CHAPTER

ORBITAL MANEUVERS

6

6.1 INTRODUCTION

Orbital maneuvers transfer a spacecraft from one orbit to another. Orbital changes can be dramatic, such as the transfer from a low earth parking orbit to an interplanetary trajectory. They can also be quite small, as in the final stages of the rendezvous of one spacecraft with another. Changing orbits requires the firing of onboard rocket engines. We will be concerned primarily with impulsive maneuvers in which the rockets fire in relatively short bursts to produce the required velocity change (delta-v).

We start with the classical, energy-efficient Hohmann transfer maneuver and generalize it to the bielliptic Hohmann transfer to see if even more efficiency can be obtained. The phasing maneuver, a form of Hohmann transfer, is considered next. This is followed by a study of non-Hohmann transfer maneuvers with and without rotation of the apse line. We then analyze chase maneuvers, which requires solving Lambert's problem as explained in Chapter 5. Energy-demanding chase maneuvers may be impractical for low earth orbits, but they are necessary for interplanetary missions, as we shall see in Chapter 8. After having focused on impulsive transfers between coplanar orbits, we finally turn our attention to plane change maneuvers and their delta-v requirements, which can be very large.

The chapter concludes with a brief consideration of some orbital transfers in which the propulsion system delivers the impulse during a finite (perhaps very long) time interval instead of instantaneously. This makes it difficult to obtain closed-form solutions, so we illustrate the use of the numerical integration techniques presented in Chapter 1 as an alternative.

6.2 IMPULSIVE MANEUVERS

Impulsive maneuvers are those in which brief firings of onboard rocket motors change the magnitude and direction of the velocity vector instantaneously. During an impulsive maneuver, the position of the spacecraft is considered to be fixed; only the velocity changes. The impulsive maneuver is an idealization by means of which we can avoid having to solve the equations of motion (Eq. 2.22) with the rocket thrust included. The idealization is satisfactory for those cases in which the position of the spacecraft changes only slightly during the time that the maneuvering rockets fire. This is true for high-thrust rockets with burn times that are short compared with the coasting time of the vehicle.

Each impulsive maneuver results in a change $\Delta\mathbf{v}$ in the velocity of the spacecraft. $\Delta\mathbf{v}$ can represent a change in the magnitude (pumping maneuver) or the direction (cranking maneuver) of the velocity

Orbital Mechanics for Engineering Students. https://doi.org/10.1016/B978-0-08-102133-0.00006-4

vector, or both. The magnitude Δv of the velocity increment is related to Δm, the mass of propellant consumed, by the ideal rocket equation (see Eq. 13.30).

$$\frac{\Delta m}{m} = 1 - e^{\frac{\Delta v}{I_{sp} g_0}} \tag{6.1}$$

where m is the mass of the spacecraft before the burn, g_0 is the sea level standard acceleration of gravity, and I_{sp} is the specific impulse of the propellants. Specific impulse is defined as follows:

$$I_{sp} = \frac{\text{Thrust}}{\text{Sea-level weight rate of fuel consumption}}$$

Specific impulse has units of seconds, and it is a measure of the performance of a rocket propulsion system. I_{sp} for some common propellant combinations is shown in Table 6.1. Fig. 6.1 is a graph of Eq. (6.1) for a range of specific impulses. Note that for Δv's on the order of 1 km/s or higher, the required propellant exceeds 25% of the spacecraft mass before the burn.

There are presently no refueling stations in space, so a mission's delta-v schedule must be carefully planned to minimize the propellant mass carried aloft in favor of payload.

Table 6.1 Some typical specific impulses

Propellant	I_{sp} (s)
Cold gas	50
Monopropellant hydrazine	230
Solid propellant	290
Nitric acid/monomethylhydrazine	310
Liquid oxygen/liquid hydrogen	455
Ion propulsion	>3000

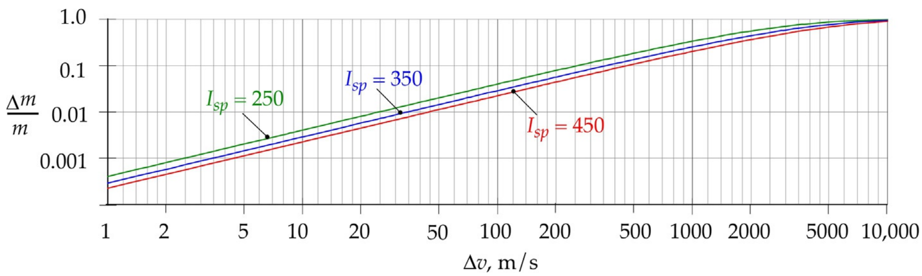


FIG. 6.1

Propellant mass fraction versus Δv for typical specific impulses.

6.3 HOHMANN TRANSFER

Walter Hohmann (1880–1945) was a German engineer whose interest in early rocketry led him to discover the orbital transfer maneuver that now bears his name. In a book published in 1925 (Hohmann, 1925), he showed that the Hohmann transfer is the most energy-efficient two-impulse maneuver for transferring between two coplanar circular orbits sharing a common focus. The Hohmann transfer is an elliptical orbit tangent to both circles on its apse line, as illustrated in Fig. 6.2. The periapsis and apoapsis of the transfer ellipse are the radii of the inner and outer circles, respectively. Obviously, only one-half of the ellipse is flown during the maneuver, which can occur in either direction, from the inner to the outer circle, or vice versa.

It may be helpful in sorting out orbit transfer strategies to use the fact that the energy of an orbit depends only on its semimajor axis a. Recall that for an ellipse (Eq. 2.80), the specific energy is negative,

$$\varepsilon = -\frac{\mu}{2a}$$

Increasing the energy requires reducing its magnitude, to make ε less negative. Therefore, the larger the semimajor axis, the more energy the orbit has. In Fig. 6.2, the energies increase as we move from the inner circle to the outer circle.

Starting at A on the inner circle, a velocity increment Δv_A in the direction of flight is required to boost the vehicle onto the higher energy elliptical trajectory. After coasting from A to B, another forward velocity increment Δv_B places the vehicle on the still higher energy, outer circular orbit. Without the latter delta-v burn, the spacecraft would, of course, remain on the Hohmann transfer ellipse and return to A. The total energy expenditure is reflected in the total delta-v requirement, $\Delta v_{\text{total}} = \Delta v_A + \Delta v_B$.

The same total delta-v is required if the transfer begins at B on the outer circular orbit. Since moving to the lower energy inner circle requires lowering the energy of the spacecraft, the Δv's must be accomplished by retrofires. That is, the thrust of the maneuvering rocket is directed opposite to the flight direction to act as a brake on the motion. Since Δv represents the same propellant expenditure regardless of the direction the thruster is aimed, when summing up Δv's, we are concerned only with their magnitudes.

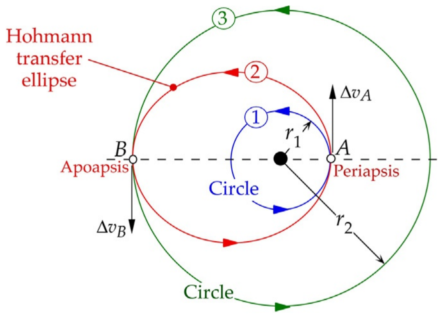


FIG. 6.2

Hohmann transfer.

Recall that the eccentricity of an elliptical orbit is found from its radius to periapsis r_p and its radius to apoapsis r_a by means of Eq. (2.84),

$$e = \frac{r_a - r_p}{r_a + r_p}$$

The radius to periapsis is given by Eq. (2.50),

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e}$$

Combining these last two expressions yields

$$r_p = \frac{h^2}{\mu}\frac{1}{1+\dfrac{r_a - r_p}{r_a + r_p}} = \frac{h^2}{\mu}\frac{r_a + r_p}{2r_a}$$

Solving for the angular momentum h, we get

$$h = \sqrt{2\mu}\sqrt{\frac{r_a r_p}{r_a + r_p}} \tag{6.2}$$

This is a useful formula for analyzing Hohmann transfers, because knowing h we can find the apsidal velocities from Eq. (2.31). Note that for circular orbits ($r_a = r_p$), Eq. (6.2) yields

$$h = \sqrt{\mu r} \quad \text{(circular orbit)}$$

Alternatively, one may prefer to compute the velocities by means of the energy equation (Eq. 2.81) in the form

$$v = \sqrt{2\mu}\sqrt{\frac{1}{r} - \frac{1}{2a}} \tag{6.3}$$

This of course yields Eq. (2.63) for circular orbits.

EXAMPLE 6.1

A 2000-kg spacecraft is in a 480 km by 800 km earth orbit (orbit 1 in Fig. 6.3). Find

(a) The Δv required at perigee A to place the spacecraft in a 480 km by 16,000 km transfer ellipse (orbit 2).
(b) The Δv (apogee kick) required at B of the transfer orbit to establish a circular orbit of 16,000 km altitude (orbit 3).
(c) The total required propellant if the specific impulse is 300 s.

Solution

Since we know the perigee and apogee of all three of the orbits, let us first use Eq. (6.2) to calculate their angular momenta.

Orbit 1: $r_p = 6378 + 480 = 6858\,\text{km} \qquad r_a = 6378 + 800 = 7178\,\text{km}$

$$\therefore h_1 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{7178 \cdot 6858}{7178 + 6858}} = 52{,}876.5\,\text{km/s}^2 \tag{a}$$

Orbit 2: $r_p = 6378 + 480 = 6858\,\text{km} \qquad r_a = 6378 + 16,000 = 22,378\,\text{km}$

$$\therefore h_2 = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{22{,}378 \cdot 6858}{22{,}378 + 6858}} = 64{,}689.5\,\text{km/s}^2 \tag{b}$$

Orbit 3: $r_a = r_p = 22,378\,\text{km}$

$$\therefore h_3 = \sqrt{398{,}600 \cdot 22{,}378} = 94{,}445.1\,\text{km/s}^2 \tag{c}$$

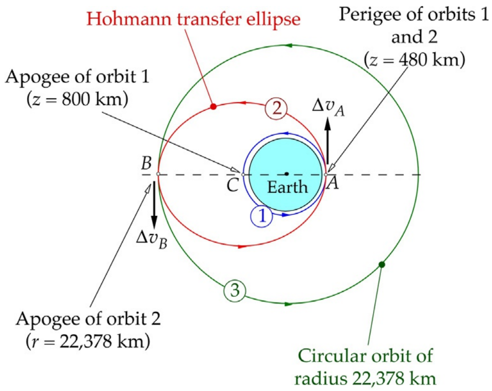


FIG. 6.3

Hohmann transfer between two earth orbits.

(a) The speed on orbit 1 at point A is

$$v_A)_1 = \frac{h_1}{r_A} = \frac{52,876}{6858} = 7.71019\,\text{km/s}$$

The speed on orbit 2 at point A is

$$v_A)_2 = \frac{h_2}{r_A} = \frac{64,689.5}{6858} = 9.43271\,\text{km/s}$$

Therefore, the delta-v required at point A is

$$\Delta v_A = v_A)_2 - v_A)_1 = \boxed{1.7225\,\text{km/s}}$$

(b) The speed on orbit 2 at point B is

$$v_B)_2 = \frac{h_2}{r_B} = \frac{64,689.5}{22,378} = 2.89076\,\text{km/s}$$

The speed on orbit 3 at point B is

$$v_B)_3 = \frac{h_3}{r_B} = \frac{94,445.1}{22,378} = 4.22044\,\text{km/s}$$

Hence, the apogee kick required at point B is

$$\Delta v_B = v_B)_3 - v_B)_2 = \boxed{1.3297\,\text{km/s}}$$

(c) The total delta-v requirement for this Hohmann transfer is

$$\Delta v_{\text{total}} = |\Delta v_A| + |\Delta v_B| = 1.7225 + 1.3297 = 3.0522\,\text{km/s}$$

According to Eq. (6.1) (converting velocity to m/s),

$$\frac{\Delta m}{m} = 1 - e^{-\frac{3052.2}{300 \cdot 9.807}} = 0.64563$$

Therefore, the mass of propellant expended is

$$\boxed{\Delta m = 0.64563 \cdot 2000 = 1291.3\,\text{kg}}$$

In the previous example the initial orbit of the Hohmann transfer sequence was an ellipse rather than a circle. Since no real orbit is perfectly circular, we must generalize the notion of a Hohmann transfer to include two-impulse transfers between elliptical orbits that are coaxial (i.e., share the same apse line), as shown in Fig. 6.4. The transfer ellipse must be tangent to both the initial and target ellipses 1 and 2. As can be seen, there are two such transfer orbits, 3 and 3′. It is not immediately obvious which of the two requires the lowest energy expenditure.

To find out which is the best transfer orbit in general, we must calculate the individual total delta-v requirement for orbits 3 and 3′. This requires finding the velocities at A, A', B, and B' for each pair of orbits having those points in common. We employ Eq. (6.2) to evaluate the angular momentum of each of the four orbits in Fig. 6.4.

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_A r_{A'}}{r_A + r_{A'}}} \quad h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_{B'}}{r_B + r_{B'}}} \quad h_3 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} \quad h_{3'} = \sqrt{2\mu}\sqrt{\frac{r_{A'} r_{B'}}{r_{A'} + r_{B'}}}$$

From these we obtain the velocities,

$$\begin{aligned} v_A)_1 &= \frac{h_1}{r_A} & v_A)_3 &= \frac{h_3}{r_A} \\ v_B)_2 &= \frac{h_2}{r_B} & v_B)_3 &= \frac{h_3}{r_B} \\ v_{A'})_1 &= \frac{h_1}{r_{A'}} & v_{A'})_{3'} &= \frac{h_{3'}}{r_{A'}} \\ v_{B'})_2 &= \frac{h_2}{r_{B'}} & v_{B'})_{3'} &= \frac{h_{3'}}{r_{B'}} \end{aligned}$$

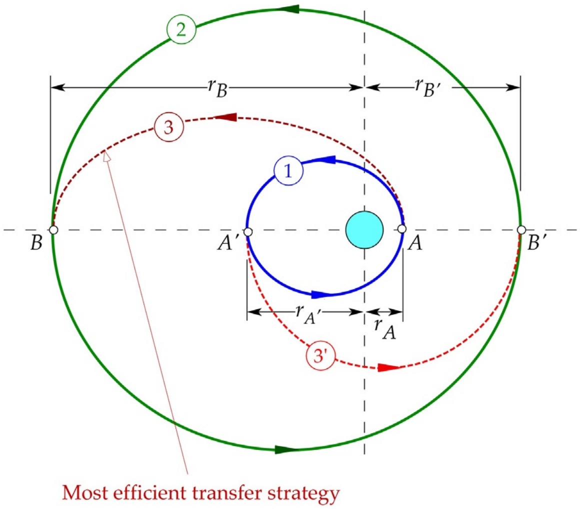
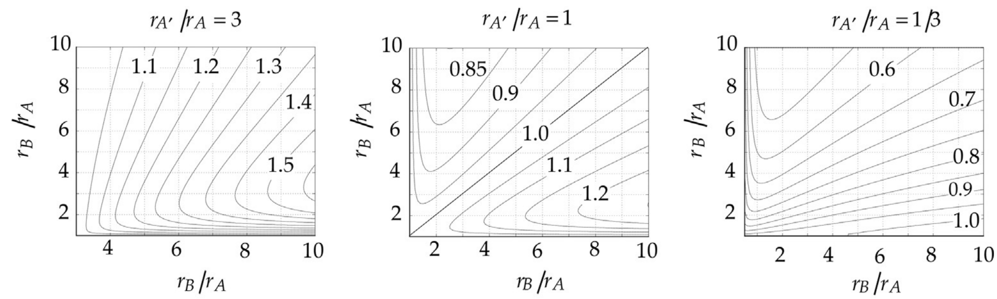


FIG. 6.4

Hohmann transfers between coaxial elliptical orbits. In this illustration, $r_{A'}/r_A = 3$, $r_B/r_A = 8$, and $r_{B'}/r_A = 4$.



(a) A is periapsis of orbit 1 (b) Orbit 1 is circular (c) A is apoapsis of orbit 1

FIG. 6.5

Contour plots of $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3$ for different relative sizes of the ellipses in Fig. 6.4. Note that $r_B > r_{A'}$ and $r_{B'} > r_A$.

These lead to the delta-v's

$$\Delta v_A = |v_A)_3 - v_A)_1| \quad \Delta v_B = |v_B)_2 - v_B)_3| \quad \Delta v_{A'} = |v_{A'})_{3'} - v_{A'})_1| \quad \Delta v_{B'} = |v_{B'})_2 - v_{B'})_{3'}|$$

and, finally, to the total delta-v requirement for the two possible transfer trajectories,

$$\Delta v_{\text{total}})_3 = \Delta v_A + \Delta v_B \quad \Delta v_{\text{total}})_{3'} = \Delta v_{A'} + \Delta v_{B'}$$

If $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3 > 1$, then orbit 3 is the most efficient. On the other hand, if $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3 < 1$, then orbit $3'$ is more efficient than orbit 3.

Three contour plots of $\Delta v_{\text{total}})_{3'}/\Delta v_{\text{total}})_3$ are shown in Fig. 6.5, for three different shapes of inner orbit 1 of Fig. 6.4. Fig. 6.5a is for $r_{A'}/r_A = 3$, which is the situation represented in Fig. 6.4, in which point A is the periapsis of the initial ellipse. In Fig. 6.5b $r_{A'}/r_A = 1$, which means the starting ellipse is a circle. Finally, in Fig. 6.5c $r_{A'}/r_A = 1/3$, which corresponds to an initial orbit of the same shape as orbit 1 in Fig. 6.4, but with point A being the apoapsis instead of periapsis.

Fig. 6.5a, for which $r_{A'} > r_A$, implies that if point A is the periapsis of orbit 1, then transfer orbit 3 is the most efficient. Fig. 6.5c, for which $r_{A'} < r_A$, shows that if point A' is the periapsis of orbit 1, then transfer orbit $3'$ is the most efficient. Together, these results lead us to conclude that it is most efficient for the transfer orbit to begin at the periapsis on inner orbit 1, where its kinetic energy is greatest, regardless of the shape of the outer target orbit. If the starting orbit is a circle, then Fig. 6.5b shows that transfer orbit $3'$ is the most efficient if $r_{B'} > r_B$. That is, from an inner circular orbit, the transfer ellipse should terminate at apoapsis of the outer target ellipse, where the speed is slowest.

If Hohmann transfer is in the reverse direction (i.e., to a lower energy inner orbit), the above analysis still applies, since the same total delta-v is required whether the Hohmann transfer runs forward or backward. Thus, from an outer circle or ellipse, to an inner ellipse the most energy-efficient transfer ellipse terminates at periapsis of the inner target orbit. If the inner orbit is a circle, the transfer ellipse should start at apoapsis of the outer ellipse.

We close this section with an illustration of the careful planning required for one spacecraft to rendezvous with another at the end of a Hohmann transfer.

EXAMPLE 6.2

A spacecraft returning from a lunar mission approaches earth on a hyperbolic trajectory. At its closest approach A it is at an altitude of 5000 km, traveling at 10 km/s. At A retrorockets are fired to lower the spacecraft into a 500-km-altitude circular orbit, where it is to rendezvous with a space station. Find the location of the space station at retrofire so that rendezvous will occur at B (Fig. 6.6).

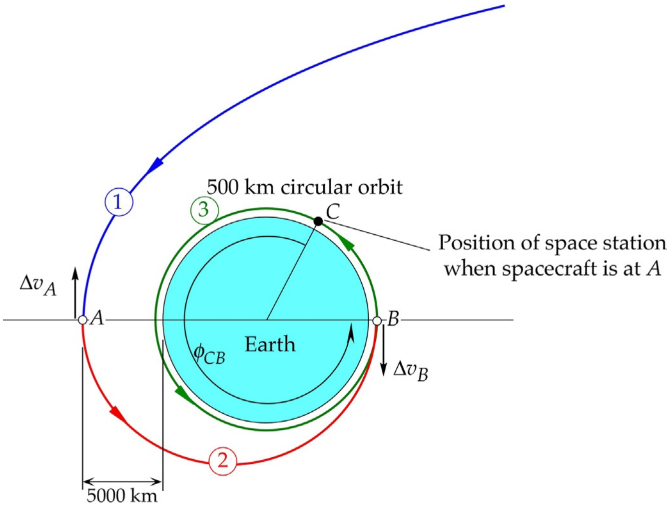


FIG. 6.6

Relative position of spacecraft and space station at beginning of the transfer ellipse.

Solution

The time of flight from A to B is one-half the period T_2 of elliptical transfer orbit 2. While the spacecraft coasts from A to B, the space station coasts through the angle ϕ_{CB} from C to B. Hence, this mission has to be carefully planned and executed, going all the way back to lunar departure, so that the two vehicles meet at B.

According to Eq. (2.83), to find the period T_2 we need to only determine the semimajor axis of orbit 2. The apogee and perigee of orbit 2 are

$$r_A = 5000 + 6378 = 11{,}378\,\text{km} \qquad r_B = 500 + 6378 = 6878\,\text{km}$$

Therefore, the semimajor axis is

$$a = \frac{1}{2}(r_A + r_B) = 9128\,\text{km}$$

From this we obtain

$$T_2 = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 9128^{3/2} = 8679.1\,\text{s} \tag{a}$$

The period of circular orbit 3 is

$$T_3 = \frac{2\pi}{\sqrt{\mu}} r_B{}^{3/2} = \frac{2\pi}{\sqrt{398{,}600}} 6878^{3/2} = 5676.8\,\text{s} \tag{b}$$

The time of flight from C to B on orbit 3 must equal the time of flight from A to B on orbit 2.

$$\Delta t_{CB} = \frac{1}{2} T_2 = \frac{1}{2} \cdot 8679.1 = 4339.5\,\text{s}$$

Since orbit 3 is a circle, its angular velocity, unlike an ellipse, is constant. Therefore, we can write

$$\frac{\phi_{CB}}{\Delta t_{CB}} = \frac{360°}{T_3} \Rightarrow \phi_{CB} = \frac{4339.5}{5676.8} \cdot 360 = \boxed{275.2°}$$

(The reader should verify that the total delta-v required to lower the spacecraft from the hyperbola into the parking orbit is 5.749 km/s. According to Eq. (6.1), that means over 85% of the returning spacecraft mass must consist of propellant!)

