CHAPTER

ORBITAL POSITION AS A FUNCTION OF TIME

3

3.1 INTRODUCTION

In Chapter 2, we found the relationship between position and true anomaly for the two-body problem. The only place time appeared explicitly was in the expression for the period of an ellipse. Obtaining position as a function of time is a simple matter for circular orbits. For elliptical, parabolic, and hyperbolic paths, we are led to the various forms of Kepler's equation relating position to time. These transcendental equations must be solved iteratively using a procedure like Newton's method, which is presented and illustrated in this chapter.

The different forms of Kepler's equation are combined into a single universal Kepler's equation by introducing universal variables. Implementation of this appealing notion is accompanied by the introduction of an unfamiliar class of functions known as Stumpff functions. The universal variable formulation is required for the Lambert and Gauss orbit determination algorithms in Chapter 5.

The road map of Appendix B may aid in grasping how the material presented here depends on that of Chapter 2.

3.2 TIME SINCE PERIAPSIS

The orbit formula, $r = (h^2/\mu)/(1 + e\cos\theta)$, gives the position of body m_2 in its orbit around m_1 as a function of the true anomaly. For many practical reasons, we need to be able to determine the position of m_2 as a function of time. For elliptical orbits, we have a formula for the period T (Eq. 2.82), but we cannot yet calculate the time required to fly between any two true anomalies. The purpose of this section is to come up with the formulas that allow us to do that calculation.

The one equation we have that relates true anomaly directly to time is Eq. (2.47), $h = r^2\dot{\theta}$, which can be written as

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

Substituting $r = (h^2/\mu)/(1 + e\cos\theta)$ we find, after separating variables,

$$\frac{\mu^2}{h^3}dt = \frac{d\theta}{(1 + e\cos\theta)^2}$$

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Integrating both sides of this equation yields

$$\frac{\mu^2}{h^3}(t-t_p)=\int_0^\theta \frac{d\vartheta}{(1+e\cos\vartheta)^2} \tag{3.1}$$

where the constant of integration t_p is the time at periapsis passage, where by definition $\theta=0$. t_p is the sixth constant of the motion that was missing in Chapter 2. The origin of time is arbitrary. It is convenient to measure time from periapsis passage, so we will usually set $t_p=0$. In that case we have

$$\frac{\mu^2}{h^3}t=\int_0^\theta \frac{d\vartheta}{(1+e\cos\vartheta)^2} \tag{3.2}$$

The integral on the right may be found in any standard mathematical handbook, such as Zwillinger (2018), in which we find

$$\int \frac{dx}{(a+b\cos x)^2}=\frac{1}{(a^2-b^2)^{3/2}}\left[2a\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)-\frac{b\sqrt{a^2-b^2}\sin x}{a+b\cos x}\right] \tag{3.3}$$

$$\int \frac{dx}{(a+b\cos x)^2}=\frac{1}{a^2}\left(\frac{1}{2}\tan\frac{x}{2}+\frac{1}{6}\tan^3\frac{x}{2}\right)\quad (b=a) \tag{3.4}$$

$$\int \frac{dx}{(a+b\cos x)^2}=\frac{1}{(b^2-a^2)^{3/2}}\left[\frac{b\sqrt{b^2-a^2}\sin x}{a+b\cos x}-a\ln\left(\frac{\sqrt{b+a}+\sqrt{b-a}\tan(x/2)}{\sqrt{b+a}-\sqrt{b-a}\tan(x/2)}\right)\right] \tag{3.5}$$

3.3 CIRCULAR ORBITS ($e=0$)

If $e=0$, the integral in Eq. (3.2) is simply $\int_0^\theta d\theta$, which means

$$t=\frac{h^3}{\mu^2}\theta$$

Recall that for a circle (Eq. 2.62), $r=h^2/\mu$. Therefore $h^3=r^{3/2}\mu^{3/2}$, so that

$$t=\frac{r^{3/2}}{\sqrt{\mu^2}}\theta$$

Finally, substituting the formula (Eq. 2.64) for the period T of a circular orbit, $T=2\pi r^{3/2}/\sqrt{\mu}$, yields

$$t=\frac{\theta}{2\pi}T$$

or

$$\theta=\frac{2\pi}{T}t$$

The reason that t is directly proportional to θ in a circular orbit is simply that the angular velocity $2\pi/T$ is constant. Therefore, the time Δt to fly through a true anomaly of $\Delta\theta$ is $(\Delta\theta/2\pi)T$.

Because the circle is symmetric about any diameter, the apse line—and therefore the periapsis—can be chosen arbitrarily (Fig. 3.1).

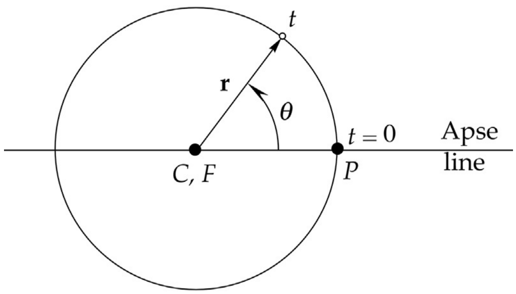


FIG. 3.1

Time since periapsis is directly proportional to true anomaly in a circular orbit.

3.4 ELLIPTICAL ORBITS ($e < 1$)

Set $a = 1$ and $b = e$ in Eq. (3.3) to obtain

$$\int_0^\theta \frac{d\vartheta}{(1+e\cos\vartheta)^2} = \frac{1}{(1-e^2)^{3/2}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

Therefore, Eq. (3.2) in this case becomes

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{3/2}}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

or

$$M_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \tag{3.6}$$

where

$$M_e = \frac{\mu^2}{h^3}\left(1-e^2\right)^{3/2} t \tag{3.7}$$

M_e is called the mean anomaly. The subscript e reminds us that this is for an ellipse and not for parabolas and hyperbolas, which have their own "mean anomaly" formulas. Eq. (3.6) is plotted in Fig. 3.2. Observe that for all values of the eccentricity e, M_e is a monotonically increasing function of the true anomaly θ.

From Eq. (2.82), the formula for the period T of an elliptical orbit, we have $\mu^2(1 - e^2)^{3/2}/h^3 = 2\pi/T$, so that the mean anomaly can be written much more simply as

$$M_e = \frac{2\pi}{T}t \tag{3.8}$$

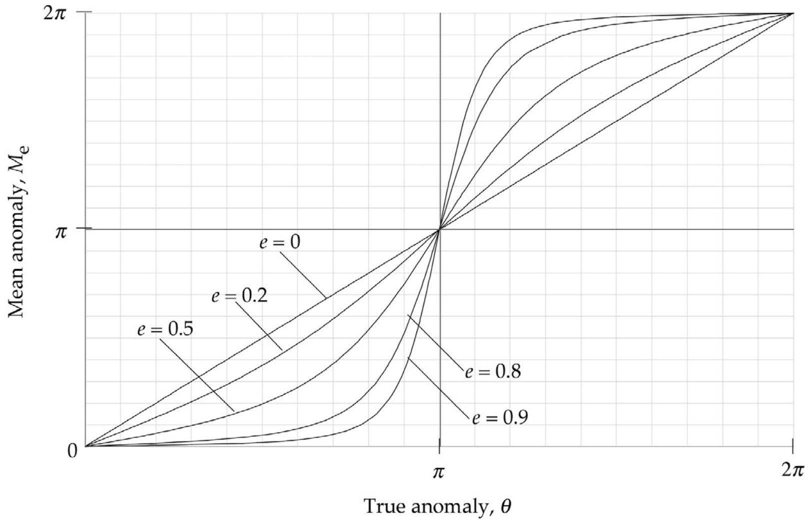


FIG. 3.2

Mean anomaly vs. true anomaly for ellipses of various eccentricities.

The angular velocity of the position vector of an elliptical orbit is not constant, but since 2π radians are swept out per period T, the ratio $2\pi/T$ is the average angular velocity, which is given the symbol n and called the mean motion,

$$n = \frac{2\pi}{T} \tag{3.9}$$

In terms of the mean motion, Eq. (3.8) can be written simpler still,

$$M_e = nt$$

The mean anomaly is the azimuth position (in radians) of a fictitious body moving around the ellipse at the constant angular speed n. For a circular orbit, the mean anomaly M_e and the true anomaly θ are identical.

It is convenient to simplify Eq. (3.6) by introducing an auxiliary angle E called the eccentric anomaly, which is shown in Fig. 3.3. This is done by circumscribing the ellipse with a concentric auxiliary circle having a radius equal to the semimajor axis a of the ellipse. Let S be that point on the ellipse whose true anomaly is θ. Through point S we pass a perpendicular to the apse line, intersecting the auxiliary circle at point Q and the apse line at point V. The angle between the apse line and the radius drawn from the center of the circle to Q on its circumference is the eccentric anomaly E. Observe that E lags the true anomaly θ from periapsis P to apoapsis A ($0 \le \theta < 180°$), whereas it leads θ from A to P ($180° \le \theta < 360°$).

To find E as a function of θ, we first observe from Fig. 3.3 that, in terms of the eccentric anomaly, $\overline{OV} = a\cos E$, whereas in terms of the true anomaly, $\overline{OV} = ae + r\cos\theta$. Thus,

$$a\cos E = ae + r\cos\theta$$

Using Eq. (2.72), $r = a(1 - e^2)/(1 + e\cos\theta)$, we can write this as

$$a\cos E = ae + \frac{a(1-e^2)\cos\theta}{1-e\cos\theta}$$

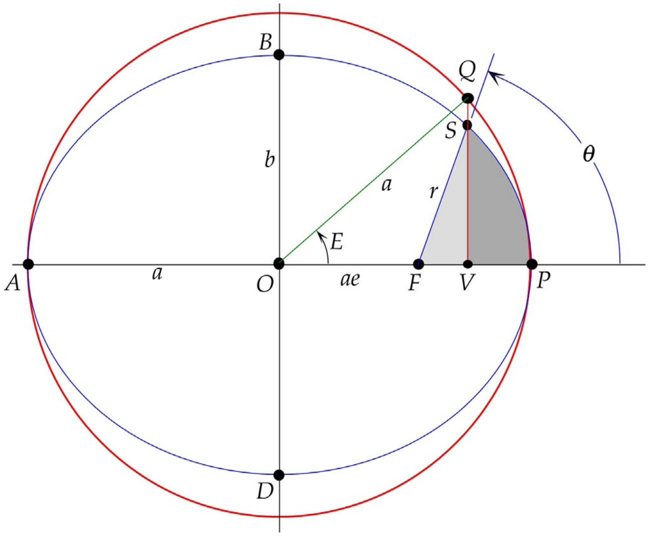


FIG. 3.3

Ellipse and the circumscribed auxiliary circle.

Simplifying the right-hand side, we get

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \tag{3.10a}$$

Solving this for $\cos\theta$ we obtain the inverse relation,

$$\cos\theta = \frac{e - \cos E}{e\cos E - 1} \tag{3.10b}$$

Substituting Eq. (3.10a) into the trigonometric identity $\sin^2 E + \cos^2 E = 1$ and solving for $\sin E$ yields

$$\sin E = \frac{\sqrt{1-e^2}\sin\theta}{1 + e\cos\theta} \tag{3.11}$$

Eq. (3.10a) would be fine for obtaining E from θ, except that, given a value of $\cos E$ between -1 and 1, there are two values of E between 0° and 360°, as illustrated in Fig. 3.4. The same comments hold for Eq. (3.11). To resolve this quadrant ambiguity, we use the following trigonometric identity:

$$\tan^2\frac{E}{2} = \frac{\sin^2 E/2}{\cos^2 E/2} = \frac{\dfrac{1-\cos E}{2}}{\dfrac{1+\cos E}{2}} = \frac{1-\cos E}{1+\cos E} \tag{3.12}$$

From Eq. (3.10a)

$$1 - \cos E = \frac{1-\cos\theta}{1+e\cos\theta}(1-e) \quad \text{and} \quad 1 + \cos E = \frac{1+\cos\theta}{1+e\cos\theta}(1+e)$$

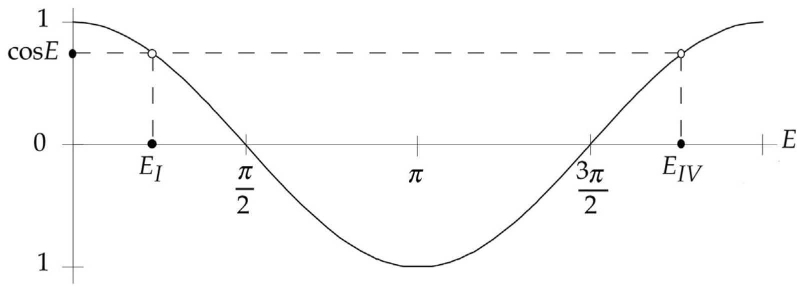


FIG. 3.4

For $0 < \cos E < 1$, E can lie in the first or fourth quadrant. For $-1 < \cos E < 0$, E can lie in the second or third quadrant.

