CHAPTER

ROCKET VEHICLE DYNAMICS 13

13.1 INTRODUCTION

In previous chapters, we have made frequent reference to delta-v maneuvers of spacecraft. These require a propulsion system of some sort whose job is to throw vehicle mass (in the form of propellants) overboard. Newton's balance of momentum principle dictates that when mass is ejected from a system in one direction, the mass left behind must acquire a velocity in the opposite direction. The familiar and oft-quoted example is the rapid release of air from an inflated toy balloon. Another is that of a diver leaping off a small boat at rest in the water, causing the boat to acquire a motion of its own. The unfortunate astronaut who becomes separated from his ship in the vacuum of space cannot with any amount of flailing of arms and legs "swim" back to safety. If he has tools or other expendable objects of equipment, accurately throwing them in the direction opposite to his spacecraft may do the trick. Spewing compressed gas from a tank attached to his back through a nozzle pointed away from the spacecraft would be a better solution.

The purpose of a rocket motor is to use the chemical energy of solid or liquid propellants to steadily and rapidly produce a large quantity of hot high-pressure gas, which is then expanded and accelerated through a nozzle. This large mass of combustion products flowing out of the nozzle at supersonic speed possesses a lot of momentum and, leaving the vehicle behind, causes the vehicle itself to acquire a momentum in the opposite direction. This is represented as the action of the force we know as thrust. The design and analysis of rocket propulsion systems is well beyond our scope.

This chapter contains a necessarily brief introduction to some of the fundamentals of rocket vehicle dynamics. The equations of motion of a launch vehicle in a gravity turn trajectory are presented first. This is followed by a simple development of the thrust equation, which brings in the concept of specific impulse. The thrust equation and the equations of motion are then combined to produce the rocket equation, which relates delta-v to propellant expenditure and specific impulse. The sounding rocket provides an important but relatively simple application of the concepts introduced to this point. After a computer simulation of a gravity turn trajectory, the chapter concludes with an elementary consideration of multistage launch vehicles.

Those seeking a more detailed introduction to the subject of rockets and rocket performance will find the texts by Wiesel (2010), Hale (1994), and Sutton and Biblarz (2017) useful.

Orbital Mechanics for Engineering Students. https://doi.org/10.1016/B978-0-08-102133-0.00013-1

13.2 EQUATIONS OF MOTION

Fig. 13.1 illustrates the trajectory of a satellite launch vehicle and the forces acting on it during the powered ascent. Rockets at the base of the booster produce the thrust $\mathbf{T}$, which acts along the vehicle's axis in the direction of the velocity vector $\mathbf{v}$. The aerodynamic drag force $\mathbf{D}$ is directed opposite to the velocity, as shown. Its magnitude is given by

$$D = qAC_D \tag{13.1}$$

where $q = \rho v^2/2$ is the dynamic pressure, in which ρ is the density of the atmosphere and v is the speed (i.e., the magnitude) of $\mathbf{v}$, A is the frontal area of the vehicle, and C_D is the coefficient of drag. C_D depends on the speed and the external geometry of the rocket. The force of gravity on the booster is $m\mathbf{g}$, where m is its mass, and $\mathbf{g}$ is the local gravitational acceleration vector, pointing toward the center of the earth. As discussed in Section 1.3, at any point of the trajectory, the velocity $\mathbf{v}$ defines the direction of the unit tangent $\hat{\mathbf{u}}_t$ to the path. The unit normal $\hat{\mathbf{u}}_n$ is perpendicular to $\mathbf{v}$ and points toward the center of curvature C. The distance of point C from the path is ρ (not to be confused with density). ρ is the radius of curvature.

In Fig. 13.1, the vehicle and its flight path are shown relative to the earth. In the interest of simplicity we will ignore the earth's spin and write the equations of motion relative to a nonrotating earth. The small-acceleration terms required to account for the earth's rotation can be added for a more refined analysis. Let us resolve Newton's second law, $\mathbf{F}_{\text{net}} = m\mathbf{a}$, into components along the path directions $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$. Recall from Section 1.3 that the acceleration along the path is

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{13.2}$$

and the normal acceleration is $a_n = v^2/\rho$ (where ρ is the radius of curvature). It was shown in Example 1.8 (Eq. 1.37) that for flight over a flat surface, $v/\rho = -\mathrm{d}\gamma/\mathrm{d}t$, in which case the normal acceleration can be expressed in terms of the flight path angle γ as

$$a_n = -v\frac{\mathrm{d}\gamma}{\mathrm{d}t}$$

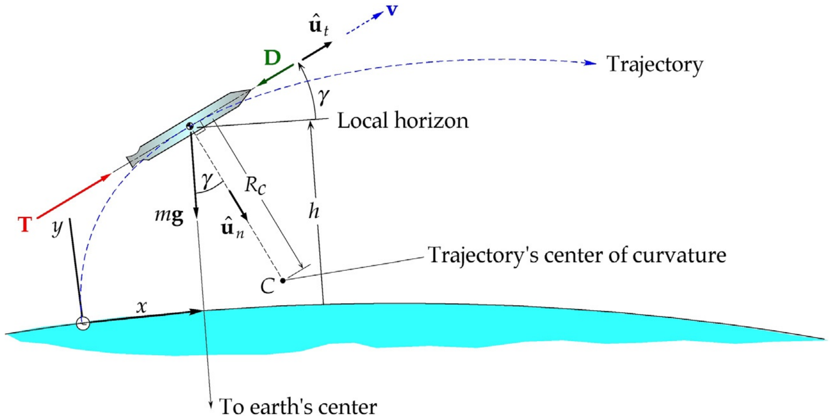


FIG. 13.1

Launch vehicle boost trajectory. γ is the flight path angle.