6.4 BIELLIPTIC HOHMANN TRANSFER

A Hohmann transfer from circular orbit 1 to circular orbit 4 in Fig. 6.7 is the dotted ellipse lying inside the outer circle, outside the inner circle, and tangent to both. A bielliptic Hohmann transfer uses two coaxial semiellipses, 2 and 3, which extend beyond the outer target orbit. Each of the two ellipses is tangent to one of the circular orbits, and they are tangent to each other at B, which is the apoapsis of both. The idea is to place B sufficiently far from the focus that the Δv_B will be very small. In fact, as r_B approaches infinity (where the orbital speed is zero), Δv_B approaches zero. For the bielliptic scheme to be more energy efficient than a Hohmann transfer, it must be true that

$$\Delta v_{\text{total}})_{\text{bielliptical}} < \Delta v_{\text{total}})_{\text{Hohmann}}$$

Let v_0 be speed in circular inner orbit 1,

$$v_0 = \sqrt{\frac{\mu}{r_A}}$$

Then calculating the total delta-v requirements of the Hohmann and bielliptic transfers leads to the following two expressions, respectively,

$$\begin{aligned} \Delta \bar{v}_{\text{H}} &= \frac{1}{\sqrt{\alpha}} - \frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha(1+\alpha)}} - 1 \\ \Delta \bar{v}_{\text{BE}} &= \sqrt{\frac{2(\alpha+\beta)}{\alpha\beta}} - \frac{1+\sqrt{\alpha}}{\sqrt{\alpha}} - \sqrt{\frac{2}{\beta(1+\beta)}}(1-\beta) \end{aligned} \tag{6.4a}$$

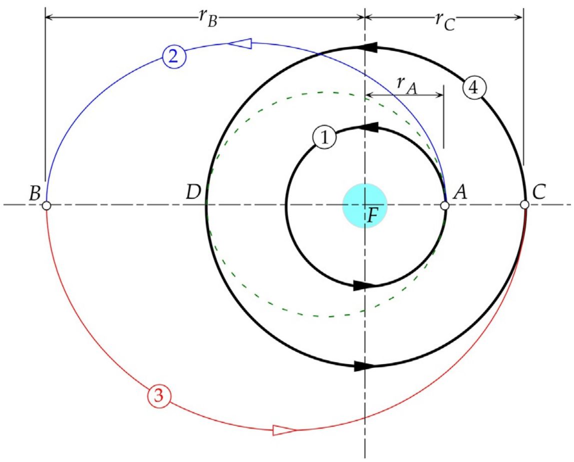


FIG. 6.7

Bielliptic transfer from inner orbit 1 to outer orbit 4.

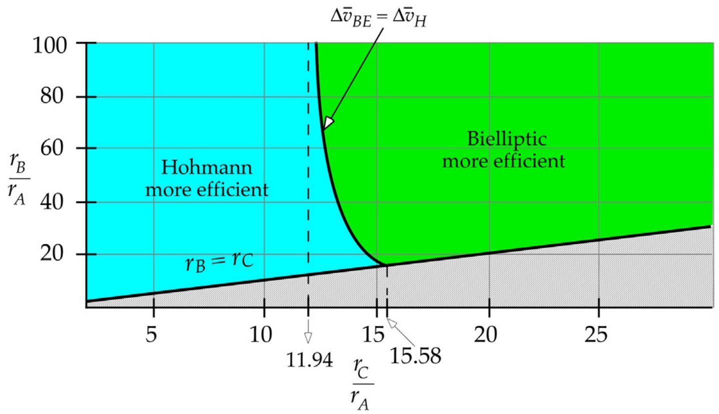


FIG. 6.8

Orbits for which a bielliptic transfer is either less efficient or more efficient than a Hohmann transfer.

where the nondimensional terms are

$$\Delta\bar{v}_H = \left.\frac{\Delta v_{\text{total}}}{v_0}\right)_{\text{Hohmann}} \qquad \Delta\bar{v}_{BE} = \left.\frac{\Delta v_{\text{total}}}{v_0}\right)_{\text{bielliptical}} \qquad \alpha = \frac{r_C}{r_A} \qquad \beta = \frac{r_B}{r_A} \tag{6.4b}$$

Plotting the difference between $\Delta\bar{v}_H$ and $\Delta\bar{v}_{BE}$ as a function of α and β reveals the regions in which the difference is positive, negative, or zero. These are shown in Fig. 6.8.

From the figure we see that if the radius of the outer circular target orbit (r_C) is less than 11.94 times that of the inner one (r_A), then the standard Hohmann maneuver is the more energy efficient. If the ratio exceeds 15.58, then the bielliptic strategy is better in that regard. Between those two ratios, large values of the apoapsis radius r_B favor bielliptic transfer, while smaller values favor Hohmann transfer.

Small gains in energy efficiency may be more than offset by the much longer flight times around bielliptic trajectories compared with the time of flight on the single semiellipse of Hohmann transfer.

EXAMPLE 6.3

Find the total delta-v requirement for bielliptic Hohmann transfer from a geocentric circular orbit of 7000 km radius to one of 105,000 km radius. Let the apogee of the first ellipse be 210,000 km. Compare the delta-v schedule and total flight time with that for an ordinary single Hohmann transfer ellipse (see Fig. 6.9).

Solution

Since

$$r_A = 7000\,\text{km} \quad r_B = 210{,}000\,\text{km} \quad \text{and} \quad r_C = r_D = 105{,}000\,\text{km}$$

we have $r_B/r_A = 30$ and $r_C/r_A = 15$, so that from Fig. 6.8 it is apparent right away that bielliptic transfer will be the more energy efficient.

To do the delta-v analysis requires analyzing each of the five orbits.

Orbit 1:

Since this is a circular orbit, we have, simply,

$$v_A)_1 = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{398{,}600}{7000}} = 7.546\,\text{km/s} \tag{a}$$

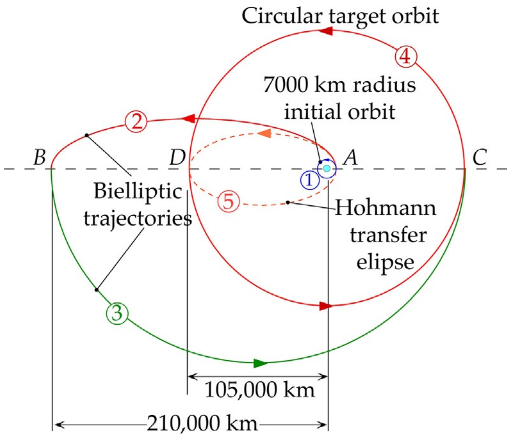


FIG. 6.9

Bielliptic transfer.

Orbit 2:

For this transfer ellipse, Eq. (6.2) yields

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2 \cdot 398,600}\sqrt{\frac{7000 \cdot 210,000}{7000 + 210,000}} = 73,487\,\mathrm{km^2/s}$$

Therefore,

$$v_A)_2 = \frac{h_2}{r_A} = \frac{73,487}{7000} = 10.498\,\mathrm{km/s} \tag{b}$$

$$v_B)_2 = \frac{h_2}{r_B} = \frac{73,487}{210,000} = 0.34994\,\mathrm{km/s} \tag{c}$$

Orbit 3:

For the second transfer ellipse, we have

$$h_3 = \sqrt{2 \cdot 398,600}\sqrt{\frac{105,000 \cdot 210,000}{105,000 + 210,000}} = 236,230\,\mathrm{km^2/s}$$

From this we obtain

$$v_B)_3 = \frac{h_3}{r_B} = \frac{236,230}{210,000} = 1.1249\,\mathrm{km/s} \tag{d}$$

$$v_C)_3 = \frac{h_3}{r_C} = \frac{236,230}{105,000} = 2.2498\,\mathrm{km/s} \tag{e}$$

Orbit 4:

The target orbit, like orbit 1, is a circle, which means

$$v_C)_4 = v_D)_4 = \sqrt{\frac{398,600}{105,000}} = 1.9484\,\mathrm{km/s} \tag{f}$$

For the bielliptic maneuver, the total delta-v is, therefore,

$$\begin{aligned}
\Delta v_{\text{total}})_{\text{bi-elliptical}} &= \Delta v_{\mathrm{A}} + \Delta v_{\mathrm{B}} + \Delta v_{\mathrm{C}} \\
&= \left|v_A)_2 - v_A)_1\right| + \left|v_B)_3 - v_B)_2\right| + \left|v_C)_4 - v_C)_3\right| \\
&= |10.498 - 7.546| + |1.1249 - 0.34994| + |1.9484 - 2.2498|
\end{aligned}$$

or

$$\boxed{\Delta v_{\text{total}})_{\text{bielliptical}} = 4.0285\text{km/s}} \tag{g}$$

The semimajor axes of transfer orbits 2 and 3 are

$$a_2 = \frac{1}{2}(7000 + 210,000) = 108,500\text{km} \qquad a_3 = \frac{1}{2}(105,000 + 210,000) = 157,500\text{km}$$

With this information and the period formula, Eq. (2.83), the time of flight for the two semiellipses of the bielliptic transfer is found to be

$$t_{\text{bielliptical}} = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}}a_2^{3/2} + \frac{2\pi}{\sqrt{\mu}}a_3^{3/2}\right) = 488,870\text{s} = \boxed{5.66\text{days}} \tag{h}$$

For the Hohmann transfer ellipse 5,

$$h_5 = \sqrt{2 \cdot 398,600}\sqrt{\frac{7000 \cdot 105,000}{7000 + 105,000}} = 72,330\text{km}^2/\text{s}$$

Hence,

$$v_A)_5 = \frac{h_5}{r_A} = \frac{72,330}{7000} = 10.333\text{km/s} \tag{i}$$

$$v_D)_5 = \frac{h_5}{r_D} = \frac{72,330}{105,000} = 0.68886\text{km/s} \tag{j}$$

It follows that

$$\begin{aligned}
\Delta v_{\text{total}})_{\text{Hohmann}} &= \left|v_A)_5 - v_A)_1\right| + \left|v_D)_5 - v_D)_1\right| \\
&= (10.333 - 7.546) + (1.9484 - 0.68886) \\
&= 2.7868 + 1.2595
\end{aligned}$$

or

$$\boxed{\Delta v_{\text{total}})_{\text{Hohmann}} = 4.0463\text{km/s}} \tag{k}$$

This is only slightly (0.44%) larger than that of the bielliptic transfer.

Since the semimajor axis of the Hohmann semiellipse is

$$\boxed{a_5 = \frac{1}{2}(7000 + 105,000) = 56,000\text{km}}$$

the time of flight from A to D is

$$t_{\text{Hohmann}} = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}}a_5^{3/2}\right) = 65,942\text{s} = \boxed{0.763\text{days}} \tag{l}$$

The time of flight of the bielliptic maneuver is over seven times longer than that of the Hohmann transfer.

6.5 PHASING MANEUVERS

A phasing maneuver is a two-impulse Hohmann transfer from and then back to the same orbit, as illustrated in Fig. 6.10. The Hohmann transfer ellipse is the phasing orbit with a period selected to return the spacecraft to the main orbit within a specified time. Phasing maneuvers are used to change the position of a spacecraft in its orbit. If two spacecraft, destined to rendezvous, are at different locations in the same orbit, then one of them may perform a phasing maneuver to catch the other one. Communications and weather satellites in geostationary earth orbit use phasing maneuvers to move to new

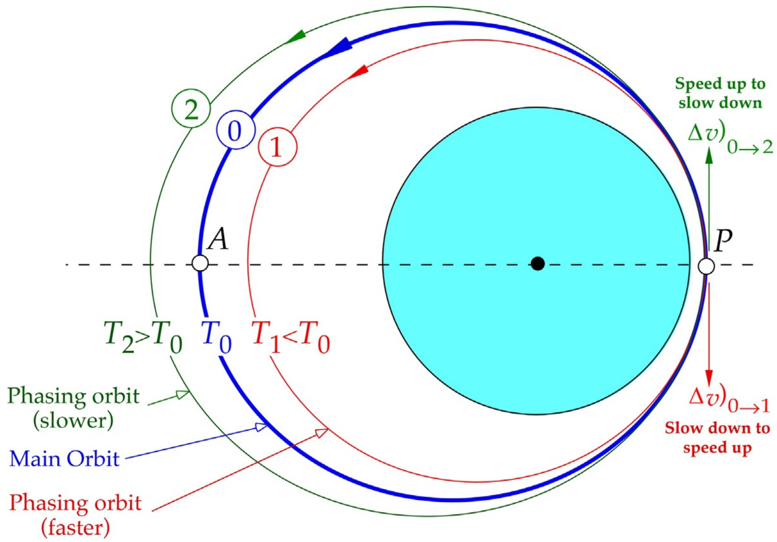


FIG. 6.10

Main orbit (0) and examples of two phasing orbits: faster (1) and slower (2). T_0 is the period of the main orbit.

locations above the equator. In that case, the rendezvous is with an empty point in space rather than with a physical target. In Fig. 6.10, phasing orbit 1 might be used to return to P in less than one period of the main orbit. This would be appropriate if the target is ahead of the chasing vehicle. Note that a retrofire is required to enter orbit 1 at P. That is, it is necessary to slow the spacecraft down to speed it up, relative to the main orbit. If the chaser is ahead of the target, then phasing orbit 2 with its longer period might be appropriate. A forward fire of the thruster boosts the spacecraft's speed to slow it down.

Once the period T of the phasing orbit is established, then Eq. (2.8) should be used to determine the semimajor axis of the phasing ellipse,

$$a = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{2/3} \tag{6.5}$$

With the semimajor axis established, the radius of point A opposite to P is obtained from the fact that $2a = r_P + r_A$. Eq. (6.2) may then be used to obtain the angular momentum.

EXAMPLE 6.4

Spacecraft at A and B are in the same orbit (1). At the instant shown in Fig. 6.11 the chaser vehicle at A executes a phasing maneuver so as to catch the target spacecraft back at A after just one revolution of the chaser's phasing orbit (2). What is the required total delta-v?

Solution

We must find the angular momenta of orbits 1 and 2 so that we can use Eq. (2.31) to find the velocities on orbits 1 and 2 at point A. (We can alternatively use energy, Eq. (2.81), to find the speeds at A.) These velocities furnish the delta-v required to leave orbit 1 for orbit 2 at the beginning of the phasing maneuver and to return to orbit 1 at the end.

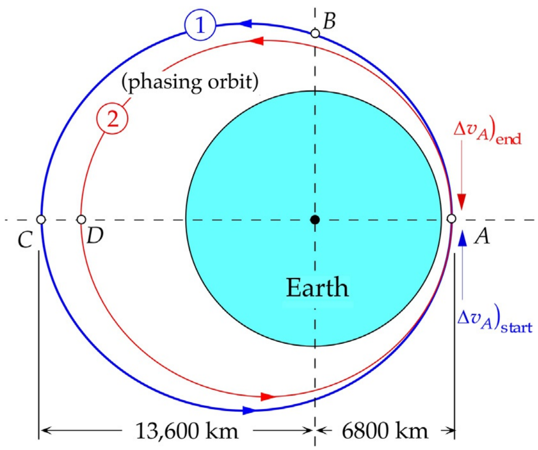


FIG. 6.11

Phasing maneuver.

Angular momentum of orbit 1

From Fig. 6.11 we observe that perigee and apogee radii of orbit 1 are, respectively,

$$r_A = 6800\,\text{km} \quad r_C = 13{,}600\,\text{km}$$

It follows from Eq. (6.2) that the orbit's angular momentum is

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_A r_C}{r_A + r_C}} = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{6800 \cdot 13{,}600}{6800 + 13{,}600}} = 60{,}116\,\text{km/s}^2$$

Angular momentum of orbit 2

The phasing orbit must have a period T_2 equal to the time it takes the target vehicle at B to coast around to point A on orbit 1. That flight time equals the period of orbit 1 minus the flight time t_{AB} from A to B. That is,

$$T_2 = T_1 - t_{AB} \tag{a}$$

The period of orbit 1 is found by computing its semimajor axis,

$$a_1 = \frac{1}{2}(r_A + r_C) = 10{,}200\,\text{km}$$

and substituting that result into Eq. (2.83),

$$T_1 = \frac{2\pi}{\sqrt{\mu}}a_1{}^{3/2} = \frac{2\pi}{\sqrt{398{,}600}}10{,}200^{3/2} = 10{,}252\,\text{s} \tag{b}$$

The flight time from the perigee A of orbit 1 to point B is obtained from Kepler's equation (Eqs. 3.8 and 3.14),

$$t_{AB} = \frac{T_1}{2\pi}(E_B - e_1 \sin E_B) \tag{c}$$

Since the eccentricity of orbit 1 is

$$e_1 = \frac{r_C - r_A}{r_C + r_A} = 0.33333 \tag{d}$$

and the true anomaly of B is 90°, it follows from Eq. (3.13b) that the eccentric anomaly of B is

$$E_B = 2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_B}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.33333}{1+0.33333}}\tan\frac{90°}{2}\right) = 1.2310\,\text{rad} \tag{e}$$

Substituting Eqs. (b), (d), and (e) into Eq. (c) yields

$$t_{AB} = \frac{10{,}252}{2\pi}(1.231 - 0.33333 \cdot \sin 1.231) = 1495.7\,\text{s}$$

It follows from Eq. (a) that

$$T_2 = 10{,}252 - 1495.7 = 8756.3\,\text{s}$$

This, together with the period formula (Eq. 2.83), yields the semimajor axis of orbit 2,

$$a_2 = \left(\frac{\sqrt{\mu}T_2}{2\pi}\right)^{2/3} = \left(\frac{\sqrt{398{,}600}\cdot 8756.2}{2\pi}\right)^{2/3} = 9182.1\,\text{km}$$

Since $2a_2 = r_A + r_D$, we find that the apogee of orbit 2 is

$$r_D = 2a_2 - r_A = 2\cdot 9182.1 - 6800 = 11{,}564\,\text{km}$$

Finally, Eq. (6.2) yields the angular momentum of orbit 2,

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_D}{r_A + r_D}} = \sqrt{2\cdot 398{,}600}\sqrt{\frac{6800\cdot 11{,}564}{6800 + 11{,}564}} = 58{,}426\,\text{km/s}^2$$

Velocities at A

Since A is the perigee of orbit 1, there is no radial velocity component there. The speed, directed entirely in the transverse direction, is found from the angular momentum formula,

$$v_A)_1 = \frac{h_1}{r_A} = \frac{60{,}116}{6800} = 8.8406\,\text{km/s}$$

Likewise, the speed at the perigee of orbit 2 is

$$v_A)_2 = \frac{h_2}{r_A} = \frac{58{,}426}{6800} = 8.5921\,\text{km/s}$$

At the beginning of the phasing maneuver, the velocity change required to drop into phasing orbit 2 is

$$\Delta v_A = v_A)_2 - v_A)_1 = 8.5921 - 8.8406 = -0.24851\,\text{km/s}$$

At the end of the phasing maneuver, the velocity change required to return to orbit 1 is

$$\Delta v_A = v_A)_1 - v_A)_2 = 8.8406 - 8.5921 = 0.24851\,\text{km/s}$$

The total delta-v required for the chaser to catch up with the target is

$$\Delta v_{\text{total}} = |-0.24851| + |0.24851| = \boxed{0.4970\,\text{km/s}}$$

The delta-v requirement for a phasing maneuver can be lowered by reducing the difference between the period of the main orbit and that of the phasing orbit. In the previous example, we could make Δv_{total} smaller by requiring the chaser to catch the target after n revolutions of the phasing orbit instead of just one. In that case, we would replace Eq. (a) of Example 6.4 by $T_2 = T_1 - t_{AB}/n$.

EXAMPLE 6.5

It is desired to shift the longitude of a GEO satellite 12° westward in three revolutions of its phasing orbit. Calculate the delta-v requirement.

Solution

This problem is illustrated in Fig. 6.12. It may be recalled from Eqs. (2.67)–(2.69) that the angular velocity of the earth, the radius to GEO, and the speed in GEO are, respectively

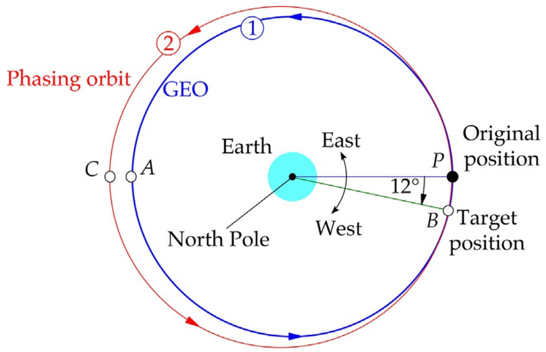


FIG. 6.12

GEO repositioning.

$$\begin{aligned}
\omega_E &= \omega_{\text{GEO}} = 72.922\left(10^{-6}\right)\text{rad/s} \\
r_{\text{GEO}} &= 42{,}164\,\text{km} \\
V_{\text{GEO}} &= 3.0747\,\text{km/s}
\end{aligned} \tag{a}$$

Let $\Delta\Lambda$ be the change in longitude in radians. Then the period T_2 of the phasing orbit can be obtained from the following formula,

$$\omega_E(3T_2) = 3\cdot 2\pi + \Delta\Lambda \tag{b}$$

which states that after three circuits of the phasing orbit, the original position of the satellite will be $\Delta\Lambda$ radians east of P. In other words, the satellite will end up $\Delta\Lambda$ radians west of its original position in GEO, as desired. From Eq. (b) we obtain,

$$T_2 = \frac{1}{3}\frac{\Delta\Lambda + 6\pi}{\omega_E} = \frac{1}{3}\frac{12^\circ\cdot\dfrac{\pi}{180^\circ} + 6\pi}{72.922\times 10^{-6}} = 87{,}121\,\text{s}$$

Note that the period of GEO is

$$T_{\text{GEO}} = \frac{2\pi}{\omega_{\text{GEO}}} = 86{,}163\,\text{s}$$

The satellite in its slower phasing orbit appears to drift westward at the rate

$$\dot{\Lambda} = \frac{\Delta\Lambda}{3T_2} = 8.0133\times 10^{-7}\,\text{rad/s} = 3.9669\,\text{degrees/day}$$

Having the period, we can use Eq. (6.5) to obtain the semimajor axis of orbit 2,

$$a_2 = \left(\frac{T_2\sqrt{\mu}}{2\pi}\right)^{2/3} = \left(\frac{87{,}121\sqrt{398{,}600}}{2\pi}\right)^{2/3} = 42{,}476\,\text{km}$$

From this we find the radius to the apogee C of the phasing orbit,

$$2a_2 = r_p + r_C \quad\Rightarrow\quad r_C = 2\cdot 42{,}476 - 42{,}164 = 42{,}788\,\text{km}$$

The angular momentum of the orbit is given by Eq. (6.2),

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2\cdot 398{,}600}\sqrt{\frac{42{,}164\cdot 42{,}748}{42{,}164 + 42{,}788}} = 130{,}120\,\text{km}^2/\text{s}$$

At P the speed in orbit 2 is

$$v_P)_2 = \frac{130{,}120}{42{,}164} = 3.0859\,\text{km/s}$$

Therefore, at the beginning of the phasing orbit,

$$\Delta v = v_P)_2 - v_{\text{GEO}} = 3.0859 - 3.0747 = 0.01126\text{km/s}$$

At the end of the phasing maneuver,

$$\Delta v = v_{\text{GEO}} - v_P)_2 = 3.0747 - 3.08597 = -0.01126\text{km/s}$$

It follows that,

$$\Delta v_{\text{total}} = |0.01126| + |-0.01126| = \boxed{0.02252\text{km/s}}$$

6.6 NON-HOHMANN TRANSFERS WITH A COMMON APSE LINE

Fig. 6.13 illustrates a transfer between two coaxial, coplanar elliptical orbits in which the transfer trajectory shares the apse line but is not necessarily tangent to either the initial or target orbit. The problem is to determine whether there exists such a trajectory joining points A and B, and, if so, to find the total delta-v requirement.

