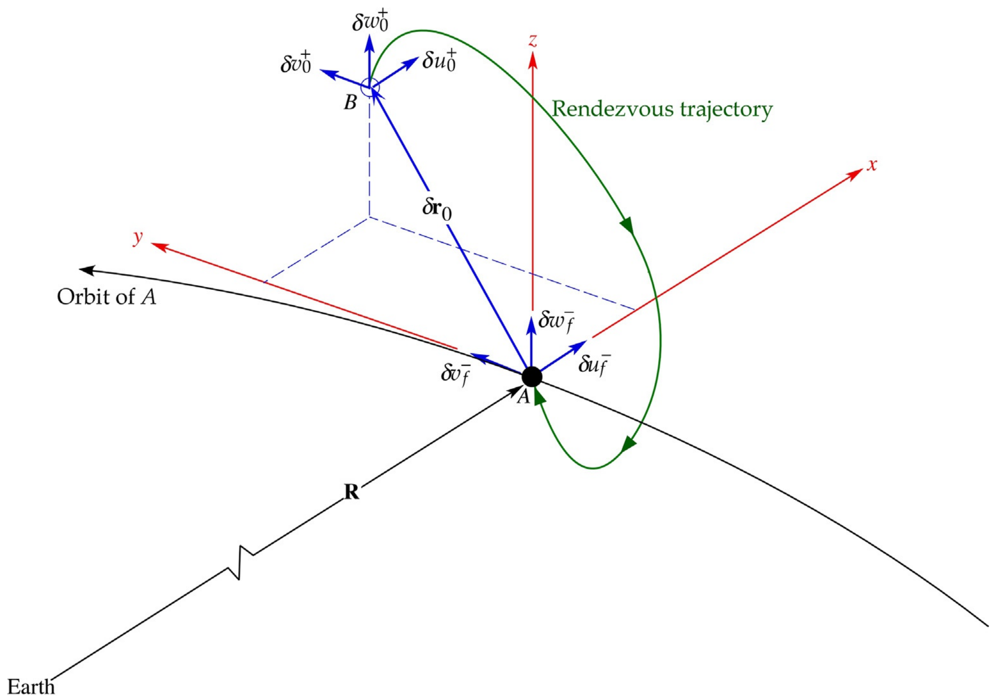



**FIG. 7.9**

Rendezvous with a target $A$ in the neighborhood of the chase vehicle $B$.

rendezvous trajectory, so that $B$ will arrive at the target in a specified time $t_f$. The delta-v required to place $B$ on the rendezvous trajectory is

$$\Delta\mathbf{v}_0 = \delta\mathbf{v}_0^+ - \delta\mathbf{v}_0^- = \left(\delta u_0^+ - \delta u_0^-\right)\hat{\mathbf{i}} + \left(\delta v_0^+ - \delta v_0^-\right)\hat{\mathbf{j}} + \left(\delta w_0^+ - \delta w_0^-\right)\hat{\mathbf{k}} \tag{7.54}$$

At time $t_f$, $B$ arrives at $A$, at the origin of the CW frame, which means $\delta\mathbf{r}_f = \delta\mathbf{r}(t_f) = \mathbf{0}$. Evaluating Eq. (7.52a) at $t_f$ we find

$$\{\mathbf{0}\} = \left[\Phi_{\mathrm{rr}}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{\mathrm{rv}}(t_f)\right]\{\delta\mathbf{v}_0^+\} \tag{7.55}$$

Solving this for $\{\delta\mathbf{v}_0^+\}$ yields

$$\{\delta\mathbf{v}_0^+\} = -\left[\Phi_{\mathrm{rv}}(t_f)\right]^{-1}\left[\Phi_{\mathrm{rr}}(t_f)\right]\{\delta\mathbf{r}_0\} \quad \left(\delta\mathbf{v}_0^+ = \delta u_0^+\hat{\mathbf{i}} + \delta v_0^+\hat{\mathbf{j}} + \delta w_0^+\hat{\mathbf{k}}\right) \tag{7.56}$$

where $\left[\Phi_{\mathrm{rv}}(t_f)\right]^{-1}$ is the matrix inverse of $\left[\Phi_{\mathrm{rv}}(t_f)\right]$. Thus, we now have the velocity $\delta\mathbf{v}_0^+$ at the beginning of the rendezvous path. We substitute Eq. (7.56) into Eq. (7.52b) to obtain the velocity $\delta\mathbf{v}_f^-$ at $t = t_f^-$, when $B$ arrives at target $A$:

$$\begin{aligned}\left\{\delta\mathbf{v}_f^-\right\} &= \left[\Phi_{\mathrm{vr}}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{\mathrm{vv}}(t_f)\right]\{\delta\mathbf{v}_0^+\} \\ &= \left[\Phi_{\mathrm{vr}}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{\mathrm{vv}}(t_f)\right]\left(-\left[\Phi_{\mathrm{rv}}(t_f)\right]^{-1}\left[\Phi_{\mathrm{rr}}(t_f)\right]\{\delta\mathbf{r}_0\}\right)\end{aligned}$$

Collecting terms we get

$$\left\{\delta\mathbf{v}_f^-\right\} = \left[\widetilde{\Phi}_{\mathrm{vr}}\right]\{\delta\mathbf{r}_0\} \quad \left(\delta\mathbf{v}_f^- = \delta u_f^-\hat{\mathbf{i}} + \delta v_f^-\hat{\mathbf{j}} + \delta w_f^-\hat{\mathbf{k}}\right) \tag{7.57a}$$

where

$$\left[\widetilde{\Phi}_{\mathrm{vr}}\right] = \left[\Phi_{\mathrm{vr}}(t_f)\right] - \left[\Phi_{\mathrm{vv}}(t_f)\right]\left[\Phi_{\mathrm{rv}}(t_f)\right]^{-1}\left[\Phi_{\mathrm{rr}}(t_f)\right] \tag{7.57b}$$

Obviously, an impulsive delta-v maneuver is required at $t = t_f$ to bring vehicle $B$ to rest relative to $A$ ($\delta\mathbf{v}_f^+ = \mathbf{0}$):

$$\Delta\mathbf{v}_f = \delta\mathbf{v}_f^+ - \delta\mathbf{v}_f^- = \mathbf{0} - \delta\mathbf{v}_f^- = -\delta\mathbf{v}_f^- \tag{7.58}$$

