CHAPTER

SPACECRAFT ATTITUDE DYNAMICS

12

12.1 INTRODUCTION

In this chapter, we apply the equations of rigid body motion presented in Chapter 11 to the study of the attitude dynamics of satellites. We begin with spin-stabilized spacecraft. Spinning a satellite around its axis is a very simple way to keep the vehicle pointed in a desired direction. We investigate the stability of a spinning satellite to show that only oblate spinners are stable over long times. Overcoming this restriction on the shape of spin-stabilized spacecraft led to the development of dual-spin vehicles, which consist of two interconnected segments rotating at different rates about a common axis. We consider the stability of that type of configuration as well. The nutation damper and its effect on the stability of spin-stabilized spacecraft are covered next.

The rest of the chapter is devoted to some of the common means of changing the attitude or motion of a spacecraft by applying external or internal forces or torques. The coning maneuver changes the attitude of a spinning spacecraft by using thrusters to apply impulsive torque, which alters the angular momentum and hence the orientation of the spacecraft. The much-used yo-yo despin maneuver reduces or eliminates the spin rate by releasing small masses attached to cords initially wrapped around the cylindrical vehicle.

An alternative to spin stabilization is three-axis stabilization by gyroscopic attitude control. In this case, the vehicle does not continuously rotate. Instead, the desired attitude is maintained by the spin of small wheels within the spacecraft. These are called reaction wheels or momentum wheels. If allowed to pivot relative to the vehicle, they are known as control moment gyros. The attitude of the vehicle can be changed by varying the speed or orientation of these internal gyros. Small thrusters may also be used to supplement gyroscopic attitude control and to hold the spacecraft orientation fixed when it is necessary to despin or reorient the gyros that have become saturated (reached their maximum spin rate or deflection) over time.

The chapter concludes with a discussion of how the earth's gravitational field by itself can stabilize the attitude of large satellites in low earth orbits.

12.2 TORQUE-FREE MOTION

Gravity is the only force acting on a satellite coasting in orbit (if we neglect secondary drag forces and the gravitational influence of bodies other than the planet being orbited). Unless the satellite is

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unusually large, the gravitational force is concentrated at the center of mass G. Since the net moment about the center of mass is zero, the satellite is torque free and according to Eq. (11.30),

$$\dot{\mathbf{H}}_G = \mathbf{0} \tag{12.1}$$

The angular momentum $\mathbf{H}_G$ about the center of mass does not depend on time. It is a vector fixed in inertial space. We will use $\mathbf{H}_G$ to define the Z axis of an inertial frame, as shown in Fig. 12.1. The xyz axes in the figure comprise the principal body frame, centered at G. The angle between the z axis and $\mathbf{H}_G$ is (by definition of the Euler angles) the nutation angle θ. Let us determine the conditions for which θ is constant. From the dot product operation, we know that

$$\cos\theta = \frac{\mathbf{H}_G}{\|\mathbf{H}_G\|} \cdot \hat{\mathbf{k}}$$

Differentiating this expression with respect to time, keeping in mind Eq. (12.1), we get

$$\frac{d\cos\theta}{dt} = \frac{\mathbf{H}_G}{\|H_G\|} \cdot \frac{d\hat{\mathbf{k}}}{dt}$$

But $d\hat{\mathbf{k}}/dt = \boldsymbol{\omega} \times \hat{\mathbf{k}}$, according to Eq. (1.52), so

$$\frac{d\cos\theta}{dt} = \frac{\mathbf{H}_G \cdot \left(\boldsymbol{\omega} \times \hat{\mathbf{k}}\right)}{\|\mathbf{H}_G\|} \tag{12.2}$$

Now,

$$\boldsymbol{\omega} \times \hat{\mathbf{k}} = \left(\omega_x \hat{\mathbf{i}} + \omega_y \hat{\mathbf{j}} + \omega_z \hat{\mathbf{k}}\right) \times \hat{\mathbf{k}} = \omega_y \hat{\mathbf{i}} - \omega_x \hat{\mathbf{j}}$$

Furthermore, we know from Eq. (11.67) that the angular momentum is related to the angular velocity in the principal body frame by the expression

$$\mathbf{H}_G = A\omega_x \hat{\mathbf{i}} + B\omega_y \hat{\mathbf{j}} + C\omega_z \hat{\mathbf{k}}$$

Thus,

$$\mathbf{H}_G \cdot \left(\boldsymbol{\omega} \times \hat{\mathbf{k}}\right) = \left(A\omega_x \hat{\mathbf{i}} + B\omega_y \hat{\mathbf{j}} + C\omega_z \hat{\mathbf{k}}\right) \cdot \left(\omega_y \hat{\mathbf{i}} - \omega_x \hat{\mathbf{j}}\right) = (A-B)\omega_x\omega_y$$

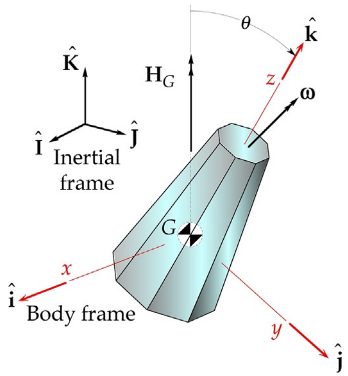


FIG. 12.1

Rotationally symmetric satellite in torque-free motion.

so that Eq. (12.2) can be written as

$$\dot{\theta} = \omega_n = -\frac{(A-B)\omega_x\omega_y}{\|\mathbf{H}_G\|\sin\theta} \tag{12.3}$$

From this, we see that the nutation rate $\dot{\theta}$ vanishes only if $A = B$. If $A \neq B$, the nutation angle θ will not in general be constant.

Relative to the body frame, Eq. (12.1) is written (cf. Eq. 1.56) as

$$\dot{\mathbf{H}}_G\big)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_G = \mathbf{0}$$

This is the Euler equation with $\mathbf{M}_G)_{\text{net}} = \mathbf{0}$, and its components are given by Eq. (11.72b),

$$\begin{aligned} A\dot{\omega}_x + (C-B)\omega_z\omega_y &= 0 \\ B\dot{\omega}_y + (A-C)\omega_x\omega_z &= 0 \\ C\dot{\omega}_z + (B-A)\omega_y\omega_x &= 0 \end{aligned} \tag{12.4}$$

In the interest of simplicity, let us consider the special case illustrated in Fig. 12.1 (namely, that in which the z axis is an axis of rotational symmetry), so that $A = B$. Then Eq. (12.4) may be written

$$\begin{aligned} A\dot{\omega}_x + (C-A)\omega_y\omega_z &= 0 \\ A\dot{\omega}_y + (A-C)\omega_z\omega_x &= 0 \\ C\dot{\omega}_z &= 0 \end{aligned} \tag{12.5}$$

From Eq. (12.5_3) we see that the body frame z component of the angular velocity is constant.

$$\omega_z = \omega_o \text{ (constant)} \tag{12.6}$$

The assumption of rotational symmetry therefore reduces the three differential equations in Eq. (12.4) to just the first two in Eq. (12.5). Substituting Eq. (12.6) into Eqs. (12.5_1) and (12.5_2) and introducing the notation

$$\lambda = \frac{A-C}{A}\omega_o \tag{12.7}$$

they can be written as

$$\begin{aligned} \dot{\omega}_x - \lambda\omega_y &= 0 \\ \dot{\omega}_y + \lambda\omega_x &= 0 \end{aligned} \tag{12.8}$$

Note that the sign of λ depends on the relative values of the principal moments of inertia A and C.

To reduce Eq. (12.8) in ω_x and ω_y to just one equation in ω_x, we first differentiate Eq. (12.8_1) with respect to time to get

$$\ddot{\omega}_x - \lambda\dot{\omega}_y = 0 \tag{12.9}$$

We then solve Eq. (12.8_2) for $\dot{\omega}_y$ and substitute the result into Eq. (12.9), which leads to

$$\ddot{\omega}_x + \lambda^2\omega_x = 0 \tag{12.10}$$

The solution of this well-known differential equation is

$$\omega_x = \omega_{xy}\sin\lambda t \tag{12.11}$$

where the constant amplitude ω_{xy} ($\omega_{xy} \neq 0$) has yet to be determined. (Without loss of generality, we have set the phase angle, the other constant of integration, equal to zero.) Substituting Eq. (12.11) back into Eq. (12.8_1) yields the solution for ω_y,

$$\omega_y = \frac{1}{\lambda}\frac{d\omega_x}{dt} = \frac{1}{\lambda}\frac{d}{dt}(\omega_{xy}\sin\lambda t)$$

or

$$\omega_y = \omega_{xy}\cos\lambda t \tag{12.12}$$

Eqs. (12.6), (12.11), and (12.12) give the components of the absolute angular velocity vector $\boldsymbol{\omega}$ along the three principal body axes,

$$\boldsymbol{\omega} = \omega_{xy}\sin\lambda t\hat{\mathbf{i}} + \omega_{xy}\cos\lambda t\hat{\mathbf{j}} + \omega_o\hat{\mathbf{k}}$$

or

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\perp + \omega_o\hat{\mathbf{k}} \tag{12.13}$$

where

$$\boldsymbol{\omega}_\perp = \boldsymbol{\omega}_{xy}\left(\sin\lambda t\hat{\mathbf{i}} + \cos\lambda t\hat{\mathbf{j}}\right) \tag{12.14}$$

$\boldsymbol{\omega}_\perp$ (omega-perp) is the component of $\boldsymbol{\omega}$ normal to the z axis. It sweeps out a circle of radius ω_{xy} in the xy plane at an angular velocity λ. Thus, $\boldsymbol{\omega}$ sweeps out a cone, as illustrated in Fig. 12.2. If ω_o is positive, then the body has an inertial counterclockwise rotation around the positive z axis ($\lambda > 0$) if $A > C$. However, an observer fixed in the body would see the world rotating in the opposite direction, clockwise around positive z, as the figure shows. Of course, the situation is reversed if $A < C$.

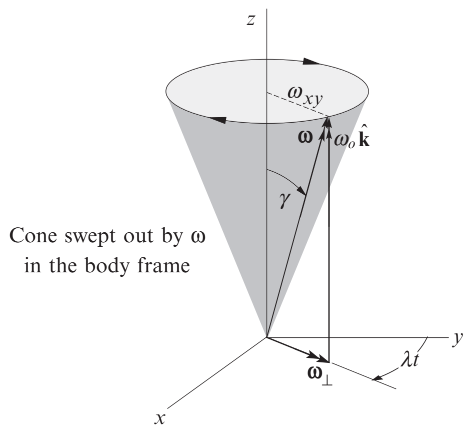


FIG. 12.2

Components of the angular velocity $\boldsymbol{\omega}$ in the body frame.

From Eq. (11.116), the three Euler orientation angles (and their rates) are related to the body frame angular velocity components ω_x, ω_y, and ω_z by

$$\begin{aligned}\omega_p &= \dot{\phi} = \frac{1}{\sin\theta}(\omega_x \sin\psi + \omega_y \cos\psi)\\ \omega_n &= \dot{\theta} = \omega_x \cos\psi - \omega_y \sin\psi\\ \omega_s &= \dot{\psi} = -\frac{1}{\tan\theta}(\omega_x \sin\psi + \omega_y \cos\psi) + \omega_z\end{aligned}$$

Substituting Eqs. (12.6), (12.11), and (12.12) into these three equations yields

$$\begin{aligned}\omega_p &= \frac{\omega_{xy}}{\sin\theta}\cos(\lambda t - \psi)\\ \omega_n &= \omega_{xy}\sin(\lambda t - \psi)\\ \omega_s &= \omega_o - \frac{\omega_{xy}}{\tan\theta}\cos(\lambda t - \psi)\end{aligned} \tag{12.15}$$

Since $A = B$, we know from Eq. (12.3) that $\omega_n = 0$. It follows from Eq. (12.15$_2$) that

$$\psi = \lambda t \tag{12.16}$$

(Actually, $\lambda t - \psi = n\pi$, $n = 0, 1, 2, \ldots$. We can set $n = 0$ without loss of generality.) Substituting Eq. (12.16) into Eqs. (12.15$_1$) and (12.15$_3$) yields

$$\omega_p = \frac{\omega_{xy}}{\sin\theta} \tag{12.17}$$

and

$$\omega_s = \omega_o - \frac{\omega_{xy}}{\tan\theta} \tag{12.18}$$

We have thus obtained the Euler angle rates ω_p and ω_s in terms of the components of the angular velocity $\boldsymbol{\omega}$ in the body frame.

Differentiating Eq. (12.16) with respect to time shows that

$$\lambda = \dot{\psi} = \omega_s \tag{12.19}$$

That is, the rate λ at which $\boldsymbol{\omega}$ rotates around the body's z axis equals the spin rate. Substituting the spin rate for λ in Eq. (12.7) shows that ω_s is related to ω_o alone,

$$\omega_s = \frac{A - C}{A}\omega_o \tag{12.20}$$

Observe that ω_s and ω_o are opposite in sign if $A < C$.

Eliminating ω_s from Eqs. (12.18) and (12.20) yields the relationship between the magnitudes of the orthogonal components of the angular velocity in Eq. (12.13),

$$\omega_{xy} = \frac{C}{A}\omega_o \tan\theta \tag{12.21}$$

A similar relationship exists between ω_p and ω_s, which generally are *not* orthogonal. Substitute Eq. (12.21) into Eq. (12.17) to obtain

$$\omega_o = \frac{A}{C}\omega_p \cos\theta \tag{12.22}$$

Placing this result in Eq. (12.20) leaves an expression involving only ω_p and ω_s, from which we get a useful formula relating the precession of a torque-free body to its spin,

$$\omega_p = \frac{C}{A-C}\frac{\omega_s}{\cos\theta} \tag{12.23}$$

Observe that if $A > C$ (i.e., the body is prolate, like a soup can or an American football), then ω_p has the same sign as ω_s, which means the precession is prograde. For an oblate body (like a tuna fish can or a frisbee), $A < C$ and the precession is retrograde.

The components of angular momentum along the body frame axes are obtained from the body frame components of $\boldsymbol{\omega}$,

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + A\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}}$$

or

$$\mathbf{H}_G = \mathbf{H}_\perp + C\omega_o\hat{\mathbf{k}} \tag{12.24}$$

where

$$\mathbf{H}_\perp = A\omega_{xy}\left(\sin\omega_s t\hat{\mathbf{i}} + \cos\omega_s t\hat{\mathbf{j}}\right) = A\boldsymbol{\omega}_\perp \tag{12.25}$$

Since $\omega_o\hat{\mathbf{k}}$ and $C\omega_o\hat{\mathbf{k}}$ are collinear, as are $\boldsymbol{\omega}_\perp$ and $A\boldsymbol{\omega}_\perp$, it follows that $\hat{\mathbf{k}}$, $\boldsymbol{\omega}$, and $\mathbf{H}_G$ all lie on the same plane. $\mathbf{H}_G$ and $\boldsymbol{\omega}$ both rotate around the z axis at the same rate ω_s. These details are illustrated in Fig. 12.3. See how the precession and spin angular velocities, ω_p and ω_s, add up vectorially to give $\boldsymbol{\omega}$. Note also that from the point of view of inertial space, where $\mathbf{H}_G$ is fixed, $\boldsymbol{\omega}$ and $\hat{\mathbf{k}}$ rotate around $\mathbf{H}_G$ with angular velocity ω_p.

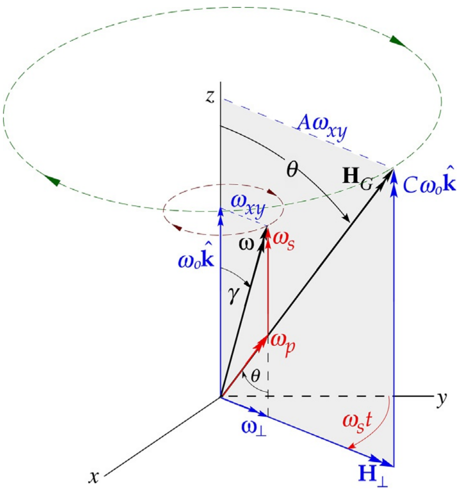


FIG. 12.3

Angular velocity and angular momentum vectors in the body frame ($A > C$).

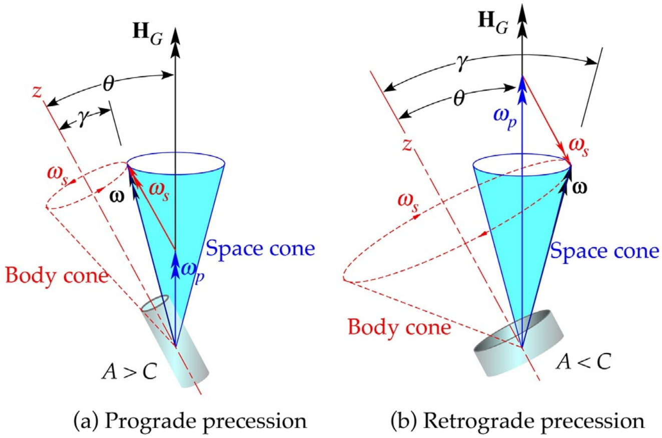


FIG. 12.4

Space and body cones for a rotationally symmetric body in torque-free motion. (a) Prolate body and (b) oblate body.

Let γ be the angle between the angular velocity vector $\boldsymbol{\omega}$ and the spin axis z, as shown in Figs. 12.2 and 12.3. γ is sometimes referred to as the wobble angle. From the figures, it is clear that $\tan\gamma = \omega_{xy}/\omega_0$ and $\tan\theta = A\omega_{xy}/C\omega_0$. It follows that

$$\tan\theta = \frac{A}{C}\tan\gamma \tag{12.26}$$

From this, we conclude that if $A > C$, then $\gamma < \theta$, whereas $C > A$ means $\gamma > \theta$. That is, the angular velocity vector $\boldsymbol{\omega}$ lies between the z axis and the angular momentum vector $\mathbf{H}_G$ when $A > C$ (prolate body). On the other hand, when $C > A$ (oblate body), $\mathbf{H}_G$ lies between the z axis and $\boldsymbol{\omega}$. These two situations are illustrated in Fig. 12.4, which also shows the body cone and space cone. The space cone is swept out in inertial space by the angular velocity vector as it rotates with angular velocity ω_p around $\mathbf{H}_G$, whereas the body cone is the trace of $\boldsymbol{\omega}$ in the body frame as it rotates with angular velocity ω_s about the z axis. From inertial space, the motion may be visualized as the body cone rolling on the space cone, with the line of contact being the angular velocity vector. From the body frame, it appears as though the space cone rolls on the body cone. Fig. 12.4 graphically confirms our deduction from Eq. (12.23) (namely, that precession and spin are in the same direction for prolate bodies and opposite in direction for oblate shapes).

Finally, we know from Eqs. (12.24) and (12.25) that the magnitude $\|\mathbf{H}_G\|$ of the angular momentum is

$$\|\mathbf{H}_G\| = H_G = \sqrt{A^2\omega_{xy}{}^2 + C^2\omega_o{}^2}$$

Using Eqs. (12.17) and (12.22), we can write this as

$$\mathbf{H}_G = \sqrt{A^2(\omega_p\sin\theta)^2 + C^2\left(\frac{A}{C}\omega_p\cos\theta\right)^2} = \sqrt{A^2\omega_p(\sin^2\theta + \cos^2\theta)}$$

so that we obtain a surprisingly simple formula for the magnitude of the angular momentum in torque-free motion,

$$H_G = A\omega_p \tag{12.27}$$

EXAMPLE 12.1

The cylindrical shell in Fig. 12.5 is rotating in torque-free motion about its longitudinal axis. If the axis is wobbling slightly, determine the ratios of l/r for which the precession will be prograde or retrograde.

Solution

Fig. 11.10 shows the moments of inertia of a thin-walled circular cylinder,

$$C = mr^2 \qquad A = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$$

According to Eq. (12.23) and Fig. 12.4, direct or prograde precession exists if $A > C$. That is, if

$$\frac{1}{2}mr^2 + \frac{1}{12}ml^2 > mr^2$$

or

$$\frac{1}{12}ml^2 > \frac{1}{2}mr^2$$

Thus,

$$\boxed{\begin{array}{lll} l > 2.45r & \Rightarrow & \text{Direct precession} \\ l < 2.45r & \Rightarrow & \text{Rectrograde precession} \end{array}}$$

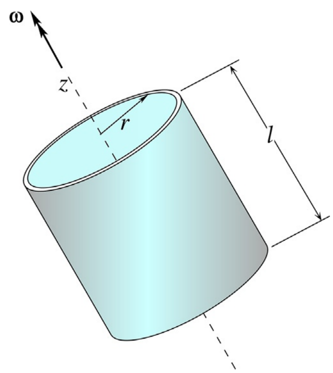


FIG. 12.5

Cylindrical shell in torque-free motion.

EXAMPLE 12.2

In the previous example, let $r = 1$ m, $l = 3$ m, $m = 100$ kg, and let the nutation angle θ be 20°. How long does it take the cylinder to precess through 180° if the spin rate is 2π rad/min?

Solution

Since $l > 2.45r$, the precession is direct. Furthermore,

$$\begin{aligned} C &= mr^2 &&= 100 \cdot 1^2 &&= 100 \text{ kg} \cdot \text{m}^2 \\ A &= \frac{1}{2}mr^2 + \frac{1}{12}ml^2 &&= \frac{1}{2} \cdot 100 \cdot 1^2 + \frac{1}{12}100 \cdot 3^2 &&= 125 \text{ kg} \cdot \text{m}^2 \end{aligned}$$

Thus, Eq. (12.23) yields

$$\omega_p = \frac{C}{A - C}\frac{\omega_s}{\cos\theta} = \frac{100}{125 - 100}\frac{2\pi}{\cos 20°} = 26.75 \text{ rad/min}$$

At this rate, the time for the spin axis to precess through an angle of 180° is

$$t = \frac{\pi}{\omega_p} = \boxed{0.1175 \text{ min}}$$

EXAMPLE 12.3

What is the torque-free motion of a spacecraft for which $A = B = C$?

Solution

If $A = B = C$, the spacecraft is spherically symmetric. Any orthogonal triad at the center of mass G is a principal body frame, so $\mathbf{H}_G$ and $\boldsymbol{\omega}$ are collinear,

$$\mathbf{H}_G = C\boldsymbol{\omega}$$

Substituting this and $\mathbf{M}_G)_{\text{net}} = \mathbf{0}$ into the Euler equations (Eq. 11.72) yields

$$C\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (C\boldsymbol{\omega}) = \mathbf{0}$$

That is, $\boldsymbol{\omega}$ is constant. The angular velocity vector of a spherically symmetric spacecraft in torque-free motion is fixed in magnitude and direction. We considered this problem in more detail in Example 11.25.

EXAMPLE 12.4

The inertial components of the angular momentum of a torque-free rigid body are

$$\mathbf{H}_G = 320\hat{\mathbf{I}} - 375\hat{\mathbf{J}} + 450\hat{\mathbf{K}} \ \left(\text{kg} \cdot \text{m}^2/\text{s}\right) \tag{a}$$

The Euler angles are

$$\phi = 20° \qquad \theta = 50° \qquad \psi = 75° \tag{b}$$

If the inertia tensor in the body-fixed principal frame is

$$[\mathbf{I}_G] = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \left(\text{kg} \cdot \text{m}^2\right) \tag{c}$$

calculate the inertial components of the (absolute) angular acceleration.

Solution

Substituting the Euler angles from Eq. (b) into Eq. (11.104), we obtain the direction cosine matrix of the transformation from the inertial frame to the body-fixed frame,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.03086 & 0.6720 & 0.7399 \\ -0.9646 & -0.1740 & 0.1983 \\ 0.2620 & -0.7198 & 0.6428 \end{bmatrix} \tag{d}$$

We use this to obtain the components of $\mathbf{H}_G$ in the body frame,

$$\begin{aligned} \{\mathbf{H}_G\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{H}_G\}_X &= \begin{bmatrix} 0.03086 & 0.6720 & 0.7399 \\ -0.9646 & -0.1740 & 0.1983 \\ 0.2620 & -0.7198 & 0.6428 \end{bmatrix} \begin{Bmatrix} 320 \\ -375 \\ 450 \end{Bmatrix} \\ &= \begin{Bmatrix} 90.86 \\ -154.2 \\ 643.0 \end{Bmatrix} \left(\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}\right) \end{aligned} \tag{e}$$

In the body frame $\{\mathbf{H}_G\}_x = [\mathbf{I}_G]\{\boldsymbol{\omega}\}_x$, where $\{\boldsymbol{\omega}\}_x$ comprises the components of angular velocity in the body frame. Thus,

$$\begin{Bmatrix} 90.86 \\ -154.2 \\ 643.0 \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \{\boldsymbol{\omega}\}_x$$

or, solving for $\{\boldsymbol{\omega}\}_x$,

$$\{\boldsymbol{\omega}\}_x = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix}^{-1} \begin{Bmatrix} 90.86 \\ -154.2 \\ 643.0 \end{Bmatrix} = \begin{Bmatrix} 0.09086 \\ -0.07709 \\ 0.2144 \end{Bmatrix} (\mathrm{rad/s}) \tag{f}$$

Euler's equations of motion (Eq. 11.72a) may be written for the case at hand as

$$[\mathbf{I}_G]\{\boldsymbol{\alpha}\}_x + \{\boldsymbol{\omega}\}_x \times \left([\mathbf{I}_G]\{\boldsymbol{\omega}\}_x\right) = \{\mathbf{0}\} \tag{g}$$

where $\{\boldsymbol{\alpha}\}_x$ is the absolute acceleration in body frame components. Substituting Eqs. (c) and (f) into this expression, we get

$$\begin{gathered} \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \{\boldsymbol{\alpha}\}_x + \begin{Bmatrix} 0.09086 \\ -0.07709 \\ 0.2144 \end{Bmatrix} \times \left(\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \begin{Bmatrix} 0.09086 \\ -0.07709 \\ 0.2144 \end{Bmatrix} \right) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \\ \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \{\boldsymbol{\alpha}\}_x + \begin{Bmatrix} -16.52 \\ -38.95 \\ -7.005 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \end{gathered}$$

so that, finally,

$$\{\boldsymbol{\alpha}\}_x = -\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 3000 \end{bmatrix}^{-1} \begin{Bmatrix} -16.52 \\ -38.95 \\ -7.005 \end{Bmatrix} = \begin{Bmatrix} 0.01652 \\ 0.01948 \\ 0.002335 \end{Bmatrix} \left(\mathrm{rad/s}^2\right) \tag{h}$$

These are the components of the angular acceleration in the body frame (xyz). To transform them into the inertial frame (XYZ) we use

$$\begin{aligned} \{\boldsymbol{\alpha}\}_X &= [\mathbf{Q}]_{xX}\{\boldsymbol{\alpha}\}_x = \left([\mathbf{Q}]_{Xx}\right)^T\{\boldsymbol{\alpha}\}_x \\ &= \begin{bmatrix} 0.03086 & -0.9646 & 0.2620 \\ 0.6720 & -0.1740 & -0.7198 \\ 0.7399 & 0.1983 & 0.6428 \end{bmatrix} \begin{Bmatrix} 0.01652 \\ 0.01948 \\ 0.002335 \end{Bmatrix} = \begin{Bmatrix} -0.01766 \\ 0.006033 \\ 0.01759 \end{Bmatrix} \left(\mathrm{rad/s}^2\right) \end{aligned}$$

That is,

$$\boxed{\boldsymbol{\alpha} = -0.01766\hat{\mathbf{i}} + 0.006033\hat{\mathbf{j}} + 0.01759\hat{\mathbf{k}}\ \left(\mathrm{rad/s}^2\right)}$$

12.3 STABILITY OF TORQUE-FREE MOTION

Let a rigid body be in torque-free motion with its angular velocity vector directed along the principal body z axis, so that $\boldsymbol{\omega} = \omega_o \hat{\mathbf{k}}$, where ω_o is constant. The nutation angle is zero, and there is no precession. Let us perturb the motion slightly, as illustrated in Fig. 12.6, so that

$$\omega_x = \delta\omega_x \qquad \omega_y = \delta\omega_y \qquad \omega_z = \omega_o + \delta\omega_z \tag{12.28}$$

As in Chapter 7, δ means a very small quantity. In this case, $\delta\omega_x \ll \omega_o$ and $\delta\omega_y \ll \omega_o$. Thus, the angular velocity vector has become slightly inclined to the z axis. For torque-free motion, $M_G)_x = M_G)_y = M_G)_z = 0$, so that the Euler equations (Eq. 11.72b) are

$$\begin{aligned} A\dot{\omega}_x + (C-B)\omega_y\omega_z &= 0 \\ B\dot{\omega}_y + (A-C)\omega_x\omega_z &= 0 \\ C\dot{\omega}_z + (B-A)\omega_x\omega_y &= 0 \end{aligned} \tag{12.29}$$

Observe that we have not assumed $A = B$, as we did in the previous section. Substituting Eq. (12.28) into Eq. (12.29) and keeping in mind our assumption that $\dot{\omega}_o = 0$, we get

$$\begin{aligned} A\delta\dot{\omega}_x + (C-B)\omega_o\delta\omega_y + (C-B)\delta\omega_y\delta\omega_z &= 0 \\ B\delta\dot{\omega}_y + (A-C)\omega_o\delta\omega_x + (C-B)\delta\omega_x\delta\omega_z &= 0 \\ C\delta\dot{\omega}_z + (B-A)\delta\omega_x\delta\omega_y &= 0 \end{aligned} \tag{12.30}$$

Neglecting all the products of the $\delta\omega$s (because they are arbitrarily small), Eq. 12.30 becomes

$$\begin{aligned} A\delta\dot{\omega}_x + (C-B)\omega_o\delta\omega_y &= 0 \\ B\delta\dot{\omega}_y + (A-C)\omega_o\delta\omega_x &= 0 \\ C\delta\dot{\omega}_z &= 0 \end{aligned} \tag{12.31}$$

Eq. (12.31_3) implies that $\delta\omega_z$ is constant.

