APPENDIX

GRAVITATIONAL POTENTIAL OF A SPHERE

E

Fig. E.1 shows a point mass m with Cartesian coordinates (x, y, z) as well a system of N point masses m_1, $m_2, m_3, \cdots, m_N$. The ith one of these particles has mass m_i and coordinates (x_i, y_i, z_i). The total mass of the N particles is M,

$$M = \sum_{i=1}^{N} m_i \tag{E.1}$$

The position vector drawn from m_i to m is $\mathbf{r}_i$ and the unit vector in the direction of $\mathbf{r}_i$ is

$$\hat{\mathbf{u}}_i = \frac{\mathbf{r}_i}{r_i}$$

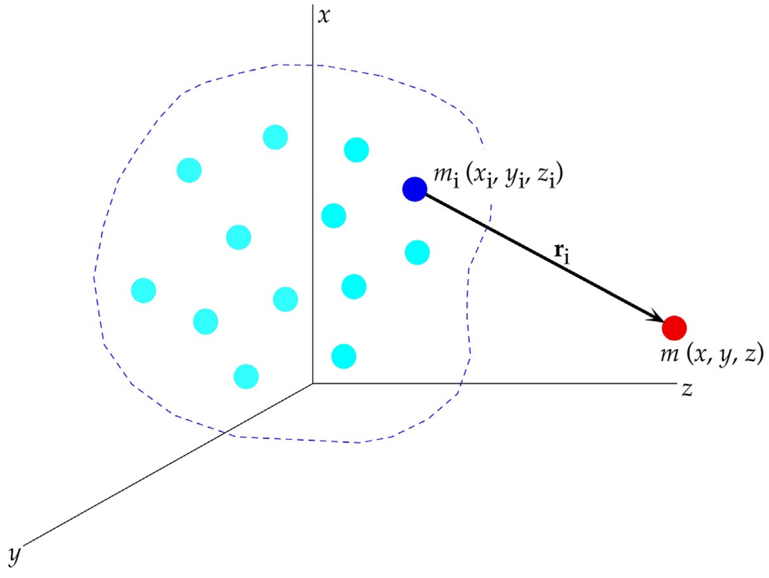

FIG. E.1

A system of point masses and a neighboring test mass *m*.

The gravitational force exerted on m by m_i is opposite in direction to $\mathbf{r}_i$, and is given by

$$\mathbf{F}_i = -\frac{Gmm_i}{r_i^2}\hat{\mathbf{u}}_i = -\frac{Gmm_i}{r_i^3}\mathbf{r}_i$$

The potential energy of this force is

$$V_i = -G\frac{mm_i}{r_i} \tag{E.2}$$

The total gravitational potential energy of the system due to the gravitational attraction of all the N particles is

$$V = \sum_{i=1}^{N} V_i \tag{E.3}$$

Therefore, the total force of gravity $\mathbf{F}$ on the mass m is

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) \tag{E.4}$$

where ∇ is the gradient operator.

Consider the solid sphere of mass M and radius R_0 illustrated in Fig. E.2. Instead of a discrete system as above, we have a continuum with mass density ρ. Each "particle" is a differential element $\mathrm{d}M = \rho \mathrm{d}v$ of the total mass M. Eq. (E.1) becomes

$$M = \iiint_v \rho \, \mathrm{d}v \tag{E.5}$$

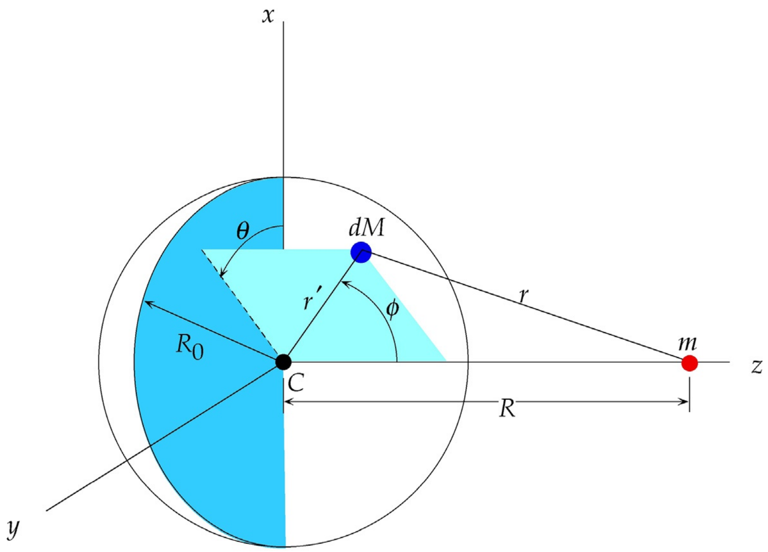

FIG. E.2

Sphere with a spherically symmetric mass distribution.

