CHAPTER

DYNAMICS OF POINT MASSES 1

1.1 INTRODUCTION

This chapter serves as a self-contained reference on the kinematics and dynamics of point masses as well as some basic vector operations and numerical integration methods. The notation and concepts summarized here will be used in the following chapters. Those familiar with the vector-based dynamics of particles can simply page through the chapter and then refer back to it later as necessary. Those who need a bit more in the way of review will find that the chapter contains all the material they need to follow the development of orbital mechanics topics in the upcoming chapters.

We begin with a review of vectors and some vector operations, after which we proceed to the problem of describing the curvilinear motion of particles in three dimensions. The concepts of force and mass are considered next, along with Newton's inverse-square law of gravitation. This is followed by a presentation of Newton's second law of motion ("force equals mass times acceleration") and the important concept of angular momentum.

As a prelude to describing motion relative to moving frames of reference, we develop formulas for calculating the time derivatives of moving vectors. These are applied to the computation of relative velocity and acceleration. Example problems illustrate the use of these results, as does a detailed consideration of how the earth's rotation and curvature influence our measurements of velocity and acceleration. This brings in the curious concept of Coriolis force. Embedded in exercises at the end of the chapter is practice in verifying several fundamental vector identities that will be employed frequently throughout the book.

The chapter concludes with an introduction to numerical methods, which can be called upon to solve the equations of motion when an analytical solution is not possible.

1.2 VECTORS

A vector is an object that is specified by both a magnitude and a direction. We represent a vector graphically by a directed line segment (i.e., an arrow pointing in the direction of the vector). The end opposite the arrow is called the tail. The length of the arrow is proportional to the magnitude of the vector. Velocity is a good example of a vector. We say that a car is traveling eastward at 80 km/h. The direction is east and the magnitude, or speed, is 80 km/h. We will use boldface type to represent vector quantities and plain type to denote scalars. Thus, whereas B is a scalar, $\mathbf{B}$ is a vector.

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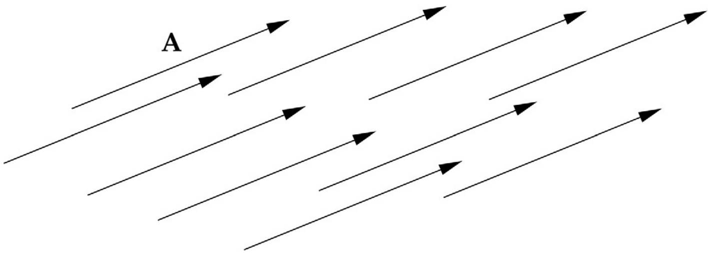


FIG. 1.1

All of these vectors may be denoted **A**, since their magnitudes and directions are the same.

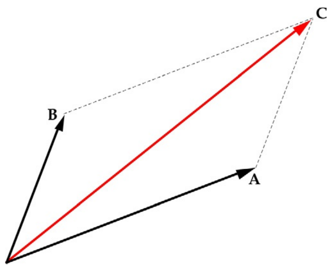


FIG. 1.2

Parallelogram rule of vector addition. $\mathbf{A} + \mathbf{B} = \mathbf{C}$.

Observe that a vector is specified solely by its magnitude and direction. If **A** is a vector, then all vectors having the same physical dimensions, the same length, and pointing in the same direction as **A** are denoted **A**, regardless of their line of action, as illustrated in Fig. 1.1. Shifting a vector parallel to itself does not mathematically change the vector. However, the parallel shift of a vector might produce a different physical effect. For example, an upward 5-kN load (force vector) applied to the tip of an airplane wing gives rise to quite a different stress and deflection pattern in the wing than the same load acting at the wing's midspan.

The magnitude of a vector **A** is denoted $\|\mathbf{A}\|$, or, simply A.

Multiplying a vector **B** by the reciprocal of its magnitude produces a vector that points in the direction of **B**, but it is dimensionless and has a magnitude of one. Vectors having dimensionless magnitude are called unit vectors. We put a hat (ˆ) over the letter representing a unit vector. Then we can tell simply by inspection that, for example, $\hat{\mathbf{u}}$ is a unit vector, as are $\hat{\mathbf{B}}$ and $\hat{\mathbf{e}}$.

It is convenient to denote the unit vector in the direction of the vector **A** as $\hat{\mathbf{u}}_A$. As pointed out above, we obtain this vector from **A** as follows:

$$\hat{\mathbf{u}}_A = \frac{\mathbf{A}}{A} \tag{1.1}$$

Likewise, $\hat{\mathbf{u}}_C = \mathbf{C}/C$, $\hat{\mathbf{u}}_F = \mathbf{F}/F$, etc.

The sum or *resultant* of two vectors is defined by the parallelogram rule (Fig. 1.2). Let **C** be the sum of the two vectors **A** and **B**. To form that sum using the parallelogram rule, the vectors **A** and **B** are

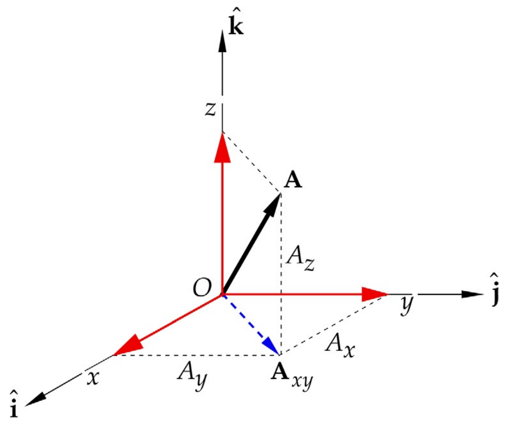


FIG. 1.3

Three-dimensional, right-handed Cartesian coordinate system.

shifted parallel to themselves (leaving them unaltered) until the tail of **A** touches the tail of **B**. Drawing dotted lines through the head of each vector parallel to the other completes a parallelogram. The diagonal from the tails of **A** and **B** to the opposite corner is the resultant **C**. By construction, vector addition is commutative; that is,

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \tag{1.2}$$

A Cartesian coordinate system in three dimensions consists of three mutually perpendicular axes, labeled x, y, and z, which intersect at the origin O. We will always use a right-handed Cartesian coordinate system, which means if you wrap the fingers of your right hand around the z axis, with the thumb pointing in the positive z direction, your fingers will be directed from the x axis toward the y axis. Fig. 1.3 illustrates such a system. Note that the unit vectors along the x, y, and z axes are, respectively, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

In terms of its Cartesian components, and in accordance with the above summation rule, a vector **A** is written in terms of its components A_x, A_y, and A_z as

$$\mathbf{A}=A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}+A_z\hat{\mathbf{k}} \tag{1.3}$$

The projection of **A** on the xy plane is a vector denoted $\mathbf{A}_{xy}$. It follows that

$$\mathbf{A}_{xy}=A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}$$

According to the Pythagorean theorem, the magnitude of **A** in terms of its Cartesian components is

$$\mathbf{A}=\sqrt{A_x^2+A_y^2+A_z^2} \tag{1.4}$$

From Eqs. (1.1) and (1.3), the unit vector in the direction of **A** is

$$\hat{\mathbf{u}}_{\mathrm{A}}=\cos\theta_x\hat{\mathbf{i}}+\cos\theta_y\hat{\mathbf{j}}+\cos\theta_z\hat{\mathbf{k}} \tag{1.5}$$

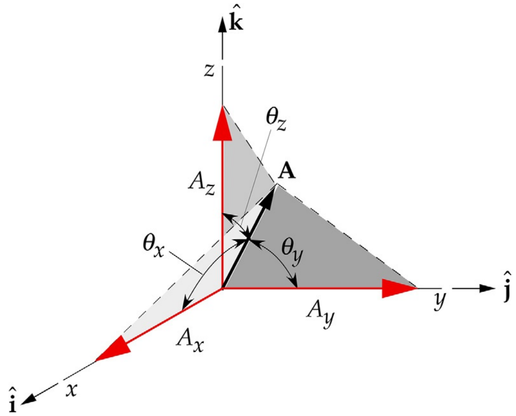


FIG. 1.4

Direction angles in three dimensions.

where

$$\cos\theta_x = \frac{A_x}{A} \qquad \cos\theta_y = \frac{A_y}{A} \qquad \cos\theta_z = \frac{A_z}{A} \tag{1.6}$$

The direction angles θ_x, θ_y, and θ_z are illustrated in Fig. 1.4, and they are measured between the vector and the positive coordinate axes. Note carefully that the sum of θ_x, θ_y, and θ_z is not in general known a priori and cannot be assumed to be, say, 180 degrees.

EXAMPLE 1.1

Calculate the direction angles of the vector $\mathbf{A} = \hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$.

Solution

First, compute the magnitude of **A** by means of Eq. (1.4),

$$\mathbf{A} = \sqrt{1^2 + (-4)^2 + 8^2} = 9$$

Then Eq. (1.6) yields

$$\theta_x = \cos^{-1}\left(\frac{A_x}{A}\right) = \cos^{-1}\left(\frac{1}{9}\right) \Rightarrow \boxed{\theta_x = 83.62\ \text{degrees}}$$

$$\theta_y = \cos^{-1}\left(\frac{A_y}{A}\right) = \cos^{-1}\left(\frac{-4}{9}\right) \Rightarrow \boxed{\theta_y = 116.4\ \text{degrees}}$$

$$\theta_z = \cos^{-1}\left(\frac{A_z}{A}\right) = \cos^{-1}\left(\frac{8}{9}\right) \Rightarrow \boxed{\theta_z = 27.27\ \text{degrees}}$$

Observe that $\theta_x + \theta_y + \theta_z = 227.3$ degrees.

Multiplication and division of two vectors are undefined operations. There are no rules for computing the product **AB** and the ratio **A/B**. However, there are two well-known binary operations on vectors: the *dot product* and the *cross product*. The dot product of two vectors is a scalar defined as follows:

$$\mathbf{A}\cdot\mathbf{B}=AB\cos\theta \tag{1.7}$$

where θ is the angle between the heads of the two vectors, as shown in Fig. 1.5. Clearly,

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A} \tag{1.8}$$

If two vectors are perpendicular to each other, then the angle between them is 90 degrees. It follows from Eq. (1.7) that their dot product is zero. Since the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ of a Cartesian coordinate system are mutually orthogonal and of magnitude 1, Eq. (1.7) implies that

$$\begin{aligned}\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=1\\ \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{k}}=0\end{aligned} \tag{1.9}$$

Using these properties, it is easy to show that the dot product of the vectors **A** and **B** may be found in terms of their Cartesian components as

$$\mathbf{A}\cdot\mathbf{B}=A_xB_x+A_yB_y+A_zB_z \tag{1.10}$$

If we set $\mathbf{B}=\mathbf{A}$, then it follows from Eqs. (1.4) and (1.10) that

$$A=\sqrt{\mathbf{A}\cdot\mathbf{A}} \tag{1.11}$$

The dot product operation is used to project one vector onto the line of action of another. We can imagine bringing the vectors tail to tail for this operation, as illustrated in Fig. 1.6. If we drop a perpendicular line from the tip of **B** onto the direction of **A**, then the line segment B_A is the orthogonal projection of **B** onto the line of action of **A**. B_A stands for the scalar projection of **B** onto **A**. From trigonometry, it is obvious from the figure that

$$B_A=B\cos\theta$$

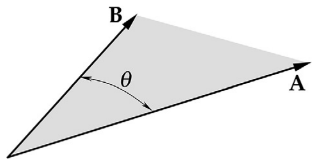


FIG. 1.5

The angle between two vectors brought tail to tail by parallel shift.

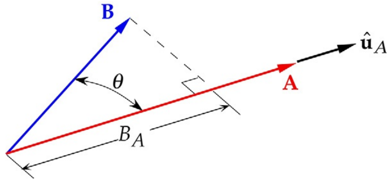


FIG. 1.6

Projecting the vector **B** onto the direction of **A**.

Let $\hat{\mathbf{u}}_A$ be the unit vector in the direction of $\mathbf{A}$. Then,

$$\mathbf{B}\cdot\hat{\mathbf{u}}_A = \|\mathbf{B}\|\overbrace{\|\hat{\mathbf{u}}_A\|}^{=1}\cos\theta = B\cos\theta$$

Comparing this expression with the preceding one leads to the conclusion that

$$B_A = \mathbf{B}\cdot\hat{\mathbf{u}}_A = \mathbf{B}\cdot\frac{\mathbf{A}}{A} \tag{1.12}$$

where $\hat{\mathbf{u}}_A$ is given by Eq. (1.1). Likewise, the projection of $\mathbf{A}$ onto $\mathbf{B}$ is given by

$$A_B = \mathbf{A}\cdot\frac{\mathbf{B}}{B}$$

Observe that $A_B = B_A$ only if $\mathbf{A}$ and $\mathbf{B}$ have the same magnitude.

EXAMPLE 1.2

Let $\mathbf{A} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 18\hat{\mathbf{k}}$ and $\mathbf{B} = 42\hat{\mathbf{i}} - 69\hat{\mathbf{j}} + 98\hat{\mathbf{k}}$. Calculate

(a) the angle between $\mathbf{A}$ and $\mathbf{B}$;
(b) the projection of $\mathbf{B}$ in the direction of $\mathbf{A}$;
(c) the projection of $\mathbf{A}$ in the direction of $\mathbf{B}$.

Solution

First, we make the following individual calculations.

$$\mathbf{A}\cdot\mathbf{B} = (1)(42) + (6)(-69) + (18)(98) = 1392 \tag{a}$$

$$A = \sqrt{(1)^2 + (6)^2 + (18)^2} = 19 \tag{b}$$

$$B = \sqrt{(42)^2 + (-69)^2 + (98)^2} = 127 \tag{c}$$

(a) According to Eq. (1.7), the angle between $\mathbf{A}$ and $\mathbf{B}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

Substituting Eqs. (a), (b), and (c) yields

$$\theta = \cos^{-1}\left(\frac{1392}{19\cdot 127}\right) = \boxed{54.77\text{ degrees}}$$

(b) From Eq. (1.12), we find the projection of $\mathbf{B}$ onto $\mathbf{A}$.

$$B_A = \mathbf{B}\cdot\frac{\mathbf{A}}{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{A}$$

Substituting Eqs. (a) and (b) we get

$$B_A = \frac{1392}{19} = \boxed{73.26}$$

(c) The projection of $\mathbf{A}$ onto $\mathbf{B}$ is

$$A_B = \mathbf{A}\cdot\frac{\mathbf{B}}{B} = \frac{\mathbf{A}\cdot\mathbf{B}}{B}$$

Substituting Eqs. (a) and (c) we obtain

$$A_B = \frac{1392}{127} = \boxed{10.96}$$

The cross product of two vectors yields another vector, which is computed as follows:

$$\mathbf{A} \times \mathbf{B} = (AB \sin\theta)\hat{\mathbf{n}}_{AB} \tag{1.13}$$

where θ is the angle between the heads of **A** and **B**, and $\hat{\mathbf{n}}_{AB}$ is the unit vector normal to the plane defined by the two vectors. The direction of $\hat{\mathbf{n}}_{AB}$ is determined by the right-hand rule. That is, curl the fingers of the right hand from the first vector (**A**) toward the second vector (**B**), and the thumb shows the direction of $\hat{\mathbf{n}}_{AB}$ (Fig. 1.7). If we use Eq. (1.13) to compute **B** × **A**, then $\hat{\mathbf{n}}_{AB}$ points in the opposite direction, which means

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}) \tag{1.14}$$

Therefore, unlike the dot product, the cross product is not commutative.

The cross product is obtained analytically by resolving the vectors into Cartesian components.

$$\mathbf{A} \times \mathbf{B} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) \times \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right) \tag{1.15}$$

Since the set $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ is a mutually perpendicular triad of unit vectors, Eq. (1.13) implies that

$$\begin{array}{lll} \hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0 & \hat{\mathbf{j}} \times \hat{\mathbf{j}} = 0 & \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} & \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} & \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \end{array} \tag{1.16}$$

Expanding the right-hand side of Eq. (1.15), substituting Eq. (1.16), and making use of Eq. (1.14) leads to

$$\mathbf{A} \times \mathbf{B} = \left(A_yB_z - A_zB_y\right)\hat{\mathbf{i}} - \left(A_xB_z - A_zB_x\right)\hat{\mathbf{j}} + \left(A_xB_y - A_yB_x\right)\hat{\mathbf{k}} \tag{1.17}$$

It may be seen that the right-hand side is the determinant of the matrix

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

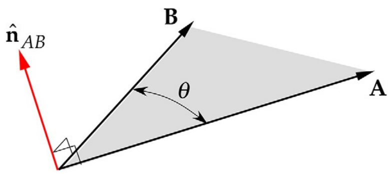


FIG. 1.7

$\hat{\mathbf{n}}_{AB}$ is normal to both **A** and **B** and defines the direction of the cross product **A** × **B**.

Thus, Eq. (1.17), can be written as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \tag{1.18}$$

where the two vertical bars stand for the determinant. Obviously, the rule for computing the cross product, though straightforward, is a bit lengthier than that for the dot product. Remember that the dot product yields a scalar whereas the cross product yields a vector.

The cross product provides an easy way to compute the normal to a plane. Let **A** and **B** be any two vectors lying in the plane, or, let any two vectors be brought tail to tail to define a plane, as shown in Fig. 1.7. The vector $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ is normal to the plane of **A** and **B**. Therefore, $\hat{\mathbf{n}}_{AB}=\mathbf{C}/C$, or

$$\hat{\mathbf{n}}_{AB}=\frac{\mathbf{A}\times\mathbf{B}}{\|\mathbf{A}\times\mathbf{B}\|} \tag{1.19}$$

EXAMPLE 1.3

Let $\mathbf{A}=-3\hat{\mathbf{i}}+7\hat{\mathbf{j}}+9\hat{\mathbf{k}}$ and $\mathbf{B}=6\hat{\mathbf{i}}-5\hat{\mathbf{j}}+8\hat{\mathbf{k}}$. Find a unit vector that lies in the plane of **A** and **B** and is perpendicular to **A**.

Solution

The plane of vectors **A** and **B** is determined by parallel-shifting the vectors so that they meet tail to tail. Calculate the vector $\mathbf{D}=\mathbf{A}\times\mathbf{B}$.

$$\mathbf{D}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -3 & 7 & 9\\ 6 & -5 & 8\end{vmatrix}=101\hat{\mathbf{i}}+78\hat{\mathbf{j}}-27\hat{\mathbf{k}}$$

Note that **A** and **B** are both normal to **D**. We next calculate the vector $\mathbf{C}=\mathbf{D}\times\mathbf{A}$.

$$\mathbf{C}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 101 & 78 & -27\\ -3 & 7 & 9\end{vmatrix}=891\hat{\mathbf{i}}-828\hat{\mathbf{j}}+941\hat{\mathbf{k}}$$

C is normal to **D** as well as to **A**. **A**, **B**, and **C** are all perpendicular to **D**. Therefore, they are coplanar. Thus, **C** is not only perpendicular to **A**, but it also lies in the plane of **A** and **B**. Therefore, the unit vector we are seeking is the unit vector in the direction of **C**. That is

$$\hat{\mathbf{u}}_C=\frac{\mathbf{C}}{C}=\frac{891\hat{\mathbf{i}}-828\hat{\mathbf{j}}+941\hat{\mathbf{k}}}{\sqrt{891^2+(-828)^2+941^2}}$$

$$\boxed{\hat{\mathbf{u}}_C=0.5794\hat{\mathbf{i}}-0.5384\hat{\mathbf{j}}+0.6119\hat{\mathbf{k}}}$$

In the chapters to follow, we will often encounter the vector triple product, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. By resolving **A**, **B**, and **C** into their Cartesian components, it can easily be shown that the vector triple product can be expressed in terms of just the dot products of these vectors as follows:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{1.20}$$

Because of the appearance of the letters on the right-hand side, this is often referred to as the "bac–cab rule."

EXAMPLE 1.4

If $\mathbf{F} = \mathbf{E} \times \{\mathbf{D} \times [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]\}$, use the bac–cab rule to reduce this expression to one involving only dot products.

Solution

First, we invoke the bac–cab rule to obtain

$$\mathbf{F} = \mathbf{E} \times \left\{ \mathbf{D} \times \overbrace{[\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})]}^{\text{bac}-\text{cab rule}} \right\}$$

Expanding and collecting terms leads to

$$\mathbf{F} = (\mathbf{A} \cdot \mathbf{C})[\mathbf{E} \times (\mathbf{D} \times \mathbf{B})] - (\mathbf{A} \cdot \mathbf{B})[\mathbf{E} \times (\mathbf{D} \times \mathbf{C})]$$

We next apply the bac–cab rule twice on the right-hand side.

$$\mathbf{F} = (\mathbf{A} \cdot \mathbf{C})\overbrace{[\mathbf{D}(\mathbf{E} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{E} \cdot \mathbf{D})]}^{\text{bac}-\text{cab rule}} - (\mathbf{A} \cdot \mathbf{B})\overbrace{[\mathbf{D}(\mathbf{E} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{E} \cdot \mathbf{D})]}^{\text{bac}-\text{cab rule}}$$

Expanding and collecting terms yields the sought-for result.

$$\boxed{\mathbf{F} = [(\mathbf{A} \cdot \mathbf{C})(\mathbf{E} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{C})]\mathbf{D} - (\mathbf{A} \cdot \mathbf{C})(\mathbf{E} \cdot \mathbf{D})\mathbf{B} + (\mathbf{A} \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{D})\mathbf{C}}$$

Another useful vector identity is the "interchange of the dot and the cross":

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \tag{1.21}$$

It is so-named because interchanging the operations in the expression $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ yields $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$. The parentheses in Eq. (1.21) are required to show which operation must be carried out first, according to the rules of vector algebra. (For example, $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$, the cross product of a scalar and a vector, is undefined.) It is easy to verify Eq. (1.21) by substituting $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$, $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$, and $\mathbf{C} = C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}$ and observing that both sides of the equal sign reduce to the same expression.