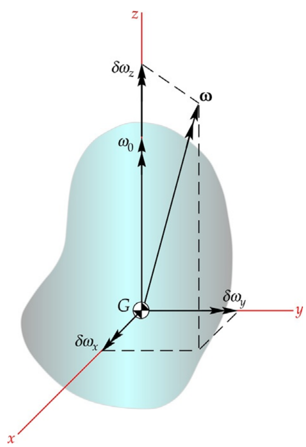


FIG. 12.6

Principal body axes of a rigid body rotating primarily about the body z axis.

Differentiating Eq. (12.31_1) with respect to time, we get

$$A\delta\ddot{\omega}_x + (C-B)\omega_o\delta\dot{\omega}_y = 0 \tag{12.32}$$

Solving Eq. (12.31_2) for $\delta\dot{\omega}_y$ yields $\delta\dot{\omega}_y = -[(A-C)/B]\omega_o\delta\omega_x$, and substituting this into Eq. (12.32) gives

$$\delta\ddot{\omega}_x - \frac{(A-C)(C-B)}{AB}\omega_o^2\delta\omega_x = 0 \tag{12.33}$$

Likewise, differentiating Eq. (12.31_2) and then substituting $\delta\dot{\omega}_x$ from Eq. (12.31_1) yields

$$\delta\ddot{\omega}_y - \frac{(A-C)(C-B)}{AB}\omega_o^2\delta\omega_y = 0 \tag{12.34}$$

If we define

$$k = \frac{(A-C)(B-C)}{AB}\omega_o^2 \tag{12.35}$$

then both Eqs. (12.33) and (12.34) may be written in the form

$$\delta\ddot{\omega} + k\delta\omega = 0 \tag{12.36}$$

If $k > 0$, then $\delta\omega = c_1 e^{i\sqrt{k}t} + c_2 e^{-i\sqrt{k}t}$, which means $\delta\omega_x$ and $\delta\omega_y$ vary sinusoidally with small amplitude. The motion is therefore bounded and neutrally stable. That means the amplitude does not die out with time, but it does not exceed the small amplitude of the perturbation. Observe from Eq. (12.35) that $k > 0$ if C is larger than both A and B or if C is smaller than both A and B. This means that the spin axis (z axis) is either the major axis of inertia or the minor axis of inertia. That is, if the spin axis is either the major or minor axis of inertia, the motion is stable. The stability is neutral for a rigid body, because there is no damping.

On the other hand, if $k < 0$, then $\delta\omega = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}$, which means that the initially small perturbations $\delta\omega_x$ and $\delta\omega_y$ increase without bound. The motion is unstable. From Eq. (12.35) we see that $k < 0$ if either $A > C > B$ or $A < C < B$. This means that the spin axis is the intermediate axis of inertia. If the spin axis is the intermediate axis of inertia, the motion is unstable.

EXAMPLE 12.5

A homogeneous, box-shaped satellite in torque-free motion has mass moments of inertia $A = 1000\ \text{kg}\cdot\text{m}^2$, $B = 300\ \text{kg}\cdot\text{m}^2$, and $C = 800\ \text{kg}\cdot\text{m}^2$, relative to its body-fixed principal xyz axes. The angular velocity relative to the body-fixed frame is

$$\boldsymbol{\omega} = 1.0\hat{\mathbf{k}} + \boldsymbol{\omega}_{xy}\ (\text{rad/s}) \tag{a}$$

where $\boldsymbol{\omega}_{xy} = \left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)\left(10^{-8}\right)$. Therefore, the spacecraft is an intermediate-axis spinner with an extremely small (essentially zero) transverse component of angular velocity. Solve the Euler equations to verify that the transverse perturbations will grow in time, so that the motion of Eq. (a) is unstable (Fig. 12.7).

Solution

First note that according to Eq. (11.67), the angular momentum vector starts out as

$$\mathbf{H}_G = C\omega_z\hat{\mathbf{k}} = 800\hat{\mathbf{k}}\ \text{kg}\cdot\text{m}^2/\text{s}$$

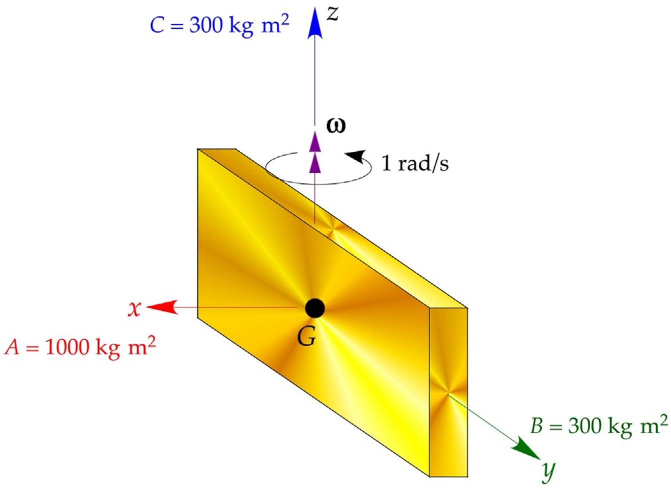


FIG. 12.7

Spacecraft as an intermediate axis spinner.

if we neglect $\boldsymbol{\omega}_{xy}$. Because the net external moment is zero, $\mathbf{H}_G$ must remain constant and aligned with the inertial z direction. The development of a significant transverse component of angular velocity will result in the spin axis (the z axis) precessing around $\mathbf{H}_G$, as shown in Fig. 12.4.

The Euler equations of motion (Eq. 12.4) for this case are

$$\dot{\omega}_x = \frac{B-C}{A}\omega_y\omega_z = -0.5000\omega_y\omega_z \quad \dot{\omega}_y = -0.6667\omega_z\omega_x \quad \dot{\omega}_z = 0.8750\omega_x\omega_y \tag{b}$$

We use the methods of Section 1.8 to numerically integrate this set of coupled, nonlinear, ordinary differential equations. To obtain Eq. (1.95), we set $y_1 = \omega_x$, $y_2 = \omega_y$, and $y_3 = \omega_z$, and write the system (b) as $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, where

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \quad \mathbf{f}(\mathbf{y}) = \begin{Bmatrix} -0.5000y_2y_3 \\ -0.6667y_3y_1 \\ 0.875y_1y_2 \end{Bmatrix} \tag{c}$$

Observe that the time t does not appear explicitly here. The initial conditions are

$$\mathbf{y}_0 = \lfloor (10^{-8}) \ \ (10^{-8}) \ \ 1.0 \rfloor^T \tag{d}$$

The following MATLAB code integrates Eq. (b) over a 450-s time interval and plots the results.

```
y0    = [1.e-8; 1.e-8; 1.0000]; % Set the initial conditions
tspan = linspace(0,450,1000);   % Set the solution time interval
[t,y] = ode45(@f, tspan, w0);   % Integrate the equations dydt in 'f' below
figure('color','w')
%...Plot the angular velocity histories:
subplot(2,1,1)
  plot(t,y(:,3)); grid on; axis([0 450 -1.2 1.2])
  xlabel('time (s)')
  ylabel('\omega_z (rad/s)')
subplot(2,1,2)
  plot(t,sqrt(y(:,1).^2 + y(:,2).^2)); grid on; axis([0 450 -0.2 1.2])
  xlabel('time (s)')
 ylabel('\omega_{xy} (rad/s)')
```

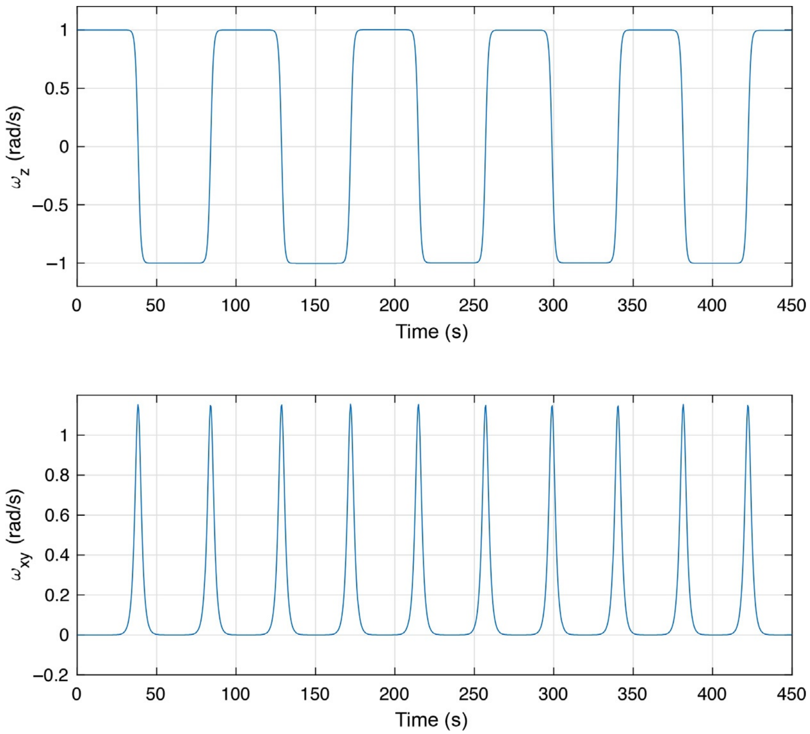


FIG. 12.8

Histories of the axial (*upper*) and transverse (*lower*) components of angular velocity for the spinning body in Fig. 12.7.

```
%~~~~~~~~~~~~~~~~~~~~~~~~~~
function dydt = f(t,y)
%------------------------------
%...Angular velocities in:
  wx = y(1); wy = y(2); wz = y(3);
%...Angular accelerations out:
  wx_dot = -0.5000*wy*wz;
  wy_dot = -0.6667*wz*wx;
  wz_dot =  0.8750*wx*wy;
  dydt   = [wx_dot; wy_dot; wz_dot];
end %f
```

The variations of ω_z and the transverse component $\omega_{xy} = \sqrt{\omega_x{}^2 + \omega_y{}^2}$ are shown in Fig. 12.8. Clearly, the spin is not asymptotically stable since ω_z is not constant but changes sign about once every 43 s. That means the body and its embedded spin axis periodically reverse their spatial orientation, as illustrated in Fig. 12.9. Likewise, the amplitude of the transverse angular velocity rises from zero to over 1 rad/s during the time when the body flips upside down. The complex transitional motion is dictated by the fact that the angular momentum vector must retain its original orientation in inertial space.

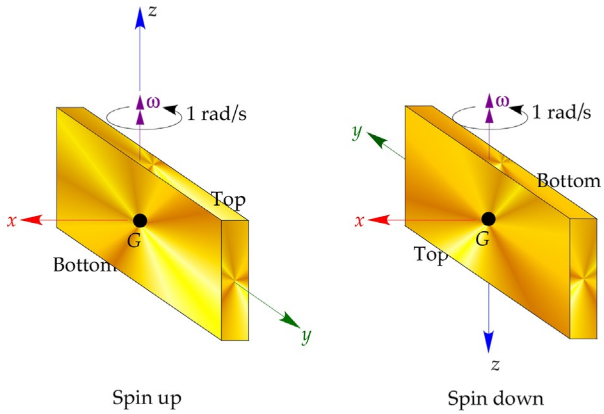


FIG. 12.9

The two semistable states of spin of the body in Fig. 12.7.

If the angular velocity vector of a satellite lies in the direction of its major axis of inertia, the satellite is called a major-axis spinner or oblate spinner. A minor-axis spinner or prolate spinner has its minor axis of inertia aligned with the angular velocity vector. Intermediate-axis spinners are unstable, causing a continual 180° reorientation of the spin axis, if the satellite is a rigid body. However, the flexibility inherent in any real satellite leads to an additional instability, as we shall now see.

Consider again the rotationally symmetric satellite in torque-free motion discussed in Section 12.2. From Eqs. (12.24) and (12.25), we know that angular momentum $\mathbf{H}_G$ is given by

$$\mathbf{H}_G = A\boldsymbol{\omega}_\perp + C\omega_z\hat{\mathbf{k}} \tag{12.37}$$

Hence,

$$\mathrm{H}_G{}^2 = A^2\omega_\perp{}^2 + C^2\omega_z{}^2 \qquad (\omega_\perp = \omega_{xy}) \tag{12.38}$$

Differentiating this equation with respect to time yields

$$\frac{\mathrm{dH}_G{}^2}{\mathrm{d}t} = A^2\frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t} + 2C^2\omega_z\dot{\omega}_z \tag{12.39}$$

But, according to Eq. (12.1), $\mathbf{H}_G$ is constant, so that $\mathrm{dH}_G^2/\mathrm{d}t = 0$ and Eq. (12.39) can be written

$$\frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t} = -2\frac{C^2}{A^2}\omega_z\dot{\omega}_z \tag{12.40}$$

The rotary kinetic energy of a rotationally symmetric body ($A = B$) is found using Eq. (11.81),

$$T_R = \frac{1}{2}A\omega_x{}^2 + \frac{1}{2}A\omega_y{}^2 + \frac{1}{2}C\omega_z{}^2 = \frac{1}{2}A\left(\omega_x{}^2 + \omega_y{}^2\right) + \frac{1}{2}C\omega_z{}^2$$

From Eq. (12.13), we know that $\omega_x{}^2 + \omega_y{}^2 = \omega_\perp{}^2$, which means

$$T_R = \frac{1}{2}A\omega_\perp{}^2 + \frac{1}{2}C\omega_z{}^2 \tag{12.41}$$

The time derivative of T_R is, therefore,

$$\dot{T}_R = \frac{1}{2}A\frac{d\omega_\perp{}^2}{dt} + C\omega_z\dot{\omega}_z$$

Solving this for $\dot{\omega}_z$, we get

$$\dot{\omega}_z = \frac{1}{C\omega_z}\left(\dot{T}_R - \frac{1}{2}A\frac{d\omega_\perp{}^2}{dt}\right)$$

Substituting this expression for $\dot{\omega}_z$ into Eq. (12.40) and solving for $d\omega_\perp{}^2/dt$ yields

$$\frac{d\omega_\perp{}^2}{dt} = 2\frac{C}{A}\frac{\dot{T}_R}{C-A} \tag{12.42}$$

Real bodies are not completely rigid, and their flexibility, however slight, gives rise to small dissipative effects, which cause the kinetic energy to decrease over time. That is,

$$\dot{T}_R < 0 \quad \text{For spacecraft with dissipation} \tag{12.43}$$

Substituting this inequality into Eq. (12.42) leads us to conclude that

$$\begin{aligned} &\frac{d\omega_\perp{}^2}{dt} < 0 \quad \text{if } C > A \quad \text{(oblate spinner)} \\ &\frac{d\omega_\perp{}^2}{dt} > 0 \quad \text{if } C < A \quad \text{(prolate spinner)} \end{aligned} \tag{12.44}$$

If $d\omega_\perp{}^2/dt$ is negative, the spin is asymptotically stable. Should a nonzero value of $\omega_\perp$ develop for some reason, it will drift back to zero over time, so that once again the angular velocity lies completely in the spin direction. On the other hand, if $d\omega_\perp{}^2/dt$ is positive, the spin is unstable. $\omega_\perp$ does not damp out, and the angular velocity vector drifts away from the spin axis as $\omega_\perp$ increases without bound. We pointed out above that spin about a minor axis of inertia is stable with respect to small disturbances. Now we see that only major-axis spin is stable in the long run if dissipative mechanisms exist.

For some additional insight into this phenomenon, solve Eq. (12.38) for $\omega_\perp{}^2$,

$$\omega_\perp{}^2 = \frac{H_G{}^2 - C^2\omega_z{}^2}{A^2}$$

and substitute this result into the expression for kinetic energy (Eq. 12.41) to obtain

$$T_R = \frac{1}{2}\frac{H_G{}^2}{A} + \frac{1}{2}\frac{(A-C)C}{A}\omega_z{}^2 \tag{12.45}$$

According to Eq. (12.24),

$$\omega_z = \frac{H_G)_z}{C} = \frac{H_G\cos\theta}{C}$$

Substituting this into Eq. (12.45) yields the kinetic energy as a function of just the inclination angle θ,

$$T_R = \frac{1}{2}\frac{H_G{}^2}{A}\left(1 + \frac{A-C}{C}\cos^2\theta\right) \tag{12.46}$$

The extreme values of T_R occur at $\theta = 0$ or $\theta = \pi$,

$$T_R = \frac{1}{2}\frac{H_G{}^2}{C} \quad \text{(major axis spinner)}$$

and $\theta = \pi/2$,

$$T_R = \frac{1}{2}\frac{H_G^2}{A} \quad \text{(minor axis spinner)}$$

Clearly, the kinetic energy of a torque-free satellite is smallest when the spin is around the major axis of inertia. We may think of a satellite with dissipation ($dT_R/dt < 0$) as seeking the state of minimum kinetic energy, which occurs when it spins about its major axis.

EXAMPLE 12.6

A rigid spacecraft is modeled by the solid cylinder B, which has a mass of 300 kg, and the slender rod R, which passes through the cylinder and has a mass of 30 kg. Which of the principal axes x, y, and z can be an axis about which stable torque-free rotation can occur (Fig. 12.10)?

Solution

For the solid cylinder B, we have

$$r_B = 0.5 \text{ m} \quad l_B = 1.0 \text{ m} \quad m_B = 300 \text{ kg}$$

The principle moments of inertia about the center of mass are found in Fig. 11.10a,

$$I_B)_x = \frac{1}{4}m_B r_B{}^2 + \frac{1}{12}m_B l_B{}^2 = 43.75 \text{ kg}\cdot\text{m}^2$$
$$I_B)_y = I_B)_x = 43.75 \text{ kg}\cdot\text{m}^2$$
$$I_B)_z = \frac{1}{2}m_B r_B^2 = 37.5 \text{ kg}\cdot\text{m}^2$$

The properties of the transverse slender rod are

$$l_R = 1.0 \text{ m} \quad m_R = 30 \text{ kg}$$

Fig. 11.10a, with $r = 0$, yields the moments of inertia,

$$I_R)_y = 0$$

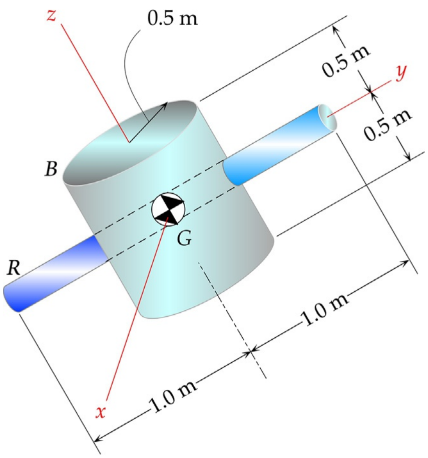


FIG. 12.10

Built-up satellite structure.

$$I_R)_z = I_R)_x = \frac{1}{12} m_A r_A^2 = 10.0 \text{ kg}\cdot\text{m}^2$$

The moments of inertia of the assembly is the sum of the moments of inertia of the cylinder and the rod,

$$\begin{aligned} I_x &= I_B)_x + I_R)_x = 53.75 \text{ kg}\cdot\text{m}^2 \\ I_y &= I_B)_y + I_R)_y = 43.75 \text{ kg}\cdot\text{m}^2 \\ I_z &= I_B)_z + I_R)_z = 47.50 \text{ kg}\cdot\text{m}^2 \end{aligned}$$

Since I_z is clearly the intermediate mass moment of inertia, rotation about the z axis is unstable. With energy dissipation, rotation is stable in the long term only about the major axis, which in this case is the x axis.

12.4 DUAL-SPIN SPACECRAFT

If a satellite is to be spin-stabilized, it must be an oblate spinner. The diameter of the spacecraft is restricted by the cross-section of the launch vehicle's upper stage, and its length is limited by stability requirements. Therefore, oblate spinners cannot take full advantage of the payload volume available in a given launch vehicle, which after all are slender, prolate shapes for aerodynamic reasons. The dual-spin design permits spin stabilization of a prolate shape.

The axisymmetric, dual-spin configuration, or gyrostat, consists of an axisymmetric rotor and a smaller axisymmetric platform joined together along a common longitudinal spin axis at a bearing, as shown in Fig. 12.11. The platform and rotor have their own components of angular velocity, ω_p and ω_r, respectively, along the spin axis direction $\hat{\mathbf{k}}$. The platform spins at a much slower rate than the rotor. The assembly acts like a rigid body as far as transverse rotations are concerned (i.e., the rotor and the platform have the transverse angular velocity $\omega_\perp$ in common). An electric motor integrated into the axle bearing connecting the two components acts to overcome frictional torque that would otherwise eventually cause the relative angular velocity between the rotor and platform to go to zero. If that

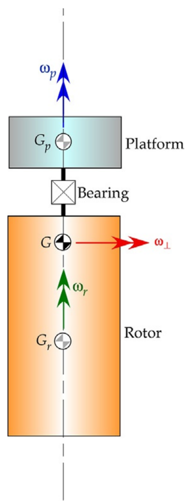


FIG. 12.11

Axisymmetric, dual-spin spacecraft.

should happen, the satellite would become a single-spin unit, probably an unstable prolate spinner, since the rotor of a dual-spin spacecraft is likely to be prolate.

The first dual-spin satellite was OSO-I (Orbiting Solar Observatory), which NASA launched in 1962. It was a major-axis spinner. The first prolate dual-spin spacecraft was the two-story-tall TACSAT I (Tactical Communications Satellite). It was launched into geosynchronous orbit by the US Air Force in 1969. Typical of many of today's communications satellites, TACSAT's platform rotated at 1 rev/day to keep its antennas pointing toward the earth. The rotor spun at about 1 rev/s. Of course, the axis of the spacecraft was normal to the plane of its orbit. The first dual-spin interplanetary spacecraft was Galileo, which we discussed briefly in Section 8.9. Galileo's platform was completely despun to provide a fixed orientation for cameras and other instruments. The rotor spun at 3 rpm.

The equations of motion of a dual-spin spacecraft will be developed later on in Section 12.9. Let us determine the stability of the motion by following the same "energy sink" procedure employed in the previous section for a single-spin-stabilized spacecraft. The angular momentum of the dual-spin configuration about the spacecraft center of mass G is the sum of the angular momenta of the rotor (r) and the platform (p) about G,

$$\mathbf{H}_G = \mathbf{H}_G^{(p)} + \mathbf{H}_G^{(r)} \tag{12.47}$$

The angular momentum of the platform about the spacecraft center of mass is

$$\mathbf{H}_G^{(p)} = C_p \omega_p \hat{\mathbf{k}} + A_p \boldsymbol{\omega}_\perp \tag{12.48}$$

where C_p is the moment of inertia of the platform about the spacecraft spin axis, and A_p is its transverse moment of inertia about G (not G_p). Likewise, for the rotor,

$$\mathbf{H}_G^{(r)} = C_r \omega_r \hat{\mathbf{k}} + A_r \boldsymbol{\omega}_\perp \tag{12.49}$$

where C_r and A_r are its longitudinal and transverse moments of inertia about axes through G. Substituting Eqs. (12.48) and (12.49) into Eq. (12.47) yields

$$\mathbf{H}_G = \left(C_r \omega_r + C_p \boldsymbol{\omega}_p\right)\hat{\mathbf{k}} + A_\perp \boldsymbol{\omega}_\perp \tag{12.50}$$

where $A_\perp$ is the total transverse moment of inertia,

$$A_\perp = A_p + A_r$$

From this, it follows that

$$H_G{}^2 = \left(C_r \omega_r + C_p \omega_p\right)^2 + A_\perp{}^2 \omega_\perp{}^2$$

For torque-free motion, $\dot{\mathbf{H}}_G = \mathbf{0}$, so that $dH_G{}^2/dt = 0$, or

$$2\left(C_r \omega_r + C_p \omega_p\right)\left(C_r \dot{\omega}_r + C_p \dot{\omega}_p\right) + A_\perp{}^2 \frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t} = 0 \tag{12.51}$$

Solving this for $\mathrm{d}\omega_\perp{}^2/\mathrm{d}t$ yields

$$\frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t} = -\frac{2}{A_\perp{}^2}\left(C_r \omega_r + C_p \omega_p\right)\left(C_r \dot{\omega}_r + C_p \dot{\omega}_p\right) \tag{12.52}$$

The total rotational kinetic energy T of the dual-spin spacecraft is that of the rotor plus that of the platform,

$$T = \frac{1}{2}C_r \omega_r{}^2 + \frac{1}{2}C_p \omega_p{}^2 + \frac{1}{2}A_\perp \omega_\perp{}^2$$

Differentiating this expression with respect to time and solving for $\mathrm{d}\omega_\perp{}^2/\mathrm{d}t$ yields

$$\frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t}=\frac{2}{A_\perp}\left(\dot{T}-C_r\omega_r\dot{\omega}_r-C_p\omega_p\dot{\omega}_p\right) \tag{12.53}$$

where $\dot{T}$ is the sum of the power $P^{(r)}$ dissipated in the rotor and the power $P^{(p)}$ dissipated in the platform,

$$\dot{T}=P^{(r)}+P^{(p)} \tag{12.54}$$

Substituting Eq. (12.54) into Eq. (12.53) we find

$$\frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t}=\frac{2}{A_\perp}\left(P^{(r)}-C_r\omega_r\dot{\omega}_r+P^{(p)}-C_p\omega_p\dot{\omega}_p\right) \tag{12.55}$$

Equating the two expressions for $\mathrm{d}\omega_\perp{}^2/\mathrm{d}t$ in Eqs. (12.52) and (12.55) yields

$$\frac{2}{A_\perp}\left(\dot{T}-C_r\omega_r\dot{\omega}_r-C_p\omega_p\dot{\omega}_p\right)=-\frac{2}{A_\perp{}^2}\left(C_r\omega_r+C_p\omega_p\right)\left(C_r\dot{\omega}_r+C_p\dot{\omega}_p\right)$$

Solve this for $\dot{T}$ to obtain

$$\dot{T}=\frac{C_r}{A_\perp}\left[(A_\perp-C_r)\omega_r-C_p\omega_p\right]\dot{\omega}_r+\frac{C_p}{A_\perp}\left[\left(A_\perp-C_p\right)\omega_p-C_r\omega_r\right]\dot{\omega}_p \tag{12.56}$$

Following Likins (1967), we identify the terms containing $\dot{\omega}_r$ and $\dot{\omega}_p$ as the power dissipation in the rotor and platform, respectively. That is, comparing Eqs. (12.54) and (12.56),

$$P^{(r)}=\frac{C_r}{A_\perp}\left[(A_\perp-C_r)\omega_r-C_p\omega_p\right]\dot{\omega}_r \tag{12.57a}$$

$$P^{(p)}=\frac{C_p}{A_\perp}\left[\left(A_\perp-C_p\right)\omega_p-C_r\omega_r\right]\dot{\omega}_p \tag{12.57b}$$

Solving these two expressions for $\dot{\omega}_r$ and $\dot{\omega}_p$, respectively, yields

$$\dot{\omega}_r=\frac{A_\perp}{C_r}\frac{P^{(r)}}{(A_\perp-C_r)\omega_r-C_p\omega_p} \tag{12.58a}$$

$$\dot{\omega}_p=\frac{A_\perp}{C_p}\frac{P^{(p)}}{\left(A_\perp-C_p\right)\omega_p-C_r\omega_r} \tag{12.58b}$$

Substituting these results into Eq. (12.55) leads to

$$\frac{\mathrm{d}\omega_\perp{}^2}{\mathrm{d}t}=\frac{2}{A_\perp}\left[\frac{P^{(r)}}{C_p\dfrac{\omega_p}{\omega_r}-(A_\perp-C_r)}+\frac{P^{(p)}}{C_r-\left(A_\perp-C_p\right)\dfrac{\omega_p}{\omega_r}}\right]\left(C_r+C_p\frac{\omega_p}{\omega_r}\right) \tag{12.59}$$

As pointed out above, for geosynchronous dual-spin communication satellites,

$$\frac{\omega_p}{\omega_r}\approx\frac{2\pi\ \mathrm{rad/day}}{2\pi\ \mathrm{rad/s}}\approx 10^{-5}$$

whereas for interplanetary dual-spin spacecraft, $\omega_p=0$. Therefore, there is an important class of spin-stabilized spacecraft for which $\omega_p/\omega_r\approx 0$. For a despun platform wherein ω_p is zero (or nearly so), Eq. (12.59) yields

$$\frac{d{\omega_\perp}^2}{dt} = \frac{2}{A_\perp}\left[P^{(p)} + \frac{C_r}{C_r - A_\perp}P^{(r)}\right] \tag{12.60}$$

If the rotor is oblate ($C_r > A_\perp$), then, since $P^{(r)}$ and $P^{(p)}$ are both negative, it follows from Eq. (12.60) that $d{\omega_\perp}^2/dt < 0$. That is, the oblate dual-spin configuration with a despun platform is unconditionally stable. In practice, however, the rotor is likely to be prolate ($C_r < A_\perp$), so that

$$\frac{C_r}{C_r - A_\perp}P^{(r)} > 0$$

In that case, $d{\omega_\perp}^2/dt < 0$ only if the dissipation $P^{(p)}$ in the platform is significantly greater than that of the rotor. Specifically, for a prolate design, it must be true that

$$\left|P^{(p)}\right| > \left|\frac{C_r}{C_r - A_\perp}P^{(r)}\right|$$

The platform dissipation rate $P^{(p)}$ can be augmented by adding nutation dampers, which are discussed in the next section.