Note that in Eqs. (7.54) and (7.58) we are using the difference between relative velocities to calculate delta-v, which is the difference in absolute velocities. To show that this is valid use Eq. (1.75) to write

$$\begin{aligned}\mathbf{v}^- &= \mathbf{v}_0^- + \boldsymbol{\Omega}^- \times \mathbf{r}_{\mathrm{rel}}^- + \mathbf{v}_{\mathrm{rel}}^- \\ \mathbf{v}^+ &= \mathbf{v}_0^+ + \boldsymbol{\Omega}^+ \times \mathbf{r}_{\mathrm{rel}}^+ + \mathbf{v}_{\mathrm{rel}}^+\end{aligned} \tag{7.59}$$

Since the target is passive, the impulsive maneuver has no effect on its state of motion, which means $\mathbf{v}_0^+ = \mathbf{v}_0^-$ and $\boldsymbol{\Omega}^+ = \boldsymbol{\Omega}^-$. Furthermore, by definition of an impulsive maneuver, there is no change in the position; that is, $\mathbf{r}_{\mathrm{rel}}^+ = \mathbf{r}_{\mathrm{rel}}^-$. It follows from Eq. (7.59) that

$$\mathbf{v}^+ - \mathbf{v}^- = \mathbf{v}_{\mathrm{rel}}^+ - \mathbf{v}_{\mathrm{rel}}^- \quad \text{or} \quad \Delta\mathbf{v} = \Delta\mathbf{v}_{\mathrm{rel}}$$

## EXAMPLE 7.4

A space station and another spacecraft are in earth orbits with the following parameters:

| | Space station | Spacecraft |
|---|---|---|
| Perigee × apogee (altitude) | 300 km circular | 320.06 km × 513.86 km |
| Period (computed using above data) | 1.5086 h | 1.5484 h |
| True anomaly, $\theta$ | 60° | 349.65° |
| Inclination, $i$ | 40° | 40.130° |
| RA of ascending node, $\Omega$ | 20° | 19.819° |
| Argument of perigee, $\omega$ | 0° (arbitrary) | 70.662° |

Compute the total delta-v required for an 8-h, two-impulse rendezvous trajectory.

### Solution

We substitute the given data into Algorithm 4.5 to obtain the state vectors of the two spacecraft in the geocentric equatorial frame.

*Space station:*

$$\mathbf{R} = 1622.39\hat{\mathbf{I}} + 5305.10\hat{\mathbf{J}} + 3717.44\hat{\mathbf{K}}\ (\text{km})$$
$$\mathbf{V} = -7.29936\hat{\mathbf{I}} + 0.492329\hat{\mathbf{J}} + 2.48304\hat{\mathbf{K}}\ (\text{km/s})$$

*Spacecraft:*

$$\mathbf{r} = 1612.75\hat{\mathbf{I}} + 5310.19\hat{\mathbf{J}} + 3750.33\hat{\mathbf{K}}\ (\text{km})$$
$$\mathbf{v} = -7.35170\hat{\mathbf{I}} + 0.463828\hat{\mathbf{J}} + 2.46906\hat{\mathbf{K}}\ (\text{km/s})$$

The space station reference frame unit vectors (at this instant) are, by definition,

$$\hat{\mathbf{i}} = \frac{\mathbf{R}}{\|\mathbf{R}\|} = 0.242945\hat{\mathbf{I}} + 0.794415\hat{\mathbf{J}} + 0.556670\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = \frac{\mathbf{V}}{\|\mathbf{V}\|} = -0.944799\hat{\mathbf{I}} + 0.063725\hat{\mathbf{J}} + 0.321394\hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}} = 0.219846\hat{\mathbf{I}} - 0.604023\hat{\mathbf{J}} + 0.766044\hat{\mathbf{K}}$$

Therefore, the direction cosine matrix of the transformation from the geocentric equatorial frame into the space station frame is (at this instant)

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix}$$

The position vector of the spacecraft relative to the space station (in the geocentric equatorial frame) is

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{R} = -9.64015\hat{\mathbf{I}} + 5.08235\hat{\mathbf{J}} + 32.8822\hat{\mathbf{K}}(\text{km})$$

The relative velocity is given by the formula (Eq. 7.8)

$$\delta\mathbf{v} = \mathbf{v} - \mathbf{V} - \mathbf{\Omega}_{\text{space station}} \times \delta\mathbf{r}$$

where $\mathbf{\Omega}_{\text{space station}} = n\hat{\mathbf{k}}$ and $n$, the mean motion of the space station, is

$$n = \frac{V}{R} = \frac{7.7258}{6678} = 0.00115691\,\text{rad/s} \tag{a}$$

Thus,

$$\delta\mathbf{v} = \overbrace{-7.35170\hat{\mathbf{I}} + 0.463828\hat{\mathbf{J}} + 2.46906\hat{\mathbf{K}}}^{\mathbf{v}} - \overbrace{\left(-7.29936\hat{\mathbf{I}} + 0.492329\hat{\mathbf{J}} + 2.48304\hat{\mathbf{K}}\right)}^{\mathbf{V}}$$

$$\overbrace{-(0.00115691)\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.219846 & -0.604023 & 0.766044 \\ -9.64015 & 5.08235 & 32.8822 \end{vmatrix}}^{\boldsymbol{\Omega}_{\text{space station}} \times \delta\mathbf{r}}$$

so that

$$\delta\mathbf{v} = -0.024854\hat{\mathbf{I}} - 0.01159370\hat{\mathbf{J}} - 0.00853575\hat{\mathbf{K}}\,(\text{km/s})$$

In space station coordinates the relative position vector $\delta\mathbf{r}_0$ at the beginning of the rendezvous maneuver is

$$\{\delta\mathbf{r}_0\} = [\mathbf{Q}]_{Xx}\{\delta\mathbf{r}\} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{Bmatrix} -9.64015 \\ 5.08235 \\ 32.8822 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix} (\text{km}) \qquad \text{(b)}$$

Likewise, the relative velocity $\delta\mathbf{v}_0^-$ just before launch into the rendezvous trajectory is

$$\{\delta\mathbf{v}_0^-\} = [\mathbf{Q}]_{Xx}\{\delta\mathbf{v}\} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{Bmatrix} -0.024854 \\ -0.0115937 \\ -0.00853575 \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.02000 \\ 0.02000 \\ -0.005000 \end{Bmatrix} (\text{km/s})$$