1.3 KINEMATICS

To track the motion of a particle P through Euclidean space, we need a frame of reference, consisting of a clock and a nonrotating Cartesian coordinate system. The clock keeps track of time t, and the xyz axes of the Cartesian coordinate system are used to locate the spatial position of the particle. In nonrelativistic mechanics, a single "universal" clock serves for all possible Cartesian coordinate systems. So when we refer to a frame of reference, we need to think only of the mutually orthogonal axes themselves.

The unit of time used throughout this book is the second (s). The unit of length is the meter (m), but the kilometer (km) will be the length unit of choice when large distances and velocities are involved. Conversion factors between kilometers, miles, and nautical miles are listed in Table A.3.

Given a frame of reference, the position of the particle P at a time t is defined by the position vector $\mathbf{r}(t)$ extending from the origin O of the frame out to P itself, as illustrated in Fig. 1.8. The components of $\mathbf{r}(t)$ are just the x, y, and z coordinates,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

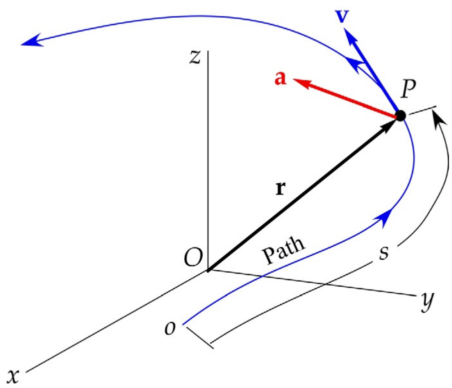


FIG. 1.8

Position, velocity, and acceleration vectors.

The distance of P from the origin is the magnitude or length of $\mathbf{r}$, denoted $\|\mathbf{r}\|$ or just r,

$$\|\mathbf{r}\| = r = \sqrt{x^2 + y^2 + z^2}$$

As in Eq. (1.11), the magnitude of $\mathbf{r}$ can also be computed by means of the dot product operation,

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$$

The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of the particle are the first and second time derivatives of the position vector,

$$\begin{aligned}
\mathbf{v}(t) &= \frac{\mathrm{d}x(t)}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}y(t)}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}y(t)}{\mathrm{d}t}\hat{\mathbf{k}} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}} \\
\mathbf{a}(t) &= \frac{\mathrm{d}v_x(t)}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}v_y(t)}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}v_z(t)}{\mathrm{d}t}\hat{\mathbf{k}} = a_x(t)\hat{\mathbf{i}} + a_y(t)\hat{\mathbf{j}} + a_z(t)\hat{\mathbf{k}}
\end{aligned}$$

The derivatives of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are zero since axes of the Cartesian frame have fixed directions. It is convenient to represent the time derivative by means of an overhead dot. In this shorthand notation, if ( ) is any quantity, then

$$\dot{(\ )} = \frac{\mathrm{d}(\ )}{\mathrm{d}t} \qquad \ddot{(\ )} = \frac{\mathrm{d}^2(\ )}{\mathrm{d}t^2} \qquad \dddot{(\ )} = \frac{\mathrm{d}^3(\ )}{\mathrm{d}t^3} \qquad \text{etc.}$$

Thus, for example,

$$\begin{aligned}
&\mathbf{v} = \dot{\mathbf{r}} \\
&\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \\
&v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z} \\
&a_x = \dot{v}_x = \ddot{x} \qquad a_y = \dot{v}_y = \ddot{y} \qquad a_z = \dot{v}_z = \ddot{z}
\end{aligned}$$

The locus of points that a particle occupies as it moves through space is called its path or trajectory. If the path is a straight line, then the motion is rectilinear. Otherwise, the path is curved, and the motion is called curvilinear. The velocity vector $\mathbf{v}$ is tangent to the path. If $\hat{\mathbf{u}}_t$ is the unit vector tangent to the trajectory, then

$$\mathbf{v} = v\hat{\mathbf{u}}_t \tag{1.22}$$

where the speed v is the magnitude of the velocity $\mathbf{v}$. The distance $\mathrm{d}s$ that P travels along its path in the time interval $\mathrm{d}t$ is obtained from the speed by

$$\mathrm{d}s = v\,\mathrm{d}t$$

In other words,

$$v = \dot{s}$$

The distance s, measured along the path from some starting point, is what the odometers in our automobiles record. Of course, $\dot{s}$, our speed along the road, is indicated by the dial of the speedometer.

Note carefully that $v \neq \dot{r}$ (i.e., the magnitude of the derivative of $\mathbf{r}$ does not equal the derivative of the magnitude of $\mathbf{r}$).

EXAMPLE 1.5

The position vector in meters is given as a function of time in seconds as

$$\mathbf{r} = \left(8t^2 + 7t + 6\right)\hat{\mathbf{i}} + \left(5t^3 + 4\right)\hat{\mathbf{j}} + \left(0.3t^4 + 2t^2 + 1\right)\hat{\mathbf{k}}\ (\mathrm{m}) \tag{a}$$

At $t = 10$ s, calculate (a) v (the magnitude of the derivative of $\mathbf{r}$) and (b) $\dot{r}$ (the derivative of the magnitude of $\mathbf{r}$).

Solution

(a) The velocity $\mathbf{v}$ is found by differentiating the given position vector with respect to time,

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (16t + 7)\hat{\mathbf{i}} + 15t^2\hat{\mathbf{j}} + \left(1.2t^3 + 4t\right)\hat{\mathbf{k}}$$

The magnitude of this vector is the square root of the sum of the squares of its components,

$$v = \sqrt{1.44t^6 + 234.6t^4 + 272t^2 + 224t + 49}$$

Evaluating this at $t = 10$ s, we get

$$\boxed{v = 1953.3\,\mathrm{m/s}}$$

(b) Calculating the magnitude of $\mathbf{r}$ in Eq. (a) leads to

$$r = \sqrt{0.09t^8 + 26.2t^6 + 68.6t^4 + 152t^3 + 149t^2 + 84t + 53}$$

The time derivative of this expression is

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{0.36t^7 + 78.6t^5 + 137.2t^3 + 228t^2 + 149t + 42}{\sqrt{0.09t^8 + 26.2t^6 + 68.6t^4 + 152t^3 + 149t^2 + 84t + 53}}$$

Substituting $t = 10$ s yields

$$\boxed{\dot{r} = 1935.5\,\mathrm{m/s}}$$

If $\mathbf{v}$ is given, then we can find the components of the unit tangent $\hat{\mathbf{u}}_t$ in the Cartesian coordinate frame of reference by means of Eq. (1.22):

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{v} = \frac{v_x}{v}\hat{\mathbf{i}} + \frac{v_y}{v}\hat{\mathbf{j}} + \frac{v_z}{v}\hat{\mathbf{k}} \quad \left(v = \sqrt{v_x^2 + v_y^2 + v_z^2}\right) \tag{1.23}$$

The acceleration may be written as

$$\mathbf{a} = a_t\hat{\mathbf{u}}_t + a_n\hat{\mathbf{u}}_n \tag{1.24}$$

where a_t and a_n are the tangential and normal components of acceleration, given by

$$a_t = \dot{v}\,(=\ddot{s}) \qquad a_n = \frac{v^2}{\rho} \tag{1.25}$$

where ρ is the radius of curvature, which is the distance from the particle P to the center of curvature of the path at that point. The unit principal normal $\hat{\mathbf{u}}_n$ is perpendicular to $\hat{\mathbf{u}}_t$ and points toward the center of curvature C, as shown in Fig. 1.9. Therefore, the position of C relative to P, denoted $\mathbf{r}_{C/P}$, is

$$\mathbf{r}_{C/P} = \rho\hat{\mathbf{u}}_n \tag{1.26}$$

The orthogonal unit vectors $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ form a plane called the osculating plane. The unit normal to the osculating plane is $\hat{\mathbf{u}}_b$, the binormal, and it is obtained from $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ by taking their cross product:

$$\hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n \tag{1.27}$$

From Eqs. (1.22), (1.24), and (1.27), we have

$$\mathbf{v}\times\mathbf{a} = v\hat{\mathbf{u}}_t \times (a_t\hat{\mathbf{u}}_t + a_n\hat{\mathbf{u}}_n) = va_n(\hat{\mathbf{u}}_t\times\hat{\mathbf{u}}_n) = va_n\hat{\mathbf{u}}_b = \|\mathbf{v}\times\mathbf{a}\|\hat{\mathbf{u}}_b$$

That is, an alternative to Eq. (1.27) for calculating the binormal vector is

$$\hat{\mathbf{u}}_b = \frac{\mathbf{v}\times\mathbf{a}}{\|\mathbf{v}\times\mathbf{a}\|} \tag{1.28}$$

Note that $\hat{\mathbf{u}}_t$, $\hat{\mathbf{u}}_n$, and $\hat{\mathbf{u}}_b$ form a right-handed triad of orthogonal unit vectors. That is

$$\hat{\mathbf{u}}_b\times\hat{\mathbf{u}}_t = \hat{\mathbf{u}}_n \qquad \hat{\mathbf{u}}_t\times\hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b \qquad \hat{\mathbf{u}}_n\times\hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t \tag{1.29}$$

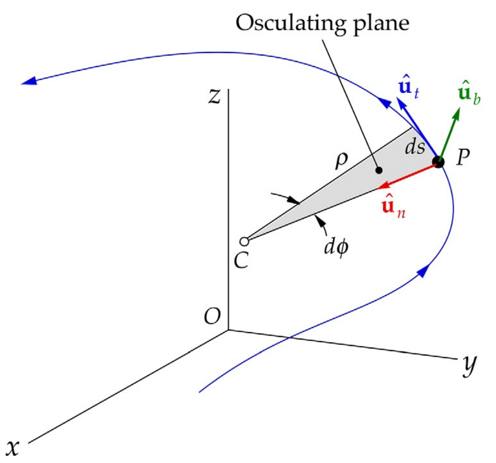


FIG. 1.9

Orthogonal triad of unit vectors associated with the moving point *P*.

The center of curvature lies in the osculating plane. When the particle P moves an incremental distance ds, the radial from the center of curvature to the path sweeps out a small angle, dϕ, measured in the osculating plane. The relationship between this angle and ds is

$$ds = \rho d\phi$$

so that $\dot{s} = \rho\dot{\phi}$, or

$$\dot{\phi} = \frac{v}{\rho} \tag{1.30}$$

EXAMPLE 1.6

Relative to a Cartesian coordinate system, the position, velocity, and acceleration of a particle P at a given instant are

$$\mathbf{r} = 250\hat{\mathbf{i}} + 630\hat{\mathbf{j}} + 430\hat{\mathbf{k}}\,(\mathrm{m}) \tag{a}$$

$$\mathbf{v} = 90\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 170\hat{\mathbf{k}}\,(\mathrm{m/s}) \tag{b}$$

$$\mathbf{a} = 16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}\,\left(\mathrm{m/s^2}\right) \tag{c}$$

Find the coordinates of the center of curvature at that instant.

Solution

The coordinates of the center of curvature C are the components of its position vector $\mathbf{r}_C$. Consulting Fig. 1.9, we observe that

$$\mathbf{r}_C = \mathbf{r} + \rho\hat{\mathbf{u}}_n \tag{d}$$

where $\mathbf{r}$ is the position vector of the point P, ρ is the radius of curvature, and $\hat{\mathbf{u}}_n$ is the unit principal normal vector. The position vector $\mathbf{r}$ is given in Eq. (a), but ρ and $\hat{\mathbf{u}}_n$ are unknowns at this point. We must use the geometry of Fig. 1.9 to find them.

We begin by seeking the value of $\hat{\mathbf{u}}_n$, using the first of Eqs. (1.29),

$$\hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t \tag{e}$$

The unit tangent vector $\hat{\mathbf{u}}_t$ is found at once from the velocity vector in Eq. (b) by means of Eq. 1.23,

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{v}$$

where

$$v = \sqrt{90^2 + 125^2 + 170^2} = 229.4\ \mathrm{m/s} \tag{f}$$

Thus,

$$\hat{\mathbf{u}}_t = \frac{90\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 170\hat{\mathbf{k}}}{229.4} = 0.39233\hat{\mathbf{i}} + 0.54490\hat{\mathbf{j}} + 0.74106\hat{\mathbf{k}} \tag{g}$$

To find the binormal $\hat{\mathbf{u}}_b$ we insert the given velocity and acceleration vectors into Eq. (1.28),

$$\hat{\mathbf{u}}_b = \frac{\mathbf{v}\times\mathbf{a}}{\|\mathbf{v}\times\mathbf{a}\|} = \frac{\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 90 & 125 & 170 \\ 16 & 125 & 30 \end{vmatrix}}{\|\mathbf{v}\times\mathbf{a}\|} = \frac{-17{,}500\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 9250\hat{\mathbf{k}}}{\sqrt{(-17{,}500)^2 + 20^2 + 9250^2}} \tag{h}$$

$$= -0.88409\hat{\mathbf{i}} + 0.0010104\hat{\mathbf{i}} + 0.46731\hat{\mathbf{k}}$$

Substituting Eqs. (g) and (h) back into Eq. (e) finally yields the unit principal normal

$$\hat{\mathbf{u}}_n = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -0.88409 & 0.0010104 & 0.46731 \\ 0.39233 & 0.5449 & 0.74106 \end{vmatrix} = -0.25389\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.48214\hat{\mathbf{k}} \tag{i}$$

The only unknown remaining in Eq. (d) is ρ, for which we appeal to Eq. (1.25),

$$\rho = \frac{v^2}{a_n} \tag{j}$$

The normal acceleration a_n is calculated by projecting the acceleration vector $\mathbf{a}$ onto the direction of the unit normal $\hat{\mathbf{u}}_n$,

$$a_n = \mathbf{a} \cdot \hat{\mathbf{u}}_n = \left(16\hat{\mathbf{i}} + 125\hat{\mathbf{j}} + 30\hat{\mathbf{k}}\right) \cdot \left(-0.25389\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.48214\hat{\mathbf{k}}\right) = 86.287\,\text{m/s}^2 \tag{k}$$

Putting the values of v and a_n from Eqs. (f) and (k) into Eq. (j) yields the radius of curvature,

$$\rho = \frac{229.4^2}{86.287} = 609.89\,\text{m} \tag{l}$$

Upon substituting Eqs. (a), (i), and (l) into Eq. (d), we obtain the position vector of the center of curvature C,

$$\begin{aligned} \mathbf{r}_C &= \left(250\hat{\mathbf{i}} + 630\hat{\mathbf{j}} + 430\hat{\mathbf{k}}\right) + 609.89\left(-0.25389\hat{\mathbf{i}} + 0.8385\hat{\mathbf{j}} - 0.48214\hat{\mathbf{k}}\right) \\ &= 95.159\hat{\mathbf{i}} + 1141.4\hat{\mathbf{j}} + 135.95\hat{\mathbf{k}}\ (\text{m}) \end{aligned}$$

Therefore, the coordinates of C are

$$\boxed{x = 95.16\,\text{m} \quad y = 1141\,\text{m} \quad z = 136.0\,\text{m}}$$

1.4 MASS, FORCE, AND NEWTON'S LAW OF GRAVITATION

Mass, like length and time, is a primitive physical concept: it cannot be defined in terms of any other physical concept. Mass is simply the quantity of matter. More practically, mass is a measure of the inertia of a body. Inertia is an object's resistance to changing its state of motion. The larger its inertia (the greater its mass), the more difficult it is to set a body into motion or bring it to rest. The unit of mass is the kilogram (kg).

Force is the action of one physical body on another, either through direct contact or through a distance. Gravity is an example of force acting through a distance, as are magnetism and the force between charged particles. The gravitational force F_g between two masses m_1 and m_2 having a distance r between their centers is

$$F_g = G\frac{m_1 m_2}{r^2} \tag{1.31}$$

This is Newton's law of gravity, in which G, the universal gravitational constant, has the value $G = 6.6742(10^{-11})\text{m}^3/(\text{kg}\cdot\text{s}^2)$. Due to the inverse-square dependence on distance, the force of gravity rapidly diminishes with the amount of separation between the two masses. In any case, the force of gravity is minuscule unless at least one of the masses is extremely big.

The force of a large mass (such as the earth) on a mass many orders of magnitude smaller (such as a person) is called weight, W. If the mass of the large object is M and that of the relatively tiny one is m, then the weight of the small body is

$$W = G\frac{Mm}{r^2} = m\left(\frac{GM}{r^2}\right)$$

or

$$W = mg \tag{1.32}$$

where

$$g = \frac{GM}{r^2} \tag{1.33}$$

g has units of acceleration (m/s^2) and is called the acceleration of gravity. If planetary gravity is the only force acting on a body, then the body is said to be in free fall. The force of gravity draws a freely falling object toward the center of attraction (e.g., center of the earth) with an acceleration g. Under ordinary conditions, we sense our own weight by feeling contact forces acting on us in opposition to the force of gravity. In free fall, there are, by definition, no contact forces, so there can be no sense of weight. Even though the weight is not zero, a person in free fall experiences weightlessness, or the absence of gravity.

Let us evaluate Eq. (1.33) at the surface of the earth, whose radius according to Table A.1 is 6378 km. Letting g_0 represent the standard sea level value of g, we get

$$g_0 = \frac{GM}{R_E^2} \tag{1.34}$$

In SI units,

$$g_0 = 9.807\,\text{m/s}^2 \tag{1.35}$$

Substituting Eq. (1.34) into Eq. (1.33) and letting z represent the distance above the earth's surface, so that $r = R_E + z$, we obtain

$$g = g_0 \frac{R_E^2}{(R_E + z)^2} = \frac{g_0}{(1 + z/R_E)^2} \tag{1.36}$$

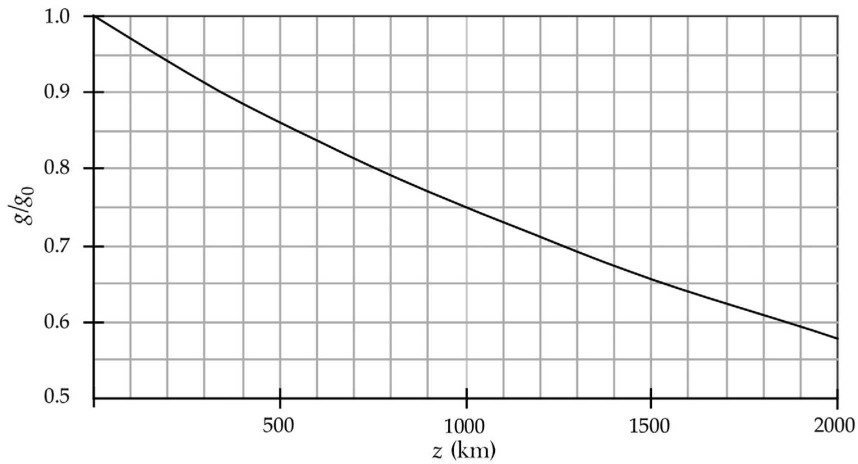


FIG. 1.10

Variation of the acceleration of gravity with altitude.

Commercial airliners cruise at altitudes on the order of 10 km (6 miles). At that height, Eq. (1.36) reveals that g (and hence weight) is only three-tenths of a percent less than its sea level value. Thus, under ordinary conditions, we ignore the variation of g with altitude. A plot of Eq. (1.36) out to a height of 2000 km (the upper limit of low earth orbit operations) is shown in Fig. 1.10. The variation of g over that range is significant. Even so, at space station altitude (400 km), weight is only about 10% less than it is on the earth's surface. The astronauts experience weightlessness, but they clearly are not weightless.

EXAMPLE 1.7

Show that in the absence of an atmosphere, the shape of a low-altitude ballistic trajectory is a parabola. Assume the acceleration of gravity g is constant and neglect the earth's curvature.

Solution

Fig. 1.11 shows a projectile launched at $t = 0$ s with a speed v_0 at a flight path angle γ_0 from the point with coordinates (x_0, y_0). Since the projectile is in free fall after launch, its only acceleration is that of gravity in the negative y direction:

$$\ddot{x} = 0$$
$$\ddot{y} = -g$$

Integrating with respect to time and applying the initial conditions leads to

$$x = x_0 + (v_0 \cos\gamma_0)t \tag{a}$$

$$y = y_0 + (v_0 \sin\gamma_0)t - \frac{1}{2}gt^2 \tag{b}$$

Solving Eq. (a) for t and substituting the result into Eq. (b) yields

$$\boxed{y = y_0 + (x - x_0)\tan\gamma_0 - \frac{1}{2}\frac{g}{v_0^2\cos^2\gamma_0}(x - x_0)^2} \tag{c}$$

This is the equation of a second-degree curve, a parabola, as sketched in Fig. 1.11.