For the despun prolate dual-spin configuration, Eqs. (12.58) imply

$$\dot{\omega}_r = \frac{P^{(r)}}{(A_\perp - C_r)}\frac{A_\perp}{C_r\omega_r} \qquad \dot{\omega}_p = -\frac{P^{(p)}}{C_p}\frac{A_\perp}{C_r\omega_r}$$

Clearly, the signs of $\dot{\omega}_r$ and $\dot{\omega}_p$ are opposite. If $\omega_r > 0$, then dissipation causes the spin rate of the rotor to decrease and that of the platform to increase. Were it not for the action of the motor on the shaft connecting the two components of the spacecraft, eventually $\omega_p = \omega_r$. That is, the relative motion between the platform and rotor would cease and the dual-spinner would become an unstable single-spin spacecraft. Setting $\omega_p = \omega_r$ in Eq. (12.59) yields

$$\frac{d{\omega_\perp}^2}{dt} = 2\frac{C_r + C_p}{A_\perp}\frac{P^{(r)} + P^{(p)}}{\left(C_r + C_p\right) - A_\perp}$$

which is the same as Eq. (12.42), the energy sink conclusion for a single-spinner.

12.5 NUTATION DAMPER

Nutation dampers are passive means of dissipating energy. A common type consists essentially of a tube filled with viscous fluid and containing a mass attached to springs, as illustrated in Fig. 12.12. Dampers may contain just fluid, only partially filling the tube so that it can slosh around. In either case, the purpose is to dissipate energy through fluid friction. The wobbling of the spacecraft due to nonalignment of the angular velocity with the principal spin axis induces accelerations throughout the satellite, giving rise to the sloshing of fluids and the, stretching and flexing of nonrigid components, etc., all of which dissipate energy to one degree or another. Nutation dampers are added to deliberately increase energy dissipation, which is desirable for stabilizing oblate single-spinners and dual-spin spacecraft (Fig. 12.12).

Let us focus on the motion of the mass within the nutation damper of Fig. 12.12 to gain some insight into how relative motion and deformation are induced by the satellite's precession. Note that point P is the center of mass of the rigid satellite body itself. The center of mass G of the satellite-damper mass combination lies between P and m, as shown in Fig. 12.9. We suppose that the tube is lined up with the

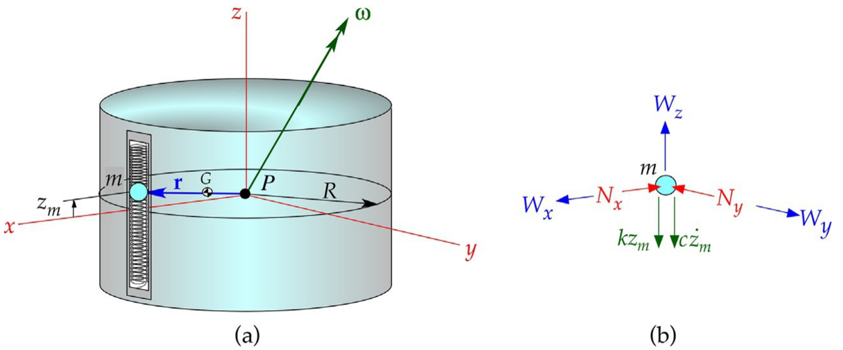


FIG. 12.12

(a) Precessing oblate spacecraft with a nutation damper aligned with the z axis. (b) Free-body diagram of the moving mass in the nutation damper.

z axis of the body-fixed xyz frame, as shown. The mass m in the tube is therefore constrained by the tube walls to move only in the z direction. When the springs are undeformed, the mass lies in the xy plane. In general, the position vector of m in the body frame is

$$\mathbf{r} = R\hat{\mathbf{i}} + z_m\hat{\mathbf{k}} \tag{12.61}$$

where z_m is the z coordinate of m, and R is the distance of the damper from the centerline of the spacecraft. The velocity and acceleration of m relative to the satellite are, therefore,

$$\mathbf{v}_{\text{rel}} = \dot{z}_m\hat{\mathbf{k}} \tag{12.62}$$

$$\mathbf{a}_{\text{rel}} = \ddot{z}_m\hat{\mathbf{k}} \tag{12.63}$$

The absolute angular velocity $\boldsymbol{\omega}$ of the satellite (and, therefore, of the body-fixed frame) is

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{12.64}$$

Recall Eq. (11.73), which states that when $\boldsymbol{\omega}$ is given in a body frame, we find the absolute angular acceleration by taking the time derivative of $\boldsymbol{\omega}$, holding the unit vectors fixed. Thus,

$$\dot{\boldsymbol{\omega}} = \dot{\omega}_x\hat{\mathbf{i}} + \dot{\omega}_y\hat{\mathbf{j}} + \dot{\omega}_z\hat{\mathbf{k}} \tag{12.65}$$

The absolute acceleration of m is found using Eq. (1.70), which for the case at hand becomes

$$\mathbf{a} = \mathbf{a}_P + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{12.66}$$

where $\mathbf{a}_P$ is the absolute acceleration of the reference point P. Substituting Eqs. (12.61)–(12.65) into Eq. (12.66), carrying out the vector operations, combining terms, and simplifying leads to the following expressions for the three components of the inertial acceleration of m,

$$\begin{aligned}
a_x &= a_P)_x - R\left(\omega_y{}^2 + \omega_z{}^2\right) + z_m\dot{\omega}_y + z_m\omega_x\omega_z + 2\dot{z}_m\omega_y \\
a_y &= a_P)_y + R\dot{\omega}_z + R\omega_x\omega_y - z_m\dot{\omega}_x + z_m\omega_y\omega_z - 2\dot{z}_m\omega_x \\
a_z &= a_P)_z - z_m\left(\omega_x{}^2 + \omega_y{}^2\right) - R\dot{\omega}_y + R\omega_x\omega_z + \ddot{z}_m
\end{aligned} \tag{12.67}$$

Fig. 12.9b shows the free body diagram of the damper mass m. In the x and y directions, the forces on m are the components of the force of gravity (W_x and W_y) and the components N_x and N_y of the force of contact with the smooth walls of the damper tube. The directions assumed for these components are, of course, arbitrary. In the z direction, we have the z component W_z of the weight, plus the force of the springs and the viscous drag of the fluid. The spring force $(-kz_m)$ is directly proportional and opposite in direction to the displacement z_m. k is the net spring constant. Viscous drag $(-c\dot{z}_m)$ is directly proportional and opposite in direction to the velocity $\dot{z}_m$ of m relative to the tube. c is the damping constant. Thus, the three components of the net force on the damper mass m are

$$\begin{aligned} F_{\text{net}})_x &= W_x - N_x \\ F_{\text{net}})_y &= W_y - N_y \\ F_{\text{net}})_z &= W_z - kz_m - c\dot{z}_m \end{aligned} \tag{12.68}$$

Substituting Eqs. (12.67) and (12.68) into Newton's second law, $\mathbf{F}_{\text{net}} = m\mathbf{a}$, yields

$$\begin{aligned} N_x &= mR\left(\omega_y^2 + \omega_z^2\right) - mz_m\dot{\omega}_y - mz_m\omega_x\omega_y - 2m\dot{z}_m\omega_y + \overbrace{\left[W_x - ma_P)_x\right]}^{=0} \\ N_y &= -mR\dot{\omega}_z - mR\omega_x\omega_y + mz_m\dot{\omega}_x - mz_m\omega_y\omega_z + 2m\dot{z}_m\omega_x + \overbrace{\left[W_y - ma_P)_y\right]}^{=0} \\ m\ddot{z}_m + c\dot{z}_m &+ \left[k - m\left(\omega_x^2 + \omega_y^2\right)\right]z_m = mR\left(\dot{\omega}_y - \omega_x\omega_z\right) + \overbrace{\left[W_z - ma_P)_z\right]}^{=0} \end{aligned} \tag{12.69}$$

The last terms in brackets in each of these expressions vanish if the acceleration of gravity is the same at m as at the reference point P of the spacecraft. This will be true unless the satellite is of enormous size.

If the damper mass m is vanishingly small compared with the mass M of the rigid spacecraft body, then it will have little effect on the rotary motion. If the rotational state is that of an axisymmetric satellite in torque-free motion, then we know from Eqs. (12.13), (12.14), and (12.19) that

$$\begin{array}{lll} \omega_x = \omega_{xy}\sin\omega_s t & \omega_y = \omega_{xy}\cos\omega_s t & \omega_z = \omega_o \\ \dot{\omega}_x = \omega_{xy}\omega_s\cos\omega_s t & \dot{\omega}_y = -\omega_{xy}\omega_s\sin\omega_s t & \dot{\omega}_z = 0 \end{array}$$

in which case Eq. (12.69) becomes

$$\begin{aligned} N_x &= mR\left(\omega_o^2 + \omega_{xy}^2\cos^2\omega_s t\right) + m(\omega_s - \omega_o)\omega_{xy}z_m\sin\omega_s t - 2m\omega_{xy}\dot{z}_m\cos\omega_s t \\ N_y &= -mR\omega_{xy}^2\cos\omega_s t\sin\omega_s t + m(\omega_s - \omega_o)\omega_{xy}z_m\cos\omega_s t + 2m\omega_{xy}\dot{z}_m\sin\omega_s t \\ m\ddot{z}_m &+ c\dot{z}_m + \left(k - m\omega_{xy}^2\right)z_m = -mR(\omega_s + \omega_o)\omega_{xy}\sin\omega_s t \end{aligned} \tag{12.70}$$

Eq. (12.70_3) is that of a single-degree-of-freedom, damped oscillator with a sinusoidal forcing function, which was discussed in Section 1.8. The precession produces a force of amplitude $m(\omega_o + \omega_s)\omega_{xy}R$ and frequency ω_s, which causes the damper mass m to oscillate back and forth in the tube such that (see the steady-state part of Eq. 1.114a)

$$z_m = \frac{mR\omega_{xy}(\omega_s + \omega_o)}{\left[k - m\left(\omega_s^2 + \omega_{xy}^2\right)\right]^2(c\omega_s)^2}\left\{c\omega_s\cos\omega_s t - \left[k - m\left(\omega_s^2 + \omega_{xy}^2\right)\sin\omega_s t\right]\right\}$$

Observe that the contact forces N_x and N_y depend exclusively on the amplitude and frequency of the precession. If the angular velocity lines up with the spin axis, so that $\omega_{xy} = 0$ (precession vanishes), then

$$\left.\begin{aligned} N_x &= m\omega_o{}^2 R \\ N_y &= 0 \\ z_m &= 0 \end{aligned}\right\} \quad \text{No precession}$$

If precession is eliminated so that there is pure spin around the principal axis, then the time-varying motions and forces vanish throughout the spacecraft, which thereafter rotates as a rigid body with no energy dissipation.

Now, the whole purpose of a nutation damper is to interact with the rotational motion of the spacecraft so as to damp out any tendencies to precess. Therefore, its mass should not be ignored in the equations of motion of the spacecraft. We will derive the equations of motion of the rigid spacecraft with a nutation damper to show how rigid body mechanics is brought to bear upon the problem, and, simply, to discover precisely what we are up against even in this extremely simplified system. We will continue to use P as the origin of our body frame. Since a moving mass has been added to the rigid spacecraft and since we are not using the center of mass of the system as our reference point, we cannot use the Euler equations. Applicable to the case at hand is Eq. (11.33), according to which the equation of rotational motion of the system of satellite plus damper is

$$\dot{\mathbf{H}}_P\big)_{\text{rel}} + \mathbf{r}_{G/P} \times (M+m)\mathbf{a}_{P/G} = \mathbf{M}_G\big)_{\text{net}} \tag{12.71}$$

The angular momentum of the satellite body plus that of the damper mass, relative to point P on the spacecraft, is

$$\mathbf{H}_P\big)_{\text{rel}} = \overbrace{A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}}}^{\text{body of the spacecraft}} + \overbrace{\mathbf{r} \times m\dot{\mathbf{r}}}^{\text{damper mass}} \tag{12.72}$$

where the position vector $\mathbf{r}$ is given by Eq. (12.61). According to Eq. (1.56),

$$\dot{\mathbf{r}} = \left.\frac{d\mathbf{r}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{r} = \dot{z}_m\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ R & 0 & z \end{vmatrix} = \omega_y z_m\hat{\mathbf{i}} + (\omega_z R - \omega_x z_m)\hat{\mathbf{j}} + \left(\dot{z}_m - \omega_y R\right)\hat{\mathbf{k}}$$

After substituting this into Eq. (12.72) and collecting terms, we obtain

$$\begin{aligned} \mathbf{H}_P\big)_{\text{rel}} = {} & \left[\left(A + m z_m{}^2\right)\omega_x - mRz_m\omega_z\right]\hat{\mathbf{i}} + \left[\left(B + mR^2 + mz_m{}^2\right)\omega_y - mR\dot{z}_m\right]\hat{\mathbf{j}} \\ & + \left[\left(C + mR^2\right)\omega_z - mRz_m\omega_x\right]\hat{\mathbf{k}} \end{aligned} \tag{12.73}$$

To calculate $\dot{\mathbf{H}}_P\big)_{\text{rel}}$ we again use Eq. (1.56),

$$\dot{\mathbf{H}}_P\big)_{\text{rel}} = \left.\frac{\mathbf{H}_P)_{\text{rel}}}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_P)_{\text{rel}}$$

Substituting Eq. (12.73) and carrying out the operations on the right leads eventually to

$$\begin{aligned} \dot{\mathbf{H}}_P\big)_{\text{rel}} = {} & \left[\left(A + mz_m{}^2\right)\dot{\omega}_x - mRz_m\dot{\omega}_z + \left(C - B - mz_m{}^2\right)\omega_y\omega_z - mRz_m\omega_x\omega_y + 2mz_m\dot{z}_m\omega_x\right]\hat{\mathbf{i}} \\ & + \left\{\left(B + mR^2 + mz_m{}^2\right)\dot{\omega}_y + mRz_m\left(\omega_x{}^2 - \omega_z{}^2\right) + \left[A + mz_m{}^2 - \left(C + mR^2\right)\right]\omega_x\omega_z \right. \\ & \left. + 2mz_m\dot{z}_m\omega_y - mR\ddot{z}_m\right\}\hat{\mathbf{j}} \\ & + \left[-mRz_m\dot{\omega}_x + \left(C + mR^2\right)\dot{\omega}_z + \left(B + mR^2 - A\right)\omega_x\omega_y + mRz_m\omega_y\omega_z - 2mR\dot{z}_m\omega_x\right]\hat{\mathbf{k}} \end{aligned} \tag{12.74}$$

To calculate the second term on the left of Eq. (12.71), we keep in mind that P is the center of mass of the body of the satellite and first determine the position vector of the center of mass G of the vehicle plus damper relative to P,

$$(M+m)\mathbf{r}_{G/P} = M\cdot\mathbf{0} + m\mathbf{r} \tag{12.75}$$

where $\mathbf{r}$, the position of the damper mass m relative to P, is given by Eq. (12.61). Thus,

$$\mathbf{r}_{G/P} = \frac{m}{m+M}\mathbf{r} = \mu\mathbf{r} = \mu\left(R\hat{\mathbf{i}} + z_m\hat{\mathbf{k}}\right) \tag{12.76}$$

in which

$$\mu = \frac{m}{m+M} \tag{12.77}$$

Thus,

$$\mathbf{r}_{G/P}\times(M+m)\mathbf{a}_{P/G} = \left(\frac{m}{M+m}\right)\mathbf{r}\times(M+m)\mathbf{a}_{P/G} = \mathbf{r}\times m\mathbf{a}_{P/G} \tag{12.78}$$

The acceleration of P relative to G is found with the aid of Eq. (1.60),

$$\mathbf{a}_{P/G} = -\ddot{\mathbf{r}}_{G/P} = -\mu\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t} = -\mu\left[\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathrm{rel}} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathrm{rel}}\right] \tag{12.79}$$

where

$$\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathrm{rel}} = \frac{\mathrm{d}R}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}z_m}{\mathrm{d}t}\hat{\mathbf{k}} = \dot{z}_m\hat{\mathbf{k}} \tag{12.80}$$

and

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathrm{rel}} = \frac{\mathrm{d}^2R}{\mathrm{d}t^2}\hat{\mathbf{i}} + \frac{\mathrm{d}^2z_m}{\mathrm{d}t^2}\hat{\mathbf{k}} = \ddot{z}_m\hat{\mathbf{k}} \tag{12.81}$$

Substituting Eqs. (12.61), (12.64), (12.65), (12.80), and (12.81) into Eq. (12.79) yields

$$\begin{aligned}\mathbf{a}_{P/G} &= \mu\left[-z_m\dot{\omega}_y + R\left(\omega_y^2+\omega_z^2\right) - z_m\omega_x\omega_z - 2\dot{z}_m\omega_y\right]\hat{\mathbf{i}}\\ &\quad+\mu\left(z_m\dot{\omega}_x - R\dot{\omega}_z - R\omega_x\omega_y - z_m\omega_y\omega_z + 2\dot{z}_m\omega_x\right)\hat{\mathbf{j}}\\ &\quad+\mu\left[R\dot{\omega}_y + z_m\left(\omega_x^2+\omega_y^2\right) - R\omega_x\omega_z - \ddot{z}_m\right]\hat{\mathbf{k}}\end{aligned} \tag{12.82}$$

We move this expression into Eq. (12.78) to get

$$\begin{aligned}\mathbf{r}_{G/P}\times(M+m)\mathbf{a}_{P/G} &= \mu m\left[-z_m^2\dot{\omega}_x - 2z_m\dot{z}_m\omega_x + Rz_m\left(\omega_x\omega_y+\dot{\omega}_z\right) + z_m^2\omega_y\omega_z\right]\hat{\mathbf{i}}\\ &\quad+\mu m\left[-\left(R^2+z_m^2\right)\dot{\omega}_y - 2z_m\dot{z}_m\omega_y + Rz_m\left(\omega_z^2-\omega_x^2\right) + \left(R^2-z_m^2\right)\omega_x\omega_z + R\ddot{z}_m\right]\hat{\mathbf{j}}\\ &\quad+\mu m\left(Rz_m\dot{\omega}_x - R^2\dot{\omega}_z + 2R\dot{z}_m\omega_x - R^2\omega_x\omega_y - Rz_m\omega_y\omega_z\right)\hat{\mathbf{k}}\end{aligned}$$

Placing this result and Eq. (12.74) in Eq. (12.71) and using the fact that $\mathbf{M}_G)_{\mathrm{net}} = \mathbf{0}$ yields a vector equation whose three components are

$$\begin{aligned}&A\dot{\omega}_x + (C-B)\omega_y\omega_z + (1-\mu)m\left[z_m^2\left(\dot{\omega}_x-\omega_y\omega_z\right) - Rz_m\left(\dot{\omega}_z+\omega_x\omega_y\right) + 2z_m\dot{z}_m\omega_x\right] = 0\\ &\left[\left(B+mR^2\right)-\mu mR^2\right]\dot{\omega}_y + \left[\left(A+\mu mR^2\right)-\left(C+mR^2\right)\right]\omega_x\omega_z\\ &\quad+(1-\mu)m\left[z_m^2\left(\omega_x\omega_z+\dot{\omega}_y\right) + 2z_m\dot{z}_m\omega_y - R\ddot{z}_m + Rz_m\left(\omega_x^2-\omega_z^2\right)\right] = 0\\ &\left[\left(C+mR^2\right)-\mu mR^2\right]\dot{\omega}_z + \left[\left(B+mR^2\right)-\left(A+\mu mR^2\right)\right]\omega_x\omega_y\\ &\quad+(1-\mu)mR\left[z_m\left(\omega_y\omega_z-\dot{\omega}_x\right) - 2\dot{z}_m\omega_x\right] = 0\end{aligned} \tag{12.83}$$

These are three equations in the four unknowns ω_x, ω_y, ω_z, and z_m. The fourth equation is that of the motion of the damper mass m in the z direction,

$$W_z - kz_m - c\dot{z}_m = ma_z \tag{12.84}$$

where a_z is given by Eq. (12.67$_3$), in which $a_P)_z = a_P)_z - a_G)_z + a_G)_z = a_{P/G})_z + a_G)_z$, so that

$$a_z = a_{P/G}\big)_z + a_G\big)_z - z_m\left(\omega_x{}^2 + \omega_y{}^2\right) - R\dot{\omega}_y + R\omega_x\omega_z + \ddot{z}_m \tag{12.85}$$

Substituting the z component of Eq. (12.82) into this expression and that result into Eq. (12.84) leads (with $W_z = ma_G)_z$) to

$$(1-\mu)m\ddot{z}_m + c\dot{z}_m + \left[k - (1-\mu)m\left(\omega_x{}^2 + \omega_y{}^2\right)\right]z_m = (1-\mu)mR\left[\dot{\omega}_y - \omega_x\omega_z\right] \tag{12.86}$$

Compare Eq. (12.69$_3$) with this expression, which is the fourth equation of motion we need.

Eqs. (12.83) and (12.86) are a rather complicated set of nonlinear, second-order differential equations that must be solved (numerically) to obtain a precise description of the motion of the semirigid spacecraft. The procedures of Section 1.8 may be employed. To study the stability of Eqs. (12.83) and (12.86), we can linearize them in much the same way as we did in Section 12.3. (Note that Eqs. 12.83 reduce to Eqs. 12.29 when $m = 0$.) With that as our objective, we assume that the spacecraft is in pure spin with angular velocity ω_o about the z axis and that the damper mass is at rest ($z_m = 0$). This motion is slightly perturbed, in such a way that

$$\omega_x = \delta\omega_x \qquad \omega_y = \delta\omega_y \qquad \omega_z = \omega_o + \delta\omega_z \qquad z_m = \delta z_m \tag{12.87}$$

It will be convenient for this analysis to introduce operator notation for the time derivative, $D = \mathrm{d}/\mathrm{d}t$. Thus, given a function of time $f(t)$, for any integer n, $D^n f = \mathrm{d}^n f/\mathrm{d}t^n$ and $D^0 f(t) = f(t)$. Then, the various time derivatives throughout the equations will, in accordance with Eq. (12.87), be replaced as follows:

$$\dot{\omega}_x = D\delta\omega_x \qquad \dot{\omega}_y = D\delta\omega_y \qquad \dot{\omega}_z = D\delta\omega_z \qquad \dot{z}_m = D\delta z_m \qquad \ddot{z}_m = D^2\delta z_m \tag{12.88}$$

Substituting Eqs. (12.87) and (12.88) into Eqs. (12.83) and (12.86) and retaining only those terms that are at most linear in the small perturbations leads to

$$\begin{aligned} AD\delta\omega_x + (C-B)\omega_o\delta\omega_y &= 0 \\ \left[A - C - (1-\mu)mR^2\right]\omega_o\delta\omega_x + \left[B + (1-\mu)mR^2\right]D\delta\omega_y - (1-\mu)mR(D^2 + \omega_o{}^2)\delta z_m &= 0 \\ \left[C + (1-\mu)mR^2\right]D\delta\omega_z &= 0 \\ (1-\mu)mR\omega_o\delta\omega_x - (1-\mu)mRD\delta\omega_y + \left[(1-\mu)mD^2 + cD + k\right]\delta z_m &= 0 \end{aligned} \tag{12.89}$$

$\delta\omega_z$ appears only in the third equation, which states that $\delta\omega_z =$ constant. The first, second, and fourth equations may be combined in matrix notation,

$$\begin{bmatrix} AD & (C-B)\omega_o & 0 \\ \left[A - C - (1-\mu)mR^2\right]\omega_o & \left[B + (1-\mu)mR^2\right]D & -(1-\mu)mR(D^2 + \omega_o{}^2) \\ (1-\mu)mR\omega_o & -(1-\mu)mRD & (1-\mu)mD^2 + cD + k \end{bmatrix} \begin{Bmatrix} \delta\omega_x \\ \delta\omega_y \\ \delta z_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12.90}$$

This is a set of three linear differential equations in the perturbations $\delta\omega_x$, $\delta\omega_y$, and δz_m. We will not try to solve them, since all we are really interested in is the stability of the satellite-damper system. It can be shown that the determinant Δ of the 3-by-3 matrix in Eq. (12.90) is

$$\Delta = a_4D^4 + a_3D^3 + a_2D^2 + a_1D + a_o \tag{12.91}$$

in which the coefficients of the characteristic equation $\Delta = 0$ are

$$\begin{aligned}
a_4 &= (1-\mu)mAB \\
a_3 &= cA\left[B+(1-\mu)mR^2\right] \\
a_2 &= k\left[B+(1-\mu)mR^2\right]A+(1-\mu)m\left[(A-C)(B-C)-(1-\mu)AmR^2\right]{\omega_o}^2 \\
a_1 &= c\left\{\left[A-C-(1-\mu)mR^2\right](B-C)\right\}{\omega_o}^2 \\
a_0 &= k\left\{\left[A-C-(1-\mu)mR^2\right](B-C)\right\}{\omega_o}^2+\left[(B-C)(1-\mu)^2\right]m^2R^2{\omega_o}^4
\end{aligned} \tag{12.92}$$

According to the Routh-Hurwitz stability criteria (see any text on control systems, e.g., Palm, 1983) the motion represented by Eq. (12.90), is asymptotically stable if and only if the signs of all of the following quantities, defined in terms of the coefficients of the characteristic equation, are the same

$$r_1 = a_4 \quad r_2 = a_3 \quad r_3 = a_2 - \frac{a_4 a_1}{a_3} \quad r_4 = a_1 - \frac{{a_3}^2 a_0}{a_3 a_2 - a_4 a_1} \quad r_5 = a_0 \tag{12.93}$$

EXAMPLE 12.7

A satellite is spinning about the z axis of its principal body frame at 2π rad/s. The principal moments of inertia about its center of mass are

$$A = 300\ \text{kg}\cdot\text{m}^2 \quad B = 400\ \text{kg}\cdot\text{m}^2 \quad C = 500\ \text{kg}\cdot\text{m}^2 \tag{a}$$

For the nutation damper, the following properties are given:

$$R = 1\ \text{m} \quad \mu = 0.01 \quad m = 10\ \text{kg} \quad k = 10{,}000\ \text{N/m} \quad c = 150\ \text{N s/m} \tag{b}$$

Use the Routh-Hurwitz stability criteria to assess the stability of the satellite as a major-axis spinner, a minor-axis spinner, and an intermediate-axis spinner.