The radials r_A and r_B are already known, as are the true anomalies θ_A and θ_B. Because of the common apse line assumption, θ_A and θ_B are the true anomalies of points A and B on the transfer orbit as well. Applying the orbit equation to A and B on the transfer orbit yields

$$r_A = \frac{h_2}{\mu}\frac{1}{1 + e\cos\theta_A} \qquad r_B = \frac{h_2}{\mu}\frac{1}{1 + e\cos\theta_B}$$

Solving these two equations for e and h, we get

$$e = \frac{r_A - r_B}{r_A\cos\theta_A - r_B\cos\theta_B} \tag{6.6a}$$

$$h = \sqrt{\mu r_A r_B}\sqrt{\frac{\cos\theta_A - \cos\theta_B}{r_A\cos\theta_A - r_B\cos\theta_B}} \tag{6.6b}$$

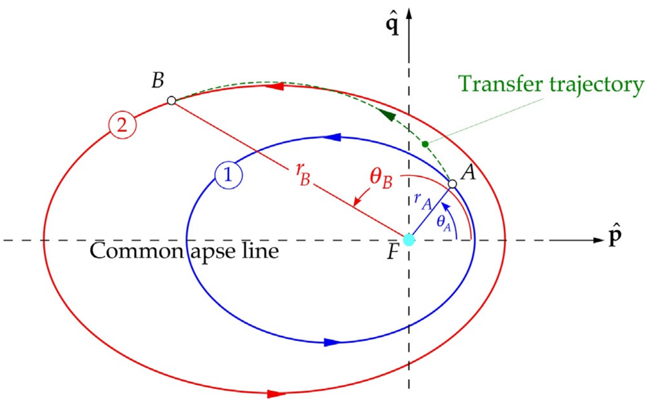


FIG. 6.13

Non-Hohmann transfer between two coaxial elliptical orbits.

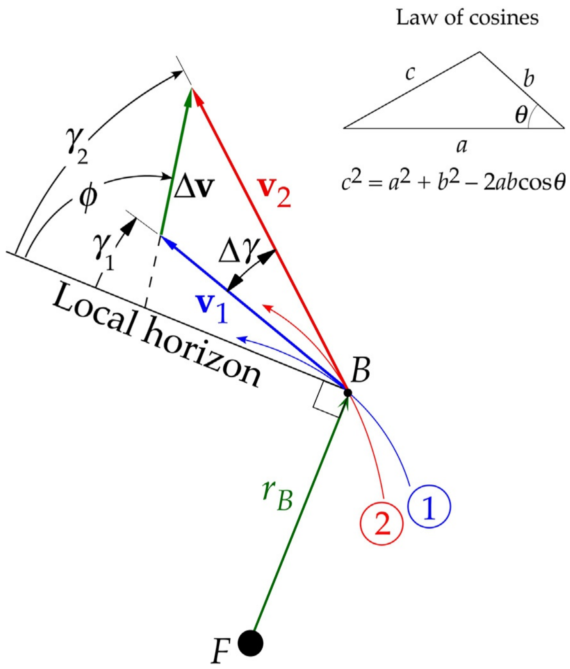


FIG. 6.14

Vector diagram of the change in velocity and flight path angle at the intersection of two orbits (plus a reminder of the law of cosines).

With these orbital elements, the transfer orbit is determined and the velocity may be found at any true anomaly. Note that for a Hohmann transfer, in which $\theta_A = 0$ and $\theta_B = \pi$, Eqs. 6.6 become

$$e = \frac{r_B - r_A}{r_B - r_A} \quad h = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} \quad \text{(Hohmann transfer)} \tag{6.7}$$

When a delta-v calculation is done for an impulsive maneuver at a point that is not on the apse line, care must be taken to include the change in direction as well as the magnitude of the velocity vector. Fig. 6.14 shows a point where an impulsive maneuver changes the velocity vector from $\mathbf{v}_1$ on orbit 1 to $\mathbf{v}_2$ on coplanar orbit 2. The difference in length of the two vectors shows the change in the speed, and the difference in the flight path angles γ_2 and γ_1 indicates the change in the direction. It is important to observe that the Δv we seek is the magnitude of the change in the velocity vector, not the change in its magnitude (speed). That is, from Eq. (1.11),

$$\Delta v = \|\Delta\mathbf{v}\| = \sqrt{(\mathbf{v}_2 - \mathbf{v}_1)\cdot(\mathbf{v}_2 - \mathbf{v}_1)}$$

Expanding under the radical we get

$$\Delta v = \sqrt{\mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 - 2\mathbf{v}_1\cdot\mathbf{v}_2}$$

Again, according to Eq. (1.11), $\mathbf{v}_1 \cdot \mathbf{v}_1 = {v_1}^2$ and $\mathbf{v}_2 \cdot \mathbf{v}_2 = {v_2}^2$. Furthermore, since $\gamma_2 - \gamma_1$ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, Eq. (1.7) implies that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = v_1 v_2 \cos \Delta\gamma$$

where $\Delta\gamma = \gamma_2 - \gamma_1$. Therefore,

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos \Delta\gamma} \quad \text{(impulsive maneuver, coplanar orbits)} \tag{6.8}$$

This is the familiar law of cosines from trigonometry. Only if $\Delta\gamma = 0$, which means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel (as in a Hohmann transfer), is it true that $\Delta v = |v_2 - v_1|$. If $v_2 = v_1 = v$, then Eq. (6.8) yields

$$\Delta v = v\sqrt{2(1 - \cos \Delta\gamma)} \quad \text{(pure rotation of the velocity vector in the orbital plane)} \tag{6.9}$$

Therefore, fuel expenditure is required to change the direction of the velocity even if its magnitude remains the same.

The direction of $\Delta\mathbf{v}$ in Fig. 6.14 shows the required alignment of the thruster that produces the impulse. The orientation of $\Delta\mathbf{v}$ relative to the local horizon is found by replacing v_r and $v_\perp$ in Eq. (2.51) with Δv_r and $\Delta v_\perp$, so that

$$\tan\phi = \frac{\Delta v_r}{\Delta v_\perp} \tag{6.10}$$

where ϕ is the angle from the local horizon to the $\Delta\mathbf{v}$ vector.

Finally, recall from Eq. (2.57) that the specific mechanical energy of a spacecraft is,

$$\varepsilon = \frac{\mathbf{v} \cdot \mathbf{v}}{2} - \frac{\mu}{r}$$

An impulsive maneuver changes the velocity $\mathbf{v}$ but not the position vector $\mathbf{r}$. It follows that

$$\Delta\varepsilon = \frac{(\mathbf{v} + \Delta\mathbf{v}) \cdot (\mathbf{v} + \Delta\mathbf{v})}{2} - \frac{\mathbf{v} \cdot \mathbf{v}}{2} = \mathbf{v} \cdot \Delta\mathbf{v} + \frac{1}{2}\Delta v^2$$

The angle between $\mathbf{v}$ and $\Delta\mathbf{v}$ is $\Delta\gamma$ (Fig. 6.14, with $\mathbf{v}_1 = \mathbf{v}$). Therefore, $\mathbf{v} \cdot \Delta\mathbf{v} = v\Delta v \cos \Delta\gamma$ and we obtain

$$\Delta\varepsilon = v\Delta v \cos \Delta\gamma + \frac{1}{2}\Delta v^2 \tag{6.11}$$

This shows that, for a given $\boldsymbol{\Delta}v$, the change in specific energy is larger when the spacecraft is moving fastest and when $\boldsymbol{\Delta}v$ is aligned with the original velocity ($\Delta\gamma \approx 0$). The larger the $\Delta\varepsilon$ associated with a given Δv, the more efficient the maneuver. As we know, a spacecraft has its greatest speed at periapsis.

EXAMPLE 6.6

A geocentric satellite in orbit 1 of Fig. 6.15 executes a delta-v maneuver at A, which places it on orbit 2, for reentry at D. Calculate $\boldsymbol{\Delta}v$ at A and its direction relative to the local horizon.

Solution

From the figure we see that

$$r_B = 20{,}000\,\text{km} \qquad r_C = 10{,}000\,\text{km} \qquad r_D = 6378\,\text{km}$$

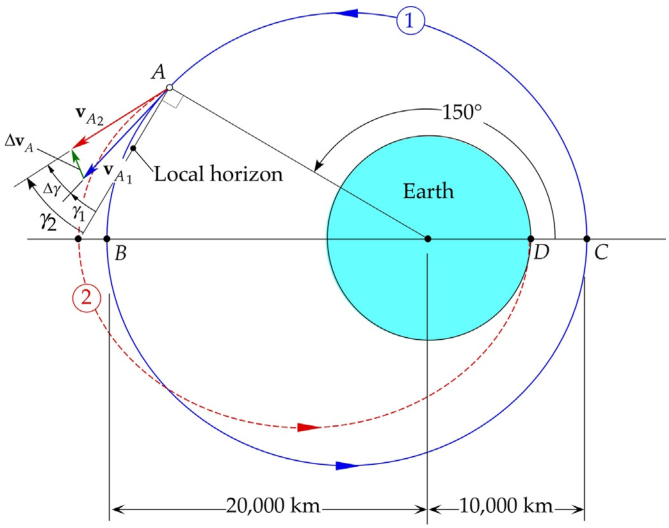


FIG. 6.15

Non-Hohmann transfer with a common apse line.

Orbit 1:
The eccentricity is

$$e_1 = \frac{r_B - r_C}{r_B + r_C} = 0.33333$$

The angular momentum is obtained from Eq. (6.2), noting that point C is perigee:

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2 \cdot 398{,}600}\sqrt{\frac{20{,}000 \cdot 10{,}000}{20{,}000 + 10{,}000}} = 72{,}902 \text{km}^2/\text{s}$$

With the angular momentum and the eccentricity, we can use the orbit equation to find the radial coordinate of point A,

$$r_A = \frac{72{,}902^2}{398{,}600}\frac{1}{1 + 0.33333 \cdot \cos 150°} = 18{,}744 \text{km}$$

Eqs. (2.31) and (2.49) yield the transverse and radial components of velocity at A on orbit 1,

$$v_{\perp_A})_1 = \frac{h_1}{r_A} = 3.8893 \text{km/s}$$

$$v_{r_A})_1 = \frac{\mu}{h_1} e_1 \sin 150° = 0.91127 \text{km/s}$$

From these we find the speed at A,

$$v_A)_1 = \sqrt{v_{\perp_A})_1{}^2 + v_{r_A})_1{}^2} = 3.9946 \text{km/s}$$

and the flight path angle,

$$\gamma_1 = \tan^{-1}\frac{v_{r_A})_1}{v_{\perp_A})_1} = \tan^{-1}\frac{0.91127}{3.8893} = 13.187°$$

Orbit 2:

The radius and true anomaly of points A and D on orbit 2 are known. From Eqs. (6.6) we find

$$e_2 = \frac{r_D - r_A}{r_D\cos\theta_D - r_A\cos\theta_A} = \frac{6378 - 18{,}744}{6378\cos 0 - 18{,}744\cos 150°} = 0.5469$$

$$h_2 = \sqrt{\mu r_A r_D}\sqrt{\frac{\cos\theta_D - \cos\theta_A}{r_D\cos\theta_D - r_A\cos\theta_A}} = \sqrt{398{,}600\cdot 18{,}744\cdot 6378}\sqrt{\frac{\cos 0 - \cos 150°}{6378\cos 0 - 18{,}744\cos 150°}}$$
$$= 62{,}711\,\text{km}^2/\text{s}$$

Now we can calculate the radial and perpendicular components of velocity on orbit 2 at point A.

$$v_{\perp_A})_2 = \frac{h_2}{r_A} = 3.3456\,\text{km/s}$$
$$v_{r_A})_2 = \frac{\mu}{h_2}e_2\sin 150° = 1.7381\,\text{km/s}$$

Hence, the speed and flight path angle at A on orbit 2 are

$$v_A)_2 = \sqrt{v_{\perp_A})_2{}^2 + v_{r_A})_2{}^2} = 3.7702\,\text{km/s}$$
$$\gamma_2 = \tan^{-1}\frac{v_{r_A})_2}{v_{\perp_A})_2} = \tan^{-1}\frac{1.7381}{3.3456} = 27.453°$$

The change in the flight path angle as a result of the impulsive maneuver is

$$\Delta\gamma = \gamma_2 - \gamma_1 = 27.453° - 13.187° = 14.266°$$

With this we can use Eq. (6.8) to finally obtain Δv_A,

$$\Delta v_A = \sqrt{v_A)_1{}^2 + v_A)_2{}^2 - 2v_A)_1 v_A)_2\cos\Delta\gamma} = \sqrt{3.9946^2 + 3.7702^2 - 2\cdot 3.9946\cdot 3.7702\cdot\cos 14.266°}$$

$$\boxed{\Delta v_A = 0.9896\,\text{km/s}}$$

Note that Δv_A is the magnitude of the change in velocity vector $\Delta\mathbf{v}_A$ at A. That is not the same as the change in the magnitude of the velocity (i.e., the change in speed), which is

$$v_A)_2 - v_A)_1 = 3.7702 - 3.9946 = -0.2244\,\text{km/s}$$

To find the orientation of $\Delta\mathbf{v}_A$, we use Eq. (6.10),

$$\tan\phi = \frac{\Delta v_{r_A}}{\Delta v_{\perp_A}} = \frac{v_{r_A})_2 - v_{r_A})_1}{v_{\perp_A})_2 - v_{\perp_A})_1} = \frac{1.7381 - 0.9113}{3.3456 - 3.8893} = -1.5207$$

so that

$$\boxed{\phi = 123.3°}$$

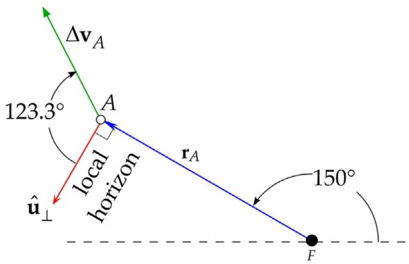


FIG. 6.16

Orientation of $\Delta\mathbf{v}_A$ to the local horizon.

This angle is illustrated in Fig. 6.16. Before firing, the spacecraft would have to be rotated so that the centerline of the rocket motor coincides with the line of action of $\Delta \mathbf{v}_A$, with the nozzle aimed in the opposite direction.

6.7 APSE LINE ROTATION

Fig. 6.17 shows two intersecting orbits that have a common focus, but their apse lines are not collinear. A Hohmann transfer between them is clearly impossible. The opportunity for transfer from one orbit to the other by a single impulsive maneuver occurs where they intersect, at points I and J in this case. As can be seen from the figure, the rotation η of the apse line is the difference between the true anomalies of the point of intersection, measured from periapsis of each orbit. That is,

$$\eta = \theta_1 - \theta_2 \tag{6.12}$$

We will consider two cases of apse line rotation.

The first case is that in which the apse line rotation η is given as well as the orbital parameters e and h of both orbits. The problem is then to find the true anomalies of I and J relative to both orbits. The radius of the point of intersection I is given by either of the following:

$$r_I)_1 = \frac{h_1^2}{\mu}\frac{1}{1+e_1\cos\theta_1} \qquad r_I)_2 = \frac{h_2^2}{\mu}\frac{1}{1+e_2\cos\theta_2}$$

Since $r_I)_1 = r_I)_2$, we can equate these two expressions and rearrange terms to get

$$e_1 h_2^2 \cos\theta_1 - e_2 h_1^2 \cos\theta_2 = h_1^2 - h_2^2$$

Setting $\theta_2 = \theta_1 - \eta$ and using the trig identity $\cos(\theta_1 - \eta) = \cos\theta_1 \cos\eta + \sin\theta_1 \sin\eta$ leads to an equation for θ_1,

$$a\cos\theta_1 + b\sin\theta_1 = c \tag{6.13a}$$

where

$$a = e_1 h_2^2 - e_2 h_1^2 \cos\eta \qquad b = -e_2 h_1^2 \sin\eta \qquad c = h_1^2 - h_2^2 \tag{6.13b}$$

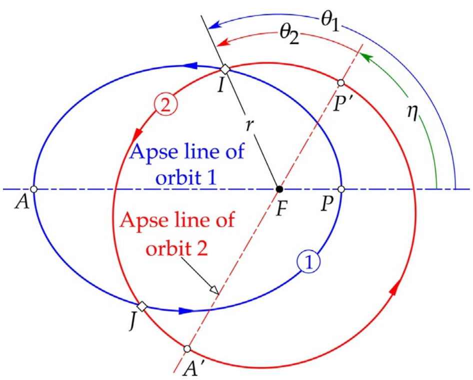


FIG. 6.17

Two intersecting orbits whose apse lines do not coincide.

Eq. (6.13a) has two roots (see Problem 3.12), corresponding to the two points of intersection I and J of the two orbits:

$$\theta_1 = \phi \pm \cos^{-1}\left(\frac{c}{a}\cos\phi\right) \tag{6.14a}$$

where

$$\phi = \tan^{-1}\frac{b}{a} \tag{6.14b}$$

Having found θ_1 we obtain θ_2 from Eq. (6.12). Δv for the impulsive maneuver may then be computed as illustrated in the following example.

EXAMPLE 6.7

An earth satellite is in an 8000 km by 16,000 km radius orbit (orbit 1 of Fig. 6.18). Calculate the delta-v and the true anomaly θ_1 required to obtain a 7000 km by 21,000 km radius orbit (orbit 2) whose apse line is rotated 25° counterclockwise. Indicate the orientation ϕ of $\Delta\mathbf{v}$ to the local horizon.

Solution

The eccentricities of the two orbits are

$$\begin{aligned} e_1 &= \frac{r_{A_1} - r_{P_1}}{r_{A_1} + r_{P_1}} = \frac{16{,}000 - 8000}{16{,}000 + 8000} = 0.33333 \\ e_2 &= \frac{r_{A_2} - r_{P_2}}{r_{A_2} + r_{P_2}} = \frac{21{,}000 - 7000}{21{,}000 + 7000} = 0.5 \end{aligned} \tag{a}$$

The orbit equation yields the angular momenta

$$\begin{aligned} r_{P_1} &= \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos(0)} \Rightarrow 8000 = \frac{h_1^2}{398{,}600}\frac{1}{1 + 0.33333} \Rightarrow h_1 = 65{,}205\,\text{km}^2/\text{s} \\ r_{P_2} &= \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos(0)} \Rightarrow 7000 = \frac{h_2^2}{398{,}600}\frac{1}{1 + 0.5} \Rightarrow h_2 = 64{,}694\,\text{km}^2/\text{s} \end{aligned} \tag{b}$$

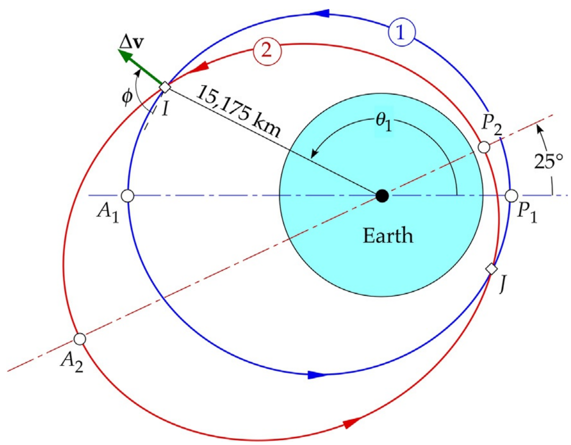


FIG. 6.18

Δv produces a rotation of the apse line.

Using these orbital parameters and the fact that $\eta = 25°$, we calculate the terms in Eq. (6.13b):

$$\begin{aligned}
a &= e_1 h_2^2 - e_2 h_1^2 \cos\eta = 0.3333 \cdot 64{,}694^2 - 0.5 \cdot 65{,}205^2 \cdot \cos 25° = -5.3159\left(10^8\right)\text{km}^4/\text{s}^2\\
b &= -e_2 h_1^2 \sin\eta \qquad = -0.5 \cdot 65{,}205^2 \sin 25° \qquad = -8.9843\left(10^8\right)\text{km}^4/\text{s}^2\\
c &= h_2^2 - h_2^2 \qquad = 65{,}205^2 - 64{,}694^2 \qquad = 6.6433\left(10^7\right)\text{km}^4/\text{s}^2
\end{aligned}$$

Then Eq. (6.14) yields

$$\phi = \tan^{-1}\frac{-8.9843\left(10^8\right)}{-5.3159\left(10^8\right)} = 59.388°$$

$$\theta_1 = 59.388° \pm \cos^{-1}\left[\frac{6.6433\left(10^7\right)}{-5.3159\left(10^8\right)}\cos 59.388°\right] = 59.388° \pm 93.649°$$

Thus, the true anomaly of point I, the point of interest, is

$$\theta_1 = 153.04° \tag{c}$$

(For point J, $\theta_1 = 325.74°$.)

With the true anomaly available, we can evaluate the radial coordinate of the maneuver point,

$$r = \frac{h_1{}^2}{\mu}\frac{1}{1 + e_1 \cos 153.04°} = 15{,}175\,\text{km}$$

The velocity components and flight path angle for orbit 1 at point I are

$$\begin{aligned}
v_{\perp_1} &= \frac{h_1}{r} = \frac{65{,}205}{15{,}175} = 4.2968\,\text{km/s}\\
v_{r_1} &= \frac{\mu}{h_1} e_1 \sin 153.04° = \frac{398{,}600}{65{,}205} \cdot 0.33333 \cdot \sin 153.04° = 0.92393\,\text{km/s}\\
\gamma_1 &= \tan^{-1}\frac{v_{r_1}}{v_{\perp_1}} = 12.135°
\end{aligned}$$

The speed of the satellite in orbit 1 is, therefore,

$$v_1 = \sqrt{v_{r_1}{}^2 + v_{\perp_1}{}^2} = 4.3950\,\text{km/s}$$

Likewise, for orbit 2,

$$\begin{aligned}
v_{\perp_2} &= \frac{h_2}{r} = \frac{64{,}694}{15{,}175} = 4.2631\,\text{km/s}\\
v_{r_2} &= \frac{\mu}{h_2} e_2 \sin(153.04° - 25°) = \frac{398{,}600}{64{,}694} \cdot 0.5 \cdot \sin 128.04° = 2.4264\,\text{km/s}\\
\gamma_2 &= \tan^{-1}\frac{v_{r_2}}{v_{\perp_2}} = 29.647°\\
v_2 &= \sqrt{v_{r_2}{}^2 + v_{\perp_2}{}^2} = 4.9053\,\text{km/s}
\end{aligned}$$

Eq. (6.8) is used to find Δv,

$$\begin{aligned}
\Delta v &= \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos(\gamma_2 - \gamma_1)}\\
&= \sqrt{4.3950^2 + 4.9053^2 - 2 \cdot 4.3950 \cdot 4.9053 \cos(29.647° - 12.135°)}\\
&\boxed{\Delta v = 1.503\,\text{km/s}}
\end{aligned}$$

The angle ϕ that the vector $\mathbf{\Delta v}$ makes with the local horizon is given by Eq. (6.10),

$$\phi = \tan^{-1}\frac{\Delta v_r}{\Delta v_\perp} = \tan^{-1}\frac{v_{r_2} - v_{r_1}}{v_{\perp_2} - v_{\perp_1}} = \tan^{-1}\frac{2.4264 - 0.92393}{4.2631 - 4.2968} = \boxed{91.28°}$$

The second case of apse line rotation is that in which the impulsive maneuver takes place at a given true anomaly θ_1 on orbit 1. The problem is to determine the angle of rotation η and the eccentricity e_2 of the new orbit.