where dv is the volume element, and v is the total volume of the sphere. In this case, Eq. (E.2) becomes

$$\mathrm{d}V = -G\frac{m\,\mathrm{d}M}{r} = -Gm\frac{\rho\,\mathrm{d}v}{r}$$

where r is the distance from the differential mass dM to the finite point mass m. Eq. (E.3) is replaced by

$$V = -Gm\iiint\limits_{v}\frac{\rho\,\mathrm{d}v}{r} \tag{E.6}$$

Let the mass of the sphere have a spherically symmetric distribution, which means that the mass density ρ depends only on r', the distance from the center C of the sphere. An element of mass dM has spherical coordinates (r', θ, ϕ), where the angle θ is measured in the xy plane of a Cartesian coordinate system with origin at C, as shown in Fig. E.2. In spherical coordinates the volume element is

$$\mathrm{d}v = r'^2 \sin\phi\ \mathrm{d}\phi\ \mathrm{d}r'\ \mathrm{d}\theta \tag{E.7}$$

Therefore Eq. (E.5) becomes

$$M = \int\limits_{\theta=0}^{2\pi}\int\limits_{r'=0}^{R_0}\int\limits_{\phi=0}^{\pi}\rho r'^2 \sin\phi\ \mathrm{d}\phi\ \mathrm{d}r'\ \mathrm{d}\theta = \left(\int\limits_0^{2\pi}\mathrm{d}\theta\right)\left(\int\limits_0^{\pi}\sin\phi\ \mathrm{d}\phi\right)\left(\int\limits_0^{R_0}\rho r'^2\ \mathrm{d}r'\right) = (2\pi)(2)\left(\int\limits_0^{R_0}\rho r'^2\ \mathrm{d}r'\right)$$

so that the mass of the sphere is given by

$$M = 4\pi\int\limits_{r'=0}^{R_0}\rho r'^2\ \mathrm{d}r' \tag{E.8}$$

Substituting Eq. (E.7) into Eq. (E.6) yields

$$V = -Gm\int\limits_{\theta=0}^{2\pi}\int\limits_{r'=0}^{R_0}\int\limits_{\phi=0}^{\pi}\frac{\rho r'^2 \sin\phi\ \mathrm{d}\phi\ \mathrm{d}r'\ \mathrm{d}\theta}{r} = -2\pi Gm\left[\int\limits_0^{R_0}\left(\int\limits_0^{\pi}\frac{\sin\phi\ \mathrm{d}\phi}{r}\right)\rho r'^2\ \mathrm{d}r'\right] \tag{E.9}$$

The distance r is found by using the law of cosines,

$$r = \left(R^2 + r'^2 - 2r'R\cos\phi\right)^{1/2}$$

where R is the distance from the center of the sphere to the mass m. Differentiating this equation with respect to ϕ, holding r' constant, yields

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{1}{2}\left(R^2 + r'^2 - 2r'R\cos\phi\right)^{-1/2}(2r'R\sin\phi\ \mathrm{d}\phi) = \frac{r'R\sin\phi}{r}$$

so that

$$\sin\phi\ \mathrm{d}\phi = \frac{r\,\mathrm{d}r}{r'R}$$

It follows that

$$\int\limits_{\phi=0}^{\pi}\frac{\sin\phi\ \mathrm{d}\phi}{r} = \frac{1}{r'R}\int\limits_{R-r'}^{R+r'}\mathrm{d}r = \frac{2}{R}$$

Substituting this result along with Eq. (E.8) into Eq. (E.9) yields

$$V = -\frac{GMm}{R} \tag{E.10}$$

We conclude that the gravitational potential energy, and hence (from Eq. E.4) the gravitational force, of a sphere with a spherically symmetric mass distribution M is the same as that of a point mass M located at the center of the sphere.