Therefore,

$$\tan^2\frac{E}{2} = \frac{1-e}{1+e}\cdot\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-e}{1+e}\tan^2\frac{\theta}{2}$$

where the last step required applying the trig identity in Eq. (3.12) to the term $(1 - \cos\theta)/(1 + \cos\theta)$. Finally, therefore, we obtain

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2} \tag{3.13a}$$

or

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) \tag{3.13b}$$

Observe from Fig. 3.5 that for any value of $\tan(E/2)$, there is only one value of E between 0° and 360°. There is no quadrant ambiguity.

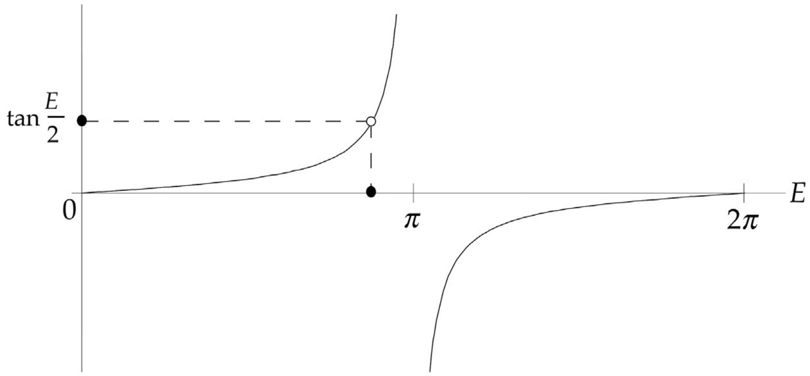


FIG. 3.5

To any value of $\tan(E/2)$, there corresponds a unique value of E in the range 0 to 2π.

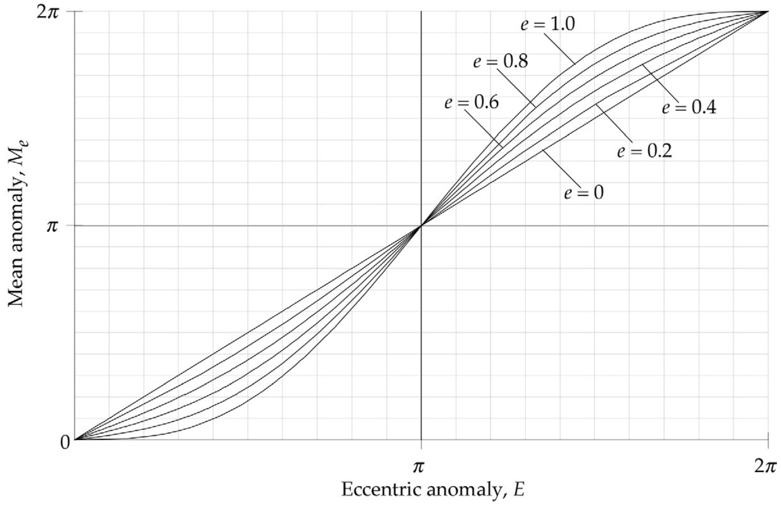


FIG. 3.6

Plot of Kepler's equation for an elliptical orbit.

Substituting Eqs. (3.11) and (3.13b) into Eq. (3.6) yields Kepler's equation,

$$M_e = E - e\sin E \tag{3.14}$$

This monotonically increasing relationship between mean anomaly and eccentric anomaly is plotted for several values of eccentricity in Fig. 3.6.

Given the true anomaly θ, we calculate the eccentric anomaly E using Eq. (3.13). Substituting E into Kepler's formula, Eq. (3.14) yields the mean anomaly directly. From the mean anomaly and the period T we find the time (since periapsis) from Eq. (3.8),

$$t = \frac{M_e}{2\pi}T \tag{3.15}$$

On the other hand, if we are given the time, then Eq. (3.15), yields the mean anomaly M_e. Substituting M_e into Kepler's equation, we get the following expression for the eccentric anomaly:

$$E - e\sin E = M_e$$

We cannot solve this transcendental equation directly for E. A rough value of E might be read from Fig. 3.6. However, an accurate solution requires an iterative, trial-and-error procedure.

Newton's method, or one of its variants, is one of the more common and efficient ways of finding the root of a well-behaved function. To find a root of the equation $f(x) = 0$ in Fig. 3.7, we estimate it to be x_i and evaluate the function $f(x)$ and its first derivative $f'(x)$ at that point. We then extend the tangent to the curve at $f(x_i)$ until it intersects the x axis at x_{i+1}, which becomes our updated estimate of the root. The intercept x_{i+1} is found by setting the slope of the tangent line equal to the slope of the curve at x_i; that is,

$$f'(x_i) = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

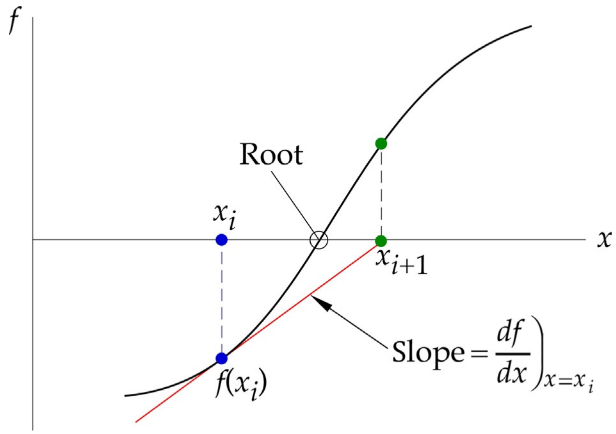


FIG. 3.7

Newton's method for finding a root of $f(x) = 0$.

from which we obtain

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{3.16}$$

The process is repeated, using x_{i+1} to estimate x_{i+2}, and so on, until the root has been found to the desired level of precision.

To apply Newton's method to the solution of Kepler's equation, we form the function

$$f(E) = E - e\sin E - M_e$$

and seek the value of eccentric anomaly that makes $f(E) = 0$. Since

$$f'(E) = 1 - e\cos E$$

for this problem, Eq. (3.16) becomes

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M_e}{1 - e\cos E_i} \tag{3.17}$$

ALGORITHM 3.1

Solve Kepler's equation for the eccentric anomaly E given the eccentricity e and the mean anomaly M_e. See Appendix D.11 for the implementation of this algorithm in MATLAB.

1. Choose an initial estimate of the root E as follows (Prussing and Conway, 2013). If $M_e < \pi$, then $E = M_e + e/2$. If $M_e > \pi$, then $E = M_e - e/2$. Remember that the angles E and M_e are in radians. (When using a handheld calculator, be sure that it is in the radian mode.)
2. At any given step, having obtained E_i from the previous step, calculate $f(E_i) = E_i - e \sin E_i - M_e$ and $f'(E_i) = 1 - e \cos E_i$.
3. Calculate $\text{ratio}_i = f(E_i)/f'(E_i)$.

4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of E,

$$E_{i+1} = E_i - \text{ratio}_i$$

Return to Step 2.

5. If $|\text{ratio}_i|$ is less than the tolerance, then accept E_i as the solution to within the chosen accuracy.

EXAMPLE 3.1

A geocentric elliptical orbit has a perigee radius of 9600 km and an apogee radius of 21,000 km, as shown in Fig. 3.8. Calculate the time to fly from perigee P to a true anomaly of 120°.

Solution

Before anything else, let us find the primary orbital parameters e and h. The eccentricity is readily obtained from the perigee and apogee radii by means of Eq. (2.84),

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{21{,}000 - 9600}{21{,}000 + 9600} = 0.37255 \tag{a}$$

We find the angular momentum using the orbit equation,

$$9600 = \frac{h^2}{398{,}600}\frac{1}{1 + 0.37255\cos(0)} \Rightarrow h = 72{,}472\,\text{km}^2/\text{s}$$

With h and e, the period of the orbit is obtained from Eq. (2.82),

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398{,}600^2}\left(\frac{72{,}472}{\sqrt{1-0.37255^2}}\right)^3 = 18{,}834\,\text{s} \tag{b}$$

Eq. (3.13b) yields the eccentric anomaly from the true anomaly,

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.37255}{1+0.37255}}\tan\frac{120°}{2}\right) = 1.7281\,\text{rad}$$

Then Kepler's equation (Eq. 3.14) is used to find the mean anomaly,

$$M_e = 1.7281 - 0.37255\sin 1.7281 = 1.3601\,\text{rad}$$

Finally, the time follows from Eq. (3.15),

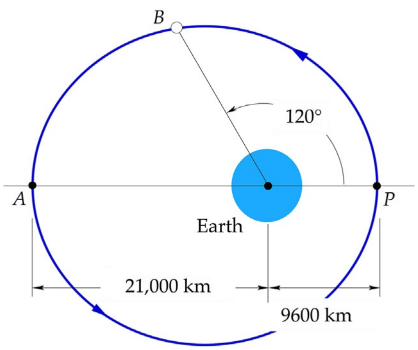


FIG. 3.8

Geocentric elliptical orbit.

$$t = \frac{M_e}{2\pi} T = \frac{1.3601}{2\pi} \cdot 18{,}834 = \boxed{4077\text{s}\ (1.132\text{h})}$$

EXAMPLE 3.2

In the previous example, find the true anomaly at 3 h after perigee passage.

Solution

Since the time (10,800 s) is greater than one-half the period, the true anomaly must be greater than 180°.

First, we use Eq. (3.8) to calculate the mean anomaly for $t = 10{,}800\,\text{s}$.

$$M_e = 2\pi \frac{t}{T} = 2\pi \frac{10{,}800}{18{,}830} = 3.6029\,\text{rad} \tag{a}$$

Kepler's equation, $E - e\sin(E) = M_e$ (with all angles in radians) is then employed to find the eccentric anomaly. This transcendental equation will be solved using Algorithm 3.1 with an error tolerance of 10^{-6}. Since $M_e > \pi$, a good starting value for the iteration is $E_0 = M_e - e/2 = 3.4166$. Executing the algorithm yields the following steps:

Step 1:

$$\begin{aligned}
E_0 &= 3.4166 \\
f(E_0) &= -0.085124 \\
f'(E_0) &= 1.3585 \\
\text{ratio} &= \frac{-0.085124}{1.3585} = -0.062658 \\
|\text{ratio}| &> 10^{-6} \qquad \therefore \text{ repeat}
\end{aligned}$$

Step 2:

$$\begin{aligned}
E_1 &= 3.4166 - (-0.062658) = 3.4793 \\
f(E_1) &= -0.0002134 \\
f'(E_1) &= 1.3515 \\
\text{ratio} &= \frac{-0.0002134}{1.3515} = -1.5778 \times 10^{-4} \\
|\text{ratio}| &> 10^{-6} \qquad \therefore \text{repeat}
\end{aligned}$$

Step 3:

$$\begin{aligned}
E_2 &= 3.4793 - \left(-1.5778 \times 10^{-4}\right) = 3.4794 \\
f(E_2) &= -1.5366 \times 10^{-9} \\
f'(E_2) &= 1.3515 \\
\text{ratio} &= \frac{-1.5366 \times 10^{-9}}{1.3515} = -1.137 \times 10^{-9} \\
|\text{ratio}| &< 10^{-6} \qquad \therefore \text{stop}
\end{aligned}$$

Convergence to even more than the desired accuracy occurred after just two iterations. With $E = 3.4794$, the true anomaly is found from Eq. (3.13a) to be

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \sqrt{\frac{1+0.37255}{1-0.37255}}\tan\frac{3.4794}{2} = -8.6721 \quad \Rightarrow \quad \boxed{\theta = 193.2°}$$

EXAMPLE 3.3

Let a satellite be in a 500 km by 5000 km orbit with its apse line parallel to the line from the earth to the sun, as shown in Fig. 3.9. Find the time that the satellite is in the earth's shadow if

(a) the apogee is toward the sun

(b) the perigee is toward the sun.