The impulsive maneuver creates a change in the radial and transverse velocity components at point I of orbit 1. From the angular momentum formula, $h = rv_\perp$, we obtain the angular momentum of orbit 2,

$$h_2 = r(v_\perp + \Delta v_\perp) = h_1 + r\Delta v_\perp \tag{6.15}$$

The formula for radial velocity, $v_r = (\mu/h)e \sin\theta$, applied to orbit 2 at point I, where $v_{r_2} = v_{r_1} + \Delta v_r$ and $\theta_2 = \theta_1 - \eta$, yields

$$v_{r_1} + \Delta v_r = \frac{\mu}{h_2} e_2 \sin\theta_2$$

Substituting Eq. (6.15) into this expression and solving for $\sin\theta_2$ leads to

$$\sin\theta_2 = \frac{1}{e_2}\frac{(h_1 + r\Delta v_\perp)(\mu e_1 \sin\theta_1 + h_1 \Delta v_r)}{\mu h_1} \tag{6.16}$$

From the orbit equation, we have at point I

$$r = \frac{{h_1}^2}{\mu}\frac{1}{1 + e_1\cos\theta_1} \quad \text{(orbit 1)}$$
$$r = \frac{{h_2}^2}{\mu}\frac{1}{1 + e_2\cos\theta_2} \quad \text{(orbit 2)}$$

Equating these two expressions for r, substituting Eq. (6.15), and solving for $\cos\theta_2$, yields

$$\cos\theta_2 = \frac{1}{e_2}\frac{(h_1 + r\Delta v_\perp)^2 e_1\cos\theta_1 + (2h_1 + r\Delta v_\perp)r\Delta v_\perp}{h_1^2} \tag{6.17}$$

Finally, by substituting Eqs. (6.16) and (6.17) into the trigonometric identity $\tan\theta_2 = \sin\theta_2/\cos\theta_2$ we obtain a formula for θ_2 that does not involve the eccentricity e_2,

$$\tan\theta_2 = \frac{h_1}{\mu}\frac{(h_1 + r\Delta v_\perp)(\mu e_1\sin\theta_1 + h_1\Delta v_r)}{(h_1 + r\Delta v_\perp)^2 e_1\cos\theta_1 + \left(2h_1 + r\Delta v|_\perp\right)r\Delta v_\perp} \tag{6.18a}$$

Eq. (6.18a) can be simplified a bit by replacing $\mu e_1 \sin\theta_1$ with $h_1 v_{r_1}$ and h_1 with $rv_{\perp_1}$, so that

$$\tan\theta_2 = \frac{(v_{\perp_1} + \Delta v_\perp)(v_{r_1} + \Delta v_r)}{(v_{\perp_1} + \Delta v_\perp)^2 e_1\cos\theta_1 + (2v_{\perp_1} + \Delta v_\perp)\Delta v_\perp(\mu/r)} \tag{6.18b}$$

Eqs. (6.18) show how the apse line rotation, $\eta = \theta_1 - \theta_2$, is completely determined by the components of $\Delta\mathbf{v}$ imparted at the true anomaly θ_1. Notice that if $\Delta v_r = -v_{r_1}$, then $\theta_2 = 0$, which means that the maneuver point is on the apse line of the new orbit.

After solving Eq. (6.18a) or (6.18b), we substitute θ_2 into either Eq. (6.16) or (6.17) to calculate the eccentricity e_2 of orbit 2. Therefore, with h_2 from Eq. (6.15), the rotated orbit 2 is completely specified.

If the impulsive maneuver takes place at the periapsis of orbit 1, so that $\theta_1 = v_r = 0$, and if it is also true that $\Delta v_\perp = 0$, then Eq. (6.18b) yields

$$\tan\eta = \frac{rv_{\perp_1}}{\mu e_1}\Delta v_r \quad \text{(with radial impulse at periapsis)}$$

Thus, if the velocity vector is given an outward radial component at periapsis, then $\eta < 0$, which means the apse line of the resulting orbit is rotated clockwise relative to the original one. That makes sense, since having acquired $v_r > 0$ means that the spacecraft is now flying away from its new periapsis. Likewise, applying an inward radial velocity component at periapsis rotates the apse line counterclockwise.

EXAMPLE 6.8

An earth satellite in orbit 1 of Fig. 6.19 undergoes the indicated delta-v maneuver at its perigee. Determine the rotation η of its apse line as well as the new perigee and apogee.

Solution

From Fig. 6.19 the apogee and perigee of orbit 1 are

$$r_{A_1} = 17{,}000\,\text{km} \qquad r_{P_1} = 7000\,\text{km}$$

Therefore, the eccentricity of orbit 1 is

$$e_1 = \frac{r_{A_1} - r_{P_1}}{r_{A_1} + r_{P_1}} = 0.41667 \tag{a}$$

As usual, we use the orbit equation to find the angular momentum,

$$r_{P_1} = \frac{h_1{}^2}{\mu}\frac{1}{1+e_1\cos(0)} \Rightarrow 7000 = \frac{h_1{}^2}{398{,}600}\frac{1}{1+0.41667} \Rightarrow h_1 = 62{,}871\,\text{km}^2/\text{s}$$

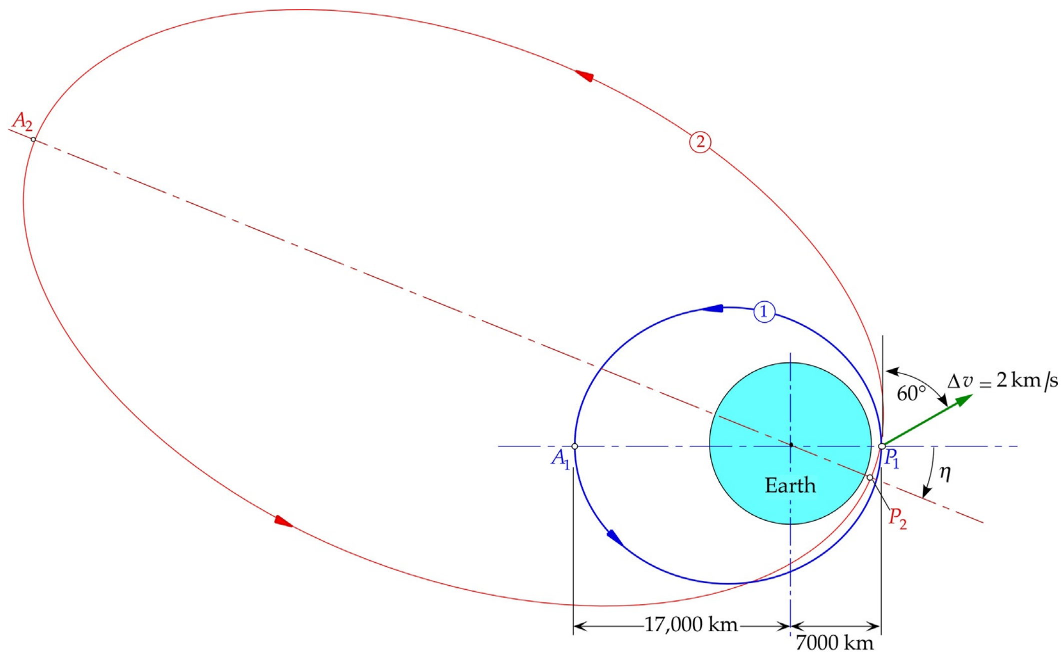


FIG. 6.19

Apse line rotation maneuver.

At the maneuver point P_1, the angular momentum formula and the fact that P_1 is the perigee of orbit 1 ($\theta_1 = 0$) imply that

$$\begin{aligned} v_{\perp_1} &= \frac{h_1}{r_{P_1}} = \frac{62{,}871}{7000} = 8.9816\,\text{km/s} \\ v_{r_1} &= 0 \end{aligned} \tag{b}$$

From Fig. 6.19 it is clear that

$$\begin{aligned} \Delta v_\perp &= \Delta v \cos 60° = 1\,\text{km/s} \\ \Delta v_r &= \Delta v \sin 60° = 1.7321\,\text{km/s} \end{aligned} \tag{c}$$

The angular momentum of orbit 2 is given by Eq. (6.15),

$$h_2 = h_1 + r\Delta v_\perp = 62{,}871 + 7000 \cdot 1 = 69{,}871\,\text{km}^2/\text{s}$$

To compute θ_2, we use Eq. (6.18b) together with Eqs. (a)–(c):

$$\begin{aligned} \tan\theta_2 &= \frac{(v_{\perp_1} + \Delta v_\perp)(v_{r_1} + \Delta v_r)}{(v_{\perp_1} + \Delta v_\perp)^2 e_1 \cos\theta_1 + (2v_{\perp_1} + \Delta v_\perp)\Delta v_\perp} \frac{{v_{\perp_1}}^2}{(\mu/r_{P_1})} \\ &= \frac{(8.9816 + 1)(0 + 1.7321)}{(8.9816+1)^2 \cdot 0.41667 \cdot \cos(0) + (2 \cdot 8.9816 + 1) \cdot 1} \frac{8.9816^2}{(398{,}600/7000)} \\ &= 0.4050 \end{aligned}$$

It follows that $\theta_2 = 22.05°$, so that Eq. (6.12) yields

$$\boxed{\eta = -22.05°}$$

This means that the rotation of the apse line is clockwise, as indicated in Fig. 6.19.

From Eq. (6.17) we obtain the eccentricity of orbit 2,

$$\begin{aligned} e_2 &= \frac{(h_1 + r_{P_1}\Delta v_\perp)^2 e_1 \cos\theta_1 + (2h_1 + r_{P_1}\Delta v_\perp) r_{P_1}\Delta v_\perp}{h_1^2 \cos\theta_2} \\ &= \frac{(62{,}871 + 7000.1)^2 \cdot 0.41667 \cdot \cos(0) + (2 \cdot 62{,}871 + 7000 \cdot 1) \cdot 7000 \cdot 1}{62{,}871^2 \cdot \cos 22.05°} \\ &= 0.80883 \end{aligned}$$

With this and the angular momentum we find using the orbit equation that the perigee and apogee radii of orbit 2 are

$$\begin{aligned} r_{P_2} &= \frac{{h_2}^2}{\mu}\frac{1}{1+e_2} = \frac{69{,}871^2}{398{,}600}\frac{1}{1+0.80883} = \boxed{6771.1\,\text{km}} \\ r_{A_2} &= \frac{69{,}871^2}{398{,}600}\frac{1}{1-0.80883} = \boxed{64{,}069\,\text{km}} \end{aligned}$$

6.8 CHASE MANEUVERS

Whereas Hohmann transfers and phasing maneuvers are leisurely, energy-efficient procedures that require some preconditions (e.g., coaxial elliptical, orbits) to work, a chase or intercept trajectory is one that answers the question, "How do I get from point *A* to point *B* in space in a given amount of time?" The nature of the orbit lies in the answer to the question rather than being prescribed at the outset. Intercept trajectories near a planet are likely to require delta-v's beyond the capabilities of today's technology, so they are largely of theoretical rather than practical interest. We might refer to them as "star wars maneuvers." Chase trajectories can be found as solutions to Lambert's problem (Section 5.3), which is useful and practical for interplanetary mission design (Chapter 8).

EXAMPLE 6.9

Spacecraft B and C are both in the geocentric elliptical orbit (1) shown in Fig. 6.20, from which it can be seen that the true anomalies are $\theta_B = 45°$ and $\theta_C = 150°$. At the instant shown, spacecraft B executes a delta-v maneuver, embarking upon a trajectory (2), which will intercept and rendezvous with vehicle C in precisely one hour. Find the orbital parameters (e and h) of the intercept trajectory and the total delta-v required for the chase maneuver.

Solution

First, we must determine the parameters of orbit 1 in the usual way. The eccentricity is found using the orbit's perigee and apogee, shown in Fig. 6.20,

$$e_1 = \frac{18,900 - 8100}{18,900 + 8100} = 0.4000$$

From Eq. (6.2),

$$h_1 = \sqrt{2 \cdot 398,600}\sqrt{\frac{8100 \cdot 18,900}{8100 + 18,900}} = 67,232 \text{km}^2/\text{s}$$

Using Eq. (2.82) yields the period,

$$T_1 = \frac{2\pi}{\mu^2}\left(\frac{h_1}{\sqrt{1-e_1^2}}\right)^3 = \frac{2\pi}{398,600^2}\left(\frac{67,232}{\sqrt{1-0.4^2}}\right)^3 = 15,610 \text{s}$$

In perifocal coordinates (Eq. 2.119) the position vector of B is

$$\mathbf{r}_B = \frac{h_1{}^2}{\mu}\frac{1}{1+e_1\cos\theta_B}(\cos\theta_B\hat{\mathbf{p}} + \sin\theta_B\hat{\mathbf{q}}) = \frac{67,232^2}{398,600}\frac{1}{1+0.4\cos 45°}(\cos 45°\hat{\mathbf{p}} + \sin 45°\hat{\mathbf{q}})$$

or

$$\mathbf{r}_B = 6250.6\hat{\mathbf{p}} + 6250.6\hat{\mathbf{q}}\ (\text{km}) \tag{a}$$

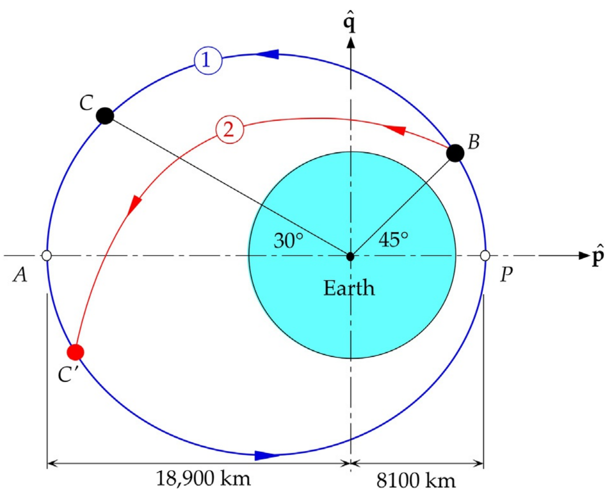


FIG. 6.20

Intercept trajectory (2) required for B to catch C in 1 h.

Likewise, according to Eq. (2.125), the velocity at B on orbit 1 is

$$\mathbf{v}_B)_1 = \frac{\mu}{h}[-\sin\theta_B\hat{\mathbf{p}} + (e+\cos\theta_B)\hat{\mathbf{q}}] = \frac{398{,}600}{67{,}232}[-\sin 45°\hat{\mathbf{p}} + (0.4+\cos 45°)\hat{\mathbf{q}}]$$

so that

$$\mathbf{v}_B)_1 = -4.1922\hat{\mathbf{p}} + 6.5637\hat{\mathbf{q}}\ (\text{km/s}) \tag{b}$$

Now we need to move spacecraft C along orbit 1 to the position C' that it will occupy one hour later, when it will presumably be met by spacecraft B. To do that, we must first calculate the time since perigee passage at C. Since we know the true anomaly, the eccentric anomaly follows from Eq. (3.13b),

$$E_C = 2\tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_C}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.4}{1+0.4}}\tan\frac{150°}{2}\right) = 2.3646\,\text{rad}$$

Substituting this value into Kepler's equation (Eqs. 3.8 and 3.14) yields the time since perigee passage,

$$t_C = \frac{T_1}{2\pi}(E_C - e_1\sin E_C) = \frac{15{,}610}{2\pi}(2.3646 - 0.4\cdot\sin 2.3646) = 5178\,\text{s}$$

An hour later ($\Delta t = 3600\,\text{s}$), the spacecraft will be in intercept position at C',

$$t_{C'} = t_C + \Delta t = 5178 + 3600 = 8778\,\text{s}$$

The corresponding mean anomaly is

$$M_e)_{C'} = 2\pi\frac{t_{C'}}{T_1} = 2\pi\frac{8778}{15{,}610} = 3.5331\,\text{rad}$$

With this value of the mean anomaly, Kepler's equation becomes

$$E_{C'} - e_1\sin E_{C'} = 3.5331$$

Applying Algorithm 3.1 to the solution of this equation we get

$$E_{C'} = 3.4223\,\text{rad}$$

Substituting this result into Eq. (3.13a) yields the true anomaly at C',

$$\tan\frac{\theta_{C'}}{2} = \sqrt{\frac{1+04}{1-04}}\tan\frac{3.4223}{2} = -10.811 \Rightarrow \theta_{C'} = 190.57°$$

We are now able to calculate the perifocal position and velocity vectors at C' on orbit 1

$$\begin{aligned}\mathbf{r}_{C'} &= \frac{67{,}232^2}{398{,}600}\frac{1}{1+0.4\cos 190.57°}(\cos 190.57°\hat{\mathbf{p}} + \sin 190.57°\hat{\mathbf{q}}) \\ &= -18{,}372\hat{\mathbf{p}} - 3428.1\hat{\mathbf{q}}\ (\text{km})\end{aligned}$$

$$\begin{aligned}\mathbf{v}_{C'})_1 &= \frac{398{,}600}{67{,}232}[-\sin 190.57°\hat{\mathbf{p}} + (0.4+\cos 190.57°)\hat{\mathbf{q}}] \\ &= 1.0875\hat{\mathbf{p}} - 3.4566\hat{\mathbf{q}}\ (\text{km/s})\end{aligned} \tag{c}$$

The intercept trajectory connecting points B and C' is found by solving Lambert's problem. Substituting $\mathbf{r}_B$ and $\mathbf{r}_{C'}$ along with $\Delta t = 3600\,\text{s}$, into Algorithm 5.2 yields

$$\mathbf{v}_B)_2 = -8.1349\hat{\mathbf{p}} + 4.0506\hat{\mathbf{q}}\ (\text{km/s}) \tag{d}$$

$$\mathbf{v}_{C'})_2 = -3.4745\hat{\mathbf{p}} - 4.7943\hat{\mathbf{q}}\ (\text{km/s}) \tag{e}$$

These velocities are most easily obtained by running the following MATLAB script, which executes Algorithm 5.2 by means of the function M-file *lambert.m* (Appendix D.25).

```
clear
global mu
deg      = pi/180;
mu       = 398600;
e        = 0.4;
h        = 67232;
theta1   = 45*deg;
theta2   = 190.57*deg;
delta_t  = 3600;
rB       = h^2/mu/(1 + e * cos(theta1))...
           * [cos(theta1),sin(theta1),0];
rC_prime = h^2/mu/(1 + e * cos(theta2))...
           * [cos(theta2),sin(theta2),0];
string   = 'pro';
[vB2 vC_prime_2] = lambert(rB, rC_prime, delta_t, string)
```

From Eqs. (b) and (d) we find

$$\Delta \mathbf{v}_B = \mathbf{v}_B)_2 - \mathbf{v}_B)_1 = -3.9326\hat{\mathbf{p}} - 2.5132\hat{\mathbf{q}}\,(\text{km/s})$$

whereas Eqs. (c) and (e) yield

$$\Delta \mathbf{v}_{C'} = \mathbf{v}_{C'})_1 - \mathbf{v}_{C'})_2 = -4.5620\hat{\mathbf{p}} + 1.3376\hat{\mathbf{q}}\,(\text{km/s})$$

The anticipated, extremely large delta-v requirement for this chase maneuver is the sum of the magnitudes of these two vectors,

$$\Delta v = \|\Delta \mathbf{v}_B\| + \|\Delta \mathbf{v}_{C'}\| = 4.6755 + 4.7540 = \boxed{9.430\,\text{km/s}}$$

We know that orbit 2 is an ellipse, because the magnitude of $\mathbf{v}_B)_2$ (9.088 km/s) is less than the escape speed ($\sqrt{2\mu/r_B} = 9.496\,\text{km/s}$) at B. To pin it down a bit more, we can use $\mathbf{r}_B$ and $\mathbf{v}_B)_2$ to obtain the orbital elements from Algorithm 4.2, which yields

$$\boxed{h_2 = 76{,}167\,\text{km}^2/\text{s}}$$
$$\boxed{e_2 = 0.8500}$$
$$a_2 = 52{,}449\,\text{km}$$
$$\theta_B)_2 = 319.52°$$

These may be found quickly by running the following MATLAB script, in which the M-function *coe_from_sv.m* implements Algorithm 4.2 (see Appendix D.18):

```
clear
global mu
mu  = 398600;
rB  = [6250.6,6250.6,0];
vB2 = [-8.1349,4.0506,0];
orbital_elements = coe_from_sv(rB, vB2);
```

The details of the intercept trajectory and the delta-v maneuvers are shown in Fig. 6.21. A far less dramatic though more leisurely (and realistic) way for B to catch up with C would be to use a phasing maneuver.

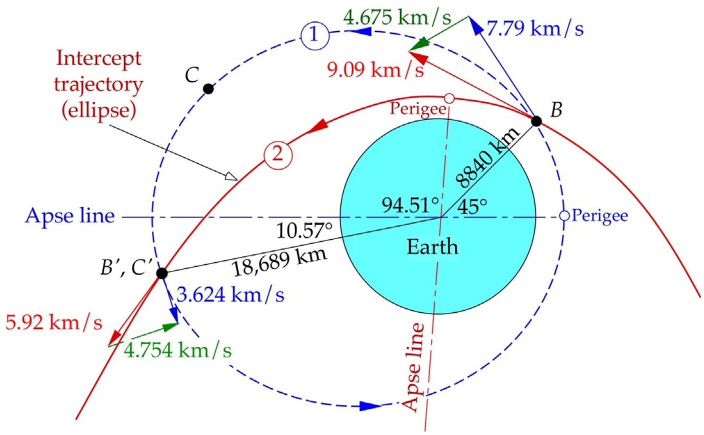


FIG. 6.21

Details of the large elliptical orbit, a portion of which serves as the intercept trajectory.

6.9 PLANE CHANGE MANEUVERS

Orbits having a common focus F need not, and generally do not, lie in a common plane. Fig. 6.22 shows two such orbits and their line of intersection BD. A and P denote the apoapses and periapses. Since the common focus lies in every orbital plane, it must lie on the line of intersection of any two orbits. For a spacecraft in orbit 1 to change its plane to that of orbit 2 by means of a single delta-v maneuver

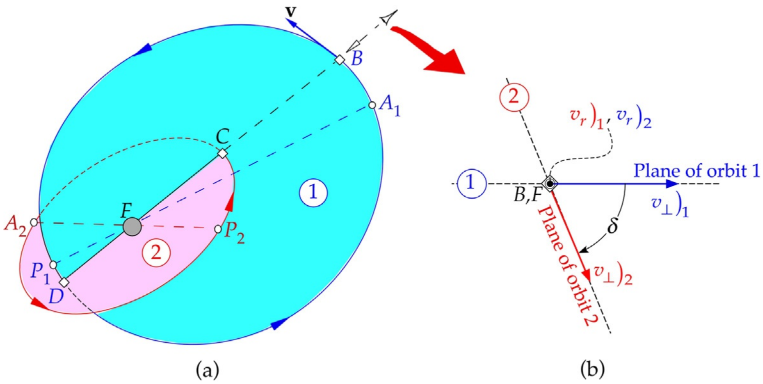


FIG. 6.22

(a) Two noncoplanar orbits about F. (b) A view down the line of intersection of the two orbital planes.