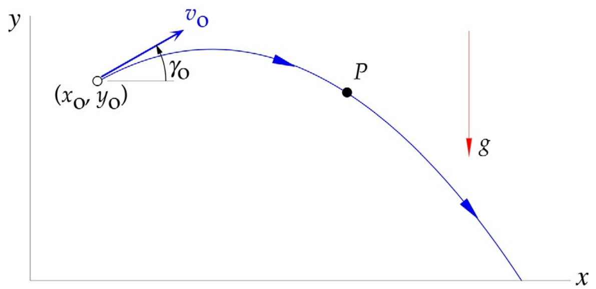


FIG. 1.11

Flight of a low-altitude projectile in free fall (no atmosphere).

EXAMPLE 1.8

An airplane flies a parabolic trajectory like that in Fig. 1.11 so that the passengers will experience free fall (weightlessness). What is the required variation of the flight path angle γ with speed v? Ignore the curvature of the earth.

Solution

Fig. 1.12 reveals that for a "flat" earth, $d\gamma = -d\phi$. That is,

$$\dot{\gamma} = -\dot{\phi}$$

It follows from Eq. (1.30) that

$$\rho\dot{\gamma} = -v \tag{1.37}$$

The normal acceleration a_n is just the component of the gravitational acceleration g in the direction of the unit principal normal to the curve (from P toward C). From Fig. 1.12, then,

$$a_n = g\cos\gamma \tag{a}$$

Substituting the second of Eqs. (1.25) into Eq. (a) and solving for the radius of curvature yields

$$\rho = \frac{v^2}{g\cos\gamma} \tag{b}$$

Combining Eqs. (1.37) and (b), we find the time rate of change of the flight path angle,

$$\boxed{\dot{\gamma} = -\frac{g\cos\gamma}{v}}$$

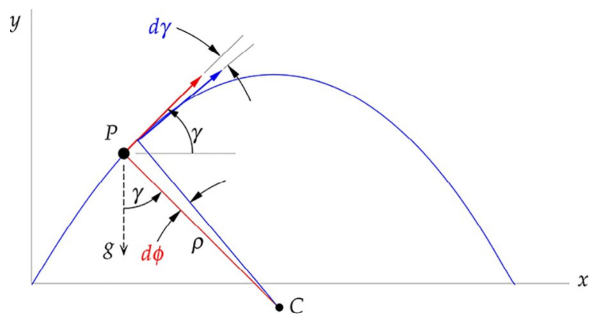


FIG. 1.12

Relationship between $d\gamma$ and $d\phi$ for a "flat" earth.

1.5 NEWTON'S LAW OF MOTION

Force is not a primitive concept like mass because it is intimately connected with the concepts of motion and inertia. In fact, the only way to alter the motion of a body is to exert a force on it. The degree to which the motion is altered is a measure of the force. Newton's second law of motion quantifies this. If the resultant or net force on a body of mass m is $\mathbf{F}_{\text{net}}$, then

$$\mathbf{F}_{\text{net}} = m\mathbf{a} \tag{1.38}$$

In this equation, **a** is the absolute acceleration of the center of mass. The absolute acceleration is measured in a frame of reference that itself has neither translational nor rotational acceleration relative to the fixed stars. Such a reference is called an absolute or inertial frame of reference.

Force is related to the primitive concepts of mass, length, and time by Newton's second law. The unit of force, appropriately, is the Newton, which is the force required to impart an acceleration of 1 m/s^2 to a mass of 1 kg. A mass of 1 kg therefore weighs 9.807 N at the earth's surface. The kilogram is not a unit of force.

Confusion can arise when mass is expressed in units of force, as frequently occurs in US engineering practice. In common parlance either the pound or the ton (2000 lb) is more likely to be used to express the mass. The pound of mass is officially defined precisely in terms of the kilogram, as shown in Table A.3. Since 1 lb of mass weighs 1 lb of force where the standard sea level acceleration of gravity (Eq. 1.35) exists, we can use Newton's second law to relate the pound of force to the Newton:

$$1\,\text{lb (force)} = 0.4536\,\text{kg} \times 9.807\,\text{m/s}^2 = 4.448\,\text{N}$$

The slug is the quantity of matter accelerated at 1 ft/s^2 by a force of 1 lb. We can again use Newton's second law to relate the slug to the kilogram. Noting the relationship between feet and meters in Table A.3, we find

$$1\ \text{slug} = \frac{1\,\text{lb}}{1\,\text{ft/s}^2} = \frac{4.448\,\text{N}}{0.3048\,\text{m/s}^2} = 14.59\,\frac{\text{kg}\cdot\text{m/s}^2}{\text{m/s}^2} = 14.59\,\text{kg}$$

EXAMPLE 1.9

On a NASA mission, the space shuttle *Atlantis* orbiter was reported to weigh 239,255 lb just prior to liftoff. On orbit 18 at an altitude of about 350 km, the orbiter's weight was reported to be 236,900 lb. (a) What was the mass, in kilograms, of *Atlantis* on the launchpad and in orbit? (b) If no mass was lost between launch and orbit 18, what would have been the weight of *Atlantis*, in pounds?

Solution

(a) The given data illustrate the common use of weight in pounds as a measure of mass. The "weights" given are actually the mass in pounds of mass. Therefore, prior to launch

$$m_{\text{launch pad}} = 239{,}255\,\text{lb (mass)} \times \frac{0.4536\,\text{kg}}{1\,\text{lb(mass)}} = 108{,}500\,\text{kg}$$

In orbit,

$$m_{\text{orbit 18}} = 236{,}900\,\text{lb (mass)} \times \frac{0.4536\,\text{kg}}{1\,\text{lb(mass)}} = 107{,}500\,\text{kg}$$

The decrease in mass is the propellant expended by the orbital maneuvering and reaction control rockets on the orbiter.

(b) Since the space shuttle launchpad at the Kennedy Space Center is essentially at sea level, the launchpad weight of *Atlantis* in pounds (force) was numerically equal to its mass in pounds (mass). With no change in mass, the force of gravity at 350 km would be, according to Eq. (1.36),

$$W = 239{,}255\,\text{lb(force)} \times \left(\frac{1}{1 + \dfrac{350}{6378}}\right)^2 = 215{,}000\,\text{lb(force)}$$

The integral of a force $\mathbf{F}$ over a time interval is called the impulse of the force,

$$\mathcal{I} = \int_{t_1}^{t_2} \mathbf{F} \mathrm{d}t \tag{1.39}$$

Impulse is a vector quantity. From Eq. (1.38) it is apparent that if the mass is constant, then

$$\mathcal{I}_{\text{net}} = \int_{t_1}^{t_2} m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \mathrm{d}t = m\mathbf{v}_2 - m\mathbf{v}_1 \tag{1.40}$$

That is, the net impulse on a body yields a change $m\Delta\mathbf{v}$ in its linear momentum, so that

$$\Delta\mathbf{v} = \frac{\mathcal{I}_{\text{net}}}{m} \tag{1.41}$$

If $\mathbf{F}_{\text{net}}$ is constant, then $\mathcal{I}_{\text{net}} = \mathbf{F}_{\text{net}}\Delta t$, in which case Eq. (1.41) becomes

$$\Delta\mathbf{v} = \frac{\mathbf{F}_{\text{net}}}{m}\Delta t \quad (\text{if } \mathbf{F}_{\text{net}} \text{ is constant}) \tag{1.42}$$

Let us conclude this section by introducing the concept of angular momentum. The moment of the net force about O in Fig. 1.13 is

$$\left.\mathbf{M}_O\right)_{\text{net}} = \mathbf{r} \times \mathbf{F}_{\text{net}}$$

Substituting Eq. (1.38) yields

$$\left.\mathbf{M}_O\right)_{\text{net}} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{1.43}$$

But, keeping in mind that the mass is constant,

$$\mathbf{r} \times m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\mathbf{v}) - \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times m\mathbf{v}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\mathbf{v}) - (\mathbf{v} \times m\mathbf{v})$$

Since $\mathbf{v} \times m\mathbf{v} = m(\mathbf{v} \times \mathbf{v}) = \mathbf{0}$, it follows that Eq. (1.43) can be written

$$\left.\mathbf{M}_O\right)_{\text{net}} = \frac{\mathrm{d}\mathbf{H}_O}{\mathrm{d}t} \tag{1.44}$$

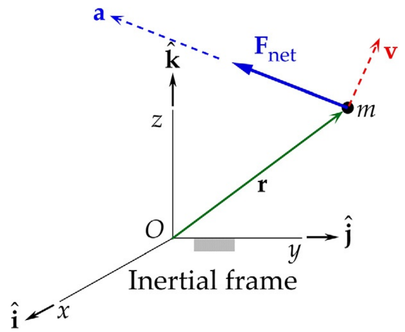


FIG. 1.13

The absolute acceleration of a particle is in the direction of the net force.

where $\mathbf{H}_O$ is the angular momentum about O,

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{1.45}$$

Thus, just as the net force on a particle changes its linear momentum $m\mathbf{v}$, the moment of that force about a fixed point changes the moment of its linear momentum about that point. Integrating Eq. (1.44) with respect to time yields

$$\int_{t_1}^{t_2} \mathbf{M}_O)_{\text{net}} = \mathbf{H}_O)_2 - \mathbf{H}_O)_1 \tag{1.46}$$

The integral on the left is the net angular impulse. This angular impulse-momentum equation is the rotational analog of the linear impulse-momentum relation given above in Eq. (1.40).

EXAMPLE 1.10

A particle of mass m is attached to point O by an inextensible string of length l, as illustrated in Fig. 1.14. Initially, the string is slack when m is moving to the left with a speed v_0 in the position shown. Calculate (a) the speed of m just after the string becomes taut and (b) the average force in the string over the small time interval Δt required to change the direction of the particle's motion.

Solution

(a) Initially, the position and velocity of the particle are

$$\mathbf{r}_1 = c\hat{\mathbf{i}} + d\hat{\mathbf{j}} \quad \mathbf{v}_1 = -v_0\hat{\mathbf{i}}$$

The angular momentum about O is

$$\mathbf{H}_1 = \mathbf{r}_1 \times m\mathbf{v}_1 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ c & d & 0 \\ -mv_o & 0 & 0 \end{vmatrix} = mv_o d\hat{\mathbf{k}} \tag{a}$$

Just after the string becomes taut,

$$\mathbf{r}_2 = -\sqrt{l^2 - d^2}\,\hat{\mathbf{i}} + d\hat{\mathbf{j}} \quad \mathbf{v}_2 = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \tag{b}$$

and the angular momentum is

$$\mathbf{H}_2 = \mathbf{r}_2 \times m\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sqrt{l^2 - d^2} & d & 0 \\ mv_x & mv_y & 0 \end{vmatrix} = \left(-mv_x d - mv_y\sqrt{l^2 - d^2}\right)\hat{\mathbf{k}} \tag{c}$$

Initially, the force exerted on m by the slack string is zero. When the string becomes taut, the force exerted on m passes through O. Therefore, the moment of the net force on m about O remains zero. According to Eq. (1.46),

$$\mathbf{H}_2 = \mathbf{H}_1$$

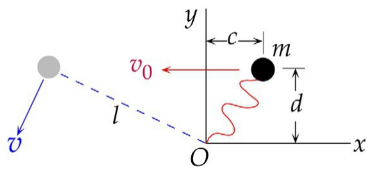


FIG. 1.14

Particle attached to O by an inextensible string.

Substituting Eqs. (a) and (c) yields

$$v_x d + \sqrt{l^2 - d^2}\, v_y = -v_o d \tag{d}$$

The string is inextensible, so the component of the velocity of m along the string must be zero:

$$\mathbf{v}_2 \cdot \mathbf{r}_2 = 0$$

Substituting $\mathbf{v}_2$ and $\mathbf{r}_2$ from Eq. (b) and solving for v_y, we get

$$v_y = v_x \sqrt{\frac{l^2}{d^2} - 1} \tag{e}$$

Solving Eqs. (d) and (e) for v_x and v_y leads to

$$v_x = -\frac{d^2}{l^2} v_o \quad v_y = -\sqrt{1 - \frac{d^2}{l^2}}\, \frac{d}{l} v_o \tag{f}$$

Thus, the speed, $v = \sqrt{{v_x}^2 + {v_y}^2}$, after the string becomes taut is

$$\boxed{v = \frac{d}{l} v_o}$$

(b) From Eq. (1.40), the impulse on m during the time it takes the string to become taut is

$$\boldsymbol{\mathcal{I}} = m(\mathbf{v}_2 - \mathbf{v}_1) = m\left[\left(-\frac{d^2}{l^2} v_o \hat{\mathbf{i}} - \sqrt{1 - \frac{d^2}{l^2}}\, \frac{d}{l} v_o \hat{\mathbf{j}}\right) - \left(-v_o \hat{\mathbf{i}}\right)\right] = \left(1 - \frac{d^2}{l^2}\right) m v_o \hat{\mathbf{i}} - \sqrt{1 - \frac{d^2}{l^2}}\, \frac{d}{l} m v_o \hat{\mathbf{j}}$$

The magnitude of this impulse, which is directed along the string, is

$$\mathcal{I} = \|\boldsymbol{\mathcal{I}}\| = \sqrt{1 - \frac{d^2}{l^2}}\, m v_o$$

Hence, the average force in the string during the small time interval Δt required to change the direction of the velocity vector turns out to be

$$\boxed{F_{\text{avg}} = \frac{\mathcal{I}}{\Delta t} = \sqrt{1 - \frac{d^2}{l^2}}\, \frac{m v_o}{\Delta t}}$$

1.6 TIME DERIVATIVES OF MOVING VECTORS

Fig. 1.15(a) shows a vector **A** inscribed in a rigid body B that is in motion relative to an inertial frame of reference (a rigid, Cartesian coordinate system, which is fixed relative to the fixed stars). The magnitude of **A** is fixed. The body B is shown at two times, separated by the differential time interval dt. At time t + dt, the orientation of vector **A** differs slightly from that at time t, but its magnitude is the same. According to one of the many theorems of the prolific 18th-century Swiss mathematician Leonhard Euler (1707–1783), there is a unique axis of rotation about which B, and therefore **A**, rotates during the differential time interval. If we shift the two vectors **A**(t) and **A**(t + dt) to the same point on the axis of rotation, so that they are tail to tail, as shown in Fig. 1.15(b), we can assess the difference d**A** between them caused by the infinitesimal rotation. Remember that shifting a vector to a parallel line does not change the vector. The rotation of the body B is measured in the plane perpendicular to the instantaneous axis of rotation. The amount of rotation is the angle dθ through which a line element normal to the rotation axis turns in the time interval dt. In Fig. 1.15(b) that line element is

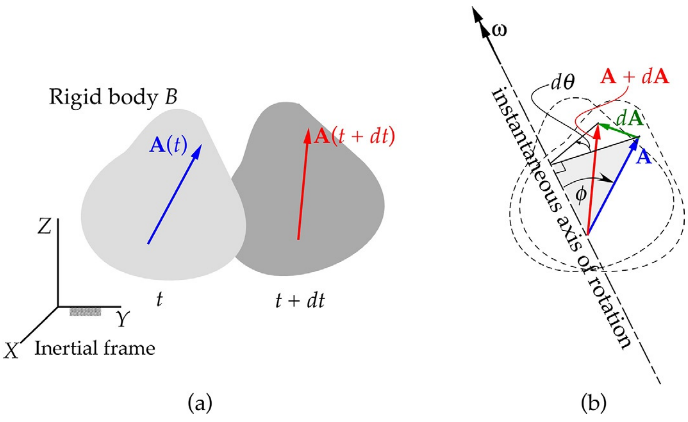


FIG. 1.15

Displacement of a rigid body. (a) Change in orientation of an embedded vector **A**. (b) Differential rotation of **A** about the instantaneous rotation axis.

the component of **A** normal to the axis of rotation. We can express the difference d**A** between **A**(t) and **A**(t + dt) as

$$\mathrm{d}\mathbf{A} = \overbrace{[(\|\mathbf{A}\| \cdot \sin\phi)\,\mathrm{d}\theta]}^{\text{magnitude of }\mathrm{d}\mathbf{A}}\,\hat{\mathbf{n}} \tag{1.47}$$

where $\hat{\mathbf{n}}$ is the unit normal to the plane defined by **A** and the axis of rotation, and it points in the direction of the rotation. The angle ϕ is the inclination of **A** to the rotation axis. By definition,

$$\mathrm{d}\theta = \|\boldsymbol{\omega}\|\mathrm{d}t \tag{1.48}$$

where $\boldsymbol{\omega}$ is the angular velocity vector, which points along the instantaneous axis of rotation, and its direction is given by the right-hand rule. That is, wrapping the right hand around the axis of rotation, with the fingers pointing in the direction of dθ, results in the thumb defining the direction of $\boldsymbol{\omega}$. This is evident in Fig. 1.15(b). It should be pointed out that the time derivative of $\boldsymbol{\omega}$ is the angular acceleration, usually given the symbol $\boldsymbol{\alpha}$. Thus,

$$\boldsymbol{\alpha} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{1.49}$$

Substituting Eq. (1.48) into Eq. (1.47), we get

$$\mathrm{d}\mathbf{A} = \|\mathbf{A}\| \cdot \sin\phi \cdot \|\boldsymbol{\omega}\|\mathrm{d}t \cdot \hat{\mathbf{n}} = (\|\boldsymbol{\omega}\| \cdot \|\mathbf{A}\| \cdot \sin\phi)\hat{\mathbf{n}}\,\mathrm{d}t \tag{1.50}$$

By definition of the cross product, $\boldsymbol{\omega} \times \mathbf{A}$ is the product of the magnitude of $\boldsymbol{\omega}$, the magnitude of **A**, the sine of the angle between $\boldsymbol{\omega}$ and **A**, and the unit vector normal to the plane of $\boldsymbol{\omega}$ and **A**, in the rotation direction. That is,

$$\boldsymbol{\omega} \times \mathbf{A} = \|\boldsymbol{\omega}\| \cdot \|\mathbf{A}\| \cdot \sin\phi \cdot \hat{\mathbf{n}} \tag{1.51}$$

Substituting Eq. (1.51) into Eq. (1.50) yields

$$\mathrm{d}\mathbf{A} = \boldsymbol{\omega} \times \mathbf{A}\mathrm{d}t$$

Dividing through by dt, we finally obtain

$$\boxed{\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}=\boldsymbol{\omega}\times\mathbf{A}} \quad \left(\text{if } \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{A}\|=0\right) \tag{1.52}$$

Eq. (1.52) is a formula we can use to compute the time derivative of any vector of constant magnitude.

EXAMPLE 1.11

Calculate the second time derivative of a vector **A** of constant magnitude, expressing the result in terms of $\boldsymbol{\omega}$ and its derivatives and **A**.

Solution

Differentiating Eq. (1.52) with respect to time, we get

$$\frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}\times\mathbf{A})=\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\mathbf{A}+\boldsymbol{\omega}\times\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$$

Using Eqs. (1.49) and (1.52), this can be written

$$\boxed{\frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}t^2}=\boldsymbol{\alpha}\times\mathbf{A}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})} \tag{1.53}$$

EXAMPLE 1.12

Calculate the third derivative of a vector **A** of constant magnitude, expressing the result in terms of $\boldsymbol{\omega}$ and its derivatives and **A**.

Solution

$$\begin{aligned}
\frac{\mathrm{d}^3\mathbf{A}}{\mathrm{d}t^3}&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\alpha}\times\mathbf{A}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]\\
&=\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}\times\mathbf{A})+\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]\\
&=\left(\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\times\mathbf{A}+\boldsymbol{\alpha}\times\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)+\left[\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\omega}\times\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}\times\mathbf{A})\right]\\
&=\left[\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\times\mathbf{A}+\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})\right]+\left[\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\mathbf{A}+\boldsymbol{\omega}\times\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)\right]\\
&=\left[\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\times\mathbf{A}+\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})\right]+\{\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\omega}\times[\boldsymbol{\alpha}\times\mathbf{A}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]\}\\
&=\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\times\mathbf{A}+\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\omega}\times(\boldsymbol{\alpha}\times\mathbf{A})+\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]\\
&=\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\times\mathbf{A}+2\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\omega}\times(\boldsymbol{\alpha}\times\mathbf{A})+\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}^3\mathbf{A}}{\mathrm{d}t^3}=\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\times\mathbf{A}+2\boldsymbol{\alpha}\times(\boldsymbol{\omega}\times\mathbf{A})+\boldsymbol{\omega}\times[\boldsymbol{\alpha}\times\mathbf{A}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})]}$$

Let XYZ be a rigid inertial frame of reference and xyz a rigid moving frame of reference, as shown in Fig. 1.16. The moving frame can be moving (translating and rotating) freely on its own accord, or it can

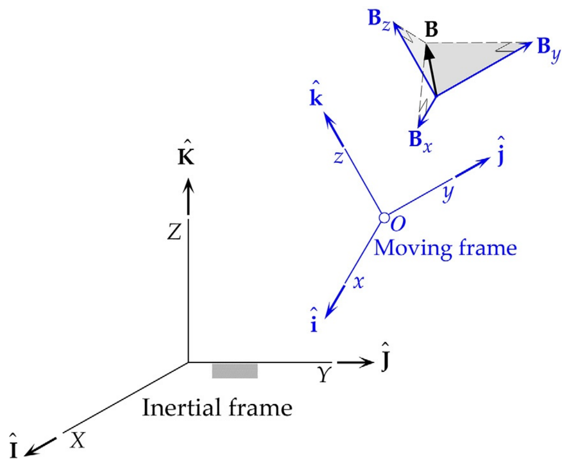


FIG. 1.16

Fixed (inertial) and moving rigid frames of reference.

be attached to a physical object, such as a car, an airplane, or a spacecraft. Kinematic quantities measured relative to the fixed inertial frame will be called absolute (e.g., absolute acceleration), and those measured relative to the moving system will be called relative (e.g., relative acceleration). The unit vectors along the inertial *XYZ* system are $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$, whereas those of the moving *xyz* system are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. The motion of the moving frame is arbitrary, and its absolute angular velocity is $\boldsymbol{\Omega}$. If, however, the moving frame is rigidly attached to an object, so that it not only translates but also rotates with it, then the frame is called a body frame and the axes are referred to as body axes. A body frame clearly has the same angular velocity as the body to which it is bound.