Solution

The data in Eq. (a) are for a major-axis spinner. Substituting into Eqs. (12.92) and (12.93), we find

$$\begin{aligned}
r_1 &= +1.188\left(10^6\right)\ \text{kg}^3\cdot\text{m}^4 \\
r_2 &= +18.44\left(10^6\right)\ \text{kg}^3\cdot\text{m}^4/\text{s} \\
r_3 &= +1.228\left(10^9\right)\ \text{kg}^3\cdot\text{m}^4/\text{s}^2 \\
r_4 &= +92{,}820\,\text{kg}^3\cdot\text{m}^4/\text{s}^3 \\
r_5 &= +8.271\left(10^9\right)\ \text{kg}^3\cdot\text{m}^4/\text{s}^4
\end{aligned} \tag{c}$$

Since every r is positive, spin about the major axis is asymptotically stable. As we know from Section 12.3, without the damper the motion is neutrally stable.

For spin about the minor axis,

$$A = 500\ \text{kg}\cdot\text{m}^2 \quad B = 400\ \text{kg}\cdot\text{m}^2 \quad C = 300\ \text{kg}\cdot\text{m}^2 \tag{d}$$

For these moment of inertia values, we obtain

$$\begin{aligned}
r_1 &= +1.980\left(10^6\right)\ \text{kg}^3\cdot\text{m}^4 \\
r_2 &= +30.74\left(10^6\right)\ \text{kg}^3\cdot\text{m}^4/\text{s} \\
r_3 &= +2.048\left(10^9\right)\ \text{kg}^3\cdot\text{m}^4/\text{s}^2 \\
r_4 &= -304{,}490\,\text{kg}^3\cdot\text{m}^4/\text{s}^3 \\
r_5 &= +7.520\left(10^9\right)\ \text{kg}^3\cdot\text{m}^4/\text{s}^4
\end{aligned} \tag{e}$$

Since the rs are not all of the same sign, spin about the minor axis is not asymptotically stable. Recall that for the rigid satellite, such a motion was neutrally stable.

Finally, for spin about the intermediate axis,

$$A = 300\ \text{kg}\cdot\text{m}^2 \quad B = 500\ \text{kg}\cdot\text{m}^2 \quad C = 400\ \text{kg}\cdot\text{m}^2 \tag{f}$$

We know this motion is unstable, even without the nutation damper, but doing the Routh-Hurwitz stability check anyway, we get

$$\begin{aligned} r_1 &= +1.485\left(10^6\right)\text{kg}^3\cdot\text{m}^4 \\ r_2 &= +22.94\left(10^6\right)\text{kg}^3\cdot\text{m}^4/\text{s} \\ r_3 &= +1.529\left(10^9\right)\text{kg}^3\cdot\text{m}^4/\text{s}^2 \\ r_4 &= -192{,}800\,\text{kg}^3\cdot\text{m}^4/\text{s}^3 \\ r_5 &= -4.323\left(10^9\right)\text{kg}^3\cdot\text{m}^4/\text{s}^4 \end{aligned} \tag{g}$$

The motion, as we expected, is not stable.

12.6 CONING MANEUVER

Like the use of nutation dampers, the coning maneuver is an example of the attitude control of spinning spacecraft. In this case, the angular momentum is changed by the use of onboard thrusters (small rockets) to apply pure torques.

Consider a spacecraft in pure spin with angular velocity ω_0 about its body-fixed z axis, which is an axis of rotational symmetry. The angular momentum is $\mathbf{H}_G)_0 = C\omega_0\hat{\mathbf{k}}$. Suppose we wish to maintain the magnitude of the angular momentum but change its direction by rotating the spin axis through an angle θ, as illustrated in Fig. 12.13. Recall from Section 11.4 that to change the angular momentum of the spacecraft requires applying an external moment,

$$\Delta\mathbf{H}_G = \int_0^{\Delta t} \mathbf{M}_G \mathrm{d}t$$

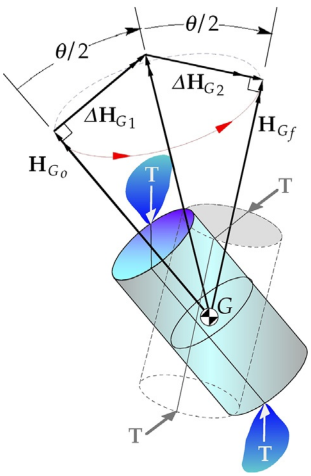


FIG. 12.13

Impulsive coning maneuver.

Thrusters may be used to provide the external impulsive torque required to produce an angular momentum increment $\Delta\mathbf{H}_G)_1$ normal to the spin axis. Since the spacecraft is spinning, this induces coning (precession) of the spacecraft about an axis at an angle of $\theta/2$ to the direction of $\mathbf{H}_G)_0$. Since the external couple is normal to the z axis, the maneuver produces no change in the z component of the angular velocity, which remains ω_0. However, after the impulsive moment, the angular velocity comprises a spin component ω_s and a precession component ω_p. Whereas before the impulsive moment $\omega_s = \omega_0$, afterward, during coning, the spin component is given by Eq. (12.20),

$$\omega_s = \frac{A-C}{A}\omega_0$$

The precession rate is given by Eq. (12.22),

$$\omega_p = \frac{C}{A}\frac{\omega_0}{\cos(\theta/2)} \tag{12.94}$$

Note that before the impulsive maneuver, the magnitude of the angular momentum is $C\omega_0$. Afterward, it has increased to

$$H_G = A\omega_p = \frac{C\omega_0}{\cos(\theta/2)}$$

After precessing 180°, an angular momentum increment $\Delta\mathbf{H}_G)_2$ normal to the spin axis and in the same direction relative to the spacecraft as the initial torque impulse, with $\|\Delta\mathbf{H}_G)_2\| = \|\Delta\mathbf{H}_G)_1\|$, stabilizes the spin vector in the desired direction. Since the spin rate ω_s is not in general the same as the precession rate ω_p, the second angular impulse must be delivered by another pair of thrusters, that have rotated into the position to apply the torque impulse in the proper direction. With only one pair of thrusters, both the spin axis and the spacecraft must rotate through 180° in the same time interval, which means $\omega_p = \omega_s$. That is,

$$\frac{A-C}{A}\omega_0 = \frac{C}{A}\frac{\omega_0}{\cos(\theta/2)}$$

This requires the deflection angle to be

$$\theta = 2\cos^{-1}\left(\frac{C}{A-C}\right)$$

and limits the values of the moments of inertia A and C to those that do not cause the magnitude of the cosine to exceed unity.

The time required for an angular reorientation θ using a single coning maneuver is found by simply dividing the precession angle, π rad, by the precession rate ω_p,

$$t_1 = \frac{\pi}{\omega_p} = \pi\frac{A}{C\omega_0}\cos\frac{\theta}{2} \tag{12.95}$$

Propellant expenditure is reflected in the magnitude of the individual angular momentum increments, in obvious analogy to delta-v calculations for orbital maneuvers. The total delta-H required for the single coning maneuver is therefore given by

$$\Delta H_{\text{total}} = \left\|\Delta\mathbf{H}_G)_1\right\| + \left\|\Delta\mathbf{H}_G)_2\right\| = 2\left(\left\|\mathbf{H}_G)_0\right\|\tan\frac{\theta}{2}\right) \tag{12.96}$$

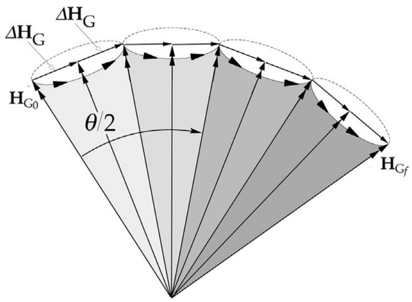


FIG. 12.14

A sequence of small coning maneuvers.

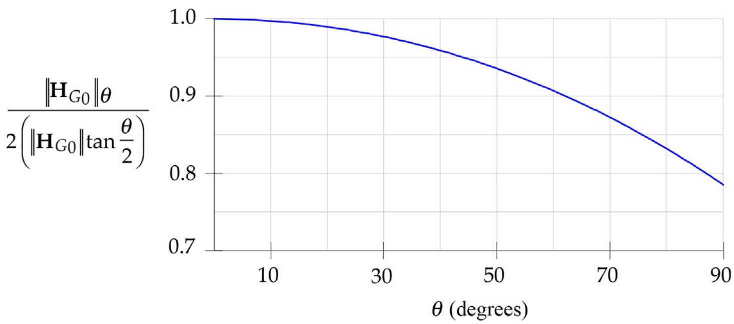


FIG. 12.15

Ratio of delta-*H* for a sequence of small coning maneuvers to that for a single coning maneuver, as a function of the angle of swing of the spin axis.

Fig. 12.14 illustrates the fact that ΔH_{total} can be reduced by using a sequence of small coning maneuvers (small θs) rather than one big θ. The large number of small ΔHs approximates a circular arc of radius $\|\mathbf{H}_G)_0\|$, subtended by the angle θ. Therefore, approximately,

$$\Delta H_{\text{total}} = 2\left(\|\mathbf{H}_G)_0\|\frac{\theta}{2}\right) = \|\mathbf{H}_G)_0\|\theta \tag{12.97}$$

This expression becomes more precise as the number of intermediate maneuvers increases. Fig. 12.15 reveals the extent to which the multiple coning maneuver strategy reduces energy requirements. The difference is quite significant for large reorientation angles.

One of the prices to be paid for the reduced energy of the multiple coning maneuver is time. (The other is the complexity mentioned above, to say nothing of the risk involved in repeating the maneuver over and over again.) From Eq. (12.95), the time required for n small-angle coning maneuvers through a total angle of θ is

$$t_n = n\pi\frac{A}{C\omega_0}\cos\frac{\theta}{2n} \tag{12.98}$$

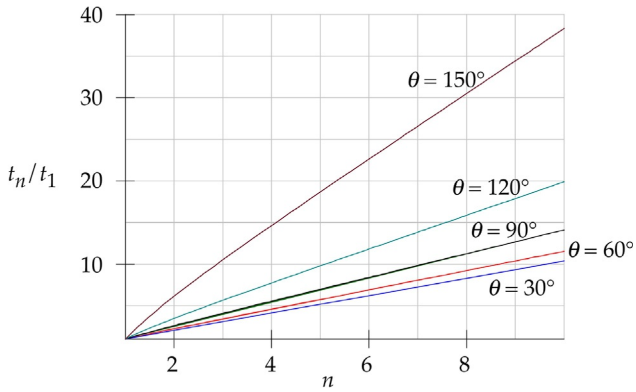


FIG. 12.16

Time for a coning maneuver vs. the number of intermediate steps.

The ratio of this to the time t_1 required for a single coning maneuver is

$$\frac{t_n}{t_1} = n\frac{\cos\dfrac{\theta}{2n}}{\cos\dfrac{\theta}{2}} \tag{12.99}$$

The time is directly proportional to the number of intermediate coning maneuvers, as illustrated in Fig. 12.16.

12.7 ATTITUDE CONTROL THRUSTERS

As mentioned above, thrusters are small jets mounted in pairs on a spacecraft to control its rotational motion about the center of mass. These thruster pairs may be mounted in principal planes (planes normal to the principal axes) passing through the center of mass. Fig. 12.17 illustrates a pair of thrusters for producing a torque about the positive y axis. These would be accompanied by another pair of reaction motors pointing in the opposite directions to exert torque in the negative y direction. If the position vectors of the thrusters relative to the center of mass are $\mathbf{r}$ and $-\mathbf{r}$, and if $\mathbf{T}$ is their thrust, then the impulsive moment they exert during a brief time interval Δt is

$$\mathbf{M} = \mathbf{r}\times\mathbf{T}\Delta t + (-\mathbf{r})\times(-\mathbf{T}\Delta t) = 2\mathbf{r}\times\mathbf{T}\Delta t \tag{12.100}$$

If the angular velocity was initially zero, then after the firing, according to Eq. (11.31), the angular momentum becomes

$$\mathbf{H} = 2\mathbf{r}\times\mathbf{T}\Delta t \tag{12.101}$$

For $\mathbf{H}$ in the principal y direction, as in the figure, the corresponding angular velocity acquired by the vehicle is, from Eq. (11.67),

$$\omega_y = \frac{\|\mathbf{H}\|}{B} \tag{12.102}$$

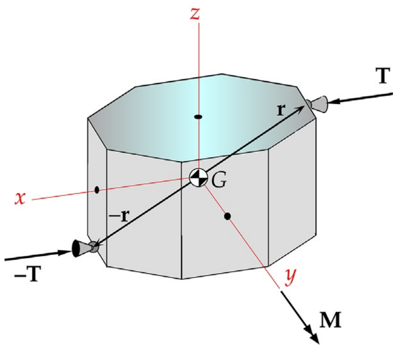


FIG. 12.17

Pair of attitude control thrusters mounted in the *xz* plane of the principal body frame.

EXAMPLE 12.8

A spacecraft of mass m and with the dimensions shown in Fig. 12.18 is spinning without precession at the rate ω_0 about the z axis of the principal body frame. At the instant shown in part (a) of the figure, the spacecraft initiates a coning maneuver to swing its spin axis through 90°, so that at the end of the maneuver the vehicle is oriented as illustrated in Fig. 12.18a. Calculate the total delta-H required and compare it with that required for the same reorientation without coning. Motion is to be controlled exclusively by the pairs of attitude thrusters shown, all of which have identical thrust T.

Solution

According to Fig. 11.10c, the moments of inertia about the principal body axes are

$$A = B = \frac{1}{12}m\left[w^2 + \left(\frac{w}{3}\right)^2\right] = \frac{5}{54}mw^2 \qquad C = \frac{1}{12}m\left(w^2 + w^2\right) = \frac{1}{6}mw^2$$

The initial angular momentum $\mathbf{H}_G)_1$ points in the spin direction, along the positive z axis of the body frame,

$$\mathbf{H}_G)_1 = C\omega_z\hat{\mathbf{k}} = \frac{1}{6}mw^2\omega_0\hat{\mathbf{k}}$$

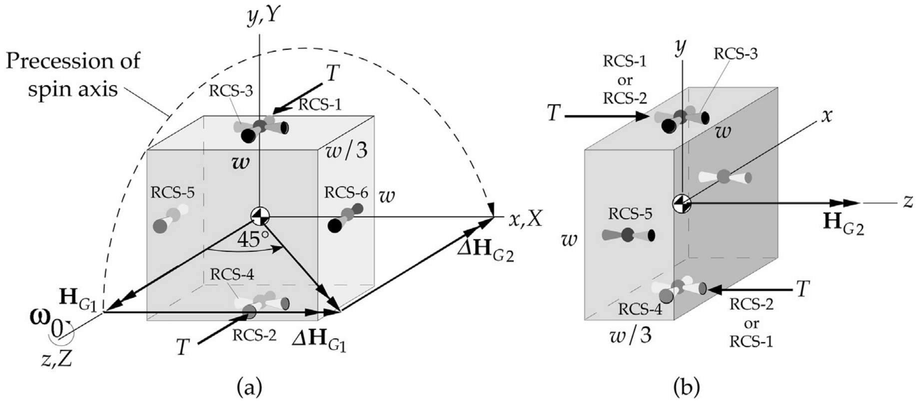


FIG. 12.18

(a) Initial orientation of spinning spacecraft. (b) Final configuration, with spin axis rotated 90°.

We can presume that in the initial orientation, the body frame happens to coincide instantaneously with the inertial frame *XYZ*. The coning motion is initiated by briefly firing the pair of thrusters RCS-1 and RCS-2, aligned with the body z axis and lying in the yz plane. The impulsive torque will cause a change $\Delta\mathbf{H}_G)_1$ in angular momentum directed normal to the plane of the thrusters, in the positive body x direction. The resultant angular momentum vector must lie at 45° to the x and z axes, bisecting the angle between the initial and final angular momenta. Thus,

$$\left\|\Delta\mathbf{H}_G)_1\right\| = \left\|\mathbf{H}_G)_1\right\|\tan 45° = \frac{1}{6}mw^2\omega_0$$

After the coning is under way, the body axes of course move away from the *XYZ* frame. Since the spacecraft is oblate ($C > A$), the precession of the spin axis will be opposite to the spin direction, as indicated in Fig. 12.15. When the spin axis, after 180° of precession, lines up with the X axis, the thrusters must fire again for the same duration as before so as to produce the angular momentum change $\Delta\mathbf{H}_G)_2$, equal in magnitude but perpendicular to $\Delta\mathbf{H}_G)_1$, so that

$$\mathbf{H}_G)_1 + \Delta\mathbf{H}_G)_1 + \Delta\mathbf{H}_G)_2 = \mathbf{H}_G)_2$$

where

$$\mathbf{H}_G)_2 = \left\|\mathbf{H}_G)_1\right\|\hat{\mathbf{I}} = \frac{1}{6}mw^2\omega_0\hat{\mathbf{k}}$$

For this to work, the plane of thrusters RCS-1 and RCS-2 (the yz plane) must be parallel to the XY plane when they fire, as illustrated in Fig. 12.18b. Since the thrusters can fire fore or aft, it does not matter which of them ends up on the top or bottom. The vehicle must therefore spin through an integral number n of half rotations while it processes to the desired orientation. That is, the total spin angle ψ between the initial and final configurations is

$$\psi = n\pi = \omega_s t \tag{a}$$

where ω_s is the spin rate, and t is the time for the proper final configuration to be achieved. In the meantime, the precession angle ϕ must be π or 3π or 5π, or, in general,

$$\phi = (2m-1)\pi = \omega_p t \tag{b}$$

where m is an integer, and t is, of course, the same as that in Eq. (a). Eliminating t from both Eqs. (a) and (b) yields

$$n\pi = (2m-1)\pi\frac{\omega_s}{\omega_p}$$

Substituting Eq. (12.23), with $\theta = \pi/4$, gives

$$n = (1-2m)\frac{4}{9}\frac{1}{\sqrt{2}} \tag{c}$$

Obviously, this equation cannot be valid if both m and n are integers. However, by tabulating n as a function of m, we find that when $m = 18$, $n = -10.999$. The minus sign simply reminds us that spin and precession are in opposite directions. Thus, the 18th time that the spin axis lines up with the X axis the thrusters may be fired to almost perfectly align the angular momentum vector with the body z axis. The slight misalignment due to the fact that $|n|$ is not precisely 11 would probably occur in reality anyway. Passive or active nutation damping can drive this deviation to zero.

Since $\|\mathbf{H}_G)_1\| = \|\mathbf{H}_G)_2\|$, we conclude that

$$\Delta H_{\text{total}} = 2\left(\frac{1}{6}mw^2\omega_0\right) = \frac{2}{3}mw^2\omega_0 \tag{d}$$

An obvious alternative to the coning maneuver is to use thrusters RCS-3 and RCS-4 to despin the craft completely, thrusters RCS-5 and RCS-6 to initiate roll around the y axis and stop it after 90°, and then RCS-3 and RCS-4 to respin the spacecraft to ω_0 around the z axis. The combined delta-H for the first and last steps equals that of Eq. (d). Additional fuel expenditure is required to start and stop the roll around the y axis. Hence, the coning maneuver is more fuel efficient.

12.8 YO-YO DESPIN MECHANISM

A simple, inexpensive way to despin an axisymmetric spacecraft is to deploy small masses attached to cords wound around the girth of the spacecraft near the transverse plane through the center of mass. As the masses unwrap in the direction of the spacecraft angular velocity, they exert centrifugal force through the cords on the periphery of the vehicle, creating a moment opposite to the spin direction, thereby slowing down the rotational motion. The cord forces are internal to the system of spacecraft plus weights, so that as the strings unwind, the total angular momentum must remain constant. Since the total moment of inertia increases as the yo-yo masses spiral farther away, the angular velocity must drop. Not only angular momentum but also rotational kinetic energy is conserved during this process. Yo-yo despin devices were introduced early in unmanned space flight (e.g., 1959 Transit 1-A) and continued to be used thereafter (e.g., 1996 Mars Pathfinder, 1998 Mars Climate Orbiter, 1999 Mars Polar Lander, 2003 Mars Exploration Rover, and 2007 Dawn spacecraft).

The problem is to determine the length of cord required to reduce the spacecraft angular velocity a specified amount. Because it is easier than solving the equations of motion, we will apply the principles of conservation of energy and angular momentum to the system comprising the spacecraft and the yo-yo masses. To maintain the position of the center of mass, two identical yo-yo masses are wound around the spacecraft in a symmetric fashion, as illustrated in Fig. 12.19. Both masses are released simultaneously by explosive bolts and unwrap in the manner shown (for only one of the weights) in the figure. In so doing, the point of tangency T moves around the circumference toward the split hinge device where the cord is attached to the spacecraft. When T and T' reach the hinges H and H', the cords automatically separate from the spacecraft.

Let each yo-yo weight have mass $m/2$. By symmetry, we need to track only one of the masses, to which we can ascribe the total mass m. Let the xyz system be a body frame rigidly attached to the spacecraft, as shown in Fig. 12.19. As usual, the z axis lies in the spin direction, pointing out of the page. The x axis is directed from the center of mass of the system through the initial position of the yo-yo mass.

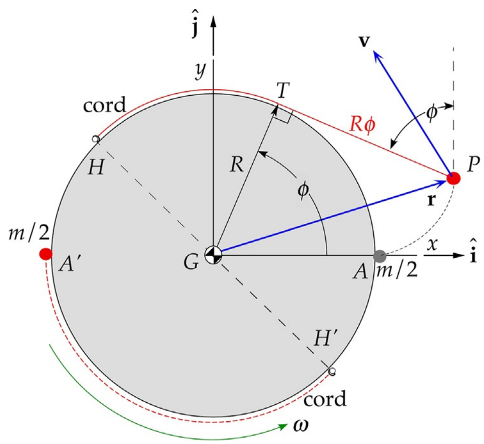


FIG. 12.19

Two identical string and mass systems wrapped symmetrically around the periphery of an axisymmetric spacecraft. For simplicity, only one is shown being deployed.

The spacecraft and the yo-yo masses, prior to release, are rotating as a single rigid body with angular velocity $\boldsymbol{\omega}_0 = \omega_0 \hat{\mathbf{k}}$. The moment of inertia of the spacecraft, excluding the yo-yo mass, is C, so that the angular momentum of the spacecraft by itself is $C\omega_0$. The concentrated yo-yo masses are fastened at a distance R from the spin axis, so that their total moment of inertia is mR^2. Therefore, the initial angular momentum of the satellite plus yo-yo system is

$$\mathrm{H}_G)_0 = C\omega_0 + mR^2\omega_0$$

It will be convenient to write this as

$$\mathrm{H}_G)_0 = KmR^2\omega_0 \tag{12.103}$$

where the nondimensional factor K is defined as

$$K = 1 + \frac{C}{mR^2} \tag{12.104}$$

$R\sqrt{mK/(m+M)}$ is the system's initial radius of gyration, where M is the spacecraft mass.

$$T_0 = \frac{1}{2}C{\omega_0}^2 + \frac{1}{2}mR^2{\omega_0}^2 = \frac{1}{2}KmR^2{\omega_0}^2 \tag{12.105}$$

At any state between the release of the weights and the release of the cords at the hinges, the velocity of the yo-yo mass must be found to compute the new angular momentum and kinetic energy. Observe that when the string has unwrapped an angle ϕ, the free length of string (between the point of tangency T and the yo-yo mass P) is $R\phi$. From the geometry shown in Fig. 12.19, the position vector of the mass relative to the body frame is seen to be

$$\begin{aligned}\mathbf{r} &= \overbrace{\left(R\cos\phi\hat{\mathbf{i}} + R\sin\phi\hat{\mathbf{j}}\right)}^{\mathbf{r}_{T/G}} + \overbrace{\left(R\phi\sin\phi\hat{\mathbf{i}} - R\phi\cos\phi\hat{\mathbf{j}}\right)}^{\mathbf{r}_{P/T}} \\ &= (R\cos\phi + R\phi\sin\phi)\hat{\mathbf{i}} + (R\sin\phi - R\phi\cos\phi)\hat{\mathbf{j}}\end{aligned} \tag{12.106}$$

Since $\mathbf{r}$ is measured in the moving reference, the absolute velocity $\mathbf{v}$ of the yo-yo mass is found using Eq. (1.56),

$$\mathbf{v} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathrm{rel}} + \boldsymbol{\Omega} \times \mathbf{r} \tag{12.107}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the xyz axes, which, of course, is the angular velocity $\boldsymbol{\omega}$ of the spacecraft at that instant,

$$\boldsymbol{\Omega} = \boldsymbol{\omega} \tag{12.108}$$

To calculate $\mathrm{d}\mathbf{r}/\mathrm{d}t)_{\mathrm{rel}}$, we hold $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ constant in Eq. (12.106), obtaining

$$\begin{aligned}\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathrm{rel}} &= \left(-R\dot{\phi}\sin\phi + R\dot{\phi}\sin\phi + R\phi\dot{\phi}\cos\phi\right)\hat{\mathbf{i}} + \left(R\dot{\phi}\cos\phi - R\dot{\phi}\cos\phi + R\phi\dot{\phi}\sin\phi\right)\hat{\mathbf{j}} \\ &= R\phi\dot{\phi}\cos\phi\hat{\mathbf{i}} + R\phi\dot{\phi}\sin\phi\hat{\mathbf{j}}\end{aligned}$$

Thus,

$$\mathbf{v} = R\phi\dot{\phi}\cos\phi\hat{\mathbf{i}} + R\phi\dot{\phi}\sin\phi\hat{\mathbf{j}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega \\ R\cos\phi + R\phi\sin\phi & R\sin\phi - R\phi\cos\phi & 0 \end{vmatrix}$$

or

$$\mathbf{v} = \left[R\phi\left(\omega+\dot{\phi}\right)\cos\phi - R\omega\sin\phi\right]\hat{\mathbf{i}} + \left[R\omega\cos\phi + R\phi\left(\omega+\dot{\phi}\right)\sin\phi\right]\hat{\mathbf{j}} \tag{12.109}$$

From this, we find the speed of the yo-yo weights,

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = R\sqrt{\omega^2 + \left(\omega+\dot{\phi}\right)^2\phi^2} \tag{12.110}$$

The angular momentum of the spacecraft plus the weights at an intermediate stage of the despin process is

$$\begin{aligned}\mathbf{H}_G &= C\omega\hat{\mathbf{k}} + \mathbf{r}\times m\mathbf{v} \\ &= C\omega\hat{\mathbf{k}} + m\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ R\cos\phi + R\phi\sin\phi & R\sin\phi - R\phi\cos\phi & \omega \\ R\phi\left(\omega+\dot{\phi}\right)\cos\phi - R\omega\sin\phi & R\omega\cos\phi + R\phi\left(\omega+\dot{\phi}\right)\sin\phi & 0 \end{vmatrix}\end{aligned}$$

Carrying out the cross product, combining terms, and simplifying leads to

$$H_G = C\omega + mR^2\left[\omega + \left(\omega+\dot{\phi}\right)\phi^2\right]$$

which, using Eq. (12.104), can be written as

$$H_G = mR^2\left[K\omega + \left(\omega+\dot{\phi}\right)\phi^2\right] \tag{12.111}$$

The kinetic energy of the spacecraft plus the yo-yo mass is

$$T = \frac{1}{2}C\omega^2 + \frac{1}{2}mv^2$$

Substituting the speed from Eq. (12.110) and making use again of Eq. (12.104), we find

$$T = \frac{1}{2}mR^2\left[K\omega^2 + \left(\omega+\dot{\phi}\right)^2\phi^2\right] \tag{12.112}$$

By the conservation of angular momentum, $H_G = H_{G_0}$, we obtain from Eqs. (12.103) and (12.111),

$$mR^2\left[K\omega + \left(\omega+\dot{\phi}\right)\phi^2\right] = KmR^2\omega_0$$

which we can write as

$$K(\omega_0 - \omega) = \left(\omega+\dot{\phi}\right)\phi^2 \quad \text{Conservation of angular momentum} \tag{12.113}$$

Eqs. (12.105) and (12.112) and the conservation of kinetic energy, $T = T_0$, combine to yield

$$\frac{1}{2}mR^2\left[K\omega^2 + \left(\omega+\dot{\phi}\right)^2\phi^2\right] = \frac{1}{2}KmR^2\omega_0^2$$

or

$$K\left({\omega_0}^2 - \omega^2\right) = \left(\omega+\dot{\phi}\right)^2\phi^2 \quad \text{Conservation of energy} \tag{12.114}$$

Since $\omega_0{}^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 + \omega)$, this can be written as

$$K(\omega_0 - \omega)(\omega_0 + \omega) = \left(\omega + \dot{\phi}\right)^2 \phi^2$$

Replacing the factor $K(\omega_0 - \omega)$ on the left using Eq. (12.113) yields

$$\left(\omega + \dot{\phi}\right)\phi^2(\omega_0 + \omega) = \left(\omega + \dot{\phi}\right)^2 \phi^2$$

After canceling terms, we find $\omega_0 + \omega = \omega + \dot{\phi}$, or, simply

$$\dot{\phi} = \omega_0 \quad \text{Conservation of energy and momentum} \tag{12.115}$$

In other words, the cord unwinds at a constant rate (relative to the spacecraft), equal to the vehicle's initial angular velocity. Thus, at any time t after the release of the weights,

$$\phi = \omega_0 t \tag{12.116}$$

By substituting Eq. (12.115) back into Eq. (12.113),

$$K(\omega_0 - \omega) = (\omega + \omega_0)\phi^2$$

we find that

$$\phi = \sqrt{K\frac{\omega_0 - \omega}{\omega_0 + \omega}} \quad \text{Partial despin} \tag{12.117}$$

Recall that the unwrapped length l of the cord is $R\phi$, which means

$$l = R\sqrt{K\frac{\omega_0 - \omega}{\omega_0 + \omega}} \quad \text{Partial despin} \tag{12.118}$$

We use Eq. (12.118) to find the length of the cord required to despin the spacecraft from ω_0 to ω. To remove all of the spin ($\omega = 0$),

$$\phi = \sqrt{K} \quad \Rightarrow \quad l = R\sqrt{K} \quad \text{Complete despin} \tag{12.119}$$

Surprisingly, the length of the cord required to reduce the angular velocity to zero is independent of the initial angular velocity.