The Clohessy-Wiltshire matrices, for $t = t_f = 8\,\text{h} = 28{,}800\,\text{s}$ and $n = 0.00115691\,\text{rad/s}$ (from Eq. (a)), are

$$[\Phi_{rr}] = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 4.97849 & 0 & 0 \\ -194.242 & 1.000 & 0 \\ 0 & 0 & -0.326163 \end{bmatrix}$$

$$[\Phi_{rv}] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1 - \cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}$$

$$[\Phi_{vr}] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0.00328092 & 0 & 0 \\ -0.00920550 & 0 & 0 \\ 0 & 0 & -0.00109364 \end{bmatrix}$$

$$[\Phi_{vv}] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} -0.326164 & 1.89063 & 0 \\ -1.89063 & -4.30466 & 0 \\ 0 & 0 & -0.326164 \end{bmatrix}$$

From Eqs. (7.56) and (b) we find $\delta\mathbf{v}_0^+$:

$$\begin{Bmatrix}\delta u_0^+\\ \delta v_0^+\\ \delta w_0^+\end{Bmatrix} = -\begin{bmatrix}817.102 & 2292.60 & 0\\ -2292.60 & -83131.6 & 0\\ 0 & 0 & 817.103\end{bmatrix}^{-1}\begin{bmatrix}4.97849 & 0 & 0\\ -194.242 & 1.000 & 0\\ 0 & 0 & -0.326163\end{bmatrix}\begin{Bmatrix}20\\ 20\\ 20\end{Bmatrix}$$
$$= -\begin{bmatrix}817.102 & 2292.60 & 0\\ -2292.60 & -83131.6 & 0\\ 0 & 0 & 817.103\end{bmatrix}^{-1}\begin{bmatrix}99.5698\\ -3864.84\\ -6.52386\end{bmatrix} = \begin{Bmatrix}0.00930458\\ -0.0467472\\ 0.00798343\end{Bmatrix}(\text{km/s}) \tag{c}$$

From Eq. (7.52b), evaluated at $t = t_f$, we have

$$\left\{\delta\mathbf{v}_f^-\right\} = \left[\Phi_{vr}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{vv}(t_f)\right]\left\{\delta\mathbf{v}_0^+\right\}$$

Substituting Eqs. (b) and (c),

$$\begin{Bmatrix}\delta u_f^-\\ \delta v_f^-\\ \delta w_f^-\end{Bmatrix} = \begin{bmatrix}0.00328092 & 0 & 0\\ -0.00920550 & 0 & 0\\ 0 & 0 & -0.00109364\end{bmatrix}\begin{Bmatrix}20\\ 20\\ 20\end{Bmatrix}$$
$$+\begin{bmatrix}-0.326164 & 1.89063 & 0\\ -1.89063 & -4.30466 & 0\\ 0 & 0 & -0.326164\end{bmatrix}\begin{Bmatrix}0.00930458\\ -0.0467472\\ 0.00798343\end{Bmatrix}$$

$$\begin{Bmatrix}\delta u_f^-\\ \delta v_f^-\\ \delta w_f^-\end{Bmatrix} = \begin{Bmatrix}-0.0257978\\ -0.000470870\\ -0.0244767\end{Bmatrix}(\text{km/s}) \tag{d}$$

The delta-v at the beginning of the rendezvous maneuver is found as

$$\{\Delta\mathbf{v}_0\} = \left\{\delta\mathbf{v}_0^+\right\} - \left\{\delta\mathbf{v}_0^-\right\} = \begin{Bmatrix}0.00930458\\ -0.0467472\\ 0.00798343\end{Bmatrix} - \begin{Bmatrix}-0.02\\ 0.02\\ -0.005\end{Bmatrix} = \begin{Bmatrix}0.0293046\\ -0.0667472\\ 0.0129834\end{Bmatrix}(\text{km/s})$$

The delta-v at the conclusion of the maneuver is

$$\left\{\Delta\mathbf{v}_f\right\} = \left\{\delta\mathbf{v}_f^+\right\} - \left\{\delta\mathbf{v}_f^-\right\} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} - \begin{Bmatrix}-0.0257978\\ -0.000470870\\ -0.0244767\end{Bmatrix} = \begin{Bmatrix}0.0257978\\ 0.000470870\\ 0.0244767\end{Bmatrix}(\text{km/s})$$

The total delta-v requirement is

$$\Delta v_{\text{total}} = \left\|\Delta\mathbf{v}_0\right\| + \left\|\Delta\mathbf{v}_f\right\| = 0.0740440 + 0.0355649 = 0.109609\,\text{km/s} = \boxed{109.6\,\text{m/s}}$$

From Eq. (7.52a) we have, for $0 < t < t_f$,

$$\begin{Bmatrix}\delta x(t)\\ \delta y(t)\\ \delta z(t)\end{Bmatrix} = \begin{bmatrix}4-3\cos nt & 0 & 0\\ 6(\sin nt - nt) & 1 & 0\\ 0 & 0 & \cos nt\end{bmatrix}\begin{Bmatrix}20\\ 20\\ 20\end{Bmatrix}$$
$$+\begin{bmatrix}\dfrac{1}{n}\sin nt & \dfrac{2}{n}(1-\cos nt) & 0\\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) & 0\\ 0 & 0 & \dfrac{1}{n}\sin nt\end{bmatrix}\begin{Bmatrix}0.00930458\\ -0.0467472\\ 0.00798343\end{Bmatrix}$$

Substituting $n$ from Eq. (a) we obtain the relative position vector as a function of time. It is plotted in Fig. 7.10.

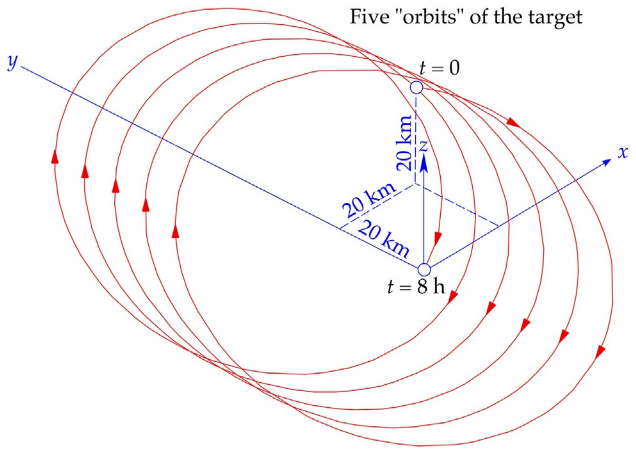



**FIG. 7.10**

Rendezvous trajectory of the chase vehicle relative to the target.