Solution

We start by using the given data to find the primary orbital parameters, e and h. The eccentricity is obtained from Eq. (2.84),

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{(6378 + 5000) - (6378 + 500)}{(6378 + 5000) + (6378 + 500)} = 0.24649 \tag{a}$$

The orbit equation can then be used to find the angular momentum

$$r_p = \frac{h^2}{\mu} \frac{1}{1 + e\cos(0)} \Rightarrow 6878 = \frac{h^2}{398{,}600} \frac{1}{1 + 0.24649} \Rightarrow h = 58{,}458\,\text{km}^2/\text{s}$$

The semimajor axis may be found from Eq. (2.71),

$$a = \frac{h^2}{\mu} \frac{1}{1 - e^2} = \frac{58{,}458^2}{398{,}600} \frac{1}{1 - 0.24649^2} = 9128\,\text{km} \tag{b}$$

or from the fact that $a = (r_p + r_a)/2$. The period of the orbit follows from Eq. (2.83),

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 9128^{3/2} = 86791.1\,\text{s}\,(2.4109\,\text{h})$$

(a) If the apogee is toward the sun, as in Fig. 3.9, then the satellite is in earth's shadow between points a and b on its orbit. These are two of the four points of intersection of the orbit with the lines that are parallel to the earth-sun line, and lie at a distance R_E from the center of the earth. The true anomaly of b is therefore given by $\sin\theta = R_E/r$, where r is the radial position of the satellite. It follows that the radius of b is

$$r = \frac{R_E}{\sin\theta} \tag{c}$$

From Eq. (2.72), we also have

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{d}$$

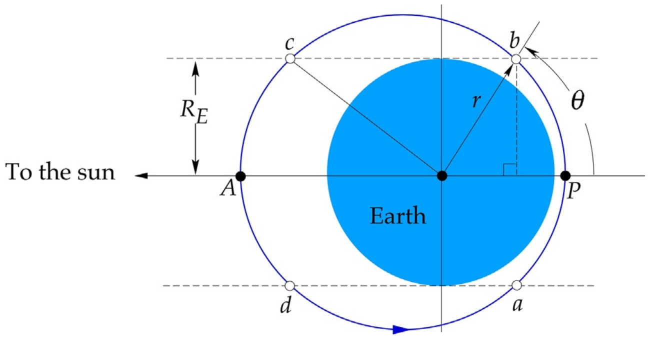


FIG. 3.9

Satellite passing in and out of the earth's shadow.

Substituting Eq. (c) into Eq. (d), collecting terms, and simplifying yields an equation in θ,

$$e\cos\theta - \left(1-e^2\right)\frac{a}{R_E}\sin\theta + 1 = 0 \tag{e}$$

Substituting Eqs. (a) and (b) together with $R_E = 6378\,\text{km}$ into Eq. (e) yields

$$0.24649\cos\theta - 1.3442\sin\theta = -1$$

This equation is of the form

$$A\cos\theta + B\sin\theta = C$$

It has two roots, which are given by (see Problem 3.12):

$$\theta = \tan^{-1}\frac{B}{A} \pm \cos^{-1}\left[\frac{C}{A}\cos\left(\tan^{-1}\frac{B}{A}\right)\right]$$

For the case at hand,

$$\theta = \tan^{-1}\frac{-1.3442}{0.24649} \pm \cos^{-1}\left[\frac{-1}{0.24649}\cos\left(\tan^{-1}\frac{-1.3442}{0.24649}\right)\right] = -79.607° \pm 137.03°$$

That is

$$\theta_b = 57.423°$$
$$\theta_c = -216.64°\,(+143.36°)$$

For apogee toward the sun, the flight from perigee to point b will be in shadow. To find the time of flight from perigee to point b, we first compute the eccentric anomaly of b using Eq. (3.13b):

$$E_b = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_b}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.24649}{1+0.24649}}\tan\frac{57.423°}{2}\right) = 0.80521\,\text{rad}$$

From this we find the mean anomaly using Kepler's equation,

$$M_e = E - e\sin E = 0.80521 - 0.24649\sin 0.80521 = 0.62749\,\text{rad}$$

Finally, Eq. (3.5) yields the time at b,

$$t_b = \frac{M_e}{2\pi}T = \frac{0.62749}{2\pi}8679.1 = 866.77\,\text{s}$$

The total time in shadow, from a to b, during which the satellite passes through perigee, is

$$\boxed{t = 2t_b = 1734\,\text{s}\ (28.98\,\text{min})} \tag{f}$$

(b) If the perigee is toward the sun, then the satellite is in shadow near apogee, from point c ($\theta_c = 143.36°$) to d on the orbit. Following the same procedure as above, we obtain (see Problem 3.13),

$$E_c = 2.3364\,\text{rad}$$
$$M_c = 2.1587\,\text{rad}$$
$$t_c = 2981.8\,\text{s}$$

The total time in shadow, from c to d, is

$$\boxed{t = T - 2t_c = 8679.1 - 2(2981.8) = 2716\,\text{s}\,(45.26\,\text{min})}$$

The time is longer than that given by Eq. (f) since the satellite travels slower near apogee.

We have observed that there is no closed-form solution for the eccentric anomaly E in Kepler's equation, $E - e \sin E = M_e$. However, there exist infinite series solutions. One of these, due to Lagrange (Battin, 1987), is a power series in the eccentricity e,

$$E = M_e + \sum_{n=1}^{\infty} a_n e^n \tag{3.18}$$

where the coefficients a_n are given by the somewhat intimidating expression

$$a_n = \frac{1}{2^{n-1}} \sum_{k=0}^{floor(n/2)} (-1)^k \frac{1}{(n-k)!k!} (n-2k)^{n-1} \sin[(n-2k)M_e] \tag{3.19}$$

where $floor(x)$ means x rounded to the next lowest integer (e.g., $floor(0.5) = 0$, $floor(\pi) = 3$). If e is sufficiently small, then the Lagrange series converges. This means that by including enough terms in the summation, we can obtain E to any desired degree of precision. Unfortunately, if e exceeds 0.6627434193, the series diverges, which means taking more and more terms yields worse and worse results for some values of M.

The limiting value for the eccentricity was discovered by the French mathematician Pierre-Simone Laplace (1749–1827) and is called the Laplace limit.

In practice, we must truncate the Lagrange series to a finite number of terms N, so that

$$E = M_e + \sum_{n=1}^{N} a_n e^n \tag{3.20}$$

For example, setting $N = 3$ and calculating each a_n by means of Eq. (3.19) leads to

$$E = M_e + e \sin M_e + \frac{e^2}{2} \sin 2M_e + \frac{e^3}{8} (3 \sin 3M_e - \sin M_e) \tag{3.21}$$

For small values of the eccentricity e, this yields good agreement with the exact solution of Kepler's equation (plotted in Fig. 3.6). However, as we approach the Laplace limit, the accuracy degrades unless more terms of the series are included. Fig. (3.10) shows that for an eccentricity of 0.65, just below the Laplace limit, Eq. (3.21) ($N = 3$) yields a solution that oscillates around the exact solution but is fairly close to it everywhere. Setting $N = 10$ in Eq. (3.20) produces a curve that, at the given scale, is indistinguishable from the exact solution. On the other hand, for an eccentricity of 0.90, far above the Laplace limit, Fig. 3.11 reveals that Eq. (3.21) is a poor approximation to the exact solution, and using $N = 10$ makes matters even worse.

Another infinite series for E (Battin, 1987) is given by

$$E = M_e + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin nM_e \tag{3.22}$$

where the coefficients J_n are Bessel functions of the first kind, attributable to German astronomer Friedrich Bessel (1784–1846). They are defined by the infinite series

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+k} \tag{3.23}$$

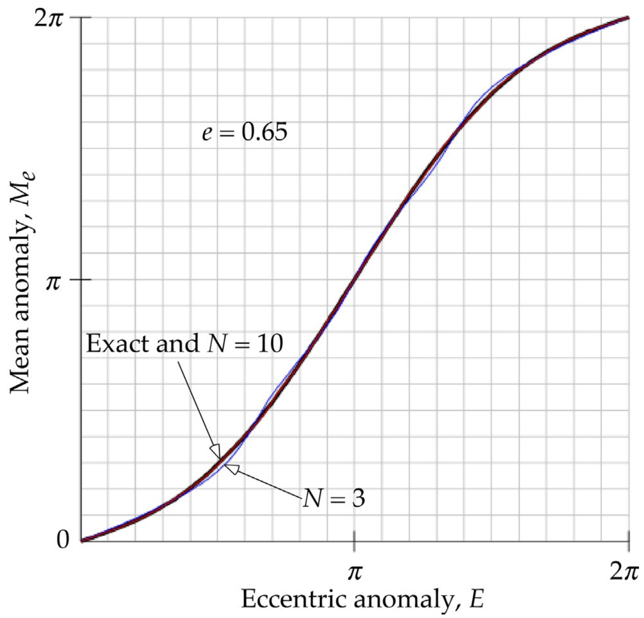


FIG. 3.10

Comparison of the exact solution of Kepler's equation with the truncated Lagrange series solution ($N = 3$ and $N = 10$) for an eccentricity of 0.65.

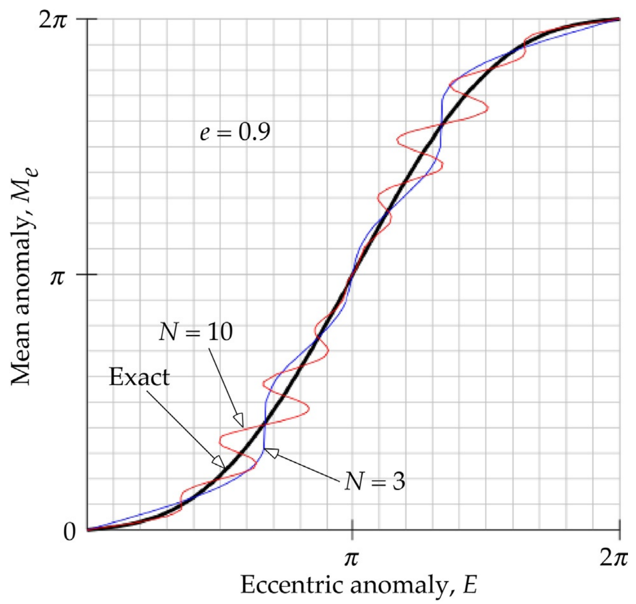


FIG. 3.11

Comparison of the exact solution of Kepler's equation with the truncated Lagrange series solution ($N = 3$ and $N = 10$) for an eccentricity of 0.90.

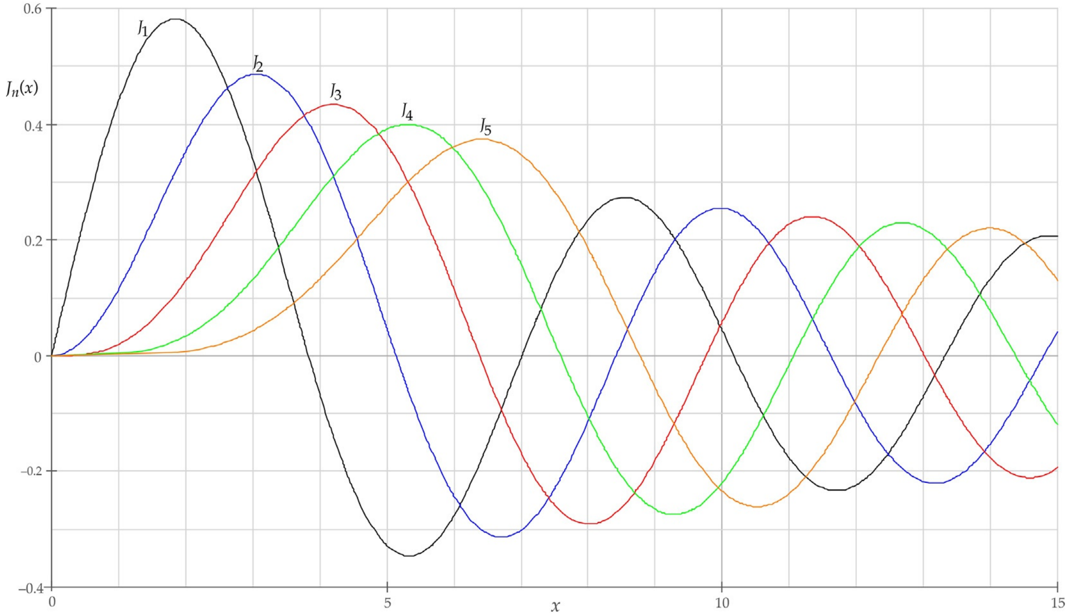


FIG. 3.12

Bessel functions of the first kind.

J_1 through J_5 are plotted in Fig. 3.12. Clearly, they are oscillatory in appearance and tend toward zero with increasing x.