We can solve Eq. (12.117) for ω in terms of ϕ,

$$\omega = \left(\frac{2K}{K + \phi^2} - 1\right)\omega_0 \tag{12.120}$$

By means of Eq. (12.116), this becomes an expression for the angular velocity as a function of time,

$$\omega = \left(\frac{2K}{K + \omega_0{}^2 t^2} - 1\right)\omega_0 \tag{12.121}$$

Alternatively, since $\phi = l/R$, Eq. (12.120) yields the angular velocity as a function of the cord length,

$$\omega = \left(\frac{2KR^2}{KR^2 + l^2} - 1\right)\omega_0 \tag{12.122}$$

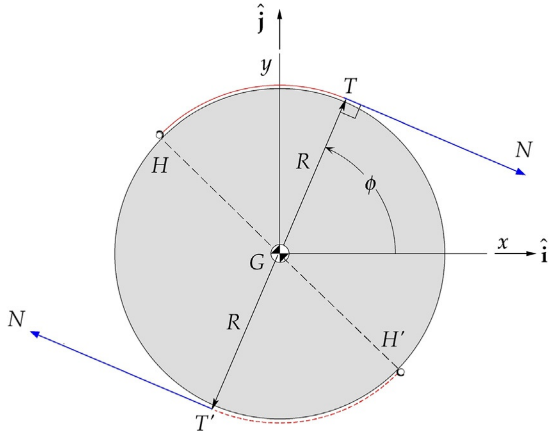


FIG. 12.20

Free body diagram of the satellite during the despin process.

Differentiating ω with respect to time in Eq. (12.121) gives us an expression for the angular acceleration of the spacecraft,

$$\alpha = \frac{d\omega}{dt} = -\frac{4K\omega_0{}^3 t}{(K + \omega_0{}^2 t^2)^2} \tag{12.123}$$

whereas integrating ω with respect to time yields the angle rotated by the spacecraft since release of the yo-yo mass,

$$\theta = 2\sqrt{K}\tan^{-1}\frac{\omega_0 t}{\sqrt{K}} - \omega_0 t = 2\sqrt{K}\tan^{-1}\frac{\phi}{\sqrt{K}} - \phi \tag{12.124}$$

For complete despin, this expression, together with Eq. (12.119), yields

$$\theta = \sqrt{K}\left(\frac{\pi}{2} - 1\right) \tag{12.125}$$

From the free body diagram of the spacecraft shown in Fig. 12.20, it is clear that the torque exerted by the yo-yo weights is

$$M_G)_z = -2RN \tag{12.126}$$

where N is the tension in the cord. From the Euler equations of motion (Eq. 11.72b)

$$M_G)_z = C\alpha \tag{12.127}$$

Combining Eqs. (12.123), (12.126), and (12.127) leads to a formula for tension in the yo-yo cables,

$$N = \frac{C}{R}\frac{2K\omega_0{}^3 t}{(K + \omega_0{}^2 t^2)^2} = \frac{C\omega_0{}^2}{R}\frac{2K\phi}{\left(K + \phi^2\right)^2} \tag{12.128}$$

12.8.1 RADIAL RELEASE

Finally, we note that instead of releasing the yo-yo masses when the cables are tangent at the split hinges (H and H'), they can be forced to pivot about the hinge and released when the string is directed radially outward, as illustrated in Fig. 12.18. The above analysis must be then extended to include the pivoting of the cord around the hinges. It turns out that in this case, the length of the cord as a function of the final angular velocity is

$$l = R\left(\sqrt{\frac{[(\omega_0 - \omega)K + \omega]^2}{(\omega_0{}^2 - \omega^2)K + \omega^2}} - 1\right) \quad \text{Partial despin, radial release} \tag{12.129}$$

so that for $\omega = 0$ (Fig. 12.21),

$$l = R\left(\sqrt{K} - 1\right) \quad \text{Complete despin, radial release} \tag{12.130}$$

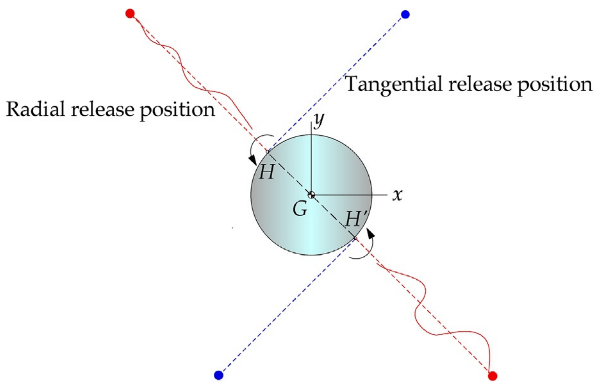


FIG. 12.21

Radial vs. tangential release of yo-yo masses.

EXAMPLE 12.9

A satellite is to be completely despun using a two-mass yo-yo device with tangential release. Assume the spin axis moment of inertia of the satellite is $C = 200\ \text{kg}\cdot\text{m}^2$ and the initial spin rate is $\omega_0 = 5\ \text{rad/s}$. The total yo-yo mass is 4 kg, and the radius of the spacecraft is 1 m. Find

(a) the required cord length l;
(b) the time t to despin;
(c) the maximum tension in the yo-yo cables;
(d) the speed of the masses at release;
(e) the angle rotated by the satellite during despin; and
(f) the cord length required for radial release.

Solution

(a) From Eq. (12.104),

$$K = 1 + \frac{C}{mR^2} = 1 + \frac{200}{4 \cdot 1^2} = 51 \tag{a}$$

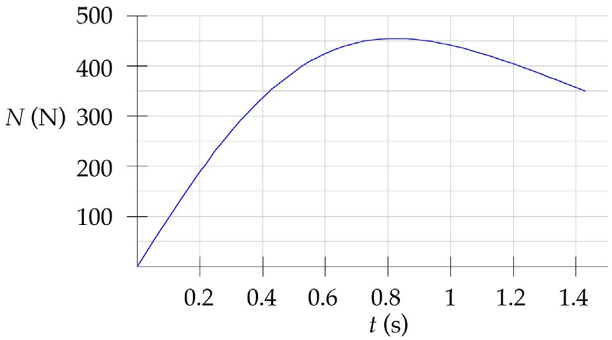


FIG. 12.22

Variation of cable tension N up to the point of release.

From Eq. (12.119) it follows that the cord length required for complete despin is

$$l = R\sqrt{K} = 1 \cdot \sqrt{51} = \boxed{7.1441 \text{ m}} \tag{b}$$

(b) The time for complete despin is obtained from Eqs. (12.116) and (12.119),

$$\omega_0 t = \sqrt{K} \quad \Rightarrow \quad t = \frac{\sqrt{K}}{\omega_0} = \frac{\sqrt{51}}{5} = \boxed{1.4283 \text{ s}}$$

(c) A graph of Eq. (12.128) is shown in Fig. 12.19, from which we see that

$$\boxed{\text{The maximum tension is 455 N}}$$

which occurs at 0.825 s (Fig. 12.22).

(d) From Eq. (12.110), the speed of the yo-yo masses is

$$v = R\sqrt{\omega^2 + \left(\omega + \dot{\phi}\right)^2 \phi^2}$$

According to Eq. (12.115), $\dot{\phi} = \omega_0$, and at the time of release ($\omega = 0$) Eq. (12.117) states that $\phi = \sqrt{K}$. Thus,

$$v = R\sqrt{\omega^2 + (\omega + \omega_0)^2 \sqrt{K}^2} = 1 \cdot \sqrt{0^2 + (0+5)^2 \sqrt{51}^2} = \boxed{35.71 \text{ m/s}}$$

(e) The angle through which the satellite rotates before coming to rotational rest is given by Eq. (12.125),

$$\theta = \sqrt{K}\left(\frac{\pi}{2} - 1\right) = \sqrt{51}\left(\frac{\pi}{2} - 1\right) = \boxed{4.076 \text{ rad}\,(233.5°)}$$

(f) Allowing the cord to detach radially reduces the cord length required for complete despin from 7.141 m to (Eq. 12.130)

$$l = R\left(\sqrt{K} - 1\right) = 1 \cdot \left(\sqrt{51} - 1\right) = \boxed{6.141 \text{ m}}$$

12.9 GYROSCOPIC ATTITUDE CONTROL

Momentum exchange systems (gyros) are used to control the attitude of a spacecraft without throwing consumable mass overboard, as occurs with the use of thruster jets. A momentum exchange system is illustrated schematically in Fig. 12.23. n flywheels, labeled 1, 2, 3, etc., are attached to the body of the

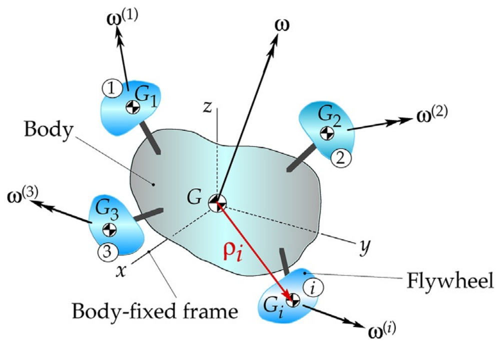


FIG. 12.23

Several attitude control flywheels, each with their own angular velocity, attached to the body of a spacecraft.

spacecraft at various locations. The mass of flywheel i is m_i. The mass of the body of the spacecraft is m_0. The total mass of the entire system (the "vehicle") is m,

$$m = m_0 + \sum_{i=1}^{n} m_i$$

The vehicle's center of mass is G, through which pass the three axes xyz of the vehicle's body-fixed frame. The center of mass G_i of each flywheel is connected rigidly to the spacecraft, but the wheel, driven by electric motors, rotates more or less independently, depending on the type of gyro. The body of the spacecraft has the angular velocity vector $\boldsymbol{\omega}$. The angular velocity vector of the ith flywheel is $\boldsymbol{\omega}^{(i)}$, and it differs from that of the body of the spacecraft unless the gyro is "caged." A caged gyro has no spin relative to the spacecraft, in which case $\boldsymbol{\omega}^{(i)} = \boldsymbol{\omega}$.

According to Eq. (11.39b), the angular momentum of the body itself relative to G is

$$\left\{\mathbf{H}_G^{(\text{body})}\right\} = \left[\mathbf{I}_G^{(\text{body})}\right]\{\boldsymbol{\omega}\} \tag{12.131}$$

where $\mathbf{I}_G^{(\text{body})}$ is the moment of inertia tensor of the body about G and $\boldsymbol{\omega}$ is the angular velocity of the body.

Eq. (11.27) gives the angular momentum of flywheel i relative to G as

$$\mathbf{H}_G^{(i)} = \mathbf{H}_{G_i}^{(i)} + \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i \tag{12.132}$$

$\mathbf{H}_{G_i}^{(i)}$ is the angular momentum vector of the flywheel i about its own center of mass G_i. Its components in the body frame are found from the expression

$$\left\{\mathbf{H}_{G_i}^{(i)}\right\} = \left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}^{(i)}\right\} \tag{12.133}$$

where $\mathbf{I}_{G_i}^{(i)}$ is the moment of inertia tensor of the flywheel about its own center of mass G_i, relative to axes that are parallel to the body-fixed xyz axes. Since a momentum wheel might be one that pivots on gimbals relative to the body frame, the inertia tensor $\mathbf{I}_{G_i}^{(i)}$ may be time dependent. The vector $\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i$ in Eq. (12.132) is the angular momentum of the concentrated mass m_i of the flywheel about the *system* center of mass G. According to Eq. (11.59), the components of $\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i$ in the body frame are given by

$$\{\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i m_i\} = \left[\mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\} \tag{12.134}$$

where $\mathbf{I}_{m_G}^{(i)}$, the moment of inertia tensor of the point mass m_i about G, is given by Eq. (11.44). Using Eqs. (12.133) and (12.134), Eq. (12.132) can be written as

$$\left\{\mathbf{H}_G^{(i)}\right\} = \left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}^{(i)}\right\} + \left[\mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\} \tag{12.135}$$

The total angular momentum of the system in Fig. 12.20 about G is that of the body plus all of the n flywheels,

$$\mathbf{H}_G = \mathbf{H}_G^{(\text{body})} + \sum_{i=1}^{n} \mathbf{H}_G^{(i)}$$

Substituting Eqs. (12.131) and (12.135), we obtain

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(\text{body})}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left(\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}^{(i)}\right\} + \left[\mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\}\right)$$

or

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(\text{body})} + \sum_{i=1}^{n} \mathbf{I}_{m_G}^{(i)}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}^{(i)}\right\} \tag{12.136}$$

Let

$$\mathbf{I}_G^{(v)} = \mathbf{I}_G^{(\text{body})} + \sum_{i=1}^{n} \mathbf{I}_{m_G}^{(i)} \tag{12.137}$$

where $\mathbf{I}_G^{(v)}$ is the time-independent total moment of inertia of the vehicle v (i.e., that of the body *plus* the concentrated masses of all the flywheels). Thus,

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}^{(i)}\right\} \tag{12.138}$$

If $\boldsymbol{\omega}_{\text{rel}}^{(i)}$ is the angular velocity of the ith flywheel relative to the spacecraft, then its inertial angular velocity $\boldsymbol{\omega}^{(i)}$ is given by Eq. (11.5),

$$\boldsymbol{\omega}^{(i)} = \boldsymbol{\omega} + \boldsymbol{\omega}_{\text{rel}}^{(i)} \tag{12.139}$$

where $\boldsymbol{\omega}$ is the inertial angular velocity of the spacecraft body. Substituting Eq. (12.139) into Eq. (12.138) yields

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega} + \boldsymbol{\omega}_{\text{rel}}^{(i)}\right\}$$

or

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)} + \sum_{i=1}^{n} \mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\right\} \tag{12.140}$$

An alternative form of this expression may be obtained by substituting Eq. (12.137):

$$\begin{aligned}\{\mathbf{H}_G\} &= \left[\mathbf{I}_G^{(\text{body})} + \sum_{i=1}^{n}\mathbf{I}_{m_G}^{(i)} + \sum_{i=1}^{n}\mathbf{I}_{G_i}^{(i)}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\right\} \\ &= \left[\mathbf{I}_G^{(\text{body})} + \sum_{i=1}^{n}\left(\mathbf{I}_{G_i}^{(i)} + \mathbf{I}_{m_G}^{(i)}\right)\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\right\}\end{aligned} \tag{12.141}$$

But, according to the parallel axis theorem (Eq. 11.61),

$$\mathbf{I}_G^{(i)} = \mathbf{I}_{G_i}^{(i)} + \mathbf{I}_{m_G}^{(i)}$$

where $\mathbf{I}_G^{(i)}$ is the moment of inertia of the ith flywheel around the center of mass of the body of the spacecraft. Hence, we can write Eq. (12.141) as

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(\text{body})} + \sum_{i=1}^{n}\mathbf{I}_G^{(i)}\right]\{\boldsymbol{\omega}\} + \sum_{i=1}^{n}\left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\right\} \tag{12.142}$$

The equation of motion of the system is given by Eqs. (11.30) and (1.56),

$$\mathbf{M}_G)_{\text{net external}} = \left.\frac{d\mathbf{H}_G}{dt}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_G \tag{12.143}$$

If $\mathbf{M}_G)_{\text{net external}} = \mathbf{0}$, then $\mathbf{H}_G = \text{constant}$.

EXAMPLE 12.10

A disk is attached to a plate at their common center of mass (Fig. 12.24). Between the two is a motor mounted on the plate, which drives the disk into rotation relative to the plate. The system rotates freely in the xy plane in gravity-free space. The moments of inertia of the plate and the disk about the z axis through G are I_p and I_w, respectively. Determine the change in the relative angular velocity ω_{rel} of the disk required to cause a given change in the inertial angular velocity ω of the plate.

Solution

The plate plays the role of the body of a spacecraft and the disk is a momentum wheel. At any given time, the angular momentum of the system about G is given by Eq. (12.142),

$$H_G = (I_p + I_w)\omega + I_w\omega_{\text{rel}}$$

At a later time (denoted by primes), after the torquing motor is activated, the angular momentum is

$$H'_G = (I_p + I_w)\omega' + I_w\omega'_{\text{rel}}$$

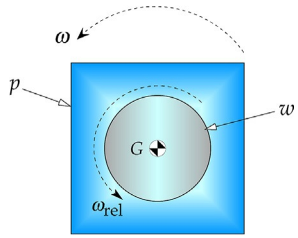


FIG. 12.24

Plate p and disk w attached at their common center of mass G.

Since the torque is internal to the system, we have conservation of angular momentum, $H_G{}' = H_G$, which means

$$\left(I_p + I_w\right)\omega' + I_w\omega'_{\text{rel}} = \left(I_p + I_w\right)\omega + I_w\omega_{\text{rel}}$$

Rearranging terms we get

$$I_w\left(\omega'_{\text{rel}} - \omega_{\text{rel}}\right) = -\left(I_p + I_w\right)\left(\omega' - \omega\right)$$

Letting $\Delta\omega = \omega' - \omega$, this can be written as

$$\boxed{\Delta\omega_{\text{rel}} = -\left(1 + \frac{I_p}{I_w}\right)\Delta\omega}$$

The change $\Delta\omega_{\text{rel}}$ in the relative rotational velocity of the disk is due to the torque applied to the disk at G by the motor mounted on the plate. An equal torque in the opposite direction is applied to the plate, producing the angular velocity change $\Delta\omega$ opposite in direction to $\Delta\omega_{\text{rel}}$.

Notice that if $I_p \gg I_w$, which is true in an actual spacecraft, then the change in angular velocity of the momentum wheel must be very much larger than the required change in angular velocity of the body of the spacecraft.

EXAMPLE 12.11

Use Eq. (12.142) to obtain the equations of motion of a torque-free, axisymmetric dual-spin satellite, such as the one shown in Fig. 12.25.

Solution

In this case, we have only one "reaction wheel" (namely, the platform p). The "body" is the rotor r. In Eq. (12.142), we make the following substitutions ($\leftarrow$ means "is replaced by"):

$$\boldsymbol{\omega} \leftarrow \boldsymbol{\omega}^{(r)}$$

$$\boldsymbol{\omega}_{\text{rel}}^{(i)} \leftarrow \boldsymbol{\omega}_{\text{rel}}^{(p)}$$

$$\mathbf{I}_G^{(\text{body})} \leftarrow \mathbf{I}_G^{(r)}$$

$$\sum_{i=1}^{n} \mathbf{I}_G^{(i)} \leftarrow \mathbf{I}_G^{(p)}$$

$$\sum_{i=1}^{n} \left[\mathbf{I}_{G_i}^{(i)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(i)}\right\} \leftarrow \left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\right\}$$

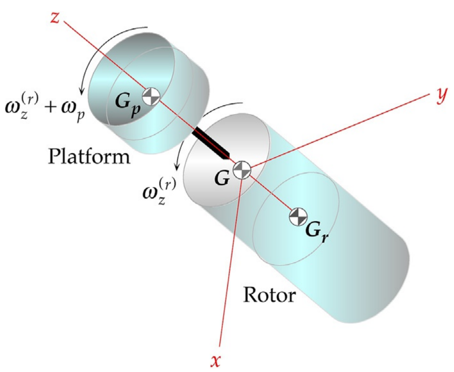


FIG. 12.25

Dual-spin spacecraft.

so that Eq. (12.142) becomes

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(r)} + \mathbf{I}_G^{(p)}\right]\left\{\boldsymbol{\omega}^{(r)}\right\} + \left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\right\} \tag{a}$$

Since $\mathbf{M}_G)_{\text{net external}} = \mathbf{0}$, Eq. (12.143) yields

$$\left[\mathbf{I}_G^{(r)} + \mathbf{I}_G^{(p)}\right]\left\{\dot{\boldsymbol{\omega}}^{(r)}\right\} + \left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\dot{\boldsymbol{\omega}}_{\text{rel}}^{(p)}\right\} + \left\{\boldsymbol{\omega}^{(r)}\right\} \times \left(\left[\mathbf{I}_G^{(r)} + \mathbf{I}_G^{(p)}\right]\left\{\boldsymbol{\omega}^{(r)}\right\} + \left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\right\}\right) = \{\mathbf{0}\} \tag{b}$$

The components of the matrices and vectors in Eq. (b) relative to the principal xyz body frame axes attached to the rotor are

$$\left[\mathbf{I}_G^{(r)}\right] = \begin{bmatrix} A_r & 0 & 0 \\ 0 & A_r & 0 \\ 0 & 0 & C_r \end{bmatrix} \quad \left[\mathbf{I}_G^{(p)}\right] = \begin{bmatrix} A_p & 0 & 0 \\ 0 & A_p & 0 \\ 0 & 0 & C_p \end{bmatrix} \quad \left[\mathbf{I}_{G_p}^{(p)}\right] = \begin{bmatrix} \overline{A}_p & 0 & 0 \\ 0 & \overline{A}_p & 0 \\ 0 & 0 & C_p \end{bmatrix} \tag{c}$$

and

$$\left\{\boldsymbol{\omega}^{(r)}\right\} = \begin{Bmatrix} \omega_x^{(r)} \\ \omega_y^{(r)} \\ \omega_z^{(r)} \end{Bmatrix} \quad \left\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\right\} = \begin{Bmatrix} 0 \\ 0 \\ \omega_p \end{Bmatrix} \tag{d}$$

where A_r, C_r, A_p, and C_p are the rotor and platform principal moments of inertia about the vehicle center of mass G, and $\overline{A}_p$ is the moment of inertia of the platform about its own center of mass G_p. We also used the fact that $\overline{C}_p = C_p$, which of course is due to the fact that G and G_p both lie on the z axis. This notation is nearly identical to that employed in our consideration of the stability of dual-spin satellites in Section 12.4 (wherein $\omega_r = \omega_z^{(r)}$ and $\boldsymbol{\omega}_\perp = \omega_x^{(r)}\hat{\mathbf{i}} + \omega_y^{(r)}\hat{\mathbf{j}}$). Substituting Eqs. (c) and (d) into each of the four terms in Eq. (b), we get

$$\left[\mathbf{I}_G^{(r)} + \mathbf{I}_G^{(p)}\right]\left\{\dot{\boldsymbol{\omega}}^{(r)}\right\} = \begin{bmatrix} A_r + A_p & 0 & 0 \\ 0 & A_r + A_p & 0 \\ 0 & 0 & C_r + C_p \end{bmatrix}\begin{Bmatrix} \dot{\boldsymbol{\omega}}_x^{(r)} \\ \dot{\boldsymbol{\omega}}_y^{(r)} \\ \dot{\boldsymbol{\omega}}_z^{(r)} \end{Bmatrix} = \begin{Bmatrix} (A_r + A_p)\dot{\boldsymbol{\omega}}_x^{(r)} \\ (A_r + A_p)\dot{\boldsymbol{\omega}}_y^{(r)} \\ (C_r + C_p)\dot{\boldsymbol{\omega}}_z^{(r)} \end{Bmatrix} \tag{e}$$

$$\left\{\boldsymbol{\omega}^{(r)}\right\} \times \left[\mathbf{I}_G^{(r)} + \mathbf{I}_G^{(p)}\right]\left\{\boldsymbol{\omega}^{(r)}\right\} = \begin{Bmatrix} \omega_x^{(r)} \\ \omega_y^{(r)} \\ \omega_z^{(r)} \end{Bmatrix} \times \begin{Bmatrix} (A_r + A_p)\omega_x^{(r)} \\ (A_r + A_p)\omega_y^{(r)} \\ (C_r + C_p)\omega_z^{(r)} \end{Bmatrix} = \begin{Bmatrix} \left[(C_p - A_p) + (C_r - A_r)\right]\omega_y^{(r)}\omega_z^{(r)} \\ \left[(A_p - C_p) + (A_r - C_r)\right]\omega_x^{(r)}\omega_z^{(r)} \\ 0 \end{Bmatrix} \tag{f}$$

$$\left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\dot{\boldsymbol{\omega}}_{\text{rel}}^{(p)}\right\} = \begin{bmatrix} \overline{A}_p & 0 & 0 \\ 0 & \overline{A}_p & 0 \\ 0 & 0 & C_p \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \dot{\omega}_p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ C_p\dot{\omega}_p \end{Bmatrix} \tag{g}$$

$$\left\{\boldsymbol{\omega}^{(r)}\right\} \times \left[\mathbf{I}_{G_p}^{(p)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(p)}\right\} = \begin{Bmatrix} \omega_x^{(r)} \\ \omega_y^{(r)} \\ \omega_z^{(r)} \end{Bmatrix} \times \begin{bmatrix} \overline{A}_p & 0 & 0 \\ 0 & \overline{A}_p & 0 \\ 0 & 0 & C_p \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \omega_p \end{Bmatrix} = \begin{Bmatrix} C_p\omega_y^{(r)}\omega_p \\ -C_p\omega_x^{(r)}\omega_p \\ 0 \end{Bmatrix} \tag{h}$$

With these four expressions, Eq. (b) becomes

$$\begin{Bmatrix} (A_r + A_p)\dot{\omega}_x^{(r)} \\ (A_r + A_p)\dot{\omega}_y^{(r)} \\ (C_r + C_p)\dot{\omega}_z^{(r)} \end{Bmatrix} + \begin{Bmatrix} \left[(C_p - A_p) + C_r + A_r\right]\omega_y^{(r)}\omega_z^{(r)} \\ \left[(A_p - C_p) + A_r + C_r\right]\omega_x^{(r)}\omega_z^{(r)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ C_p\dot{\omega}_p \end{Bmatrix} + \begin{Bmatrix} C_p\omega_y^{(r)}\omega_p \\ -C_p\omega_x^{(r)}\omega_p \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{i}$$

Summing the four vectors on the left-hand side and then extracting the three components of the vector equation, finally yields the three scalar equations of motion of the dual-spin satellite in the body frame,

$$\boxed{\begin{aligned} A\dot{\omega}_x^{(r)} + (C - A)\omega_y^{(r)}\omega_z^{(r)} + C_p\omega_y^{(r)}\omega_p &= 0 \\ A\dot{\omega}_y^{(r)} + (A - C)\omega_x^{(r)}\omega_z^{(r)} - C_p\omega_x^{(r)}\omega_p &= 0 \\ C\dot{\omega}_z^{(r)} + C_p\dot{\omega}_p &= 0 \end{aligned}} \tag{j}$$

where A and C are the combined transverse and axial moments of inertia of the dual-spin vehicle about its center of mass,

$$A = A_r + A \qquad C = C_r + C_p \tag{k}$$

The three equations (j) involve four unknowns, $\omega_x^{(r)}$, $\omega_y^{(r)}$, $\omega_z^{(r)}$, and ω_p. A fourth equation is required to account for the means of providing the relative velocity ω_p between the platform and the rotor. Friction in the axle bearing between the platform and the rotor would eventually cause ω_p to go to zero, as pointed out in Section 12.4. We may assume that the electric motor in the bearing acts to keep ω_p constant at a specified value, so that $\dot{\omega}_p = 0$. Then, Eq. $(\text{j})_3$ implies that $\omega_z^{(r)}$ is constant as well. Thus, ω_p and $\omega_z^{(r)}$ are removed from our list of unknowns, leaving $\omega_x^{(r)}$ and $\omega_y^{(r)}$ to be governed by the first two equations in Eq. (j). Note that we actually employed Eq. $(\text{j})_3$ in the solution of Example 12.10.