Let $\mathbf{B}$ be any time-dependent vector. Resolved into components along the inertial frame of reference, it is expressed analytically as

$$\mathbf{B} = B_X\hat{\mathbf{I}} + B_Y\hat{\mathbf{J}} + B_Z\hat{\mathbf{K}}$$

where B_X, B_Y, and B_Z are functions of time. Since $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ are fixed, the time derivative of $\mathbf{B}$ is simply

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \frac{\mathrm{d}B_X}{\mathrm{d}t}\hat{\mathbf{I}} + \frac{\mathrm{d}B_Y}{\mathrm{d}t}\hat{\mathbf{J}} + \frac{\mathrm{d}B_Z}{\mathrm{d}t}\hat{\mathbf{K}}$$

$\mathrm{d}B_X/\mathrm{d}t$, $\mathrm{d}B_Y/\mathrm{d}t$, and $\mathrm{d}B_Z/\mathrm{d}t$ are the components of the absolute time derivative of $\mathbf{B}$.

$\mathbf{B}$ may also be resolved into components along the moving *xyz* frame, so that, at any instant,

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{1.54}$$

Using this expression to calculate the time derivative of $\mathbf{B}$ yields

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \frac{\mathrm{d}B_x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}B_y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}B_z}{\mathrm{d}t}\hat{\mathbf{k}} + B_x\frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t} + B_y\frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t} + B_z\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} \tag{1.55}$$

The orthogonal unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are not fixed in space but are continuously changing direction; therefore, their time derivatives are not zero. They obviously have a constant magnitude (unity) and, being attached to the xyz frame, they all have the angular velocity $\mathbf{\Omega}$. It follows from Eq. (1.52) that

$$\frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t}=\mathbf{\Omega}\times\hat{\mathbf{i}} \quad \frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t}=\mathbf{\Omega}\times\hat{\mathbf{j}} \quad \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}=\mathbf{\Omega}\times\hat{\mathbf{k}}$$

Substituting these on the right-hand side of Eq. (1.55) yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}&=\frac{\mathrm{d}B_x}{\mathrm{d}t}\hat{\mathbf{i}}+\frac{\mathrm{d}B_y}{\mathrm{d}t}\hat{\mathbf{j}}+\frac{\mathrm{d}B_z}{\mathrm{d}t}\hat{\mathbf{k}}+B_x\left(\mathbf{\Omega}\times\hat{\mathbf{i}}\right)+B_y\left(\mathbf{\Omega}\times\hat{\mathbf{j}}\right)+B_z\left(\mathbf{\Omega}\times\hat{\mathbf{k}}\right)\\&=\frac{\mathrm{d}B_x}{\mathrm{d}t}\hat{\mathbf{i}}+\frac{\mathrm{d}B_y}{\mathrm{d}t}\hat{\mathbf{j}}+\frac{\mathrm{d}B_z}{\mathrm{d}t}\hat{\mathbf{k}}+\left(\mathbf{\Omega}\times B_x\hat{\mathbf{i}}\right)+\left(\mathbf{\Omega}\times B_y\hat{\mathbf{j}}\right)+\left(\mathbf{\Omega}\times B_z\hat{\mathbf{k}}\right)\\&=\frac{\mathrm{d}B_x}{\mathrm{d}t}\hat{\mathbf{i}}+\frac{\mathrm{d}B_y}{\mathrm{d}t}\hat{\mathbf{j}}+\frac{\mathrm{d}B_z}{\mathrm{d}t}\hat{\mathbf{k}}+\mathbf{\Omega}\times\left(B_x\hat{\mathbf{i}}+B_y\hat{\mathbf{j}}+B_z\hat{\mathbf{k}}\right)\end{aligned}$$

In view of Eq. (1.54), this can be written as

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}=\left.\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}+\mathbf{\Omega}\times\mathbf{B} \tag{1.56}$$

where

$$\left.\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}=\frac{\mathrm{d}B_x}{\mathrm{d}t}\hat{\mathbf{i}}+\frac{\mathrm{d}B_y}{\mathrm{d}t}\hat{\mathbf{j}}+\frac{\mathrm{d}B_z}{\mathrm{d}t}\hat{\mathbf{k}} \tag{1.57}$$

$\mathrm{d}\mathbf{B}/\mathrm{d}t)_{\mathrm{rel}}$ is the time derivative of $\mathbf{B}$ relative to the moving frame. Eq. (1.56) shows how the absolute time derivative is obtained from the relative time derivative. Clearly, $\mathrm{d}\mathbf{B}/\mathrm{d}t=\mathrm{d}\mathbf{B}/\mathrm{d}t)_{\mathrm{rel}}$ only when the moving frame is in pure translation ($\mathbf{\Omega}=\mathbf{0}$).

Eq. (1.56) can be used recursively to compute higher order time derivatives. Thus, differentiating Eq. (1.56) with respect to t, we get

$$\frac{\mathrm{d}^2\mathbf{B}}{\mathrm{d}t^2}=\left.\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}+\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t}\times\mathbf{B}+\mathbf{\Omega}\times\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$$

Using Eq. (1.56) in the last term yields

$$\frac{\mathrm{d}^2\mathbf{B}}{\mathrm{d}t^2}=\left.\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}+\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t}\times\mathbf{B}+\mathbf{\Omega}\times\left[\left.\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}+\mathbf{\Omega}\times\mathbf{B}\right] \tag{1.58}$$

Eq. (1.56) also implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}=\left.\frac{\mathrm{d}^2\mathbf{B}}{\mathrm{d}t^2}\right)_{\mathrm{rel}}+\mathbf{\Omega}\times\left.\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}} \tag{1.59}$$

where

$$\left.\frac{\mathrm{d}^2\mathbf{B}}{\mathrm{d}t^2}\right)_{\mathrm{rel}}=\frac{\mathrm{d}^2B_x}{\mathrm{d}t^2}\hat{\mathbf{i}}+\frac{\mathrm{d}^2B_y}{\mathrm{d}t^2}\hat{\mathbf{j}}+\frac{\mathrm{d}^2B_z}{\mathrm{d}t^2}\hat{\mathbf{k}}$$

Substituting Eq. (1.59) into Eq. (1.58) yields

$$\frac{\mathrm{d}^2\mathbf{B}}{\mathrm{d}t^2}=\left[\left.\frac{\mathrm{d}^2\mathbf{B}}{\mathrm{d}t^2}\right)_{\mathrm{rel}}+\mathbf{\Omega}\times\left.\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}\right]+\frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t}\times\mathbf{B}+\mathbf{\Omega}\times\left[\left.\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_{\mathrm{rel}}+\mathbf{\Omega}\times\mathbf{B}\right]$$

Collecting terms, this becomes

$$\frac{d^2\mathbf{B}}{dt^2} = \left.\frac{d^2\mathbf{B}}{dt^2}\right)_{\text{rel}} + \dot{\boldsymbol{\Omega}} \times \mathbf{B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{B}) + 2\boldsymbol{\Omega} \times \left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} \tag{1.60}$$

where $\dot{\boldsymbol{\Omega}} \equiv d\boldsymbol{\Omega}/dt$ is the absolute angular acceleration of the xyz frame.

Formulas for higher order time derivatives are found in a similar fashion.

1.7 RELATIVE MOTION

Let P be a particle in arbitrary motion. The absolute position vector of P is $\mathbf{r}$ and the position of P relative to the moving frame is $\mathbf{r}_{\text{rel}}$. If $\mathbf{r}_O$ is the absolute position of the origin of the moving frame, then it is clear from Fig. 1.17 that

$$\mathbf{r} = \mathbf{r}_O + \mathbf{r}_{\text{rel}} \tag{1.61}$$

Since $\mathbf{r}_{\text{rel}}$ is measured in the moving frame,

$$\mathbf{r}_{\text{rel}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{1.62}$$

where x, y, and z are the coordinates of P relative to the moving reference.

The absolute velocity $\mathbf{v}$ of P is $d\mathbf{r}/dt$, so that from Eq. (1.61) we have

$$\mathbf{v} = \mathbf{v}_O + \frac{d\mathbf{r}_{\text{rel}}}{dt} \tag{1.63}$$

where $\mathbf{v}_O = d\mathbf{r}_O/dt$ is the (absolute) velocity of the origin of the xyz frame. From Eq. (1.56), we can write

$$\frac{d\mathbf{r}_{\text{rel}}}{dt} = \mathbf{v}_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} \tag{1.64}$$

where $\mathbf{v}_{\text{rel}}$ is the velocity of P relative to the xyz frame (so that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are held fixed):

$$\mathbf{v}_{\text{rel}} = \left.\frac{d\mathbf{r}_{\text{rel}}}{dt}\right)_{\text{rel}} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \tag{1.65}$$

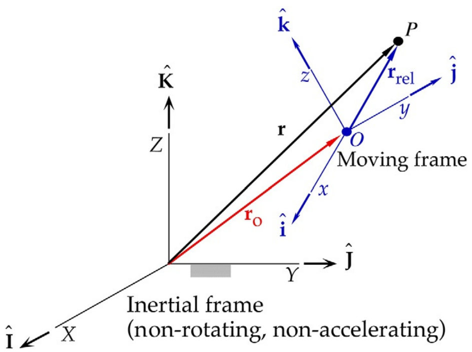


FIG. 1.17

Absolute and relative position vectors.

Substituting Eq. (1.64) into Eq. (1.63) yields

$$\boxed{\mathbf{v} = \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}} \tag{1.66}$$

The absolute acceleration $\mathbf{a}$ of P is $\mathrm{d}\mathbf{v}/\mathrm{d}t$, so that from Eq. (1.63) we have

$$\mathbf{a} = \mathbf{a}_O + \frac{\mathrm{d}^2\mathbf{r}_{\text{rel}}}{\mathrm{d}t^2} \tag{1.67}$$

where $\mathbf{a}_O = \mathrm{d}\mathbf{v}_O/\mathrm{d}t$ is the absolute acceleration of the origin of the xyz frame. We evaluate the second term on the right using Eq. (1.60).

$$\frac{\mathrm{d}^2\mathbf{r}_{\text{rel}}}{\mathrm{d}t^2} = \left.\frac{\mathrm{d}^2\mathbf{r}_{\text{rel}}}{\mathrm{d}t^2}\right)_{\text{rel}} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\Omega} \times \left.\frac{\mathrm{d}\mathbf{r}_{\text{rel}}}{\mathrm{d}t}\right)_{\text{rel}} \tag{1.68}$$

Since $\mathbf{v}_{\text{rel}} = \mathrm{d}\mathbf{r}_{\text{rel}}/\mathrm{d}t)_{\text{rel}}$ and $\mathbf{a}_{\text{rel}} = \mathrm{d}^2\mathbf{r}_{\text{rel}}/\mathrm{d}t^2)_{\text{rel}}$, this can be written

$$\frac{\mathrm{d}^2\mathbf{r}_{\text{rel}}}{\mathrm{d}t^2} = \mathbf{a}_{\text{rel}} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} \tag{1.69}$$

Upon substituting this result into Eq. (1.67), we find

$$\boxed{\mathbf{a} = \mathbf{a}_O + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}} \tag{1.70}$$

The cross product $2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}}$ is called the Coriolis acceleration after Gustave Gaspard de Coriolis (1792–1843), the French mathematician who introduced this term (Coriolis, 1835). Because of the number of terms on the right, Eq. (1.70) is sometimes referred to as the five-term acceleration formula.

EXAMPLE 1.13

At a given instant, the absolute position, velocity, and acceleration of the origin O of a moving frame are

$$\left.\begin{aligned} \mathbf{r}_O &= 100\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}\ (\text{m}) \\ \mathbf{v}_O &= -50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}\ (\text{m/s}) \\ \mathbf{a}_O &= -15\hat{\mathbf{I}} + 40\hat{\mathbf{J}} + 25\hat{\mathbf{K}}\left(\text{m/s}^2\right) \end{aligned}\right\}(\text{given}) \tag{a}$$

The angular velocity and acceleration of the moving frame are

$$\left.\begin{aligned} \boldsymbol{\Omega} &= 1.0\hat{\mathbf{I}} - 0.4\hat{\mathbf{J}} + 0.6\hat{\mathbf{K}}(\text{rad/s}) \\ \dot{\boldsymbol{\Omega}} &= -1.0\hat{\mathbf{I}} + 0.3\hat{\mathbf{J}} - 0.4\hat{\mathbf{K}}\left(\text{rad/s}^2\right) \end{aligned}\right\}(\text{given}) \tag{b}$$

The unit vectors of the moving frame are

$$\left.\begin{aligned} \hat{\mathbf{i}} &= 0.5571\hat{\mathbf{I}} + 0.7428\hat{\mathbf{J}} + 0.3714\hat{\mathbf{K}} \\ \hat{\mathbf{j}} &= -0.06331\hat{\mathbf{I}} + 0.4839\hat{\mathbf{J}} - 0.8728\hat{\mathbf{K}} \\ \hat{\mathbf{k}} &= -0.8280\hat{\mathbf{I}} + 0.4627\hat{\mathbf{J}} + 0.3166\hat{\mathbf{K}} \end{aligned}\right\}(\text{given}) \tag{c}$$

The absolute position, velocity, and acceleration of P are

$$\left.\begin{aligned} \mathbf{r} &= 300\hat{\mathbf{I}} - 100\hat{\mathbf{J}} + 150\hat{\mathbf{K}}(\text{m}) \\ \mathbf{v} &= 70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}(\text{m/s}) \\ \mathbf{a} &= 7.5\hat{\mathbf{I}} - 8.5\hat{\mathbf{J}} + 6.0\hat{\mathbf{K}}\left(\text{m/s}^2\right) \end{aligned}\right\}(\text{given}) \tag{d}$$

Find (a) the velocity $\mathbf{v}_{\text{rel}}$ and (b) the acceleration $\mathbf{a}_{\text{rel}}$ of P relative to the moving frame.

Solution

Let us first use Eq. (c) to solve for $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, and $\hat{\mathbf{K}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ (three equations in three unknowns):

$$\begin{aligned}\hat{\mathbf{I}} &= 0.5571\hat{\mathbf{i}} - 0.06331\hat{\mathbf{j}} - 0.8280\hat{\mathbf{k}}\\ \hat{\mathbf{J}} &= 0.7428\hat{\mathbf{i}} + 0.4839\hat{\mathbf{j}} + 0.4627\hat{\mathbf{k}}\\ \hat{\mathbf{K}} &= 0.3714\hat{\mathbf{i}} - 0.8728\hat{\mathbf{j}} + 0.3166\hat{\mathbf{k}}\end{aligned} \tag{e}$$

(a) The relative position vector is

$$\mathbf{r}_{\text{rel}} = \mathbf{r} - \mathbf{r}_O = \left(300\hat{\mathbf{I}} - 100\hat{\mathbf{J}} + 150\hat{\mathbf{K}}\right) - \left(100\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}\right) = 200\hat{\mathbf{I}} - 300\hat{\mathbf{J}} - 150\hat{\mathbf{K}}\ (\text{m}) \tag{f}$$

From Eq. (1.66), the relative velocity vector is

$$\begin{aligned}\mathbf{v}_{\text{rel}} &= \mathbf{v} - \mathbf{v}_O - \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}\\ &= \left(70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}\right) - \left(-50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}\right) - \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 1.0 & -0.4 & 0.6\\ 200 & -300 & -150\end{vmatrix}\\ &= \left(70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}\right) - \left(-50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}\right) - \left(240\hat{\mathbf{I}} + 270\hat{\mathbf{J}} - 220\hat{\mathbf{K}}\right)\end{aligned}$$

or

$$\mathbf{v}_{\text{rel}} = -120\hat{\mathbf{I}} - 275\hat{\mathbf{J}} + 210\hat{\mathbf{K}}\ (\text{m/s}) \tag{g}$$

To obtain the components of the relative velocity along the axes of the moving frame, substitute Eq. (e) into Eq. (g),

$$\begin{aligned}\mathbf{v}_{\text{rel}} = &-120\left(0.5571\hat{\mathbf{i}} - 0.06331\hat{\mathbf{j}} - 0.8280\hat{\mathbf{k}}\right)\\ &-275\left(0.7428\hat{\mathbf{i}} + 0.4839\hat{\mathbf{j}} + 0.4627\hat{\mathbf{k}}\right) + 210\left(0.3714\hat{\mathbf{i}} - 0.8728\hat{\mathbf{j}} + 0.3166\hat{\mathbf{k}}\right)\end{aligned}$$

so that

$$\boxed{\mathbf{v}_{\text{rel}} = -193.1\hat{\mathbf{i}} - 308.8\hat{\mathbf{j}} + 38.60\hat{\mathbf{k}}\ (\text{m/s})} \tag{h}$$

Alternatively, in terms of the unit vector $\hat{\mathbf{u}}_v$ in the direction of $\mathbf{v}_{\text{rel}}$,

$$\boxed{\mathbf{v}_{\text{rel}} = 366.2\hat{\mathbf{u}}_v\,(\text{m/s}) \quad \left(\hat{\mathbf{u}}_v = -0.5272\hat{\mathbf{i}} - 0.8432\hat{\mathbf{j}} + 0.1005\hat{\mathbf{k}}\right)} \tag{i}$$

(b) To find the relative acceleration, we use the five-term acceleration formula, Eq. (1.70):

$$\begin{aligned}\mathbf{a}_{\text{rel}} &= \mathbf{a} - \mathbf{a}_O - \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) - 2(\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}})\\ &= \mathbf{a} - \mathbf{a}_O - \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -1.0 & 0.3 & -0.4\\ 200 & -300 & -150\end{vmatrix} - \boldsymbol{\Omega} \times \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 1.0 & -0.4 & 0.6\\ 200 & -300 & -150\end{vmatrix} - 2\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 1.0 & -0.4 & 0.6\\ -120 & -275 & 210\end{vmatrix}\\ &= \mathbf{a} - \mathbf{a}_O - \left(-165\hat{\mathbf{I}} - 230\hat{\mathbf{J}} + 240\hat{\mathbf{K}}\right) - \begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 1.0 & -0.4 & 0.6\\ 240 & 270 & -220\end{vmatrix} - \left(162\hat{\mathbf{I}} - 564\hat{\mathbf{J}} - 646\hat{\mathbf{K}}\right)\\ &= \left(7.5\hat{\mathbf{I}} - 8.5\hat{\mathbf{J}} + 6\hat{\mathbf{K}}\right) - \left(-15\hat{\mathbf{I}} + 40\hat{\mathbf{J}} + 25\hat{\mathbf{K}}\right) - \left(-165\hat{\mathbf{I}} - 230\hat{\mathbf{J}} + 240\hat{\mathbf{K}}\right)\\ &\quad - \left(-74\hat{\mathbf{I}} + 364\hat{\mathbf{J}} + 366\hat{\mathbf{K}}\right) - \left(162\hat{\mathbf{I}} - 564\hat{\mathbf{J}} - 646\hat{\mathbf{K}}\right)\end{aligned}$$

$$\mathbf{a}_{\text{rel}} = 99.5\hat{\mathbf{I}} + 381.5\hat{\mathbf{J}} + 21.0\hat{\mathbf{K}}\ (\text{m/s})^2 \tag{j}$$

The components of the relative acceleration along the axes of the moving frame are found by substituting Eq. (e) into Eq. (j):

$$\begin{aligned}\mathbf{a}_{\text{rel}} = &\,99.5\left(0.5571\hat{\mathbf{i}} - 0.06331\hat{\mathbf{j}} - 0.8282\hat{\mathbf{k}}\right)\\ &+381.5\left(0.7428\hat{\mathbf{i}} + 0.4839\hat{\mathbf{j}} + 0.4627\hat{\mathbf{k}}\right) + 21.0\left(0.3714\hat{\mathbf{i}} - 0.8728\hat{\mathbf{j}} + 0.3166\hat{\mathbf{k}}\right)\end{aligned}$$

$$\boxed{\mathbf{a}_{\text{rel}} = 346.6\hat{\mathbf{i}} + 160.0\hat{\mathbf{j}} + 100.8\hat{\mathbf{k}}\ (\text{m/s}^2)} \tag{k}$$

Or, in terms of the unit vector $\hat{\mathbf{u}}_a$ in the direction of $\mathbf{a}_{\text{rel}}$,

$$\boxed{\mathbf{a}_{\text{rel}} = 394.8\hat{\mathbf{u}}_a(\text{m/s}^2) \quad \left(\hat{\mathbf{u}}_a = 0.8778\hat{\mathbf{i}} + 0.4052\hat{\mathbf{j}} + 0.2553\hat{\mathbf{k}}\right)} \tag{l}$$

Fig. 1.18 shows the nonrotating inertial frame of reference XYZ with its origin at the center C of the earth, which we shall assume to be a sphere. That assumption will be relaxed in Chapter 5. Embedded in the earth and rotating with it is the orthogonal $x'y'z'$ frame, also centered at C, with the z' axis parallel to Z, the earth's axis of rotation. The x' axis intersects the equator at the prime meridian (0 degree longitude), which passes through Greenwich in London, England. The angle between X and x' is θ_G, and the rate of increase of θ_G is just the angular velocity Ω of the earth. P is a particle (e.g., an airplane or spacecraft), which is moving in an arbitrary fashion above the surface of the earth. $\mathbf{r}_{\text{rel}}$ is the position vector of P relative to C in the rotating $x'y'z'$ system. At a given instant, P is directly over point O, which lies on the earth's surface at longitude Λ and latitude ϕ. Point O coincides instantaneously with the origin of what is known as a topocentric-horizon coordinate system xyz. For our purposes, x and y are measured positive eastward and northward along the local latitude and meridian,

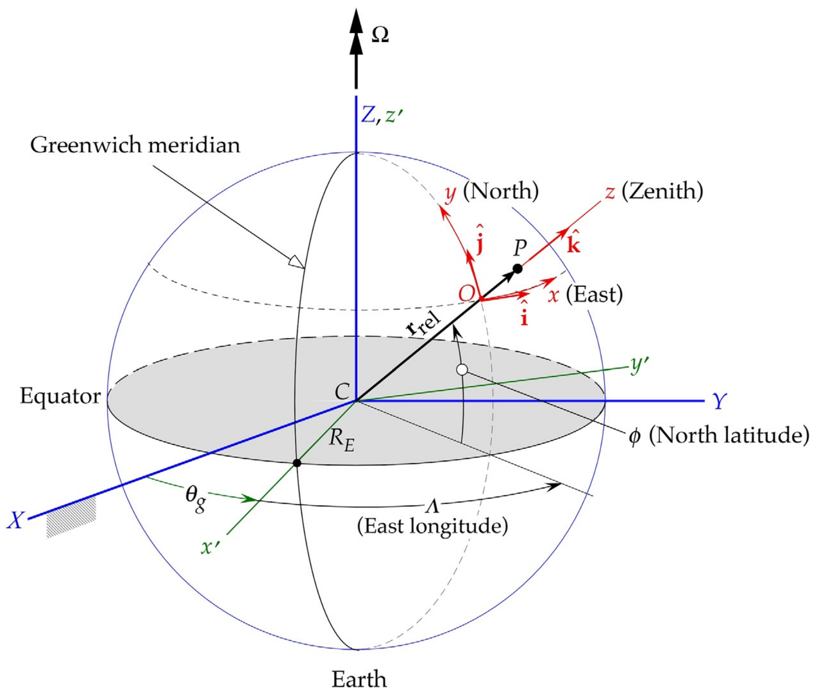


FIG. 1.18

Earth-centered inertial frame (XYZ); earth-centered noninertial $x'y'z'$ frame embedded in and rotating with the earth; and a noninertial, topocentric-horizon frame xyz attached to a point O on the earth's surface.

respectively, through O. The tangent plane to the earth's surface at O is the local horizon. The z axis is the local vertical (straight up), and it is directed radially outward from the center of the earth. The unit vectors of the xyz frame are $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, as indicated in Fig. 1.18. Keep in mind that O remains directly below P, so that as P moves, so do the xyz axes. Thus, the $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ triad, which comprises the unit vectors of a spherical coordinate system, varies in direction as P changes location, thereby accounting for the curvature of the earth.