(cranking maneuver), it must do so when it is on the line of intersection of the orbital planes. Those two opportunities occur only at points B and D in Fig. 6.22a.

A view down the line of intersection, from B toward D, is shown in Fig. 6.22b. Here we can see in true view the dihedral angle δ between the two planes. The transverse component of velocity $\mathbf{v}_\perp$ at B is evident in this perspective, whereas the radial component $\mathbf{v}_r$, lying as it does on the line of intersection, is normal to the view plane (thus appearing as a dot). It is apparent that changing the plane of orbit 1 requires simply rotating $\mathbf{v}_\perp$ around the intersection line, through the dihedral angle. If $v_\perp$ and v_r remain unchanged in the process, then we have a rigid body rotation of the orbit. That is, except for its new orientation in space, the orbit remains unchanged. If the magnitudes of $\mathbf{v}_r$ and $\mathbf{v}_\perp$ change in the process, then the rotated orbit acquires a new size and shape.

To find the delta-v associated with a plane change, let $\mathbf{v}_1$ be the velocity before and $\mathbf{v}_2$ the velocity after the impulsive maneuver. Then

$$\mathbf{v}_1 = v_{r_1}\hat{\mathbf{u}}_r + v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}$$
$$\mathbf{v}_2 = v_{r_2}\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2}$$

where $\hat{\mathbf{u}}_r$ is the radial unit vector directed along the line of intersection of the two orbital planes. $\hat{\mathbf{u}}_r$ does not change during the maneuver. As we know, the transverse unit vector $\hat{\mathbf{u}}_\perp$ is perpendicular to $\hat{\mathbf{u}}_r$ and lies in the orbital plane. Therefore, it rotates through the dihedral angle δ from its initial orientation $\hat{\mathbf{u}}_{\perp_1}$ to its final orientation $\hat{\mathbf{u}}_{\perp_2}$.

The change $\Delta\mathbf{v}$ in the velocity vector is

$$\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = (v_{r_2} - v_{r_1})\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}$$

The magnitude Δv is found by taking the dot product of $\Delta\mathbf{v}$ with itself,

$$\Delta v^2 = \Delta\mathbf{v}\cdot\Delta\mathbf{v} = \left[(v_{r_2} - v_{r_1})\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}\right]\cdot\left[(v_{r_2} - v_{r_1})\hat{\mathbf{u}}_r + v_{\perp_2}\hat{\mathbf{u}}_{\perp_2} - v_{\perp_1}\hat{\mathbf{u}}_{\perp_1}\right]$$

Carrying out the dot products while noting that $\hat{\mathbf{u}}_r\cdot\hat{\mathbf{u}}_r = \hat{\mathbf{u}}_{\perp_1}\cdot\hat{\mathbf{u}}_{\perp_1} = \hat{\mathbf{u}}_{\perp_2}\cdot\hat{\mathbf{u}}_{\perp_2} = 1$ and $\hat{\mathbf{u}}_r\cdot\hat{\mathbf{u}}_{\perp_1} = \hat{\mathbf{u}}_{\perp_1}\cdot\hat{\mathbf{u}}_{\perp_2} = 0$ yields

$$\Delta v^2 = (v_{r_2} - v_{r_1})^2 + {v_{\perp_1}}^2 + {v_{\perp_2}}^2 - 2v_{\perp_1}v_{\perp_2}(\hat{\mathbf{u}}_{\perp_1}\cdot\hat{\mathbf{u}}_{\perp_2})$$

But $\hat{\mathbf{u}}_{\perp_1}\cdot\hat{\mathbf{u}}_{\perp_2} = \cos\delta$, so that we finally obtain a general formula for Δv with plane change,

$$\boxed{\Delta v = \sqrt{(v_{r_2} - v_{r_1})^2 + {v_{\perp_1}}^2 + {v_{\perp_2}}^2 - 2v_{\perp_1}v_{\perp_2}\cos\delta}} \tag{6.19}$$

From the definition of the flight path angle (cf. Fig. 2.12),

$$v_{r_1} = v_1\sin\gamma_1 \qquad v_{\perp_1} = v_1\cos\gamma_1$$
$$v_{r_2} = v_2\sin\gamma_2 \qquad v_{\perp_2} = v_2\cos\gamma_2$$

Substituting these relations into Eq. (6.19), expanding and collecting terms, and using the trigonometric identities,

$$\sin^2\gamma_1 + \cos^2\gamma_1 = \sin^2\gamma_2 + \cos^2\gamma_2 = 1$$
$$\cos(\gamma_2 - \gamma_1) = \cos\gamma_2\cos\gamma_1 + \sin\gamma_2\sin\gamma_1$$

leads to another version of Eq. (6.19),

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2[\cos\Delta\gamma - \cos\gamma_2\cos\gamma_1(1 - \cos\delta)]} \tag{6.20}$$

where $\Delta\gamma = \gamma_2 - \gamma_1$. If there is no plane change ($\delta = 0$), then $\cos\delta = 1$ and Eq. (6.20) reduces to

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\Delta\gamma} \quad \text{(No plane change)}$$

which is the cosine law we have been using to compute Δv in coplanar maneuvers. Therefore, not surprisingly, Eq. (6.19) contains Eq. (6.8) as a special case.

To keep Δv at a minimum, it is clear from Eq. (6.19) that the radial velocity should remain unchanged during a plane change maneuver. For the same reason, it is apparent that the maneuver should occur where $v_\perp$ is smallest, which is at apoapsis. Fig. 6.23 illustrates a plane change maneuver at the apoapsis of both orbits. In this case $v_{r_1} = v_{r_2} = 0$, so that $v_{\perp_1} = v_1$ and $v_{\perp_2} = v_2$, thereby reducing Eq. (6.19) to

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\delta} \tag{6.21}$$

Eq. (6.21) is for a speed change accompanied by a plane change, as illustrated in Fig. 6.24a. Using the trigonometric identity

$$\cos\delta = 1 - 2\sin^2\frac{\delta}{2}$$

we can rewrite Eq. (6.21) as follows,

$$\Delta v_{\mathrm{I}} = \sqrt{(v_2 - v_1)^2 + 4v_1 v_2 \sin^2\frac{\delta}{2}} \quad \text{(Rotation about the common apse line)} \tag{6.22}$$

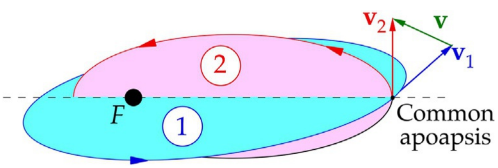


FIG. 6.23

Impulsive plane change maneuver at apoapsis.

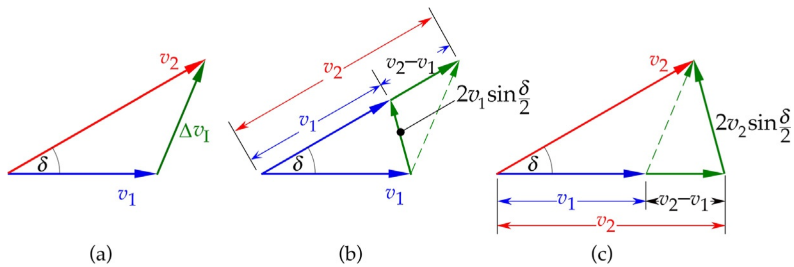


FIG. 6.24

The orbital plane rotates about the common apse line. (a) Speed change accompanied by plane change. (b) Plane change followed by speed change. (c) Speed change followed by plane change.

If there is no change in the speed, so that $v_2 = v_1$, then Eq. (6.22) yields

$$\Delta v_\delta = 2v\sin\frac{\delta}{2} \quad \text{(Pure rotation of the velocity vector)} \tag{6.23}$$

The subscript δ reminds us that this is the delta-v for a pure rotation of the velocity vector through the angle δ.

Another plane change strategy, illustrated in Fig. 6.24b, is to rotate the velocity vector and then change its magnitude. In that case, the delta-v is

$$\Delta v_{\mathrm{II}} = 2v_1 \sin\frac{\delta}{2} + |v_2 - v_1|$$

Yet another possibility is to change the speed first, and then rotate the velocity vector (Fig. 6.24c). Then

$$\Delta v_{\mathrm{III}} = |v_2 - v_1| + 2v_2 \sin\frac{\delta}{2}$$

Since the sum of the lengths of any two sides of a triangle must be greater than the length of the third side, it is evident from Fig. 6.24 that both Δv_{II} and Δv_{III} are greater than Δv_{I}. It follows that plane change accompanied by speed change is the most efficient of the above three maneuvers.

Eq. (6.23), the delta-v formula for pure rotation of the velocity vector, is plotted in Fig. 6.25, which shows why significant plane changes are so costly in terms of propellant expenditure. For example, a plane change of just 24° requires a delta-v equal to that needed for an escape trajectory (41.4% velocity boost). A 60° plane change requires a delta-v equal to the speed of the spacecraft itself, which in earth orbit operations is about 7.5 km/s. For such a maneuver in LEO, the most efficient chemical propulsion system would require that well over 80% of the spacecraft mass consist of propellant. The space shuttle was capable of a plane change in orbit of only about 3°, a maneuver which would exhaust its entire orbital-maneuvering fuel capacity. Orbit plane adjustments are therefore made during the powered ascent phase when the energy is available to do so.

For some missions, however, plane changes must occur in orbit. A common example is the maneuvering of GEO satellites into position. These must orbit the earth in the equatorial plane, but it is impossible to throw a satellite directly into an equatorial orbit from a launch site that is not on the equator. That is not difficult to understand when we realize that the plane of the orbit must contain the center of the earth (the focus) as well as the point at which the satellite is inserted into orbit, as illustrated in Fig. 6.26. So if the insertion point is anywhere but on the equator, the plane of the orbit will be tilted

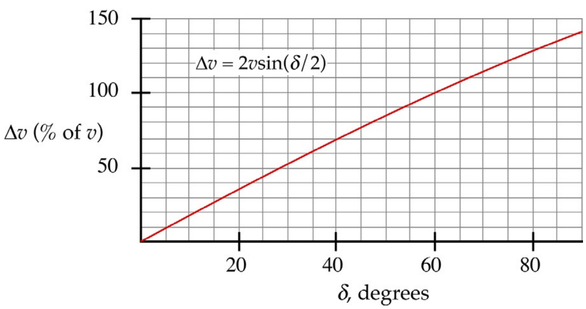


FIG. 6.25

Δv required to rotate the velocity vector through an angle δ.

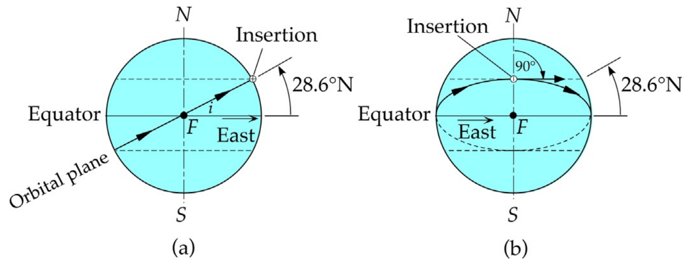


FIG. 6.26

Two views of the orbit of a satellite launched directly east at 28.6°N latitude. (a) Edge-on view of the orbital plane. (b) View toward insertion point meridian.

away from the earth's equator. As we know from Chapter 4, the angle between the equatorial plane and the plane of the orbiting satellite is called the inclination i.

Launching a satellite due east takes full advantage of the earth's rotational velocity, which is 0.46 km/s (about 1000 miles per hour) at the equator and diminishes toward the poles according to the formula

$$v_{\text{rotational}} = v_{\text{equatorial}} \cos\phi$$

where ϕ is the latitude. Fig. 6.26 shows a spacecraft launched due east into low earth orbit at a latitude of 28.6°N, which is the latitude of the Kennedy Space Flight Center (KSC). As can be seen from the figure, the inclination of the orbit will be 28.6°. One-fourth of the way around the earth the satellite will cross the equator. Halfway around the earth it reaches its southernmost latitude, $\phi = 28.6°\,\text{S}$. It then heads north, crossing over the equator at the three-quarters point, and returning after one complete revolution to $\phi = 28.6°\,\text{N}$.

Launch azimuth A is the flight direction at insertion, measured clockwise from north on the local meridian. Thus $A = 90°$ is due east. If the launch direction is not directly eastward, then the orbit will have an inclination greater than the launch latitude, as illustrated in Fig. 6.27 for $\phi = 28.6°\,\text{N}$. Northeasterly ($0 < A < 90°$) or southeasterly ($90° < A < 180°$) launches take only partial advantage of the earth's rotational speed and both produce an inclination i greater than the launch latitude but less than 90°. Since these orbits have an eastward velocity component, they are called prograde orbits. Launches

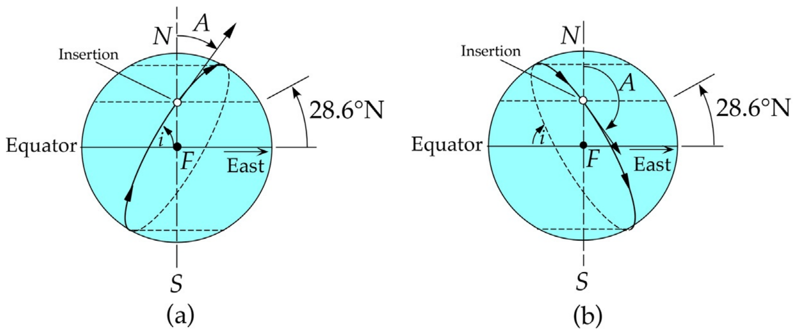


FIG. 6.27

(a) Northeasterly launch ($0 < A < 90°$) from a latitude of 28.6°N. (b) Southeasterly launch ($90° < A < 270°$).

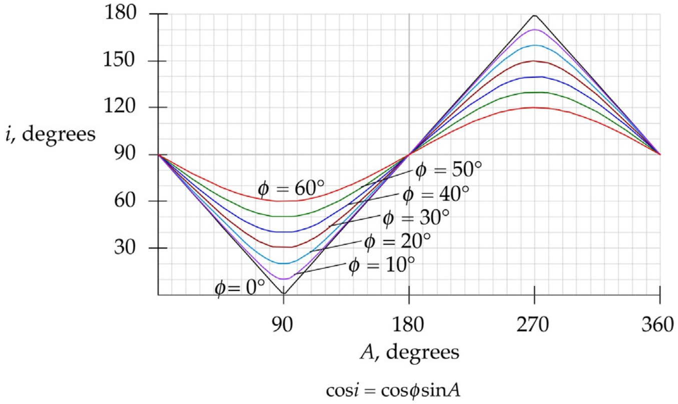


FIG. 6.28

Orbit inclination i versus launch azimuth A for several latitudes ϕ.

to the west produce retrograde orbits with inclinations between 90° and 180°. Launches directly north or directly south result in polar orbits.

Spherical trigonometry is required to obtain the relationship between orbital inclination i, launch platform latitude ϕ, and launch azimuth A. It turns out that

$$\cos i = \cos\phi \sin A \tag{6.24}$$

From this, we verify, for example, that $i = \phi$ when $A = 90°$, as pointed out above. A plot of this relation is presented in Fig. 6.28, while Fig. 6.29 illustrates the orientation of orbits for a range of launch azimuths at $\phi = 28°$.

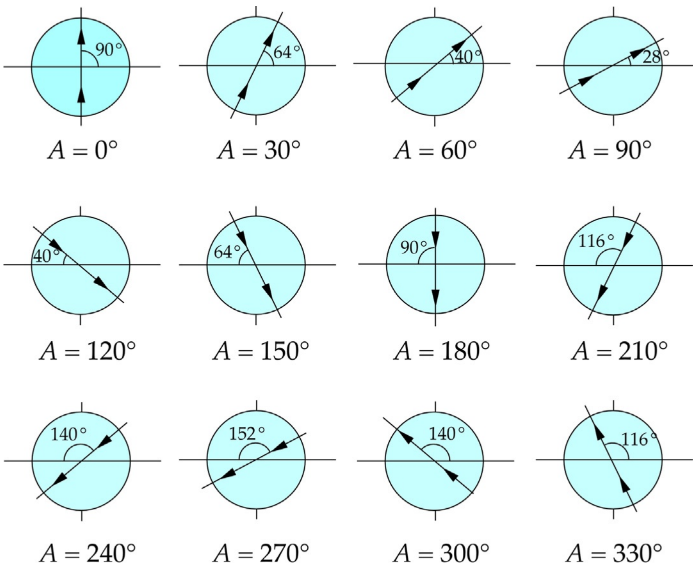


FIG. 6.29

Variation of orbit inclinations with launch azimuth at $\phi = 28^\circ$. Note the retrograde orbits for $A > 180^\circ$.

EXAMPLE 6.10

Determine the required launch azimuth for the sun-synchronous satellite of Example 4.9 if it is launched from Vandenburgh AFB on the California coast (latitude = 34.5°N).

Solution

In Example 4.9 the inclination of the sun-synchronous orbit was determined to be 98.43°. Eq. (6.24) is used to calculate the launch azimuth,

$$\sin A = \frac{\cos i}{\cos l} = \frac{\cos 98.43°}{\cos 34.5°} = -0.1779$$

From this $A = 190.2°$, a launch to the south, or $A = 349.8°$, a launch to the north.

Fig. 6.30 shows the effect that the choice of launch azimuth has on the sun-synchronous orbit of Example 6.10. It does not change the fact that the orbit is retrograde; it simply determines whether the ascending node will be in the same hemisphere as the launch site or on the opposite side of the earth. Actually, a launch to the north from Vandenburgh is not an option because of the safety hazard to the populated land lying below the ascent trajectory. Launches to the south, over open water, are not a hazard. Working this problem for Kennedy Space Center (latitude 28.6°N) yields nearly the same values of A. Since safety considerations on the Florida east coast limit launch azimuths to between 35° and 120°, polar and sun-synchronous satellites cannot be launched from the eastern test range.

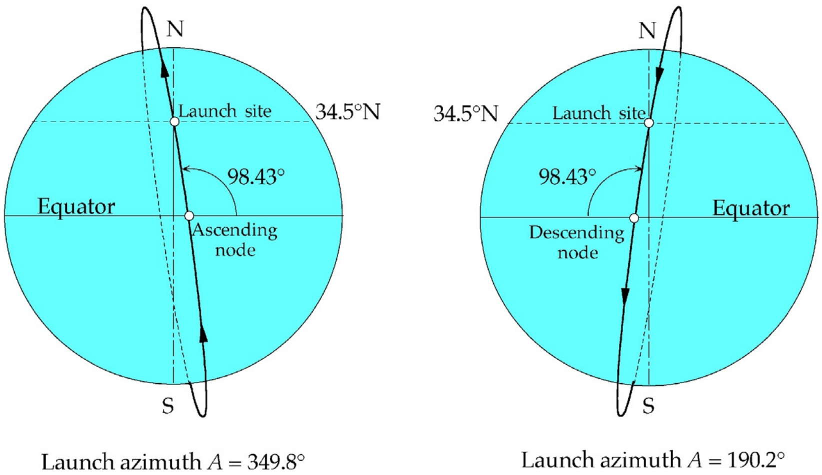


FIG. 6.30

Effect of launch azimuth on the position of the orbit.

EXAMPLE 6.11

Find the delta-v required to transfer a satellite from a circular, 300-km-altitude low earth orbit of 28° inclination to a geostationary equatorial orbit. Circularize and change the inclination at altitude. Compare that delta-v requirement with the one in which the plane change is done in the low earth orbit.

Solution

Fig. 6.31 shows the 28° inclined low earth parking orbit (1), the coplanar Hohmann transfer ellipse (2), and the coplanar GEO orbit (3). From the figure we see that

$$r_B = 6678\,\text{km} \qquad r_C = 42{,}164\,\text{km}$$

Orbit 1:
For this circular orbit the speed at B is

$$v_B)_1 = \sqrt{\frac{\mu}{r_B}} = \sqrt{\frac{398{,}600}{6678}} = 7.7258\,\text{km/s}$$

Orbit 2:
We first obtain the angular momentum by means of Eq. (6.2),

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = 67{,}792\,\text{km/s}$$

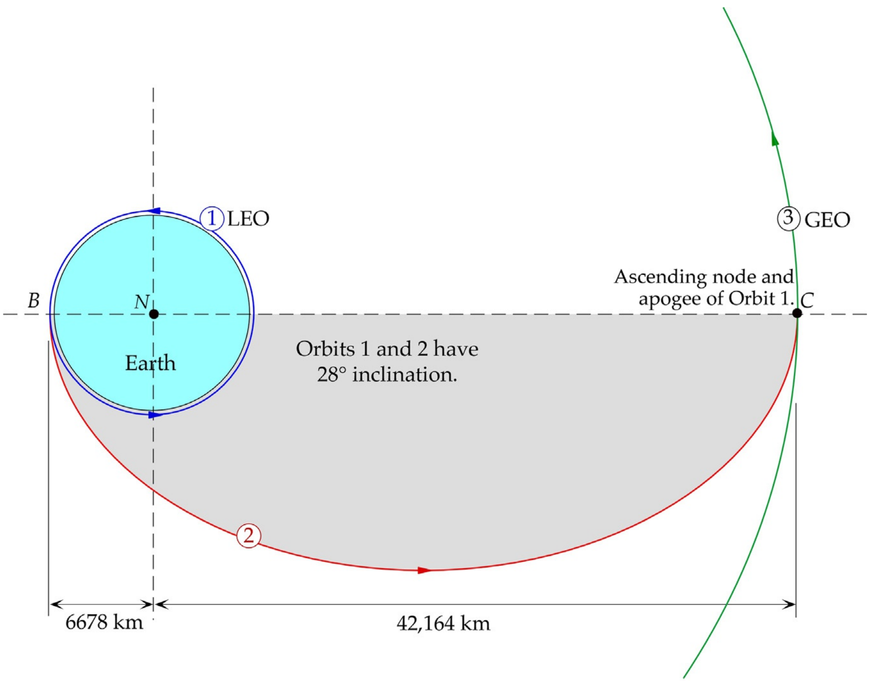


FIG. 6.31

Transfer from LEO to GEO in an orbit of 28° inclination.