EXAMPLE 12.12

A spacecraft has three identical momentum wheels with their spin axes aligned with the vehicle's principal body axes. The spin axes of momentum wheels 1, 2, and 3 are aligned with the x, y, and z axes, respectively. The inertia tensors of the rotationally symmetric momentum wheels about their centers of mass are, therefore,

$$\left[\mathbf{I}_{G_1}^{(1)}\right] = \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix} \qquad \left[\mathbf{I}_{G_2}^{(2)}\right] = \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix} \qquad \left[\mathbf{I}_{G_3}^{(3)}\right] = \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix} \tag{a}$$

The spacecraft moment of inertia tensor about the vehicle (v) center of mass is

$$\left[\mathbf{I}_{G}^{(v)}\right] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{b}$$

Calculate the spin accelerations of the momentum wheels in the presence of external torque.

Solution

For $n = 3$, Eq. (12.140) becomes

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)} + \mathbf{I}_{G_1}^{(1)} + \mathbf{I}_{G_2}^{(2)} + \mathbf{I}_{G_3}^{(3)}\right]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_1}^{(1)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(1)}\right\} + \left[\mathbf{I}_{G_2}^{(2)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(2)}\right\} + \left[\mathbf{I}_{G_3}^{(3)}\right]\left\{\boldsymbol{\omega}_{\text{rel}}^{(3)}\right\} \tag{c}$$

The absolute angular velocity $\boldsymbol{\omega}$ of the spacecraft and the angular velocities $\boldsymbol{\omega}_{\text{rel}}^{(1)}$, $\boldsymbol{\omega}_{\text{rel}}^{(2)}$, $\boldsymbol{\omega}_{\text{rel}}^{(3)}$ of the three flywheels relative to the spacecraft are

$$\{\boldsymbol{\omega}\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \qquad \left\{\boldsymbol{\omega}_{\text{rel}}^{(1)}\right\} = \begin{Bmatrix} \omega^{(1)} \\ 0 \\ 0 \end{Bmatrix} \qquad \left\{\boldsymbol{\omega}_{\text{rel}}^{(2)}\right\} = \begin{Bmatrix} 0 \\ \omega^{(2)} \\ 0 \end{Bmatrix} \qquad \left\{\boldsymbol{\omega}_{\text{rel}}^{(3)}\right\} = \begin{Bmatrix} 0 \\ 0 \\ \omega^{(3)} \end{Bmatrix} \tag{d}$$

Substituting Eqs. (a), (b), and (d) into Eq. (c) yields

$$\begin{aligned} \{\mathbf{H}_G\} = {} & \left(\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix}\right)\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \\ & + \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}\begin{Bmatrix} \omega^{(1)} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix}\begin{Bmatrix} 0 \\ \omega^{(2)} \\ 0 \end{Bmatrix} + \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \omega^{(3)} \end{Bmatrix} \end{aligned}$$

or

$$\{\mathbf{H}_G\} = \begin{bmatrix} A+I+2J & 0 & 0 \\ 0 & B+I+2J & 0 \\ 0 & 0 & C+I+2J \end{bmatrix}\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{Bmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \omega^{(3)} \end{Bmatrix} \tag{e}$$

Substituting this expression for $\{\mathbf{H}_G\}$ into Eq. (12.143), we get

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \dot{\omega}^{(1)} \\ \dot{\omega}^{(2)} \\ \dot{\omega}^{(3)} \end{Bmatrix} + \begin{bmatrix} A+I+2J & 0 & 0 \\ 0 & B+I+2J & 0 \\ 0 & 0 & C+I+2J \end{bmatrix} \begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix}$$
$$+ \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \times \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \omega^{(3)} \end{Bmatrix} + \begin{bmatrix} A+I+2J & 0 & 0 \\ 0 & B+I+2J & 0 \\ 0 & 0 & C+I+2J \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \right) = \begin{Bmatrix} M_G)_x \\ M_G)_y \\ M_G)_z \end{Bmatrix} \tag{f}$$

Expanding and collecting terms yield the time rates of change of the flywheel spins (relative to the spacecraft) in terms of those of the spacecraft absolute angular velocity components,

$$\begin{aligned} \dot{\omega}^{(1)} &= \frac{M_G)_x}{I} + \frac{B-C}{I}\omega_y\omega_z - \left(1 + \frac{A}{I} + 2\frac{J}{I}\right)\dot{\omega}_x + \omega^{(2)}\omega_z - \omega^{(3)}\omega_y \\ \dot{\omega}^{(2)} &= \frac{M_G)_y}{I} + \frac{C-A}{I}\omega_x\omega_z - \left(1 + \frac{B}{I} + 2\frac{J}{I}\right)\dot{\omega}_y + \omega^{(3)}\omega_x - \omega^{(1)}\omega_z \\ \dot{\omega}^{(3)} &= \frac{M_G)_z}{I} + \frac{A-B}{I}\omega_x\omega_y - \left(1 + \frac{C}{I} + 2\frac{J}{I}\right)\dot{\omega}_z + \omega^{(1)}\omega_y - \omega^{(2)}\omega_x \end{aligned} \tag{g}$$

EXAMPLE 12.13

A communication satellite is in a circular earth orbit of period T. The body z axis lies on the outward radial from the earth's center to the spacecraft, so the angular velocity about the body y axis is $2\pi/T$. The angular velocities about the body x and z axes are zero. The attitude control system consists of three momentum wheels 1, 2, and 3 aligned with the principal x, y, and z axes of the satellite. A variable torque is applied to each wheel by its own electric motor. At time $t = 0$, the angular velocities of the three wheels relative to the spacecraft are all zero. A small, constant environmental torque $\mathbf{M}_0$ acts on the spacecraft. Determine the axial torques $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$ that the three motors must exert on their wheels so that the angular velocity $\boldsymbol{\omega}$ of the satellite will remain constant. The moment of inertia tensors of the reaction wheels about their centers of mass are given by Eq. (a) of Example 12.12 (Fig. 12.26).

Solution

The absolute angular velocity vector of the xyz frame is given by

$$\boldsymbol{\omega} = \omega_0\hat{\mathbf{j}} \tag{a}$$

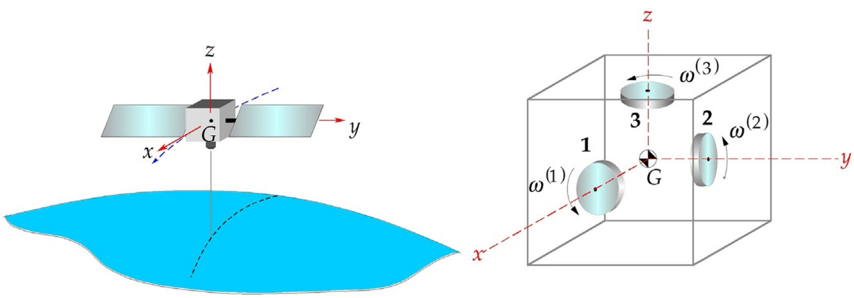


FIG. 12.26

Three-axis stabilized satellite.

where $\omega_0 = 2\pi/T$, a constant. At any instant, the absolute angular velocities of the three reaction wheels are, accordingly,

$$\begin{aligned}\boldsymbol{\omega}^{(1)} &= \omega^{(1)}\hat{\mathbf{i}} + \omega_0\hat{\mathbf{j}}\\ \boldsymbol{\omega}^{(2)} &= \omega^{(2)}\hat{\mathbf{j}} + \omega_0\hat{\mathbf{j}}\\ \boldsymbol{\omega}^{(3)} &= \omega^{(3)}\hat{\mathbf{k}} + \omega_0\hat{\mathbf{j}}\end{aligned} \tag{b}$$

From Eq. (a), it is clear that $\omega_x = \omega_z = \dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Therefore, Eqs. (g) of Example 12.12 become, for the case at hand,

$$\begin{aligned}\dot{\omega}^{(1)} &= \frac{M_G)_x}{I} + \frac{B-C}{I}\cdot\omega_0\cdot(0) - \left(1+\frac{A}{I}+2\frac{J}{I}\right)\cdot(0) + \omega^{(2)}\cdot(0) - \omega^{(3)}\cdot\omega_0\\ \dot{\omega}^{(2)} &= \frac{M_G)_y}{I} + \frac{A-C}{I}\cdot(0)\cdot(0) - \left(1+\frac{B}{I}+2\frac{J}{I}\right)\cdot(0) + \omega^{(3)}\cdot(0) - \omega^{(1)}\cdot(0)\\ \dot{\omega}^{(3)} &= \frac{M_G)_z}{I} + \frac{A-B}{I}\cdot(0)\cdot\omega_0 - \left(1+\frac{C}{I}+2\frac{J}{I}\right)\cdot(0) + \omega^{(1)}\cdot\omega_0 - \omega^{(2)}\cdot(0)\end{aligned}$$

which reduce to the following set of three first-order differential equations:

$$\begin{aligned}\dot{\omega}^{(1)} + \omega_0\omega^{(3)} &= \frac{M_G)_x}{I}\\ \dot{\omega}^{(2)} &= \frac{M_G)_y}{I}\\ \dot{\omega}^{(3)} - \omega_0\omega^{(1)} &= \frac{M_G)_z}{I}\end{aligned} \tag{c}$$

Eq. $(c)_2$ implies that $\omega^{(2)} = M_G)_y t/I + \text{constant}$, and since $\omega^{(2)} = 0$ at $t = 0$, this means that for time t thereafter,

$$\omega^{(2)} = \frac{M_G)_y}{I}t \tag{d}$$

Differentiating Eq. $(c)_3$ with respect to t and solving for $\dot{\omega}^{(1)}$ yields $\dot{\omega}^{(1)} = \ddot{\omega}^{(3)}/\omega_0$. Substituting this result into Eq. $(c)_1$ we get

$$\ddot{\omega}^{(3)} + {\omega_0}^2\omega^{(3)} = \frac{\omega_0 M_G)_x}{I}$$

The well-known solution of this familiar differential equation is

$$\omega^{(3)} = a\cos\omega_0 t + b\sin\omega_0 t + \frac{M_G)_x}{I\omega_0}$$

where a and b are constants of integration. According to the problem statement, $\omega^{(3)} = 0$ when $t = 0$. This initial condition requires $a = -M_G)_x/\omega_0 I$, so that

$$\omega^{(3)} = b\sin\omega_0 t + \frac{M_G)_x}{I\omega_0}(1-\cos\omega_0 t) \tag{e}$$

From this, we obtain $\dot{\omega}^{(3)} = b\omega_0\cos\omega_0 t + \left[M_G)_x/I\right]\sin\omega_0 t$, which, when substituted into Eq. $(c)_3$, yields

$$\omega^{(1)} = b\cos\omega_0 t + \frac{M_G)_x}{I\omega_0}\sin\omega_o t - \frac{M_G)_z}{I\omega_0} \tag{f}$$

Since $\omega^{(1)} = 0$ at $t = 0$, this implies $b = M_G)_z/I\omega_0$. In summary, therefore, the angular velocities of wheels 1, 2, and 3 relative to the satellite are

$$\begin{aligned}\omega^{(1)} &= \frac{M_G)_x}{I\omega_0}\sin\omega_o t + \frac{M_G)_z}{I\omega_0}(\cos\omega_0 t - 1)\\ \omega^{(2)} &= \frac{M_G)_y}{I}t\\ \omega^{(3)} &= \frac{M_G)_z}{I\omega_0}\sin\omega_o t + \frac{M_G)_x}{I\omega_0}(1-\cos\omega_0 t)\end{aligned} \tag{g}$$

The angular momenta of the reaction wheels are

$$\begin{aligned}\mathbf{H}_{G_1}^{(1)} &= I\omega_x^{(1)}\hat{\mathbf{i}} + J\omega_y^{(1)}\hat{\mathbf{j}} + J\omega_z^{(1)}\hat{\mathbf{k}} \\ \mathbf{H}_{G_2}^{(2)} &= J\omega_x^{(2)}\hat{\mathbf{i}} + I\omega_y^{(2)}\hat{\mathbf{j}} + J\omega_z^{(2)}\hat{\mathbf{k}} \\ \mathbf{H}_{G_3}^{(3)} &= J\omega_x^{(3)}\hat{\mathbf{i}} + J\omega_y^{(3)}\hat{\mathbf{j}} + I\omega_z^{(3)}\hat{\mathbf{k}}\end{aligned} \tag{h}$$

According to Eq. (b), the components of the flywheels' angular velocities are

$$\begin{array}{lll}\omega_x^{(1)} = \omega^{(1)} & \omega_y^{(1)} = \omega_0 & \omega_z^{(1)} = 0 \\ \omega_x^{(2)} = 0 & \omega_y^{(2)} = \omega^{(2)} + \omega_0 & \omega_z^{(2)} = 0 \\ \omega_x^{(3)} = 0 & \omega_y^{(3)} = \omega_0 & \omega_z^{(3)} = \omega^{(3)}\end{array}$$

so that Eq. (h) becomes

$$\begin{aligned}\mathbf{H}_{G_1}^{(1)} &= I\omega^{(1)}\hat{\mathbf{i}} + J\omega_0\hat{\mathbf{j}} \\ \mathbf{H}_{G_2}^{(2)} &= I\left(\omega^{(2)} + \omega_0\right)\hat{\mathbf{j}} \\ \mathbf{H}_{G_3}^{(3)} &= J\omega_0\hat{\mathbf{j}} + I\omega^{(3)}\hat{\mathbf{k}}\end{aligned} \tag{i}$$

Substituting Eq. (g) into these expressions yields the angular momenta of the wheels as a function of time,

$$\begin{aligned}\mathbf{H}_{G_1}^{(1)} &= \left[\frac{M_G)_x}{\omega_0}\sin\omega_0 t + \frac{M_G)_z}{\omega_0}(\cos\omega_0 t - 1)\right]\hat{\mathbf{i}} + J\omega_0\hat{\mathbf{j}} \\ \mathbf{H}_{G_2}^{(2)} &= \left[M_G)_y t + I\omega_0\right]\hat{\mathbf{j}} \\ \mathbf{H}_{G_3}^{(3)} &= J\omega_0\hat{\mathbf{j}} + \left[\frac{M_G)_z}{\omega_0}\sin\omega_0 t + \frac{M_G)_x}{\omega_0}(1 - \cos\omega_0 t)\right]\hat{\mathbf{k}}\end{aligned} \tag{j}$$

The torque on the reaction wheels is found by applying the Euler equation to each one. Thus, for wheel 1

$$\begin{aligned}\mathbf{M}_{G_1})_{\text{net}} &= \left.\frac{\mathrm{d}\mathbf{H}_{G_1}^{(1)}}{\mathrm{d}t}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_{G_1}^{(1)} \\ &= \left[M_G)_x\cos\omega_0 t - M_G)_x\sin\omega_0 t\right]\hat{\mathbf{i}} + \left[M_G)_x(1 - \cos\omega_0 t) - M_G)_x\sin\omega_0 t\right]\hat{\mathbf{k}}\end{aligned}$$

Since the axis of wheel 1 is in the x direction, the torque is the x component of this moment (the z component being a gyroscopic bending moment),

$$\boxed{C^{(1)} = M_G)_x\cos\omega_0 t - M_G)_z\sin\omega_0 t}$$

For wheel 2,

$$\mathbf{M}_{G_2})_{\text{net}} = \left.\frac{\mathrm{d}\mathbf{H}_{G_2}^{(2)}}{\mathrm{d}t}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_{G_2}^{(2)} = M_G)_y\hat{\mathbf{j}}$$

Thus,

$$\boxed{C^{(2)} = M_G)_y}$$

Finally, for wheel 3,

$$\begin{aligned}\mathbf{M}_{G_3})_{\text{net}} &= \left.\frac{\mathrm{d}\mathbf{H}_{G_3}^{(3)}}{\mathrm{d}t}\right)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_{G_3}^{(3)} \\ &= \left[M_G)_x(1 - \cos\omega_0 t) + M_G)_z\sin\omega_0 t\right]\hat{\mathbf{i}} + \left[M_G)_x\sin\omega_0 t + M_G)_z\cos\omega_0 t\right]\hat{\mathbf{k}}\end{aligned}$$

For this wheel, the torque direction is the z axis, so

$$\boxed{C^{(3)} = M_G)_x\sin\omega_0 t + M_G)_z\cos\omega_0 t}$$

The external torques on the spacecraft of the previous example may be due to thruster misalignment or they may arise from environmental effects such as gravity gradients, solar pressure, or interaction with the earth's magnetic field. The example assumed that these torques were constant, which is the simplest means of introducing their effects, but they actually vary with time. In any case, their magnitudes are extremely small, typically $<10^{-3}\,\mathrm{N \cdot m}$ for ordinary-sized, unmanned spacecraft. Eq. $(\mathrm{g})_2$ of the example reveals that a small torque normal to the satellite's orbital plane will cause the angular velocity of momentum wheel 2 to slowly but constantly increase. Over a long-enough period of time, the angular velocity of the gyro might approach its design limits, whereupon it is said to be saturated. At that point, attitude jets on the satellite would have to be fired to produce a torque around the y axis while the wheel is "caged" (i.e., its angular velocity reduced to zero or to its nonzero bias value). Finally, note that if all of the external torques were zero, none of the momentum wheels in the example would be required. The constant angular velocity $\boldsymbol{\omega} = (2\pi/T)\hat{\mathbf{j}}$ of the vehicle, once initiated, would continue unabated.

So far, we have dealt with momentum wheels, which are characterized by the fact that their axes are rigidly aligned with the principal axes of the spacecraft, as shown in Fig. 12.27. The speed of the electrically driven wheels is varied to produce the required rotation rates of the vehicle in response to external torques. Depending on the spacecraft, the nominal speed of a momentum wheel may be from zero to several thousand revolutions per minute.

Momentum wheels that are free to pivot on one or more gimbals are called control moment gyros. Fig. 12.28 illustrates a double-gimbaled control moment gyro. These gyros spin at several thousand revolutions per minute. The motor-driven speed of the flywheel is constant, and moments are exerted on the vehicle when torquers (electric motors) tilt the wheel about a gimbal axis. The torque direction is normal to the gimbal axis.

Set $n = 1$ in Eq. (12.140) and replace i with w (representing "wheel") to obtain

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_w}^{(w)}\right]\left(\{\boldsymbol{\omega}\} + \left\{\boldsymbol{\omega}_{\mathrm{rel}}^{(w)}\right\}\right) \tag{12.144}$$

The relative angular velocity of the rotor is

$$\boldsymbol{\omega}_{\mathrm{rel}}^{(w)} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_n + \boldsymbol{\omega}_s \tag{12.145}$$

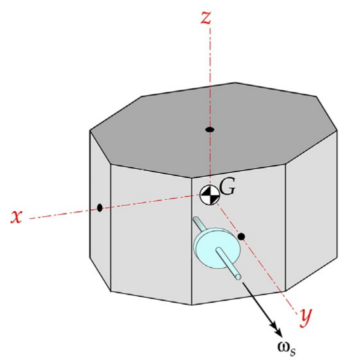


FIG. 12.27

Momentum wheel aligned with a principal body axis.

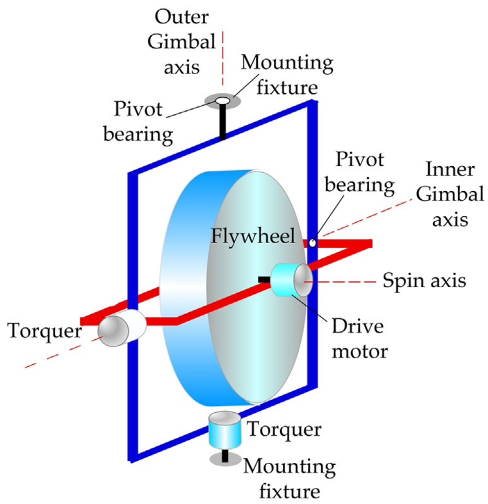


FIG. 12.28

Two-gimbal control moment gyro.

where $\boldsymbol{\omega}_p$, $\boldsymbol{\omega}_n$, and $\boldsymbol{\omega}_s$ are the precession, nutation, and spin angular velocities of the gyro relative to the vehicle. Substituting Eq. (12.145) into Eq. (12.144) yields

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_w}^{(w)}\right]\{\boldsymbol{\omega} + \boldsymbol{\omega}_p + \boldsymbol{\omega}_n + \boldsymbol{\omega}_s\} \tag{12.146}$$

The spin rate of the gyro is three or more orders of magnitude greater than any of the other rates. That is, under conditions in which a control moment gyro is designed to operate,

$$\|\boldsymbol{\omega}_s\| \gg \|\boldsymbol{\omega}\| \qquad \|\boldsymbol{\omega}_s\| \gg \|\boldsymbol{\omega}_p\| \qquad \|\boldsymbol{\omega}_s\| \gg \|\boldsymbol{\omega}_p\|$$

Therefore,

$$\{\mathbf{H}_G\} \approx \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_{G_w}^{(w)}\right]\{\boldsymbol{\omega}_s\} \tag{12.147}$$

Since the spin axis of a gyro is an axis of symmetry, about which the moment of inertia is $C^{(w)}$, this can be written as

$$\{\mathbf{H}_G\} = \left[\mathbf{I}_G^{(v)}\right]\{\boldsymbol{\omega}\} + C^{(w)}\omega_s\{\hat{\mathbf{n}}_s\} \tag{12.148}$$

where

$$\left[\mathbf{I}_G^{(v)}\right] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

and $\hat{\mathbf{n}}_s$ is the unit vector along the spin axis, as illustrated in Fig. 12.29. Relative to the body frame axes of the spacecraft, the components of $\hat{\mathbf{n}}_s$ appear as follows:

$$\hat{\mathbf{n}}_s = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{12.149}$$

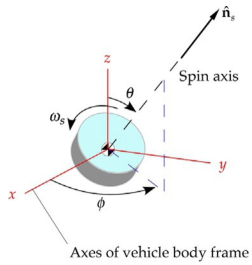


FIG. 12.29

Inclination angles of the spin vector of a gyro.

Thus, Eq. (12.148) becomes

$$\{\mathbf{H}_G\} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} + C^{(w)}\omega_s \begin{Bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{Bmatrix} = \begin{Bmatrix} A\omega_x + C^{(w)}\omega_s \sin\theta\cos\phi \\ B\omega_y + C^{(w)}\omega_s \sin\theta\sin\phi \\ C\omega_z + C^{(w)}\omega_s \cos\theta \end{Bmatrix} \tag{12.150}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{H}_G\} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} A\omega_x + C^{(w)}\omega_s \sin\theta\cos\phi \\ B\omega_y + C^{(w)}\omega_s \sin\theta\sin\phi \\ C\omega_z + C^{(w)}\omega_s \cos\theta \end{Bmatrix} + \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \times \begin{Bmatrix} A\omega_x + C^{(w)}\omega_s \sin\theta\cos\phi \\ B\omega_y + C^{(w)}\omega_s \sin\theta\sin\phi \\ C\omega_z + C^{(w)}\omega_s \cos\theta \end{Bmatrix} \tag{12.151}$$

Expanding the right-hand side, collecting terms, and setting the result equal to the net external moment, we find

$$\begin{aligned} &A\dot{\omega}_x + C^{(w)}\omega_s\dot{\theta}\cos\phi\cos\theta - C^{(w)}\omega_s\dot{\phi}\sin\phi\sin\theta + C^{(w)}\dot{\omega}_s\cos\phi\sin\theta \\ &\quad + \left(C^{(w)}\omega_s\cos\theta + C\omega_z\right)\omega_y - \left(C^{(w)}\omega_s\sin\phi\sin\theta + B\omega_y\right)\omega_z = M_G)_x \end{aligned} \tag{12.152a}$$

$$\begin{aligned} &B\dot{\omega}_y + C^{(w)}\omega_s\dot{\theta}\sin\phi\cos\theta + C^{(w)}\omega_s\dot{\phi}\cos\phi\sin\theta + C^{(w)}\dot{\omega}_s\sin\phi\sin\theta \\ &\quad - \left(C^{(w)}\omega_s\cos\theta + C\omega_z\right)\omega_x + \left(C^{(w)}\omega_s\cos\phi\sin\theta + A\omega_x\right)\omega_z = M_G)_y \end{aligned} \tag{12.152b}$$

$$\begin{aligned} &C\dot{\omega}_z + C^{(w)}\omega_s\dot{\theta}\sin\theta + C^{(w)}\dot{\omega}_s\cos\theta - \left(C^{(w)}\omega_s\cos\phi\sin\theta + A\omega_x\right)\omega_y \\ &\quad + \left(C^{(w)}\omega_s\sin\phi\sin\theta + B\omega_y\right)\omega_x = M_G)_z \end{aligned} \tag{12.152c}$$

Additional gyros are accounted for by adding the spin inertia, spin rate, and inclination angles for each one into Eqs. (12.152).

EXAMPLE 12.14

A satellite is in torque-free motion, $\mathbf{M}_G)_{\text{net}} = \mathbf{0}$. A nongimbaled gyro (momentum wheel) is aligned with the vehicle's x axis and is spinning at the rate $\omega_s)_0$. The spacecraft angular velocity is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}}$. If the spin of the gyro is increased at the rate $\dot{\omega}_s$, find the angular acceleration of the spacecraft.

Solution

Using Fig. 12.29 as a guide, we set $\phi = 0$ and $\theta = 90°$ to align the spin axis with the x axis. Since there is no gimbaling, $\dot{\theta} = \dot{\phi} = 0$. Eqs. (12.152) then yield

$$A\dot{\omega}_x + C^{(w)}\dot{\omega}_s = 0$$
$$B\dot{\omega}_y = 0$$
$$C\dot{\omega}_z = 0$$

Clearly, the angular velocities around the y and z axes remain zero, whereas

$$\boxed{\dot{\omega}_x = \frac{C^{(w)}}{A}\dot{\omega}_s}$$

Thus, a change in the vehicle's roll rate around the x axis can be initiated by accelerating the momentum wheel in the opposite direction (see Example 12.10).

EXAMPLE 12.15

A satellite is in torque-free motion. A control moment gyro, spinning at the constant rate ω_s, is gimbaled about the spacecraft y and z axes, with $\phi = 0$ and $\theta = 90°$ (cf. Fig. 12.29). The spacecraft angular velocity is $\boldsymbol{\omega} = \omega_z\hat{\mathbf{k}}$. If the spin axis of the gyro, initially along the x direction, is rotated around the y axis at the rate $\dot{\theta}$, what is the resulting angular acceleration of the spacecraft?

Solution

Substituting $\omega_x = \omega_y = \dot{\omega}_y = \dot{\phi} = 0$ and $\theta = 90°$ into Eqs. (12.152a)–(12.152c) gives

$$A\dot{\omega}_x = 0$$
$$B\dot{\omega}_y + C^{(w)}\omega_s\left(\omega_z + \dot{\phi}\right) = 0$$
$$C\dot{\omega}_z - H^{(w)}\dot{\theta} = 0$$

Thus, the components of vehicle angular acceleration are

$$\boxed{\dot{\omega}_x = 0 \qquad \dot{\omega}_y = -\frac{C^{(w)}}{B}\omega_s\left(\omega_z + \dot{\phi}\right) \qquad \dot{\omega}_z = \frac{C^{(w)}}{C}\omega_s\dot{\theta}}$$

We see that pitching the gyro at the rate $\dot{\theta}$ around the vehicle y axis alters only ω_z, leaving ω_x unchanged. However, to keep $\omega_y = 0$ clearly requires that $\dot{\phi} = -\omega_z$. In other words, for the control moment gyro to control the angular velocity about only one vehicle axis, it must therefore be able to precess around that axis (the z axis in this case). That is why the control moment gyro must have two gimbals.