Let us find the absolute velocity and acceleration of P. It is convenient to first obtain the velocity and acceleration of P relative to the nonrotating earth, and then use Eqs. (1.66) and (1.70) to calculate their inertial values.

The relative position vector can be written

$$\mathbf{r}_{\text{rel}} = (R_E + z)\hat{\mathbf{k}} \tag{1.71}$$

where R_E is the radius of the earth, and z is the height of P above the earth (i.e., its altitude). The time derivative of $\mathbf{r}_{\text{rel}}$ is the velocity $\mathbf{v}_{\text{rel}}$ relative to the nonrotating earth,

$$\mathbf{v}_{\text{rel}} = \frac{\mathrm{d}\mathbf{r}_{\text{rel}}}{\mathrm{d}t} = \dot{z}\hat{\mathbf{k}} + (R_E + z)\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} \tag{1.72}$$

To calculate $\mathrm{d}\hat{\mathbf{k}}/\mathrm{d}t$, we must use Eq. (1.52). The angular velocity $\boldsymbol{\omega}$ of the xyz frame relative to the nonrotating earth is found in terms of the rates of change of latitude ϕ and longitude Λ,

$$\boldsymbol{\omega} = -\dot{\phi}\hat{\mathbf{i}} + \dot{\Lambda}\cos\phi\hat{\mathbf{j}} + \dot{\Lambda}\sin\phi\hat{\mathbf{k}} \tag{1.73}$$

Thus,

$$\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} = \boldsymbol{\omega}\times\hat{\mathbf{k}} = \dot{\Lambda}\cos\phi\hat{\mathbf{i}} + \dot{\phi}\hat{\mathbf{j}} \tag{1.74}$$

Let us also record the following for future use:

$$\frac{\mathrm{d}\hat{\mathbf{j}}}{\mathrm{d}t} = \boldsymbol{\omega}\times\hat{\mathbf{j}} = -\dot{\Lambda}\sin\phi\hat{\mathbf{i}} - \dot{\phi}\hat{\mathbf{k}} \tag{1.75}$$

$$\frac{\mathrm{d}\hat{\mathbf{i}}}{\mathrm{d}t} = \boldsymbol{\omega}\times\hat{\mathbf{i}} = -\dot{\Lambda}\sin\phi\hat{\mathbf{j}} - \dot{\Lambda}\cos\phi\hat{\mathbf{k}} \tag{1.76}$$

Substituting Eq. (1.74) into Eq. (1.72) yields the velocity in the nonrotating frame resolved along the topocentric-horizon axes,

$$\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{1.77a}$$

where

$$\dot{x} = (R_E + z)\dot{\Lambda}\cos\phi \qquad \dot{y} = (R_E + z)\dot{\phi} \tag{1.77b}$$

It is convenient to use these results to express the rates of change of latitude and longitude in terms of the components of relative velocity over the earth's surface,

$$\dot{\phi} = \frac{\dot{y}}{R_E + z} \qquad \dot{\Lambda} = \frac{\dot{x}}{(R_E + z)\cos\phi} \tag{1.78}$$

The time derivatives of these two expressions are

$$\ddot{\phi} = \frac{(R_E + z)\ddot{y} - \dot{y}\dot{z}}{(R_E + z)^2} \qquad \ddot{\Lambda} = \frac{(R_E + z)\ddot{x}\cos\phi - (\dot{z}\cos\phi - \dot{y}\sin\phi)\dot{x}}{(R_E + z)^2\cos^2\phi} \tag{1.79}$$

The acceleration of P relative to the nonrotating earth is found by taking the time derivative of $\mathbf{v}_{\text{rel}}$. From Eqs. (1.77a) and (1.77b) we thereby obtain

$$\begin{aligned}
\mathbf{a}_{\text{rel}} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} + \dot{x}\frac{d\hat{\mathbf{i}}}{dt} + \dot{y}\frac{d\hat{\mathbf{j}}}{dt} + \dot{z}\frac{d\hat{\mathbf{k}}}{dt} \\
&= [\dot{z}\dot{\Lambda}\cos\phi + (R_E + z)\ddot{\Lambda}\cos\phi - (R_E + z)\dot{\phi}\dot{\Lambda}\sin\phi]\hat{\mathbf{i}} + [\dot{z}\dot{\phi} + (R_E + z)\ddot{\phi}]\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \\
&\quad + (R_E + z)\dot{\Lambda}\cos\phi\left(\boldsymbol{\omega} \times \hat{\mathbf{i}}\right) + (R_E + z)\dot{\phi}\left(\boldsymbol{\omega} \times \hat{\mathbf{j}}\right) + \dot{z}\left(\boldsymbol{\omega} \times \hat{\mathbf{k}}\right)
\end{aligned}$$

Substituting Eq. (1.74) through Eq. (1.76) together with Eqs. (1.78) and (1.79) into this expression yields, upon simplification,

$$\mathbf{a}_{\text{rel}} = \left[\ddot{x} + \frac{\dot{x}(\dot{z} - \dot{y}\tan\phi)}{R_E + z}\right]\hat{\mathbf{i}} + \left(\ddot{y} + \frac{\dot{y}\dot{z} + \dot{x}^2\tan\phi}{R_E + z}\right)\hat{\mathbf{j}} + \left(\ddot{z} - \frac{\dot{x}^2 + \dot{y}^2}{R_E + z}\right)\hat{\mathbf{k}} \tag{1.80}$$

Observe that the curvature of the earth's surface is neglected by letting $R_E + z$ become infinitely large, in which case

$$\left.\mathbf{a}_{\text{rel}}\right)_{\text{neglecting earth's curvature}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}$$

That is, for a "flat earth," the components of the relative acceleration vector are just the derivatives of the components of the relative velocity vector.

For the absolute velocity we have, according to Eq. (1.66),

$$\mathbf{v} = \mathbf{v}_C + \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}} \tag{1.81}$$

From Fig. 1.18, it can be seen that $\hat{\mathbf{K}} = \cos\phi\hat{\mathbf{j}} + \sin\phi\hat{\mathbf{k}}$, which means the angular velocity of the earth is

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{K}} = \Omega\cos\phi\hat{\mathbf{j}} + \Omega\sin\phi\hat{\mathbf{k}} \tag{1.82}$$

Substituting this, together with Eqs. (1.71) and (1.77a) and the fact that $\mathbf{v}_C = \mathbf{0}$, into Eq. (1.81) yields

$$\mathbf{v} = [\dot{x} + \Omega(R_E + z)\cos\phi]\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{1.83}$$

From Eq. (1.70) the absolute acceleration of P is

$$\mathbf{a} = \mathbf{a}_C + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

Since $\mathbf{a}_C = \dot{\boldsymbol{\Omega}} = \mathbf{0}$, we find, upon substituting Eqs. (1.71), (1.77a), (1.80), and (1.82), that

$$\begin{aligned}
\mathbf{a} = &\left[\ddot{x} + \frac{\dot{x}(\dot{z} - \dot{y}\tan\phi)}{R_E + z} + 2\Omega(\dot{z}\cos\phi - \dot{y}\sin\phi)\right]\hat{\mathbf{i}} \\
&+ \left\{\ddot{y} + \frac{\dot{y}\dot{z} + \dot{x}^2\tan\phi}{R_E + z} + \Omega\sin\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{j}} \\
&+ \left\{\ddot{z} - \frac{\dot{x}^2 + \dot{y}^2}{R_E + z} - \Omega\cos\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{k}}
\end{aligned} \tag{1.84}$$

Some special cases of Eqs. (1.83) and (1.84) follow.

Straight and level, unaccelerated flight: $\dot{z} = \ddot{z} = \ddot{x} = \ddot{y} = 0$

$$\mathbf{v} = [\dot{x} + \Omega(R_E + z)\cos\phi]\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} \tag{1.85a}$$

$$\begin{aligned}\mathbf{a} = &-\left[\frac{\dot{x}\dot{y}\tan\phi}{R_E + z} + 2\Omega\dot{y}\sin\phi\right]\hat{\mathbf{i}} + \left\{\frac{\dot{x}^2\tan\phi}{R_E + z} + \Omega\sin\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{j}} \\ &-\left\{\frac{\dot{x}^2 + \dot{y}^2}{R_E + z} + \Omega\cos\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{k}}\end{aligned} \tag{1.85b}$$

Flight due north (y) at a constant speed and altitude: $\dot{z} = \ddot{z} = \dot{x} = \ddot{x} = \ddot{y} = 0$

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} \tag{1.86a}$$

$$\mathbf{a} = -2\Omega\dot{y}\sin\phi\hat{\mathbf{i}} + \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} - \left[\frac{\dot{y}^2}{R_E + z} + \Omega^2(R_E + z)\cos^2\phi\right]\hat{\mathbf{k}} \tag{1.86b}$$

Flight due east (x) at a constant speed and altitude: $\dot{z} = \ddot{z} = \ddot{x} = \dot{y} = \ddot{y} = 0$

$$\mathbf{v} = [\dot{x} + \Omega(R_E + z)\cos\phi]\hat{\mathbf{i}} \tag{1.87a}$$

$$\begin{aligned}\mathbf{a} = &\left\{\frac{\dot{x}^2\tan\phi}{R_E + z} + \Omega\sin\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{j}} \\ &-\left\{\frac{\dot{x}^2}{R_E + z} + \Omega\cos\phi[\Omega(R_E + z)\cos\phi + 2\dot{x}]\right\}\hat{\mathbf{k}}\end{aligned} \tag{1.87b}$$

Flight straight up (z): $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = 0$

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} + \dot{z}\hat{\mathbf{k}} \tag{1.88a}$$

$$\mathbf{a} = 2\Omega(\dot{z}\cos\phi)\hat{\mathbf{i}} + \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} + \left[\ddot{z} - \Omega^2(R_E + z)\cos^2\phi\right]\hat{\mathbf{k}} \tag{1.88b}$$

Stationary: $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} \tag{1.89a}$$

$$\mathbf{a} = \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} - \Omega^2(R_E + z)\cos^2\phi\hat{\mathbf{k}} \tag{1.89b}$$

EXAMPLE 1.14

An airplane of mass 70,000 kg is traveling due north at a latitude 30°N, at an altitude of 10 km (32,800 ft), with a speed of 300 m/s (671 mph). Calculate (a) the components of the absolute velocity and acceleration along the axes of the topocentric-horizon reference frame and (b) the net force on the airplane. Assume the winds aloft are zero.

Solution

(a) First, using the sidereal rotation period of the earth in Table A.1, we note that the earth's angular velocity is

$$\Omega = \frac{2\pi\,\text{radians}}{\text{sidereal day}} = \frac{2\pi\,\text{radians}}{23.93\text{h}} = \frac{2\pi\,\text{radians}}{86{,}160\,\text{s}} = 7.292 \times 10^{-5}\,\text{radians/s}$$

From Eq. (1.86a), the absolute velocity is

$$\mathbf{v} = \Omega(R_E + z)\cos\phi\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} = \left[\left(7.292 \times 10^{-5}\right) \cdot (6378 + 10) \cdot 10^3 \cos 30°\right]\hat{\mathbf{i}} + 300\hat{\mathbf{j}}$$

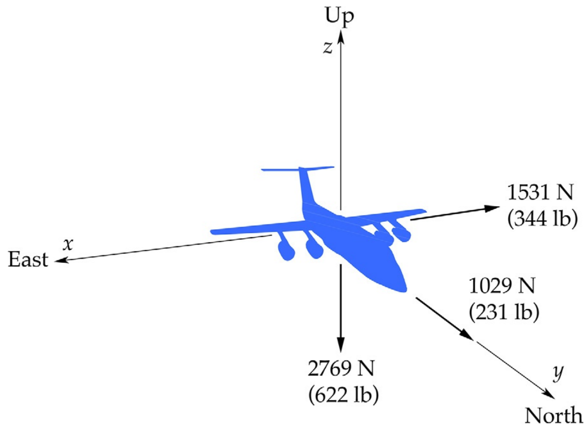


FIG. 1.19

Components of the net force on the airplane.

or

$$\mathbf{v} = 403.4\hat{\mathbf{i}} + 300\hat{\mathbf{j}}\ (\text{m/s})$$

The 403.4 m/s (901 mph) component of velocity to the east (x direction) is due entirely to the earth's rotation.

From Eq. (1.86b), the absolute acceleration is

$$\begin{aligned}
\mathbf{a} &= -2\Omega\dot{y}\sin\phi\hat{\mathbf{i}} + \Omega^2(R_E + z)\sin\phi\cos\phi\hat{\mathbf{j}} - \left[\frac{\dot{y}^2}{R_E + z} + \Omega^2(R_E + z)\cos^2\phi\right]\hat{\mathbf{k}} \\
&= -2\left(7.292\times10^{-5}\right)\cdot 300\cdot \sin 30^\circ\hat{\mathbf{i}} \\
&\quad + \left(7.292\times10^{-5}\right)^2\cdot(6378+10)\cdot10^3\cdot\sin30^\circ\cdot\cos30^\circ\hat{\mathbf{j}} \\
&\quad - \left[\frac{300^2}{(6378+10)\cdot10^3} + \left(7.292\times10^{-5}\right)^2\cdot(6378+10)\cdot10^3\cdot\cos^2 30^\circ\right]\hat{\mathbf{k}}
\end{aligned}$$

or

$$\boxed{\mathbf{a} = -0.02187\hat{\mathbf{i}} + 0.01471\hat{\mathbf{j}} - 0.03956\hat{\mathbf{k}}\ (\text{m/s}^2)}$$

The westward (negative x) acceleration of $0.02187\,\text{m/s}^2$ is the Coriolis acceleration.

(b) Since the acceleration in part (a) is the absolute acceleration, we can use it in Newton's law to calculate the net force on the airplane,

$$\begin{aligned}
\mathbf{F}_{\text{net}} = m\mathbf{a} &= 70{,}000\left(-0.02187\hat{\mathbf{i}} + 0.01471\hat{\mathbf{j}} - 0.03956\hat{\mathbf{k}}\right) \\
&= \boxed{-1531\hat{\mathbf{i}} + 1029\hat{\mathbf{j}} - 2769\hat{\mathbf{k}}\ (\text{N})}
\end{aligned}$$

Fig. 1.19 shows the components of this relatively small force. The forward (y) and downward (negative z) forces are in the directions of the airplane's centripetal acceleration, caused by the earth's rotation and, in the case of the downward force, by the earth's curvature as well. The westward force is in the direction of the Coriolis acceleration, which is due to the combined effects of the earth's rotation and the motion of the airplane. These net external forces must exist if the airplane is to fly in the prescribed path.

In the vertical direction, the net force is that of the upward lift L of the wings plus the downward weight W of the aircraft, so that

$$F_{\text{net}})_z = L - W = -2769 \ \Rightarrow\ L = W - 2769\text{N}$$

Thus, the effect of the earth's rotation and curvature is to apparently produce an outward *centrifugal force*, reducing the weight of the airplane a bit, in this case by about 0.4%. The fictitious centrifugal force also increases the apparent drag in the flight direction by 1029 N. That is, in the flight direction

$$F_{\text{net}})_y = T - D = 1029\,\text{N}$$

where T is the thrust and D is the drag. Hence

$$T = D + 1029\,(\text{N})$$

The 1531-N force to the left, produced by crabbing the airplane very slightly in that direction, is required to balance the fictitious Coriolis force, which would otherwise cause the airplane to deviate to the right of its flight path.

1.8 NUMERICAL INTEGRATION

Analysis of the motion of a spacecraft leads to ordinary differential equations with time as the independent variable. It is often impractical if not impossible to solve them exactly. Therefore, the ability to solve differential equations numerically is important. In this section, we will take a look at a few common numerical integration schemes and investigate their accuracy and stability by applying them to some problems that do have an analytical solution.

Particle mechanics is based on Newton's second law (Eq. 1.38), which may be written as

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} \tag{1.90}$$

This is a second-order, ordinary differential equation for the position vector $\mathbf{r}$ as a function of time. Depending on the complexity of the force function $\mathbf{F}$, there may or may not be a closed-form, analytical solution of Eq. (1.90). In the most trivial case, the force vector $\mathbf{F}$ and the mass m are constant, which means we can use elementary calculus to integrate Eq. (1.90) twice to get

$$\mathbf{r} = \frac{\mathbf{F}}{2m}t^2 + \mathbf{C}_1 t + \mathbf{C}_2 \quad (\mathbf{F}\text{ and }m\text{ constant}) \tag{1.91}$$

$\mathbf{C}_1$ and $\mathbf{C}_2$ are the two vector constants of integration. Since each vector has three components, there are a total of six scalar constants of integration. If the position and velocity are both specified at time $t = 0$ to be $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, respectively, then we have an initial value problem. Applying the initial conditions to Eq. (1.91), we find $\mathbf{C}_1 = \dot{\mathbf{r}}_0$ and $\mathbf{C}_2 = \mathbf{r}_0$, which means

$$\mathbf{r} = \frac{\mathbf{F}}{2m}t^2 + \dot{\mathbf{r}}_0 t + \mathbf{r}_0 \quad (\mathbf{F}\text{ and }m\text{ constant})$$

On the other hand, we may know the position $\mathbf{r}_0$ at $t = 0$ and the velocity $\dot{\mathbf{r}}_f$ at a later time $t = t_f$. These are boundary conditions and this is an example of a boundary value problem. Applying the boundary conditions to Eq. (1.91) yields $\mathbf{C}_1 = \dot{\mathbf{r}}_f - (\mathbf{F}/m)t_f$ and $\mathbf{C}_2 = \mathbf{r}_0$, which means

$$\mathbf{r} = \frac{\mathbf{F}}{2m}t^2 + \left(\dot{\mathbf{r}}_f - \frac{\mathbf{F}}{m}t_f\right)t + \mathbf{r}_0 \quad (\mathbf{F}\text{ and }m\text{ constant})$$

For the remainder of this section we will focus on the numerical solution of initial value problems only.