It turns out that, unlike the Lagrange series, the Bessel function series solution converges for all values of the eccentricity <1. Fig. 3.13 shows how the truncated Bessel series solutions

$$E = M_e + \sum_{n=1}^{N} \frac{2}{n} J_n(ne) \sin nM_e \tag{3.24}$$

for $N = 3$ and $N = 10$ compare with the exact solution of Kepler's equation for the very large elliptical eccentricity of $e = 0.99$. It can be seen that the case $N = 3$ yields a poor approximation for all but a few values of M_e. Increasing the number of terms in the series to $N = 10$ obviously improves the approximation, and adding even more terms will make the truncated series solution indistinguishable from the exact solution at the given scale.

Observe that we can combine Eqs. (3.10) and (2.72) as follows to obtain the orbit equation for the ellipse in terms of the eccentric anomaly:

$$r = \frac{a(1-e^2)}{1 + e\cos\theta} = \frac{a(1-e^2)}{1 + e\left(\dfrac{e - \cos E}{e\cos E - 1}\right)}$$

From this it is easy to see that

$$r = a(1 - e\cos E) \tag{3.25}$$

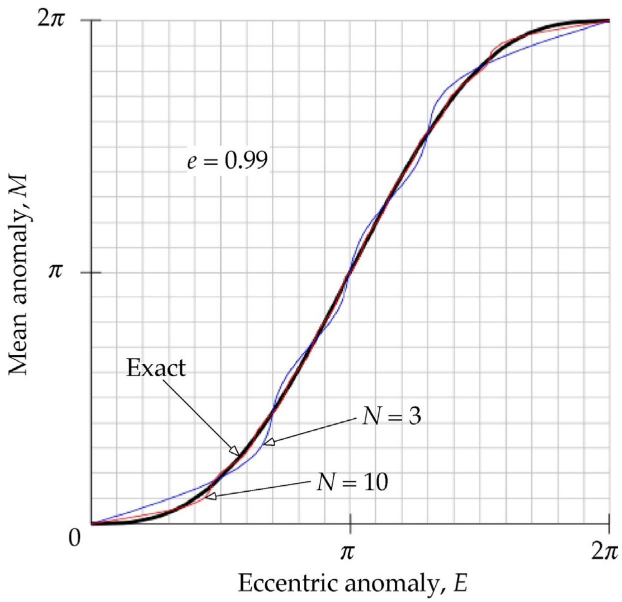


FIG. 3.13

Comparison of the exact solution of Kepler's equation with the truncated Bessel series solution ($N = 3$ and $N = 10$) for an eccentricity of 0.99.

In Eq. (2.86), we defined the true anomaly-averaged radius $\bar{r}_\theta$ of an elliptical orbit. Alternatively, the time-averaged radius $\bar{r}_t$ of an elliptical orbit is defined as

$$\bar{r}_t = \frac{1}{T}\int_0^T r\,\mathrm{d}t \tag{3.26}$$

According to Eqs. (3.14) and (3.15),

$$t = \frac{T}{2\pi}(E - e\sin E)$$

Therefore,

$$\mathrm{d}t = \frac{T}{2\pi}(1 - e\cos E)\mathrm{d}E$$

Upon using this relationship to change the variable of integration from t to E and substituting Eq. (3.25), Eq. (3.26) becomes

$$\begin{aligned}\bar{r}_t &= \frac{1}{T}\int_0^{2\pi}[a(1-e\cos E)]\left[\frac{T}{2\pi}(1-e\cos E)\right]\mathrm{d}E \\ &= \frac{a}{2\pi}\int_0^{2\pi}(1-e\cos E)^2\mathrm{d}E \\ &= \frac{a}{2\pi}\int_0^{2\pi}\left(1-2e\cos E+e^2\cos^2 E\right)\mathrm{d}E \\ &= \frac{a}{2\pi}\left(2\pi-0+e^2\pi\right)\end{aligned}$$

so that

$$\bar{r}_t = a\left(1+\frac{e^2}{2}\right) \quad \text{Time-averaged radius of an elliptical orbit} \tag{3.27}$$

Comparing this result with Eq. (2.87) reveals, as we should have expected, that $\bar{r}_t > \bar{r}_\theta$. In fact, combining Eqs. (2.87) and (3.27) yields

$$\bar{r}_\theta = a\sqrt{3-2\frac{\bar{r}_t}{a}} \tag{3.28}$$

3.5 PARABOLIC TRAJECTORIES ($e = 1$)

For the parabola, Eq. (3.2) becomes

$$\frac{\mu^2}{h^3}t = \int_0^\theta \frac{\mathrm{d}\vartheta}{(1+\cos\vartheta)^2} \tag{3.29}$$

Setting $a = b = 1$ in Eq. (3.4) yields

$$\int_0^\theta \frac{\mathrm{d}\vartheta}{(1+\cos\vartheta)^2} = \frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{6}\tan^3\frac{\theta}{2}$$

Therefore, Eq. (3.29) may be written as

$$M_p = \frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{6}\tan^3\frac{\theta}{2} \tag{3.30}$$

where

$$M_p = \frac{\mu^2 t}{h^3} \tag{3.31}$$

M_p is dimensionless, and it may be thought of as the "mean anomaly" for the parabola. Eq. (3.30) is plotted in Fig. 3.14. Eq. (3.30) is also known as Barker's equation, after Thomas Barker (1722–1809), a British meteorologist.

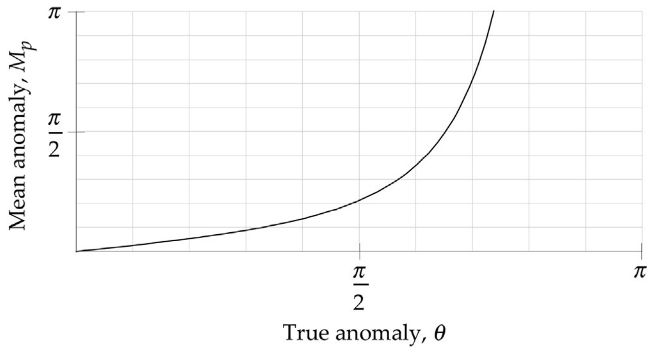


FIG. 3.14

Graph of Barker's equation.

Given the true anomaly θ, we find the time directly from Eqs. (3.30) and (3.31). If time is the given variable, then we must solve the cubic equation

$$\frac{1}{6}\left(\tan\frac{\theta}{2}\right)^3 + \frac{1}{2}\tan\frac{\theta}{2} - M_p = 0$$

which has but one real root, namely,

$$\tan\frac{\theta}{2} = z - \frac{1}{z} \tag{3.32a}$$

where

$$z = \left(3M_p + \sqrt{1 + \left(3M_p\right)^2}\right)^{1/3} \tag{3.32b}$$

EXAMPLE 3.4

A geocentric parabola has a perigee velocity of 10 km/s. How far is the satellite from the center of the earth 6 h after perigee passage?

Solution

The first step is to find the orbital parameters e and h. Of course, we know that $e = 1$. To get the angular momentum, we can use the given perigee speed and Eq. (2.90) (the energy equation) to find the perigee radius,

$$r_p = \frac{2\mu}{v_p^2} = \frac{2 \cdot 398{,}600}{10^2} = 7972\,\text{km}$$

It follows from Eq. (2.31) that the angular momentum is

$$h = r_p v_p = 7972 \cdot 10 = 79{,}720\,\text{km}^2/\text{s}$$

We can now calculate the parabolic mean anomaly by means of Eq. (3.31),

$$M_p = \frac{\mu^2 t}{h^3} = \frac{398{,}600^2 \cdot (6 \cdot 3600)}{79{,}720^3} = 6.7737\,\text{rad}$$

Therefore, $3M_p = 20.321\,\text{rad}$, which, when substituted into Eqs. (3.32), yields the true anomaly,

$$z = \left(20.321 + \sqrt{1 + 20.321^2}\right)^{1/3} = 3.4388$$
$$\tan\frac{\theta}{2} = 3.4388 - \frac{1}{3.4388} = 3.1480 \Rightarrow \theta = 144.75°$$

Finally, we substitute the true anomaly into the orbit equation to find the radius,

$$r = \frac{79{,}720^2}{398{,}600}\frac{1}{1 + \cos(144.75°)} = \boxed{86{,}899\,\text{km}}$$

3.6 HYPERBOLIC TRAJECTORIES ($e > 1$)

Setting $a = 1$ and $b = e$ in Eq. (3.5) yields

$$\int_0^\theta \frac{d\vartheta}{(1 + e\cos\vartheta)^2} = \frac{1}{e^2 - 1}\left\{\frac{e\sin\theta}{1 + e\cos\theta} - \frac{1}{\sqrt{e^2 - 1}}\ln\left[\frac{\sqrt{e+1} + \sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1} - \sqrt{e-1}\tan(\theta/2)}\right]\right\}$$

Therefore, for the hyperbola, Eq. (3.1) becomes

$$\frac{\mu^2}{h^3}t = \frac{1}{e^2 - 1}\frac{e\sin\theta}{1 + e\cos\theta} - \frac{1}{(e^2 - 1)^{3/2}}\ln\left[\frac{\sqrt{e+1} + \sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1} - \sqrt{e-1}\tan(\theta/2)}\right]$$

Multiplying both sides by $(e^2 - 1)^{3/2}$, we get

$$M_h = \frac{e\sqrt{e^2 - 1}\sin\theta}{1 + e\cos\theta} - \ln\left[\frac{\sqrt{e+1} + \sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1} - \sqrt{e-1}\tan(\theta/2)}\right] \tag{3.33}$$

where

$$M_h = \frac{\mu^2}{h^3}(e^2 - 1)^{3/2}t \tag{3.34}$$

M_h is the hyperbolic mean anomaly. Eq. (3.33) is plotted in Fig. 3.15. Recall that θ cannot exceed θ_∞ (Eq. 2.97).

We can simplify Eq. (3.33) by introducing an auxiliary angle analogous to the eccentric anomaly E for the ellipse. Consider a point on a hyperbola whose polar coordinates are r and θ. Referring to Fig. 3.16, let x be the horizontal distance of the point from the center C of the hyperbola, and let y be its distance above the apse line. The ratio y/b defines the hyperbolic sine of the dimensionless variable F that we will use as the hyperbolic eccentric anomaly. That is, we define F to be such that

$$\sinh F = \frac{y}{b} \tag{3.35}$$

In view of the equation of a hyperbola,

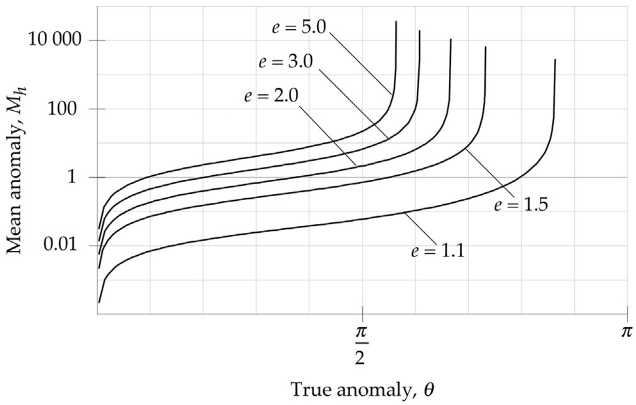


FIG. 3.15

Plots of Eq. (3.33) for several different eccentricities.

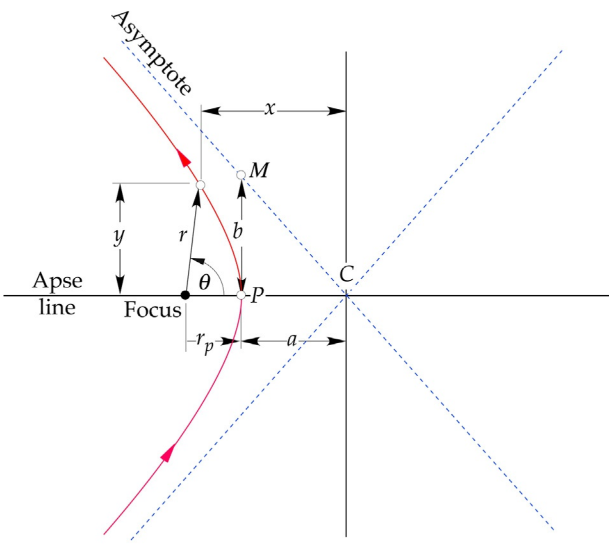


FIG. 3.16

Hyperbola parameters.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

it is consistent with the definition of sinh$F$ to define the hyperbolic cosine as

$$\cosh F=\frac{x}{a} \tag{3.36}$$

(It should be recalled that $\sinh x=(e^x-e^{-x})/2$ and $\cosh x=(e^x+e^{-x})/2$, and that, therefore, $\cosh^2 x-\sinh^2 x=1$.)

