CHAPTER

LUNAR TRAJECTORIES

9

9.1 INTRODUCTION

The orbit of the moon around the earth is an ellipse having a small eccentricity ($e = 0.0549$) and perigee and apogee radii of $r_p = 363{,}400\,\text{km}$ and $r_a = 405{,}500\,\text{km}$, respectively. Therefore, the semimajor axis a of the moon's orbit is

$$a = \frac{r_a + r_p}{2} = 384{,}400\,\text{km} \tag{9.1}$$

A circular orbit of this radius has the same period as the moon's elliptical orbit (see Eq. 2.83). Therefore, to simplify our analysis, let us assume that the moon's path around the earth is a circle of radius D, where

$$D = 384{,}400\,\text{km} \tag{9.2}$$

Recalling that the earth's gravitational parameter is

$$\mu_e = 398{,}600\,\text{km}^3/\text{s}^2$$

Eq. (2.63) gives the circular orbital speed v_m of the moon as

$$v_m = \sqrt{\frac{\mu_e}{D}} = 1.0183\,\text{km/s} \tag{9.3}$$

Imagine that a spacecraft has been placed in a circular earth orbit of 320 km altitude (radius $r_c = 6698\,\text{km}$) and that its orbit is coplanar with that of the moon. The speed of the vehicle in this circular parking orbit is

$$v_c = \sqrt{\frac{\mu_e}{r_c}} = 7.7143\,\text{km/s}$$

To most efficiently transfer the spacecraft from the low earth orbit out to lunar orbit requires a Hohmann transfer (namely, orbit 2 in Fig. 9.1).

At its perigee orbit 2 is tangent to the circular low earth orbit, so that $r_p = 6698\,\text{km}$, and at apogee it is tangent to the moon's orbit, which means that $r_a = 384{,}400\,\text{km}$. Therefore, the semimajor axis of this transfer ellipse is

$$a = \frac{r_a + r_p}{2} = 195{,}549\,\text{km}$$

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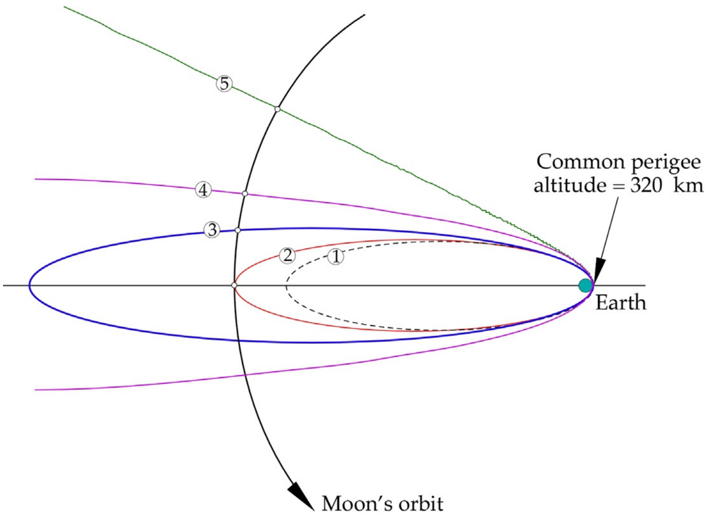

FIG. 9.1

Several strategies for reaching the moon's orbit from low earth orbit. The perigee speeds of the orbits (in kilometers per second) are: (1) 10.8, (2) 10.815, (3) 10.85, (4) 10.9, and (5) 11.2. Clearly, the energy of orbit 1 is too low to reach the moon.

According to Eq. (2.84), the eccentricity e of the orbit is

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.96575$$

and its period T is given by Eq. (2.83),

$$T = \frac{2\pi}{\sqrt{\mu_e}} a^{3/2} = 8.6059\left(10^5\right)\text{s} = 239.05\text{h}$$

If follows that the time of flight t_F for this Hohmann transfer is

$$t_F = \frac{T}{2} = 119.52\text{h}\,(4.98\,\text{days})$$

The angular momentum of the transfer ellipse is given by Eq. (6.2),

$$h = \sqrt{2\mu_e}\sqrt{\frac{r_a r_p}{r_a + r_p}} = 72{,}444\,\text{km}^2/\text{s}$$

We use the angular momentum to find the speeds v_p and v_a at perigee and apogee, respectively,

$$v_p = \frac{h}{r_p} = 10.815\,\text{km/s} \qquad v_a = \frac{h}{r_a} = 0.18846\,\text{km/s}$$

The delta-v required to transfer from the initial circular parking orbit to the Hohmann transfer trajectory at its perigee is

$$\Delta v_p = v_p - v_c = 10.815 - 7.7143 = 3.1007\,\text{km/s}$$

The delta-v required to finally transfer from the Hohmann ellipse to the moon's orbit is

$$\Delta v_a = v_m - v_a = 1.0183 - 0.18846 = 0.82984\,\mathrm{km/s}$$

The total delta-v is the sum of these two,

$$\Delta v_{\text{total}} = 3.1007 + 0.82984 = 3.9305\,\mathrm{km/s}$$

To reduce the flight time to lunar orbit from the same departure point and the same departure flight path angle (zero degrees), we must increase the injection speed to a value greater than 10.815 km/s. Let us choose orbit 3 in Fig. 9.1, for which $v_p = 10.85\,\mathrm{km/s}$, which is still below the escape speed of 10.91 km/s (see Eq. 2.91). The angular momentum of orbit 3 is

$$h = r_p v_p = 6698 \cdot 10.85 = 72{,}673\,\mathrm{km^2/s}$$

Solving Eq. (2.50) for the eccentricity e yields

$$e = \frac{h^2}{\mu_e r_p} - 1 = 0.97819$$

The new semimajor axis a is obtained from Eq. (2.71),

$$a = \frac{h^2}{\mu_e}\frac{1}{1-e^2} = 307{,}104\,\mathrm{km}$$

and from this we find the period

$$T = \frac{2\pi}{\sqrt{\mu_e}} a^{3/2} = 470.48\,\mathrm{h}$$

Next, we set $r = 384{,}400\,\mathrm{km}$ in Eq. (2.45) and solve for the true anomaly at which transfer ellipse 3 first crosses the moon's orbit,

$$\theta = \cos^{-1}\left[\frac{1}{e}\left(\frac{h^2}{\mu_e r} - 1\right)\right] = 170.77^\circ$$

From Eqs. (3.13b), (3.14), and (3.15), the flight time t_F (since perigee) to lunar orbit crossing is

$$t_F = \frac{T}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - e\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)\right]\right\}$$

Substituting $\theta = 170.77^\circ$, $e = 0.97819$, and $T = 470.48$ h yields

$$t_F = 66.343\,\mathrm{h}$$

This is a little over half the time required to reach the moon's orbit on the Hohmann transfer ellipse (orbit 2).

The radial (v_r) and transverse ($v_\perp$) components of spacecraft velocity on orbit 3 at the lunar orbit crossing are found using Eqs. (2.49) and (2.31),

$$v_r = \frac{\mu_e}{h} e\sin\theta = \frac{398{,}600}{72{,}673} \cdot 0.97819 \sin 170.77^\circ = 0.86025\,\mathrm{km/s}$$

$$v_\perp = \frac{h}{r} = \frac{72{,}673}{384{,}400} = 0.18906\,\mathrm{km/s}$$

Therefore, the speed is

$$v = \sqrt{{v_r}^2 + {v_\perp}^2} = 0.88078\,\text{km/s}$$

and, according to Eq. (2.51), the flight path angle γ is

$$\gamma = \tan^{-1}\left(\frac{v_r}{v_\perp}\right) = 77.605°$$

From Eq. (9.3) we know that the speed v_m in the circular lunar orbit is 1.0183 km/s, and the flight path angle is zero. The delta-v required to transfer to the lunar orbit from orbit 3 is given by Eq. (6.8),

$$\Delta v_F = \sqrt{v^2 + {v_m}^2 - 2vv_m\cos\Delta\gamma} = \sqrt{0.88078^2 + 1.0183^2 - 2\cdot 0.88078\cdot 1.0183\cos(0 - 77.605°)}$$
$$= 1.1949\,\text{km/s}$$

We previously calculated the speed of the initial circular parking orbit as 7.7143 km/s. Therefore, the necessary delta-v for translunar injection on orbit 3 is

$$\Delta v_0 = 10.85 - 7.7143 = 3.1357\,\text{km/s}$$

It follows that the total delta-v required to enter the moon's earth orbit after the 66-h flight is

$$\Delta v_{\text{total}} = \Delta v_0 + \Delta v_F = 3.1357 + 1.1949 = 4.3306\,\text{km/s}$$

This is about ten percent more than the total delta-v that we found for the nearly 120 hour Hohmann transfer strategy (orbit 2).

Increasing the perigee speed to 10.9 and 11.2 km/s yields, respectively, orbits 4 and 5 in Fig. 9.1. These orbits have still smaller flight times but larger delta-v requirements. On the other hand, orbit 1, with a perigee speed of only 10.8 km/s, cannot reach lunar orbit.

An obvious omission from our brief analysis so far is the moon itself. The goal of a lunar mission is not to simply reach the moon's orbit but to go into orbit around the moon or to either impact or land on its surface. Thus, as a spacecraft approaches the moon's orbit, the moon should be nearing the same position. That means lunar gravity will increasingly affect the spacecraft's trajectory, bending it more and more toward the moon. In this chapter we show how the method of patched conics, employed in Chapter 8 for interplanetary trajectories, can be applied to lunar trajectories.

We conclude this chapter with a numerical integration approach to lunar trajectory analysis.

9.2 COPLANAR PATCHED CONIC LUNAR TRAJECTORIES

In this section we shall for simplicity continue to assume that the moon's orbit is a circle and, furthermore, that the spacecraft's translunar trajectory lies in the moon's orbital plane, as illustrated in Fig. 9.2, which is a not-to-scale view looking down on that plane. Our approach is similar to those of Bate et al. (1971), Chobotov (1996), and Brown (1998). The x axis of the earth-centered, nonrotating xyz coordinate system is directed toward the position of the moon at the instant the spacecraft crosses the moon's sphere of influence (SOI), so that the moon's position vector $\mathbf{r}_m$ at that instant is

$$\mathbf{r}_m = D\hat{\mathbf{i}} \tag{9.4}$$

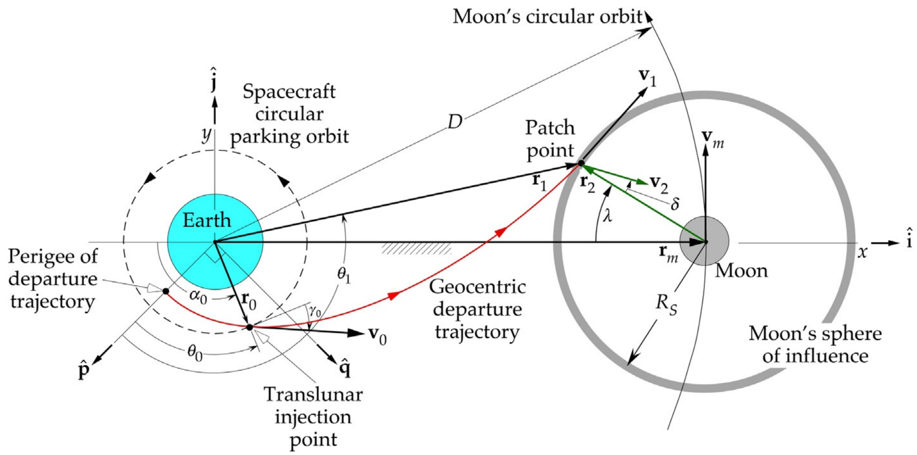

FIG. 9.2

Coplanar translunar trajectory from earth orbit to crossing of the moon's sphere of influence. The earth-centered *xy* axes do not rotate. Not to scale.

The z axis points out of the plane, and the y axis completes the right-handed triad, which means the moon's circular velocity $\mathbf{v}_m$ at this instant is

$$\mathbf{v}_m = v_m\hat{\mathbf{j}} \tag{9.5}$$

We calculated the circular orbital speed v_m of the moon in Eq. (9.3).

As shown in Fig. 9.2, the position vector $\mathbf{r}_0$ of the spacecraft at translunar injection (TLI), when it departs earth orbit, is

$$\mathbf{r}_0 = -r_0\cos\alpha_0\hat{\mathbf{i}} - r_0\sin\alpha_0\hat{\mathbf{j}} \tag{9.6}$$

where r_0 is the distance from the earth, and α_0 is its angular position relative to the earth-moon line. When the spacecraft arrives at the moon's SOI, its position vector $\mathbf{r}_2$ relative to the moon is

$$\mathbf{r}_2 = -R_S\cos\lambda\hat{\mathbf{i}} + R_S\sin\lambda\hat{\mathbf{j}} \tag{9.7}$$

where λ is the lunar arrival angle. The radius R_S of the SOI is given by Eq. (8.34),

$$R_S = D(m_m/m_e)^{2/5} \tag{9.8}$$

where m_m and m_e are the mass of the moon and the earth, respectively. Since $m_e = 5.974(10^{24})$ kg and $m_m = 7.348(10^{22})$ kg, it follows that

$$R_S = 0.172D = 66{,}183\,\text{km} \tag{9.9}$$

From Fig. 9.2 it is clear that the position vector $\mathbf{r}_1$ of the patch point relative to the earth is

$$\mathbf{r}_1 = \mathbf{r}_m + \mathbf{r}_2 \tag{9.10}$$

Substituting Eqs. (9.3) and (9.7) into this expression yields

$$\mathbf{r}_1 = (D - R_S\cos\lambda)\hat{\mathbf{i}} + R_S\sin\lambda\hat{\mathbf{j}} \tag{9.11}$$

Clearly, the position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ of both the TLI point and the patch point are known if we provide values for the parking orbit radius r_0 and the angles α_0 and λ. We can then find the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ at those same two points by means of Eqs. (5.28) and (5.29),

$$\mathbf{v}_0 = \frac{1}{g}(\mathbf{r}_1 - f\mathbf{r}_0) \tag{9.12a}$$

$$\mathbf{v}_1 = \frac{1}{g}(\dot{g}\mathbf{r}_1 - \mathbf{r}_0) \tag{9.12b}$$

The Lagrange coefficients f, g, and $\dot{g}$ are listed in Eqs. (5.30),

$$f = 1 - \frac{\mu_e r_1}{h_1{}^2}(1 - \cos\Delta\theta) \tag{9.13a}$$

$$g = \frac{r_0 r_1}{h_1}\sin\Delta\theta \tag{9.13b}$$

$$\dot{g} = 1 - \frac{\mu_e r_0}{h_1{}^2}(1 - \cos\Delta\theta) \tag{9.13c}$$

where h_1 is the angular momentum of the departure trajectory and $\Delta\theta$ is the difference $\theta_1 - \theta_0$ between the true anomalies of the position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$. We will refer to this difference as the "sweep angle." The sweep angle is found by means of Eq. (5.23),

$$\cos\Delta\theta = \hat{\mathbf{u}}_{r_0} \cdot \hat{\mathbf{u}}_{r_1} \tag{9.14}$$

where the radial unit vectors at each end of the departure trajectory are

$$\hat{\mathbf{u}}_{r_0} = \frac{\mathbf{r}_0}{r_0} \qquad \hat{\mathbf{u}}_{r_1} = \frac{\mathbf{r}_1}{r_1} \tag{9.15}$$

It remains to find an expression for the angular momentum h_1 that appears in Eqs. (9.13a)–(9.13c). To that end, we first use Eq. (9.12a) to calculate the radial component of the TLI velocity,

$$v_{r_0} = \hat{\mathbf{u}}_{r_0} \cdot \mathbf{v}_0 = \hat{\mathbf{u}}_{r_0} \cdot \frac{1}{g}(r_1\hat{\mathbf{u}}_{r_1} - fr_0\hat{\mathbf{u}}_{r_0}) = \frac{1}{g}(r_1\hat{\mathbf{u}}_{r_0} \cdot \mathbf{u}_{r_1} - fr_0\hat{\mathbf{u}}_{r_0} \cdot \hat{\mathbf{u}}_{r_0})$$

so that, with the aid of Eq. (9.14), we get

$$v_{r_0} = \frac{1}{g}(r_1\cos\Delta\theta - fr_0) \tag{9.16}$$

Combining Eqs. (2.31) and (2.51), we can calculate the radial component of the velocity $\mathbf{v}_0$ by the formula

$$v_{r_0} = \frac{h_1}{r_0}\tan\gamma_0$$

where γ_0 is the flight path angle at TLI. Substituting this expression into Eq. (9.16) yields

$$\frac{h_1}{r_0}\tan\gamma_0 = \frac{1}{g}(r_1\cos\Delta\theta - fr_0) \tag{9.17}$$

Replacing f and g by their expressions in Eqs. (9.13a)–(9.13c) leads to

$$\tan\gamma_0 = \frac{1}{r_1\sin\Delta\theta}\left\{r_1\cos\Delta\theta - \left[1 - \frac{\mu_e r_1}{h_1{}^2}(1 - \cos\Delta\theta)\right]r_0\right\}$$

Finally, solving this equation for the angular momentum h_1 yields

$$h_1 = \sqrt{\mu_e r_0}\sqrt{\frac{1-\cos\Delta\theta}{\dfrac{r_0}{r_1}+\sin\Delta\theta\tan\gamma_0-\cos\Delta\theta}} \tag{9.18}$$

Clearly, by specifying the initial and final position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ (and hence, $\Delta\theta$) as well as the initial flight path angle γ_0, we are able to find the angular momentum h_1 of the translunar trajectory. Then Eqs. (9.12a), (9.12b), and (9.13a)–(9.13c) provide the initial and final velocities, $\mathbf{v}_0$ and $\mathbf{v}_1$, so that the orbit is completely determined. Note that the radial components of the velocities at the end points of the translunar trajectory are

$$v_{r_0} = \mathbf{v}_0\cdot\hat{\mathbf{u}}_{r_0} \qquad v_{r_1} = \mathbf{v}_1\cdot\hat{\mathbf{u}}_{r_1} \tag{9.19}$$

Since $h_1 = r_0 v_\perp)_0 = v_0 r_0\cos\gamma_0$ and $v_{esc})_0 = \sqrt{2\mu_e/r_0}$, Eq. (9.18) can be written alternatively as

$$\left.\frac{v}{v_{esc}}\right)_0 = \frac{1}{\sqrt{2}\cos\gamma_0}\sqrt{\frac{1-\cos\Delta\theta}{\dfrac{r_0}{r_1}+\sin\Delta\theta\tan\gamma_0-\cos\Delta\theta}}$$

This equation limits the range of the sweep angle $\Delta\theta$ to those values for which the denominator in the radical is positive. That is,

$$\frac{r_0}{r_1}+\sin\Delta\theta\tan\gamma_0-\cos\Delta\theta > 0 \tag{9.20}$$

Within that range, only those $\Delta\theta'$s for which $v/v_{esc})_0 < 1$ yield elliptical orbits. For example, if $r_0 = 6700\,\text{km}$ and $r_1 = 335{,}000\,\text{km}$ $(r_0/r_1 = 0.02)$, the allowable values of γ_0 and $\Delta\theta$ lie within the triangular shaded region shown in Fig. 9.3. Observe that larger sweep angles require smaller initial flight path angles.