In general, the function $\mathbf{F}$ in Eq. (1.90) is not constant but is instead a function of time t, position $\mathbf{r}$, and velocity $\dot{\mathbf{r}}$. That is, $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$. Let us resolve the vector $\mathbf{r}$ and its derivatives as well as the force $\mathbf{F}$ into their Cartesian components in three-dimensional space:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \quad \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \quad \mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

The three components of Eq. (1.90) are

$$\ddot{x} = \frac{F_x(t, \mathbf{r}, \dot{\mathbf{r}})}{m} \quad \ddot{y} = \frac{F_y(t, \mathbf{r}, \dot{\mathbf{r}})}{m} \quad \ddot{z} = \frac{F_z(t, \mathbf{r}, \dot{\mathbf{r}})}{m} \tag{1.92}$$

These are three second-order differential equations. For the purpose of numerical solution, they must be reduced to six first-order differential equations. This is accomplished by introducing six auxiliary variables y_1 through y_6, defined as follows:

$$\begin{aligned} y_1 &= x \quad & y_2 &= y \quad & y_3 &= z \\ y_4 &= \dot{x} \quad & y_5 &= \dot{y} \quad & y_6 &= \dot{z} \end{aligned} \tag{1.93}$$

In terms of these auxiliary variables, the position and velocity vectors are

$$\mathbf{r} = y_1\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + y_3\hat{\mathbf{k}} \quad \dot{\mathbf{r}} = y_4\hat{\mathbf{i}} + y_5\hat{\mathbf{j}} + y_6\hat{\mathbf{k}}$$

Taking the derivative $\mathrm{d}/\mathrm{d}t$ of each of the six expressions in Eq. (1.93) yields

$$\begin{aligned} \mathrm{d}y_1/\mathrm{d}t &= \dot{x} \quad & \mathrm{d}y_2/\mathrm{d}t &= \dot{y} \quad & \mathrm{d}y_3/\mathrm{d}t &= \dot{z} \\ \mathrm{d}y_4/\mathrm{d}t &= \ddot{x} \quad & \mathrm{d}y_5/\mathrm{d}t &= \ddot{y} \quad & \mathrm{d}y_6/\mathrm{d}t &= \ddot{z} \end{aligned}$$

Upon substituting Eqs. (1.92) and (1.93), we arrive at the six first-order differential equations

$$\begin{aligned} \dot{y}_1 &= y_4 \\ \dot{y}_2 &= y_5 \\ \dot{y}_3 &= y_6 \\ \dot{y}_4 &= \frac{F_x(t, y_1, y_2, y_3, y_4, y_5, y_6)}{m} \\ \dot{y}_5 &= \frac{F_y(t, y_1, y_2, y_3, y_4, y_5, y_6)}{m} \\ \dot{y}_6 &= \frac{F_z(t, y_1, y_2, y_3, y_4, y_5, y_6)}{m} \end{aligned} \tag{1.94}$$

These equations are coupled because the right-hand side of each one contains variables that belong to other equations as well. Eq. (1.94) can be written more compactly in vector notation as

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{1.95}$$

where the column vectors $\mathbf{y}$, $\dot{\mathbf{y}}$, and $\mathbf{f}$ are

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} \quad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \dot{y}_6 \end{Bmatrix} \quad \mathbf{f} = \begin{Bmatrix} y_4 \\ y_5 \\ y_6 \\ F_x(t, \mathbf{y})/m \\ F_y(t, \mathbf{y})/m \\ F_z(t, \mathbf{y})/m \end{Bmatrix} \tag{1.96}$$

Note that in this case $\mathbf{f}(t, \mathbf{y})$ is shorthand for $\mathbf{f}(t, y_1, y_2, y_3, y_4, y_5, y_6)$. Any set of one or more ordinary differential equations of any order can be cast in the form of Eq. (1.95).

EXAMPLE 1.15

Write the third-order nonlinear differential equation

$$\dddot{x} - x\ddot{x} + \dot{x}^2 = 0 \tag{a}$$

as three first-order differential equations.

Solution

Introducing the three auxiliary variables

$$y_1 = x \qquad y_2 = \dot{x} \qquad y_3 = \ddot{x} \tag{b}$$

we take the derivative of each one to get

$$\begin{aligned} dy_1/dt &= dx/dt = \dot{x} \\ dy_2/dt &= d\dot{x}/dt = \ddot{x} \\ dy_3/dt &= d\ddot{x}/dt = \dddot{x} \overbrace{=}^{\text{From (a)}} x\ddot{x} - \dot{x}^2 \end{aligned}$$

Substituting Eq. (b) on the right of these expressions yields

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_1 y_3 - y_2^2 \end{aligned} \tag{c}$$

This is a system of three first-order, coupled ordinary differential equations. It is an autonomous system, since time t does not appear explicitly on the right-hand side. The three equations can therefore be written compactly as $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$.

Before discussing some numerical integration schemes, it will be helpful to review the concept of the Taylor series, named after the English mathematician Brook Taylor (1685–1731). Recall from calculus that if we know the value of a function $g(t)$ at time t and wish to approximate its value at a neighboring time $t + h$, we can use the Taylor series to express $g(t + h)$ as an infinite power series in h,

$$g(t+h) = g(t) + c_1 h + c_2 h^2 + c_3 h^3 + \cdots + c_n h^n + O\left(h^{n+1}\right) \tag{1.97}$$

The coefficients c_m are found by taking successively higher order derivatives of $g(t)$ according to the formula

$$c_m = \frac{1}{m!}\frac{d^m g(t)}{dt^m} \tag{1.98}$$

$O(h^{n+1})$ ("order of h to the $n+1$") means that the remaining terms of this infinite series all have h^{n+1} as a factor. In other words,

$$\lim_{h \to 0} \frac{O(h^{n+1})}{h^{n+1}} = c_{n+1}$$

$O(h^{n+1})$ is the truncation error due to retaining only terms up to h^n. The order of a Taylor series expansion is the highest power of h retained. The more terms of the Taylor series that we keep, the more accurate will be the representation of the function $g(t + h)$ in the neighborhood of t. Reducing h lowers the truncation error. For example, if we reduce h to $h/2$, then $O(h^n)$ goes down by a factor of $(1/2)^n$.

EXAMPLE 1.16

Expand the function $\sin(t+h)$ in a Taylor series about $t=1$. Plot the Taylor series of order 1, 2, 3, and 4 and compare them with $\sin(1+h)$ for $-2<h<2$.

Solution

The nth-order Taylor power series expansion of $\sin(t+h)$ is written

$$\sin(t+h)=p_n(h)$$

where, according to Eqs. (1.97) and (1.98), the polynomial p_n is given by

$$p_n(h)=\sum_{m=0}^{n}\frac{h^m}{m!}\frac{\mathrm{d}^m\sin t}{\mathrm{d}t^m}$$

Thus, the zeroth- through fourth-order Taylor series polynomials in h are

$$\begin{aligned}
p_0&=\frac{h^0}{0!}\frac{\mathrm{d}^0\sin t}{\mathrm{d}t^0}=\sin t\\
p_1&=p_0+\frac{h}{1!}\frac{\mathrm{d}\sin t}{\mathrm{d}t}=\sin t+h\cos t\\
p_2&=p_1+\frac{h^2}{2!}\frac{\mathrm{d}^2\sin t}{\mathrm{d}t^2}=\sin t+h\cos t-\frac{h^2}{2}\sin t\\
p_3&=p_2+\frac{h^3}{3!}\frac{\mathrm{d}^3\sin t}{\mathrm{d}t^3}=\sin t+h\cos t-\frac{h^2}{2}\sin t-\frac{h^3}{6}\cos t\\
p_4&=p_3+\frac{h^4}{4!}\frac{\mathrm{d}^4\sin t}{\mathrm{d}t^4}=\sin t+h\cos t-\frac{h^2}{2}\sin t-\frac{h^3}{6}\cos t+\frac{h^4}{24}\sin t
\end{aligned}$$

For $t=1$, p_1 through p_4 as well as $\sin(t+h)$ are plotted in Fig. 1.20. As expected, we see that the higher degree Taylor polynomials for $\sin(1+h)$ lie closer to $\sin(1+h)$ over a wider range of h.

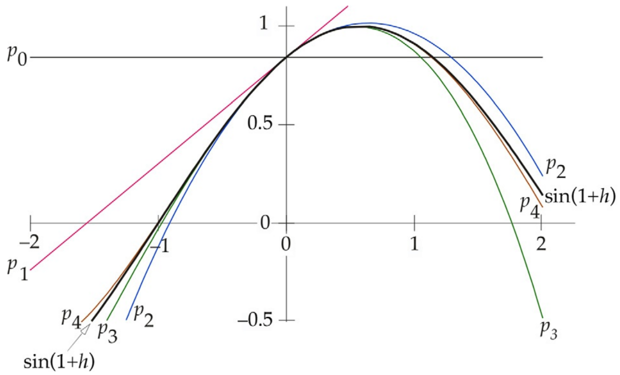


FIG. 1.20

Plots of zeroth- to fourth-order Taylor series expansions of $\sin(1+h)$.

The numerical integration schemes that we shall examine are designed to solve first-order ordinary differential equations of the form shown in Eq. (1.95). To obtain a numerical solution of $\dot{\mathbf{y}}=\mathbf{f}(t,\mathbf{y})$ over the time interval t_0 to t_f, we divide or "mesh" the interval into N discrete times t_1, t_2, t_3, … , t_N, where $t_1=t_0$ and $t_N=t_f$. The step size h is the difference between two adjacent times on the mesh (i.e., $h = t_{i+1} - t_i$). h may be constant for all steps across the entire time span t_0 to t_f. Modern methods have adaptive step size control in which h varies from step to step to provide better accuracy and efficiency.

Let us denote the values of $\mathbf{y}$ and $\dot{\mathbf{y}}$ at time t_i as $\mathbf{y}_i$ and $\mathbf{f}_i$, respectively, where $\mathbf{f}_i = \mathbf{f}(t_i, \mathbf{y}_i)$. In an initial value problem, the values of all components of $\mathbf{y}$ at the initial time t_0 together with Eq. (1.95) provide the information needed to determine $\mathbf{y}$ at the subsequent discrete times.

1.8.1 RUNGE-KUTTA METHODS

The Runge-Kutta (*RK*) methods were originally developed by the German mathematicians Carl Runge (1856–1927) and Martin Kutta (1867–1944). In the explicit, single-step *RK* methods, $\mathbf{y}_{i+1}$ at $t_i + h$ is obtained from $\mathbf{y}_i$ at t_i by the formula

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\boldsymbol{\phi}\left(t_{i,}\mathbf{y}_i, h\right) \tag{1.99}$$

The increment function $\boldsymbol{\phi}$ is an average of the derivative $\mathrm{d}\mathbf{y}/\mathrm{d}t$ over the time interval t_i to $t_i + h$. This average is obtained by evaluating the derivative $\mathbf{f}(t, \mathbf{y})$ at several points or "stages" within the time interval. The order of an *RK* method reflects the accuracy to which $\boldsymbol{\phi}$ is computed, compared with a Taylor series expansion. An *RK* method of order p is called an *RKp* method. An *RKp* method is as accurate in computing $\mathbf{y}_i$ from Eq. (1.99) as is the pth-order Taylor series

$$\mathbf{y}(t_i + h) = \mathbf{y}_i + \mathbf{c}_1 h + \mathbf{c}_2 h^2 + \cdots \mathbf{c}_p h^p \tag{1.100}$$

An attractive feature of the *RK* schemes is that only the first derivative $\mathbf{f}(t, \mathbf{y})$ is required, and it is available from the differential equation itself (Eq. (1.95)). By contrast, the pth-order Taylor series expansion in Eq. (1.100) requires computing all derivatives of $\mathbf{y}$ through order p.

The higher the *RK* order, the more stages there are and the more accurate is $\boldsymbol{\phi}$. The number of stages equals the order of the *RK* method if the order is less than 5. If the number of stages is s, then there are s times $\tilde{t}$ within the interval t_i to $t_i + h$ at which we evaluate the derivatives $\mathbf{f}(t, \mathbf{y})$. These times are given by specifying numerical values of the nodes a_m in the expression

$$\tilde{t}_m = t_i + a_m h \quad m = 1, 2, \ldots, s$$

At each of these times the value of $\tilde{\mathbf{y}}$ is obtained by providing numerical values for the coupling coefficients b_{mn} in the formula

$$\tilde{\mathbf{y}}_m = \mathbf{y}_i + h\sum_{n=1}^{m-1} b_{mn}\tilde{\mathbf{f}}_n \quad m = 1, 2, \ldots, s \tag{1.101}$$

The vector of derivatives $\tilde{\mathbf{f}}_m$ is evaluated at stage m by substituting $\tilde{t}_m$ and $\tilde{\mathbf{y}}_m$ into Eq. (1.95),

$$\tilde{\mathbf{f}}_m = \mathbf{f}\left(\tilde{t}_m, \tilde{\mathbf{y}}_m\right) \quad m = 1, 2, \ldots, s \tag{1.102}$$

The increment function $\boldsymbol{\phi}$ is a weighted sum of the derivatives $\tilde{\mathbf{f}}_m$ over the s stages within the time interval t_i to $t_i + h$,

$$\boldsymbol{\phi} = \sum_{m=1}^{s} c_m \tilde{\mathbf{f}}_m \tag{1.103}$$

The coefficients c_m are known as the weights. Substituting Eq. (1.103) into Eq. (1.99) yields

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\sum_{m=1}^{s} c_m \widetilde{\mathbf{f}}_m \tag{1.104}$$

The numerical values of the coefficients a_m, b_{mn}, and c_m depend on which *RK* method is being used. It is convenient to write these coefficients as arrays, so that

$$\{\mathbf{a}\} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{Bmatrix} \quad [\mathbf{b}] = \begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & \\ \vdots & \vdots & \cdots & \\ b_{s1} & b_{s2} & \cdots & b_{s,s-1} \end{bmatrix} \quad \{\mathbf{c}\} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{Bmatrix} \tag{1.105}$$

where s is the number of stages, and $[\mathbf{b}]$ is undefined if $s = 1$. The nodes $\{\mathbf{a}\}$, coupling coefficients $[\mathbf{b}]$, and weights $\{\mathbf{c}\}$ for a given *RK* method are not necessarily unique, although research favors the choice of some sets over others. Details surrounding the derivation of these coefficients as well as in-depth discussions of not only *RK* but also the numerous other common numerical integration techniques may be found in numerical analysis textbooks, such as the one by Butcher (2008).

For *RK* orders 1–4, we list below the commonly used values of the coefficients (Eq. 1.105), the resulting formula for the derivatives $\widetilde{\mathbf{f}}$ at each stage (Eq. 1.102), and the formula for the difference $\mathbf{y}_{i+1} - \mathbf{y}_i$ (Eq. 1.104). These *RK* schemes all use a uniform step size h.

RK1 (Euler's method)

$$\begin{aligned} \{\mathbf{a}\} &= \{0\} \quad \{\mathbf{c}\} = \{1\} \\ \widetilde{\mathbf{f}}_1 &= \mathbf{f}(t_i, y_i) \\ \mathbf{y}_{i+1} &= \mathbf{y}_i + h\widetilde{\mathbf{f}}_1 \end{aligned} \tag{1.106}$$

RK2 (Heun's method)

$$\begin{aligned} &\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [\mathbf{b}] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \{\mathbf{c}\} = \begin{Bmatrix} 1/2 \\ 1/2 \end{Bmatrix} \\ &\widetilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \quad \widetilde{\mathbf{f}}_2 = \mathbf{f}\left(t_i + h, \mathbf{y}_i + h\widetilde{\mathbf{f}}_1\right) \\ &\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(\frac{1}{2}\widetilde{\mathbf{f}}_1 + \frac{1}{2}\widetilde{\mathbf{f}}_2\right) \end{aligned} \tag{1.107}$$

RK3

$$\begin{aligned} &\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1/2 \\ 1 \end{Bmatrix} \quad [\mathbf{b}] = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \\ -1 & 2 \end{bmatrix} \quad \{\mathbf{c}\} = \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix} \\ &\widetilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \quad \widetilde{\mathbf{f}}_2 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{y}_i + \frac{1}{2}h\widetilde{\mathbf{f}}_1\right) \quad \widetilde{\mathbf{f}}_3 = \mathbf{f}\left(t_i + h, \mathbf{y}_i + h\left[-\widetilde{\mathbf{f}}_1 + 2\widetilde{\mathbf{f}}_2\right]\right) \\ &\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(\frac{1}{6}\widetilde{\mathbf{f}}_1 + \frac{2}{3}\widetilde{\mathbf{f}}_2 + \frac{1}{6}\widetilde{\mathbf{f}}_3\right) \end{aligned} \tag{1.108}$$

RK4

$$\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1 \end{Bmatrix} \quad [\mathbf{b}] = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \{\mathbf{c}\} = \begin{Bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{Bmatrix}$$

$$\tilde{\mathbf{f}}_1 = \mathbf{f}(t_i, \mathbf{y}_i) \quad \tilde{\mathbf{f}}_2 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{y}_i + \frac{1}{2}h\tilde{\mathbf{f}}_1\right) \quad \tilde{\mathbf{f}}_3 = \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{y}_i + \frac{1}{2}h\tilde{\mathbf{f}}_2\right) \tag{1.109}$$

$$\tilde{\mathbf{f}}_4 = \mathbf{f}\left(t_i + h, \mathbf{y}_i + h\tilde{\mathbf{f}}_3\right)$$

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(\frac{1}{6}\tilde{\mathbf{f}}_1 + \frac{1}{3}\tilde{\mathbf{f}}_2 + \frac{1}{3}\tilde{\mathbf{f}}_3 + \frac{1}{6}\tilde{\mathbf{f}}_4\right)$$

Observe that in each of the four cases the sum of the components of $\{\mathbf{c}\}$ is 1 and the sum of each row of $[\mathbf{b}]$ equals the value in that row of $\{\mathbf{a}\}$. This is a characteristic of the *RK* methods.

ALGORITHM 1.1

Given the vector $\mathbf{y}$ at time t, the derivatives $\mathbf{f}(t, \mathbf{y})$, and the step size h, use one of the methods *RK1* through *RK4* to find $\mathbf{y}$ at time $t + h$. See Appendix D.2 for a MATLAB implementation of this algorithm in the form of the function *rk1_4.m*. *rk1_4.m* executes any of the four *RK* methods according to whether the variable *rk* passed to the function has the value 1, 2, 3, or 4.

1. Evaluate the derivatives $\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2, \ldots, \tilde{\mathbf{f}}_s$ at stages 1 through s by means of Eq. (1.102).
2. Use Eq. (1.104) to compute $\mathbf{y}(t+h) = \mathbf{y}(t) + h\sum_{m=1}^{s}\tilde{\mathbf{f}}_m$.

Repeat these steps to obtain $\mathbf{y}$ at subsequent times $t + 2h$, $t + 3h$, etc.

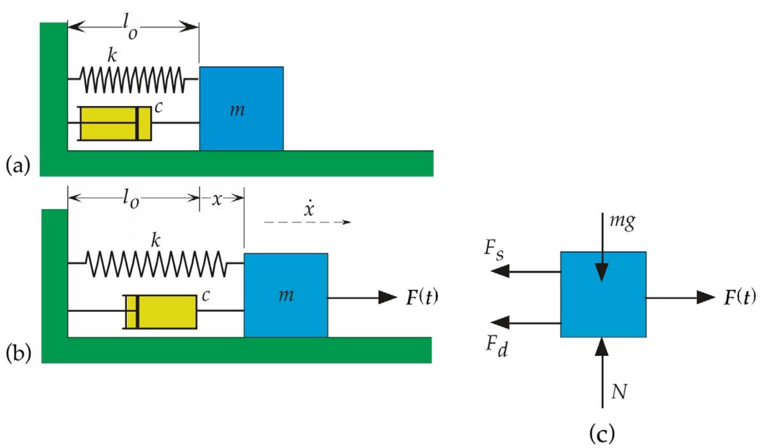


FIG. 1.21

A damped spring-mass system with a forcing function applied to the mass. (a) At rest. (b) In motion under the action of the applied force $F(t)$. (c) Free body diagram at any instant.

Let us employ the *RK* methods and Algorithm 1.1 to solve for the motion of the well-known viscously damped spring-mass system pictured in Fig. 1.21. The spring has an unstretched length l_0 and a spring constant k. The viscous damping coefficient is c and the mass of the block, which slides on a frictionless surface, is m. A forcing function $F(t)$ is applied to the mass. From the free body diagram in part (c) of the figure, we obtain the equation of motion of this one-dimensional system in the x direction.

$$-F_s - F_d + F(t) = m\ddot{x} \tag{1.110}$$

where F_s and F_d are the forces of the spring and dashpot, respectively. Since $F_s = kx$ and $F_d = c\dot{x}$, Eq. (1.110), after dividing through by the mass, can be rewritten as

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \tag{1.111}$$

The spring rate k and the mass m determine the natural circular frequency of vibration of the system, $\omega_n = \sqrt{k/m}$ (radians per second). Furthermore, the damping coefficient c may be expressed as $c = 2\zeta m\omega_n$, where ζ is the dimensionless damping factor ($\zeta \geq 0$). Making these substitutions in Eq. (1.111), we get the standard form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \frac{F(t)}{m} \tag{1.112}$$

If the forcing function is sinusoidal with amplitude F_0 and circular frequency ω, then Eq. (1.112) becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \frac{F_0}{m}\sin\omega t \tag{1.113}$$

This second-order ordinary differential equation has a closed-form solution, which is found using procedures taught in a differential equations course. If the system is underdamped, which means $\zeta < 1$, then it can be verified by substitution that the solution of Eq. (1.113) is

$$\begin{aligned} x &= e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) \\ &\quad + \frac{F_0/m}{(\omega_n{}^2 - \omega^2)^2 + (2\omega\omega_n\zeta)}\left[\left(\omega_n{}^2 - \omega^2\right)\sin\omega t - 2\omega\omega_n\zeta\cos\omega t\right] \end{aligned} \tag{1.114a}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. The initial conditions determine the values of the coefficients A and B. If at $t = 0$, $x = x_0$, and $\dot{x} = \dot{x}_0$, it turns out that

$$\begin{aligned} A &= \zeta\frac{\omega_n}{\omega_d}x_0 + \frac{\dot{x}_0}{\omega_d} + \frac{\omega^2 + \left(2\zeta^2 - 1\right)\omega_n^2}{(\omega_n{}^2 - \omega^2)^2 + (2\omega\omega_n\zeta)^2}\frac{\omega}{\omega_d}\frac{F_0}{m} \\ B &= x_0 + \frac{2\omega\omega_n\zeta}{(\omega_n{}^2 - \omega^2)^2 + (2\omega\omega_n\zeta)^2}\frac{F_0}{m} \end{aligned} \tag{1.114b}$$

The transient term with the exponential factor in Eq. (1.114a) dies out eventually, leaving only the steady-state solution, which persists as long as the forcing function acts.