From Fig. 3.16 we see that $y = r\sin\theta$. Substituting this into Eq. (3.35), along with $r = a(e^2 - 1)/(1 + e\cos\theta)$ (Eq. 2.104) and $b = a\sqrt{e^2-1}$ (Eq. 2.106), we get

$$\sinh F = \frac{1}{b} r\sin\theta = \frac{1}{a\sqrt{e^2-1}}\frac{a(e^2-1)}{1+e\cos\theta}\sin\theta$$

so that

$$\sinh F = \frac{\sin\theta\sqrt{e^2-1}}{1+e\cos\theta} \tag{3.37}$$

This can be used to obtain F in terms of the true anomaly,

$$F = \sinh^{-1}\left(\frac{\sin\theta\sqrt{e^2-1}}{1+e\cos\theta}\right) \tag{3.38}$$

Using the formula $\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$ we can, after simplifying the algebra, write Eq. (3.38) as

$$F = \ln\left(\frac{\sin\theta\sqrt{e^2-1}+\cos\theta+e}{1+e\cos\theta}\right)$$

Substituting the trigonometric identities,

$$\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} \qquad \cos\theta = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$$

and doing some more algebra yields

$$F = \ln\left[\frac{1+e+(e-1)\tan^2(\theta/2)+2\tan(\theta/2)\sqrt{e^2-1}}{1+e+(1-e)\tan^2(\theta/2)}\right]$$

Fortunately, but not too obviously, the numerator and the denominator in the brackets have a common factor, so that this expression for the hyperbolic eccentric anomaly reduces to

$$F = \ln\left[\frac{\sqrt{e+1}+\sqrt{e-1}\tan(\theta/2)}{\sqrt{e+1}-\sqrt{e-1}\tan(\theta/2)}\right] \tag{3.39}$$

Substituting Eqs. (3.37) and (3.39) into Eq. (3.33) yields Kepler's equation for the hyperbola,

$$M_h = e\sinh F - F \tag{3.40}$$

This equation is plotted for several different eccentricities in Fig. 3.17.

If we substitute the expression for $\sinh F$ (Eq. 3.37) into the hyperbolic trig identity

$$\cosh^2 F - \sinh^2 F = 1$$

we get

$$\cosh^2 F = 1 + \left(\frac{\sin\theta\sqrt{e^2-1}}{1+e\cos\theta}\right)^2$$

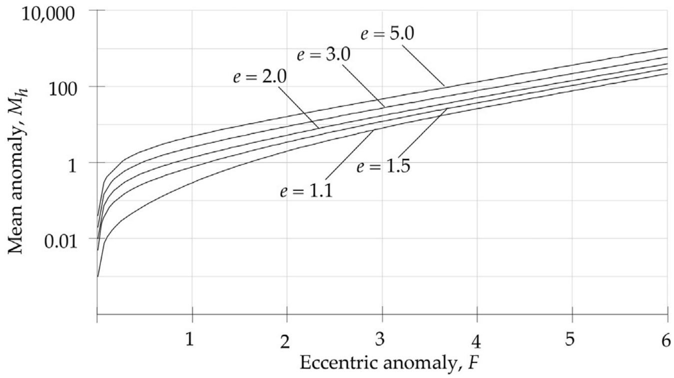


FIG. 3.17

Plot of Kepler's equation for the hyperbola.

A few steps of algebra lead to

$$\cosh^2 F = \left(\frac{\cos\theta + e}{1 + e\cos\theta}\right)^2$$

so that

$$\cosh F = \frac{\cos\theta + e}{1 + e\cos\theta} \tag{3.41a}$$

Solving this for $\cos\theta$, we obtain the inverse relation,

$$\cos\theta = \frac{\cosh F - e}{1 - e\cosh F} \tag{3.41b}$$

The hyperbolic tangent is found in terms of the hyperbolic sine and cosine by the formula

$$\tanh F = \frac{\sinh F}{\cosh F}$$

In mathematical handbooks, we can find the hyperbolic trig identity,

$$\tanh\frac{F}{2} = \frac{\sinh F}{1 + \cosh F} \tag{3.42}$$

Substituting Eqs. (3.37) and (3.41a) into this formula and simplifying yields

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\frac{\sin\theta}{1 + \cos\theta} \tag{3.43}$$

Interestingly enough, Eq. (3.42) holds for ordinary trig functions, too; that is,

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$$

Therefore, Eq. (3.43) can be written

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2} \tag{3.44a}$$

This is a somewhat simpler alternative to Eq. (3.39) for computing eccentric anomaly from true anomaly, and it is a whole lot simpler to invert:

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} \tag{3.44b}$$

If time is the given quantity, then Eq. (3.40)—a transcendental equation—must be solved for F by an iterative procedure, as was the case for the ellipse. To apply Newton's procedure to the solution of Kepler's equation for the hyperbola, we form the function

$$f(F) = e\sinh F - F - M_h$$

and seek the value of F that makes $f(F) = 0$. Since

$$f'(F) = e\cosh F - 1$$

Eq. (3.16) becomes

$$F_{i+1} = F_i - \frac{e\sinh F_i - F_i - M_h}{e\cosh F_i - 1} \tag{3.45}$$

All quantities in this formula are dimensionless (radians, not degrees).

ALGORITHM 3.2

Solve Kepler's equation for the hyperbola for the hyperbolic eccentric anomaly F given the eccentricity e and the hyperbolic mean anomaly M_h. See Appendix D.12 for the implementation of this algorithm in MATLAB.

1. Choose an initial estimate of the eccentric anomaly F.
   a. For hand computations, read a rough value of F_0 (no more than two significant figures) from Fig. 3.17 to keep the number of iterations to a minimum.
   b. In computer software, let $F_0 = M_h$, an inelegant choice that may result in many iterations but will nevertheless rapidly converge on today's high-speed desktops and laptops.
2. At any given step, having obtained F_i from the previous step, calculate $f(F_i) = e \sinh F_i - F_i - M_h$ and $f'(F_i) = e \cosh F_i - 1$.
3. Calculate $\text{ratio}_i = f(F_i)/f'(F_i)$.
4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of F_i. Return to Step 2.
5. If $|\text{ratio}_i|$ is less than the tolerance, then accept F_i as the solution to within the desired accuracy.

EXAMPLE 3.5

A geocentric trajectory has a perigee velocity of 15 km/s and a perigee altitude of 300 km. Find
(a) the radius and the time when the true anomaly is 100°;
(b) the position and speed 3 h later.

Solution

We first calculate the primary orbital parameters e and h. The angular momentum is calculated from Eq. (2.31) and the given perigee data:

$$h = r_p v_p = (6378 + 300) \cdot 15 = 100{,}170\,\text{km}^2/\text{s}$$

The eccentricity is found by evaluating Eq. (2.50), the orbit equation, $r = (h^2/\mu)/(1 + e\cos\theta)$, at perigee ($\theta = 0°$):

$$6378 + 300 = \frac{100{,}170^2}{398{,}600}\frac{1}{1+e} \Rightarrow e = 2.7696$$

(a) Since $e > 1$, the trajectory is a hyperbola. Note that the true anomaly of the asymptote of the hyperbola is, according to Eq. (2.97),

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{2.7696}\right) = 111.17°$$

Evaluating the orbit equation at the given true anomaly, $\theta = 100°$, yields

$$r = \frac{100{,}170^2}{398{,}600}\frac{1}{1 + 2.7696\cos 100°} = \boxed{48{,}497\ \text{km}}$$

To find the time since perigee passage at $\theta = 100°$, we first use Eq. (3.44a) to calculate the hyperbolic eccentric anomaly,

$$\tanh\frac{F}{2} = \sqrt{\frac{2.7696 - 1}{2.7696 + 1}}\tan\frac{100°}{2} = 0.81653 \Rightarrow F = 2.2927\,\text{rad}$$

Kepler's equation for the hyperbola then yields the mean anomaly,

$$M_h = e\sinh F - F = 2.7696\sinh 2.2927 - 2.2927 = 11.279\,\text{rad}$$

The time since perigee passage is found from Eq. (3.34),

$$t = \frac{h^3}{\mu^2}\frac{1}{\left(e^2-1\right)^{3/2}}M_h = \frac{100{,}170^3}{398{,}600^2}\frac{1}{\left(2.7696^2-1\right)^{3/2}}11.279 = \boxed{4141.4\ \text{s}}$$

(b) After 3 h, the time since perigee passage is

$$t = 4141.4 + 3 \cdot 3600 = 14{,}941\,\text{s}\ (4.15\,\text{h})$$

The corresponding mean anomaly, from Eq. (3.34), is

$$M_h = \frac{398{,}600^2}{100{,}170^3}\left(2.7696^2 - 1\right)^{3/2}(14{,}941) = 40.690\,\text{rad}$$

We will use Algorithm 3.2 with an error tolerance of 10^{-6} to find the hyperbolic eccentric anomaly F. Referring to Fig. 3.17, we see that for $M_h = 40.69$ and $e = 2.7696$, F lies between 3 and 4. Let us arbitrarily choose $F_0 = 3$ as our initial estimate of F. Executing the algorithm yields the following steps:

$$F_0 = 3$$

Step 1:

$$\begin{aligned} f(F_0) &= -15.944494 \\ f'(F_0) &= 26.883397 \\ \text{ratio} &= -0.59309818 \\ F_1 &= 3-(-0.59309818) = 3.5930982 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat.} \end{aligned}$$

Step 2:

$$\begin{aligned} f(F_1) &= 6.0114484 \\ f'(F_1) &= 49.370747 \\ \text{ratio} &= -0.12176134 \\ F_2 &= 3.5930982-(-0.12176134) = 3.4713368 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat.} \end{aligned}$$

Step 3:

$$\begin{aligned} f(F_2) &= 0.35812370 \\ f'(F_2) &= 43.605527 \\ \text{ratio} &= 8.2128052 \times 10^{-3} \\ F_3 &= 3.4713368-\left(8.2128052 \times 10^{-3}\right) = 3.4631240 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat.} \end{aligned}$$

Step 4:

$$\begin{aligned} f(F_3) &= 1.4973128 \times 10^{-3} \\ f'(F_3) &= 43.241398 \\ \text{ratio} &= 3.4626836 \times 10^{-5} \\ F_4 &= 3.4631240-\left(3.4626836 \times 10^{-5}\right) = 3.4630894 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat.} \end{aligned}$$

Step 5:

$$\begin{aligned} f(F_4) &= 2.6470781 \times 10^{-3} \\ f'(F_4) &= 43.239869 \\ \text{ratio} &= 6.1218459 \times 10^{-10} \\ F_5 &= 3.4630894-\left(6.1218459 \times 10^{-10}\right) = 3.4630894 \\ |\text{ratio}| &< 10^{-6} \quad \therefore \text{accept } F = 3.4631 \text{ as the solution.} \end{aligned}$$

We substitute this final value of F into Eq. (3.44b) to find the true anomaly,

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{2.7696+1}{2.7696-1}}\tanh\frac{3.4631}{2} = 1.3708 \Rightarrow \theta = 107.78^\circ$$

With the true anomaly, the orbital equation yields the radial coordinate at the final time

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{100{,}170^2}{398{,}600}\frac{1}{1+2.7696\cos 107.78^\circ} = \boxed{163{,}180\,\text{km}}$$

The velocity components are obtained from Eq. (2.31),

$$v_\perp = \frac{h}{r} = \frac{100{,}170}{163{,}180} = 0.61386\,\text{km/s}$$

and Eq. (2.49),

$$v_r = \frac{\mu}{h}e\sin\theta = \frac{398{,}600}{100{,}170}2.7696\sin 107.78^\circ = 10.496\,\text{km/s}$$

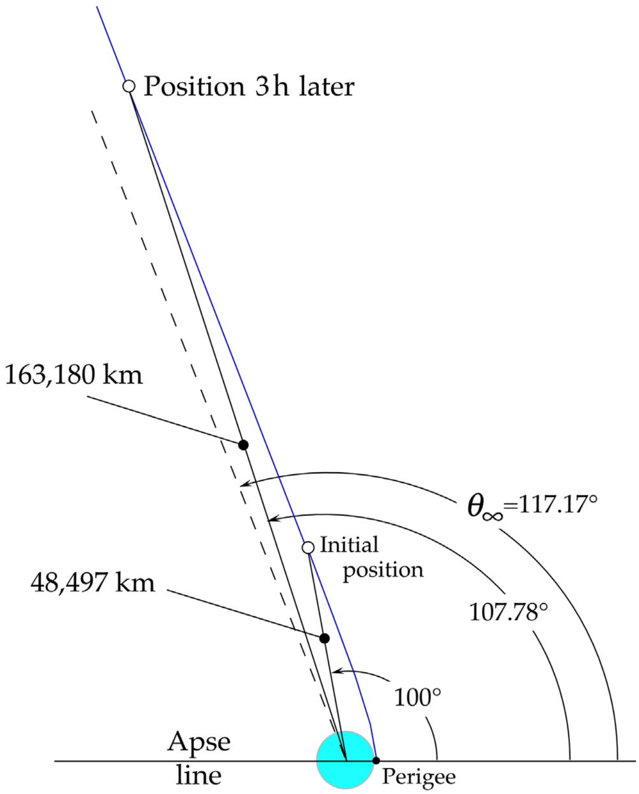


FIG. 3.18

Given and computed data for Example 3.5.