If prior to departing for the moon the spacecraft is in a circular parking orbit of radius r_0, then its speed is $v_c = \sqrt{\mu_e/r_0}$ (see Eq. 2.63). The magnitude of the TLI velocity in Eq. (9.12a) is

$$v_0 = \|\mathbf{v}_0\| = \frac{1}{g}\sqrt{f^2 r_0{}^2 + r_1{}^2 - 2 f r_0 r_1\cos\Delta\theta} \tag{9.21}$$

This speed is reached by supplying a delta-v given by Eq. (6.2),

$$\Delta v_0 = \sqrt{v_c{}^2 + v_0{}^2 - 2 v_c v_0\cos\gamma_0} \tag{9.22}$$

The eccentricity vector of the translunar trajectory can be obtained from Eq. (4.10),

$$\mathbf{e}_1 = \frac{1}{\mu_e}\left[\left(v_0{}^2 - \frac{\mu_e}{r_0}\right)\mathbf{r}_0 - r_0 v_{r_0}\mathbf{v}_0\right] \tag{9.23}$$

The magnitude e_1 of this vector is the orbit's eccentricity, and since the orbit must be an ellipse, e_1 must be less than unity. From the angular momentum h_1 and the eccentricity e_1 we obtain the semimajor axis and the period by means of Eqs. (2.71) and (2.83), respectively,

$$a_1 = \frac{h_1{}^2}{\mu_e}\frac{1}{1-e_1{}^2} \tag{9.24}$$

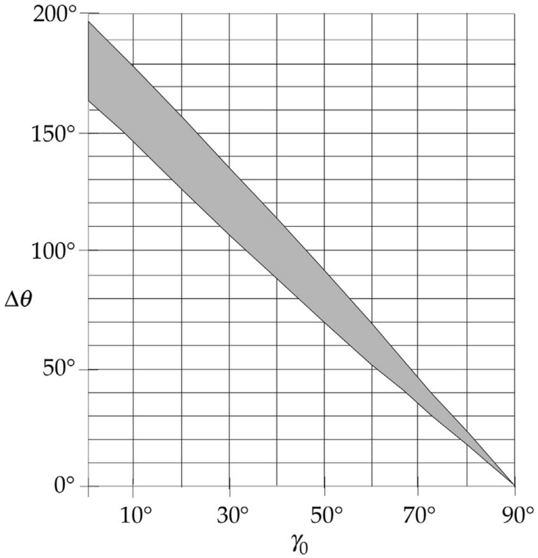

FIG. 9.3

Allowable values of flight path angle γ_0 and sweep angle $\Delta\theta$ if $r_0/r_1 = 0.02$.

$$T_1 = 2\pi\sqrt{\frac{{a_1}^3}{\mu_e}} \tag{9.25}$$

The perifocal unit vectors (see Figs. 2.29 and 2.30) of the translunar trajectory are

$$\hat{\mathbf{p}}_1 = \frac{\mathbf{e}_1}{e_1} \qquad \hat{\mathbf{w}}_1 = \frac{\mathbf{r}_1 \times \mathbf{v}_1}{h_1} \qquad \mathbf{q}_1 = \hat{\mathbf{w}}_1 \times \hat{\mathbf{p}}_1 \tag{9.26}$$

The true anomaly θ_0 at TLI is the angle between perigee and the radial $\mathbf{r}_0$. Therefore, we obtain θ_0 from the formula

$$\theta_0 = \cos^{-1}(\hat{\mathbf{p}}_1 \cdot \mathbf{r}_0/r_0) \tag{9.27}$$

This true anomaly is less than 180°, because the spacecraft is flying outbound toward apogee, which lies beyond the patch point. Since the trajectory is an ellipse, we find the time t_0 at TLI by means of the sequence of calculations listed in Eqs. (3.13b), (3.14), and (3.15), which yield

$$t_0 = \frac{T_1}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_0}{2}\right) - e_1\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_0}{2}\right)\right]\right\} \tag{9.28}$$

Having found the true anomaly at TLI, the true anomaly θ_1 at the patch point follows from

$$\theta_1 = \theta_0 + \Delta\theta \tag{9.29}$$

where the sweep angle $\Delta\theta$ was obtained from Eq. (9.14). The time t_1 at the patch point follows by replacing θ_0 with θ_1 in Eq. (9.28),

$$t_1 = \frac{T_1}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_1}{2}\right) - e_1\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_1}{2}\right)\right]\right\} \tag{9.30}$$

The total flight time Δt_1 from TLI to the SOI is

$$\Delta t_1 = t_1 - t_0 \tag{9.31}$$

When the spacecraft arrives at the patch point, its velocity $\mathbf{v}_2$ relative to the moon is

$$\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_m \tag{9.32}$$

where the geocentric velocities $\mathbf{v}_m$ and $\mathbf{v}_1$ are found in Eqs. (9.5) and (9.12b), respectively.

Within the SOI, the spacecraft moves under the influence of the moon's gravity exclusively (according to the patched conic approximation). At the patch point, its specific angular momentum $\mathbf{h}_2$ *relative to the moon* is

$$\mathbf{h}_2 = \mathbf{r}_2 \times \mathbf{v}_2 \qquad h_2 = \|\mathbf{h}_2\| \tag{9.33}$$

where $\mathbf{r}_2$ is given by Eq. (9.7). We know that all points on this coasting orbit around the moon have the same angular momentum $\mathbf{h}_2$. According to Eq. (2.40) the eccentricity vector of the lunar approach trajectory is

$$\mathbf{e}_2 = \frac{\mathbf{v}_2 \times \mathbf{h}_2}{\mu_m} - \hat{\mathbf{u}}_{r_2} \tag{9.34}$$

where $\mu_m = 4902.8\,\text{km}^3/\text{s}^2$, and $\hat{\mathbf{u}}_{r_2} = \mathbf{r}_2/r_2 = -\cos\lambda\hat{\mathbf{i}} + \sin\lambda\hat{\mathbf{j}}$. The magnitude of $\mathbf{e}_2$ is the orbit's eccentricity e_2, and since the approach trajectory is a hyperbola, e_2 must exceed unity. Observe that if $\hat{\mathbf{h}}_2 \cdot \hat{\mathbf{k}} < 0$, then the motion around the moon is retrograde (clockwise), whereas $\hat{\mathbf{h}}_2 \cdot \hat{\mathbf{k}} > 0$ means the motion is posigrade. For rectilinear motion directly toward the center of the moon, $\hat{\mathbf{h}}_2 \cdot \hat{\mathbf{k}} = 0$. At the patch point, the angle between the probe's relative velocity vector $\mathbf{v}_2$ and the moon's position $(-\mathbf{r}_1)$ relative to the probe is labeled δ in Fig. 9.2. This deviation angle may be computed from the fact that

$$\cos\delta = (-\mathbf{r}_1/r_1)\cdot(\mathbf{v}_2/v_2) \tag{9.35}$$

Clearly, $\delta = 0$ is another indication that the probe is destined for lunar impact.

Eq. (2.50) gives us the perilune radius r_{p_2} of the approach hyperbola,

$$r_{p_2} = \frac{h_2{}^2}{\mu_m}\frac{1}{1+e_2} \tag{9.36}$$

To get the perilune altitude z_{p_2}, we subtract the moon's radius $R_m = 1737\,\text{km}$,

$$z_{p_2} = r_{p_2} - R_m \tag{9.37}$$

If $z_{p_2} < 0$, then the spacecraft impacts the lunar surface. From Eq. (9.36) and the fact that $h_2 = r_{p_2}v_{p_2}$, we find that the perilune speed of the lunar approach orbit is

$$v_{p_2} = \sqrt{1+e_2}\sqrt{\frac{\mu_m}{r_{p_2}}} \tag{9.38}$$

If the objective is to enter a circular orbit at perilune, then the delta-v required at that point is

$$\Delta v_2 = \sqrt{\frac{\mu_m}{r_{p_2}}} - v_{p_2} = \left(1 - \sqrt{1+e_2}\right)\sqrt{\frac{\mu_m}{r_{p_2}}}$$

The total delta-v for this scenario, starting at TLI, is

$$\Delta v = \Delta v_0 + \Delta v_2 \tag{9.39}$$

The unit vectors of the approach hyperbola's perifocal frame are

$$\hat{\mathbf{p}}_2 = \frac{\mathbf{e}_2}{e_2} \qquad \hat{\mathbf{w}}_2 = \frac{\mathbf{h}_2}{h_2} \qquad \hat{\mathbf{q}}_2 = \hat{\mathbf{w}}_2 \times \hat{\mathbf{p}}_2 \tag{9.40}$$

According to Eq. (4.13a), the true anomaly θ_2 of the patch point *relative to perilune* is

$$\theta_2 = 360° - \cos^{-1}(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{u}}_{r_2}) \tag{9.41}$$

because the spacecraft is flying toward perilune ($v_{r_2} < 0$). Given the true anomaly θ_2, we find the time t_2 at the patch point of the hyperbolic lunar orbit by means of the following calculation (see Eqs. 3.34, 3.40, and 3.44a),

$$t_2 = \frac{{h_2}^3}{{\mu_m}^2({e_2}^2-1)^{3/2}}\left\{e_2 \sinh\left[2\tanh^{-1}\left(\sqrt{\frac{e_2-1}{e_2+1}}\tan\frac{\theta_2}{2}\right)\right] - 2\tanh^{-1}\left(\sqrt{\frac{e_2-1}{e_2+1}}\tan\frac{\theta_2}{2}\right)\right\} \tag{9.42}$$

According to Eq. (3.2), time is zero at the periapsis of an orbit. In the present case, as the spacecraft flies from the patch point to perilune, time increases from t_2 to zero. Hence, t_2 must be negative. The flight time from patch point to perilune is

$$\Delta t_2 = 0 - t_2 \quad (\Delta t_2 > 0) \tag{9.43}$$

The total flight time Δt from translunar injection to perilune is

$$\Delta t = \Delta t_1 + \Delta t_2 \tag{9.44}$$

Keep in mind that we describe the motion of the spacecraft within the moon's SOI relative to a rotating frame of reference that is attached to the moon with its origin at the center of the moon. Let us label the axes of this rotating moon-fixed frame $x'y'z'$ to distinguish them from the xyz axes of the earth-fixed system shown in Fig. 9.2. (The z axes of both frames are normal to the moon's orbital plane and therefore coincide.) As illustrated in Fig. 9.4, the x' axis lies on the rotating earth–moon radial and is directed away from the earth. The y' axis is perpendicular to x' and points in the direction of the moon's velocity relative to the earth. The angle ϕ between the x and x' axes increases at a rate equal to the moon's angular velocity, and it is zero at the instant the spacecraft crosses the moon's SOI.

According to Eq. (2.119), the position vector of the spacecraft relative to the rotating moon-fixed $x'y'z'$ frame is

$$\mathbf{r}_{\text{rel}} = r\cos\theta\hat{\mathbf{p}}_2 + r\sin\theta\hat{\mathbf{q}}_2 \qquad r = {h_2}^2/[\mu_m(1+e\cos\theta)] \tag{9.45}$$

where $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{q}}_2$ are given by Eq. (9.40). The angle between $\hat{\mathbf{p}}_2$ and the x' axis (the argument of perilune) is ω_p, which is constant. Therefore,

$$\begin{aligned}\hat{\mathbf{p}}_2 &= \cos\omega_p\hat{\mathbf{i}}' + \sin\omega_p\hat{\mathbf{j}}' \\ \hat{\mathbf{q}}_2 &= -\sin\omega_p\hat{\mathbf{i}}' + \cos\omega_p\hat{\mathbf{j}}'\end{aligned} \tag{9.46}$$

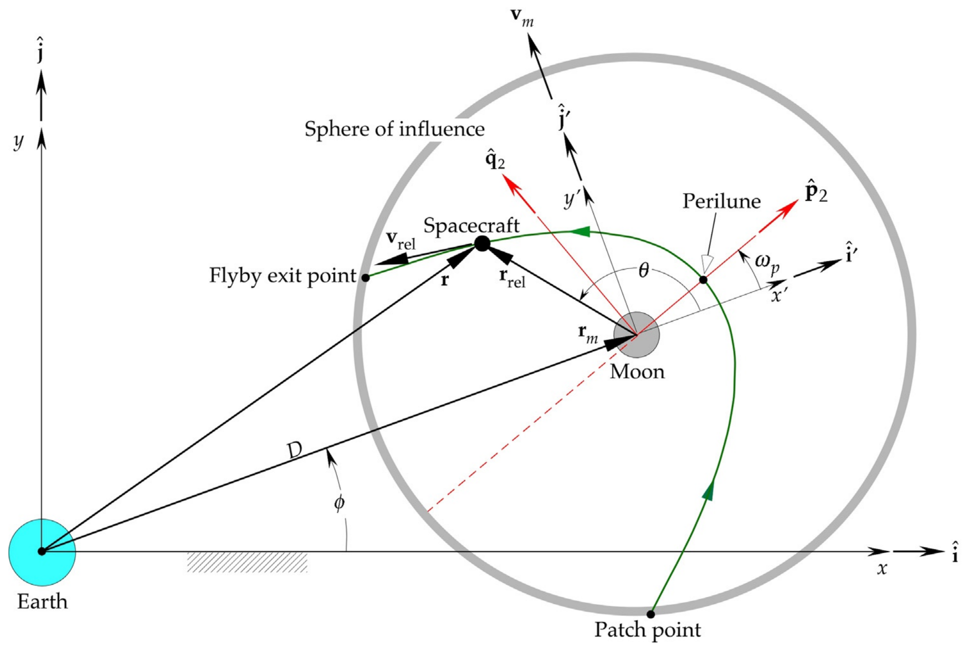

FIG. 9.4

Within the sphere of influence, the orbit and its apse line appear fixed relative to the moon. Not to scale.

Substituting these into Eq. (9.45) yields

$$\mathbf{r}_{rel} = r_{rel})_{x'}\hat{\mathbf{i}}' + r_{rel})_{y'}\hat{\mathbf{j}}'$$

where

$$r_{rel})_{x'} = r\cos\theta\cos\omega_p - r\sin\theta\sin\omega_p$$
$$r_{rel})_{y'} = r\cos\theta\sin\omega_p + r\sin\theta\cos\omega_p$$

In matrix notation

$$\{\mathbf{r}_{rel}\}_{x'y'z'} = \begin{Bmatrix} r_{rel})_{x'} \\ r_{rel})_{y'} \\ r_{rel})_{z'} \end{Bmatrix} = \begin{Bmatrix} r\cos\theta\cos\omega_p - r\sin\theta\sin\omega_p \\ r\cos\theta\sin\omega_p + r\sin\theta\cos\omega_p \\ 0 \end{Bmatrix}$$

At a given instant, the direction cosine matrix $[\mathbf{Q}]$ of the transformation from xyz into $x'y'z'$ is found in Eq. (4.34),

$$[\mathbf{Q}] = [\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.47}$$

The components $r_{rel})_x$ and $r_{rel})_y$ of the relative position vector resolved along the fixed xyz axes are obtained from those projected onto the rotating $x'y'z'$ frame by means of Eq. (4.31),

$$\{\mathbf{r}_{rel}\}_{xyz} = [\mathbf{Q}]^T \{\mathbf{r}_{rel}\}_{x'y'z'} = \begin{Bmatrix} r_{rel})_x \\ r_{rel})_y \\ 0 \end{Bmatrix} \tag{9.48}$$

so that, in vector notation,

$$\mathbf{r}_{rel} = r_{rel})_x \hat{\mathbf{i}} + r_{rel})_y \hat{\mathbf{j}}$$

At any instant, during its hyperbolic orbit around the moon, the position of the spacecraft relative to the earth is

$$\mathbf{r} = \mathbf{r}_m + \mathbf{r}_{rel} \tag{9.49}$$

The components of each vector in this expression are along the earth-fixed xyz frame.

According to Eq. (2.125), the velocity of the spacecraft relative to the moon is

$$\mathbf{v}_{rel} = -\frac{\mu_m}{h_2} \sin\theta \hat{\mathbf{p}}_2 + \frac{\mu_m}{h_2} (e_2 + \cos\theta) \hat{\mathbf{q}}_2$$

Substituting Eq. (9.46), we get

$$\mathbf{v}_{rel} = v_{rel})_{x'} \hat{\mathbf{i}}' + v_{rel})_{y'} \hat{\mathbf{j}}'$$

where

$$v_{rel})_{x'} = -\frac{\mu_m}{h_2} \sin\theta \cos\omega_p - \frac{\mu_m}{h_2} (e_2 + \cos\theta) \sin\omega_p$$

$$v_{rel})_{y'} = -\frac{\mu_m}{h_2} \sin\theta \sin\omega_p + \frac{\mu_m}{h_2} (e_2 + \cos\theta) \cos\omega_p$$

In matrix notation

$$\{\mathbf{v}_{rel}\}_{x'y'z'} = \begin{Bmatrix} v_{rel})_{x'} \\ v_{rel})_{y'} \\ v_{rel})_{z'} \end{Bmatrix} = \begin{Bmatrix} -\frac{\mu_m}{h_2} \sin\theta \cos\omega_p - \frac{\mu_m}{h_2} (e_2 + \cos\theta) \sin\omega_p \\ -\frac{\mu_m}{h_2} \sin\theta \sin\omega_p + \frac{\mu_m}{h_2} (e_2 + \cos\theta) \cos\omega_p \\ 0 \end{Bmatrix}$$

The components $v_{rel})_x$ and $v_{rel})_y$ of this relative velocity vector in the fixed xyz system are obtained from those in the rotating $x'y'z'$ frame as in Eq. (9.48), by the operation

$$\{\mathbf{v}_{rel}\}_{xyz} = [\mathbf{Q}]^T \{\mathbf{v}_{rel}\}_{x'y'z'} = \begin{Bmatrix} v_{rel})_x \\ v_{rel})_y \\ 0 \end{Bmatrix} \tag{9.50}$$

so that

$$\mathbf{v}_{rel} = v_{rel})_x \hat{\mathbf{i}} + v_{rel})_y \hat{\mathbf{j}}$$

According to Eq. (1.66), the absolute velocity of the spacecraft within the SOI is

$$\overbrace{\mathbf{v}}^{\text{absolute velocity}} = \overbrace{\mathbf{v}_m}^{\text{velocity of origin of moving frame}} + \overbrace{\boldsymbol{\omega}_m}^{\text{angular velocity of moving frame}} \times \overbrace{\mathbf{r}_{rel}}^{\text{position vector relative to moving frame}} + \overbrace{\mathbf{v}_{rel}}^{\text{velocity relative to moving frame}} \tag{9.51}$$

Since we are assuming that the position vector $\mathbf{r}_m$ of the moon's orbit has constant magnitude, it follows from Eq. (1.52) that $\mathbf{v}_m = \boldsymbol{\omega}_m \times \mathbf{r}_m$. Therefore, $\mathbf{v} = \boldsymbol{\omega}_m \times (\mathbf{r}_m + \mathbf{r}_{\text{rel}}) + \mathbf{v}_{\text{rel}}$, or

$$\mathbf{v} = \boldsymbol{\omega}_m \times \mathbf{r} + \mathbf{v}_{\text{rel}} \tag{9.52}$$

where $\mathbf{r}$ is given by Eq. (9.49). Within the moon's SOI we use this formula to determine the absolute velocity of the spacecraft from its velocity relative to the moon. All vector components in Eq. (9.52) are along the earth-fixed xyz frame.