## EXAMPLE 7.5

A target and a chase vehicle are in the same 300-km circular earth orbit. The chaser is 2 km behind the target when the chaser initiates a two-impulse rendezvous maneuver so as to rendezvous with the target in 1.49 h. Find the total delta-v requirement.

### Solution

For the circular reference orbit

$$v=\sqrt{\frac{\mu}{r}}=\sqrt{\frac{398,600}{6378+300}}=7.7258\,\mathrm{km/s} \tag{a}$$

so that the mean motion is

$$n=\frac{v}{r}=\frac{7.7258}{6678}=0.0011569\,\mathrm{rad/s} \tag{b}$$

For this mean motion and the rendezvous trajectory time $t = 1.49$ h $= 5364$ s, the Clohessy-Wiltshire matrices are

$$[\mathbf{\Phi}_{rr}]=\left[\begin{array}{cc:c} 1.0090 & 0 & 0 \\ -37.699 & 1 & 0 \\ \hdashline 0 & 0 & 0.99700 \end{array}\right] \qquad [\mathbf{\Phi}_{rv}]=\left[\begin{array}{cc:c} -66.946 & 5.1928 & 0 \\ -5.1928 & -16360 & 0 \\ \hdashline 0 & 0 & -66.946 \end{array}\right]$$

$$[\mathbf{\Phi}_{vr}]=\left[\begin{array}{cc:c} -2.6881(10^{-4}) & 0 & 0 \\ -2.0851(10^{-5}) & 0 & 0 \\ \hdashline 0 & 0 & 8.9603(10^{-5}) \end{array}\right] \qquad [\mathbf{\Phi}_{vv}]=\left[\begin{array}{cc:c} 0.99700 & -0.15490 & 0 \\ 0.15490 & 0.98798 & 0 \\ \hdashline 0 & 0 & 0.99700 \end{array}\right] \tag{c}$$

The initial and final positions of the chaser in the CW frame are

$$\{\delta\mathbf{r}_0\} = \begin{Bmatrix} 0 \\ -2 \\ 0 \end{Bmatrix}(\text{km}) \quad \{\delta\mathbf{r}_f\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}(\text{km}) \tag{d}$$

Since $\delta z_0 = \delta w_0 = 0$ there is no motion in the $z$ direction $[\delta z(t) = 0]$, so we need employ only the upper left 2 by 2 corners of the Clohessy-Wiltshire matrices and treat this as a two-dimensional problem in the plane of the reference orbit. Thus solving the first CW equation, $\{\delta\mathbf{r}_f\} = [\Phi_{rr}]\{\delta\mathbf{r}_0\} + [\Phi_{rv}]\{\delta\mathbf{v}_0^+\}$, for $\{\delta\mathbf{v}_0^+\}$ we get

$$\begin{aligned}\{\delta\mathbf{v}_0^+\} &= -[\Phi_{rv}]^{-1}[\Phi_{rr}]\{\delta\mathbf{r}_0\} = -\begin{bmatrix} -0.014937 & -4.7412(10^{-6}) \\ 4.7412(10^{-6}) & -6.1124(10^{-5}) \end{bmatrix}\begin{bmatrix} 1.0090 & 0 \\ -37.699 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \\ &= \begin{Bmatrix} 9.4824(10^{-6}) \\ -1.2225(10^{-4}) \end{Bmatrix}\end{aligned}$$

or

$$\delta\mathbf{v}_0^+ = -9.4824(10^{-6})\hat{\mathbf{i}} + 1.2225(10^{-4})\hat{\mathbf{j}}(\text{km/s}) \tag{e}$$

Therefore, the second CW equation, $\{\delta\mathbf{v}_f^-\} = [\Phi_{vr}]\{\delta\mathbf{r}_0\} + [\Phi_{vv}]\{\delta\mathbf{v}_0^+\}$, yields

$$\begin{aligned}\{\delta\mathbf{v}_f^-\} &= \begin{bmatrix} -2.6881(10^{-4}) & 0 \\ -2.0851(10^{-5}) & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -2 \end{Bmatrix} + \begin{bmatrix} 0.99700 & -0.15490 \\ 0.15490 & 0.98798 \end{bmatrix}\begin{Bmatrix} -9.4824(10^{-6}) \\ -1.2225(10^{-4}) \end{Bmatrix} \\ &= \begin{Bmatrix} 9.4824(10^{-6}) \\ -1.2225(10^{-4}) \end{Bmatrix}\end{aligned}$$

or

$$\delta\mathbf{v}_f^- = 9.4824(10^{-6})\hat{\mathbf{i}} - 1.2225(10^{-4})\hat{\mathbf{j}}(\text{km/s}) \tag{f}$$

Since the chaser is in the same circular orbit as the target, its relative velocity is initially zero; that is, $\delta\mathbf{v}_0^- = \mathbf{0}$. (See also Eq. 7.68 at the end of the next section.) Thus,

$$\begin{aligned}\Delta\mathbf{v}_0 = \delta\mathbf{v}_0^+ - \delta\mathbf{v}_0^- &= \left(-9.4824\times10^{-6}\hat{\mathbf{i}} - 1.2225\times10^{-4}\hat{\mathbf{j}}\right) - \mathbf{0} \\ &= -9.4824\times10^{-6}\hat{\mathbf{i}} + 1.2225\times10^{-4}\hat{\mathbf{j}}(\text{km/s})\end{aligned}$$

which implies

$$\|\Delta\mathbf{v}_0\| = 0.1226\,\text{m/s} \tag{g}$$

At the end of the rendezvous maneuver, $\delta\mathbf{v}_f^+ = \mathbf{0}$, so that

$$\begin{aligned}\Delta\mathbf{v}_f = \delta\mathbf{v}_f^+ - \delta\mathbf{v}_f^- &= \mathbf{0} - \left(9.4824\times10^{-6}\hat{\mathbf{i}} - 1.2225\times10^{-4}\hat{\mathbf{j}}\right) \\ &= -9.4824\times10^{-6}\hat{\mathbf{i}} + 1.2225\times10^{-4}\hat{\mathbf{j}}(\text{km/s})\end{aligned}$$

Therefore,

$$\|\Delta\mathbf{v}_f\| = 0.1226\text{m/s} \tag{h}$$

The total delta-v required is

$$\Delta v_{\text{total}} = \|\Delta\mathbf{v}_0\| + \|\Delta\mathbf{v}_f\| = \boxed{0.2452\text{ m/ s}} \tag{i}$$

The coplanar rendezvous trajectory relative to the CW frame is sketched in Fig. 7.11. Notice that in the CW frame circular orbits appear as straight lines parallel to the $y$ axis. This is due to the linearization we did, based on Eq. (7.15).