EXAMPLE 1.17

Plot Eq. (1.114a). from $t = 0$ to $t = 110$ s if $m = 1$ kg, $\omega_n = 1$ rad/s, $\zeta = 0.03$, $F_0 = 1$ N, $\omega = 0.4$rad/s and the initial conditions are $x = \dot{x} = 0$.

Solution

Substituting the given values into Eq. (1.114) yields

$$x = e^{-0.03t}[0.03399\cos(0.9995t) - 0.4750\sin(0.9995t)] + [1.190\sin(0.4t) - 0.03399\cos(0.4t)] \tag{1.115}$$

This function is plotted over the time span 0–110 s in Fig. 1.22. Observe that after about 80 s, the transient has damped out and the system vibrates at the same frequency as the forcing function (although slightly out of phase due to the small viscosity).

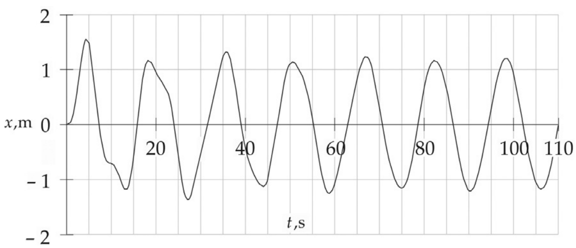


FIG. 1.22

Over time only the steady-state solution of Eq. (1.123) remains.

EXAMPLE 1.18

Solve Eq. (1.113) numerically, using the *RK* method and the data of Example 1.17. Compare the *RK* solution with the exact one, given by Eq. (1.115).

Solution

We must first reduce Eq. (1.113) to two first-order differential equations by introducing the two auxiliary variables

$$y_1 = x(t) \tag{a}$$

$$y_2 = \dot{x}(t) \tag{b}$$

Differentiating Eq. (a) we find

$$\dot{y}_1 = \dot{x}(t) = y_2(t) \tag{c}$$

Differentiating Eq. (b) and using Eq. (1.113) yields

$$\dot{y}_2 = \ddot{x}(t) = \frac{F_0}{m}\sin\omega t - \omega_n^2 y_1(t) - 2\zeta\omega_n y_2(t) \tag{d}$$

Systems (c) and (d) can be written compactly in standard vector notation as

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{e}$$

where

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \quad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} \quad \mathbf{f}(t, \mathbf{y}) = \begin{Bmatrix} y_2(t) \\ \dfrac{F_0}{m}\sin\omega t - {\omega_n}^2 y_1(t) - 2\zeta\omega_n y_2(t) \end{Bmatrix} \tag{1.116}$$

Eq. (1.116) is what we need to implement Algorithm 1.1 for this problem.

We will use the two MATLAB functions listed in Appendix D.2 (namely, *Example_1_18.m* and *rk1_4.m*). *Example_1_18.m* passes the data of Example 1.17 to the function *rk1_4.m*, which executes Algorithm 1.1 for *RK1*, *RK2*, *RK3*, and *RK4* over the time interval from 0 to 110 s. In each case, the problem is solved for two different values of the time step h. The subfunction *rates* within *Example_1_18.m* calculates the derivatives $\mathbf{f}(t, \mathbf{y})$ given in Eq. (1.116_3). The exact solution (Eq. 1.115) along with the four *RK* solutions is nondimensionalized and plotted at each time step in Fig. 1.23.

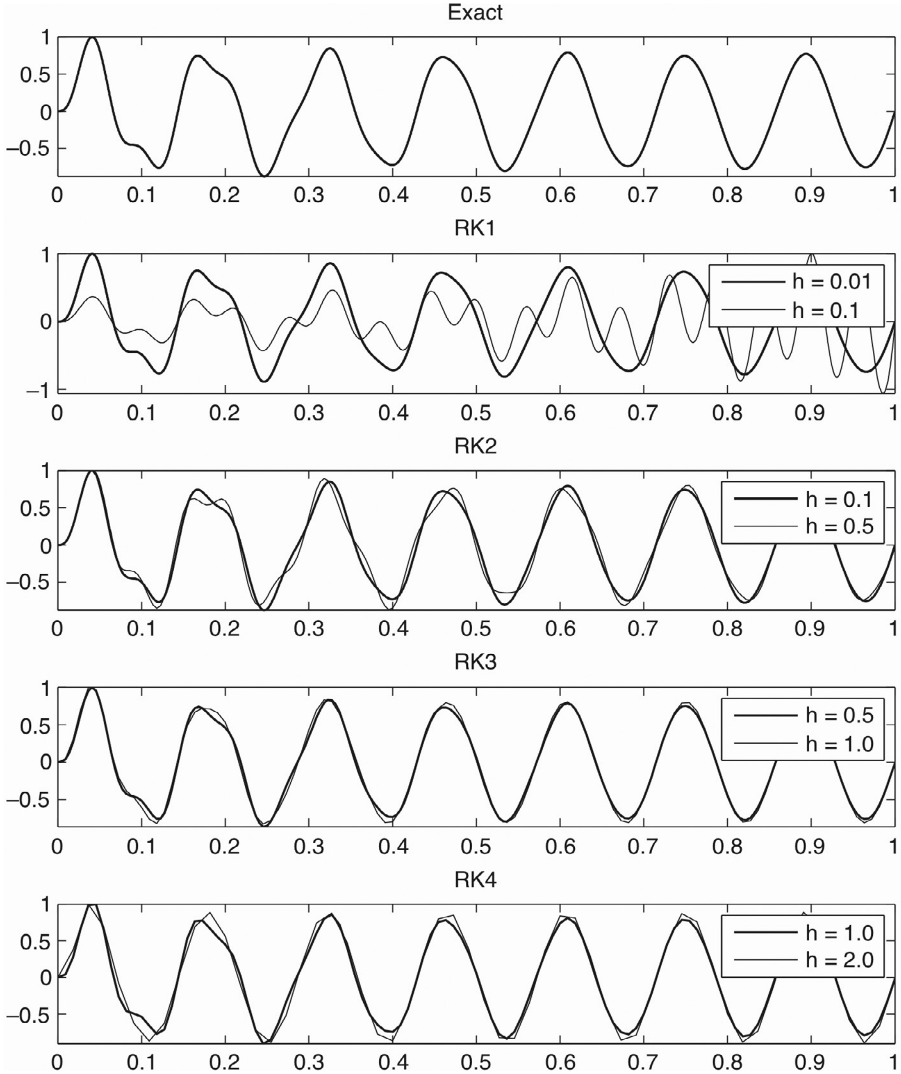


FIG. 1.23

x/x_{max} versus t/t_{max} for the *RK1* through *RK4* solutions of Eq. (1.123) using the data of Example 1.17. The exact solution is at the top.

We see that all the *RK* solutions agree closely with the analytical one for a sufficiently small step size. The figure shows, as expected, that to obtain accuracy, the uniform step size h must be reduced as the order of the *RK* method is reduced. Likewise, the figure suggests that a step size that yields inaccurate results for a given *RK* order may work just fine for the next higher order procedure.

1.8.2 HEUN'S PREDICTOR-CORRECTOR METHOD

As we have seen, the *RK1* method (Eq. 1.106) uses just $\tilde{\mathbf{f}}_1$, the derivative of $\mathbf{y}$ at the beginning of the time interval, to approximate the value of $\mathbf{y}$ at the end of the interval. The use of Eq. (1.106) for approximate numerical integration of nonlinear functions was introduced by Leonhard Euler in 1768 and is therefore known as Euler's method. *RK2* (Eq. 1.107) improves the accuracy by using the average of the derivatives $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$ at each end of the time interval. The predictor-corrector method due originally to the German mathematician Karl Heun (1859–1929) employs this idea.

First, we use *RK1* to estimate the value of $\mathbf{y}$ at t_{i+1}, labeling that approximation $\mathbf{y}^*_{i+1}$:

$$\mathbf{y}^*_{i+1} = \mathbf{y}_i + h\mathbf{f}\left(t_i, \mathbf{y}_i\right) \quad \text{(predictor)} \tag{1.117a}$$

$\mathbf{y}^*_{i+1}$ is then used to compute the derivative $\mathbf{f}$ at $t + h$, whereupon the average of the two derivatives is used to correct the estimate

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\frac{\mathbf{f}\left(t_i, \mathbf{y}_i\right) + \mathbf{f}\left(t_i + h, \mathbf{y}^*_{i+1}\right)}{2} \quad \text{(corrector)} \tag{1.117b}$$

We can iteratively improve the estimate of $\mathbf{y}_{i+1}$ by making the substitution $\mathbf{y}^*_{i+1} \leftarrow \mathbf{y}_{i+1}$ (where $\leftarrow$ means "is replaced by") and computing a new value of $\mathbf{y}_{i+1}$ from Eq. (1.117b). That process is repeated until the difference between $\mathbf{y}_{i+1}$ and $\mathbf{y}^*_{i+1}$ becomes acceptably small.

ALGORITHM 1.2

Given the vector $\mathbf{y}$ at time t and the derivatives $\mathbf{f}(t,\mathbf{y})$, use Heun's method to find $\mathbf{y}$ at time $t + h$. See Appendix D.3 for a MATLAB implementation of this algorithm (*heun.m*):

1. Evaluate the vector of derivatives $\mathbf{f}(t,\mathbf{y})$.
2. Compute the predictor $\mathbf{y}^*(t + h) = \mathbf{y}(t) + \mathbf{f}(t, \mathbf{y})h$.
3. Compute the corrector $\mathbf{y}(t+h) = \mathbf{y}(t) + \frac{h}{2}\{\mathbf{f}(t, \mathbf{y}) + \mathbf{f}[t+h, \mathbf{y}^*(t+h)]\}$.
4. Make the substitution $\mathbf{y}^*(t + h) \leftarrow \mathbf{y}(t + h)$ and use Step 3 to recompute $\mathbf{y}(t + h)$.
5. Repeat Step 4 until $\mathbf{y}(t + h) \approx \mathbf{y}^*(t + h)$ to within a given tolerance.

Repeat these steps to obtain $\mathbf{y}$ at subsequent times $t + 2h$, $t + 3h$, etc.

EXAMPLE 1.19

Employ Heun's method to solve Eq. (1.113) using the data provided in Example 1.17. Use two different time steps, $h = 1$ s and $h = 0.1$ s, and compare the results.

Solution

We use the MATLAB functions *Example_1_19.m* and *heun.m* listed in Appendix D.3. The function *Example_1_19.m* passes the given data to the function *heun.m*, which uses the subfunction *rates* within *Example_1_19.m* to compute the

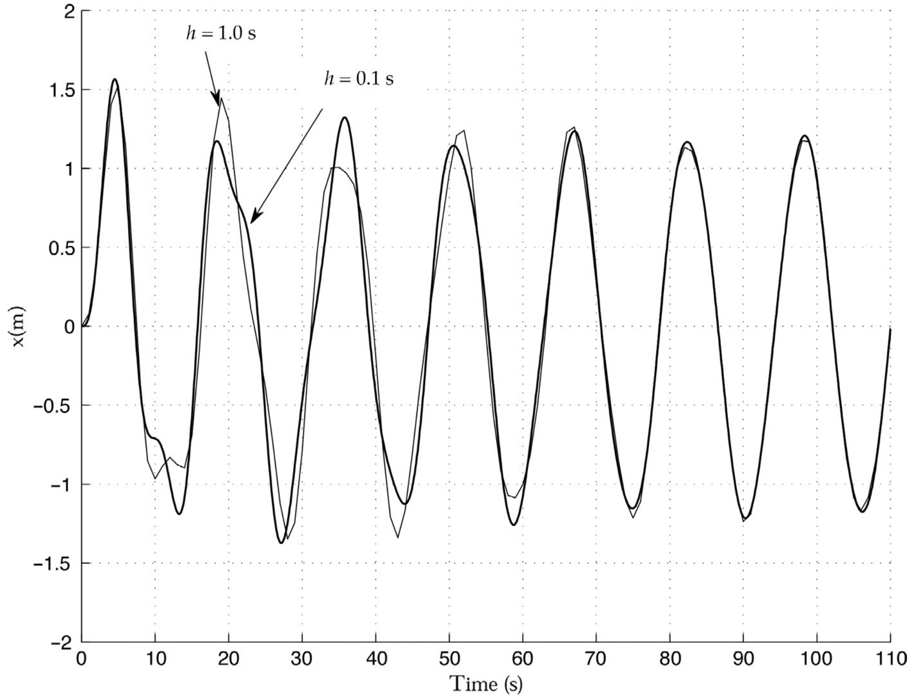


FIG. 1.24

Numerical solution of Eq. (1.123) using Heun's method with two different step sizes.

derivatives $\mathbf{f}(t, \mathbf{y})$ in Eq. (1.116_3). *heun.m* executes Algorithm 1.2 over the time interval from 0 to 110 s, once for $h = 1$ s and again for $h = 0.1$ s, and plots the output in each case, as illustrated in Fig. 1.24.

The graph shows that for $h = 0.1$ s, Heun's method yields a curve identical to the exact solution (whereas the *RK1* method diverged for this time step in Fig. 1.23). Even for the rather large time step $h = 1$ s, the Heun solution, although it starts out a bit ragged, proceeds after 60 s (about the time the transient dies out) to settle down and coincide thereafter very well with the exact solution. For this problem, Heun's method is a decidedly better choice than *RK1* and competes with *RK2* and *RK3*.

1.8.3 RUNGE-KUTTA WITH VARIABLE STEP SIZE

Using a constant step size to integrate a differential equation can be inefficient. The value of h in those regions where the solution varies slowly should be larger than in regions where the variation is more rapid, which requires h to be small to maintain accuracy. Methods for automatically adjusting the step size have been developed. They involve combining two adjacent-order *RK* methods into one and using the difference between the higher and lower order solution to estimate the truncation error in the lower order solution. The step size h is adjusted to keep the truncation error in bounds.

A common example is the embedding of *RK4* into *RK5* to produce the *RKF4(5)* method. The *F* is added in recognition of E. Fehlberg's contribution to this extension of the *RK* method. The procedure has six stages, and the Fehlberg coefficients are (Fehlberg, 1969)

$$
\{\mathbf{a}\} = \begin{Bmatrix} 0 \\ 1/4 \\ 3/8 \\ 12/13 \\ 1 \\ 1/2 \end{Bmatrix}
$$
$$
[\mathbf{b}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 \\ 3/32 & 9/32 & 0 & 0 & 0 \\ 1932/2197 & -7200/2197 & 7296/2197 & 0 & 0 \\ 439/216 & -8 & 3680/513 & -845/4104 & 0 \\ -8/27 & 2 & -3544/2565 & 1859/4104 & -11/40 \end{bmatrix} \tag{1.118}
$$

$$
\{\mathbf{c}^*\} = \begin{Bmatrix} 25/216 \\ 0 \\ 1408/2565 \\ 2197/4104 \\ -1/5 \\ 0 \end{Bmatrix} \quad \{\mathbf{c}\} = \begin{Bmatrix} 16/135 \\ 0 \\ 6656/12825 \\ 28561/56430 \\ -9/50 \\ 2/55 \end{Bmatrix} \tag{1.119}
$$

Using asterisks to indicate that *RK4* is the lower order of the two, we have from Eq. (1.104)

$$
\mathbf{y}_{i+1}^* = \mathbf{y}_i + h\left(c_1^*\widetilde{\mathbf{f}}_1 + c_2^*\widetilde{\mathbf{f}}_2 + c_3^*\widetilde{\mathbf{f}}_3 + c_4^*\widetilde{\mathbf{f}}_4 + c_5^*\widetilde{\mathbf{f}}_5 + c_6^*\widetilde{\mathbf{f}}_6\right) \quad \text{Low-order solution}\,(RK4) \tag{1.120}
$$

$$
\mathbf{y}_{i+1} = \mathbf{y}_i + h\left(c_1\widetilde{\mathbf{f}}_1 + c_2\widetilde{\mathbf{f}}_2 + c_3\widetilde{\mathbf{f}}_3 + c_3\widetilde{\mathbf{f}}_4 + c_5\widetilde{\mathbf{f}}_5 + c_6\widetilde{\mathbf{f}}_6\right) \quad \text{High-order solution}\,(RK5) \tag{1.121}
$$

where, from Eqs. (1.100), (1.101), and (1.102), the derivatives at the six stages are

$$
\begin{aligned}
\widetilde{\mathbf{f}}_1 &= \mathbf{f}(t_i, \mathbf{y}_i) \\
\widetilde{\mathbf{f}}_2 &= \mathbf{f}\left(t_i + a_2 h, \mathbf{y}_i + h b_{21}\widetilde{\mathbf{f}}_1\right) \\
\widetilde{\mathbf{f}}_3 &= \mathbf{f}\left(t_i + a_3 h, \mathbf{y}_i + h\left[b_{31}\widetilde{\mathbf{f}}_1 + b_{32}\widetilde{\mathbf{f}}_2\right]\right) \\
\widetilde{\mathbf{f}}_4 &= \mathbf{f}\left(t_i + a_4 h, \mathbf{y}_i + h\left[b_{41}\widetilde{\mathbf{f}}_1 + b_{42}\widetilde{\mathbf{f}}_2 + b_{43}\widetilde{\mathbf{f}}_3\right]\right) \\
\widetilde{\mathbf{f}}_5 &= \mathbf{f}\left(t_i + a_5 h, \mathbf{y}_i + h\left[b_{51}\widetilde{\mathbf{f}}_1 + b_{52}\widetilde{\mathbf{f}}_2 + b_{53}\widetilde{\mathbf{f}}_3 + b_{54}\widetilde{\mathbf{f}}_4\right]\right) \\
\widetilde{\mathbf{f}}_6 &= \mathbf{f}\left(t_i + a_6 h, \mathbf{y}_i + h\left[b_{61}\widetilde{\mathbf{f}}_1 + b_{62}\widetilde{\mathbf{f}}_2 + b_{63}\widetilde{\mathbf{f}}_3 + b_{64}\widetilde{\mathbf{f}}_4 + b_{65}\widetilde{\mathbf{f}}_5\right]\right)
\end{aligned} \tag{1.122}
$$

Observe that, although the low- and high-order solutions have different weights ($\{\mathbf{c}^*\}$ and $\{\mathbf{c}\}$, respectively), they share the same nodes $\{\mathbf{a}\}$ and coupling coefficients $[\mathbf{b}]$, and, hence, the same values of the derivatives $\widetilde{\mathbf{f}}$. This is another convenient feature of the Runge-Kutta-Fehlberg (*RKF*) method.

The truncation vector $\mathbf{e}$ is the difference between the higher order solution $\mathbf{y}_{i+1}$ and the lower order solution $\mathbf{y}_{i+1}^*$,

$$
\begin{aligned}
\mathbf{e} &= \mathbf{y}_{i+1} - \mathbf{y}_{i+1}^* \\
&= h\left[\left(c_1 - c_1^*\right)\widetilde{\mathbf{f}}_1 + \left(c_2 - c_2^*\right)\widetilde{\mathbf{f}}_2 + \left(c_3 - c_3^*\right)\widetilde{\mathbf{f}}_3 + \left(c_4 - c_4^*\right)\widetilde{\mathbf{f}}_4 + \left(c_5 - c_5^*\right)\widetilde{\mathbf{f}}_5 + \left(c_6 - c_6^*\right)\widetilde{\mathbf{f}}_6\right]
\end{aligned} \tag{1.123}
$$

The number of components of **e** equals N, the number of first-order differential equations in the system (e.g., three in Example 1.15 and two in Example 1.18). The scalar truncation error e is the largest of the absolute values of the components of **e**,

$$e = \text{maximum of the set}\,(|e_1|, |e_2|, |e_3|, \cdots, |e_N|) \tag{1.124}$$

We set up a tolerance *tol*, which the truncation error cannot exceed. Instead of using the same h for every step of the numerical integration process, we can adjust the step size so as to keep the error e from exceeding *tol*. A simple strategy for adaptive step size control is to update h after each time step using a formula derived, for example, in Bond and Allman (1996),

$$h_{\text{new}} = h_{\text{old}}\left(\frac{tol}{e}\right)^{\frac{1}{p+1}} \tag{1.125}$$

where p is the lower of the two orders in an *RKFp(p+1)* method. For *RKF4(5)*, $p = 4$. According to Eq. (1.125), if $e > tol$, then $h_{\text{new}} < h_{\text{old}}$, whereas if $e < tol$, then $h_{\text{new}} > h_{\text{old}}$. A factor β is commonly added so that

$$h_{\text{new}} = h_{\text{old}}\beta\left(\frac{tol}{e}\right)^{\frac{1}{p+1}} \tag{1.126}$$

where β may be 0.8 or 0.9, depending on the computer program.