EXAMPLE 9.1

A spacecraft is in a circular earth orbit of 320 km altitude. When $\alpha_0 = 28°$, it is launched into a translunar trajectory with a flight path angle of $\gamma_0 = 6°$ (see Fig. 9.2). The lunar arrival angle is $\lambda = 55°$. Find the perilune altitude and the total flight time from TLI to perilune.

Solution

Since $r_0 = 6378 + 320 = 6698$ km, the position vector of the spacecraft at TLI is, according to Eq. (9.6),

$$\begin{aligned}\mathbf{r}_0 &= -r_0 \cos\alpha_0 \hat{\mathbf{i}} - r_0 \sin\alpha_0 \hat{\mathbf{j}} \\ &= -6698\cos 28° \hat{\mathbf{i}} - 6698 \sin 28° \hat{\mathbf{j}} \\ &= -5914.0\hat{\mathbf{i}} - 3144.5\hat{\mathbf{j}}\,(\text{km})\end{aligned}$$

so that

$$\hat{\mathbf{u}}_{r_0} = \frac{\mathbf{r}_0}{r_0} = -0.88295\hat{\mathbf{i}} - 0.46947\hat{\mathbf{j}}$$

Likewise, since $r_2 = R_S = 66{,}183$ km, it follows from Eq. (9.7) that

$$\begin{aligned}\mathbf{r}_2 &= -R_S \cos\lambda \hat{\mathbf{i}} + R_S \sin\lambda \hat{\mathbf{j}} \\ &= -66,183\cos 55° \hat{\mathbf{i}} + 66,183 \sin 55° \hat{\mathbf{j}} \\ &= -37,961\hat{\mathbf{i}} + 54,214\hat{\mathbf{j}}\,(\text{km})\end{aligned}$$

$$\therefore \hat{\mathbf{u}}_{r_2} = \frac{\mathbf{r}_2}{r_2} = -0.57358\hat{\mathbf{i}} + 0.81915\hat{\mathbf{j}} \tag{a}$$

From Eq. (9.2) we know that $\mathbf{r}_m = 384{,}400\hat{\mathbf{i}}\,(\text{km})$ at SOI encounter, so that, according to Eq. (9.10), the position vector $\mathbf{r}_1$ of the patch point relative to the earth is

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{r}_m + \mathbf{r}_2 \\ &= 384,400\hat{\mathbf{i}} + \left(-37,961\hat{\mathbf{i}} + 54,214\hat{\mathbf{j}}\right) \\ &= 346,440\hat{\mathbf{i}} + 54,214\hat{\mathbf{j}}\,(\text{km}) \\ r_1 &= 350,655\,\text{km}\end{aligned}$$

$$\therefore \hat{\mathbf{u}}_{r_1} = \frac{\mathbf{r}_1}{r_1} = 0.98798\hat{\mathbf{i}} + 0.15461\hat{\mathbf{j}}$$

Using Eq. (9.14) we find

$$\cos\Delta\theta = \overbrace{\left(-0.88295\hat{\mathbf{i}} - 0.46947\hat{\mathbf{j}}\right)}^{\hat{\mathbf{u}}_{r_0}} \cdot \overbrace{\left(0.98798\hat{\mathbf{i}} + 0.15461\hat{\mathbf{j}}\right)}^{\hat{\mathbf{u}}_{r_1}} = -0.94491$$

which means the sweep angle is

$$\Delta\theta = 160.89°$$

We can now use Eq. (9.18) to find the angular momentum h_1 of the translunar orbit,

$$
\begin{aligned}
h_1 &= \sqrt{\mu_e r_0}\sqrt{\frac{1-\cos\Delta\theta}{\dfrac{r_0}{r_1}+\sin\Delta\theta\tan\gamma_0-\cos\Delta\theta}} \\
&= \sqrt{398{,}600\cdot 6698}\sqrt{\frac{1-(-0.94491)}{\dfrac{6698}{350{,}655}+\sin 160.89^\circ\cdot\tan 6^\circ-(-0.94491)}} \\
&= 72{,}117\,\mathrm{km^2/s}
\end{aligned}
$$

Eqs. (9.13a)–(9.13c) then yield the values of the Lagrange coefficients,

$$
\begin{aligned}
f &= 1-\frac{\mu_e r_1}{h_1{}^2}(1-\cos\Delta\theta) = 1-\frac{398{,}600\cdot 350{,}655}{72{,}117^2}[1-(-0.94491)] = -51.269 \\
g &= \frac{r_0 r_1}{h_1}\sin\Delta\theta = \frac{6698\cdot 350{,}655}{72{,}117}\sin 160.89^\circ = 10{,}660\,\mathrm{s} \\
\dot{g} &= 1-\frac{\mu_e r_0}{h_1{}^2}(1-\cos\Delta\theta) = 1-\frac{398{,}600\cdot 6698}{72{,}117^2}[1-(-0.94491)] = 0.0015816
\end{aligned}
$$

From Eq. (9.12a) and (9.12b) we finally obtain the spacecraft velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ at the beginning and the end of the geocentric departure trajectory:

$$
\begin{aligned}
\mathbf{v}_0 &= \overbrace{\frac{1}{10{,}660}\left[\left(346{,}440\hat{\mathbf{i}}+54{,}214\hat{\mathbf{j}}\right)-(-51.269)\left(-5914.0\hat{\mathbf{i}}-3144.5\hat{\mathbf{j}}\right)\right]}^{\frac{1}{g}(\mathbf{r}_1-f\mathbf{r}_0)} \\
&= 4.05556\hat{\mathbf{i}}-10.0379\hat{\mathbf{j}}\,(\mathrm{km/s}) \\
v_0 &= 10.826\,\mathrm{km/s} \\
v_{r_0} &= \mathbf{v}_0\cdot\hat{\mathbf{u}}_{r_0} = 1.1316\,\mathrm{km/s} > 0
\end{aligned}
\tag{b}
$$

$$
\begin{aligned}
\mathbf{v}_1 &= \overbrace{\frac{1}{10{,}660}\left[0.0015816\left(346{,}440\hat{\mathbf{i}}+54{,}214\hat{\mathbf{j}}\right)-\left(-5914.0\hat{\mathbf{i}}-3144.5\hat{\mathbf{j}}\right)\right]}^{\frac{1}{g}(\dot{g}\mathbf{r}_1-\mathbf{r}_0)} \\
&= 0.60618\hat{\mathbf{i}}+0.30302\hat{\mathbf{j}}\,(\mathrm{km/s}) \\
v_1 &= 0.67770\,\mathrm{km/s} \\
v_{r_1} &= \mathbf{v}_1\cdot\hat{\mathbf{u}}_{r_1} = 0.64574\,\mathrm{km/s} > 0
\end{aligned}
$$

It follows from Eq. (9.23) that the eccentricity vector is

$$
\begin{aligned}
\mathbf{e}_1 &= \frac{1}{\mu_e}\left[\left(v_0{}^2-\frac{\mu_e}{r_0}\right)\mathbf{r}_0-r_0 v_{r_0}\mathbf{v}_0\right] \\
&= \frac{1}{398{,}600}\left[\left(10.826^2-\frac{398{,}600}{6698}\right)\left(-5914.0\hat{\mathbf{i}}-3.1445\hat{\mathbf{j}}\right)-6698\cdot 1.1316\left(4.0556\hat{\mathbf{i}}-10.038\hat{\mathbf{j}}\right)\right] \\
&= -0.93315\hat{\mathbf{i}}-0.26428\hat{\mathbf{j}} \\
\therefore e_1 &= 0.96985
\end{aligned}
$$

From Eq. (9.26), the perifocal unit vectors of the elliptical translunar trajectory are

$$
\begin{aligned}
\hat{\mathbf{p}}_1 &= \frac{\mathbf{e}_1}{e_1} = \frac{-0.93315\hat{\mathbf{i}}-0.26428\hat{\mathbf{j}}}{0.96985} = -0.96216\hat{\mathbf{i}}-0.27249\hat{\mathbf{j}} \\
\hat{\mathbf{w}}_1 &= \frac{\mathbf{r}_0\times\mathbf{v}_0}{h_1} = \frac{72{,}117\hat{\mathbf{k}}}{72{,}117} = \hat{\mathbf{k}} \\
\hat{\mathbf{q}}_1 &= \hat{\mathbf{w}}_1\times\hat{\mathbf{p}}_1 = \hat{\mathbf{k}}\times\left(-0.96216\hat{\mathbf{i}}-0.27249\hat{\mathbf{j}}\right) = 0.27249\hat{\mathbf{i}}-0.96216\hat{\mathbf{j}}
\end{aligned}
$$

Eqs. (9.24) and (9.25) yield the semimajor axis a_1 and the period T_1,

$$a_1 = \frac{h_1{}^2}{\mu_e}\frac{1}{1-e_1{}^2} = \frac{72{,}117^2}{398{,}600}\frac{1}{1-0.96985^2} = 219{,}714\,\text{km}$$

$$T_1 = 2\pi\sqrt{\frac{a_1{}^3}{\mu_e}} = 2\pi\sqrt{\frac{219{,}714^3}{398{,}600}} = 1.0249\left(10^6\right)\text{s} = 11.863\,\text{days}$$

According to Eq. (9.27), the true anomaly θ_0 of the injection point is

$$\begin{aligned}
\theta_0 &= \cos^{-1}\left(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{u}}_{r_0}\right)\\
&= \cos^{-1}\left[\left(-0.96216\hat{\mathbf{i}} - 0.27249\hat{\mathbf{j}}\right)\cdot\left(-0.88295\hat{\mathbf{i}} - 0.46957\hat{\mathbf{j}}\right)\right]\\
&= \cos^{-1}0.97746\\
\therefore \theta_0 &= 12.187°
\end{aligned}$$

From this we find the time t_0 by means of Eq. (9.28),

$$\begin{aligned}
t_0 &= \frac{T_1}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_0}{2}\right) - e_1\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_0}{2}\right)\right]\right\}\\
&= \frac{1.0249\left(10^6\right)}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-0.96985}{1+0.96985}}\tan\frac{12.187°}{2}\right)\right.\\
&\qquad\qquad \left. - 0.96985\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-0.96985}{1+0.96985}}\tan\frac{12.187°}{2}\right)\right]\right\}\\
&= 130.37\,\text{s}
\end{aligned}$$

Since we know that the sweep angle is 160.89°, it follows from Eq. (9.29) that the true anomaly θ_1 at the patch point is

$$\theta_1 = \overbrace{12.187° + 160.89°}^{\theta_0+\Delta\theta} = 173.08°$$

Then Eq. (9.30) yields t_1, the time at the patch point:

$$\begin{aligned}
t_1 &= \frac{T_1}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_1}{2}\right) - e_1\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_1}{2}\right)\right]\right\}\\
&= \frac{1.0249\left(10^6\right)}{2\pi}\left\{2\tan^{-1}\left(\sqrt{\frac{1-0.96985}{1+0.96985}}\tan\frac{173.08°}{2}\right)\right.\\
&\qquad\qquad \left. -0.96985\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-0.96985}{1+0.96985}}\tan\frac{173.08°}{2}\right)\right]\right\}\\
&= 239{,}370\,\text{s}
\end{aligned}$$

From Eq. (9.31) we obtain the flight time Δt_1 from TLI to the patch point,

$$\Delta t_1 = \overbrace{239{,}370 - 130.38}^{t_1 - t_0} = 239{,}236\,\text{s} = 66.454\,\text{h}$$

The counterclockwise angular velocity ω_m of the moon in its assumed circular orbit around the earth is

$$\omega_m = \frac{v_m}{D} = \frac{1.0183\,\text{km/s}}{384{,}400\,\text{km}} = 2.6491\left(10^{-6}\right)\text{rad/s} = 0.5464\,\text{deg/h} \tag{c}$$

Multiplying this by the time interval Δt_1 yields the moon's lead angle of 36.31°, as shown in Fig. 9.5. The moon moves through this angle as the spacecraft flies to the patch point along its geocentric departure trajectory.

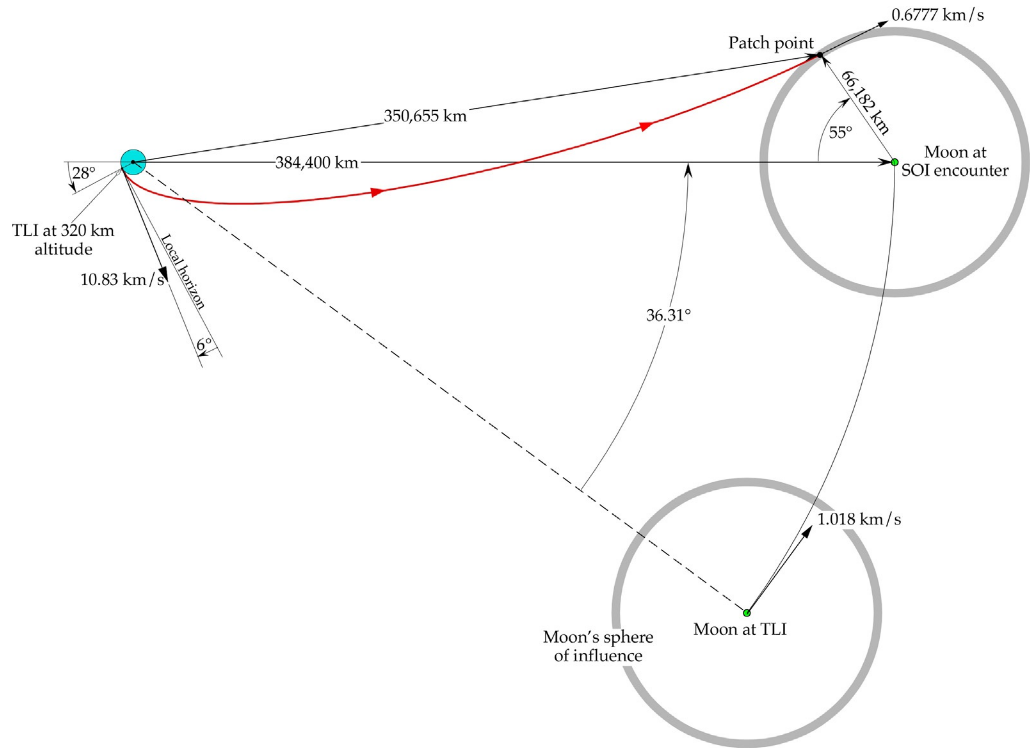

FIG. 9.5

Translunar trajectory of the spacecraft and concomitant motion of the moon. Drawn to scale.

At the patch point, the velocity $\mathbf{v}_2$ of the spacecraft relative to the moon is obtained from Eqs. (9.5) and (9.32)

$$\mathbf{v}_2 = \overbrace{(0.60618\hat{\mathbf{i}} + 0.30302\hat{\mathbf{j}})}^{\mathbf{v}_1} - \overbrace{1.0183\hat{\mathbf{j}}}^{\mathbf{v}_m} = 0.60618\hat{\mathbf{i}} - 0.71528\hat{\mathbf{j}}\,(\text{km/s})$$

$$v_2 = 0.93759\,\text{km/s}$$

We use $\hat{\mathbf{u}}_{r_2}$ from Eq. (a) above to calculate the radial component v_{r_2} of the relative velocity at the patch point,

$$v_{r_2} = \overbrace{\left(0.60618\hat{\mathbf{i}} - 0.71528\hat{\mathbf{j}}\right)}^{\mathbf{v}_2} \cdot \overbrace{\left(-0.57358\hat{\mathbf{i}} + 0.81915\hat{\mathbf{j}}\right)}^{\hat{\mathbf{u}}_{r_2}} = -0.93361\,\text{km/s} \tag{d}$$

The negative sign indicates, as it should, that the spacecraft is flying toward perilune. From Eq. (9.33), the probe's angular momentum relative to the moon is

$$\mathbf{h}_2 = \overbrace{\left(-37,961\hat{\mathbf{i}} + 54,214\hat{\mathbf{j}}\right)}^{\mathbf{r}_2} \times \overbrace{\left(0.60616\hat{\mathbf{i}} - 0.71528\hat{\mathbf{j}}\right)}^{\mathbf{v}_2} = -5710.78\hat{\mathbf{k}}\left(\text{km}^2/\text{s}\right)$$

$$h_2 = 5710.78\text{km}^2/\text{s}$$

The fact that $\mathbf{h}_2$ points toward the orbital plane (in the negative z direction) means that the motion of the spacecraft is retrograde (i.e., clockwise around the moon).