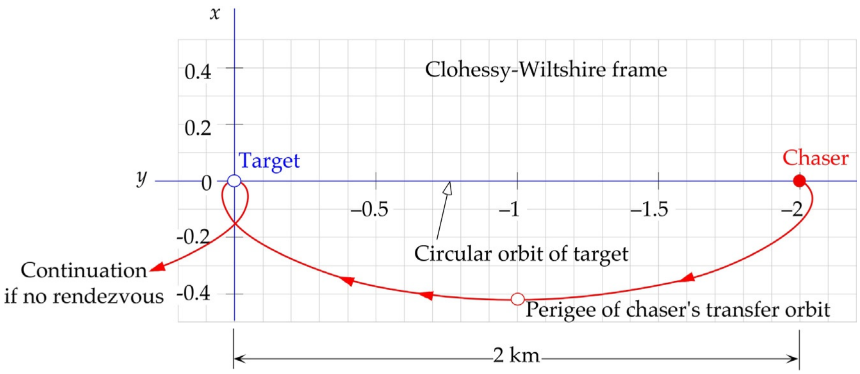



**FIG. 7.11**

Motion of the chaser relative to the target.

## 7.6 RELATIVE MOTION IN CLOSE-PROXIMITY CIRCULAR ORBITS

Fig. 7.12 shows two spacecraft in coplanar circular orbits. Let us calculate the velocity $\delta\mathbf{v}$ of the chase vehicle $B$ relative to target $A$ when they are in close proximity. "Close proximity" means that

$$\frac{\delta r}{R} << 1$$

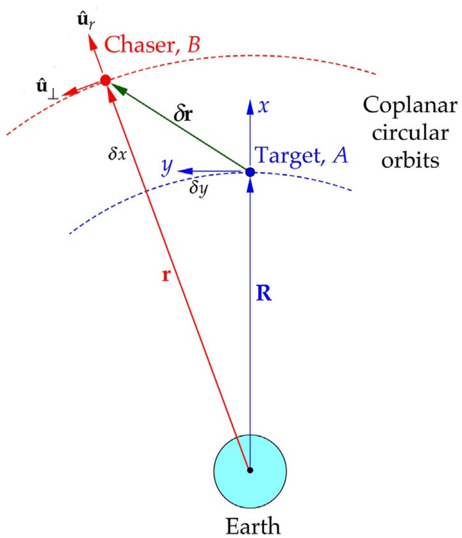



**FIG. 7.12**

Two spacecraft in close proximity.

To solve this problem we must use the relative velocity equation,

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \delta\mathbf{r} + \delta\mathbf{v} \tag{7.60}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the Clohessy-Wiltshire frame attached to $A$,

$$\boldsymbol{\Omega} = n\hat{\mathbf{k}}$$

$n$ is the mean motion of the target vehicle,

$$n = \frac{v_A}{R} \tag{7.61}$$

where, by virtue of the circular orbit,

$$v_A = \sqrt{\frac{\mu}{R}} \tag{7.62}$$

Solving Eq. (7.60) for the relative velocity $\delta\mathbf{v}$ yields

$$\delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A - \left(n\hat{\mathbf{k}}\right) \times \delta\mathbf{r} \tag{7.63}$$

Since the chase orbit is circular, we have for the first term on the right-hand side of Eq. (7.63)

$$\mathbf{v}_B = \sqrt{\frac{\mu}{r}}\hat{\mathbf{u}}_\perp = \sqrt{\frac{\mu}{r}}(\hat{\mathbf{k}} \times \hat{\mathbf{u}}_r) = \sqrt{\mu}\hat{\mathbf{k}} \times \left(\frac{1}{\sqrt{r}}\frac{\mathbf{r}}{r}\right) \tag{7.64}$$

Since, as is apparent from Fig. 7.12, $\mathbf{r} = \mathbf{R} + \delta\mathbf{r}$, we can write this expression for $\mathbf{v}_B$ as follows:

$$\mathbf{v}_B = \sqrt{\mu}\hat{\mathbf{k}} \times r^{-3/2}(\mathbf{R} + \delta\mathbf{r}) \tag{7.65}$$

Now

$$r^{-3/2} = \left(r^2\right)^{-3/4} = \left[\overbrace{R^2\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)}^{\text{See Eq.7.18}}\right]^{-3/4} = R^{-3/2}\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/4} \tag{7.66}$$

Using the binomial theorem (Eq. 5.44), and retaining terms at most linear in $\delta\mathbf{r}$, we find

$$\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/4} = 1 - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^2}$$

Substituting this into Eq. (7.66) leads to

$$r^{-3/2} = R^{-3/2} - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^{7/2}}$$

Upon substituting this result into Eq. (7.65) we get

$$\mathbf{v}_B = \sqrt{\mu}\hat{\mathbf{k}} \times (\mathbf{R} + \delta\mathbf{r})\left(R^{-3/2} - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^{7/2}}\right)$$

Retaining terms at most linear in $\delta\mathbf{r}$ we can write this as

$$\mathbf{v}_B = \hat{\mathbf{k}} \times \left\{\sqrt{\frac{\mu}{R}}\frac{\mathbf{R}}{R} + \frac{\sqrt{\mu/R}}{R}\delta\mathbf{r} - \frac{3}{2}\frac{\sqrt{\mu/R}}{R}\left[\left(\frac{\mathbf{R}}{R}\right)\cdot\delta\mathbf{r}\right]\frac{\mathbf{R}}{R}\right\}$$

Using Eqs. (7.61) and (7.62), together with the facts that $\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}$ and $\mathbf{R}/R = \hat{\mathbf{i}}$, this reduces to

$$\begin{aligned}\mathbf{v}_B &= \hat{\mathbf{k}} \times \left\{ v_A\hat{\mathbf{i}} + \frac{v_A}{R}\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right) - \frac{3}{2}\frac{v_A}{R}\left[\hat{\mathbf{i}}\cdot\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right)\hat{\mathbf{i}}\right] \right\} \\ &= v_A\hat{\mathbf{j}} + \left(-n\delta y\hat{\mathbf{i}} + n\delta x\hat{\mathbf{j}}\right) - \frac{3}{2}n\delta x\hat{\mathbf{j}}\end{aligned}$$

so that

$$\mathbf{v}_B = -n\delta y\hat{\mathbf{i}} + \left(v_A - \frac{1}{2}n\delta x\right)\hat{\mathbf{j}} \tag{7.67}$$

This is the absolute velocity of the chaser resolved into components in the target's CW frame.