12.10 GRAVITY GRADIENT STABILIZATION

Consider a satellite in circular orbit, as shown in Fig. 12.30. Let $\mathbf{r}$ be the position vector of a mass element dm relative to the center of attraction, $\mathbf{R}$ the position vector of the center of mass G, and $\boldsymbol{\rho}$ the position vector of dm relative to G. The force of gravity on dm is

$$d\mathbf{F}_g = -G\frac{M\,dm}{r^3}\mathbf{r} = -\mu\frac{\mathbf{r}}{r^3}dm \tag{12.153}$$

where M is the mass of the central body, and $\mu = GM$. The net moment of the gravitational force around G is

$$\mathbf{M}_G)_{\text{net}} = \int_m \boldsymbol{\rho} \times d\mathbf{F}_g\,dm \tag{12.154}$$

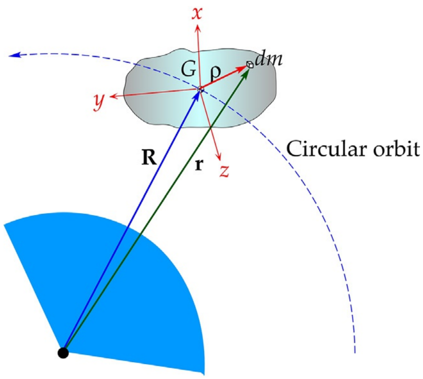


FIG. 12.30

Rigid satellite in a circular orbit. *xyz* is the principal body frame.

Since $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$ and

$$\begin{aligned}\mathbf{R} &= R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}} \\ \boldsymbol{\rho} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\end{aligned} \tag{12.155}$$

we have

$$\boldsymbol{\rho} \times \mathrm{d}\mathbf{F}_g = -\mu\frac{\mathrm{d}m}{r^3}\boldsymbol{\rho} \times (\mathbf{R} + \boldsymbol{\rho}) = -\mu\frac{\mathrm{d}m}{r^3}\boldsymbol{\rho} \times \mathbf{R} = -\mu\frac{\mathrm{d}m}{r^3}\begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ R_x & R_y & R_z\end{bmatrix}$$

Thus,

$$\boldsymbol{\rho} \times \mathrm{d}\mathbf{F}_g = -\mu\frac{\mathrm{d}m}{r^3}(R_z y - R_y z)\hat{\mathbf{i}} - \mu\frac{\mathrm{d}m}{r^3}(R_x z - R_z x)\hat{\mathbf{j}} - \mu\frac{\mathrm{d}m}{r^3}(R_y x - R_x y)\hat{\mathbf{k}}$$

Substituting this back into Eq. (12.154) yields

$$\begin{aligned}\mathbf{M}_G)_{\text{net}} = &\left(-\mu R_z\int_m \frac{y}{r^3}\mathrm{d}m + \mu R_y\int_m \frac{z}{r^3}\mathrm{d}m\right)\hat{\mathbf{i}} + \left(-\mu R_z\int_m \frac{z}{r^3}\mathrm{d}m + \mu R_z\int_m \frac{x}{r^3}\mathrm{d}m\right)\hat{\mathbf{j}} \\ &+ \left(-\mu R_y\int_m \frac{x}{r^3}\mathrm{d}m + \mu R_x\int_m \frac{y}{r^3}\mathrm{d}m\right)\hat{\mathbf{k}}\end{aligned}$$

or

$$\begin{aligned}M_G)_x &= -\mu R_z\int_m \frac{y}{r^3}\mathrm{d}m + \mu R_y\int_m \frac{z}{r^3}\mathrm{d}m \\ M_G)_y &= -\mu R_x\int_m \frac{z}{r^3}\mathrm{d}m + \mu R_z\int_m \frac{x}{r^3}\mathrm{d}m \\ M_G)_z &= -\mu R_y\int_m \frac{x}{r^3}\mathrm{d}m + \mu R_z\int_m \frac{y}{r^3}\mathrm{d}m\end{aligned} \tag{12.156}$$

Now, since $\|\boldsymbol{\rho}\| \ll \|\mathbf{R}\|$, it follows from Eq. (7.20) that

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\boldsymbol{\rho}$$

or

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\left(R_x x + R_y y + R_z z\right)$$

Therefore,

$$\int_m \frac{x}{r^3}\mathrm{d}m = \frac{1}{R^3}\int_m x\,\mathrm{d}m - \frac{3R_x}{R^5}\int_m x^2\,\mathrm{d}m - \frac{3R_y}{R^5}\int_m xy\,\mathrm{d}m - \frac{3R_z}{R^5}\int_m xz\,\mathrm{d}m$$

But the center of mass lies at the origin of the *xyz* axes, which are principal moments of inertia directions. That means

$$\int_m x\,\mathrm{d}m = \int_m xy\,\mathrm{d}m = \int_m xz\,\mathrm{d}m = 0$$

so that

$$\int_m \frac{x}{r^3}\mathrm{d}m = -\frac{3R_x}{R^5}\int_m x^2\,\mathrm{d}m \tag{12.157}$$

In a similar fashion, we can show that

$$\int_m \frac{y}{r^3}\mathrm{d}m = -\frac{3R_y}{R^5}\int_m y^2\,\mathrm{d}m \tag{12.158}$$

and

$$\int_m \frac{z}{r^3}\mathrm{d}m = -\frac{3R_z}{R^5}\int_m z^2\,\mathrm{d}m \tag{12.159}$$

Substituting these last three expressions into Eq. (12.156) leads to

$$\begin{aligned} M_G)_x &= \frac{3\mu R_y R_z}{R^5}\left(\int_m y^2\,\mathrm{d}m - \int_m z^2\,\mathrm{d}m\right) \\ M_G)_y &= \frac{3\mu R_x R_z}{R^5}\left(\int_m z^2\,\mathrm{d}m - \int_m x^2\,\mathrm{d}m\right) \\ M_G)_z &= \frac{3\mu R_x R_y}{R^5}\left(\int_m x^2\,\mathrm{d}m - \int_m y^2\,\mathrm{d}m\right) \end{aligned} \tag{12.160}$$

From Section 11.5, we recall that the moments of inertia are defined as

$$A = \int_m y^2\mathrm{d}m + \int_m z^2\mathrm{d}m \quad B = \int_m x^2\mathrm{d}m + \int_m z^2\mathrm{d}m \quad C = \int_m x^2\mathrm{d}m + \int_m y^2\mathrm{d}m \tag{12.161}$$

from which we may write

$$B - A = \int_m x^2 \mathrm{d}m - \int_m y^2 \mathrm{d}m \quad A - C = \int_m z^2 \mathrm{d}m - \int_m x^2 \mathrm{d}m \quad C - B = \int_m y^2 \mathrm{d}m - \int_m z^2 \mathrm{d}m$$

It follows that Eq. (12.160) reduce to

$$\begin{aligned} M_G)_x &= \frac{3\mu R_y R_z}{R^5}(C - B) \\ M_G)_y &= \frac{3\mu R_x R_z}{R^5}(A - C) \\ M_G)_z &= \frac{3\mu R_x R_y}{R^5}(B - A) \end{aligned} \tag{12.162}$$

These are the components, in the spacecraft body frame, of the gravitational torque produced by the variation of the earth's gravitational field over the volume of the spacecraft. To get an idea of these torque magnitudes, note first of all that R_x/R, R_y/R, and R_z/R are the direction cosines of the position vector of the center of mass, so that their magnitudes do not exceed 1. For a satellite in a low earth orbit of radius 6700 km, $3\mu/R^3 \approx 4(10^{-6})\,\mathrm{s}^{-2}$, which is therefore the maximum order of magnitude of the coefficients of the inertia terms in Eq. (12.162). The moments of inertia of the space shuttle were on the order of $10^6\ \mathrm{kg \cdot m^2}$, so the gravitational torques on that large vehicle were on the order of $1\ \mathrm{N \cdot m}$.

Substituting Eq. (12.162) into Euler's equations of motion (Eq. 11.72b), we get

$$\begin{aligned} A\dot{\omega}_x + (C - B)\omega_y\omega_z &= \frac{3\mu R_y R_z}{R^5}(C - B) \\ B\dot{\omega}_y + (A - C)\omega_z\omega_x &= \frac{3\mu R_x R_z}{R^5}(A - C) \\ C\dot{\omega}_z + (B - A)\omega_x\omega_y &= \frac{3\mu R_x R_y}{R^5}(B - A) \end{aligned} \tag{12.163}$$

Now consider the local vertical/local horizontal orbital reference frame shown in Fig. 12.31. It is actually the Clohessy-Wiltshire frame of Chapter 7, with the axes relabeled. The z' axis points radially

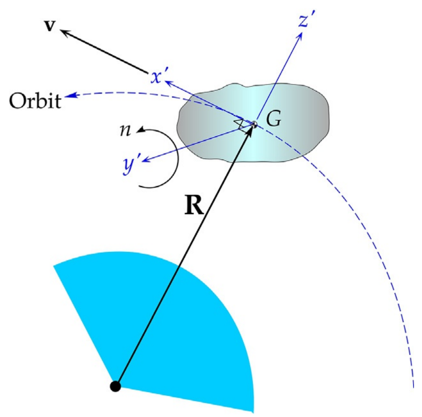


FIG. 12.31

Orbital reference frame $x'y'z'$ attached to the center of mass of the satellite.

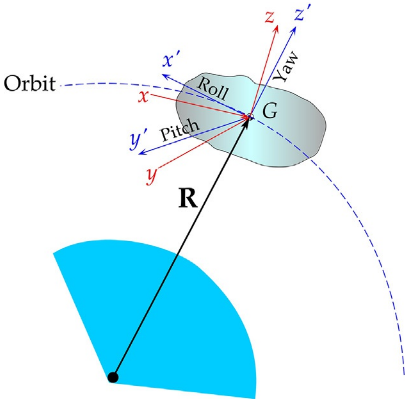


FIG. 12.32

Satellite body frame slightly misaligned with the orbital frame $x'y'z'$.

outward from the center of the earth, the x' axis is in the direction of the local horizon, and the y' axis completes the right-handed triad by pointing in the direction of the orbit normal. This frame rotates around the y' axis with an angular velocity equal to the mean motion n of the circular orbit. Suppose we align the satellite's principal body frame axes xyz with $x'y'z'$, respectively. When the body x axis is aligned with the x' direction, it is called the roll axis. The body y axis, when aligned with the y' direction, is the pitch axis. The body z axis, pointing outward from the earth in the z' direction, is the yaw axis. These directions are illustrated in Fig. 12.32. With the spacecraft aligned in this way, the body frame components of the inertial angular velocity vector $\boldsymbol{\omega}$ are $\omega_x = \omega_z = 0$ and $\omega_y = n$. The components of the position vector $\mathbf{R}$ are $R_x = R_y = 0$ and $R_z = R$. Substituting these data into Eq. (12.163) yields

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

That is, the spacecraft will orbit the planet with its principal axes remaining aligned with the orbital frame. If this motion is stable under the influence of gravity alone, without the use of thrusters, gyros, or other devices, then the spacecraft is gravity-gradient-stabilized. We need to assess the stability of this motion so that we can determine how to orient a spacecraft to take advantage of this type of passive attitude stabilization.

Let the body frame xyz be slightly misaligned with the orbital reference frame, so that the yaw, pitch, and roll angles between the xyz axes and the $x'y'z'$ axes, respectively, are very small, as suggested in Fig. 12.32. The absolute angular velocity $\boldsymbol{\omega}$ of the spacecraft is the angular velocity $\boldsymbol{\omega}_{\text{rel}}$ relative to the orbital reference frame plus the inertial angular velocity $\boldsymbol{\Omega}$ of the $x'y'z'$ frame,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{rel}} + \boldsymbol{\Omega}$$

The components of $\boldsymbol{\omega}_{\text{rel}}$ in the body frame are found using the yaw, pitch, and roll relations (Eq. 11.129). In so doing, it must be kept in mind that all angles and rates are assumed to be so small that their squares and products may be neglected. Recalling that $\sin\alpha = \alpha$ and $\cos\alpha = 1$, when $\alpha \ll 1$, we therefore obtain

$$\omega_{\mathrm{rel}})_x = \omega_{\mathrm{roll}} - \omega_{\mathrm{yaw}} \overbrace{\sin\theta_{\mathrm{pitch}}}^{=\theta_{\mathrm{pitch}}} = \dot{\psi}_{\mathrm{roll}} - \overbrace{\dot{\phi}_{\mathrm{yaw}}\theta_{\mathrm{pitch}}}^{\text{neglect product}} = \dot{\psi}_{\mathrm{roll}} \tag{12.164}$$

$$\omega_{\mathrm{rel}})_y = \omega_{\mathrm{yaw}} \overbrace{\cos\theta_{\mathrm{pitch}}}^{=1} \overbrace{\sin\psi_{\mathrm{roll}}}^{=\psi_{\mathrm{roll}}} + \omega_{\mathrm{pitch}} \overbrace{\cos\psi_{\mathrm{roll}}}^{=1} = \overbrace{\dot{\phi}_{\mathrm{yaw}}\psi_{\mathrm{roll}}}^{\text{neglect product}} + \dot{\theta}_{\mathrm{pitch}} = \dot{\theta}_{\mathrm{pitch}} \tag{12.165}$$

$$\omega_{\mathrm{rel}})_z = \omega_{\mathrm{yaw}} \overbrace{\cos\theta_{\mathrm{pitch}}}^{=1} \overbrace{\cos\psi_{\mathrm{roll}}}^{=1} - \omega_{\mathrm{pitch}} \overbrace{\sin\psi_{\mathrm{roll}}}^{=\psi_{\mathrm{roll}}} = \dot{\phi}_{\mathrm{yaw}} - \overbrace{\dot{\theta}_{\mathrm{pitch}}\psi_{\mathrm{roll}}}^{\text{neglect product}} = \dot{\phi}_{\mathrm{yaw}} \tag{12.166}$$

The orbital frame's angular velocity is the mean motion n of the circular orbit, so that

$$\mathbf{\Omega} = n\hat{\mathbf{j}}'$$

To obtain the orbital frame's angular velocity components along the body frame, we must use the transformation rule

$$\{\mathbf{\Omega}\}_x = [\mathbf{Q}]_{x'x}\{\mathbf{\Omega}\}_{x'} \tag{12.167}$$

where $[\mathbf{Q}]_{x'x}$ is given by Eq. (11.119). (Keep in mind that $x'y'z'$ are playing the role of XYZ in Fig. 11.27.) Using the above small-angle approximations in Eq. (11.119) leads to

$$[\mathbf{Q}]_{x'x} = \begin{bmatrix} 1 & \phi_{\mathrm{yaw}} & -\theta_{\mathrm{pitch}} \\ -\phi_{\mathrm{yaw}} & 1 & \psi_{\mathrm{roll}} \\ \theta_{\mathrm{pitch}} & -\psi_{\mathrm{roll}} & 1 \end{bmatrix}$$

With this, Eq. (12.167) becomes

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{bmatrix} 1 & \phi_{\mathrm{yaw}} & -\theta_{\mathrm{pitch}} \\ -\phi_{\mathrm{yaw}} & 1 & \psi_{\mathrm{roll}} \\ \theta_{\mathrm{pitch}} & -\psi_{\mathrm{roll}} & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ n \\ 0 \end{Bmatrix} = \begin{Bmatrix} n\phi_{\mathrm{yaw}} \\ n \\ -n\psi_{\mathrm{roll}} \end{Bmatrix}$$

Now we can calculate the components of the satellite's inertial angular velocity along the body frame axes,

$$\begin{aligned} \dot{\omega}_x &= \ddot{\psi}_{\mathrm{roll}} + n\dot{\phi}_{\mathrm{yaw}} \\ \omega_y &= \omega_{\mathrm{rel}})_y + \Omega_y = \dot{\theta}_{\mathrm{pitch}} + n \\ \omega_z &= \omega_{\mathrm{rel}})_z + \Omega_z = \dot{\phi}_{\mathrm{yaw}} - n\psi_{\mathrm{roll}} \end{aligned} \tag{12.168}$$

Differentiating these with respect to time, remembering that the mean motion n is constant for a circular orbit, gives the components of inertial angular acceleration in the body frame,

$$\begin{aligned} \dot{\omega}_x &= \ddot{\psi}_{\mathrm{roll}} + n\dot{\phi}_{\mathrm{yaw}} \\ \dot{\omega}_y &= \ddot{\theta}_{\mathrm{pitch}} \\ \dot{\omega}_z &= \ddot{\phi}_{\mathrm{yaw}} - n\dot{\psi}_{\mathrm{roll}} \end{aligned} \tag{12.169}$$

The position vector of the satellite's center of mass lies along the z' axis of the orbital frame,

$$\mathbf{R} = R\hat{\mathbf{k}}'$$

To obtain the components of $\mathbf{R}$ in the body frame, we once again use the transformation matrix $[\mathbf{Q}]_{x'x}$,

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} = \begin{bmatrix} 1 & \phi_{\text{yaw}} & -\theta_{\text{pitch}} \\ -\phi_{\text{yaw}} & 1 & \psi_{\text{roll}} \\ \theta_{\text{pitch}} & -\psi_{\text{roll}} & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix} = \begin{Bmatrix} -R\theta_{\text{pitch}} \\ R\psi_{\text{roll}} \\ R \end{Bmatrix} \tag{12.170}$$

Substituting Eqs. (12.168)–(12.170), together with $n = \sqrt{\mu/R^3}$ into Eq. (12.163) and setting

$$A = I_{\text{roll}} \qquad B = I_{\text{pitch}} \qquad C = I_{\text{yaw}} \tag{12.171}$$

yields

$$\begin{aligned}
&I_{\text{roll}}\left(\ddot{\psi}_{\text{roll}} + n\dot{\phi}_{\text{yaw}}\right) + \left(I_{\text{yaw}} - I_{\text{pitch}}\right)\left(\dot{\theta}_{\text{pitch}} + n\right)\left(\dot{\phi}_{\text{yaw}} - n\psi_{\text{roll}}\right) = 3\left(I_{\text{yaw}} - I_{\text{pitch}}\right)n^2\psi_{\text{roll}} \\
&I_{\text{pitch}}\ddot{\theta}_{\text{pitch}} + \left(I_{\text{roll}} - I_{\text{yaw}}\right)\left(\dot{\psi}_{\text{roll}} + n\phi_{\text{yaw}}\right)\left(\dot{\phi}_{\text{yaw}} - n\psi_{\text{roll}}\right) = -3\left(I_{\text{roll}} - I_{\text{yaw}}\right)n^2\theta_{\text{pitch}} \\
&I_{\text{yaw}}\left(\ddot{\phi}_{\text{yaw}} - n\dot{\psi}_{\text{roll}}\right) + \left(I_{\text{pitch}} - I_{\text{roll}}\right)\left(\dot{\theta}_{\text{pitch}} + n\right)\left(\dot{\psi}_{\text{roll}} + n\phi_{\text{yaw}}\right) = -3\left(I_{\text{pitch}} - I_{\text{roll}}\right)n^2\theta_{\text{pitch}}\psi_{\text{roll}}
\end{aligned}$$

Expanding terms and retaining terms at most linear in all angular quantities and their rates yields

$$I_{\text{yaw}}\ddot{\phi}_{\text{yaw}} + \left(I_{\text{pitch}} - I_{\text{roll}}\right)n^2\phi_{\text{yaw}} + \left(I_{\text{pitch}} - I_{\text{roll}} - I_{\text{yaw}}\right)n\dot{\psi}_{\text{roll}} = 0 \tag{12.172}$$

$$I_{\text{roll}}\ddot{\psi}_{\text{roll}} + \left(I_{\text{roll}} - I_{\text{pitch}} + I_{\text{yaw}}\right)n\dot{\phi}_{\text{yaw}} + 4\left(I_{\text{pitch}} - I_{\text{yaw}}\right)n^2\psi_{\text{roll}} = 0 \tag{12.173}$$

$$I_{\text{pitch}}\ddot{\theta}_{\text{pitch}} + 3\left(I_{\text{roll}} - I_{\text{yaw}}\right)n^2\theta_{\text{pitch}} = 0 \tag{12.174}$$

These are the differential equations governing the influence of gravity gradient torques on the small angles and rates of misalignment of the body frame with the orbital frame.

Eq. (12.174), governing the pitching motion around the y' axis, is not coupled to the other two equations. We make the classical assumption that the solution is of the form

$$\theta_{\text{pitch}} = Pe^{pt} \tag{12.175}$$

where P and p are constants, and P is the amplitude of the small disturbance that initiates the pitching motion. Substituting Eq. (12.175) into Eq. (12.174) yields $[I_{\text{pitch}}p^2 + 3(I_{\text{roll}} - I_{\text{yaw}})n^2]Pe^{pt} = 0$ for all values of t, which implies that the bracketed term must vanish, and that means p must have either of the two values

$$p_{1,2} = \pm i\sqrt{3\frac{\left(I_{\text{roll}} - I_{\text{yaw}}\right)n^2}{I_{\text{pitch}}}} \qquad \left(i = \sqrt{-1}\right)$$

Thus,

$$\theta_{\text{pitch}} = P_1e^{p_1t} + P_2e^{p_2t}$$

yields the stable, small-amplitude, steady-state harmonic oscillator solution only if p_1 and p_2 are imaginary. That is, if

$$I_{\text{roll}} > I_{\text{yaw}} \qquad \text{For stability in pitch} \tag{12.176}$$

The stable pitch oscillation frequency is

$$\left.\omega_f\right)_{\text{pitch}} = n\sqrt{3\frac{I_{\text{roll}} - I_{\text{yaw}}}{I_{\text{pitch}}}} \tag{12.177}$$

(If $I_{\text{yaw}} > I_{\text{roll}}$, then p_1 and p_2 are both real, one positive, the other negative. The positive root causes $\theta_{\text{pitch}} \to \infty$, which is the undesirable, unstable case.)

Let us now turn our attention to Eqs. (12.172) and (12.173), which govern yaw and roll motions under gravity gradient torque. Again, we assume the solution is exponential in form,

$$\phi_{\text{yaw}} = Ye^{qt} \qquad \psi_{\text{roll}} = \text{R}e^{qt} \tag{12.178}$$

Substituting these into Eqs. (12.172) and (12.173) yields

$$\begin{aligned}
\left[\left(I_{\text{pitch}} - I_{\text{roll}}\right)n^2 + I_{\text{yaw}}q^2\right]Y + \quad \left[\left(I_{\text{pitch}} - I_{\text{roll}} - I_{\text{yaw}}\right)nq\right]R &= 0 \\
\left[\left(I_{\text{roll}} - I_{\text{pitch}} + I_{\text{yaw}}\right)nq\right]Y + \left[4\left(I_{\text{pitch}} - I_{\text{yaw}}\right)n^2 + I_{\text{roll}}q^2\right]R &= 0
\end{aligned}$$

In the interest of simplification, we can factor I_{yaw} out of the first equation and I_{roll} out of the second one to get

$$\begin{aligned}
\left(\frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}}n^2 + q^2\right)Y + \quad \left(\frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} - 1\right)nqR &= 0 \\
\left(1 - \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}}\right)nqY + \left(4\frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}}n^2 + q^2\right)R &= 0
\end{aligned} \tag{12.179}$$

Let

$$k_Y = \frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} \qquad k_R = \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}} \tag{12.180}$$

It is easy to show from Eqs. (12.161), (12.171), and (12.180) that

$$k_Y = \frac{\left(\int_m x^2 \mathrm{d}m \Big/ \int_m y^2 \mathrm{d}m\right) - 1}{\left(\int_m x^2 \mathrm{d}m \Big/ \int_m y^2 \mathrm{d}m\right) + 1} \qquad k_R = \frac{\left(\int_m z^2 \mathrm{d}m \Big/ \int_m y^2 \mathrm{d}m\right) - 1}{\left(\int_m z^2 \mathrm{d}m \Big/ \int_m y^2 \mathrm{d}m\right) + 1}$$

which means

$$|k_Y| < 1 \qquad |k_R| < 1$$

Using the definitions in Eqs. (12.180), we can write Eq. (12.179) more compactly as

$$\begin{aligned}
(k_Y n^2 + q^2)Y + \quad (k_Y - 1)nqR &= 0 \\
(1 - k_R)nqY + (4k_R n^2 + q^2)R &= 0
\end{aligned}$$

or, using matrix notation,

$$\begin{bmatrix} k_Y n^2 + q^2 & (k_Y - 1)nq \\ (1 - k_R)nq & 4k_R n^2 + q^2 \end{bmatrix} \begin{Bmatrix} Y \\ R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{12.181}$$

To avoid the trivial solution ($Y = R = 0$), the determinant of the coefficient matrix must be zero. Expanding the determinant and collecting terms yields the characteristic equation for q,

$$q^4 + bn^2q^2 + cn^4 = 0 \tag{12.182}$$

where

$$b = 3k_R + k_Y k_R + 1 \qquad c = 4k_Y k_R \tag{12.183}$$

This quartic equation has four roots which, when substituted back into Eq. (12.178), yields

$$\phi_{\text{yaw}} = Y_1 e^{q_1 t} + Y_2 e^{q_2 t} + Y_3 e^{q_3 t} + Y_4 e^{q_4 t}$$
$$\psi_{\text{roll}} = R_1 e^{q_1 t} + R_2 e^{q_2 t} + R_3 e^{q_3 t} + R_4 e^{q_4 t}$$

For these solutions to remain finite in time, the roots $q_1, \ldots, q_4$ must all be negative (solution decays to zero) or imaginary (steady oscillation at the initial small amplitude).

To reduce Eq. (12.182) to a quadratic equation, let us introduce a new variable λ and write

$$q = \pm n\sqrt{\lambda} \tag{12.184}$$

Then, Eq. (12.182) becomes

$$\lambda^2 + b\lambda + c = 0 \tag{12.185}$$

the familiar solution of which is

$$\lambda_1 = -\frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right) \quad \lambda_2 = -\frac{1}{2}\left(b - \sqrt{b^2 - 4c}\right) \tag{12.186}$$

To guarantee that q in Eq. (12.184) does not take a positive value, we must require that λ be real and negative (so q will be imaginary). For λ to be real requires that $b > 2\sqrt{c}$, or

$$3k_R + k_Y k_R + 1 > 4\sqrt{k_Y k_R} \tag{12.187}$$

For λ to be negative requires that $b^2 > b^2 - 4c$, which will be true if $c > 0$. That is,

$$k_Y k_R > 0 \tag{12.188}$$

Eqs. (12.187) and (12.188) are the conditions required for yaw and roll stability under gravity gradient torques, to which we must add Eq. (12.176) for pitch stability. Observe that we can solve Eq. (12.180) to obtain

$$I_{\text{yaw}} = \frac{1 - k_R}{1 - k_Y k_R} I_{\text{pitch}} \quad I_{\text{roll}} = \frac{1 - k_Y}{1 - k_Y k_R} I_{\text{pitch}} \tag{12.189}$$

By means of these relationships, the pitch stability criterion, $I_{\text{roll}}/I_{\text{yaw}} > 1$, becomes

$$\frac{1 - k_Y}{1 - k_R} > 1$$

In view of the fact that $|k_R| < 1$, this means

$$k_Y < k_R \tag{12.190}$$

Fig. 12.33 shows those regions I and II on the $k_Y - k_R$ plane in which all three stability criteria (Eqs. (12.187), (12.188), and (12.190)) are simultaneously satisfied, along with the requirement that the three moments of inertia I_{pitch}, I_{roll}, and I_{yaw} are positive.

In the small sliver of region I, $k_Y < 0$ and $k_R < 0$; therefore, according to Eq. (12.189), $I_{\text{yaw}} > I_{\text{pitch}}$ and $I_{\text{roll}} > I_{\text{pitch}}$, which together with Eq. (12.176), yield $I_{\text{roll}} > I_{\text{yaw}} > I_{\text{pitch}}$. Remember that the gravity gradient spacecraft is slowly "spinning" about the minor pitch axis (normal to the orbit plane) at an angular velocity equal to the mean motion of the orbit. So this criterion makes the spacecraft a "minor-axis spinner," the roll axis (flight direction) being the major axis of inertia. With energy dissipation, we know that this orientation is not stable in the long run. On the other hand, in region II, k_Y and k_R are

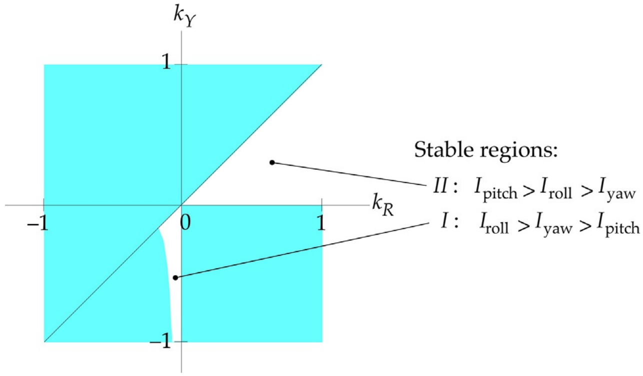


FIG. 12.33

Regions in which the values of k_Y and k_R yield neutral stability in yaw, pitch, and roll of a gravity gradient satellite.

both positive, so that Eq. (12.189) implies $I_{pitch} > I_{yaw}$ and $I_{pitch} > I_{roll}$. Thus, along with the pitch criterion ($I_{roll} > I_{yaw}$), we have $I_{pitch} > I_{roll} > I_{yaw}$. In this, the preferred, configuration, the gravity gradient spacecraft is a "major-axis spinner" about the pitch axis, and the minor yaw axis is the minor axis of inertia. It turns out that all the known gravity-gradient-stabilized moons of the solar system, like the earth's, whose "captured" rate of rotation equals the orbital period, are major-axis spinners.