The eccentricity vector $\mathbf{e}_2$ may now be calculated by using Eq. (9.34)

$$\begin{aligned}\mathbf{e}_2 &= \frac{\mathbf{v}_2 \times \mathbf{h}_2}{\mu_m} - \hat{\mathbf{u}}_{r_2} \\ &= \frac{\left(0.60618\hat{\mathbf{i}} - 0.71528\hat{\mathbf{j}}\right) \times \left(-5710.78\hat{\mathbf{k}}\right)}{4902.8} - \left(-0.57358\hat{\mathbf{i}} + 0.81915\hat{\mathbf{j}}\right) \\ &= 1.4067\hat{\mathbf{i}} - 0.11307\hat{\mathbf{j}} \\ \therefore e_2 &= 1.44127\end{aligned}$$

Since the eccentricity exceeds unity, the inbound orbit is indeed a hyperbola, relative to the moon. The perifocal unit vector $\hat{\mathbf{p}}_2$ directed from the center of the moon through perilune of the hyperbolic approach trajectory is, from Eq. (9.40),

$$\hat{\mathbf{p}}_2 = \overbrace{\left(\frac{1.4067\hat{\mathbf{i}} - 0.11307\hat{\mathbf{j}}}{1.44127}\right)}^{\mathbf{e}_2/e_2} = 0.99678\hat{\mathbf{i}} - 0.080122\hat{\mathbf{j}}$$

Substituting this into Eq. (9.41) and using the fact that $v_{r_2} < 0$, we find the true anomaly of the patch point on the lunar approach hyperbola, measured positive clockwise from perilune,

$$\begin{aligned}\theta_2 &= 360° - \cos^{-1}\overbrace{\left[\left(0.99678\hat{\mathbf{i}} - 0.080122\hat{\mathbf{j}}\right)\cdot\left(-0.57358\hat{\mathbf{i}} + 0.81915\hat{\mathbf{j}}\right)\right]}^{\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{u}}_{r_2}} \\ &= 360° - 129.60° \\ &= 230.40°\end{aligned} \tag{e}$$

We find the time relative to perilune at this point on the hyperbola by means of Eq. (9.42),

$$\begin{aligned}t_2 &= \frac{h_2{}^3}{\mu_m{}^2\left(e_2{}^2 - 1\right)^{3/2}}\left\{e_2 \sinh\left[2\tanh^{-1}\left(\sqrt{\frac{e_2 - 1}{e_2 + 1}}\tan\frac{\theta_2}{2}\right)\right] - 2\tanh^{-1}\left(\sqrt{\frac{e_2 - 1}{e_2 + 1}}\tan\frac{\theta_2}{2}\right)\right\} \\ &= \frac{5710.77^3}{4902.8^2\left(1.41126^2 - 1\right)^{3/2}}\left\{1.41126\sinh\left[2\tanh^{-1}\left(\sqrt{\frac{1.41127 - 1}{1.41127 + 1}}\tan\frac{230.40°}{2}\right)\right]\right. \\ &\qquad \left. -2\tanh^{-1}\left(\sqrt{\frac{1.41127 - 1}{1.41127 + 1}}\tan\frac{230.40°}{2}\right)\right\} \\ &= -63{,}116\,\text{s} = -17.532\,\text{h}\end{aligned} \tag{f}$$

The minus sign means that t_2 is the time *until* perilune. The elapsed time Δt_2 from patch point to perilune is $\Delta t_2 = t_{\text{perilune}} - t_2 = 0 - (-17.532\,\text{h}) = 17.532\,\text{h}$. The total time from translunar injection to perilune passage is

$$\Delta t_{\text{total}} = \overbrace{66.454}^{\Delta t_1} + \overbrace{17.532}^{\Delta t_2} = 83.986\,\text{h} = \boxed{3.4994\,\text{days}}$$

According to Eqs. (9.36) and (9.37), the perilune radius r_{p_2} and altitude are z_{p_2}

$$r_{p_2} = \frac{h_2{}^2}{\mu_m}\frac{1}{1 + e_2} = \frac{5710.8^2}{4902.8}\frac{1}{1 + 1.41127} = 2758.67\,\text{km}$$

$$z_{p_2} = \overbrace{2758.67 - 1737}^{r_{p_2} - R_m} = \boxed{1021.67\,\text{km}}$$

From Eq. (9.38) we know that the spacecraft's speed at perilune relative to the moon is

$$v_{p_2} = \sqrt{1 + e_2}\sqrt{\frac{\mu_m}{r_{p_2}}} = \sqrt{1 + 1.41127}\sqrt{\frac{4902.8}{2758.67}} = 2.07012\,\text{km/s}$$

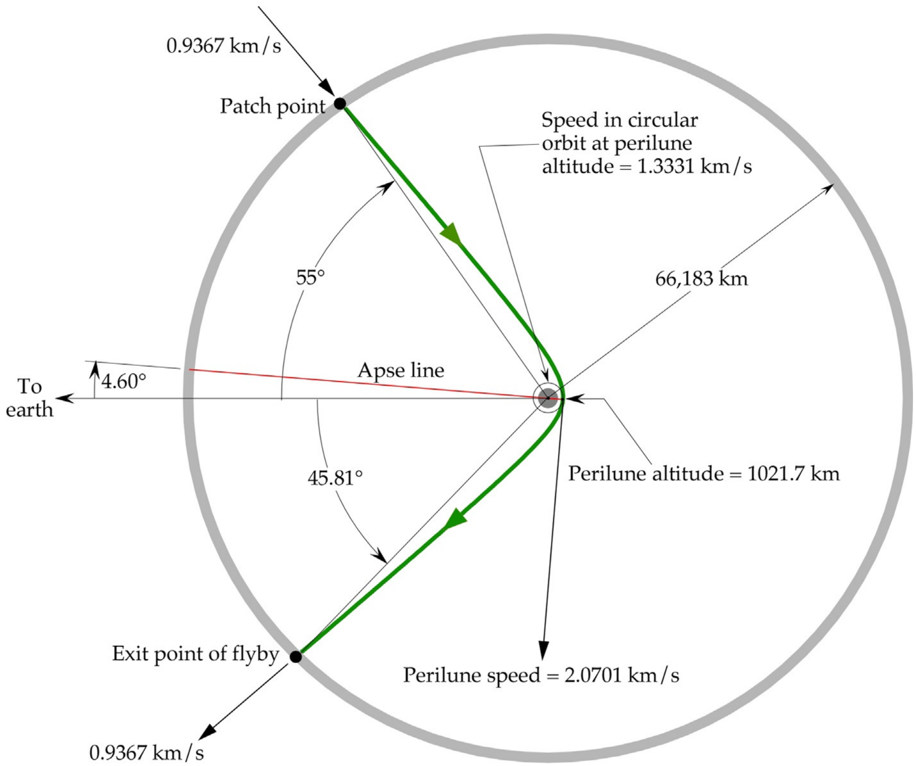

FIG. 9.6

Motion of the spacecraft within the lunar sphere of influence, relative to the moon-fixed frame of reference. Drawn to scale.

Therefore, the delta-v required at perilune to enter a circular lunar orbit of radius r_{p_2} is

$$\Delta v_2 = \sqrt{\frac{\mu_m}{r_{p_2}}} - v_{p_2} = \sqrt{\frac{4902.8}{2758.67}} - 2.07012 = -0.73698\,\text{km/s}$$

Fig. 9.6 shows the hyperbolic path of the spacecraft within the moon's SOI, relative to the moon. Also shown is the small circular capture orbit with a radius equal to that of perilune. If there is no delta-v maneuver at perilune, then the spacecraft continues on its hyperbolic path around the moon and leaves the SOI 17.548 h after perilune passage with the same relative speed at which it entered.

The moon-fixed hyperbolic flyby trajectory pictured in Fig. 9.6 is plotted relative to the stationary, earth-fixed coordinates in Fig. 9.7. The position vector of each point of the hyperbola is transformed into the earth-fixed xy frame using Eqs. (9.48) and (9.49). The result is a flyby trajectory that resembles a partial figure eight (between "SOI entrance" and "SOI exit" in Fig. 9.7) and bears little resemblance to the hyperbola as seen from the moon's perspective in Fig. 9.6.

To determine the trajectory of the spacecraft after leaving the moon's SOI on a flyby, we must first calculate the geocentric state vector at the SOI exit (namely, the position vector $\mathbf{r}_3$ and velocity vector $\mathbf{v}_3$ relative to the earth). The true anomaly on the hyperbola at SOI entrance was found in Eq. (e) to be $\theta_2 = -129.6°$, measured counterclockwise from perilune. The time of that initial SOI crossing was $t_2 = -17.532\,\text{h}$, according to Eq. (f). Therefore, the time and true anomaly of the SOI exit are

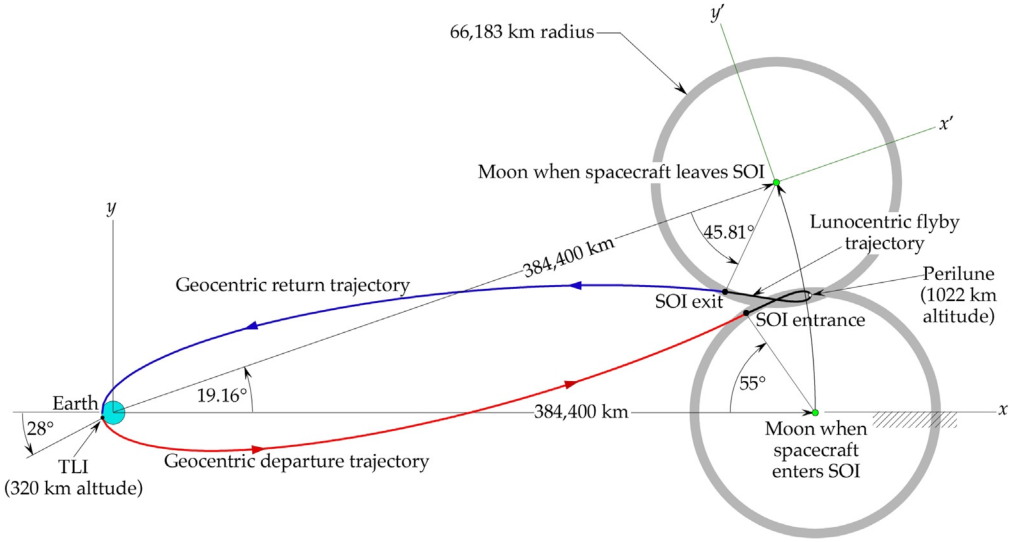

FIG. 9.7

The complete coplanar ballistic trajectory consisting of TLI, coast to moon, flyby, and coasting return to earth, all relative to the earth-fixed reference frame. Drawn to scale.

$$\begin{aligned} t_3 &= +17.532\,\text{h} = 63{,}115\,\text{s} \\ \theta_3 &= +129.6° \end{aligned} \tag{g}$$

Recall that the moon's position angle ϕ is zero at the initial SOI encounter, so that the xyz and $x'y'z'$ axes instantaneously coincide. The angular position of the moon at the end of the hyperbolic fly-around is

$$\phi = \omega_m(t_3 - t_2) = \left[2.6491\left(10^{-6}\right)\frac{\text{rad}}{\text{s}}\right](126{,}230\,\text{s}) = 0.33439\,\text{rad} = 19.159°$$

From Fig. 9.4 we see that the position vector $\mathbf{r}_m$ of the moon is

$$\mathbf{r}_m = D\left(\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}\right)$$

Substituting $D = 384{,}400\,\text{km}$ and $\phi = 19.159°$ yields

$$\mathbf{r}_m = 363{,}107\hat{\mathbf{i}} + 126{,}160\hat{\mathbf{j}}\ (\text{km}) \tag{h}$$

The position of the spacecraft relative to the moon is given by Eq. (9.45),

$$\left.\mathbf{r}_{\text{rel}}\right)_{x'y'z'} = \frac{{h_2}^2}{\mu_m}\frac{1}{1 + e_2\cos\theta_3}\left(\cos\theta_3\hat{\mathbf{p}}_2 + \sin\theta_3\hat{\mathbf{q}}_2\right)$$

where, in terms of the rotating but moon-fixed unit vectors $\hat{\mathbf{i}}'\hat{\mathbf{j}}'\hat{\mathbf{k}}'$,

$$\begin{aligned} \hat{\mathbf{p}}_2 &= \frac{\mathbf{e}_2}{e_2} = \frac{1.4067\hat{\mathbf{i}}' - 0.11307\hat{\mathbf{j}}'}{1.41127} &&= 0.99678\hat{\mathbf{i}}' - 0.080122\hat{\mathbf{j}}' \\ \hat{\mathbf{w}}_2 &= \frac{\mathbf{r}_2 \times \mathbf{v}_2}{h_2} = \frac{-5710.8\hat{\mathbf{k}}}{5710.8} &&= -\hat{\mathbf{k}} \\ \hat{\mathbf{q}}_2 &= \hat{\mathbf{w}}_2 \times \hat{\mathbf{p}}_2 = -\hat{\mathbf{k}} \times \left(0.99678\hat{\mathbf{i}}' - 0.080122\hat{\mathbf{j}}'\right) &&= -0.080122\hat{\mathbf{i}}' - 0.99678\hat{\mathbf{j}}' \end{aligned}$$

so that

$$\mathbf{r}_{\rm rel})_{x'y'z'} = \frac{5710.8^2}{4902.8}\frac{1}{1+1.4113\cos 129.6°}\left[\cos 129.6°\left(0.99678\hat{\mathbf{i}}' - 0.080122\hat{\mathbf{j}}'\right) + \sin 129.6°\left(-0.080122\hat{\mathbf{i}}' - 0.99678\hat{\mathbf{j}}'\right)\right]$$
$$= -46,133\hat{\mathbf{i}}' - 47,454\hat{\mathbf{j}}'\ (\text{km})$$

The direction cosine matrix for transforming vector components from the rotating $x'y'z'$ to the fixed xyz is given in Eq. (9.47),

$$[\mathbf{Q}] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 19.160° & \sin 19.160° & 0 \\ -\sin 19.160° & \cos 19.160° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.94461 & 0.32820 & 0 \\ -0.32820 & 0.94461 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, according to Eq. (9.48),

$$\{\mathbf{r}_{\rm rel}\}_{xyz} = [\mathbf{Q}]^T\{\mathbf{r}_{\rm rel}\}_{x'y'z'} = \begin{bmatrix} 0.94461 & -0.32820 & 0 \\ 0.32820 & 0.94461 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -46,133 \\ -47,454 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -28,003 \\ -59,967 \\ 0 \end{Bmatrix}(\text{km})$$

or

$$\mathbf{r}_{\rm rel} = -28,003\hat{\mathbf{i}} - 59,967\hat{\mathbf{j}}\,(\text{km})$$

Substituting this along with Eq. (h) into Eq. (9.49) yields the spacecraft position vector at SOI exit, relative to the earth-fixed frame

$$\mathbf{r}_3 = 335,104\hat{\mathbf{i}} + 66194\hat{\mathbf{j}}\,(\text{km}) \tag{i}$$

The velocity of the spacecraft at SOI exit relative to the moon is

$$\mathbf{v}_{\rm rel})_{x'y'z'} = \frac{\mu_m}{h_2}\left[-\sin\theta_3\hat{\mathbf{p}}_2 + (e_2 + \cos\theta_3)\hat{\mathbf{q}}_2\right]$$

That is,

$$\mathbf{v}_{\rm rel})_{x'y'z'} = \frac{4902.8}{5710.8}\left[-\sin 129.6°\left(0.99678\hat{\mathbf{i}}' - 0.080122\hat{\mathbf{j}}'\right) + (e_2 + \cos 129.6°)\left(-0.080122\hat{\mathbf{i}}' - 0.99678\hat{\mathbf{j}}'\right)\right]$$
$$= -0.71265\hat{\mathbf{i}}' - 0.60927\hat{\mathbf{j}}'\ (\text{km/s})$$

We use Eq. (9.50) to obtain the components of $\mathbf{v}_{\rm rel}$ in the xyz frame,

$$\{\mathbf{v}_{\rm rel}\}_{xyz} = [\mathbf{Q}]^T\{\mathbf{v}_{\rm rel}\}_{x'y'z'} = \begin{bmatrix} 0.94461 & -0.32820 & 0 \\ 0.32820 & 0.94461 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -0.71265 \\ -0.60927 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.47321 \\ -0.80941 \\ 0 \end{Bmatrix}(\text{km/s})$$

or

$$\mathbf{v}_{\rm rel} = -0.47321\hat{\mathbf{i}} - 0.80941\hat{\mathbf{j}}\,(\text{km/s})$$

Substituting this expression along with Eq. (i) and the angular velocity of the moon (Eq. c) into Eq. (9.52) yields

$$\mathbf{v}_3 = \boldsymbol{\omega}_m \times \mathbf{r}_3 + \mathbf{v}_{\rm rel}$$
$$= 2.6491\left(10^{-6}\right)\hat{\mathbf{k}} \times \left(335,104\hat{\mathbf{i}} + 66194\hat{\mathbf{j}}\right) + \left(-0.47321\hat{\mathbf{i}} - 0.80941\hat{\mathbf{j}}\right)$$

Therefore, the absolute velocity of the spacecraft at the SOI exit is

$$\mathbf{v}_3 = -0.64856\hat{\mathbf{i}} + 0.078302\hat{\mathbf{j}}\,(\text{km/s}) \tag{j}$$

From the state vector $(\mathbf{r}_3, \mathbf{v}_3)$, we obtain the transearth trajectory's orbital elements. That is,

$$\mathbf{h}_3 = \mathbf{r}_3 \times \mathbf{v}_3 = 69,170\hat{\mathbf{k}}\left(\text{km}^2/\text{s}\right)$$
$$h_3 = 69,170\text{km}^2/\text{s}$$
$$\text{e}_3 = \frac{1}{\mu_e}(\text{v}_3 \times \text{h}_3) - \frac{\text{r}_3}{r_3} = -0.967456\hat{\text{i}} - 0.0812406\hat{\text{j}}$$
$$e_3 = 0.970860$$

With this information, we can plot the geocentric return trajectory, as shown in Fig. 9.7. Its perigee is

$$r_{p_3} = \frac{h_3{}^2}{\mu_e}\frac{1}{1+e_3} = \frac{69,170^2}{398,600}\frac{1}{1+0.97086} = 6090.4\text{km}$$

Since this is less than the radius of the earth, the spacecraft will impact the atmosphere.

The path followed by the spacecraft in this example is called a free return trajectory because the single delta-v maneuver at TLI yields a lunar flyby followed by a return to earth.

9.3 A SIMPLIFIED LUNAR EPHEMERIS

We will employ the geocentric equatorial XYZ frame (Section 4.3) to describe the three-dimensional translunar trajectory of a spacecraft as well as the motion of the moon around the earth. The state vector of the moon at any time is found by means of a lunar ephemeris. High-precision ephemerides are found in the Jet Propulsion Laboratory's authoritative DE (*development ephemeris*) series. These currently may be accessed online at the JPL Horizons ephemeris system website (JPL Horizons Web-Interface, 2018) and within MATLAB by means of the function `planetEphemeris`. Simpson (1999) developed a simplified lunar ephemeris, which is a curve fit of JPL's 1984 DE200 ephemeris model. It is easy to use and the precision is sufficient for our needs.