Substituting Eq. (7.67) into Eq. (7.63) and using the fact that $\mathbf{v}_A = v_A\hat{\mathbf{j}}$ yields

$$\begin{aligned}\delta\mathbf{v} &= \left[-n\delta y\hat{\mathbf{i}} + \left(v_A - \frac{1}{2}n\delta x\right)\hat{\mathbf{j}}\right] - \left(v_A\hat{\mathbf{j}}\right) - \left(n_A\hat{\mathbf{k}}\right) \times \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right) \\ &= -n\delta y\hat{\mathbf{i}} + v_A\hat{\mathbf{j}} - \frac{1}{2}n\delta x\hat{\mathbf{j}} - v_A\hat{\mathbf{j}} - n\delta x\hat{\mathbf{j}} + n\delta y\hat{\mathbf{i}}\end{aligned}$$

so that

$$\delta\mathbf{v} = -\frac{3}{2}n\delta x\hat{\mathbf{j}} \tag{7.68}$$

This is the velocity of the chaser as measured in the moving reference frame of the neighboring target. Keep in mind that circular orbits were assumed at the outset.

In the Clohessy-Wiltshire frame, neighboring coplanar circular orbits appear to be straight lines parallel to the $y$ axis, which is the orbit of the origin. Fig. 7.13 illustrates this point, showing also the linear velocity variation according to Eq. (7.68).

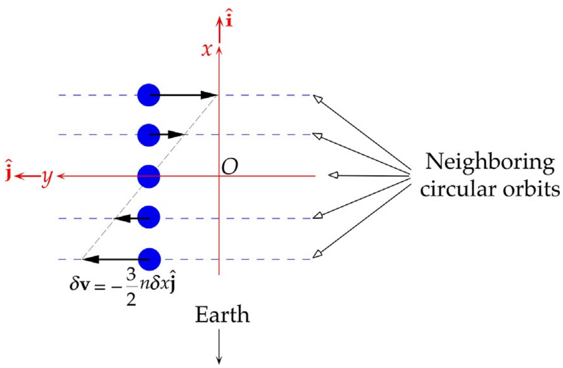


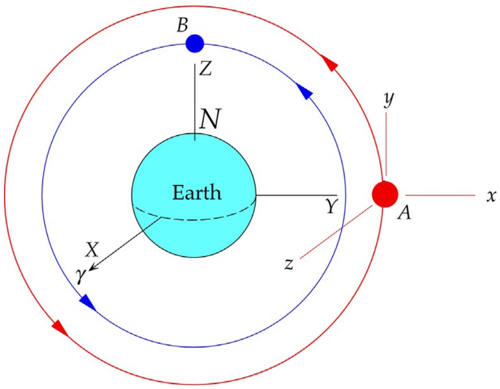

**FIG. 7.13**

Circular orbits, with relative velocity directions, in the vicinity of the Clohessy-Wiltshire frame.

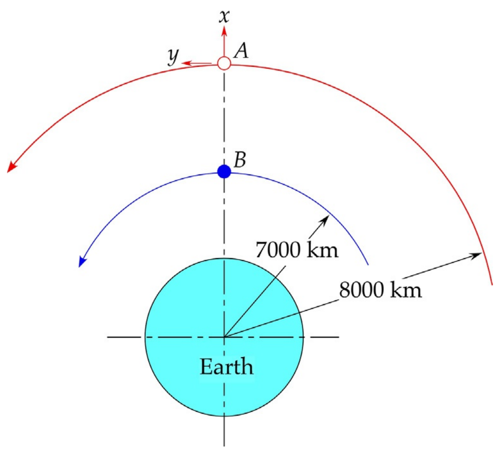
# PROBLEMS

## Section 7.3

**7.1** Two manned spacecraft, $A$ and $B$ (see the figure), are in circular polar ($i = 90°$) orbits around the earth. $A$'s orbital altitude is 300 km, $B$'s is 250 km. At the instant shown ($A$ over the equator, $B$ over the North Pole), calculate (a) the position, (b) velocity, and (c) the acceleration of $B$ relative to $A$. $A$'s $y$ axis points always in the flight direction and its $x$ axis is directed radially outward at all times.
{Ans.: (a) $\mathbf{r}_{\mathrm{rel}})_{xyz} = -6678\hat{\mathbf{i}} + 6628\hat{\mathbf{j}}(\mathrm{km})$; (b) $\mathbf{v}_{\mathrm{rel}})_{xyz} = -0.08693\hat{\mathbf{i}}(\mathrm{km/s})$;
(c) $\mathbf{a}_{\mathrm{rel}})_{xyz} = -1.140(10^{-6})\hat{\mathbf{j}}(\mathrm{km/s^2})$}




**7.2** Spacecraft $A$ and $B$ are in coplanar, circular geocentric orbits. The orbital radii are shown in the figure. When $B$ is directly below $A$, as shown, calculate $B$'s acceleration $\mathbf{a}_{\mathrm{rel}})_{xyz}$ relative to $A$.
{Ans.: $\mathbf{a}_{\mathrm{rel}})_{xyz} = -0.268\hat{\mathbf{i}}(\mathrm{m/s^2})$}



## Section 7.3

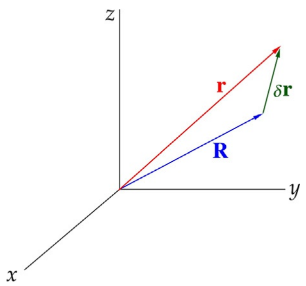
**7.3** Use the order-of-magnitude analysis in this chapter as a guide to answer the following questions:

**(a)** If $\mathbf{r} = \mathbf{R} + \delta\mathbf{r}$, express $\sqrt{r}$ (where $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$) to the first order in $\delta\mathbf{r}$ (i.e., to the first order in the components of $\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}$). In other words, find $O(\delta\mathbf{r})$, such that $\sqrt{r} = \sqrt{R} + O(\delta\mathbf{r})$, where $O(\delta\mathbf{r})$ is linear in $\delta\mathbf{r}$.