ALGORITHM 1.3

Given the vector $\mathbf{y}_i$ at time t_i, the derivative functions $\mathbf{f}(t,\mathbf{y})$, the time step h, and the tolerance *tol*, use the *RKF4(5)* method with adaptive step size control to find $\mathbf{y}_{i+1}$ at time t_{i+1}. See Appendix D.4 for *rkf45.m*, a MATLAB implementation of this algorithm.

1. Evaluate the derivatives $\widetilde{\mathbf{f}}_1$ through $\widetilde{\mathbf{f}}_6$ using Eq. (1.122).
2. Calculate the truncation vector using Eq. (1.123).
3. Compute the scalar truncation error e using Eq. (1.124).
4. If $e > tol$ then replace h by $h\beta(tol/e)^{1/5}$ and return to Step 1.
5. Replace t by $t+h$ and calculate $\mathbf{y}_{i+1}$ using Eq. (1.121).
6. Replace h by $h\beta(tol/e)^{1/5}$.

Repeat these steps to obtain $\mathbf{y}_{i+2}$, $\mathbf{y}_{i+3}$, etc.

EXAMPLE 1.20

A spacecraft S of mass m travels in a straight line away from the center C of the earth, as illustrated in Fig. 1.25. If at a distance of 6500 km from C its outbound velocity is 7.8 km/s, what will be its position and velocity 70 min later?

Solution

Solving this problem requires writing down and then integrating the equations of motion. Starting with the free body diagram of S, shown in Fig. 1.25, we find that Newton's second law (Eq. 1.38) for the spacecraft is

$$-F_g = m\ddot{x} \tag{a}$$

The variable force of gravity F_g on the spacecraft is its mass m times the local acceleration of gravity, given by Eq. (1.36). That is,

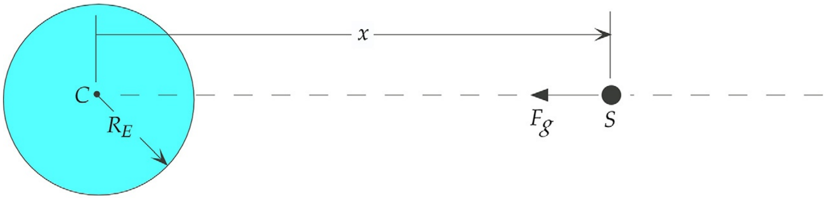


FIG. 1.25

Spacecraft *S* in rectilinear motion relative to the earth.

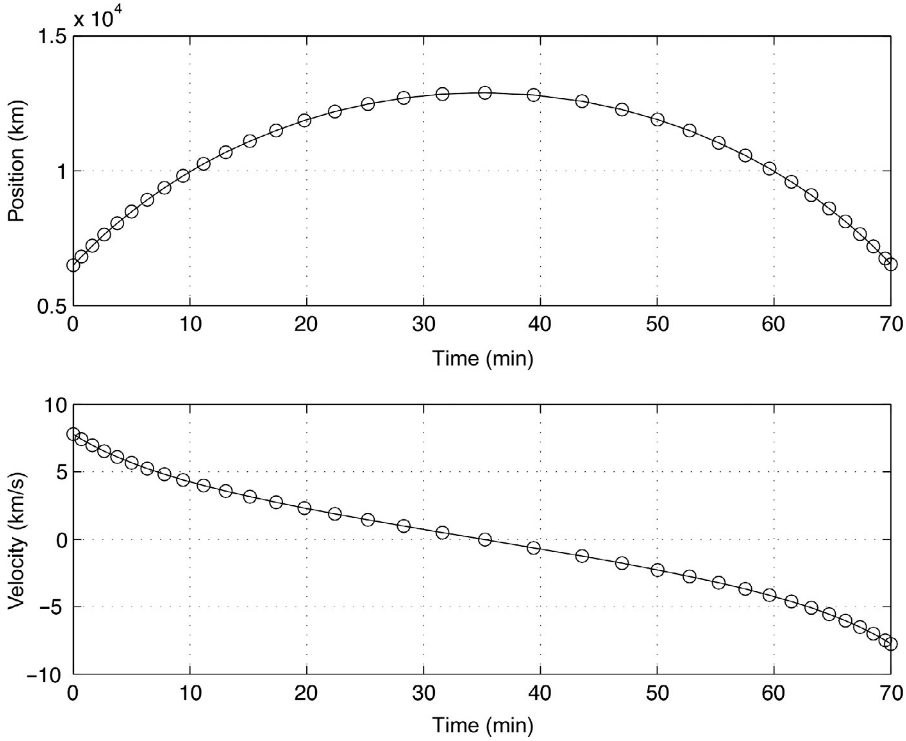


FIG. 1.26

Position and velocity versus time. The solution points are circled.

$$F_g = mg = m\frac{g_0 R_E^2}{x^2} \tag{b}$$

R_E is the earth's radius (6378 km), and g_0 is the sea level acceleration of gravity (9.807 m/s^2). Combining Eqs. (a) and (b) yields

$$\ddot{x} + \frac{g_0 R_E^2}{x^2} = 0 \tag{1.127}$$

This differential equation for the rectilinear motion of the spacecraft has an analytical solution, which we shall not go into here. Instead, we will solve it numerically using Algorithm 1.3 and the given initial conditions. For that, we must as usual introduce the auxiliary variables $y_1 = x$ and $y_2 = \dot{x}$ to obtain the two differential equations

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\frac{g_0 R_E^2}{y_1^2}\end{aligned} \tag{c}$$

The initial conditions in this case are

$$y_1(0) = 6500\text{km} \quad y_2(0) = 7.8\text{km/s}^2 \tag{d}$$

The MATLAB programs *Example_1_20.m* and *rkf45.m*, both in Appendix D.4, were used to produce Fig. 1.26, which shows the position and velocity of the spacecraft over the requested time span. *Example_1_20.m* passes the initial conditions and time span to *rkf4.m*, which uses the subroutine *rates* within *Example_1_20.m* to compute the derivatives $\dot{x}$ and $\ddot{x}$.

Fig. 1.26 reveals that the spacecraft takes 35 min to coast out to twice its original 6500 km distance from C before reversing the direction and returning 35 min later to where it started with a speed of 7.8 km/s. The nonuniform spacing between the solution points shows how *rkf4.m* controlled the step size such that h was smaller during rapid variations of the solution but larger elsewhere.

PROBLEMS

Section 1.2

1.1 Given the three vectors $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$, $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$, and $\mathbf{C} = C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}$, show analytically that

(a) $\mathbf{A}\cdot\mathbf{A} = A^2$

(b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ (interchangeability of the dot and cross)

(c) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ (the bac-cab rule)

(Hint: Simply compute the expressions on each side of the = signs and demonstrate conclusively that they are the same.) Do *not* substitute numbers to "prove" your point. Use Eqs. (1.9) and (1.16).

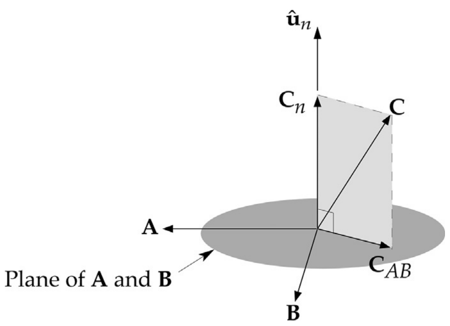
1.2 Use just the vector identities in Problem 1.1 to show that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

1.3 Let $\mathbf{A} = 8\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$, $\mathbf{B} = 9\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$, and $\mathbf{C} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$. Calculate the (scalar) projection C_{AB} of $\mathbf{C}$ onto the plane of $\mathbf{A}$ and $\mathbf{B}$ (see illustration below).

(Hint: $C^2 = C_n^2 + C_{AB}^2$)

{Ans.: $C_{AB} = 11.58$}



Section 1.3

1.4 Since $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ are perpendicular and $\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b$, use the bac-cab rule to show that $\hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t = \hat{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_n \times \hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t$, thereby verifying Eq. (1.29).

1.5 The x, y, and z coordinates (in meters) of a particle P as a function of time (in seconds) are $x = \sin 3t$, $y = \cos t$, and $z = \sin 2t$. At $t = 3\,\text{s}$, determine:

(a) The velocity $\mathbf{v}$ in Cartesian coordinates.
(b) The speed v.
(c) The unit tangent vector $\hat{\mathbf{u}}_t$.
(d) The angles θ_x, θ_y, and θ_z that $\mathbf{v}$ makes with the x, y, and z axes.
(e) The acceleration $\mathbf{a}$ in Cartesian coordinates.
(f) The unit binormal vector $\hat{\mathbf{u}}_b$.
(g) The unit normal vector $\hat{\mathbf{u}}_n$.
(h) The angles ϕ_x, ϕ_y, and ϕ_z that $\mathbf{a}$ makes with the x, y, and z axes.
(i) The tangential component a_t of the acceleration.
(j) The normal component a_n of the acceleration.
(k) The radius of curvature of the path of P.
(l) The Cartesian coordinates of the center of curvature of the path.

{Partial Ans.: (b) 2.988 m/s; (d) $\theta_x = 139.7\text{deg}$; (j) $a_n = 5.398\ \text{m/s}^2$; (l) $x_C = -0.4068\,\text{m}$}

Section 1.4

1.6 An 80-kg man and 50-kg woman stand 0.5 m from each other. What is the force of gravitational attraction between the couple?
{Ans.: 36.04 μN}

1.7 If a person's weight is W on the surface of the earth, calculate the earth's gravitational pull on that person at a distance equal to the moon's orbit.
{Ans.: $275(10^{-6})W$}

1.8 If a person's weight is W on the surface of the earth, calculate what it would be, in terms of W, at the surface of

(a) the moon;
(b) Mars;
(c) Jupiter.

{Partial Ans.: (c) $2.53W$}

Section 1.5

1.9 A satellite of mass m is in a circular orbit around the earth, whose mass is M. The orbital radius from the center of the earth is r. Use Newton's second law of motion, together with Eqs. (1.25) and (1.31), to calculate the speed v of the satellite in terms of M, r, and the gravitational constant G.
{Ans.: $v = \sqrt{GM/r}$}

1.10 If the earth takes 365.25 days to complete its circular orbit of radius $149.6(10^6)$km around the sun, use the result of Example 1.9 to calculate the mass of the sun.
{Ans.: $1.988(10^{30})$kg}

Section 1.6

1.11 $\mathbf{F}$ is a force vector of fixed magnitude embedded on a rigid body in plane motion (in the xy plane). At a given instant, $\boldsymbol{\omega} = 2\hat{\mathbf{k}}\,\text{rad/s}$, $\dot{\boldsymbol{\omega}} = -5\hat{\mathbf{k}}\,\text{rad/s}^2$, $\ddot{\boldsymbol{\omega}} = 3\hat{\mathbf{k}}\,\text{rad/s}^3$, and $\mathbf{F} = \left(15\hat{\mathbf{i}} + 10\hat{\mathbf{j}}\right)(\text{N})$. At that instant, calculate $\dddot{\mathbf{F}}$.
{Ans.: $\dddot{\mathbf{F}} = 500\hat{\mathbf{i}} + 225\hat{\mathbf{j}}\,(\text{N/s}^3)$}

Section 1.7

1.12 The absolute position, velocity, and acceleration of O are

$$\mathbf{r}_0 = -16\hat{\mathbf{I}} + 84\hat{\mathbf{J}} + 59\hat{\mathbf{K}}(\text{m})$$
$$\mathbf{v}_0 = 7\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 4\hat{\mathbf{K}}(\text{m/s})$$
$$\mathbf{a}_0 = 3\hat{\mathbf{I}} - 7\hat{\mathbf{J}} + 4\hat{\mathbf{K}}(\text{m/s}^2)$$

The angular velocity and acceleration of the moving frame are

$$\boldsymbol{\Omega} = -0.8\hat{\mathbf{I}} + 0.7\hat{\mathbf{J}} + 0.4\hat{\mathbf{K}}(\text{rad/s}) \quad \dot{\boldsymbol{\Omega}} = -0.4\hat{\mathbf{I}} + 0.9\hat{\mathbf{J}} - 1.0\hat{\mathbf{K}}(\text{rad/s}^2)$$

The unit vectors of the moving frame are

$$\hat{\mathbf{i}} = -0.15670\hat{\mathbf{I}} - 0.31235\hat{\mathbf{J}} + 0.93704\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = -0.12940\hat{\mathbf{I}} + 0.94698\hat{\mathbf{J}} + 0.29409\hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = -0.97922\hat{\mathbf{I}} - 0.075324\hat{\mathbf{J}} - 0.18831\hat{\mathbf{K}}$$

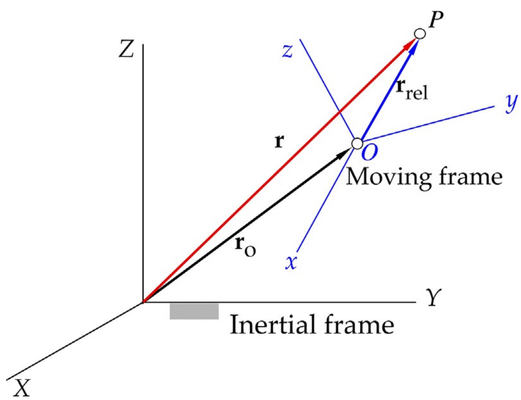
The absolute position of P is

$$\mathbf{r} = 51\hat{\mathbf{I}} - 45\hat{\mathbf{J}} + 36\hat{\mathbf{K}}(\text{m})$$

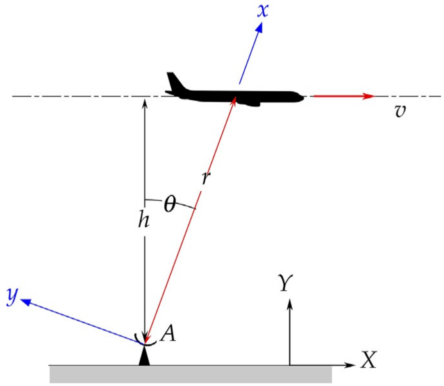
The velocity and acceleration of P relative to the moving frame are

$$\mathbf{v}_{\text{rel}} = 31\hat{\mathbf{i}} - 68\hat{\mathbf{j}} - 77\hat{\mathbf{k}}(\text{m/s}) \qquad \mathbf{a}_{\text{rel}} = 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}(\text{m/s}^2)$$

Calculate the absolute velocity $\mathbf{v}_P$ and acceleration $\mathbf{a}_P$ of P.
{Ans.: $\mathbf{v}_P = 156.4\hat{\mathbf{u}}_v(\text{m/s}) \quad \hat{\mathbf{u}}_v = 0.7790\hat{\mathbf{I}} - 0.3252\hat{\mathbf{J}} + 0.5360\hat{\mathbf{K}}$
$\mathbf{a}_P = 85.13\hat{\mathbf{u}}_a(\text{m/s}^2) \quad \hat{\mathbf{u}}_a = -0.3229\hat{\mathbf{I}} + 0.8284\hat{\mathbf{J}} - 0.4576\hat{\mathbf{K}}$}

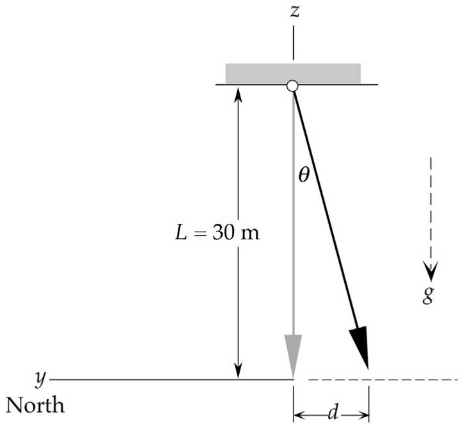


1.13 An airplane in level flight at an altitude h and a uniform speed v passes directly over a radar tracking station A. Calculate the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ of the radar antenna as well as the rate $\dot{r}$ at which the airplane is moving away from the antenna. Use the equations of this chapter (rather than polar coordinates, which you can use to check your work). Attach the inertial frame of reference to the ground and assume a nonrotating earth. Attach the moving frame to the antenna, with the x axis pointing always from the antenna toward the airplane.
{Ans.: (a) $\dot{\theta} = v\cos^2\theta/h$, (b) $\ddot{\theta} = -2v^2\cos^3\theta\sin\theta/h^2$, (c) $v_{\text{rel}} = v\sin\theta$}



1.14 At 30°N latitude, a 1000-kg (2205-lb) car travels due north at a constant speed of 100 km/h (62 mph) on a level road at sea level. Taking into account the earth's rotation, calculate:
(a) the lateral (sideways) force of the road on the car;
(b) the normal force of the road on the car.
{Ans.: (a) $F_{\text{lateral}} = 2.026\,\text{N}$, to the left (west); (b) $F_{\text{normal}} = 9784\,\text{N}$}

1.15 At 29°N latitude, what is the deviation d from the vertical of a plumb bob at the end of a 30-m string, due to the earth's rotation?
{Ans.: 44.1 mm to the south}



Section 1.8

1.16 Verify by substitution that Eq. (1.114a) is the solution of Eq. (1.113).

1.17 Verify that Eqs. (1.114b) are valid.

1.18 Numerically solve the fourth-order differential equation

$$\mathrm{d}^4y/\mathrm{d}t^4 + 2\mathrm{d}^2y/\mathrm{d}t^2 + y = 0$$

for y at $t = 20$, if the initial conditions are $y = 1$ and $\mathrm{d}y/\mathrm{d}t = \mathrm{d}^2y/\mathrm{d}t^2 = \mathrm{d}^3y/\mathrm{d}t^3 = 0$ at $t = 0$.
{Ans.: $y(20) = 9.545$}

1.19 Numerically solve the differential equation

$$\mathrm{d}^4y/\mathrm{d}t^4 + 3\mathrm{d}^3y/\mathrm{d}t^3 - 4\mathrm{d}y/\mathrm{d}t - 12y = te^{2t}$$

for y at $t = 3$, if, at $t = 0$, $y = \mathrm{d}y/\mathrm{d}t = \mathrm{d}^2y/\mathrm{d}t^2 = 0$.
{Ans.: $y(3) = 66.62$}

1.20 Numerically solve the differential equation

$$t\ddot{y} + t^2\dot{y} - 2y = 0$$

to obtain y at $t = 4$ if the initial conditions are $y = 0$ and $\dot{y} = 1$ at $t = 1$.
{Ans.: $y(4) = 1.29$}

1.21 Numerically solve the system

$$\begin{aligned}\dot{x} + \frac{1}{2}y - z &= 0\\ -\frac{1}{2}x + \dot{y} + \frac{1}{\sqrt{2}}z &= 0\\ \frac{1}{2}x - \frac{1}{\sqrt{2}}y + \dot{z} &= 0\end{aligned}$$

to obtain x, y, and z at $t = 20$. The initial conditions are $x = 1$ and $y = z = 0$ at $t = 0$.
{Ans.: $x(20) = 0.704$, $y(20) = 0.665$, $z(20) = -0.246$}

1.22 Use one of the numerical methods discussed in this section to solve Eq. (1.127) for the time required for the moon to fall to the earth after it is somehow stopped in its orbit while the earth remains fixed in space. (This will require a trial-and-error procedure known formally as a shooting method. It is not necessary for this problem to code the procedure. Simply guess a time and let the solver compute the final radius. On the basis of the deviation of that result from the earth's radius (6378 km), revise your time estimate and rerun the problem to compute a new final radius. Repeat this process in a logical fashion until your time estimate yields a final radius that is accurate to at least three significant figures.) Compare your answer with the analytical solution,

$$t = \sqrt{\frac{r_0}{2g_0R_E^2}}\left[\frac{\pi}{4}r_0 + \sqrt{r(r_0 - r)} + \frac{r_0}{2}\sin^{-1}\left(\frac{r_0 - 2r}{r_0}\right)\right]$$

where t is the time, r_0 is the initial radius, r is the final radius ($r < r_0$), g_0 is the sea level acceleration of earth's gravity, and R_E is the radius of the earth.

1.23 Use an *RK* solver such as *rkf45* in Appendix D.4 or MATLAB's *ode45* to solve the nonlinear Lorenz equations, due to the American meteorologist and mathematician E.N. Lorenz (1917–2008):

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

Start off by using the values Lorenz (1963) used in his paper (namely, $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$). For initial conditions use $x = 0$, $y = 1$, and $z = 0$ at $t = 0$. Let t range to a value of at least 20. Plot the phase trajectory $x = x(t)$, $y = y(t)$, $z = z(t)$ in three dimensions to see the now-famous "Lorenz attractor." The Lorenz equations are a simplified model of the two-dimensional convective motion within a fluid layer due to a temperature difference ΔT between the upper and lower surfaces. The equations are chaotic in nature. For one thing, this means that the solutions are extremely sensitive to the initial conditions. A minute change yields a completely different solution in the long run. Check this out yourself. (x represents the intensity of the convective motion of the fluid, y is proportional to the temperature difference between rising and falling fluid, and z represents the nonlinearity of the temperature profile across the depth. σ is a fluid property (the Prandtl number), ρ is proportional to ΔT, β is a geometrical parameter, and t is a nondimensional time.)

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