Therefore, the speed of the spacecraft is

$$v = \sqrt{v_r^2 + v_\perp^2} = \sqrt{10.494^2 + 0.61386^2} = \boxed{10.51\,\text{km/ s}}$$

Note that the hyperbolic excess speed for this orbit is

$$v_\infty = \frac{\mu}{h} e \sin\theta_\infty = \frac{398{,}600}{100{,}170} \cdot 2.7696 \cdot \sin 111.7° = 10.277\,\text{km/s}$$

The results of this analysis are shown in Fig. 3.18.

When determining orbital position as a function of time with the aid of Kepler's equation, it is convenient to have position r as a function of eccentric anomaly F. This is obtained by substituting Eq. (3.41b) into Eq. (2.104),

$$r = \frac{a(e^2-1)}{1 + e\cos\theta} = \frac{a(e^2-1)}{1 + e\left(\dfrac{\cos F - e}{1 - e\cos F}\right)}$$

This reduces to

$$r = a(e\cosh F - 1) \tag{3.46}$$

3.7 UNIVERSAL VARIABLES

The equations for elliptical and hyperbolic trajectories are very similar, as can be seen from Table 3.1. Observe, for example, that the hyperbolic mean anomaly is obtained from that of the ellipse as follows:

$$\begin{aligned}
M_h &= \frac{\mu^2}{h^3}\left(e^2 - 1\right)^{3/2} t \\
&= \frac{\mu^2}{h^3}\left[(-1)\left(1 - e^2\right)\right]^{3/2} t \\
&= \frac{\mu^2}{h^3}(-1)^{3/2}\left(1 - \mathrm{e}^2\right)^{3/2} t \\
&= \frac{\mu^2}{h^3} t(-i)\left(1 - \mathrm{e}^2\right)^{3/2} t \\
&= -i\left[\frac{\mu^2}{h^3}\left(1 - e^2\right)^{3/2} t\right] \\
&= -iM_e
\end{aligned}$$

In fact, the formulas for the hyperbola can all be obtained from those of the ellipse by replacing the variables in the ellipse equations according to the following scheme, wherein "←" means "replace by" and $i = \sqrt{-1}$:

$$\begin{aligned}
a &\leftarrow -a \\
b &\leftarrow ib \\
M_e &\leftarrow -iM_h \\
E &\leftarrow iF
\end{aligned}$$

Table 3.1 Comparison of some of the orbital formulas for the ellipse and hyperbola

	Equation	Ellipse ($e < 1$)	Hyperbola ($e > 1$)
1.	Orbit equation vs. true anomaly	$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$	Same
2.	Conic equation in Cartesian coordinates	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
3.	Semimajor axis	$a = \frac{h^2}{\mu}\frac{1}{1 - e^2}$	$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1}$
4.	Semiminor axis	$b = \sqrt{1 - e^2}$	$b = \sqrt{e^2 - 1}$
5.	Energy equation	$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$	$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$
6.	Mean anomaly	$M_e = \frac{\mu^2}{h^3}\left(1 - e^2\right)^{3/2} t$	$M_h = \frac{\mu^2}{h^3}\left(e^2 - 1\right)^{3/2} t$
7.	Kepler's equation	$M_e = E - e \sin E$	$M_h = e \sinh F - F$
8.	Orbit equation vs. eccentric anomaly	$r = a(1 - \cos E)$	$r = a(e \cosh F - 1)$

Note in this regard that $\sin(iF) = i\sinh F$ and $\cos(iF) = \cosh F$. Relations among the circular and hyperbolic trig functions are found in mathematics handbooks, such as Zwillinger (2018).

In the universal variable approach, the semimajor axis of the hyperbola is considered to have a negative value, so that the energy equation (row 5 of Table 3.1) has the same form for any type of orbit, including the parabola, for which $a = \infty$. In this formulation, the semimajor axis of any orbit is found using (row 3),

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} \tag{3.47}$$

If the position r and velocity v are known at a given point on the path, then the energy equation (row 5) is convenient for finding the semimajor axis of any orbit,

$$a = \frac{1}{\dfrac{2}{r} - \dfrac{v^2}{\mu}} \tag{3.48}$$

Kepler's equation may also be written in terms of a universal variable, or universal anomaly χ, that is valid for all orbits. See, for example, Prussing and Conway, 2013, Battin, 1987, and Bond and Allman, 1996. If t_0 is the time when the universal variable is zero, then the value of χ at time $t_0 + \Delta t$ is found by iterative solution of the universal Kepler's equation

$$\sqrt{\mu}\Delta t = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi^2 C\left(\alpha\chi^2\right) + (1-\alpha r_0)\chi^3 S\left(\alpha\chi^2\right) + r_0\chi \tag{3.49}$$

where r_0 and $v_r)_0$ are the radius and radial velocity, respectively, at $t = t_0$, and α is the reciprocal of the semimajor axis

$$\alpha = \frac{1}{a} \tag{3.50}$$

$\alpha < 0$, $\alpha = 0$, and $\alpha > 0$ for hyperbolas, parabolas, and ellipses, respectively. The units of χ are $\text{km}^{1/2}$ (so $\alpha\chi^2$ is dimensionless). The functions $C(z)$ and $S(z)$ belong to the class known as Stumpff functions, named for the German astronomer Karl Stumpff (1895–1970). They are defined by the infinite series,

$$S(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+3)!} = \frac{1}{6} - \frac{z}{120} + \frac{z^2}{5040} - \frac{z^3}{362{,}880} + \cdots \tag{3.51a}$$

$$C(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+2)!} = \frac{1}{2} - \frac{z}{24} + \frac{z^2}{720} - \frac{z^3}{40{,}320} + \cdots \tag{3.51b}$$

$C(z)$ and $S(z)$ are related to the circular and hyperbolic trig functions as follows:

$$S(z) = \begin{cases} \dfrac{\sqrt{z} - \sin\sqrt{z}}{\left(\sqrt{z}\right)^3} & (z > 0) \\[2ex] \dfrac{\sinh\sqrt{-z} - \sqrt{-z}}{\left(\sqrt{z}\right)^3} & (z < 0) \qquad (z = \alpha\chi^2) \\[2ex] \dfrac{1}{6} & (z = 0) \end{cases} \tag{3.52}$$

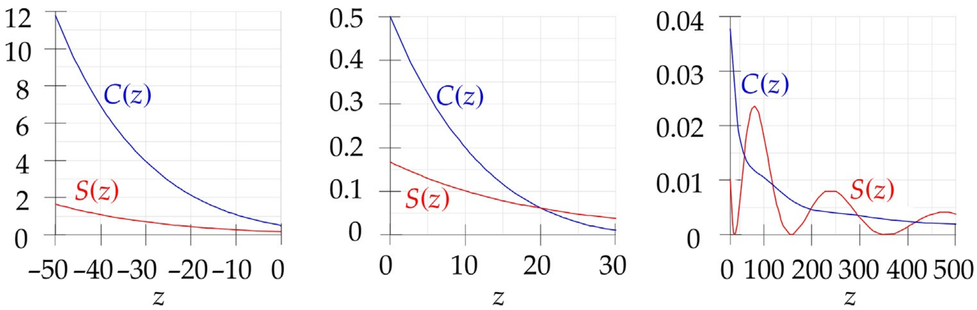


FIG. 3.19

A plot of the Stumpff functions $C(z)$ and $S(z)$.

$$C(z)=\begin{cases}\dfrac{1-\cos\sqrt{z}}{z} & (z>0)\\[2mm] \dfrac{\cosh\sqrt{-z}-1}{-z} & (z<0) \qquad (z=\alpha\chi^2)\\[2mm] \dfrac{1}{2} & (z=0)\end{cases} \tag{3.53}$$

Clearly, $z<0$, $z=0$, and $z>0$ for hyperbolas, parabolas, and ellipses, respectively. It should be pointed out that if $C(z)$ and $S(z)$ are computed by the series expansions (Eqs. 3.51), then the forms of $C(z)$ and $S(z)$, depending on the sign of z, are selected, so to speak, automatically. $C(z)$ and $S(z)$ behave as shown in Fig. 3.19. Both $C(z)$ and $S(z)$ are nonnegative functions of z. They increase without bound as z approaches $-\infty$ and tend toward zero for large positive values of z. As can be seen from Eq. (3.53), for $z>0$, $C(z)=0$ when $\cos\sqrt{z}=1$; that is, when $z=(2\pi)^2, (4\pi)^2, (6\pi)^2, \cdots$.

The price we pay for using the universal variable formulation is having to deal with the relatively unknown Stumpff functions. However, Eqs. (3.52) and (3.53) are easy to implement in both computer programs and programmable calculators. See Appendix D.13 for the implementation of these expressions in MATLAB.

To gain some insight into how Eq. (3.49) represents the Kepler equations for all the conic sections, let t_0 be the time at periapse passage and let us set $t_0=0$, as we have assumed previously. Then $\Delta t=t$, $v_r)_0=0$, and $r_0=r_p$, the periapsis radius. In that case Eq. (3.49) reduces to

$$\sqrt{\mu}t=\left(1-\alpha r_p\right)\chi^3 S(\alpha\chi^2)+r_p\chi \qquad (\text{time}=0\,\text{at periapse passage}) \tag{3.54}$$

Consider first the parabola. In this case $\alpha=0$, and $S=S(0)=1/6$, so that Eq. (3.54) becomes a cubic polynomial in χ,

$$\sqrt{\mu}t=\frac{1}{6}\chi^3+r_p\chi$$

Multiply this equation through by $\left(\sqrt{\mu}/h\right)^3$ to obtain

$$\frac{\mu^2}{h^3}t=\frac{1}{6}\left(\frac{\chi\sqrt{\mu}}{h}\right)^3+r_p\chi\left(\frac{\sqrt{\mu}}{h}\right)^3$$

Since $r_p = h^2/2\mu$ for a parabola, we can write this as

$$\frac{\mu^2}{h^3}t = \frac{1}{6}\left(\frac{\chi\sqrt{\mu}}{h}\right)^3 + \frac{1}{2}\left(\frac{\sqrt{\mu}}{h}\chi\right) \tag{3.55}$$

Upon setting $\chi = h\tan(\theta/2)/\sqrt{\mu}$, Eq. (3.55) becomes identical to Eq. (3.30), the time vs. true anomaly relation for the parabola.