Simpson's lunar ephemeris yields the geocentric equatorial coordinates X, Y, and Z of the moon in kilometers for any year in the range CE 2000 through CE 2100, according to the formula

$$X_i = \sum_{j=1}^{7} a_{ij}\sin\left(b_{ij}t + c_{ij}\right) \quad (i = 1, 2, 3) \tag{9.54}$$

where $X_1 = X$, $X_2 = Y$, $X_3 = Z$, and t is the time in Julian centuries since J2000 (see Eq. 5.49),

$$t = \frac{JD - 2,451,545}{36,525}\,(\text{centuries}) \tag{9.55}$$

and JD is the Julian date (in days). The components of the 3-by-7 matrices $[\mathbf{a}]$, $[\mathbf{b}]$, and $[\mathbf{c}]$ are

$$[\mathbf{a}] = \begin{bmatrix} 383,000 & 31,500 & 10,600 & 6,200 & 3,200 & 2,300 & 800 \\ 351,000 & 28,900 & 13,700 & 9,700 & 5,700 & 2,900 & 2,100 \\ 153,200 & 31,500 & 12,500 & 4,200 & 2,500 & 3,000 & 1,800 \end{bmatrix} (\text{km})$$

$$[\mathbf{b}] = \begin{bmatrix} 8399.685 & 70.990 & 16728.377 & 1185.622 & 7143.070 & 15613.745 & 8467.263 \\ 8399.687 & 70.997 & 8433.466 & 16728.380 & 1185.667 & 7143.058 & 15613.755 \\ 8399.672 & 8433.464 & 70.996 & 16728.364 & 1185.645 & 104.881 & 8399.116 \end{bmatrix} (\text{rad/century})$$

$$[\mathbf{c}] = \begin{bmatrix} 5.381 & 6.169 & 1.453 & 0.481 & 5.017 & 0.857 & 1.010 \\ 3.811 & 4.596 & 4.766 & 6.165 & 5.164 & 0.300 & 5.565 \\ 3.807 & 1.629 & 4.595 & 6.162 & 5.167 & 2.555 & 6.248 \end{bmatrix} (\text{rad})$$

The moon's geocentric equatorial velocity components $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ are found by simply differentiating Eq. (9.53) with respect to time,

$$\dot{X}_i = \frac{1}{t_C}\sum_{j=1}^{7} a_{ij}b_{ij}\cos\left(b_{ij}t + c_{ij}\right) \quad (i = 1, 2, 3)$$

The conversion factor t_C is required to convert kilometers per century to kilometers per second

$$t_C = \left(36,525\frac{\text{d}}{\text{cy}}\right)\cdot\left(24\frac{\text{h}}{\text{d}}\right)\cdot\left(3600\frac{\text{s}}{\text{h}}\right) = 3.15576\left(10^9\right)\frac{\text{s}}{\text{cy}}$$

A MATLAB implementation of Simpson's lunar ephemeris is listed in Appendix D.37.

EXAMPLE 9.2

Use Simpson's ephemeris to find the variation of the moon's inclination to the earth's equatorial plane from January 1, 2000, UT 12:00:00 through January 1, 2100, UT 12:00:00.

Solution

The following MATLAB script computes and plots the inclinations.

```
clear all; close all; clc
%
%...Initial and final Julian dates (jd1 & jd2):
jd1  = julian_day(2000,1,1,12,0,0); %January 1,2000 UT 12:00:00
jd2  = julian_day(2100,1,1,12,0,0); %January 1,2100 UT 12:00:00
%
%...The following initially empty column vectors will contain the lunar
%   inclinations and the Julian day for which each one is evaluated:
incl = [];
days = [];
%
%...Loop through the sequence of Julian days, one at a time:
for jd = jd1 : jd2 + 20*365.25
    days    = [days; jd - jd1];             %Store the elapsed
%                                              time in the
%                                              days array.
    [r_,v_] = simpsons_lunar_ephemeris(jd); %Compute the
%                                              position and
%                                              velocity vectors,
%                                              r_ and v_.
    h_      = cross(r_,v_);                 %h_ is the angular
%                                              momentum vector.
    h       = norm(h_);                     %h is its magnitude.
    incl    = [incl; acosd(h_(3)/h)];       %Compute the
%                                              inclination from
%                                              Eq. 4.7 and add
%                                              it to the
%                                              incl array.
end
%
%...Smooth out the high frequency content with MATLAB's
%   smoothdata function using a Savitzky-Golay filter:
```

```
incl_smooth = smoothdata(incl, 'sgolay');
%
%...Plot the computed values of incl against the elapsed time
%   in years:
plot(days/365.25,incl_smooth,'b') %(365.25 days per Julian year)
title('Variation of lunar inclination from J2000 to J2100')
axis([0 100 15 30])
xlabel('Elapsed time, years')
ylabel('Inclination, degrees')
grid on
hold on
plot([0 100],[28.5 28.5], '--r')
plot([0 100],[18.4 18.4], '--r')
```

Fig. 9.8 shows the output of the above script. Clearly, the angle between the moon's orbital plane and the earth's equator varies from 18.4° to 28.5° over a period of 18.6 years.

The moon's orbital plane is inclined to the ecliptic plane by an angle (obliquity) of 5.14°, whereas the earth's obliquity is 23.4°. Due primarily to solar gravity, the moon's orbital plane precesses

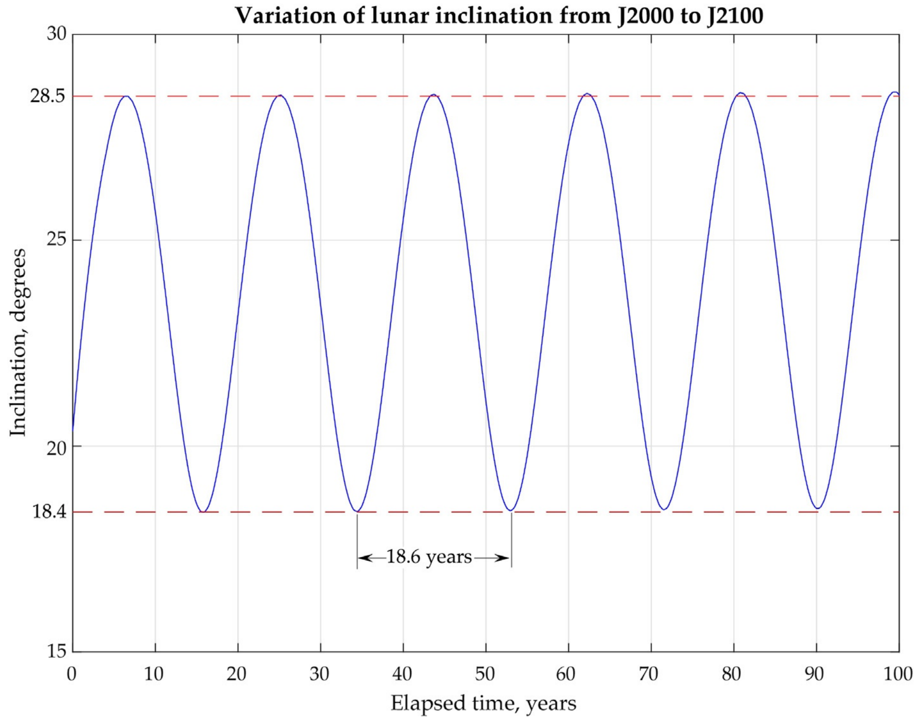

FIG. 9.8

Variation of the moon's orbital inclination with time (Simpson's ephemeris).

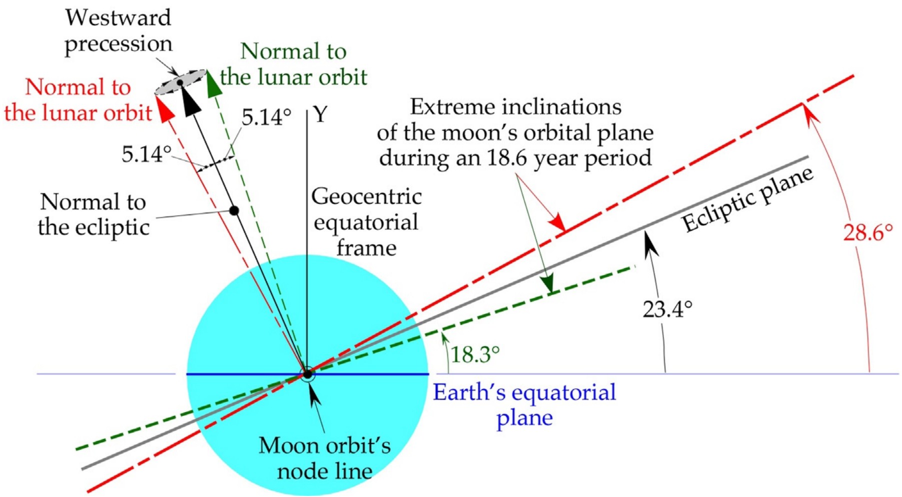

FIG. 9.9

Precession of the lunar orbit around the normal to the ecliptic plane.

Table 9.1 Apollo translunar orbit inclinations and lunar orbit inclination at the time

Manned lunar mission	Date	Orbit inclination (°)	Moon's inclination (°)
Apollo 8	December 21–27, 1968	30.6	28.4
Apollo 10	May 18–26, 1969	31.7	28.5
Apollo 11	July 16–24, 1969	31.4	28.5
Apollo 12	November 14–24, 1969	30.4	28.4
Apollo 13	April 11–17, 1970	31.8	28.4
Apollo 14	January 31–February 9, 1971	30.8	27.9
Apollo 15	July 26–August 7, 1971	29.7	27.4
Apollo 16	April 16–27, 1972	32.5	26.4
Apollo 17	December 7–19, 1972	28.5	25.5

westward at the rate of one revolution in 18.6 years. The earth's orbital plane is stationary in comparison (one revolution per 26,000 years). Thus, during the course of 18.6 years, the inclination of the moon's orbit to the earth's equatorial plane varies between (23.4° − 5.14°) and (23.4° + 5.14°), as revealed in Fig. 9.8 and further illustrated in Fig. 9.9.

According to Eq. (6.24), launching a spacecraft due east (azimuth $A = 90°$), to take full advantage of the earth's eastward rotational velocity, yields an orbit whose inclination equals the latitude ϕ of the launch site. Since the latitude of Kennedy Space Center (KSC) is 28.5°N, the smallest orbital inclination i of a launch from that site is 28.5°. Therefore, a coplanar lunar mission from KSC, like that described in Example 9.1, can only occur during that part of the 18.6-year cycle when the moon's inclination is at or near 28.5°. Table 9.1 compares the orbital inclination of each Apollo lunar mission (Orloff, 2000) with the moon's inclination at the time, which is obtained from the JPL ephemeris.

9.4 PATCHED CONIC LUNAR TRAJECTORIES IN THREE DIMENSIONS

In Section 9.2 we cast the procedure for coplanar patched conic lunar trajectory analysis in vector format. Therefore, the notation and procedure for three dimensions remains essentially the same. We simply drop the assumption that the paths of the spacecraft and the moon lie in the same plane. We obtain the position and velocity of the moon from an accurate lunar ephemeris, instead of assuming that the moon moves around the earth in a circular path. The plane of the translunar trajectory is not that of the moon's orbit, but is determined by the spacecraft's position vector $\mathbf{r}_0$ at TLI and the position vector $\mathbf{r}_m$ of the moon when the spacecraft crosses into the moon's SOI. We continue to assume that the motion of the spacecraft after TLI is ballistic, which means there are no midcourse propulsive maneuvers prior to lunar encounter. As in Section 9.2, let us do the trajectory analysis in four parts.

I. After TLI, travel a ballistic geocentric trajectory until entering the moon's SOI.
II. Determine the spacecraft trajectory inside the moon's SOI, relative to the rotating moon-fixed frame.
III. Transform the trajectory in II into the inertial geocentric equatorial frame.
IV. If flyby occurs, then determine the geocentric trajectory after departing the moon's SOI.

We will use a numerical example and simply outline the procedure, since the details were mostly covered in Section 9.2.

I. *Translunar trajectory up to the moon's SOI* (Fig. 9.10)
Recall that $\mu_e = 398,600\,\text{km}^3/\text{s}^2$.
   1. Choose values for the independent variables of the problem.
      a. Select the date for the moon's position at SOI intercept.
         May 4, 2020, 12:00:00 UT. Julian day: 2,458,974.
      b. Select the value for the arrival angle λ (i.e., the angle between the radials from moon to earth and from moon to spacecraft at the moon's SOI) (see Fig. 9.10).

$$\lambda = 50°$$

      c. Select the probe's radius r_0, right ascension α_L, declination δ_L, and flight path angle γ_0 at TLI:

$$r_0 = 6698\,\text{km} \quad \alpha_L = 40° \quad \delta_L = 10° \quad \gamma_0 = 10°$$

   2. Use an ephemeris to determine the moon's geocentric equatorial state vector $(\mathbf{r}_m, \mathbf{v}_m)$ on the date in I.1.a.

$$\mathbf{r}_m = -359{,}984\hat{\mathbf{I}} - 25{,}810.2\hat{\mathbf{J}} + 22{,}885.4\hat{\mathbf{K}}\,(\text{km}) \qquad r_m = 361{,}835\,\text{km}$$
$$\mathbf{v}_m = 0.0805809\hat{\mathbf{I}} - 0.990137\hat{\mathbf{J}} - 0.437526\hat{\mathbf{K}}\,(\text{km/s}) \qquad v_m = 1.08558\,\text{km/s}$$

   3. Calculate $\hat{\mathbf{s}}$, the unit vector along the earth-to-moon radial:

$$\hat{\mathbf{s}} = \frac{\mathbf{r}_m}{\|\mathbf{r}_m\|} \qquad \therefore \hat{\mathbf{s}} = -0.994882\hat{\mathbf{I}} - 0.0787934\hat{\mathbf{J}} + 0.0632482\hat{\mathbf{K}}$$

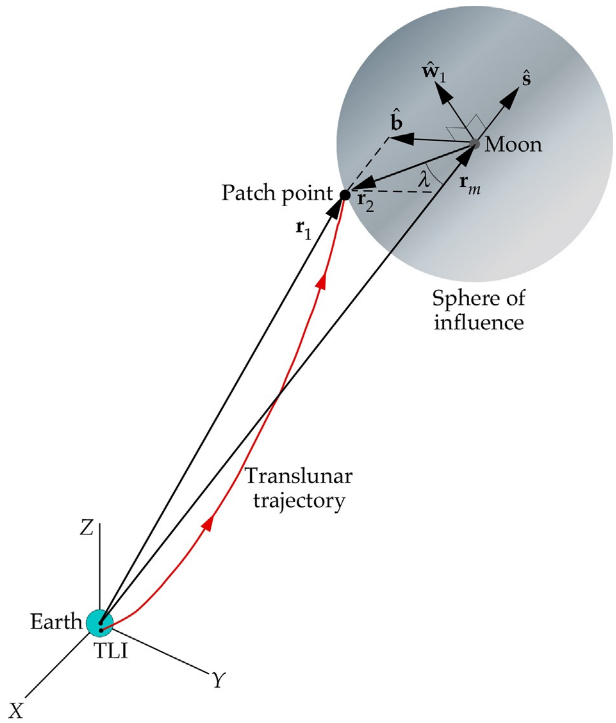

FIG. 9.10

Translunar trajectory up to encounter of moon's sphere of influence.

4. Calculate $\boldsymbol{\omega}_m$, the instantaneous angular velocity of the moon at the time of SOI intercept:

$$\boldsymbol{\omega}_m = \frac{\mathbf{r}_m \times \mathbf{v}_m}{{r_m}^2} \qquad \therefore \boldsymbol{\omega}_m = \left(0.268368\hat{\mathbf{I}} - 1.18891\hat{\mathbf{J}} + 2.740245\hat{\mathbf{K}}\right)\left(10^{-6}\right)(\text{rad/s})$$

$$\omega_m = 2.99908\left(10^{-6}\right)\text{rad/s}$$

5. Calculate the geocentric position vector $\mathbf{r}_0$ at TLI using Eqs. (4.4) and (4.5) and the data in I.1.c:

$$\mathbf{r}_0 = r_0\left(\cos\alpha_L\cos\delta_L\hat{\mathbf{I}} + \sin\alpha_L\cos\delta_L\hat{\mathbf{J}} + \sin\delta_L\hat{\mathbf{K}}\right) = 5053.02\hat{\mathbf{I}} + 4239.98\hat{\mathbf{J}} + 1163.10\hat{\mathbf{K}}\,(\text{km})$$
$$(r_0 = 6698\,\text{km})$$

6. Calculate $\hat{\mathbf{w}}_1$, the unit normal to the plane of the translunar trajectory:

$$\hat{\mathbf{w}}_1 = \frac{\mathbf{r}_0 \times \mathbf{r}_m}{\|\mathbf{r}_0 \times \mathbf{r}_m\|} \qquad \therefore \hat{\mathbf{w}}_1 = 0.0875163\hat{\mathbf{I}} - 0.359180\hat{\mathbf{J}} + 0.929156\hat{\mathbf{K}}$$

7. Calculate $\hat{\mathbf{b}}$, the unit normal to the plane of $\hat{\mathbf{s}}$ and $\hat{\mathbf{w}}_1$. $\hat{\mathbf{b}}$ lies in the plane of the translunar trajectory (see Fig. 9.10):

$$\hat{\mathbf{b}} = \frac{\hat{\mathbf{w}}_1 \times \hat{\mathbf{s}}}{\|\hat{\mathbf{w}}_1 \times \hat{\mathbf{s}}\|} \qquad \therefore \hat{\mathbf{b}} = 0.0504938\hat{\mathbf{I}} - 0.929936\hat{\mathbf{J}} - 0.364238\hat{\mathbf{K}}$$

8. Calculate $\hat{\mathbf{n}}$, the unit vector from the center of the moon to the SOI patch point:

$$\hat{\mathbf{n}} = -\cos\lambda\hat{\mathbf{s}} + \sin\lambda\hat{\mathbf{b}} \quad \therefore \hat{\mathbf{n}} = 0.678179\hat{\mathbf{I}} - 0.661725\hat{\mathbf{J}} - 0.319678\hat{\mathbf{K}}$$

9. Calculate $\mathbf{r}_2$, the position vector of the patch point relative to the moon:

$$\mathbf{r}_2 = R_S\hat{\mathbf{n}} \quad \therefore \mathbf{r}_2 = 44{,}883.7\hat{\mathbf{I}} - 43{,}794.8\hat{\mathbf{J}} - 21{,}157.2\hat{\mathbf{K}}\,(\text{km})$$
$$r_2 = 66{,}183\,\text{km}$$