In Eq. (12.177), we presented the frequency of the gravity gradient pitch oscillation. For completeness, we should also point out that the coupled yaw and roll motions have two oscillation frequencies, which are obtained from Eqs. (12.184) and (12.186),

$$\omega_{f_{yaw-roll}}\Big)_{1,2} = n\sqrt{\frac{1}{2}\left(b \pm \sqrt{b^2 - 4c}\right)} \tag{12.190}$$

Recall that b and c are found in Eq. (12.183).

We have assumed throughout this discussion that the orbit of the gravity gradient satellite is circular. Kaplan (1976) shows that the effect of a small eccentricity turns up only in the pitching motion. In particular, the natural oscillation expressed by Eq. (12.175) is augmented by a forced oscillation term,

$$\theta_{pitch} = P_1 e^{p_1 t} + P_2 e^{p_2 t} + \frac{2e \sin nt}{3\left(I_{roll} - I_{yaw}\right)/I_{pitch} - 1} \tag{12.191}$$

where e is the (small) eccentricity of the orbit. From this, we see that there is a pitch resonance. When $(I_{roll} - I_{yaw})/I_{pitch}$ approaches 1/3, the amplitude of the last term grows without bound.

EXAMPLE 12.16

A uniform, monolithic 10,000-kg slab, having the dimensions shown in Fig. 12.34, is in a circular low earth orbit. Determine the orientation of the satellite in its orbit for gravity gradient stabilization, and compute the periods of the pitch and yaw/roll oscillations in terms of the orbital period T.

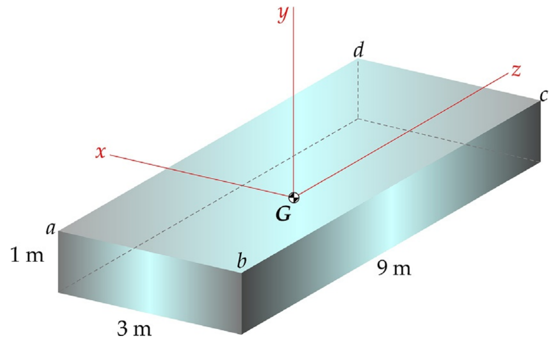


FIG. 12.34

Parallelepiped satellite.

According to Fig. 11.9C, the principal moments of inertia around the *xyz* axes through the center of mass are

$$A = \frac{10{,}000}{12}\left(1^2 + 9^2\right) = 68{,}333\ \text{kg}\cdot\text{m}^2$$
$$B = \frac{10{,}000}{12}\left(3^2 + 9^2\right) = 75{,}000\ \text{kg}\cdot\text{m}^2$$
$$C = \frac{10{,}000}{12}\left(3^2 + 1^2\right) = 8333.3\ \text{kg}\cdot\text{m}^2$$

Solution

Let us first determine whether we can stabilize this object as a minor-axis spinner. In that case,

$$I_{\text{pitch}} = C = 8333.3\ \text{kg}\cdot\text{m}^2 \quad I_{\text{yaw}} = A = 68{,}333\ \text{kg}\cdot\text{m}^2 \quad I_{\text{roll}} = B = 75{,}000\ \text{kg}\cdot\text{m}^2$$

Since $I_{\text{roll}} > I_{\text{yaw}}$, the satellite would be stable in pitch. To check yaw/roll stability, we first compute

$$k_Y = \frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} = -0.97561 \quad k_R = \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}} = -0.8000$$

We see that $k_Y k_R > 0$, which is one of the two requirements. The other one is found in Eq. (12.187), but in this case

$$1 + 3k_R + k_Y k_R - 4\sqrt{k_Y k_R} = -4.1533 < 0$$

so that the condition is not met. Hence, the object cannot be gravity-gradient-stabilized as a minor-axis spinner. As a major-axis spinner, we must have

$$I_{\text{pitch}} = B = 75{,}000\ \text{kg}\cdot\text{m}^2 \quad I_{\text{yaw}} = C = 8333.3\ \text{kg}\cdot\text{m}^2 \quad I_{\text{roll}} = A = 68{,}333\ \text{kg}\cdot\text{m}^2$$

Then $I_{\text{roll}} > I_{\text{yaw}}$, so the pitch stability condition is satisfied. Furthermore, since

$$k_Y = \frac{I_{\text{pitch}} - I_{\text{roll}}}{I_{\text{yaw}}} = 0.8000 \quad k_R = \frac{I_{\text{pitch}} - I_{\text{yaw}}}{I_{\text{roll}}} = 0.97561$$

we have

$$k_Y k_R = 0.7805 > 0$$
$$1 + 3k_R + k_Y k_R - 4\sqrt{k_Y k_R} = 1.1735 > 0$$

which means the two criteria for stability in the yaw and roll modes are met. The satellite should therefore be orbited as shown in Fig. 12.32, with its minor axis aligned with the radial from the earth's center, the plane *abcd* lying in the orbital plane, and the body *x* axis aligned with the local horizon (Fig. 12.35).

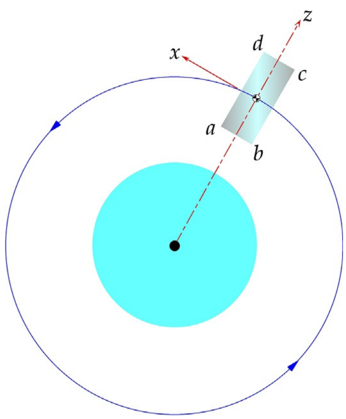


FIG. 12.35

Orientation of the parallelepiped for gravity gradient stabilization.

According to Eq. (12.177), the frequency of the pitch oscillation is

$$\omega_{f_{\text{pitch}}} = n\sqrt{3\frac{I_{\text{roll}} - I_{\text{yaw}}}{I_{\text{pitch}}}}$$
$$= n\sqrt{3\frac{68{,}333 - 8333.3}{75{,}000}} = 1.5492n$$

where n is the mean motion. Hence, the period of this oscillation, in terms of that of the orbit, is

$$T_{\text{pitch}} = \frac{2\pi}{\omega_{f_{\text{pitch}}}} = 0.6455\frac{2\pi}{n} = \boxed{0.6455T}$$

For the yaw/roll frequencies, we use Eq. (12.190),

$$\left.\omega_{f_{\text{yaw/roll}}}\right)_1 = n\sqrt{\frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right)}$$

where

$$b = 1 + 3k_R + k_Y k_R = 4.7073 \quad \text{and} \quad c = 4k_Y k_R = 3.122$$

Thus,

$$\left.\omega_{f_{\text{yaw/roll}}}\right)_1 = 2.3015n$$

Likewise,

$$\left.\omega_{f_{\text{yaw/roll}}}\right)_2 = \sqrt{\frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right)} = 1.977n$$

From these, we obtain

$$\left.T_{\text{yaw/roll}}\right)_1 = \boxed{0.5058T} \qquad \left.T_{\text{yaw/roll}}\right)_2 = \boxed{0.4345T}$$

Finally, observe that

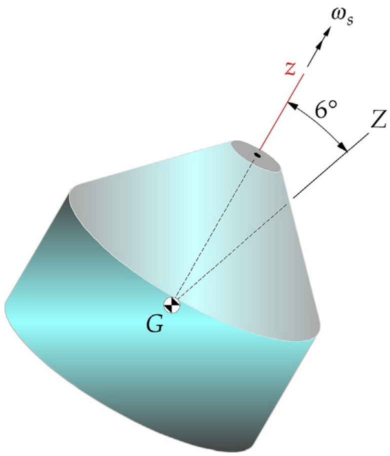
$$\frac{I_{\text{roll}} - I_{\text{yaw}}}{I_{\text{pitch}}} = 0.8$$

so that we are far from the pitch resonance condition that exists if the orbit has a small eccentricity.

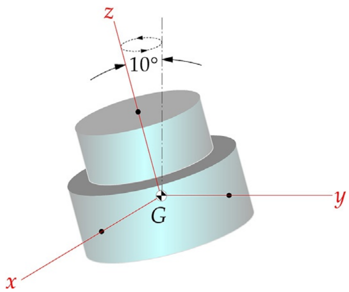
PROBLEMS

Section 12.2

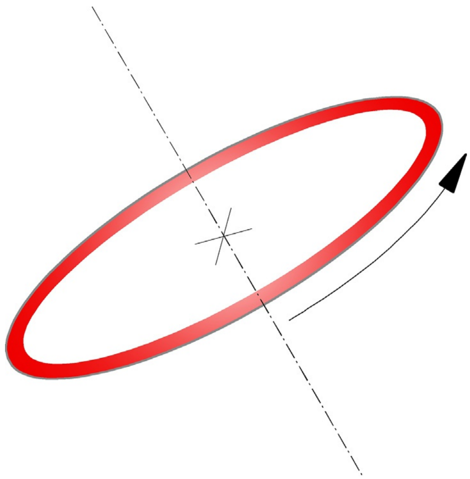
12.1 An axisymmetric satellite has axial and transverse mass moments of inertia about axes through the mass center G of $C = 1200\ \text{kg}\cdot\text{m}^2$ and $A = 2600\ \text{kg}\cdot\text{m}^2$, respectively. If it is spinning at $\omega_s = 6\ \text{rad/s}$ when it is launched, determine its angular momentum. Precession occurs about the inertial Z axis.
{Ans.: $\|\mathbf{H}_G\| = 13{,}450\ \text{kg}\cdot\text{m}^2/\text{s}$}



12.2 A spacecraft is symmetric about its body-fixed z axis. Its principal mass moments of inertia are $A = B = 300\ \text{kg}\cdot\text{m}^2$ and $C = 500\ \text{kg}\cdot\text{m}^2$. The z axis sweeps out a cone with a total vertex angle of 10° as it precesses around the angular momentum vector. If the spin velocity is 6 rad/s, compute the period of precession.
{Ans.: 0.417 s}



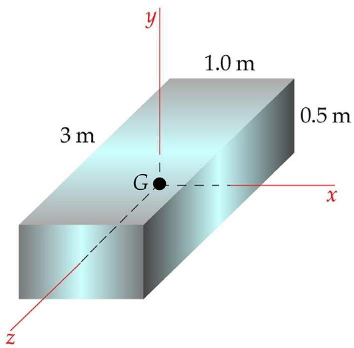
12.3 A thin ring tossed into the air with a spin velocity of ω_s has a very small nutation angle θ (in radians). What is the precession rate ω_p?
{Ans.: $\omega_p = 2\omega_s(1 + \theta^2/2)$, retrograde}



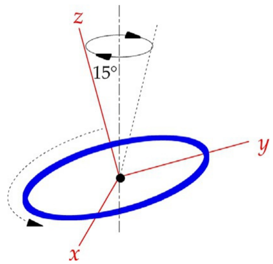
12.4 For an axisymmetric rigid satellite,

$$[\mathbf{I}_G] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 5000 \end{bmatrix} \text{kg} \cdot \text{m}^2$$

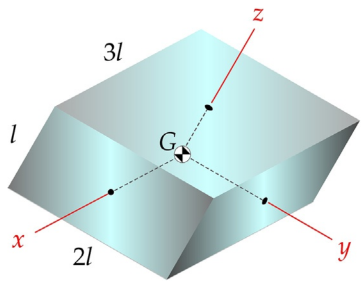
It is spinning about the body z axis in torque-free motion, precessing around the angular momentum vector $\mathbf{H}$ at the rate of 2 rad/s. Calculate the magnitude of $\mathbf{H}$.
{Ans.: $2000 \ \text{kg} \cdot \text{m}^2/\text{s}$}



12.5 At a given instant, the above box-shaped 500-kg satellite (in torque-free motion) has an absolute angular velocity $\boldsymbol{\omega} = 0.01\hat{\mathbf{i}} = .03\hat{\mathbf{j}} + 0.02\hat{\mathbf{k}}$ (rad/s). Its moments of inertia about the principal body axes xyz are $A = 385.4 \ \text{kg} \cdot \text{m}^2$, $B = 416.7 \ \text{kg} \cdot \text{m}^2$, and $C = 52.08 \ \text{kg} \cdot \text{m}^2$, respectively. Calculate the magnitude of its absolute angular acceleration.
{Ans.: $6.167(10^{-4}) \ \text{rad/s}^2$}



12.6 The above 8-kg thin ring in torque-free motion is spinning with an angular velocity of 30 rad/s and a constant nutation angle of 15°. Calculate the rotational kinetic energy if $A = B = 0.36\ \text{kg}\cdot\text{m}^2$, $C = 0.72\ \text{kg}\cdot\text{m}^2$.
{Ans.: 370.5 J}

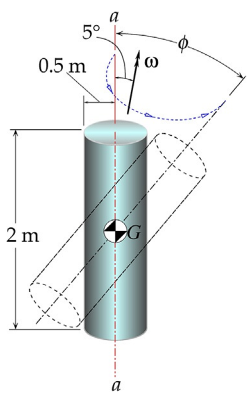


12.7 The above rectangular block has an angular velocity $\boldsymbol{\omega} = 1.5\omega_0\hat{\mathbf{i}} + 0.8\omega_0\hat{\mathbf{j}} + 0.6\omega_0\hat{\mathbf{k}}$, where ω_0 has units of radians per second.

(a) Determine the angular velocity ω of the block if it spins around the body z axis with the same rotational kinetic energy.

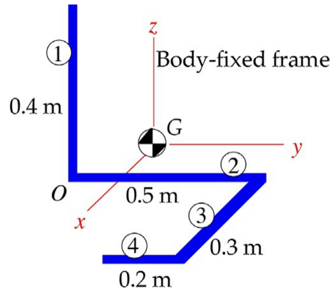
(b) Determine the angular velocity ω of the block if it spins instead around the body z axis with the same angular momentum.

{Ans.: (a) $\omega = 1.31\omega_0$; (b) $\omega = 1.04\omega_0$}



12.8 The above solid right circular cylinder of mass 500 kg is set into torque-free motion with its symmetry axis initially aligned with the fixed spatial line *a-a*. Due to an injection error, the vehicle's angular velocity vector

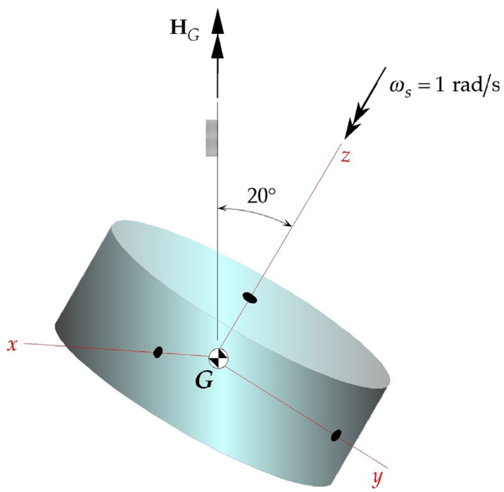
$\boldsymbol{\omega}$ is misaligned 5° (the wobble angle) from the symmetry axis. Calculate the maximum angle ϕ between fixed line *a-a* and the axis of the cylinder.
{Ans.: 30.96°}



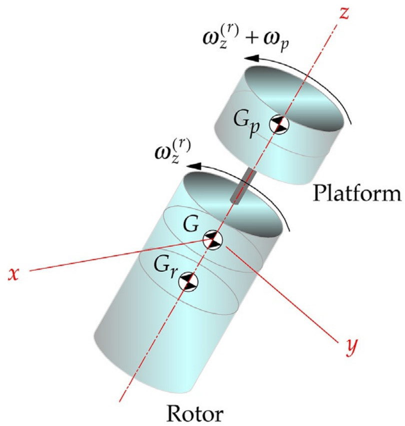
Section 12.3

12.9 For a rigid axisymmetric satellite, the mass moment of inertia about its long axis is 1000 $\text{kg}\cdot\text{m}^2$, and the moment of inertia about transverse axes through the center of mass is 5000 $\text{kg}\cdot\text{m}^2$. It is spinning about the minor principal body axis in torque-free motion at 6 rad/s with the angular velocity lined up with the angular momentum vector **H**. Over time, the energy degrades due to internal effects and the satellite is eventually spinning about a major principal body axis with the angular velocity lined up with the angular momentum vector **H**. Calculate the change in rotational kinetic energy between the two states.
{Ans.: −14.4 kJ}

12.10 Let the object in Example 11.11 (redrawn above) be a highly dissipative torque-free satellite, whose angular velocity at the instant shown is $\boldsymbol{\omega} = 10\hat{\mathbf{i}}$ rad/s. Calculate the decrease in kinetic energy after it becomes, as eventually it must, a major-axis spinner.
{Ans.: −0.487 J}

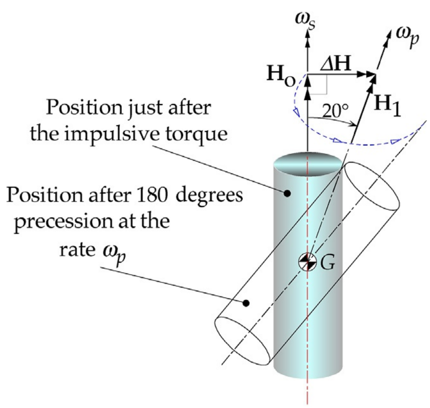


12.11 The above dissipative torque-free cylindrical satellite has the initial spin state shown. $A = B = 320$ $\text{kg}\cdot\text{m}^2$ and $C = 560$ $\text{kg}\cdot\text{m}^2$. Calculate the magnitude of the angular velocity when it reaches its stable spin state.
{Ans.: 1.419 rad/s}



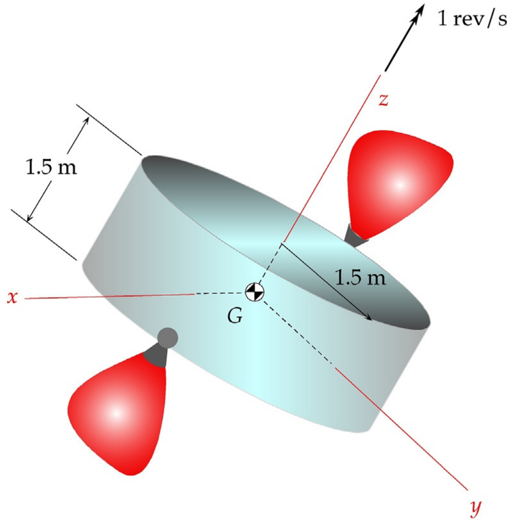
Section 12.4

12.12 For the above nonprecessing, dual-spin satellite, $C_r = 1000\ \text{kg}\cdot\text{m}^2$ and $C_p = 500\ \text{kg}\cdot\text{m}^2$. The angular velocity of the rotor is $3\hat{\mathbf{k}}$ rad/s and the angular velocity of the platform relative to the rotor is $1\hat{\mathbf{k}}$ rad/s. If the relative angular velocity of the platform is reduced to $0.5\hat{\mathbf{k}}$ rad/s, what is the new angular velocity of the rotor?
{Ans.: 3.17 rad/s}



Section 12.6

12.13 For the above rigid axisymmetric satellite, the mass moment of inertia about its long axis is $1000\ \text{kg}\cdot\text{m}^2$, and the moment of inertia about transverse axes through the center of mass is $5000\ \text{kg}\cdot\text{m}^2$. It is initially spinning about the minor principal body axis in torque-free motion at $\omega_s = 0.1$ rad/s, with the angular velocity lined up with the angular momentum vector $\mathbf{H}_0$. A pair of thrusters exert an external impulsive torque on the satellite, causing an instantaneous change $\Delta\mathbf{H}$ of angular momentum in the direction normal to $\mathbf{H}_0$, so that the new angular momentum is $\mathbf{H}_1$, at an angle of 20° to $\mathbf{H}_0$, as shown in the figure. How long does it take the satellite to precess (cone) through an angle of 180° around $\mathbf{H}_1$?
{Ans.: 147.6 s}



Section 12.7

12.14 A satellite is spinning at 0.01 rev/s. The moment of inertia of the satellite about the spin axis is $2000\ \text{kg}\cdot\text{m}^2$. Paired thrusters are located at a distance of 1.5 m from the spin axis. They deliver their thrust in pulses, each thruster producing an impulse of $15\ \text{N}\cdot\text{s}$ per pulse. At what rate will the satellite be spinning after 30 pulses?
{Ans.: 0.1174 rev/s}

12.15 A satellite has moments of inertia $A = 2000\ \text{kg}\cdot\text{m}^2$, $B = 4000\ \text{kg}\cdot\text{m}^2$, and $C = 6000\ \text{kg}\cdot\text{m}^2$ about its principal body axes *xyz*. Its angular velocity is $\boldsymbol{\omega} = 0.1\hat{\mathbf{i}} + 0.3\hat{\mathbf{j}} + 0.5\hat{\mathbf{k}}$ (rad/s). If thrusters cause the angular momentum vector to undergo the change $\Delta\mathbf{H}_G = 50\hat{\mathbf{i}} - 100\hat{\mathbf{j}} + 300\hat{\mathbf{k}}$ $(\text{kg}\cdot\text{m}^2/\text{s})$, what is the magnitude of the new angular velocity?
{Ans.: 1.045 rad/s}

12.16 The body-fixed *xyz* axes are principal axes of inertia passing through the center of mass of a 300-kg cylindrical satellite pictured above. It is spinning at 1 rev/s about the *z* axis. What impulsive torque about the *y* axis must the thrusters impart to cause the satellite to precess at 5 rev/s?
{Ans.: $6740\ \text{N}\cdot\text{m}\cdot\text{s}$}

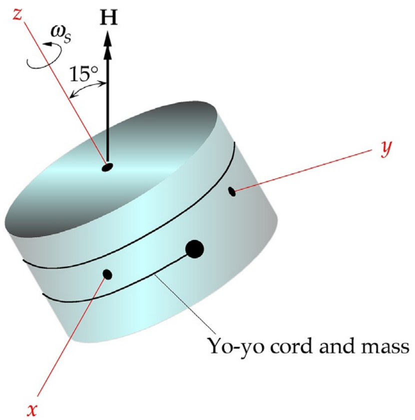
Section 12.8

12.17 A satellite is to be despun by means of a tangential release yo-yo mechanism consisting of two masses, 3 kg each, wound around the midplane of the satellite. The satellite is spinning around its axis of symmetry with an angular velocity $\omega_s = 5$ rad/s. The radius of the cylindrical satellite is 1.5 m and the moment of inertia about the spin axis is $C = 300\ \text{kg}\cdot\text{m}^2$.
(a) Find the cord length and the deployment time to reduce the spin rate to 1 rad/s.
(b) Find the cord length and time to reduce the spin rate to zero.
{Ans.: (a) $l = 5.902$ m, $t = 0.787$ s; (b) $l = 7.228$ m, $t = 0.964$ s}

12.18 A cylindrical satellite of radius 1 m is initially spinning about the axis of symmetry at the rate of 2 rad/s with a nutation angle of 15°. The principal moments of inertia are $A = B = 30\ \text{kg}\cdot\text{m}^2$ and $C = 60\ \text{kg}\cdot\text{m}^2$. An energy dissipation device is built into the satellite, so that it eventually ends up in pure spin around the *z* axis.

(a) Calculate the final spin rate about the z axis.
(b) Calculate the loss of kinetic energy.
(c) A tangential release yo-yo despin device is also included in the satellite. If the two yo-yo masses are each 7 kg, what cord length is required to completely despin the satellite? Is it wrapped in the proper direction in the figure?

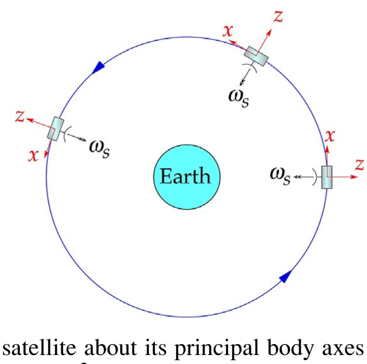
{Ans.: (a) 2.071 rad/s; (b) 8.62 J; (c) 2.3 m}



Section 12.9

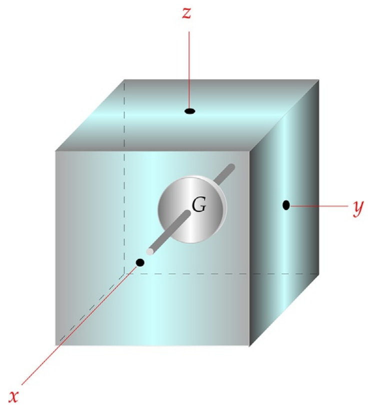
12.19 A communications satellite is in a geostationary equatorial orbit with a period of 24 h. The spin rate ω_s about its axis of symmetry is 1 rpm, and the moment of inertia about the spin axis is 550 $\text{kg}\cdot\text{m}^2$. The moment of inertia about transverse axes through the mass center G is 225 $\text{kg}\cdot\text{m}^2$. If the spin axis is initially pointed toward the earth, calculate the magnitude and direction of the applied torque $\mathbf{M}_G$ required to keep the spin axis pointed always toward the earth.

{Ans.: 0.00420 N · m, about the negative x axis}

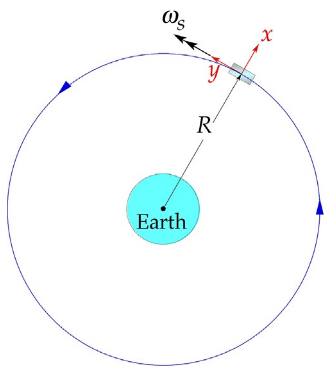


12.20 The moments of inertia of a satellite about its principal body axes xyz are $A = 1000\ \text{kg}\cdot\text{m}^2$, $B = 600\ \text{kg}\cdot\text{m}^2$, and $C = 500\ \text{kg}\cdot\text{m}^2$, respectively. The moments of inertia of a momentum wheel at the center of mass of the satellite and aligned with the x axis are $I_x = 20\ \text{kg}\cdot\text{m}^2$ and $I_y = I_z = 6\ \text{kg}\cdot\text{m}^2$. The absolute angular velocity of the satellite with the momentum wheel locked is $\boldsymbol{\omega}_0 = 0.1\hat{\mathbf{i}} + 0.05\hat{\mathbf{j}}$ rad/s. Calculate the angular velocity ω_f of the momentum wheel (relative to the satellite) required to reduce the x component of the absolute angular velocity of the satellite to 0.003 rad/s.

{Ans.: 4.95 rad/s}



12.21 A solid circular cylindrical satellite of radius 1 m, length 4 m, and mass 250 kg is in a circular earth orbit with a period of 90 min. The cylinder is spinning at 0.001 rad/s (no precession) around its axis, which is aligned with the y axis of the Clohessy-Wiltshire frame. Calculate the magnitude of the external torque required to maintain this attitude.
{Ans.: $-0.00014544\hat{\mathbf{i}}$ $(\mathrm{N}\cdot\mathrm{m})$}



Section 12.10

12.22 A satellite has principal moments of inertia $A = 300\ \mathrm{kg}\cdot\mathrm{m}^2$, $B = 400\ \mathrm{kg}\cdot\mathrm{m}^2$, and $C = 500\ \mathrm{kg}\cdot\mathrm{m}^2$. Determine the permissible orientations in a circular orbit for gravity gradient stabilization. Specify which axes may be aligned in the pitch, roll, and yaw directions. Recall that, relative to a Clohessy-Wiltshire frame at the center of mass of a satellite, yaw is about the x axis (outward radial from earth's center), roll is about the y axis (velocity vector), and pitch is about the z axis (normal to orbital plane).

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