The velocities at perigee and apogee of orbit 2 are, from the angular momentum formula,

$$v_B)_2 = \frac{h_2}{r_B} = 10.152\,\text{km/s} \qquad v_C)_2 = \frac{h_2}{r_C} = 1.6078\,\text{km/s}$$

At this point we can calculate Δv_B,

$$\Delta v_B = v_B)_2 - v_B)_1 = 10.152 - 7.7258 = 2.4258\,\text{km/s}$$

Orbit 3:
For this GEO orbit, which is circular, the speed at C is

$$v_C)_3 = \sqrt{\frac{\mu}{r_C}} = 3.0747\,\text{km/s}$$

The spacecraft in orbit 2 arrives at C with a velocity of 1.6078 km/s inclined at 28° to orbit 3. Therefore, both its orbital speed and inclination must be changed at C (Fig. 6.32). The most efficient strategy is to combine the plane change with the speed change (Eq. 6.21), so that

$$\begin{aligned}\Delta v_C &= \sqrt{v_C)_2{}^2 + v_C)_3{}^2 - 2v_C)_2 v_C)_3 \cos\Delta i} \\ &= \sqrt{1.6078^2 + 3.0747^2 - 2\cdot 1.6078\cdot 3.0747\cdot \cos 28^\circ} = 1.8191\,\text{km/s}\end{aligned}$$

Therefore, the total delta-v requirement is

$$\Delta v_{\text{total}} = \Delta v_B + \Delta v_C = 2.4258 + 1.819 = \boxed{4.2449\,\text{km/s}} \quad \text{(Plane change at } C\text{)}$$

Suppose we make the plane change at LEO instead of at GEO. In that case, Eq. (6.21) provides the initial delta-v,

$$\begin{aligned}\Delta v_B &= \sqrt{v_B)_1{}^2 + v_B)_2{}^2 - 2v_B)_1 v_B)_2 \cos\Delta i} \\ &= \sqrt{7.7258^2 + 10.152^2 - 2\cdot 7.7258\cdot 10.152\cdot \cos 28^\circ} = 4.9242\,\text{km/s}\end{aligned}$$

The spacecraft travels to C in the equatorial plane, so that when it arrives, the delta-v requirement at C is simply

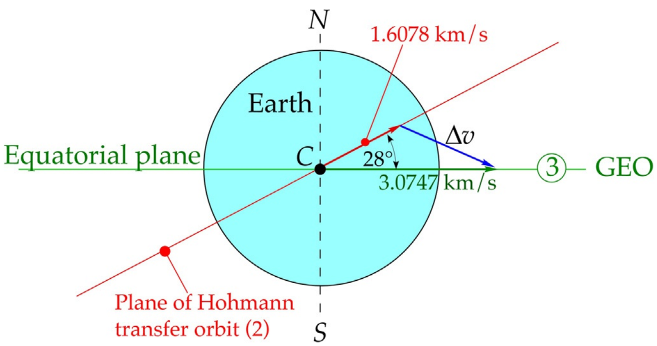


FIG. 6.32

Plane change maneuver required after the Hohmann transfer.

$$\Delta v_C = v_C)_3 - v_C)_2 = 3.0747 - 1.6078 = 1.4668\,\text{km/s}$$

Therefore, the total delta-v is

$$\Delta v_{\text{total}} = \Delta v_B + \Delta v_C = 4.9242 + 1.4668 = \boxed{6.3910\,\text{km/s}} \quad (\text{Plane change at } B)$$

This is a 50% increase over the total delta-v with plane change at GEO. Clearly, it is best to do plane change maneuvers at the largest possible distance (apoapsis) from the primary attractor, where the velocities are smallest.

EXAMPLE 6.12

Suppose in the previous example that part of the plane change, Δi, takes place at B, the perigee of the Hohmann transfer ellipse, and the remainder, $28° - \Delta i$, occurs at the apogee C. What is the value of Δi that results in the minimum Δv_{total}?

Solution

We found in Example 6.11 that if $\Delta i = 0$, then $\Delta v_{\text{total}} = 4.2449\,\text{km/s}$, whereas $\Delta i = 28°$ made $\Delta v_{\text{total}} = 6.3910\,\text{km/s}$. Here we are to determine if there is a value of Δi between 0° and 28° that yields a Δv_{total} that is smaller than either of those two.

In this case a plane change occurs at both B and C. The most efficient strategy is to combine the plane change with the speed change, so that the delta-v's at those points are (Eq. 6.21)

$$\begin{aligned}\Delta v_B &= \sqrt{v_B)_1{}^2 + v_B)_2{}^2 - 2v_B)_1 v_B)_2 \cos\Delta i} \\ &= \sqrt{7.7258^2 + 10.152^2 - 2\cdot 7.7258\cdot 10.152\cdot \cos\Delta i} \\ &= \sqrt{162.74 - 156.86\cos\Delta i}\end{aligned}$$

and

$$\begin{aligned}\Delta v_C &= \sqrt{v_C)_2{}^2 + v_C)_3{}^2 - 2v_C)_2 v_C)_3 \cos(28° - \Delta i)} \\ &= \sqrt{1.6078^2 + 3.0747^2 - 2\cdot 1.1078\cdot 3.0747\cdot \cos(28° - \Delta i)} \\ &= \sqrt{12.039 - 9.8874\cos(28° - \Delta i)}\end{aligned}$$

Thus,

$$\Delta v_{\text{total}} = \Delta v_B + \Delta v_C = \sqrt{162.74 - 156.86\cos\Delta i} + \sqrt{12.039 - 9.8874\cos(28° - \Delta i)} \qquad \text{(a)}$$

To determine if there is a Δi that minimizes Δv_{total}, we take its derivative with respect to Δi and set it equal to zero:

$$\frac{d\Delta v_{\text{total}}}{d\Delta i} = \frac{78.43\sin\Delta i}{\sqrt{162.74 - 156.86\cos\Delta i}} - \frac{4.9435\sin(28° - \Delta i)}{\sqrt{12.039 - 9.8871\cos(28° - \Delta i)}} = 0$$

This is a transcendental equation, which must be solved iteratively. The solution, as the reader may verify, is

$$\boxed{\Delta i = 2.1751°} \qquad \text{(b)}$$

That is, an inclination change of 2.1751° should occur in low earth orbit, while the rest of the plane change, 25.825°, is done at GEO. Substituting Eq. (b) into Eq. (a) yields

$$\boxed{\Delta v_{\text{total}} = 4.2207\,\text{km/s}}$$

This is only very slightly smaller (less than 1%) than the lowest v_{total} computed in Example 6.11.

EXAMPLE 6.13

A spacecraft is in a 500 km by 10,000 km altitude geocentric orbit that intersects the equatorial plane at a true anomaly of 120° (see Fig. 6.33). If the orbit's inclination to the equatorial plane is 15°, what is the minimum velocity increment required to make this an equatorial orbit?

Solution

The orbital parameters are

$$e = \frac{r_A - r_P}{r_A + r_P} = \frac{(6378 + 10,000) - (6378 + 500)}{(6378 + 10,000) + (6378 + 500)} = 0.4085$$

$$h = \sqrt{2\mu}\sqrt{\frac{r_A r_P}{r_A + r_P}} = \sqrt{2 \cdot 398,600}\sqrt{\frac{16,378 \cdot 6878}{16,378 + 6878}} = 62,141\,\text{km/s}$$

The radial position and velocity components at points B and C, on the line of intersection with the equatorial plane, are

$$r_B = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_B} = \frac{62,141^2}{398,600}\frac{1}{1 + 0.4085 \cdot \cos 120°} = 12,174\,\text{km}$$

$$v_{\perp_B} = \frac{h}{r_B} = \frac{62,141}{12,174} = 5.1043\,\text{km/s}$$

$$v_{r_B} = \frac{\mu}{h}e\sin\theta = \frac{398,600}{62,141} \cdot 0.4085 \cdot \sin 120° = 2.2692\,\text{km/s}$$

and

$$r_C = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_C} = \frac{62,141^2}{398,600}\frac{1}{1 + 0.4085 \cdot \cos 300°} = 8044.6\,\text{km}$$

$$v_{\perp_C} = \frac{h}{r_C} = \frac{62,141}{8044.6} = 7.7246\,\text{km/s}$$

$$v_{r_C} = \frac{\mu}{h}e\sin\theta_C = \frac{398,600}{62,141} \cdot 0.4085 \cdot \sin 300° = -2.2692\,\text{km/s}$$

These are all shown in Fig. 6.33.

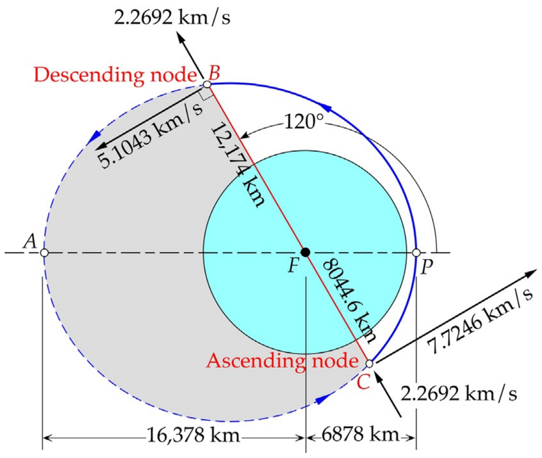


FIG. 6.33

An orbit that intersects the equatorial plane along line BC. The equatorial plane makes an angle of 15° with the plane of the page.

All we wish to do here is to rotate the plane of the orbit rigidly around the node line BC. The impulsive maneuver must occur at either B or C. Eq. (6.19) applies, and since the radial and transverse velocity components remain fixed, it reduces to

$$\Delta v = v_\perp \sqrt{2(1-\cos\delta)} = 2v_\perp \sin\frac{\delta}{2}$$

where $\delta = 15°$. For the minimum Δv, the maneuver must be done where $v_\perp$ is smallest, which is at B, the point farthest from the center of attraction F. Thus,

$$\Delta v = 2\cdot 5.1043\cdot \sin\frac{15°}{2} = \boxed{1.3325\,\text{km/s}}$$

EXAMPLE 6.14

Orbit 1 in Fig. 6.34 has angular momentum h and eccentricity e. The direction of motion is shown. Calculate the Δv required to rotate the orbit 90° about its latus rectum BC without changing h and e. The required direction of motion in orbit 2 is shown.

Solution

By symmetry, the required maneuver may occur at either B or C, and it involves a rigid body rotation of the ellipse, so that v_r and $v_\perp$ remain unaltered. Because of the directions of motion shown, the true anomalies of B on the two orbits are

$$\theta_B)_1 = -90° \quad \theta_B)_2 = +90°$$

The radial coordinate of B is

$$r_B = \frac{h^2}{\mu}\frac{1}{1+e\cos(\pm 90)} = \frac{h^2}{\mu}$$

For the velocity components at B, we have

$$v_{\perp_B})_1 = v_{\perp_B})_2 = \frac{h}{r_B} = \frac{\mu}{h}$$
$$v_{r_B})_1 = \frac{\mu}{h}e\sin\theta_B)_1 = -\frac{\mu e}{h}$$
$$v_{r_B})_2 = \frac{\mu}{h}e\sin\theta_B)_2 = \frac{\mu e}{h}$$

Substituting these into Eq. (6.19), yields

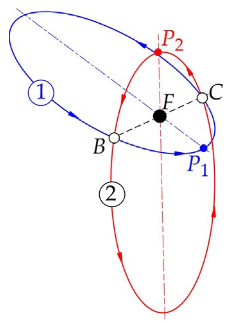


FIG. 6.34

Identical ellipses intersecting at 90° along their common latus rectum, *BC*.

$$\Delta v_B = \sqrt{\left[v_{r_B})_2 - v_{r_B})_1\right]^2 + v_{\perp_B})_1{}^2 + v_{\perp_B})_2{}^2 - 2v_{\perp_B})_1 v_{\perp_B})_2 \cos 90^\circ}$$
$$= \sqrt{\left[\frac{\mu e}{h} - \left(-\frac{\mu e}{h}\right)\right]^2 + \left(\frac{\mu}{h}\right)^2 + \left(\frac{\mu}{h}\right)^2 - 2\left(\frac{\mu}{h}\right)\left(\frac{\mu}{h}\right)(0)}$$
$$= \sqrt{4\frac{\mu^2}{h^2}e^2 + 2\frac{\mu^2}{h^2}}$$

so that

$$\boxed{\Delta v_B = \frac{\sqrt{2}\mu}{h}\sqrt{1+2e^2}} \tag{a}$$

If the motion on ellipse 2 were opposite to that shown in Fig. 6.34, then the radial velocity components at B (and C) would be in the same rather than in the opposite direction on both ellipses, so that instead of Eq. (a) we would find a smaller velocity increment,

$$\Delta v_B = \frac{\sqrt{2}\mu}{h}$$

6.10 NONIMPULSIVE ORBITAL MANEUVERS

Up to this point we have assumed that delta-v maneuvers take place in zero time, altering the velocity vector but leaving the position vector unchanged. In nonimpulsive maneuvers the thrust acts over a significant time interval and must be included in the equations of motion. According to Problem 2.3, adding an external force $\mathbf{F}$ to the spacecraft yields the following equation of relative motion:

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \frac{\mathbf{F}}{m} \tag{6.25}$$

where m is the mass of the spacecraft. This of course reduces to Eq. (2.22) when $\mathbf{F} = 0$. If the external force is a thrust T in the direction of the velocity vector $\mathbf{v}$, then $\mathbf{F} = T(\mathbf{v}/v)$ and Eq. (6.25) becomes

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \frac{T}{m}\frac{\mathbf{v}}{v} \quad (\mathbf{v} = \dot{\mathbf{r}}) \tag{6.26}$$

(Drag forces act opposite to the velocity vector, and so does thrust during a retrofire maneuver.) The Cartesian component form of Eq. (6.26) is

$$\ddot{x} = -\mu\frac{x}{r^3} + \frac{T}{m}\frac{\dot{x}}{v} \qquad \ddot{y} = -\mu\frac{y}{r^3} + \frac{T}{m}\frac{\dot{y}}{v} \qquad \ddot{z} = -\mu\frac{z}{r^3} + \frac{T}{m}\frac{\dot{z}}{v} \tag{6.27a}$$

where

$$r = \sqrt{x^2+y^2+z^2} \qquad v = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \tag{6.27b}$$

While the rocket motor is firing, the spacecraft mass decreases, because propellant combustion products are being discharged into space through the nozzle. According to elementary rocket dynamics (cf. Section 13.3), the mass decreases at a rate given by the formula

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{T}{I_{sp}g_0} \tag{6.28}$$

where T and I_{sp} are the thrust and the specific impulse of the propulsion system, and g_0 is the sea level acceleration of gravity.

If the thrust is not zero, then Eqs. (6.27a) may not have a straightforward analytical solution. In any case, they can be solved numerically using methods such as those discussed in Section 1.8. For that purpose, Eqs. (6.27a), (6.27b), and (6.28) must be rewritten as a system of linear differential equations in the form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{6.29}$$

For the case at hand, the vector **y** consists of the six components of the state vector (position and velocity vectors) plus the mass. Therefore, with the aid of Eqs. (6.27a), (6.27b), and (6.28), we have

$$\mathbf{y} = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix} \quad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{m} \end{Bmatrix} \quad \mathbf{f}(t, y) = \begin{Bmatrix} y_4 \\ y_5 \\ y_6 \\ -\mu \dfrac{y_1}{r^3} + \dfrac{T}{m}\dfrac{y_4}{v} \\ -\mu \dfrac{y_2}{r^3} + \dfrac{T}{m}\dfrac{y_5}{v} \\ -\mu \dfrac{y_3}{r^3} + \dfrac{T}{m}\dfrac{y_6}{v} \\ -\dfrac{T}{I_{sp} g_o} \end{Bmatrix} \tag{6.30}$$

The numerical solution of Eq. (6.30) is illustrated in the following examples.

EXAMPLE 6.15

Suppose the spacecraft in Example 6.1 (see Fig. 6.3) has a restartable onboard propulsion system with a thrust of 10 kN and specific impulse of 300 s. Assuming that the thrust vector remains aligned with the velocity vector, solve Example 6.1 without using impulsive (zero time) delta-v burns. Compare the propellant expenditures for the two solutions.

Solution

Refer to Fig. 6.3 as an aid to visualizing the solution procedure described below. Let us assume that the plane of Fig. 6.3 is the xy plane of an earth-centered inertial frame with the z axis directed out of the page. The apse line of orbit 1 is the x axis, which is directed to the right, and y points upward toward the top of the page.

Transfer from perigee of orbit 1 to apogee of orbit 2

According to Example 6.1, the state vector just before the first delta-v maneuver is

$$\mathbf{y}_0 = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix}_{t=0} = \begin{Bmatrix} 6858\,\text{km} \\ 0 \\ 0 \\ 0 \\ 7.7102\,\text{km/s} \\ 0 \\ 2000 \end{Bmatrix} \tag{a}$$

Using this together with an assumed burn time t_{burn}, we numerically integrate Eq. (6.29) from $t = 0$ to $t = t_{\text{burn}}$. This yields **r**, **v**, and the mass m at the start of the coasting trajectory (orbit 2). We can find the true anomaly θ at the start of orbit 2 by substituting these values of **r** and **v** into Algorithm 4.2. The spacecraft must coast through a true anomaly of $\Delta\theta = 180° - \theta$ to reach apogee. Substituting **r**, **v**, and $\Delta\theta$ into Algorithm 2.3 yields the state vector ($\mathbf{r}_a$ and $\mathbf{v}_a$) at apogee.

The apogee radius r_a is the magnitude of $\mathbf{r}_a$. If r_a does not equal the target value of 22,378 km, then we assume a new burn time and repeat the above steps to calculate a new r_a. This trial-and-error process is repeated until r_a is acceptably close to 22,378 km.

The calculations are done in the MATLAB M-function *integrate_thrust.m*, which is listed in Appendix D.30. *rkf45.m* (see Appendix D.4) was chosen as the numerical integrator. The initial conditions $\mathbf{y}_0$ in Eq. (a) are passed to *rkf45*, which solves the system of Eq. (6.29) at discrete times between 0 and t_{burn}. *rkf45.m* employs the subfunction *rates*, embedded in *integrate_thrust.m*, to calculate the vector of derivatives **f** in Eq. (6.30). Output is to the command window, and a revised burn time was entered into the code in the MATLAB editor after each calculation of r_a.

The following output of *integrate_thrust.m* shows that a burn time of 261.1127 s (4.352 min), with a propellant expenditure of 887.5 kg, is required to produce a coasting trajectory with an apogee of 22,378 km. Due to the finite burn time, the apse line in this case is rotated 8.336° counterclockwise from that in Example 6.1 (line *BCA* in Fig. 6.3). Notice that the speed boost Δv imparted by the burn is $9.38984 - 7.71020 = 1.6796$ km/s, compared with the impulsive $\Delta v_A = 1.7725$ km/s in Example 6.1.

```
Before ignition:
  Mass = 2000 kg
  State vector:
    r = [     6858,           0,     0] (km)
     Radius = 6858
    v = [        0,     7.7102,     0] (km/s)
     Speed = 7.7102

Thrust          =           10 kN
Burn time       =    261.112700 s
Mass after burn = 1.112495E+03 kg

End-of-burn state vector:
    r = [    6551.56,  2185.85,     0] (km)
     Radius = 6906.58
    v = [ -2.42229,    9.07202,     0] (km/s)
     Speed = 9.38984

Postburn trajectory:
  Eccentricity = 0.530257
  Semimajor axis = 14623.7 km
  Apogee state vector:
    r = [-2.21416E+04, -3.24453E+03,  0.00000E+00] (km)
     Radius = 22378
    v = [ 4.19390E-01, -2.86203E+00, -0.00000E+00] (km/s)
     Speed = 2.8926
```

Transfer from apogee of orbit 2 to the circular target orbit 3

The spacecraft mass and state vector at apogee, given by the above MATLAB output (under "`Postburn trajectory`"), are entered as new initial conditions in *integrate_thrust.m*, and the manual trial-and-error process described above is carried out. It is not possible to transfer from the 22,378-km apogee of orbit 2 to a circular orbit of radius 22,378 km using a single finite-time burn. Therefore, the objective in this case is to make the semimajor axis of the final orbit equal to 22,378 km. This was achieved with a burn time of 118.88 s and a propellant expenditure of 404.05 kg, and it yields a nearly circular orbit having an eccentricity of 0.00867 and an apse line rotated 80.85° clockwise from the x axis.

The computed spacecraft mass at the end of the second delta-v maneuver is 708.44 kg. Therefore, the total propellant expenditure is $2000 - 708.44 = 1291.6$ kg. This is essentially the same as the propellant requirement (1291.3 kg) calculated in Example 6.1, in which the two delta-v maneuvers were impulsive.

Let us take the dot product of both sides of Eq. (6.26) with the velocity **v**, to obtain

$$\ddot{\mathbf{r}} \cdot \mathbf{v} = -\frac{\mu}{r^3}\mathbf{r} \cdot \mathbf{v} + \frac{T}{m}\frac{\mathbf{v} \cdot \mathbf{v}}{v} \tag{6.31}$$

In Section 2.5, we showed that

$$\ddot{\mathbf{r}}\cdot\mathbf{v}=\frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t}\quad\text{and}\quad\frac{\mu}{r^3}\mathbf{r}\cdot\mathbf{v}=-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{r}\right)$$

Substituting these together with $\mathbf{v}\cdot\mathbf{v}=v^2$ into Eq. (6.31) yields the energy equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}-\frac{\mu}{r}\right)=\frac{T}{m}v \tag{6.32}$$

This equation may be applied to the approximate solution of a constant tangential thrust orbit transfer problem. If the spacecraft is in a circular orbit, then applying a very low constant thrust T in the forward direction will cause its total energy $\varepsilon = v^2/2 - \mu/r$ to slowly increase over time according to Eq. (6.32). This will raise the height after each revolution, resulting in a slow outward spiral (or inward spiral if the thrust is directed aft). If we assume that the speed at any radius of the closely spaced spiral trajectory is essentially that of a circular orbit of that radius (Wiesel, 2010), then we can replace v by $\sqrt{\mu/r}$ to obtain an approximate version of Eq. (6.32),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\mu}{r}-\frac{\mu}{r}\right)=\frac{T}{m}\sqrt{\frac{\mu}{r}}$$

Simplifying and separating variables leads to

$$\frac{\mathrm{d}(\mu/r)}{\sqrt{\mu/r}}=-2\frac{T}{m}\mathrm{d}t \tag{6.33}$$

The spacecraft mass is a function of time

$$m=m_0-\dot{m}_e t \tag{6.34}$$

where m_0 is the mass at the start of the orbit transfer ($t=0$), and $\dot{m}_e$ is the constant rate at which propellant is expended. Thus

$$\frac{\mathrm{d}(\mu/r)}{\sqrt{\mu/r}}=-2\frac{T}{m_0-\dot{m}_e t}\mathrm{d}t \tag{6.35}$$

Integrating both sides of this equation and setting $r=r_0$ when $t=0$ results in

$$\sqrt{\frac{\mu}{r}}-\sqrt{\frac{\mu}{r_0}}=\frac{T}{\dot{m}_e}\ln\left(1-\frac{\dot{m}_e}{m_0}t\right) \tag{6.36}$$

Finally, since $\dot{m}_e=-\mathrm{d}m/\mathrm{d}t$, Eq. (6.28) implies that we can replace $\dot{m}_e$ with $T/(I_{sp}g_0)$, so that

$$\sqrt{\frac{\mu}{r}}-\sqrt{\frac{\mu}{r_0}}=I_{sp}g_0\ln\left(1-\frac{T}{m_0 g_0 I_{sp}}t\right) \tag{6.37}$$

We may solve this equation for either r or t to get

$$r=\frac{\mu}{\left[\sqrt{\frac{\mu}{r_0}}+I_{sp}g_0\ln\left(1-\frac{T}{m_0 g_0 I_{sp}}t\right)\right]^2} \tag{6.38}$$

$$t=\frac{m_0 g_0 I_{sp}}{T}\left\{1-\exp\left[\frac{\sqrt{\mu}}{I_{sp}g_0}\left(\sqrt{\frac{1}{r}}-\sqrt{\frac{1}{r_0}}\right)\right]\right\} \tag{6.39}$$

where $\exp(x)=e^x$. Although this scalar analysis yields the radius in terms of the elapsed time, it does not provide us the state vector components $\mathbf{r}$ and $\mathbf{v}$.