10. Calculate $\mathbf{r}_1$, the position vector of the patch point relative to the earth:

$$\mathbf{r}_1 = \mathbf{r}_m + \mathbf{r}_2 \quad \therefore \mathbf{r}_1 = -315{,}100\hat{\mathbf{I}} - 72{,}305.0\hat{\mathbf{J}} + 1728.29\hat{\mathbf{K}}\,(\text{km})$$
$$r_1 = 323{,}294\,\text{km}$$

11. Use Eq. (9.14) to calculate the sweep angle $\Delta\theta$:

$$\cos\Delta\theta = (\mathbf{r}_0/r_0)\cdot(\mathbf{r}_1/r_1) \quad \Rightarrow \quad \Delta\theta = 151.156^\circ$$

12. Calculate the angular momentum h_1 of the translunar trajectory using Eq. (9.18):

$$h_1 = 71{,}426.1\,\text{km}^2/\text{s}$$

13. Calculate the Lagrange coefficients f, g, and $\dot{g}$ from Eqs. (9.13a)–(9.13c):

$$f = -46.3848 \quad g = 14625.9\,\text{s} \quad \dot{g} = 0.0182827$$

14. Calculate the velocity $\mathbf{v}_0$ at TLI and the velocity $\mathbf{v}_1$ at the patch point by means of Eqs. (9.12a) and (9.12b):

$$\mathbf{v}_0 = -5.51878\hat{\mathbf{I}} + 8.503129\hat{\mathbf{J}} + 3.80683\hat{\mathbf{K}}\,(\text{km/s})$$
$$\mathbf{v}_1 = -0.739368\hat{\mathbf{I}} - 0.380279\hat{\mathbf{J}} - 0.0773628\hat{\mathbf{K}}\,(\text{km/s})$$

15. Calculate the radial component of velocity v_{r_0} at TLI:

$$v_{r_0} = \mathbf{v}_0\cdot(\mathbf{r}_0/r_0) \quad \therefore v_{r_0} = 1.88031\,\text{km/s}\,(\text{positive})$$

16. Using the TLI state vector $(\mathbf{r}_0, \mathbf{v}_0)$, calculate the eccentricity vector $\mathbf{e}_1$ of the translunar trajectory from Eq. (2.40). The eccentricity, $e_1 = \|\mathbf{e}_1\|$, must be less than 1:

$$\mathbf{e}_1 = \frac{\mathbf{v}_0\times(\mathbf{r}_0\times\mathbf{v}_0)}{\mu_e} - \frac{\mathbf{r}_0}{r_0} \quad \therefore \mathbf{e}_1 = 0.906360\hat{\mathbf{I}} + 0.345541\hat{\mathbf{J}} + 0.0482052\hat{\mathbf{K}}$$
$$e_1 = 0.971190$$

17. Calculate the semimajor axis a_1 and the period T_1 of the translunar trajectory from Eqs. (9.24) and (9.25), respectively:

$$a_1 = 225{,}375\,\text{km} \quad T_1 = 1{,}064{,}806\,\text{s} = 12.3241\,\text{d}$$

18. Calculate the triad of perifocal unit vectors $\hat{\mathbf{p}}_1$, $\hat{\mathbf{q}}_1$, and $\hat{\mathbf{w}}_1$ for the translunar trajectory:

$$\hat{\mathbf{p}}_1 = \frac{\mathbf{e}_1}{e_1} \quad \therefore \hat{\mathbf{p}}_1 = 0.933246\hat{\mathbf{I}} + 0.355791\hat{\mathbf{J}} + 0.0496352\hat{\mathbf{K}}$$
$$\hat{\mathbf{w}}_1 = 0.0875163\hat{\mathbf{I}} - 0.359180\hat{\mathbf{J}} + 0.929156\hat{\mathbf{K}} \quad \text{(Calculated in Step I.6 above.)}$$
$$\hat{\mathbf{q}}_1 = \hat{\mathbf{w}}_1\times\hat{\mathbf{p}}_1 \quad \therefore \hat{\mathbf{q}}_1 = -0.348413\hat{\mathbf{I}} + 0.862788\hat{\mathbf{J}} + 0.366341\hat{\mathbf{K}}$$

19. Calculate the true anomaly θ_0 at the TLI point using Eq. (9.27) and noting from Step I.16 that $v_{r_0} > 0$ at TLI:

$$\theta_0 = 20.2998°$$

20. Calculate the time t_0 since perigee at the TLI point using Eq. (9.28):

$$t_0 = 213.532\,\text{s}$$

21. Calculate the true anomaly θ_1 at the patch point, $\theta_1 = \theta_0 + \Delta\theta$, where we found the sweep angle $\Delta\theta$ in Step I.11:

$$\theta_1 = 20.2998° + 151.156° = 171.455°$$

22. Calculate the time t_1 since perigee at the patch point using Eq. (9.30):

$$t_1 = 54.8899\,\text{h}$$

23. Calculate the flight time Δt_1 from TLI to the patch point, $\Delta t_1 = t_1 - t_0$:

$$\Delta t_1 = 54.8306\,\text{h}$$

II. *Determine the lunar approach trajectory within the moon's SOI, relative to the moon.* Recall that the gravitational parameter and the radius of the moon are $\mu_m = 4902.8\,\text{km}^3/\text{s}^2$ and $R_m = 1727$ km, respectively.

1. Calculate the velocity $\mathbf{v}_2$ of the spacecraft relative to the moon at the patch point:

$$\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_m \qquad \therefore \mathbf{v}_2 = -0.819949\hat{\mathbf{I}} + 0.609957\hat{\mathbf{J}} + 0.360164\hat{\mathbf{K}}\,(\text{km/s})$$
$$v_2 = 1.08355\,\text{km/s}$$

$\mathbf{v}_m$ was obtained in Step I.2.

2. Calculate the radial speed v_{r_2} relative to the moon at the patch point:

$$v_{r_2} = \mathbf{v}_2 \cdot (\mathbf{r}_2/r_2) \qquad \therefore v_{r_2} = -1.07483\,\text{km/s} \quad (\mathbf{r}_2 \text{ was found in Step I.9})$$

3. Calculate the angular momentum $\mathbf{h}_2$ of the trajectory relative to the moon:

$$\mathbf{h}_2 = \mathbf{r}_2 \times \mathbf{v}_2 \qquad \therefore \mathbf{h}_2 = -2868.33\hat{\mathbf{I}} + 1182.29\hat{\mathbf{J}} - 8532.32\hat{\mathbf{J}}\left(\text{km}^2/\text{s}\right)$$
$$h_2 = 9078.86\,\text{km}^2/\text{s}$$

4. Use Eq. (2.40) to calculate the eccentricity vector $\mathbf{e}_2$ of the trajectory, relative to the moon. e_2 must be greater than 1:

$$\mathbf{e}_2 = \frac{1}{\mu_m}\left(\mathbf{v}_2 \times \mathbf{h}_2 - \frac{\mathbf{r}_2}{r_2}\right) \qquad \therefore \mathbf{e}_2 = -1.82654\hat{\mathbf{I}} - 0.975939\hat{\mathbf{J}} + 0.478800\hat{\mathbf{K}}$$
$$e_2 = 2.12554$$

5. Calculate the perilune radius r_{p_2} and altitude z_{p_2} of the hyperbolic lunar approach trajectory:

$$r_{p_2} = \frac{h_2{}^2}{\mu_m}\frac{1}{1+e_2} = 5378.89\,\text{km} \qquad \therefore z_{p_2} = r_{p_2} - R_m = 3641.9\,\text{km}$$

6. Calculate the perifocal unit vectors $\hat{\mathbf{p}}_2$, $\hat{\mathbf{q}}_2$, and $\hat{\mathbf{w}}_2$ of the hyperbolic lunar approach trajectory:

$$\hat{\mathbf{p}}_2 = \frac{\mathbf{e}_2}{e_2} \qquad \therefore \hat{\mathbf{p}}_2 = -0.859326\hat{\mathbf{I}} - 0.459147\hat{\mathbf{J}} + 0.225260\hat{\mathbf{K}}$$

$$\hat{\mathbf{w}}_2 = \frac{\mathbf{h}_2}{h_2} \qquad \therefore \hat{\mathbf{w}}_2 = -0.315936\hat{\mathbf{I}} + 0.130224\hat{\mathbf{J}} - 0.939801\hat{\mathbf{K}}$$

$$\hat{\mathbf{q}}_2 = \hat{\mathbf{w}}_2 \times \hat{\mathbf{p}}_2 \quad \therefore \hat{\mathbf{q}}_2 = -0.402173\hat{\mathbf{I}} + 0.878763\hat{\mathbf{J}} + 0.256966\hat{\mathbf{K}}$$

7. Calculate the triad of orthogonal unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ directed along the rotating xyz moon-fixed Cartesian coordinate axes at the instant the spacecraft crosses the lunar SOI. Note that the z axis lies in the direction of the moon's angular velocity vector $\boldsymbol{\omega}_m$, which we computed in Step I.4.

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_m}{r_m} \qquad \therefore \hat{\mathbf{i}} = -0.994882\hat{\mathbf{I}} - 0.0787934\hat{\mathbf{J}} + 0.0632482\hat{\mathbf{K}}$$

$$\hat{\mathbf{k}} = \frac{\boldsymbol{\omega}_m}{\omega_m} \qquad \therefore \hat{\mathbf{k}} = 0.0894832\hat{\mathbf{I}} - 0.396426\hat{\mathbf{J}} + 0.913695\hat{\mathbf{K}}$$

$$\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} \qquad \therefore \hat{\mathbf{j}} = 0.0469199\hat{\mathbf{I}} - 0.914679\hat{\mathbf{J}} - 0.401448\hat{\mathbf{K}}$$

8. Use the above geocentric equatorial components of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ to calculate the instantaneous direction cosine matrix $[\mathbf{Q}]$ of the transformation from the geocentric equatorial XYZ frame to the moon-fixed xyz frame (see Eq. 4.23):

$$[\mathbf{Q}] = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \quad \therefore [\mathbf{Q}] = \begin{bmatrix} -0.994882 & -0.0787934 & 0.0632482 \\ 0.0469199 & -0.914679 & -0.401448 \\ 0.0894832 & -0.396426 & 0.913695 \end{bmatrix}$$

9. Calculate the components of the perifocal unit vectors $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{q}}_2$ (calculated in the XYZ frame in II.6) in the rotating xyz moon-fixed frame:

$$\{\hat{\mathbf{p}}_2\}_{xyz} = [\mathbf{Q}]\{\hat{\mathbf{p}}_2\}_{XYZ} \quad \therefore \{\hat{\mathbf{p}}_2\}_{xyz} = 0.905354\hat{\mathbf{i}} + 0.289223\hat{\mathbf{j}} + 0.310942\hat{\mathbf{k}}$$

$$\{\hat{\mathbf{q}}_2\}_{xyz} = [\mathbf{Q}]\{\hat{\mathbf{q}}_2\}_{XYZ} \quad \therefore \{\hat{\mathbf{q}}_2\}_{xyz} = 0.347127\hat{\mathbf{i}} - 0.925815\hat{\mathbf{j}} - 0.149563\hat{\mathbf{k}}$$

10. Calculate the time t_2 ($t_2 < 0$) at the patch point on the hyperbolic approach trajectory using Eqs. (9.41) and (9.42):

$$t_2 = -15.8112\,\text{h}$$

The elapsed time from the patch point to perilune is $\Delta t_2 = t_{perilune} - t_2 = 0 - t_2$. The total time from TLI to perilune is $\Delta t = \Delta t_1 + \Delta t_2$. If the spacecraft proceeds from perilune on a flyby, it will exit the SOI at the time $t_3 = -t_2 = 15.8112$ h by virtue of the symmetry of the flyby hyperbola, which is illustrated in Fig. 9.11.

III. *At any time t within the moon's SOI ($t_2 \le t \le t_3$), evaluate the spacecraft's geocentric equatorial position vector **r** and velocity vector **v**.* For example, let us choose $t = 0$ (perilune).

1. For the time t, use a lunar ephemeris to calculate the moon's position vector $\mathbf{r}_m$ and velocity vector $\mathbf{v}_m$ relative to the geocentric equatorial frame.

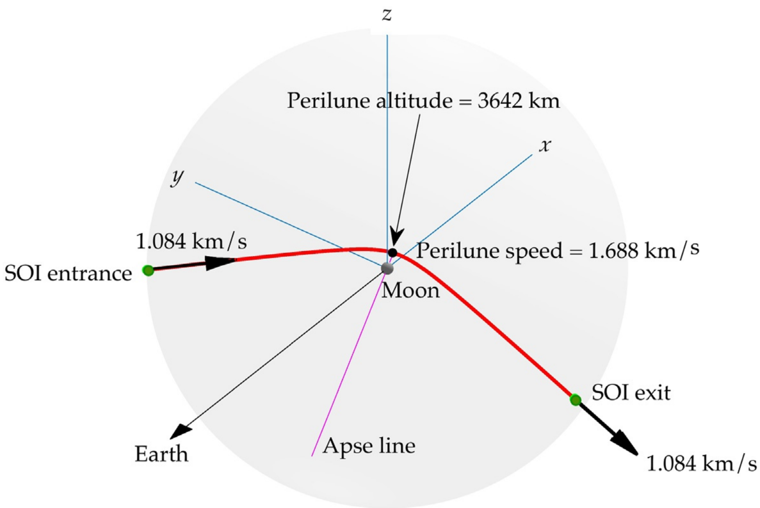

FIG. 9.11

Hyperbolic trajectory within the sphere of influence.

The perilune time $t = 0$ is 15.8112 h after passing the patch point, which means the Julian date at perilune is 2,458,974.66 and for that time the JPL Horizons ephemeris yields:

$$\mathbf{r}_m = -350,457\hat{\mathbf{I}} - 84,251.0\hat{\mathbf{J}} - 2235.54\hat{\mathbf{K}}\,(\mathrm{km}) \qquad \therefore r_m = 360,448\,\mathrm{km}$$
$$\mathbf{v}_m = 0.253698\hat{\mathbf{I}} - 0.963737\hat{\mathbf{J}} - 0.443114\hat{\mathbf{K}}\,(\mathrm{km/s}) \qquad \therefore v_m = 1.09064\,\mathrm{km/s}$$

2. Calculate the angular velocity of the moon, $\boldsymbol{\omega}_m = (\mathbf{r}_m \times \mathbf{v}_m)/{r_m}^2$, at the instant t:

$$\boldsymbol{\omega}_m = \left(0.270763\hat{\mathbf{I}} - 1.19963\hat{\mathbf{J}} + 2.76412\hat{\mathbf{K}}\right)\left(10^{-6}\right)(\mathrm{rad/s})$$
$$\therefore \omega_m = 3.02535\left(10^{-6}\right)\mathrm{rad/s}$$

3. Calculate the triad of orthogonal unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ directed along the rotating xyz moon-fixed Cartesian coordinate axes at the instant the spacecraft is at perilune:

$$\begin{aligned}
&\hat{\mathbf{i}} = \mathbf{r}_m/r_m && \hat{\mathbf{i}} = -0.972280\hat{\mathbf{I}} - 0.233739\hat{\mathbf{J}} - 0.00620221\hat{\mathbf{K}}\\
&\hat{\mathbf{k}} = \boldsymbol{\omega}_m/\omega_m && \hat{\mathbf{k}} = 0.0894979\hat{\mathbf{I}} - 0.396525\hat{\mathbf{J}} + 0.913651\hat{\mathbf{K}}\\
&\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} && \hat{\mathbf{j}} = 0.216015\hat{\mathbf{I}} - 0.887769\hat{\mathbf{J}} - 0.406453\hat{\mathbf{K}}
\end{aligned}$$

4. Use the geocentric equatorial components of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ to calculate the instantaneous direction cosine matrix $[\mathbf{Q}]$ of the transformation from the geocentric equatorial XYZ frame to the moon-fixed xyz frame (see Eq. 4.23):

$$[\mathbf{Q}] = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \quad \therefore [\mathbf{Q}] = \begin{bmatrix} -0.972280 & -0.233739 & -0.00620212 \\ 0.216015 & -0.887769 & -0.406453 \\ 0.0894979 & -0.396525 & 0.913651 \end{bmatrix}$$

5. Calculate the components of the perifocal unit vectors $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{q}}_2$ (found in the *XYZ* frame in II.6) in the rotating *xyz* moon-fixed frame:

$$\{\hat{\mathbf{p}}_2\}_{xyz} = [\mathbf{Q}]\{\hat{\mathbf{p}}_2\}_{XYZ} \quad \therefore \{\hat{\mathbf{p}}_2\}_{xyz} = 0.905354\hat{\mathbf{i}} + 0.289223\hat{\mathbf{j}} + 0.310942\hat{\mathbf{k}}$$
$$\{\hat{\mathbf{q}}_2\}_{xyz} = [\mathbf{Q}]\{\hat{\mathbf{q}}_2\}_{XYZ} \quad \therefore \{\hat{\mathbf{q}}_2\}_{xyz} = 0.347127\hat{\mathbf{i}} - 0.925815\hat{\mathbf{j}} - 0.149563\hat{\mathbf{k}}$$

6. Determine the true anomaly θ at the time t, which is perilune in this example, so that $t = 0$:

$$M = \frac{\mu_m{}^2}{h_2{}^3}\left(e_2{}^2 - 1\right)^{3/2} t \qquad M = \frac{\mu_m{}^2}{h_2{}^3}\left(e_2{}^2 - 1\right)^{3/2}(0) \qquad \therefore M = 0$$
$$e_2 \sinh F - F = M \qquad e_2 \sinh F - F = 0 \qquad \therefore F = 0$$
$$\theta = 2\tan^{-1}\left(\sqrt{\frac{e_2+1}{e_2-1}}\tanh\frac{F}{2}\right) \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{e_2+1}{e_2-1}}\tanh(0)\right) \qquad \therefore \theta = 0$$