Kepler's equation for the ellipse can be obtained by multiplying Eq. (3.54) throughout by $\left(\sqrt{\mu(1-e^2)}/h\right)^3$:

$$\frac{\mu^2}{h^3}(1-e^2)^{3/2}t = \left(\chi\frac{\sqrt{\mu}}{h}\sqrt{1-e^2}\right)^3(1-\alpha r_p)S(z) + r_p\chi\left(\frac{\sqrt{\mu}}{h}\sqrt{1-e^2}\right)^3 \quad (z=\alpha\chi^2) \tag{3.56}$$

Recall that for the ellipse, $r_p = h^2/[\mu(1+e)]$ and $\alpha = 1/a = \mu(1-e^2)/h^2$. Using these two expressions in Eq. (3.56), along with $S(z) = [\sqrt{\alpha}\chi - \sin(\sqrt{\alpha}\chi)]/(\sqrt{\alpha}\chi)^3$ (from Eq. 3.52), and working through the algebra ultimately leads to

$$M_e = \frac{\chi}{\sqrt{a}} - e\sin\left(\frac{\chi}{\sqrt{a}}\right)$$

Comparing this with Kepler's equation for an ellipse (Eq. 3.14) reveals that the relationship between the universal variable χ and the eccentric anomaly E is $\chi = \sqrt{a}E$. Similarly, it can be shown for hyperbolic orbits that $\chi = \sqrt{-a}F$. In summary, the universal anomaly χ is related to the previously encountered anomalies as follows:

$$\chi = \begin{cases} \dfrac{h}{\sqrt{\mu}}\tan\dfrac{\theta}{2} & \text{parabola} \\ \sqrt{a}E & \text{ellipse} \quad (t=0 \text{ at periapse}) \\ \sqrt{-a}F & \text{hyperbola} \end{cases} \tag{3.57}$$

When t_0 is the time at a point other than periapsis, such that Eq. (3.49) applies, then Eq. (3.57) becomes

$$\chi = \begin{cases} \dfrac{h}{\sqrt{\mu}}\left(\tan\dfrac{\theta}{2} - \tan\dfrac{\theta_0}{2}\right) & \text{parabola} \\ \sqrt{a}(E-E_0) & \text{ellipse} \\ \sqrt{-a}(F-F_0) & \text{hyperbola} \end{cases} \tag{3.58}$$

As before, we can use Newton's method to solve Eq. (3.49) for the universal anomaly χ, given the time interval Δt. To do so, we form the function

$$f(\chi) = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi^2 C(\alpha\chi^2) + (1-\alpha r_0)\chi^3 S(\alpha\chi^2) + r_0\chi - \sqrt{\mu}\Delta t \tag{3.59}$$

and its derivative

$$\begin{aligned}\frac{df(\chi)}{d\chi} &= 2\frac{r_0 v_r)_0}{\sqrt{\mu}}\chi C(z) + \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi^2\frac{dC(z)}{dz}\frac{dz}{d\chi} \\ &\quad + 3(1-\alpha r_0)\chi^2 S(z) + (1-\alpha r_0)\chi^3\frac{dS(z)}{dz}\frac{dz}{d\chi} + r_0\end{aligned} \tag{3.60}$$

where it is to be recalled that

$$z = \alpha\chi^2 \tag{3.61}$$

which means of course that

$$\frac{dz}{d\chi} = 2\alpha\chi \tag{3.62}$$

It turns out that

$$\begin{aligned}\frac{dS(z)}{dz} &= \frac{1}{2z}[C(z) - 3S(z)]\\ \frac{dC(z)}{dz} &= \frac{1}{2z}[1 - zS(z) - 2C(z)]\end{aligned} \tag{3.63}$$

Substituting Eqs. (3.61–3.63) into Eq. (3.60) and simplifying the result yields

$$\frac{df(\chi)}{d\chi} = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi\left[1 - \alpha\chi^2 S(z)\right] + (1 - \alpha r_0)\chi^2 C(z) + r_0 \tag{3.64}$$

With Eqs. (3.59) and (3.64), Newton's algorithm (Eq. 3.16) for the universal Kepler's equation becomes

$$\chi_{i+1} = \chi_i - \frac{\dfrac{r_0 v_r)_0}{\sqrt{\mu}}\chi_i{}^2 C(z_i) + (1 - \alpha r_0)\chi_i{}^3 S(z_i) + r_0\chi - \sqrt{\mu}\Delta t}{\dfrac{r_0 v_r)_0}{\sqrt{\mu}}\chi_i\left[1 - \alpha\chi_i{}^2 S(z_i)\right] + (1 - \alpha r_0)\chi_i{}^2 C(z_i) + r_0} \quad (z_i = \alpha\chi_i{}^2) \tag{3.65}$$

According to Chobotov (2002), a reasonable estimate for the starting value χ_0 is

$$\chi_0 = \sqrt{\mu}|\alpha|\Delta t \tag{3.66}$$

ALGORITHM 3.3

Solve the universal Kepler's equation for the universal anomaly χ given Δt, r_0, $v_r)_0$, and α. See Appendix D.14 for an implementation of this procedure in MATLAB.

1. Use Eq. (3.66) for an initial estimate of χ_0.
2. At any given step, having obtained χ_i from the previous step, calculate

$$f(\chi_i) = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi_i{}^2 C(z_i) + (1 - \alpha r_0)\chi_i{}^3 S(z_i) + r_0\chi_i - \sqrt{\mu}\Delta t$$

   and

$$f'(\chi_i) = \frac{r_0 v_r)_0}{\sqrt{\mu}}\chi_i\left[1 - \alpha\chi_i{}^2 S(z_i)\right] + (1 - \alpha r_0)\chi_i{}^2 C(z_i) + r_0$$

   where $z_i = \alpha\chi_i{}^2$.
3. Calculate $\text{ratio}_i = f(\chi_i)/f'(\chi_i)$.

4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of χ,

$$\chi_{i+1} = \chi_i - \text{ratio}_i$$

   Return to Step 2.
5. If $|\text{ratio}_i|$ is less than the tolerance, then accept χ_i as the solution to within the desired accuracy.

EXAMPLE 3.6

An earth satellite has an initial true anomaly of $\theta = 30°$, a radius of $r_0 = 10,000\,\text{km}$, and a speed of $v_0 = 10\,\text{km/s}$. Use the universal Kepler's equation to find the change in universal anomaly χ after 1 h and use that information to determine the true anomaly θ at that time.

Solution

Using the initial conditions, let us first determine the angular momentum and the eccentricity of the trajectory. From the orbit formula (Eq. 2.45) we have

$$h = \sqrt{\mu r_0(1+e\cos\theta_0)} = \sqrt{398,600 \cdot 10,000 \cdot (1+e\cos 30°)} = 63,135\sqrt{1+0.86602e} \tag{a}$$

This, together with the angular momentum formula (Eq. 2.31), yields

$$v_\perp)_0 = \frac{h}{r_0} = \frac{63,135\sqrt{1+0.86602e}}{10,000} = 6.3135\sqrt{1+0.86602e}$$

Using the radial velocity relation (Eq. 2.49) we find

$$v_r)_0 = \frac{\mu}{h}e\sin\theta_0 = \frac{398,600}{63,135\sqrt{1+0.86602e}}e\sin 30° = 3.1567\frac{e}{\sqrt{1+0.86602e}} \tag{b}$$

Since $v_r)_0{}^2 + v_\perp)_0{}^2 = v_0{}^2$, it follows that

$$\left(3.1567\frac{e}{\sqrt{1+0.86602e}}\right)^2 + \left(6.3135\sqrt{1+0.86602e}\right)^2 = 10^2$$

which simplifies to become $39.86e^2 - 17.563e - 60.14 = 0$. The only positive root of this quadratic equation is

$$e = 1.4682$$

Since e is greater than 1, the orbit is a hyperbola. Substituting this value of the eccentricity back into Eqs. (a) and (b) yields the angular momentum

$$h = 95,154\,\text{km}^2/\text{s}$$

as well as the initial radial speed

$$v_r)_0 = 3.0752\,\text{km/s}$$

The hyperbolic eccentric anomaly F_0 for the initial conditions may now be found from Eq. (3.44a),

$$\tanh\frac{F_0}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta_0}{2} = \sqrt{\frac{1.4682-1}{1.4682+1}}\tan\frac{30°}{2} = 0.16670$$

Solving for F_0 yields

$$F_0 = 0.23448\,\text{rad} \tag{c}$$

In the universal variable formulation, we calculate the semimajor axis of the orbit by means of Eq. (3.47),

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{95{,}154^2}{398{,}600}\frac{1}{1-1.4682^2} = -19{,}655\,\text{km} \tag{d}$$

The negative value is consistent with the fact that the orbit is a hyperbola. From Eq. (3.50) we get

$$\alpha = \frac{1}{a} = \frac{1}{-19{,}655} = -5.0878\left(10^{-5}\right)\text{km}^{-1}$$

which appears throughout the universal Kepler's equation.

We will use Algorithm 3.3 with an error tolerance of 10^{-6} to find the universal anomaly. From Eq. (3.66), our initial estimate is

$$\chi_0 = \sqrt{398{,}600}\cdot\left|-5.0878\left(10^{-5}\right)\right|\cdot 3600 = 115.6$$

Executing the algorithm yields the following steps:

$$\chi_0 = 115.6$$

Step 1:

$$\begin{aligned} f(\chi_0) &= -370{,}650.01 \\ f'(\chi_0) &= 26{,}956.300 \\ \text{ratio} &= -13.750033 \\ \chi_1 &= 115.6 - (-13.750033) = 129.35003 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat} \end{aligned}$$

Step 2:

$$\begin{aligned} f(\chi_1) &= 25{,}729.002 \\ f'(\chi_1) &= 30{,}776.401 \\ \text{ratio} &= 0.83599669 \\ \chi_2 &= 129.35003 - 0.83599669 = 128.51404 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat} \end{aligned}$$

Step 3:

$$\begin{aligned} f(\chi_2) &= 102.83891 \\ f'(\chi_2) &= 30{,}530.672 \\ \text{ratio} &= 3.3683800\left(10^{-3}\right) \\ \chi_3 &= 128.51404 - 3.3683800\left(10^{-3}\right) = 128.51067 \\ |\text{ratio}| &> 10^{-6} \quad \therefore \text{repeat} \end{aligned}$$

Step 4:

$$\begin{aligned} f(\chi_3) &= 1.6614116\left(10^{-3}\right) \\ f'(\chi_3) &= 30{,}529.686 \\ \text{ratio} &= 5.4419545\left(10^{-8}\right) \\ \chi_4 &= 128.51067 - 5.4419545\left(10^{-8}\right) = 128.51067 \\ |\text{ratio}| &< 10^{-6} \quad \therefore \text{stop} \end{aligned}$$

So we accept

$$\chi = 128.51\,\text{km}^{1/2}$$

as the solution after four iterations. Substituting this value of χ together with the semimajor axis (Eq. d) into Eq. (3.58) yields

$$F - F_0 = \frac{\chi}{\sqrt{-a}} = \frac{128.51}{\sqrt{-(-19{,}655)}} = 0.91664$$

It follows from Eq. (c) that the hyperbolic eccentric anomaly after 1 h is

$$F = 0.23448 + 0.91664 = 1.1511$$

Finally, we calculate the corresponding true anomaly using Eq. (3.44b),

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{1.4682+1}{1.4682-1}}\tanh\frac{1.1511}{2} = 1.1926$$

which means that after 1 h

$$\boxed{\theta = 100.04°}$$

Recall from Section 2.11 that the position **r** and velocity **v** on a trajectory at any time t can be found in terms of the position $\mathbf{r}_0$ and velocity $\mathbf{v}_0$ at time t_0 by means of the Lagrange f and g coefficients and their first derivatives,

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0 \tag{3.67}$$

$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \tag{3.68}$$

Eqs. (2.158) give f, g, $\dot{f}$, and $\dot{g}$ explicitly in terms of the change in true anomaly $\Delta\theta$ over the time interval $\Delta t = t - t_0$. The Lagrange coefficients can also be derived in terms of changes in the eccentric anomaly ΔE for elliptical orbits, ΔF for hyperbolas, or $\Delta\tan(\theta/2)$ for parabolas. However, if we take advantage of the universal variable formulation, we can cover all these cases with the same set of Lagrange coefficients. In terms of the universal anomaly χ and the Stumpff functions $C(z)$ and $S(z)$, the Lagrange coefficients are (Bond and Allman, 1996)

$$f = 1 - \frac{\chi^2}{r_0}C(\alpha\chi^2) \tag{3.69a}$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(\alpha\chi^2) \tag{3.69b}$$

$$\dot{f} = \frac{\sqrt{\mu}}{rr_0}\left[\alpha\chi^3 S(\alpha\chi^2) - \chi\right] \tag{3.69c}$$

$$\dot{g} = 1 - \frac{\chi^2}{r}C(\alpha\chi^2) \tag{3.69d}$$

The implementation of these four functions in MATLAB is found in Appendix D.15.

ALGORITHM 3.4

Given $\mathbf{r}_0$ and $\mathbf{v}_0$, find **r** and **v** at a time Δt later. See Appendix D.16 for an implementation of this procedure in MATLAB.

1. Use the initial conditions to find:
   a. The magnitude of $\mathbf{r}_0$ and $\mathbf{v}_0$,

$$r_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0} \qquad v_0 = \sqrt{\mathbf{v}_0 \cdot \mathbf{v}_0}$$

   b. The radial component of velocity $v_r)_0$ by projecting $\mathbf{v}_0$ onto the direction of $\mathbf{r}_0$,

$$v_r)_0 = \frac{\mathbf{v}_0 \cdot \mathbf{r}_0}{r_0}$$

c. The reciprocal α of the semimajor axis, using Eq. (3.48),

$$\alpha = \frac{2}{r_0} - \frac{{v_0}^2}{\mu}$$

The sign of α determines whether the trajectory is an ellipse ($\alpha > 0$), parabola ($\alpha = 0$), or hyperbola ($\alpha < 0$).