EXAMPLE 6.16

A 1000-kg spacecraft is in a 6678-km (300-km-altitude) circular equatorial earth orbit. Its ion propulsion system, which has a specific impulse of 10,000 s, exerts a constant tangential thrust of $2500(10^{-6})$ kN.

(a) How long will it take the spacecraft to reach GEO (42,164 km)?
(b) How much fuel will be expended?

Solution

(a) Using Eq. (6.39), and remembering to express the acceleration of gravity in km/s^2, the flight time is

$$t = \frac{1000 \cdot 0.009807 \cdot 10{,}000}{2500(10^{-6})}\left\{1 - \exp\left[\frac{\sqrt{398{,}600}}{10{,}000 \cdot 0.009807}\left(\sqrt{\frac{1}{42{,}164}} - \sqrt{\frac{1}{6678}}\right)\right]\right\}$$
$$\boxed{t = 1{,}817{,}000\,\text{s} = 21.03\,\text{days}}$$

(b) The propellant mass m_p used is

$$m_p = \dot{m}_e t = \frac{T}{I_{sp} g_0} t = \frac{2500(10^{-6})}{10{,}000 \cdot 0.009807} \cdot 1{,}817{,}000$$
$$\boxed{m_p = 46.32\,\text{kg}}$$

In Example 6.11, we found that the total delta-v for a Hohmann transfer from 6678 km to GEO radius, with no plane change, is 3.893 km/s. Assuming a typical chemical rocket specific impulse of 300 s, Eq. (6.1) reveals that the propellant requirement would be 734 kg if the initial mass is 1000 kg. This is almost 16 times that required for the hypothetical ion-propelled spacecraft of Example 6.12. Because of their efficiency (high specific impulse), ion engines—typically using xenon as the propellant—will play an increasing role in deep-space missions and satellite station keeping. However, these extremely low-thrust devices cannot replace chemical rockets in high-acceleration applications, such as launch vehicles.

EXAMPLE 6.17

What will be the orbit after the ion engine in Example 6.16 shuts down upon reaching GEO radius?

Solution

This requires a numerical solution using the MATLAB M-function *integrate_thrust.m*, listed in Appendix D.30. According to the data of Example 6.16, the initial state vector in geocentric equatorial coordinates can be written

$$\mathbf{r}_0 = 6678\hat{\mathbf{I}}\,(\text{km}) \qquad \mathbf{v}_0 = \sqrt{\frac{\mu}{r_0}}\hat{\mathbf{J}} = 7.72584\hat{\mathbf{J}}\,(\text{km/s})$$

Using these as the initial conditions, we start by assuming that the elapsed time is 21.03 days, as calculated in Example 6.16. *integrate_thrust.m* computes the final radius for that burn time and outputs the results to the command window. Depending on whether the radius is smaller or greater than 42,164 km, we reenter a slightly larger or slightly smaller time in the MATLAB editor and run the program again. Several of these manual trial-and-error steps yield the following MATLAB output:

```
Before ignition:
  Mass = 1000 kg
  State vector:
    r = [      6678,          0,    0] (km)
     Radius = 6678
```

```
    v = [        0,   7.72584,   0] (km/s)
     Speed = 7.72584

Thrust =                0.0025 kN
Burn time =          21.037600 days
Mass after burn = 9.536645E+02 kg
End-of-burn state vector:
  r = [    -19028,    -37625.9,    0] (km)
    Radius = 42163.6
  v = [ 2.71001,     -1.45129,    0] (km/s)
    Speed = 3.07415

Postburn trajectory:
  Eccentricity   = 0.0234559
  Semimajor axis = 42149.2 km
  Apogee state vector:
    r = [ 3.77273E+04, -2.09172E+04, 0.00000E+00] (km)
     Radius = 43137.9
    v = [ 1.45656E+00,  2.62713E+00, 0.00000E+00] (km/s)
     Speed = 3.0039
```

From the printout it is evident that to reach GEO radius requires the following time and propellant expenditure:

(a) $\boxed{t = 21.0376\,\text{days}}$

(b) $\boxed{m_p = 46.34\,\text{kg}}$

These are very nearly the same as the values found in the previous example. However, this numerical solution in addition furnishes the end-of-burn state vector, which shows that the postburn orbit is slightly elliptical, having an eccentricity of 0.02346 and a semimajor axis that is only 15 km less than GEO radius.

PROBLEMS

Section 6.2

6.1 A large spacecraft has a mass of 125,000 kg. Its orbital-maneuvering engines produce a thrust of 50 kN. The orbiter is in a 400-km circular earth orbit. A delta-v maneuver transfers the spacecraft to a coplanar 300 km by 400 km elliptical orbit. Neglecting propellant loss and using elementary physics (linear impulse equals change in linear momentum, distance equals speed times time):

(a) Estimate the time required for the Δv burn.

(b) Estimate the distance traveled by the spacecraft during the burn.

(c) Calculate the ratio of your answer for (b) to the circumference of the initial circular orbit.

(d) What percent of the initial mass was expelled as combustion products?

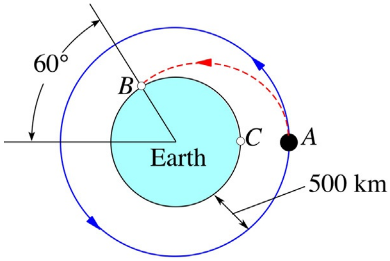
{Ans.: (a) $\Delta t = 71$ s; (b) 548 km; (c) 1.3%; (d) 1%}

6.2 A satellite traveling 8 km/s at a perigee altitude of 500 km fires a retrorocket. What delta-v is necessary to reach a minimum altitude of 200 km during the next orbit?

{Ans.: −473 m/s}

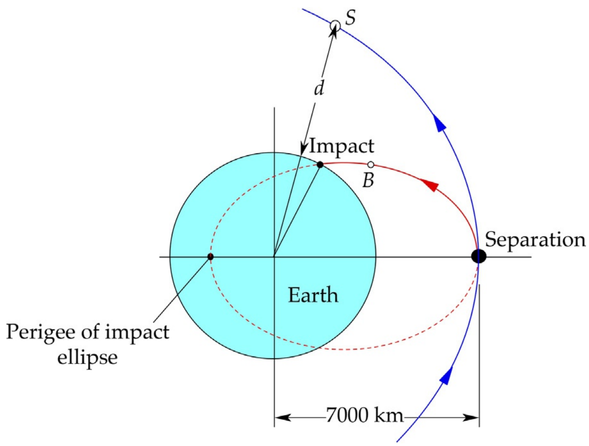
6.3 A spacecraft is in a 500-km-altitude circular earth orbit. Neglecting the atmosphere, find the delta-v required at A to impact the earth at (a) point B and (b) point C.

{Ans.: (a) −0.192 km/s; (b) −7.61 km/s}



6.4 A satellite is in a circular orbit at an altitude of 250 km above the earth's surface. If an onboard rocket provides a delta-v of 200m/s in the direction of the satellite's motion, calculate the altitude of the new orbit's apogee.
{Ans.: 981 km}

6.5 A spacecraft S is in a geocentric hyperbolic trajectory with a perigee radius of 7000 km and a perigee speed of $1.3v_{esc}$. At perigee, the spacecraft releases a projectile B with a speed of 7.1 km/s parallel to the spacecraft's velocity. How far d from the earth's surface is S at the instant B impacts the earth? Neglect the atmosphere.
{Ans.: $d = 8978$ km}



6.6 A spacecraft is in a 200-km circular earth orbit. At $t = 0$, it fires a projectile in the direction opposite to the spacecraft's motion. Some 30 min after leaving the spacecraft, the projectile impacts the earth. What delta-v was imparted to the projectile? Neglect the atmosphere.
{Ans.: $\Delta v = 77.2$ m/s}

6.7 A spacecraft is in a circular orbit of radius r and speed v around an unspecified planet. A rocket on the spacecraft is fired, instantaneously increasing the speed in the direction of motion by the amount $\Delta v = \alpha v$, where $\alpha > 0$. Calculate the eccentricity of the new orbit.
{Ans.: $e = \alpha(\alpha + 2)$}

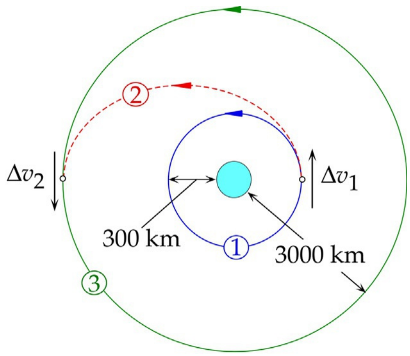
Section 6.3

6.8 A spacecraft is in a 300-km circular earth orbit. Calculate

(a) The total delta-v required for a Hohmann transfer to a 3000-km coplanar circular earth orbit.

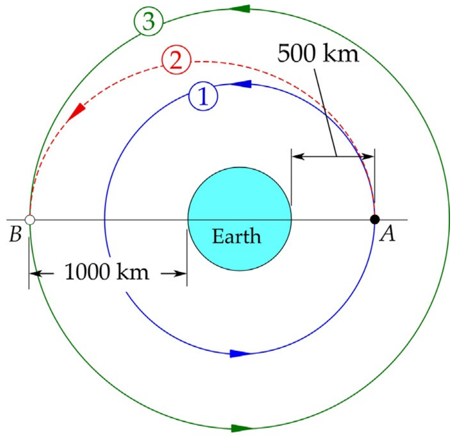
(b) The transfer orbit time.

{Ans.: (a) 1.198 km/s; (b) 59 min 39 s}



6.9 A space vehicle in a circular orbit at an altitude of 500 km above the earth executes a Hohmann transfer to a 1000-km circular orbit. Calculate the total delta-v requirement.

{Ans.: 0.2624 km/s}



6.10 Assuming the orbits of earth and Mars are circular and coplanar, calculate

(a) The time required for a Hohmann transfer from earth orbit to Mars orbit.

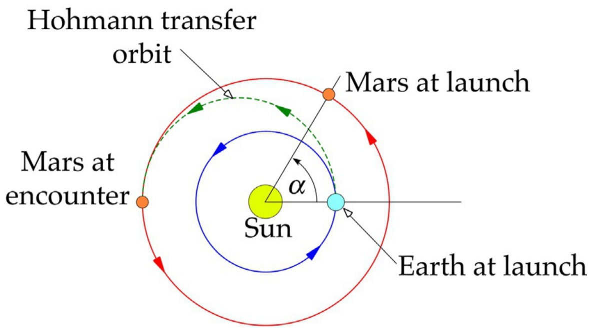
(b) The initial position of Mars (α) in its orbit relative to earth for interception to occur.

Radius of earth orbit $= 1.496(10^8)$ km.

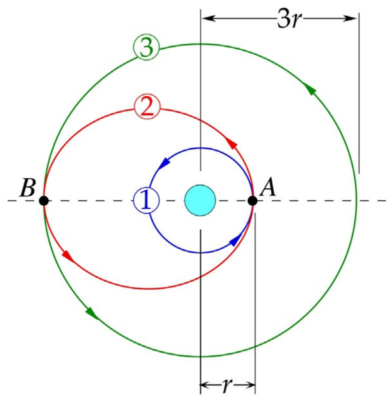
Radius of Mars orbit $= 2.279(10^8)$ km.

$\mu_{Sun} = 1.327(10^{11})\,km^3/s^2$.

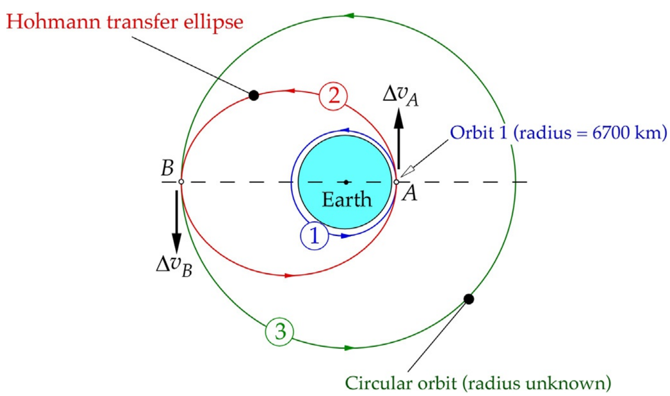
{Ans.: (a) 259 days; (b) $\alpha = 44.3°$}



6.11 Calculate the total delta-v required for a Hohmann transfer from the smaller circular orbit to the larger one.
{Ans.: $0.394v_1$, where v_1 is the speed in orbit 1}



6.12 With a Δv_A of 1.500 km/s, a spacecraft in the circular 6700-km geocentric orbit 1 initiates a Hohmann transfer to the larger circular orbit 3. Calculate Δv_B at apogee of the Hohmann transfer ellipse 2.
{Ans.: $\Delta v_B = 1.874$ km/s}

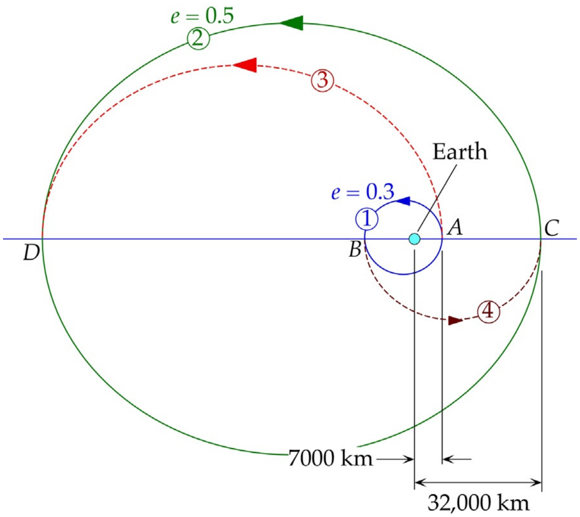


6.13 Two geocentric elliptical orbits have common apse lines and their perigees are on the same side of the earth. The first orbit has a perigee radius of $r_p = 7000$ km and $e = 0.3$, whereas for the second orbit $r_p = 32,000$ km and $e = 0.5$.

(a) Find the minimum total delta-v and the time of flight for a transfer from the perigee of the inner orbit to the apogee of the outer orbit.

(b) Do part (a) for a transfer from the apogee of the inner orbit to the perigee of the outer orbit.

{Ans.: (a) $\Delta v_{\text{total}} = 2.388$ km/s, time of flight (TOF) = 16.2 h; (b) $\Delta v_{\text{total}} = 3.611$ km/s, TOF = 4.66 h}

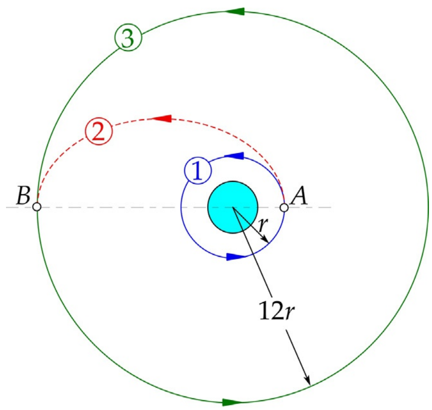


6.14 The space shuttle was launched on a 15-day mission. There were four orbits after injection, all of them at 39° inclination.

Orbit 1: 302 km by 296 km
Orbit 2 (day 11): 291 km by 259 km
Orbit 3 (day 12): 259 km circular
Orbit 4 (day 13): 255 km by 194 km

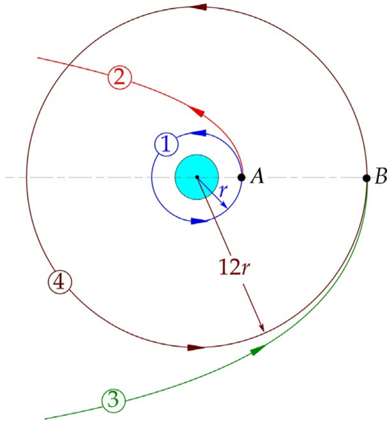
Calculate the total delta-v, which should be as small as possible, assuming Hohmann transfers.
{Ans.: $\Delta v_{\text{total}} = 43.5$ m/s }

6.15 Calculate the total delta-v required for a Hohmann transfer from a circular orbit of radius r to a circular orbit of radius $12r$.
{Ans.: $0.5342\sqrt{\mu/r}$}



Section 6.4

6.16 A spacecraft in circular orbit 1 of radius r leaves for infinity on parabolic trajectory 2 and returns from infinity on a parabolic trajectory 3 to a circular orbit 4 of radius $12r$. Find the total delta-v required for this non-Hohmann orbit change maneuver.
{Ans.: $0.5338\sqrt{\mu/r}$}

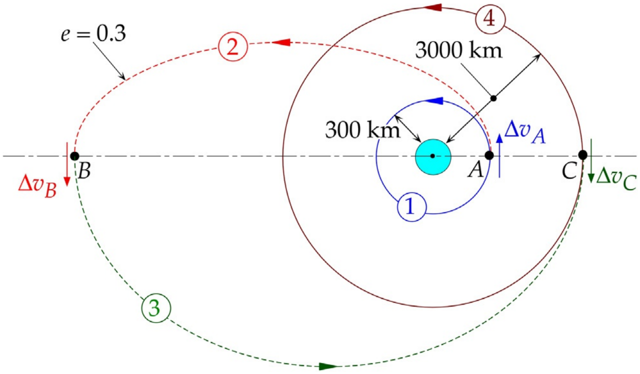


6.17 A spacecraft is in a 300-km circular earth orbit. Calculate

(a) The total delta-v required for the bielliptical transfer to the 3000-km-altitude coplanar circular orbit shown.

(b) The total transfer time.

{Ans.: (a) 2.039 km/s; (b) 2.86 h}



6.18 Verify Eq. (6.4).

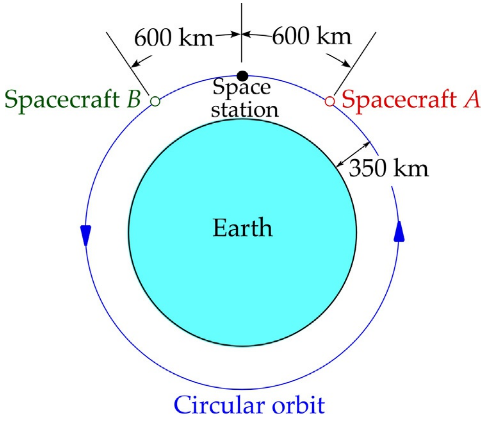
Section 6.5

6.19 The space station and spacecraft A and B are all in the same circular earth orbit of 350 km altitude. Spacecraft A is 600 km behind the space station and spacecraft B is 600 km ahead of the space station. At the same instant, both spacecraft apply a $\Delta v_\perp$ so as to arrive at the space station in one revolution of their phasing orbits.

(a) Calculate the time required for each spacecraft to reach the space station.

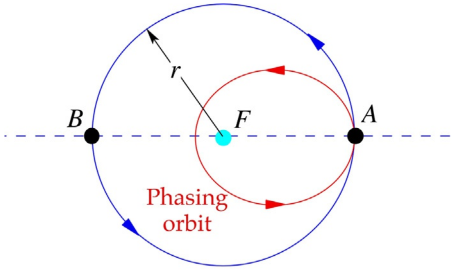
(b) Calculate the total delta-v requirement for each spacecraft.

{Ans.: (a) Spacecraft A, 90.2 min; spacecraft B, 92.8 min; (b) $\Delta v_A = 73.9$ m/s; $\Delta v_B = 71.5$ m/s}

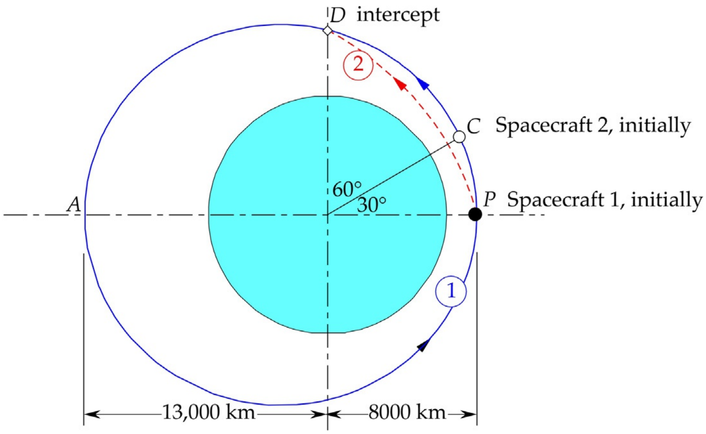


6.20 Satellites A and B are in the same circular orbit of radius r. B is 180° ahead of A. Calculate the semimajor axis of a phasing orbit in which A will rendezvous with B after just one revolution in the phasing orbit.