7. Calculate the components of the position vector $\mathbf{r}_{\text{rel}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and the velocity vector $\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$ of the spacecraft relative to the moon-fixed *xyz* frame:

$$\begin{aligned}
\{\mathbf{r}_{\text{rel}}\}_{xyz} &= \frac{h_2{}^2}{\mu_m}\frac{1}{1 + e_2\cos\theta}\left(\cos\theta\{\hat{\mathbf{p}}_2\}_{xyz} + \sin\theta\{\hat{\mathbf{q}}_2\}_{xyz}\right) \\
&= \frac{9078.86^2}{4902.8}\frac{1}{1 + 2.12554\cos(0)}\left(\cos(0)\{\hat{\mathbf{p}}_2\}_{xyz} + \sin(0)\{\hat{\mathbf{q}}_2\}_{xyz}\right) \\
&= 5378.90\{\hat{\mathbf{p}}_2\}_{xyz} \\
\therefore \mathbf{r}_{\text{rel}} &= 5378.90\left(0.905354\hat{\mathbf{i}} + 0.289223\hat{\mathbf{j}} + 0.310942\hat{\mathbf{k}}\right) \\
&= 4869.80\hat{\mathbf{i}} + 1555.70\hat{\mathbf{j}} + 1672.52\hat{\mathbf{k}}\ (\text{km}) \\
\{\mathbf{v}_{\text{rel}}\}_{xyz} &= -\frac{\mu_m}{h_2}\sin\theta\{\hat{\mathbf{p}}_2\}_{xyz} + \frac{\mu_m}{h_2}[e_2 + \cos\theta]\{\hat{\mathbf{q}}_2\}_{xyz} \\
&= -\frac{4902.8}{9078.86}\sin(0)\{\hat{\mathbf{p}}_2\}_{xyz} + \frac{4902.8}{9078.86}[2.12554 + \cos(0)]\{\hat{\mathbf{q}}_2\}_{xyz} \\
&= 1.68787\{\hat{\mathbf{q}}_2\}_{xyz} \\
\therefore \mathbf{v}_{\text{rel}} &= 1.68786\left(0.347127\hat{\mathbf{i}} - 0.925815\hat{\mathbf{j}} - 0.149563\hat{\mathbf{k}}\right) \\
&= 0.585904\hat{\mathbf{i}} - 1.56265\hat{\mathbf{j}} - 0.252443\hat{\mathbf{k}}\ (\text{km/s})
\end{aligned}$$

8. Obtain the components of the relative position vector $\mathbf{r}_{\text{rel}}$ and the relative velocity $\mathbf{v}_{\text{rel}}$ in the geocentric equatorial *XYZ* frame:

$$\{\mathbf{r}_{\text{rel}}\}_{XYZ} = [\mathbf{Q}]^T\{\mathbf{r}_{\text{rel}}\}_{xyz} = \begin{bmatrix} -0.972280 & 0.2160154 & 0.0894979 \\ -0.233739 & -0.887769 & -0.396525 \\ -0.00620212 & -0.4064527 & 0.913651 \end{bmatrix}\begin{Bmatrix} 4869.81 \\ 1555.70 \\ 1672.53 \end{Bmatrix}$$
$$\therefore \mathbf{r}_{\text{rel}} = -4249.06\hat{\mathbf{I}} - 3182.56\hat{\mathbf{J}} + 865.578\hat{\mathbf{K}}\ (\text{km})$$
$$\{\mathbf{v}_{\text{rel}}\}_{XYZ} = [\mathbf{Q}]^T\{\mathbf{v}_{\text{rel}}\}_{xyz} = \begin{bmatrix} -0.972280 & 0.2160154 & 0.0894979 \\ -0.233739 & -0.887769 & -0.396525 \\ -0.00620212 & -0.4064527 & 0.913651 \end{bmatrix}\begin{Bmatrix} 0.585904 \\ -1.56265 \\ -0.252442 \end{Bmatrix}$$
$$\therefore \mathbf{v}_{\text{rel}} = -0.929814\hat{\mathbf{I}} + 1.35043\hat{\mathbf{J}} + 0.400866\hat{\mathbf{K}}\ (\text{km/s})$$

9. Calculate the absolute position vector **r** and the absolute velocity vector **v** of the spacecraft in the geocentric equatorial *XYZ* frame:

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}_m + \mathbf{r}_{\mathrm{rel}} \\
&= \left(-350,457\hat{\mathbf{I}} - 84,251.0\hat{\mathbf{J}} - 2235.54\hat{\mathbf{K}}\right) + \left(-4249.06\hat{\mathbf{I}} - 3182.56\hat{\mathbf{J}} + 865.578\hat{\mathbf{K}}\right) \\
&= -354,706\hat{\mathbf{I}} - 87433.6\hat{\mathbf{J}} - 1369.97\hat{\mathbf{K}}\,(\mathrm{km}) \\
r &= 365,325\,\mathrm{km}
\end{aligned}$$

For the velocity we must use Eq. (9.51) instead of (9.52), because the radius of the moon's orbit is not constant as we assumed previously (which means $\mathbf{v}_m \neq \boldsymbol{\omega}_m \times \mathbf{r}_m$):

$$\overbrace{\mathbf{v}}^{\substack{\text{absolute}\\ \text{velocity of probe}}} = \overbrace{\mathbf{v}_m}^{\substack{\text{velocity of the}\\ \text{moon}}} + \overbrace{\boldsymbol{\omega}_m}^{\substack{\text{angular velocity}\\ \text{of the moon}}} \times \overbrace{\mathbf{r}_{\mathrm{rel}}}^{\substack{\text{position vector of}\\ \text{probe relative to the moon}}} + \overbrace{\mathbf{v}_{\mathrm{rel}}}^{\substack{\text{velocity of the probe}\\ \text{relative to the moon}}}$$

$\mathbf{v}_m$ and $\boldsymbol{\omega}_m$ are given in Steps III.1 and III.2, respectively, whereas $\mathbf{r}_{\mathrm{rel}}$ and $\mathbf{v}_{\mathrm{rel}}$ were calculated in Step III.8. Making these substitutions and noting that

$$\begin{aligned}
\boldsymbol{\omega}_m \times \mathbf{r}_{\mathrm{rel}} &= \left[\left(0.270763\hat{\mathbf{I}} - 1.19963\hat{\mathbf{J}} + 2.76412\hat{\mathbf{K}}\right)\left(10^{-6}\right)\right] \times \left(-4249.06\hat{\mathbf{I}} - 3182.56\hat{\mathbf{J}} + 865.578\hat{\mathbf{K}}\right) \\
&= 0.00775859\hat{\mathbf{I}} - 0.0119792\hat{\mathbf{J}} - 0.00595901\hat{\mathbf{K}}\,(\mathrm{km/s})
\end{aligned}$$

we get

$$\begin{aligned}
\mathbf{v} = {} & \overbrace{\left(0.253698\hat{\mathbf{I}} - 0.963737\hat{\mathbf{J}} - 0.443114\hat{\mathbf{K}}\right)}^{\mathbf{v}_m} + \overbrace{\left(0.00775859\hat{\mathbf{I}} - 0.0119792\hat{\mathbf{J}} - 0.00595901\hat{\mathbf{K}}\right)}^{\boldsymbol{\omega}_m \times \mathbf{r}_{\mathrm{rel}}} \\
& + \overbrace{\left(-0.929814\hat{\mathbf{I}} + 1.35043\hat{\mathbf{J}} + 0.400866\hat{\mathbf{K}}\right)}^{\mathbf{v}_{\mathrm{rel}}}
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{v} &= -0.668357\hat{\mathbf{I}} + 0.374711\hat{\mathbf{J}} - 0.0482070\hat{\mathbf{K}}\,(\mathrm{km/s}) \\
\mathbf{v} &= 0.767746\,\mathrm{km/s}
\end{aligned}$$

Repeating the calculations of **r** in Steps III.1 through III.9 for a sufficient number of times between $t_2 = -15.81$ h and $t_3 = +15.81$ h and "connecting the dots" yields the three-dimensional trace of the position vector **r** within the moon's SOI, relative to the earth. This is the portion of the curve in Fig. 9.12 between "Enter SOI" and "Exit SOI".

IV. *Determine the spacecraft's geocentric orbit upon departing the SOI after a lunar flyby.*

1. Evaluate the geocentric equatorial state vector $(\mathbf{r}_3, \mathbf{v}_3)$ at t_3 (the SOI exit), and use Algorithm 4.2 to determine the geocentric orbital elements of the lunar departure trajectory.
2. If earth is the return target, determine the delta-v required for an acceptable perigee. In this particular case, an in-track delta-v of –0.1225 km/s is required upon exiting the SOI to establish the return trajectory shown in Fig. 9.12. Just after that maneuver the state vector is

$$\begin{aligned}
\mathbf{r}_3 &= -357,478\hat{\mathbf{I}} - 77,874.4\hat{\mathbf{J}} - 16,825.7\hat{\mathbf{K}}\,(\mathrm{km}) \qquad & r_3 &= 366,249\,\mathrm{km} \\
\mathbf{v}_3 &= 0.0700313\hat{\mathbf{I}} + 0.0736792\hat{\mathbf{J}} - 0.1821388\hat{\mathbf{K}}\,(\mathrm{km/s}) \qquad & v_3 &= 0.208585\,\mathrm{km/s}
\end{aligned}$$

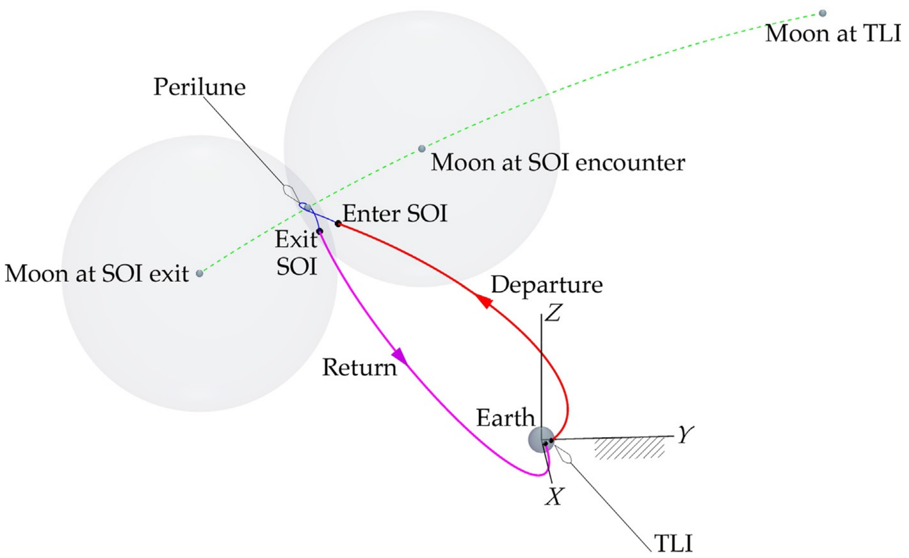

FIG. 9.12

Translunar injection followed by translunar coast, lunar flyaround, and transearth return.

and the orbital elements are

$$\theta_3 = 180.802° \quad e_3 = 0.965378 \quad h_3 = 71,192.1\,\text{km}^2/\text{s}$$
$$i_3 = 107.059° \quad \Omega_3 = 13.0981° \quad \omega_3 = 1.95243$$

Upon return to earth, the perigee altitude is 91.65 km, and its right ascension and declination are α_p = 12.52° and δ_p = 1.866°, respectively.

9.5 LUNAR TRAJECTORIES BY NUMERICAL INTEGRATION

In this section we will put aside the notion of sphere of influence and assume that a spacecraft moving within the earth–moon environment is always attracted to both bodies. For simplicity, we ignore the gravity of the sun as well as that of all other members of the solar system. This leaves us with a three-body system, as illustrated in Fig. 9.13.

The equations of motion of each member of a three-body system are given in Appendix C. As shown in Fig. 9.13, we choose the geocentric equatorial frame as our inertial reference and denote the position vector of the spacecraft by **r**. According to Section 10.10, it follows from Eq. (C.2) that the equation of motion of the spacecraft relative to the earth is Eq. (2.22), with a term **p** added to account for the acceleration due to lunar gravity. Thus,

$$\ddot{\mathbf{r}} = -\mu_e \frac{\mathbf{r}}{r^3} + \mathbf{p} \tag{9.57}$$

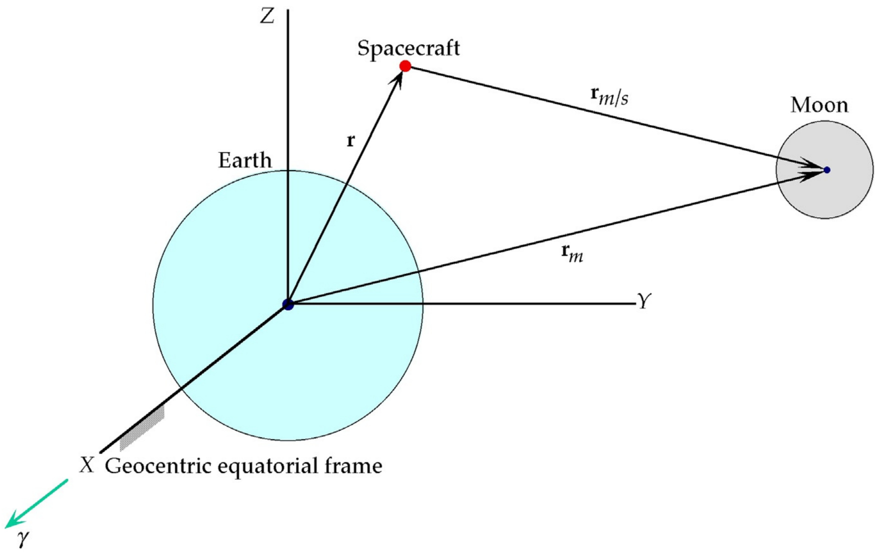

FIG. 9.13

Three-body system of earth, moon, and spacecraft.

where μ_e is earth's gravitational parameter. The moon's gravitational parameter μ_m appears in the term

$$\mathbf{p} = \mu_m \left(\frac{\mathbf{r}_{m/s}}{r_{m/s}{}^3} - \frac{\mathbf{r}_m}{r_m{}^3} \right) \tag{9.58}$$

As shown in Fig. 9.13, $\mathbf{r}_m$ is the position vector of the moon relative to the earth. $\mathbf{r}_{m/s}$ is the position vector of the moon relative to the spacecraft, so that

$$\mathbf{r}_{m/s} = \mathbf{r}_m - \mathbf{r} \tag{9.59}$$

$\mathbf{r}_m$ is obtained from an accurate lunar ephemeris, as discussed in Section 9.3. Therefore, unlike the circular restricted three-body problem in Section 2.12, we assume neither that the moon's orbit around the earth is circular nor that it is coplanar with the spacecraft's.

To numerically integrate Eq. (9.57), we follow the procedure introduced in Section 1.8 and rewrite the second-order differential equation as a system of first-order differential equations. To that end, let

$$\mathbf{y}_1 = \mathbf{r} \tag{9.60a}$$

$$\mathbf{y}_2 = \dot{\mathbf{r}} \tag{9.60b}$$

$\mathbf{y}_1$ and $\mathbf{y}_2$, the position and velocity vectors, form the state vector of the spacecraft,

$$\mathbf{y} = \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{Bmatrix} \tag{9.61}$$

It follows from Eqs. (9.60a) and (9.60b) that

$$\dot{\mathbf{y}}_1 = \mathbf{y}_2$$

Differentiating Eq. (9.60b) and substituting Eq. (9.57) along with $\mathbf{r} = \mathbf{y}_1$, we find

$$\dot{\mathbf{y}}_2 = -\mu_e \frac{\mathbf{r}}{r^3} + \mathbf{p} = -\mu_e \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|^3} + \mathbf{p}$$

Using Eqs. (9.58), (9.59), and (9.60a), the expression for the lunar gravitational acceleration $\mathbf{p}$ becomes

$$\mathbf{p} = \mu_m \left(\frac{\mathbf{r}_m(t) - \mathbf{y}_1}{\|\mathbf{r}_m(t) - \mathbf{y}_1\|^3} - \frac{\mathbf{r}_m(t)}{r_m(t)^3} \right)$$

In summary, the two first-order, nonlinear differential equations of motion are

$$\dot{\mathbf{y}}_1 = \mathbf{y}_2$$
$$\dot{\mathbf{y}}_2 = -\mu_e \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|^3} + \mu_m \left(\frac{\mathbf{r}_m(t) - \mathbf{y}_1}{\|\mathbf{r}_m(t) - \mathbf{y}_1\|^3} - \frac{\mathbf{r}_m(t)}{r_m(t)^3} \right)$$

This system of equations may be written more compactly in the standard form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{9.62}$$

where the components of the state vector $\mathbf{y}$ are found in Eq. (9.61) and the rate functions $\mathbf{f}$ are

$$\mathbf{f}(t, \mathbf{y}) = \left\{ \begin{matrix} \mathbf{y}_2 \\ -\mu_e \dfrac{\mathbf{y}_1}{\|\mathbf{y}_1\|^3} + \mu_m \left(\dfrac{\mathbf{r}_m(t) - \mathbf{y}_1}{\|\mathbf{r}_m(t) - \mathbf{y}_1\|^3} - \dfrac{\mathbf{r}_m(t)}{r_m(t)^3} \right) \end{matrix} \right\}$$

Keep in mind that the lunar position vector $\mathbf{r}_m(t)$, obtained from an ephemeris, is a *known* function of time.

Any of the well-known numerical integrators designed to solve Eq. (9.62), such as those described in Section 1.8, can be brought to bear upon the problem of determining a lunar trajectory from the initial conditions in low earth orbit.

EXAMPLE 9.3

A spacecraft in low earth orbit is launched on a ballistic lunar trajectory so as to arrive at the moon on May 4, 2020, at 12:00 UT after a 3-day flight. The conditions at TLI are (see Fig. 4.5)

Altitude:	$z = 320$ km
Right ascension:	$\alpha = 90°$
Declination:	$\delta = 15°$
Flight path angle:	$\gamma = 40°$
Speed:	$v = 10.8267$ km/s

If the objective is to simply fly around the moon, determine the perilune altitude, and find the location of the spacecraft 5.67 days after perilune passage.

