1)a) $\left[\begin{array}{cc|c} -1 & 1 & 4 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{r_2 = r_2 - (-3)r_1} \left[\begin{array}{cc|c} -1 & 1 & 4 \\ 0 & 2 & 12 \end{array}\right] \Rightarrow$

$2y = 12, \; y = 6 \Rightarrow -x + 6 = 4 \Rightarrow x = 2$

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b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 3 & 1 & 4 \\ 2 & -1 & 1 & 2 \end{array}\right] \xrightarrow{r_3 = r_3 - (2)r_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & -5 & 3 & -2 \end{array}\right] \xrightarrow{r_3 = r_3 - \left(-\frac{5}{3}\right)r_2}$

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 3+\frac{5}{3} & \frac{20}{3}-2 \end{array}\right] \Rightarrow \left(3+\frac{5}{3}\right)z = \left(\frac{20}{3}-2\right) \Rightarrow 3y + 1 = 4 \Rightarrow x + 2 - 1 = 2$

$z = 1 \qquad y = 1 \qquad x = 1$

2)a) $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{r_3 - (1)r_1} \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 1^* & 0^* & 3 \end{bmatrix} \xrightarrow{r_2 - (2)r_1} \begin{bmatrix} 3 & 1 & 2 \\ 2^* & 1 & 0 \\ 1^* & 0^* & 3 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0^* & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix} \xrightarrow{r_3 - (1)r_1} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0^* & 2 & 1 & 0 \\ 1^* & 4 & 3 & 2 \\ 0^* & 2 & 1 & -1 \end{bmatrix} \xrightarrow{r_3 - (2)r_2} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0^* & 2 & 1 & 0 \\ 1^* & 2^* & 1 & 2 \\ 0^* & 2 & 1 & -1 \end{bmatrix} \xrightarrow{r_4 - (1)r_2}$

$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0^* & 2 & 1 & 0 \\ 1^* & 2^* & 1 & 2 \\ 0^* & 1^* & 0^* & -1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

3.

3) a)

$$\underline{L}\,\underline{y}=\underline{b}=\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{bmatrix}1&&\\2&1&\\1&0&1\end{bmatrix}\underline{y}=\begin{pmatrix}y_1\\2y_1+y_2\\y_1+\quad y_3\end{pmatrix}\Rightarrow\underline{y}=\begin{pmatrix}0\\1\\3\end{pmatrix}$$

$$\underline{U}x=\underline{y}=\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{bmatrix}3&1&2\\0&1&0\\0&0&3\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3x_1+x_2+2x_3\\+x_2\\+3x_3\end{pmatrix}=\begin{pmatrix}0\\1\\3\end{pmatrix}\Rightarrow\underline{x}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

b.)

$$\underline{L}\,\underline{y}=\underline{b}=\begin{pmatrix}5\\0\\9\\-1\end{pmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&2&1&0\\0&1&0&1\end{bmatrix}\underline{y}=\begin{pmatrix}y_1\\y_2\\y_1+2y_2+y_3\\y_2+y_4\end{pmatrix}\Rightarrow\underline{y}=\begin{pmatrix}5\\0\\4\\-1\end{pmatrix}$$

$$\underline{U}\,\underline{x}=\underline{y}=\begin{pmatrix}5\\0\\4\\-1\end{pmatrix}=\begin{bmatrix}1&-1&1&2\\0&2&1&0\\0&0&1&2\\0&0&0&-1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}2x_4+x_3-x_2+x_1\\2x_2+x_3\\x_3+2x_4\\-x_4\end{pmatrix}\Rightarrow\begin{pmatrix}0\\-1\\2\\1\end{pmatrix}=\underline{x}$$

Problem 3 $\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = \left(|X_1|^p + |X_2|^p + \dots + |X_n|^p\right)^{1/p}$

$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$

Because the largest term will be infinitely larger than the next largest, when the square root is taken, only the largest will matter.

Proof

$\|x\|_p = x_n \left(\sum_{k=1}^{n} \left(\frac{x_k}{x_n}\right)^p\right)^{\frac{1}{p}} = x_n \left(1 + \sum_{k=1}^{n-1} \left(\frac{x_k}{x_n}\right)^p\right)^{\frac{1}{p}}$

$0 \leq \frac{x_k}{x_n} \leq 1$

$x_n = x_n \cdot 1^{1/p} \leq \|x\|_p \leq x_n (n \cdot 1)^{1/p} = x_n \cdot n^{1/p}$

as $p \to \infty$

by squeeze theorem, $\lim_{p\to\infty} \|x\|_p = x_n$

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imagine $V = [1, 2, 3, 3]$

$\|V\|_1 = (1^1 + 2^1 + 3^1 + 3^1)^1 = (9)^1 = 9$

$\|V\|_2 = (1^2 + 2^2 + 3^2 + 3^2)^{\frac{1}{2}} = (29)^{1/2} = 4.7958$

$\|V\|_{10} = (1^{10} + 2^{10} + 3^{10} + 3^{10})^{1/10} = (1025 + 118098)^{\frac{1}{10}} = 3.2181$

$\|V\|_{100} = (1.3e^{30} + 1.03e^{48})^{\frac{1}{100}} = (1.03e^{48})^{\frac{1}{100}} = 3.0209$

$(2.06e^{48})^{\frac{1}{100}} = 3.0419$

$(5e^{47})^{\frac{1}{100}} = 2.999$

any small factors the largest value (3) is multiplied by are brought to negligible amounts by around $p = 100$. they are further reduced as $p \to \infty$

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2) a) $X = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $\|X\|_2 = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

$\|X\|_1 = 1 + |-2| + 2 = 5$

$\|X\|_\infty = 2$

b) $X = \begin{pmatrix} 5 \\ 0 \\ 3 \\ -1 \end{pmatrix}$ $\|X\|_2 = 5.9161$

$\|X\|_1 = 9$

$\|X\|_\infty = 5$