APPENDIX

G DIRECTION COSINE MATRIX IN TERMS OF THE UNIT QUATERNION

By means of Eq. (11.154) we can rewrite the direction cosine matrix (Eq. 11.143) entirely in terms of the components of the unit quaternion $\widehat{\mathbf{q}}$.

Let us deal with each of the nine components of $[\mathbf{Q}]_{Xx}$ in turn, starting with Q_{11}. From Eq. (11.143) we have

$$Q_{11} = l^2(1-\cos\theta)+\cos\theta$$

Substituting the trig identity $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$ from Eq. (11.156), and then expanding and rearranging terms yields

$$Q_{11} = l^2 - l^2\cos^2(\theta/2) - \sin^2(\theta/2) + \left[l^2\sin^2(\theta/2)+\cos^2(\theta/2)\right]$$

Since $\cos^2(\theta/2) = 1 - \sin^2(\theta/2)$, we may write this as

$$Q_{11} = \left(l^2-1\right)\sin^2(\theta/2) + \left[l^2\sin^2(\theta/2)+\cos^2(\theta/2)\right]$$

From Eq. (11.141) we have $l^2 - 1 = -m^2 - n^2$, so that, making use of Eq. (11.154),

$$\begin{aligned} Q_{11} &= l^2\sin^2(\theta/2) - m^2\sin^2(\theta/2) - n^2\sin^2(\theta/2) + \cos^2(\theta/2) \\ &= \overbrace{l^2\sin^2(\theta/2)}^{q_1{}^2} - \overbrace{m^2\sin^2(\theta/2)}^{q_2{}^2} - \overbrace{n^2\sin^2(\theta/2)}^{q_3{}^2} + \overbrace{\cos^2(\theta/2)}^{q_4{}^2} \end{aligned}$$

Therefore,

$$Q_{11} = q_1{}^2 - q_2{}^2 - q_3{}^2 + q_4{}^2$$

Likewise, for the remaining two diagonal components of $[\mathbf{Q}]_{Xx}$ we find that

$$\begin{aligned} Q_{22} &= -q_1{}^2 + q_2{}^2 - q_3{}^2 + q_4{}^2 \\ Q_{33} &= -q_1{}^2 - q_2{}^2 + q_3{}^2 + q_4{}^2 \end{aligned}$$

For the off-diagonal components of $[\mathbf{Q}]_{Xx}$, we start with Q_{12} and observe from Eq. (11.143) that

$$Q_{12} = lm(1-\cos\theta) + n\sin\theta$$

Replacing $\sin\theta$ and $\cos\theta$ by the trig identities in Eq. (11.156), we get

$$Q_{12} = lm\left[1-\cos^2(\theta/2)+\sin^2(\theta/2)\right] + \left[2n\sin(\theta/2)\cos(\theta/2)\right]$$

Employing the identity $\cos^2(\theta/2) = 1 - \sin^2(\theta/2)$ yields

$$\begin{aligned}Q_{12} &= 2lm\sin^2(\theta/2) + 2n\sin(\theta/2)\cos(\theta/2)\\ &= 2\cdot\overbrace{l\sin(\theta/2)}^{q_1}\cdot\overbrace{m\sin(\theta/2)}^{q_2} + 2\cdot\overbrace{n\sin(\theta/2)}^{q_3}\cdot\overbrace{\cos(\theta/2)}^{q_4}\end{aligned}$$

so that

$$Q_{12} = 2(q_1q_2 + q_3q_4)$$

Following the same line of reasoning for the five remaining off-diagonal components, leads to

$$\begin{aligned}Q_{13} &= 2(q_1q_3 - q_2q_4)\\ Q_{21} &= 2(q_1q_2 - q_3q_4)\\ Q_{23} &= 2(q_2q_3 + q_1q_4)\\ Q_{31} &= 2(q_1q_3 + q_2q_4)\\ Q_{32} &= 2(q_2q_3 - q_1q_4)\end{aligned}$$

This shows that Eq. (11.157) is indeed a valid formula for the direction cosine matrix $[\mathbf{Q}]_{Xx}$ in terms of the unit quaternion $\widehat{\mathbf{q}}$.