Solution

From Eqs. (4.4) and (4.5) the geocentric equatorial position vector of the spacecraft at TLI is

$$\begin{aligned}\mathbf{r}_0 &= r_0\left(\cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}}\right) \\ &= (6378 + 320)\left(\cos 15°\cos 90°\hat{\mathbf{I}} + \cos 15°\sin 90°\hat{\mathbf{J}} + \sin 15°\hat{\mathbf{K}}\right) \\ &= 6469.77\hat{\mathbf{J}} + 1733.57\hat{\mathbf{K}}\,(\text{km})\end{aligned} \tag{a}$$

Substituting $y = 2020$, $m = 5$, $d = 4$, and $UT = 12$ into Eqs. (5.47) and (5.48) yields the Julian day of the arrival time,

$$JD = 2{,}458{,}974\,\text{days}$$

This together with Eqs. (9.54) and (9.55) determines the moon's geocentric equatorial position vector $\mathbf{r}_m$ three days after TLI,

$$\mathbf{r}_m = -358{,}887\hat{\mathbf{I}} - 32{,}072.3\hat{\mathbf{J}} + 18{,}358.9\hat{\mathbf{K}}\,(\text{km})$$
$$r_m = 360{,}785\,\text{km}$$

Both of the position vectors $\mathbf{r}_0$ and $\mathbf{r}_m$ define the initial plane of the translunar trajectory, which means that the unit normal $\hat{\mathbf{w}}$ to that orbital plane may be found by normalizing the cross product of $\mathbf{r}_0$ into $\mathbf{r}_m$,

$$\hat{\mathbf{w}} = \frac{\mathbf{r}_0 \times \mathbf{r}_m}{\|\mathbf{r}_0 \times \mathbf{r}_m\|} = 0.0723516\hat{\mathbf{I}} - 0.258141\hat{\mathbf{J}} + 0.963394\hat{\mathbf{K}}$$

The unit vector $\hat{\mathbf{u}}_r$ in the direction of $\mathbf{r}_0$ is

$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}_0}{r_0} = 0.965926\hat{\mathbf{J}} + 0.248819\hat{\mathbf{K}}$$

Let $\hat{\mathbf{u}}_\perp$ be the unit vector that is normal to both of the orthogonal vectors $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{w}}$ and therefore lies in the plane of the translunar trajectory. $\hat{\mathbf{u}}_\perp$ is simply the cross product of the unit vector $\hat{\mathbf{w}}$ into the unit vector $\hat{\mathbf{u}}_r$,

$$\hat{\mathbf{u}}_\perp = \hat{\mathbf{w}} \times \hat{\mathbf{u}}_r = -0.997379\hat{\mathbf{I}} - 0.0187260\hat{\mathbf{J}} + 0.0698863\hat{\mathbf{K}}$$

The radial and transverse components of the TLI velocity $\mathbf{v}_0$ are found from the initial speed v_0 and the flight path angle γ

$$v_{r_0} = v_0 \sin\gamma \qquad v_{\perp_0} = v_0 \cos\gamma$$

Therefore, the velocity vector $\mathbf{v}_0$ may be written

$$\begin{aligned}\mathbf{v}_0 &= v_0 \sin\gamma\,\hat{\mathbf{u}}_r + v_0 \cos\gamma\,\hat{\mathbf{u}}_\perp \\ &= 10.8267\sin 40^\circ\left(0.965926\hat{\mathbf{J}} + 0.248819\hat{\mathbf{K}}\right) \\ &\quad + 10.8267\cos 40^\circ\left(-0.997379\hat{\mathbf{I}} - 0.0187260\hat{\mathbf{J}} + 0.0698863\hat{\mathbf{K}}\right)\end{aligned}$$

or

$$\mathbf{v}_0 = -8.27203\hat{\mathbf{I}} + 6.56685\hat{\mathbf{J}} + 2.38082\hat{\mathbf{K}}\,(\text{km/s}) \tag{b}$$

The initial value of the state vector $\mathbf{y}$ in Eq. (9.61) comprises $\mathbf{r}_0$ and $\mathbf{v}_0$, as given here in Eqs. (a) and (b).

Starting with the state vector $\mathbf{y}_0$ at time t_0 and using Simpson's ephemeris for the moon (Section 9.3), we numerically integrate Eq. (9.62) to obtain the values $\mathbf{y}_i$ of the state vector at n discrete times t_i between t_0 and the final time t_f. Using MATLAB's *ode45* with $t_0 = 0$ and $t_f = 5.667$ days, and with both the relative and absolute tolerances set to 10^{-10}, we obtain the free return trajectory shown in Fig. 9.14.

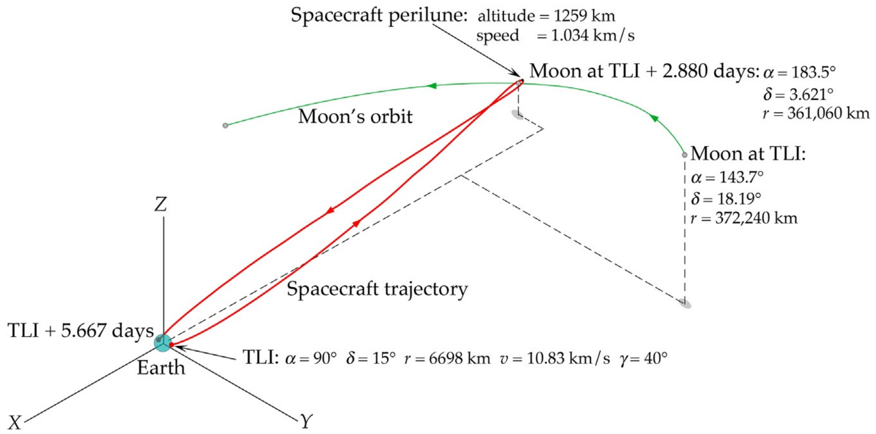

FIG. 9.14

Ballistic lunar flyaround trajectory obtained from numerical integration of the restricted three-body equations of motion.

It may be convenient to view the motion from a noninertial xyz frame with origin at the center of the earth, but having an x axis that always points to the moon. The x axis is therefore defined by the moon's instantaneous position vector $\mathbf{r}_m$. The z axis lies in the direction of the normal to the moon's orbital plane, which is defined by the cross product $\mathbf{r}_m \times \mathbf{v}_m$, where $\mathbf{v}_m$ is the moon's instantaneous velocity. The y axis is normal to the x and the z axes, according to the right-hand rule. The instantaneous unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ of the rotating frame are therefore obtained from $\mathbf{r}_m$ and $\mathbf{v}_m$ as follows:

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_m}{r_m} \qquad \hat{\mathbf{k}} = \frac{\mathbf{r}_m \times \mathbf{v}_m}{\|\mathbf{r}_m \times \mathbf{v}_m\|} \qquad \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} \tag{c}$$

As we know from Chapter 4, the direction cosine matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the geocentric equatorial XYZ frame to the rotating moon-fixed xyz frame is

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix}$$

where the rows of this matrix comprise the components of each rotating unit vector in Eq. (c) along the axes of the inertial frame. Thus, if the coordinates of a point in the geocentric inertial frame are $\lfloor X \;\; Y \;\; Z \rfloor$, then the coordinates of that same point in the moon-fixed frame are $\lfloor x \;\; y \;\; z \rfloor$, where

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [\mathbf{Q}]_{Xx} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

Applying this transformation to each point of the spacecraft and moon trajectories in Fig. 9.14 yields the trajectory shown in Fig. 9.15, in which the moon simply oscillates between the points on the x axis defined by the perigee and apogee of its noncircular orbit.

The MATLAB listing of the code for this example is found in Appendix D.38.

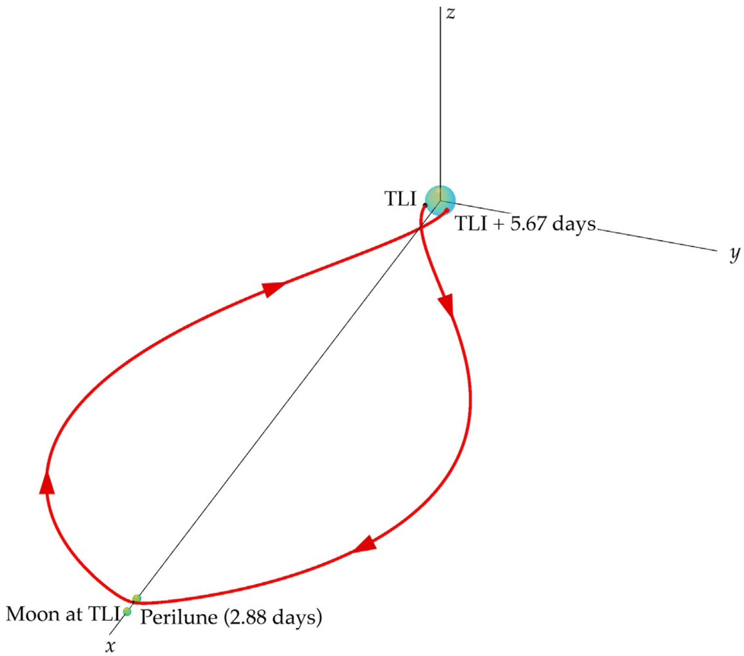

FIG. 9.15

The lunar trajectory of Fig. 9.14 viewed relative to the rotating earth-centered frame whose x axis is the earth-moon line.

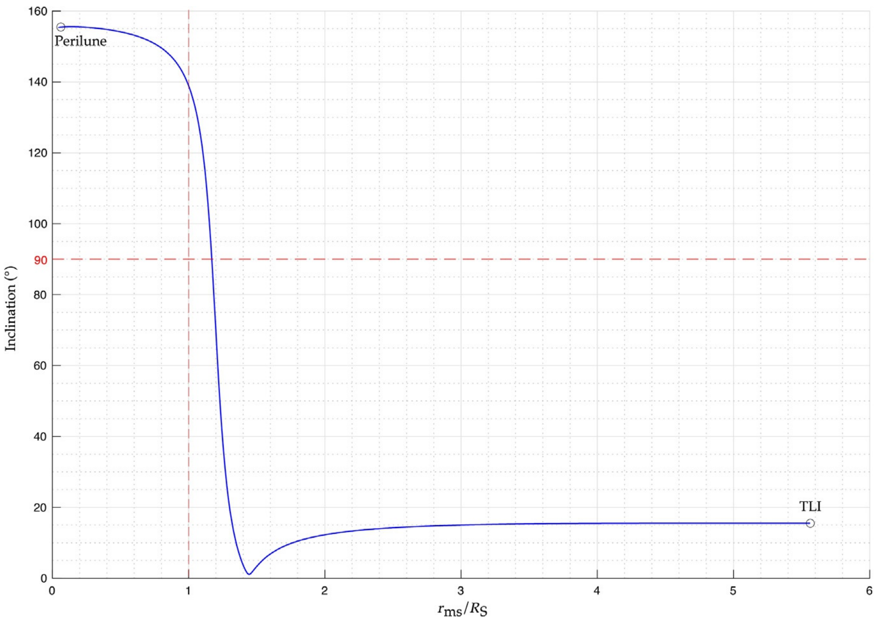

FIG. 9.16

Variation of inclination with distance from the moon for the orbit of Fig. 9.14.

The curve in Fig. 9.14 is not a Keplerian orbit because it results from the gravitational force of not one, but two bodies (the earth and the moon) on the spacecraft. Therefore, the trajectory does not lie in a single plane. Instead, each point of the trajectory has its own *osculating* plane, which is defined by the velocity and acceleration vectors of that point (see Fig. 1.9). The unit normal $\hat{\mathbf{b}}$ to the osculating plane is called the *binormal*. The binormal of each point of the orbit is found by means of the cross product operation,

$$\hat{\mathbf{b}} = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|} \tag{9.63}$$

Knowing the binormal, we can find the local inclination i,

$$i = \cos^{-1}\left(\hat{b}_Z\right) \tag{9.64}$$

just as we used Eq. (4.7) for the inclination of Keplerian orbits, in which case $\hat{\mathbf{h}}$ and $\hat{\mathbf{b}}$ coincide. Plotting Eq. (9.64) for the trajectory of Fig. 9.14, from TLI to perilune, yields Fig. 9.16, in which r_{ms} is the distance between the probe and the moon's center and R_S is the radius of the moon's SOI (Eq. 9.9). Fig. 9.16 shows that the inclination is steady at about 18° for most of the early part of the trajectory, so that the orbit is prograde relative to the earth. However, as the probe nears SOI, the inclination increases dramatically, soon exceeding 90°, at which point the orbit becomes retrograde, with the probe beginning to curve to the right, eventually passing behind the moon to reach perilune.

PROBLEMS

Section 9.2

9.1 A spacecraft in a circular 160-km earth orbit, coplanar with that of the moon, is launched at perigee onto an elliptical trajectory whose apogee lies on the moon's SOI. Calculate the Δv required to boost the spacecraft onto this trajectory.
{Ans.: 3.122 km/s}

9.2 If $\delta = 5°$ at the patch point, determine the eccentricity e and the relative speed v_2 required for the perilune altitude to be 100 km.
{Ans.: $e = 1.164$; $v_2 = 0.7653\,\text{km/s}$}

9.3 At the patch point, $v_2 = 0.7\,\text{km/s}$, $\lambda = 30°$, and $\delta = -15°$. Calculate the angle λ_p between the earth-moon line and perilune.
{Ans.: 101.4° clockwise}

9.4 Perilune of a lunar approach trajectory is 250 km. If $v_2 = 0.5\,\text{km/s}$, calculate the required value of δ and the Δv at perilune required to place the probe in a circular prograde lunar orbit.
{Ans.: $\delta = -7.745°$, $\Delta v = -673\,\text{m/s}$}

9.5 A spacecraft is launched from a circular earth orbit of 300 km altitude onto a Hohmann transfer trajectory to Mars.

(a) What is the burnout speed relative to the earth?
(b) How long does it take the probe to reach the moon's orbit on its way to Mars?
(c) What is the speed of the probe relative to the earth when it crosses the moon's orbit?
(d) Through what angle does the moon move in the time it takes for the probe to coast to lunar orbit?

{Ans.: (a) 11.04 km/s; (b) 40.51 h; (c) 2.15 km/s; (d) 22.1°}

9.6 A lunar probe in a prograde translunar trajectory arrives at the moon's SOI with $\lambda = 0°$. Its speed and flight path angle relative to the earth are 0.7 km/s and 45°, respectively.

(a) Find the perilune radius.
(b) How long does the probe remain within the SOI?

{Ans.: (a) 45,177 km; (b) 39 h}

9.7 In Fig. 9.2, the altitude of TLI is 320 km, its right ascension to the earth-moon line is $\alpha_0 = 37°$, and the flight path angle is $\gamma_0 = 10°$. If $\lambda = 45°$, use the patched conic method to find the perilune altitude for a ballistic, coplanar translunar trajectory. Assume the moon's orbit is circular.
{Ans.: 202.3 km}

9.8 In Fig. 9.2, the following are given: TLI altitude = 185 km, $\alpha_0 = 20°$, $\gamma_0 = 17.18°$, and $\lambda = -60°$. Use the patched conic method to find the perilune altitude for a ballistic, coplanar translunar trajectory, assuming the moon's orbit is circular.
{Ans.: 491.2 km}

Section 9.3

9.9 Use Simpson's lunar ephemeris to find, for November 2034:

(a) The day and the UT of the moon's perigee.
(b) The perigee's distance.
(c) The perigee's right ascension and declination.

{Partial Ans.: (b) 357,400 km}

9.10 Find the radial speed of the moon on April 30, 2025, at 06:00:00 UT.
{Ans.: 56.7 m/s}
9.11 Verify the entries in the last column of Table 9.1.

Section 9.4

9.12 A lunar probe is launched on a ballistic lunar flyaround trajectory from an earth altitude of 180 km with a flight path angle of $\gamma_0 = 13°$. TLI is located at right ascension $\alpha_0 = 42°$ and declination $\delta_0 = 9°$. Upon reaching the moon's SOI the phase angle between earth and the probe is $\lambda = 47°$, and the moon's geocentric state vector is

$$\mathbf{r}_{\text{moon}} = -387{,}639\hat{\mathbf{I}} - 4443.51\hat{\mathbf{J}} + 11{,}750.5\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{v}_{\text{moon}} = -0.0603414\hat{\mathbf{I}} - 0.955154\hat{\mathbf{J}} - 0.321928\hat{\mathbf{K}}\,(\text{km/s})$$

(a) Show that the path around the moon is retrograde.
(b) Find the perilune altitude .
(c) Determine the flight time from TLI to perilune.
{Ans.: (b) 71.2 km; (c) 3.20 days}

9.13 At TLI the geocentric state vector of a ballistic lunar probe is

$$\mathbf{r}_0 = 3340.59\hat{\mathbf{I}} + 5346.06\hat{\mathbf{J}} + 1807.63\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{v}_0 = -6.97237\hat{\mathbf{I}} + 7.92687\hat{\mathbf{J}} + 3.09093\hat{\mathbf{K}}\,(\text{km/s})$$

From TLI to the moon's SOI requires 44.101 h. If the moon's state vector at SOI encounter is the same as in Problem 9.11, calculate
(a) The lunar arrival angle λ.
(b) The additional time required to reach perilune.
(c) The perilune altitude.
{Ans.: (a) 31°; (b) 12.14 h; (c) 73.5 km}

9.14 At TLI the geocentric state vector of a ballistic lunar probe is

$$\mathbf{r}_0 = -5716.11\hat{\mathbf{I}} - 3168.49\hat{\mathbf{J}} - 571.785\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{v}_0 = 3.95652\hat{\mathbf{I}} - 9.76084\hat{\mathbf{J}} - 2.91477\hat{\mathbf{K}}\,(\text{km/s})$$

It arrives at the lunar SOI with a phase angle of $\lambda = 61°$ when the moon's state vector is

$$\mathbf{r}_{\text{moon}} = 359{,}880\hat{\mathbf{I}} - 9215.13\hat{\mathbf{J}} - 21{,}798.5\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{v}_{\text{moon}} = 0.068456\hat{\mathbf{I}} + 1.03141\hat{\mathbf{J}} + 0.35269\hat{\mathbf{K}}\,(\text{km/s})$$

Calculate the perilune altitude.
{Ans.: 100.4 km}

Section 9.5

9.15 A lunar probe is launched on a ballistic trajectory to reach the moon on May 4, 2020, at 12:00 UT after a 3-day flight from TLI to perilune. The conditions at TLI are:
Altitude: z = 180 km
Right ascension: α = 70°
Declination: δ = 20°

Flight path angle: $\gamma = 30°$
Speed$v = 10.9395$ km/s
Determine the perilune altitude and show that the path around the moon is retrograde.
{Ans.: 205 km}

9.16 A lunar probe is launched on a ballistic trajectory to reach the moon on June 13, 2035, at 12:00 UT after a 3.3-day flight from TLI to perilune. The conditions at TLI are:
Altitude: $z = 180$ km
Right ascension: $\alpha = 65°$
Declination: $\delta = 25°$
Flight path angle: $\gamma = 30°$
Speed $v = 10.9472$ km/s
Determine the perilune altitude and show that the path around the moon is retrograde.
{Ans.: 174 km. Trajectory becomes retrograde at 2.4 days after TLI}

9.17 Show that Eq. (9.63) becomes $\hat{\mathbf{b}} = \mathbf{h}/h$ for a Keplerian orbit.

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