2. With r_0, $v_r)_0$, α, and Δt, use Algorithm 3.3 to find the universal anomaly χ.
3. Substitute α, r_0, Δt, and χ into Eqs. (3.69a) and (3.69b) to obtain f and g.
4. Use Eq. (3.67) to compute $\mathbf{r}$ followed by its magnitude r.
5. Substitute α, r_0, r, and χ into Eqs. (3.69c) and (3.69d) to obtain $\dot{f}$ and $\dot{g}$.
6. Use Eq. (3.68) to compute $\mathbf{v}$.

EXAMPLE 3.7

An earth satellite moves in the xy plane of an inertial frame with origin at the earth's center. Relative to that frame, the position and velocity of the satellite at time t_0 are

$$\begin{aligned} \mathbf{r}_0 &= 7000.0\hat{\mathbf{i}} - 12,124\hat{\mathbf{j}}\,(\text{km}) \\ \mathbf{v}_0 &= 2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}\,(\text{km/s}) \end{aligned} \tag{a}$$

Compute the position and velocity vectors of the satellite 60 min later using Algorithm 3.4.

Solution

Step 1:

$$\begin{aligned} r_0 &= \sqrt{7000.0^2 + (-12,124)^2} && = 14,000\,\text{km} \\ v_r)_0 &= \frac{2.6679 \cdot 7000.0 + 4.6210 \cdot (-12,124)}{14,000} && = -2.6679\,\text{km/s} \\ \alpha &= \frac{2}{14,000} - \frac{5.3359^2}{398,600} && = 7.1429\left(10^{-5}\right)\text{km}^{-1} \end{aligned}$$

The trajectory is an ellipse, because α is positive.

Step 2:

Using the results of Step 1, Algorithm 3.3 yields

$$\chi = 253.53\,\text{km}^{1/2}$$

which means

$$z = \alpha\chi^2 = 7.1429\left(10^{-5}\right) \cdot 253.53^2 = 4.5911$$

Step 3:

Substituting the above values of χ and z into Eqs. (3.69a) and (3.69b), we find

$$f = 1 - \frac{\chi^2}{r_0}C(\alpha\chi^2) = 1 - \frac{253.53^2}{14{,}000}\overbrace{C(4.5911)}^{0.3357} = -0.54123$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(\alpha\chi^2) = 3600 - \frac{253.53^3}{\sqrt{398{,}600}}\overbrace{S(4.5911)}^{0.13233} = 184.35\,\text{s}$$

Step 4:

$$\begin{aligned}\mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0 \\ &= (-0.54123)\left(7000.0\hat{\mathbf{i}} - 12.124\hat{\mathbf{j}}\right) + 184.35\left(2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}\right) \\ &= \boxed{-3296.8\hat{\mathbf{i}} + 7413.9\hat{\mathbf{j}}\ (\text{km})}\end{aligned}$$

Therefore, the magnitude of **r** is

$$r = \sqrt{(-3296.8)^2 + (7413.9)^2} = 8113.9\,\text{km}$$

Step 5:

$$\begin{aligned}\dot{f} &= \frac{\sqrt{\mu}}{rr_0}\left[\alpha\chi^3 S(\alpha\chi^2) - \chi\right] \\ &= \frac{\sqrt{398{,}600}}{8113.9\cdot 14{,}000}\left[7.1429(10^5)\cdot 253.53^3\cdot \overbrace{S(4.5911)}^{0.13233} - 253.53\right] \\ &= -0.00055298\,\text{s}^{-1}\end{aligned}$$

$$\dot{g} = 1 - \frac{\chi^2}{r}C(\alpha\chi^2) = 1 - \frac{253.53^2}{8113.9}\overbrace{C(4.5911)}^{0.3357} = -1.6593$$

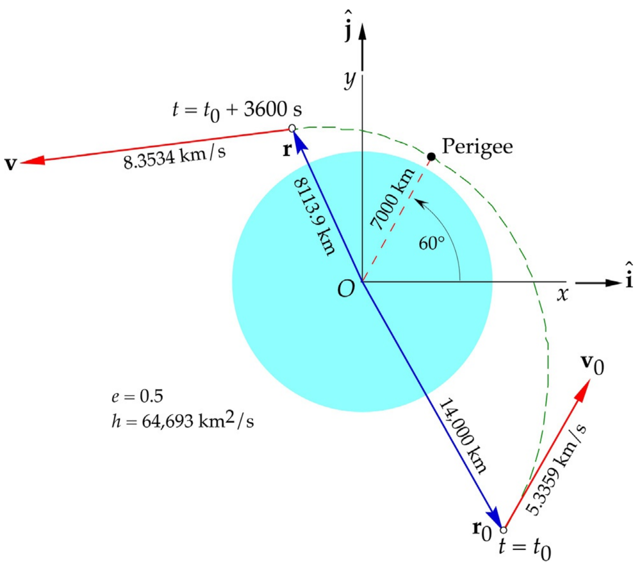


FIG. 3.20

Initial and final points on the geocentric trajectory of Example 3.7.

Step 6:

$$\begin{aligned}\mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \\ &= (-0.00055298)\left(7000.0\hat{\mathbf{i}} - 12.124\hat{\mathbf{j}}\right) + (-1.6593)\left(2.6679\hat{\mathbf{i}} + 4.6210\hat{\mathbf{j}}\right) \\ &= \boxed{-8.2977\hat{\mathbf{i}} - 0.96309\hat{\mathbf{j}}\ (\mathrm{km/s})}\end{aligned}$$

The initial and final position and velocity vectors, as well as the trajectory, are accurately illustrated in Fig. 3.20.

PROBLEMS

Section 3.2

3.1 If $f = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{6}\tan^3\frac{x}{2}$, then show that $df/dx = 1/(1+\cos x)^2$, thereby verifying the integral in Eq. (3.4).

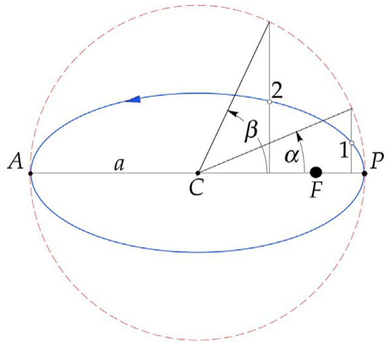
Section 3.4

3.2 Find the three positive roots of the equation $10e^{\sin x} = x^2 - 5x + 4$ to eight significant figures. Use
(a) Newton's method.
(b) Bisection method.

3.3 Find the first four nonnegative roots of the equation $\tan(x) = \tanh(x)$ to eight significant figures. Use
(a) Newton's method.
(b) Bisection method.

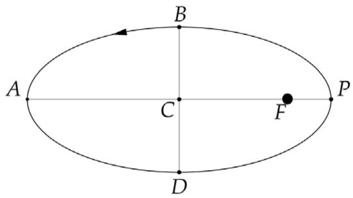
3.4 In terms of the eccentricity e, the period T, and the angles α and β (in radians), find the time t required to fly from point 1 to point 2 on the ellipse. C is the center of the ellipse.

$$\left\{\text{Ans.}: t = \frac{T}{2\pi}\left(\beta - \alpha - 2e\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right)\right\}$$



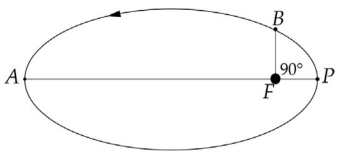
3.5 Calculate the time required to fly from P to B, in terms of the eccentricity e and the period T. B lies on the minor axis.

$$\left\{\text{Ans.}: \left(\frac{1}{4} - \frac{e}{2\pi}\right)T\right\}$$



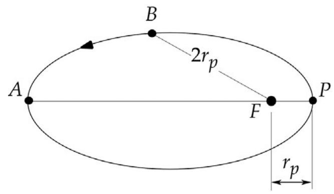
3.6 If the eccentricity of the elliptical orbit is 0.3, calculate, in terms of the period T, the time required to fly from P to B.

{Ans.: 0.156 T}



3.7 If the eccentricity of the elliptical orbit is 0.5, calculate, in terms of the period T, the time required to fly from P to B.

{Ans.: 0.170 T}



3.8 A satellite is in earth orbit for which the perigee altitude is 200 km and the apogee altitude is 600 km. Find the time interval during which the satellite remains above an altitude of 400 km.

{Ans.: 47.15 min}

3.9 An earth-orbiting satellite has a perigee radius of 7000 km and an apogee radius of 10,000 km.
(a) What true anomaly $\Delta\theta$ is swept out between $t = 0.5$ h and $t = 1.5$ h after perigee passage?
(b) What area is swept out by the position vector during that time interval?

{Ans.: (a) 128.7°; (b) $1.03(10^8)\ \text{km}^2$}

3.10 An earth-orbiting satellite has a period of 14 h and a perigee radius of 10,000 km. At time $t = 10$ h after perigee passage, determine
(a) The radial position.
(b) The speed.
(c) The radial component of the velocity.

{Ans.: (a) 42,356 km; (b) 2.303 km/s; (c) −1.271 km/s}

3.11 A satellite in earth orbit has perigee and apogee radii of $r_p = 7500$ km and $r_a = 16{,}000$ km, respectively. Find its true anomaly 40 min after passing the true anomaly of 80°.

{Ans.: 174.7°}

3.12 Show that the solution to $a\cos\theta + b\sin\theta = c$, where a, b, and c are given, is

$$\theta = \phi \pm \cos^{-1}\left(\frac{c}{a}\cos\phi\right)$$

where $\tan\phi = b/a$.

3.13 Verify the results of part (b) of Example 3.3.

Section 3.5

3.14 Calculate the time required for a spacecraft launched into a parabolic trajectory at a perigee altitude of 200 km to leave the earth's sphere of influence (see Table A.2).

{Ans.: 7.77 days}

3.15 A spacecraft on a parabolic trajectory around the earth has a perigee radius of 6600 km.

(a) How long does it take to coast from $\theta = -90°$ to $\theta = +90°$?

(b) How far is the spacecraft from the center of the earth 36 h after passing through perigee?

{Ans.: (a) 0.8897 h; (b) 304,700 km}

Section 3.6

3.16 A spacecraft on a hyperbolic trajectory around the earth has a perigee radius of 6600 km and a perigee speed of $1.2v_{\text{esc}}$.

(a) How long does it take to coast from $\theta = -90°$ to $\theta = +90°$?

(b) How far is the spacecraft from the center of the earth 24 h after passing through perigee?

{Ans.: (a) 0.9992 h; (b) 656,610 km}

3.17 A trajectory has a perigee velocity $1.1v_{\text{esc}}$ and a perigee altitude of 200 km. If at 10 a.m. the satellite is traveling toward the earth with a speed of 8 km/s, how far will it be from the earth's surface at 5 p.m. the same day?

{Ans.: 136,250 km}

3.18 An incoming object is sighted at an altitude of 100,000 km with a speed of 6 km/s and a flight path angle of −80°.

(a) Will it impact the earth or fly by?

(b) What is the time to impact or to closest approach?

{Partial Ans.: (b) 4 h 29 min}

Section 3.7

3.19 At a given instant, the radial position of an earth-orbiting satellite is 7200 km and its radial speed is 1 km/s. If the semimajor axis is 10,000 km, use Algorithm 3.3 to find the universal anomaly 60 min later. Check your result using Eq. (3.58).

3.20 At a given instant, a space object has the following position and velocity vectors relative to an earth-centered inertial frame of reference:

$$\mathbf{r}_0 = 20{,}000\hat{\mathbf{i}} - 105{,}000\hat{\mathbf{j}} - 19{,}000\hat{\mathbf{k}}\,(\text{km})$$
$$\mathbf{v}_0 = 0.9000\hat{\mathbf{i}} - 3.4000\hat{\mathbf{j}} - 1.5000\hat{\mathbf{k}}\,(\text{km/s})$$

Use Algorithm 3.4 to find **r** and **v** 2 h later.

{Ans.: $\mathbf{r} = 26{,}338\hat{\mathbf{i}} - 128{,}750\hat{\mathbf{j}} - 29{,}656\hat{\mathbf{k}}\,(\text{km})$,
$\mathbf{v} = 0.86280\hat{\mathbf{i}} - 3.2116\hat{\mathbf{j}} - 1.4613\hat{\mathbf{k}}\,(\text{km/s})$}

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