**(b)** For the special case $\mathbf{R} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\delta\mathbf{r} = 0.01\hat{\mathbf{i}} - 0.01\hat{\mathbf{j}} + 0.03\hat{\mathbf{k}}$, calculate $\sqrt{r} - \sqrt{R}$ and compare that result with $O(\delta\mathbf{r})$.

**(c)** Repeat Part (b) using $\delta\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and compare the results.

{Ans.: (a) $O(\delta\mathbf{r}) = \mathbf{R}\cdot\delta\mathbf{r}/(2R^{3/2})$; (b) $O(\delta\mathbf{r})/\left(\sqrt{r} - \sqrt{R}\right) = 0.998$ ;
(c) $O(\delta\mathbf{r})/\left(\sqrt{r} - \sqrt{R}\right) = 0.903$}




**7.4** Write the expression $r = \frac{a(1-e^2)}{1+e\cos\theta}$ as a linear function of $e$, valid for small values of $e$ ($e \ll 1$).

## Section 7.4

**7.5** Given $\ddot{x} + 9x = 10$, with the initial conditions $x = 5$ and $\dot{x} = -3$ at $t = 0$, find $x$ and $\dot{x}$ at $t = 1.2$.
{Ans.: $x(1.2) = -1.934$, $\dot{x}(1.2) = 7.853$}

**7.6** Given that

$$\ddot{x} + 10x + 2\dot{y} = 0$$
$$\ddot{y} + 3\dot{x} = 0$$

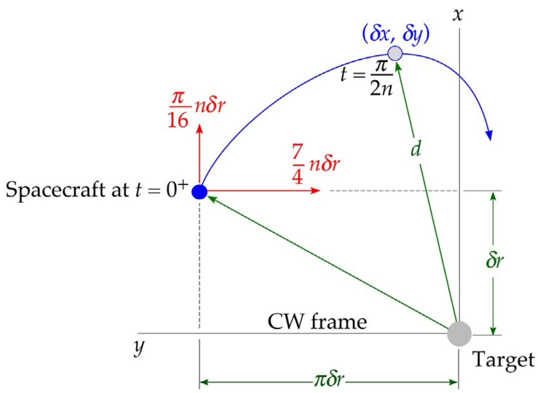
with initial conditions $x(0) = 1$, $y(0) = 2$, $\dot{x}(0) = -3$, and $\dot{y}(0) = 4$, find $x$ and $y$ at $t = 5$.
{Ans.: $x(5) = -6.460$, $y(5) = 97.31$}

**7.7** A space station is in an earth orbit with a 90-min period. At $t = 0$ a satellite has the following position and velocity components relative to a CW frame attached to the space station: $\delta\mathbf{r} = 1\hat{\mathbf{i}}(\text{km})$, $\delta\mathbf{v} = 10\hat{\mathbf{j}}(\text{m/s})$. How far is the satellite from the space station 15 min later?
{Ans.: 11.2 km}

**7.8** Spacecraft $A$ and $B$ are in the same circular earth orbit with a period of 2 h. $B$ is 6 km ahead of $A$. At $t = 0$, $B$ applies an in-track delta-v (retrofire) of 3 m/s. Using a CW frame attached to $A$, determine the distance between $A$ and $B$ at $t = 30$ min and the velocity of $B$ relative to $A$ at that instant.
{Ans.: $\delta r = 10.9$ km, $\delta v = 10.8$ m/s}

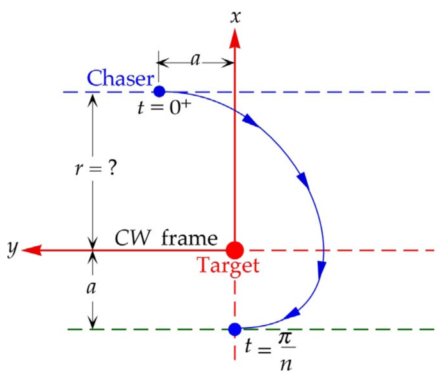
**7.9** The CW coordinates and velocities of a spacecraft upon entering a rendezvous trajectory with the target vehicle are shown. The spacecraft orbits are coplanar. Calculate the distance $d$ of the

spacecraft from the target when $t = \pi/(2n)$, where $n$ is the mean motion of the target's circular orbit.
{Ans.: $0.900\delta r$}




**7.10** At time $t = 0$ a particle is at the origin of a CW frame with a relative velocity $\delta\mathbf{v}_0 = v\hat{\mathbf{j}}$. What will be the relative speed of the particle after a time equal to one-half the orbital period of the CW frame?
{Ans.: $7v$}

**7.11** The chaser and the target are in close-proximity, coplanar circular orbits. At $t = 0$ the position of the chaser relative to the target is $\delta\mathbf{r}_0 = r\hat{\mathbf{i}} + a\hat{\mathbf{j}}$, where $a$ is given and $r$ is unknown. The relative velocity at $t = 0^+$ is $\delta\mathbf{v}_0^+ = v_0\hat{\mathbf{j}}$ ($v_0$ is unknown), and the chaser ends up at $\delta\mathbf{r}_f = -a\hat{\mathbf{i}}$ when $t = \pi/n$, where $n$ is the mean motion of the target. Use the Clohessy-Wiltshire equations to find the required value of the orbital spacing $r$.
{Ans.: $1.424a$}




**Section 7.5**

**7.12** A space station is in a circular earth orbit of radius 6600 km. An approaching spacecraft executes a delta-v burn when its position vector relative to the space station is $\delta\mathbf{r}_0 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$(km). Just before the burn the relative velocity of the spacecraft was $\delta\mathbf{v}_0^- = 5$(m/s). Calculate the total

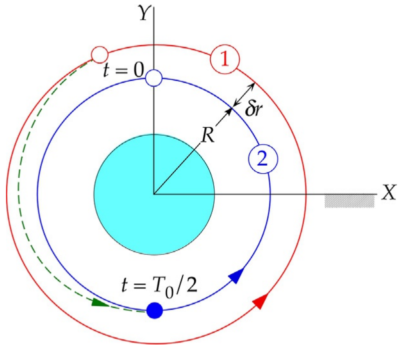
delta-v required for the spacecraft to rendezvous with the station in a time equal to one-third period of the space station's orbit.
{Ans.: 6.21 m/s}

**7.13** A space station is in circular orbit 2 of radius $R$. A spacecraft is in coplanar circular orbit 1 of radius $R + \delta r$. At $t = 0$ the spacecraft executes an impulsive maneuver to rendezvous with the space station at time $t_f =$ one-half the period $T_0$ of the space station. If $\delta u_0^+ = 0$ find

**(a)** The initial position of the spacecraft relative to the space station.

**(b)** The relative velocity of the spacecraft when it arrives at the target. Sketch the rendezvous trajectory relative to the target.