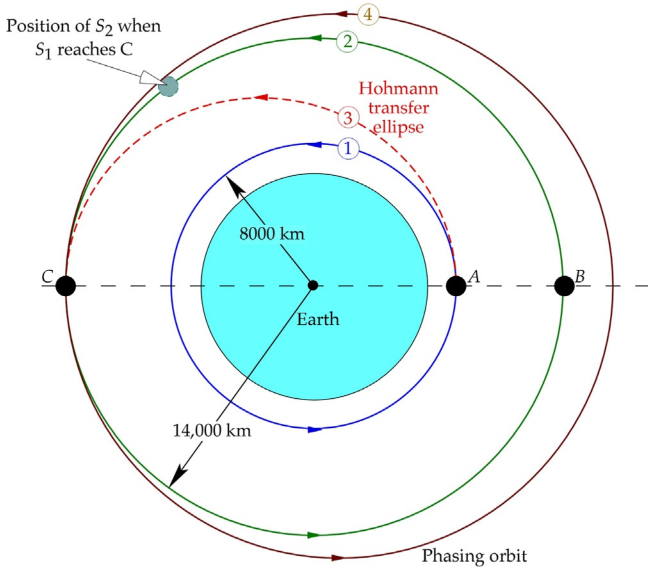
{Ans.: $a = 0.63r$}



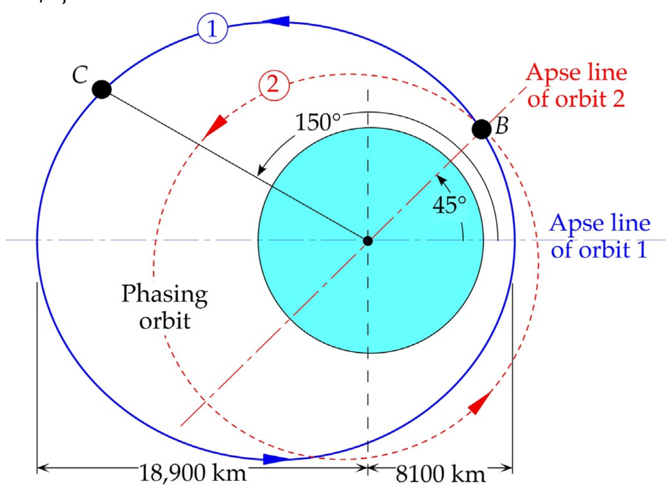
6.21 Two spacecraft are in the same elliptical earth orbit with perigee radius 8000 km and apogee radius 13,000 km. Spacecraft 1 is at perigee and spacecraft 2 is 30° ahead. Calculate the total delta-v required for spacecraft 1 to intercept and rendezvous with spacecraft 2 when spacecraft 2 has traveled 60°.
{Ans.: $\Delta v_{total} = 6.24\,\text{km/s}$}



6.22 At the instant shown, spacecraft S_1 is at point A of circular orbit 1 and spacecraft S_2 is at point B of circular orbit 2. At that instant, S_1 executes a Hohmann transfer so as to arrive at point C of orbit 2. After arriving at C, S_1 immediately executes a phasing maneuver to rendezvous with S_2 after one revolution of its phasing orbit. What is the total delta-v requirement?
{Ans.: 2.159 km/s}



6.23 Spacecraft B and C, which are in the same elliptical earth orbit 1, are located at the true anomalies shown. At this instant, spacecraft B executes a phasing maneuver so as to rendezvous with spacecraft C after one revolution of its phasing orbit 2. Calculate the total delta-v required. Note that the apse line of orbit 2 is at 45° to that of orbit 1.
{Ans.: 3.405 km/s}

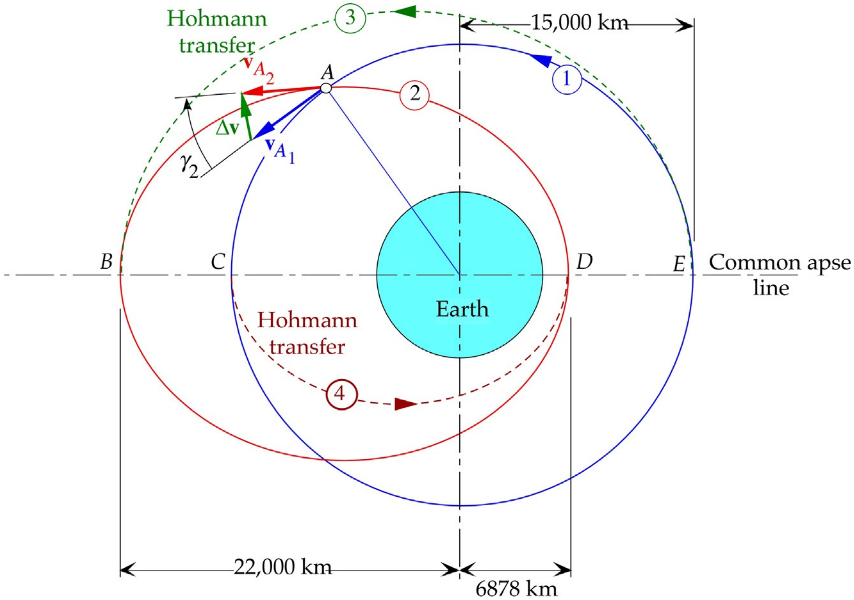


Section 6.6

6.24 **(a)** With a single delta-v maneuver, the earth orbit of a satellite is to be changed from a circle of radius 15,000 km to a collinear ellipse with perigee altitude of 500 km and apogee radius of 22,000 km. Calculate the magnitude of the required delta-v and the change in the flight path angle $\Delta\gamma$.

(b) What is the minimum total delta-v if the orbit change is accomplished instead by a Hohmann transfer?

{Ans.: (a) $\|\Delta\mathbf{v}\| = 2.77\,\text{km/s}$, $\Delta\gamma = 31.51°$; (b) $\Delta v_{\text{Hohmann}} = 1.362$ km/s}

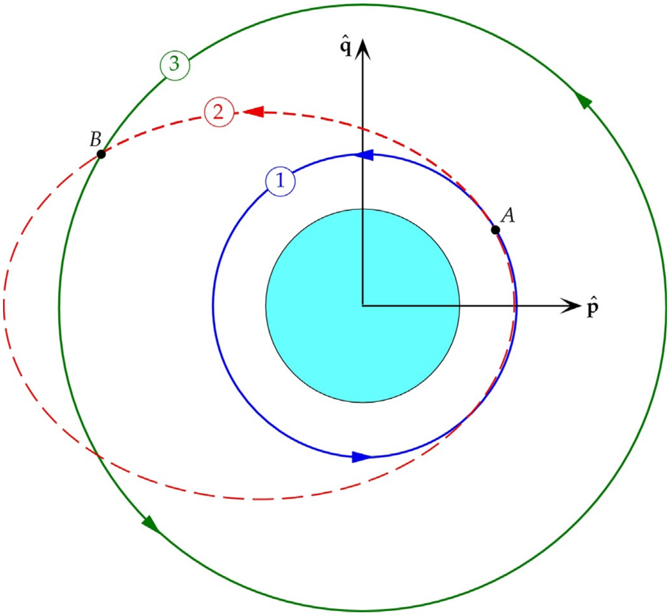


6.25 An earth satellite has a perigee altitude of 1270 km and a perigee speed of 9 km/s. It is required to change its orbital eccentricity to 0.4, without rotating the apse line, by a delta-v maneuver at $\theta = 100°$. Calculate the magnitude of the required $\Delta\mathbf{v}$ and the change in flight path angle $\Delta\gamma$.
{Ans.: $\|\Delta\mathbf{v}\| = 0.915\,\text{km/s}$; $\Delta\gamma = -8.18°$}

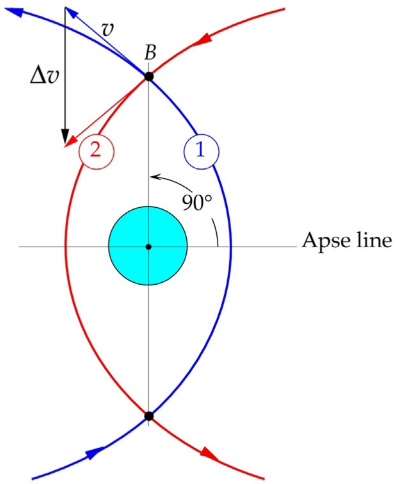
6.26 The velocities at points A and B on orbits 1, 2, and 3, respectively, are (relative to the perifocal frame)

$$\begin{aligned}
\mathbf{v}_A)_1 &= -3.7730\hat{\mathbf{p}} + 6.5351\hat{\mathbf{q}}\,(\text{km/s})\\
\mathbf{v}_A)_2 &= -3.2675\hat{\mathbf{p}} + 8.1749\hat{\mathbf{q}}\,(\text{km/s})\\
\mathbf{v}_B)_2 &= -3.2675\hat{\mathbf{p}} - 3.1442\hat{\mathbf{q}}\,(\text{km/s})\\
\mathbf{v}_B)_3 &= -2.6679\hat{\mathbf{p}} - 4.6210\hat{\mathbf{q}}\,(\text{km/s})
\end{aligned}$$

Calculate the total Δv for a transfer from orbit 1 to orbit 3 by means of orbit 2.
{Ans.: 3.310 km/s}



6.27 Trajectories 1 and 2 are ellipses with eccentricity 0.4 and the same angular momentum h. Their speed at B is v. Calculate, in terms of v, the Δv required at B to transfer from orbit 1 to orbit 2. {Ans.: $\Delta v = 0.7428v$}



Section 6.7

6.28 A satellite is in a circular earth orbit of altitude 400 km. Determine the new perigee and apogee altitudes if the satellite's onboard rocket

(a) provides a delta-v in the tangential direction of 240 m/s.

(b) provides a delta-v in the radial (outward) direction of 240 m/s.

{Ans.: (a) $z_a = 1320\,\text{km}$, $z_p = 400\,\text{km}$; (b) $z_a = 619\,\text{km}$, $z_p = 194\,\text{km}$}

6.29 At point A on its earth orbit, the radius, speed, and flight path angle of a satellite are $r_A = 12,756\,\text{km}$, $v_A = 6.5992\,\text{km/s}$, and $\gamma_A = 20°$. At point B, at which the true anomaly is 150°, an impulsive maneuver causes $\Delta v_\perp = +0.75820\,\text{km/s}$ and $\Delta v_r = 0$.

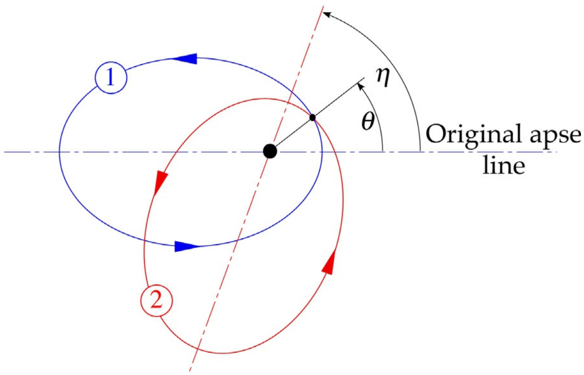
(a) What is the time of flight from A to B?

(b) What is the rotation of the apse line as a result of this maneuver?

{Ans.: (a) 2.045 h; (b) 43.39° counterclockwise}

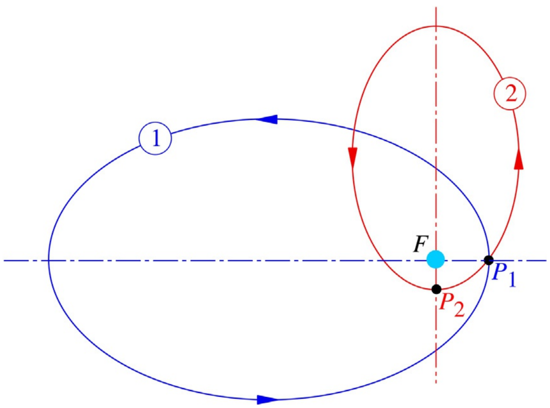
6.30 A satellite is in elliptical orbit 1. Calculate the true anomaly θ (relative to the apse line of orbit 1) of an impulsive maneuver that rotates the apse line an angle η counterclockwise but leaves the eccentricity and the angular momentum unchanged.

{Ans.: $\theta = \eta/2$}

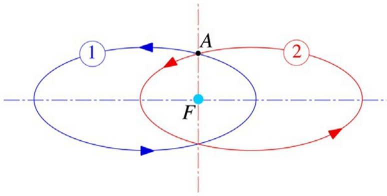


6.31 A satellite in orbit 1 undergoes a delta-v maneuver at perigee P_1 such that the new orbit 2 has the same eccentricity e, but its apse line is rotated 90° clockwise from the original one. Calculate the specific angular momentum of orbit 2 in terms of that of orbit 1 and the eccentricity e.

{Ans.: $h_2 = h_1/\sqrt{1+e}$}

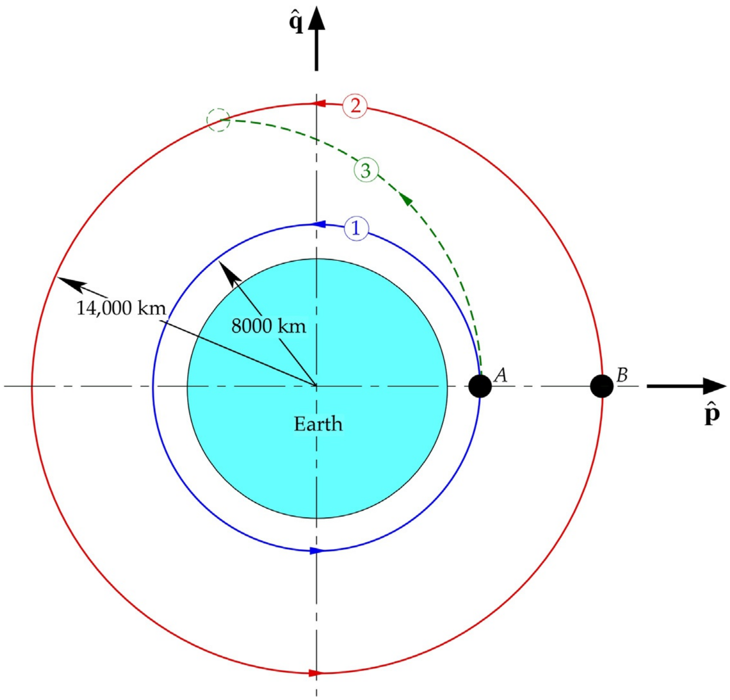


6.32 Calculate the delta-v required at A in orbit 1 for a single impulsive maneuver to rotate the apse line 180° counterclockwise (to become orbit 2), but keep the eccentricity e and the angular momentum h the same.
{Ans.: $\Delta v = 2\mu e/h$}

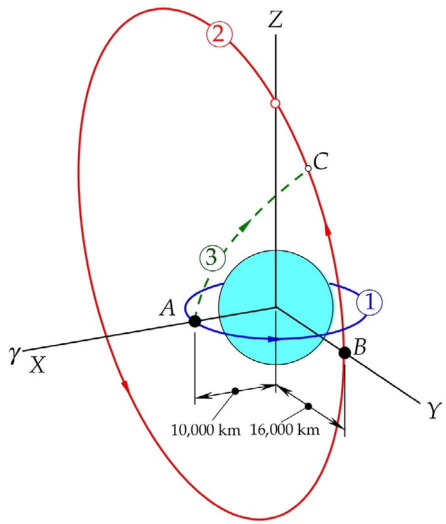


Section 6.8

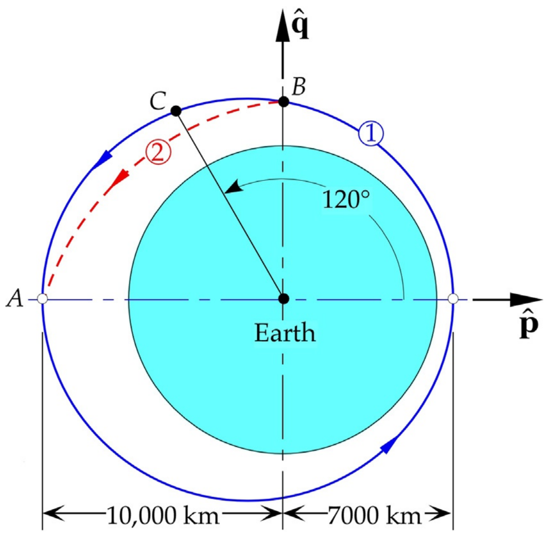
6.33 Spacecraft A and B are in concentric, coplanar circular orbits 1 and 2, respectively. At the instant shown, spacecraft A executes an impulsive delta-v maneuver to embark on orbit 3 to intercept and rendezvous with spacecraft B in a time equal to the period of orbit 1. Calculate the total delta-v required.
{Ans.: 3.795 km/s}



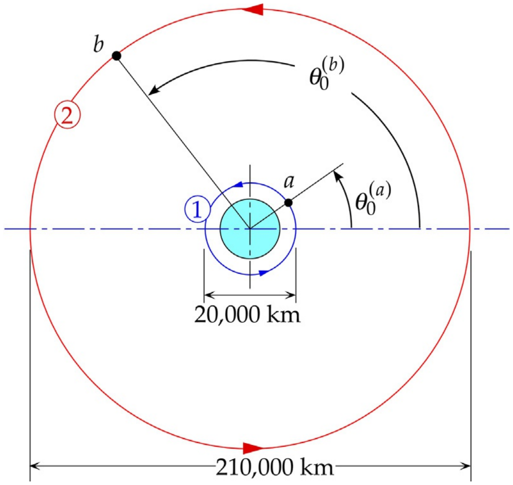
6.34 Spacecraft A is in orbit 1, a 10,000-km-radius equatorial earth orbit. Spacecraft B is in elliptical polar orbit 2, having eccentricity 0.5 and perigee radius 16,000 km. At the instant shown, both spacecraft are in the equatorial plane and B is at its perigee. At that instant, spacecraft A executes an impulsive delta-v maneuver to intercept spacecraft B 1 h later at point C. Calculate the delta-v required for A to switch to the intercept trajectory 3.
{Ans.: 8.117 km/s}



6.35 Spacecraft B and C are in the same elliptical orbit 1, characterized by a perigee radius of 7000 km and an apogee radius of 10,000 km. The spacecraft are in the positions shown when B executes an impulsive transfer to orbit 2 to catch and rendezvous with C when C arrives at apogee A. Find the total delta-v requirement.
{Ans.: 5.066 km/s}



6.36 At time $t = 0$, manned spacecraft a and unmanned spacecraft b are at the positions shown in circular earth orbits 1 and 2, respectively. For assigned values of $\theta_0^{(a)}$ and $\theta_0^{(b)}$, design a series of impulsive maneuvers by means of which spacecraft a transfers from orbit 1 to orbit 2 so as to rendezvous with spacecraft b (i.e., occupy the same position in space). The total time and total delta-v required for the transfer should be as small as possible. Consider earth's gravity only.



Section 6.9

6.37 What must the launch azimuth be if the satellite in Example 4.8 is launched from
(a) Kennedy Space Center (latitude = 28.5°N);
(b) Vandenburgh AFB (latitude = 34.5°N);
(c) Kourou, French Guiana (latitude = 5.5°N).
{Ans.: (a) 329.4° or 210.6°; (b) 327.1° or 212.9°; (c) 333.3° or 206.7°}

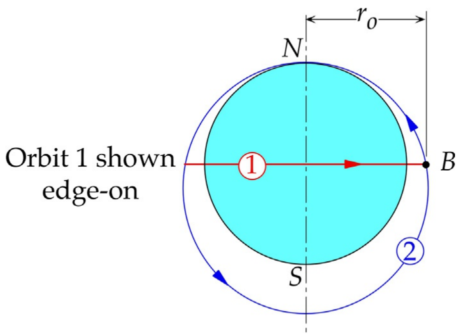
6.38 The state vector of a spacecraft in the geocentric equatorial frame is $\mathbf{r} = r\hat{\mathbf{I}}$ and $\mathbf{v} = v\hat{\mathbf{J}}$. At that instant an impulsive maneuver produces the velocity change $\Delta\mathbf{v} = 0.5v\hat{\mathbf{I}} + 0.5v\hat{\mathbf{K}}$. What is the inclination of the new orbit?
{Ans.: 26.57°}

6.39 An earth satellite has the following orbital elements: $a = 15,000$ km, $e = 0.5$, $\Omega = 45°$, $\omega = 30°$, and $i = 10°$. What minimum delta-v is required to reduce the inclination to zero?
{Ans.: 0.588 km/s}

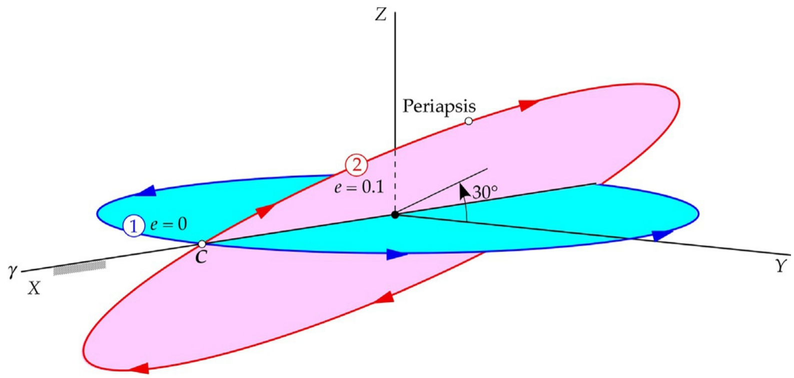
6.40 With a single impulsive maneuver, an earth satellite changes from a 400-km circular orbit inclined at 60° to an elliptical orbit of eccentricity $e = 0.5$ with an inclination of 40°. Calculate the minimum required delta-v.
{Ans.: 3.41 km/s}

6.41 An earth satellite is in an elliptical orbit of eccentricity 0.3 and angular momentum 60,000 km^2/s. Find the delta-v required for a 90° change in inclination at apogee (no change in speed).
{Ans.: 6.58 km/s}

6.42 A spacecraft is in a circular, equatorial orbit (1) of radius r_0 about a planet. At point B it impulsively transfers to polar orbit (2), whose eccentricity is 0.25 and whose perigee is directly over the north pole. Calculate the minimum delta-v required at B for this maneuver.
{Ans.: $1.436\sqrt{\mu/r_0}$}



6.43 A spacecraft is in a circular, equatorial orbit (1) of radius r_0 and speed v_0 about an unknown planet ($\mu \neq 398,600\,\text{km}^3/\text{s}^2$). At point C it impulsively transfers to orbit (2), for which the ascending node is point C, the eccentricity is 0.1, the inclination is 30°, and the argument of periapsis is 60°. Calculate, in terms of v_0, the single delta-v required at C for this maneuver.
{Ans.: $\Delta v = 0.5313v_0$}



6.44 A spacecraft is in a 300-km circular parking orbit. It is desired to increase the altitude to 600 km and change the inclination by 20°. Find the total delta-v required if

(a) the plane change is made after insertion into the 600-km orbit (so that there are a total of three delta-v burns).

(b) the plane change and insertion into the 600-km orbit are accomplished simultaneously (so that the total number of delta-v burns is two).

(c) the plane change is made upon departing the lower orbit (so that the total number of delta-v burns is two).

{Ans.: (a) 2.793 km/s; (b) 2.696 km/s; (c) 2.783 km/s}

Section 6.10

6.45 Calculate the total propellant expenditure for Problem 6.3 using finite-time delta-v maneuvers. The initial spacecraft mass is 4000 kg. The propulsion system has a thrust of 30 kN and a specific impulse of 280 s.

6.46 Calculate the total propellant expenditure for Problem 6.14 using finite-time delta-v maneuvers. The initial spacecraft mass is 4000 kg. The propulsion system has a thrust of 30 kN and a specific impulse of 280 s.

6.47 At a given instant t_0, a 1000-kg earth-orbiting satellite has the inertial position and velocity vectors

$$\mathbf{r}_0 = 436\hat{\mathbf{i}} + 6083\hat{\mathbf{j}} + 2529\hat{\mathbf{k}}\,(\text{km}) \quad \mathbf{v}_0 = -7.340\hat{\mathbf{i}} - 0.5125\hat{\mathbf{j}} + 2.497\hat{\mathbf{k}}\,(\text{km/s})$$

About 89 min later a rocket motor with $I_{sp} = 300\,\text{s}$ and 10 kN thrust aligned with the velocity vector ignites and burns for 120 s. Use numerical integration to find the maximum altitude reached by the satellite and the time it occurs.

REFERENCES

Hohmann, W., 1925. The Attainability of Celestial Bodies. R. Oldenbourg, Munich (in German).
Wiesel, W.E., 2010. Spacecraft Dynamics, third ed. Aphelion Press, Beaver Creek, OH.