{Ans.: (a) $\delta\mathbf{r}_0 = \delta r\hat{\mathbf{i}} + (3\pi\delta r/4)\hat{\mathbf{j}}$; (b) $\delta\mathbf{v}_f^- = \pi\delta r/(2T_0)\hat{\mathbf{j}}$}

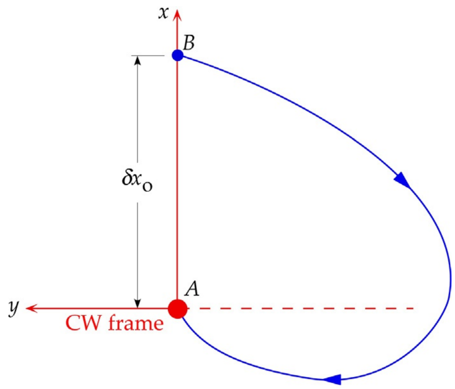



**7.14** If $\delta u_0^+ = 0$, calculate the total delta-v required for rendezvous if $\delta\mathbf{r}_0 = \delta y_0\hat{\mathbf{j}}$, $\delta\mathbf{v}_0^- = \mathbf{0}$, and $t_f =$ the period $T$ of the circular target orbit. Sketch the rendezvous trajectory relative to the target.
{Ans.: $\Delta v_{tot} = 2\delta y_o/(3T)$}

**7.15** A GEO satellite strikes some orbiting debris and is found 2 h afterward to have drifted to the position $\delta\mathbf{r} = -10\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$(km) relative to its original location. At that time the only slightly damaged satellite initiates a two-impulse maneuver to return to its original location in 6 h. Find the total delta-v for this maneuver.
{Ans.: 3.5 m/s}

**7.16** A space station is in a 245-km circular earth orbit inclined at 30°. The right ascension of its node line is 40°. Meanwhile, a spacecraft has been launched into a 280 km by 250 km orbit inclined at 30.1°, with a nodal right ascension of 40° and argument of perigee equal to 60°. When the spacecraft's true anomaly is 40° the space station is 99° beyond its node line. At that instant the spacecraft executes a delta-v burn to rendezvous with the space station in (precisely) $t_f$ hours, where $t_f$ is either selected by you or assigned by the instructor. Calculate the total delta-v required and sketch the projection of the rendezvous trajectory on the $xy$ plane of the space station coordinates.

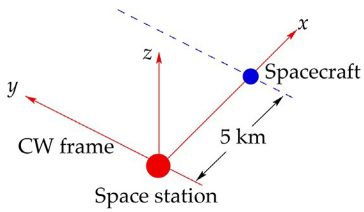
**7.17** The target $A$ is in a circular earth orbit with mean motion $n$. The chaser $B$ is directly above $A$ in a slightly larger circular orbit having the same plane as $A$. What relative initial velocity $\delta\mathbf{v}_0^+$ is required so that $B$ arrives at target $A$ at time $t_f =$ one-half the target's period?
{Ans.: $\delta\mathbf{v}_0^+ = -0.589n\delta x_0\hat{\mathbf{i}} - 1.75n\delta x_0$}




**Section 7.6**

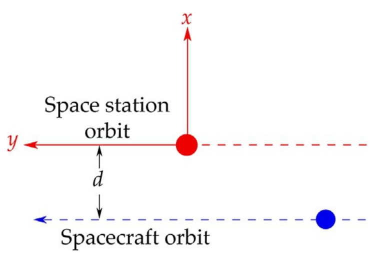
**7.18** The space station is in a circular earth orbit of radius 6600 km. Another spacecraft is also in a circular orbit in the same plane as the space station. At the instant that the position of the spacecraft relative to the space station, in Clohessy-Wiltshire coordinates, is $\delta\mathbf{r} = 5\hat{\mathbf{i}}$(km), what is the relative velocity $\delta\mathbf{v}$ of the spacecraft in meters/s?

{Ans.: 8.83 m/s}



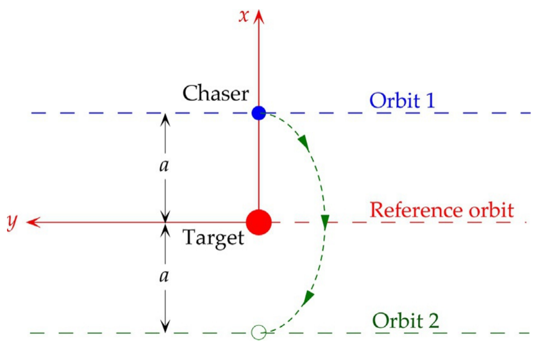

**7.19** A spacecraft and the space station are in coplanar circular orbits. The space station has an orbital radius $R$ and a mean motion $n$. The spacecraft's radius is $R - d$ ($d \ll R$). If a two-impulse rendezvous maneuver with $t_f = \pi/(4n)$ is initiated with zero relative velocity in the $x$ direction ($\delta u_0^+ = 0$), calculate the total delta-v.

{Ans.: $4.406nd$}




**7.20** The chaser and the target are in close-proximity, coplanar circular orbits. At $t = 0$ the position of the chaser relative to the target is $\delta\mathbf{r}_0 = a\hat{\mathbf{i}}$. Use the CW equations to find the total delta-v required

for the chaser to end up in circular orbit 2 at $\delta\mathbf{r}_f = -a\hat{\mathbf{i}}$ when $t = \pi/n$, where $n$ is the mean motion of the target.
{Ans.: $na$}




## REFERENCE

Clohessy, W.H., Wiltshire, R.S., 1960. Terminal guidance system for satellite rendezvous. J. Aerosol Sci. 27, 653–658.