To account for the curvature of the earth, as was done in Section 1.7, we can use polar coordinates with origin at the earth's center to show that a term must be added to this expression, so that it becomes

$$a_n = -v\frac{d\gamma}{dt} + \frac{v^2}{R_E + h}\cos\gamma \tag{13.3}$$

where R_E is the radius of the earth, and h (instead of z as in previous chapters) is the altitude of the rocket. Thus, in the direction of $\hat{\mathbf{u}}_t$, Newton's second law requires

$$T - D - mg\sin\gamma = ma_t \tag{13.4}$$

whereas in the $\hat{\mathbf{u}}_n$ direction

$$mg\cos\gamma = ma_n \tag{13.5}$$

After substituting Eqs. (13.2) and (13.3), the latter two expressions may be written:

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma \tag{13.6}$$

$$v\frac{d\gamma}{dt} = -\left(g - \frac{v^2}{R_E + h}\right)\cos\gamma \tag{13.7}$$

To these we must add the equations for downrange distance x and altitude h,

$$\frac{dx}{dt} = \frac{R_E}{R_E + h}v\cos\gamma \qquad \frac{dh}{dt} = v\sin\gamma \tag{13.8}$$

Recall that the variation of g with altitude is given by Eq. (1.36). Numerical methods must be used to solve Eqs. (13.6), (13.7), and (13.8). To do so, we must account for the variation of the thrust, booster mass, atmospheric density, the drag coefficient, and the acceleration of gravity. Of course, the vehicle mass continuously decreases as propellants are consumed to produce the thrust, which we shall discuss in the following section.

The free body diagram in Fig. 13.1 does not include a lifting force, which, if the vehicle were an airplane, would act normal to the velocity vector. Launch vehicles are designed to be strong in lengthwise compression, like a column. To save weight they are, unlike an airplane, made relatively weak in bending, shear, and torsion, which are the kinds of loads induced by lifting surfaces. Transverse lifting loads are held closely to zero during powered ascent through the atmosphere by maintaining a zero angle of attack, that is, by keeping the axis of the booster aligned with its velocity vector (the relative wind). Pitching maneuvers are done early in the launch, soon after the rocket clears the launch tower, when its speed is still low. At the high speeds acquired within a minute or so after launch, the slightest angle of attack can produce destructive transverse loads in the vehicle. The Space Shuttle orbiter had wings so that it could act as a glider after reentry into the atmosphere. However, the launch configuration of the orbiter was such that its wings were at the zero lift angle of attack throughout the ascent.

Satellite launch vehicles take off vertically and, at injection into orbit, must be flying parallel to the earth's surface. During the initial phase of the ascent, the rocket builds up speed on a nearly vertical trajectory taking it above the dense lower layers of the atmosphere. While it transitions to the thinner upper atmosphere, the trajectory bends over, trading vertical speed for horizontal speed so that the rocket can achieve orbital perigee velocity at burnout. The gradual transition from vertical to horizontal flight, as illustrated in Fig. 13.1, is caused by the force of gravity, and it is called a gravity turn trajectory.

At liftoff the rocket is vertical, and the flight path angle γ is 90°. After clearing the tower and gaining speed, vernier thrusters or gimbaling of the main engines produce a small, programmed pitchover, establishing an initial flight path angle γ_0, slightly less than 90°. Thereafter, γ will continue to decrease at a rate dictated by Eq. (13.7). (For example, if $\gamma = 85°$, $v = 110$ m/s (250 mph), and $h = 2$ km, then $d\gamma/dt = -0.44$ deg /s). As the speed v of the vehicle increases, the coefficient of $\cos\gamma$ in Eq. (13.7) decreases, which means the rate of change of the flight path angle becomes increasingly smaller, tending toward zero as the booster approaches orbital speed, $v_{\text{circular orbit}} = \sqrt{g/(R_E + h)}$. Ideally, the vehicle is flying horizontally ($\gamma = 0$) at that point.

The gravity turn trajectory is just one example of a practical trajectory, tailored for satellite boosters. On the other hand, sounding rockets fly straight up from launch through burnout. Rocket-powered guided missiles must execute high-speed pitch and yaw maneuvers as they careen toward moving targets and require a rugged structure to withstand the accompanying side loads.

13.3 THE THRUST EQUATION

To discuss rocket performance requires an expression for the thrust T in Eq. (13.6). It can be obtained by a simple one-dimensional momentum analysis. Fig. 13.2a shows a system consisting of a rocket and its propellants. The exterior of the rocket is surrounded by the static pressure p_a of the atmosphere everywhere except at the rocket nozzle exit, where the pressure is p_e. p_e acts over the nozzle exit area A_e. The value of p_e depends on the design of the nozzle. For simplicity, we assume that no other forces act on the system. At time t the mass of the system is m and the absolute velocity in its axial direction is v. The propellants combine chemically in the rocket's combustion chamber, and during the small time interval Δt a small mass Δm of combustion products is forced out of the nozzle, to the left. Because of this expulsion, the velocity of the rocket changes by the small amount Δv, to the right. The absolute velocity of Δm is v_e, assumed to be to the left. According to Newton's second law of motion,

$$(\text{Momentum of the system at } t + \Delta t) - (\text{Momentum of the system at } t) = \text{Net external impulse}$$

or

$$\left[(m - \Delta m)(v + \Delta v)\hat{\mathbf{i}} + \Delta m\left(-v_e\hat{\mathbf{i}}\right)\right] - mv\hat{\mathbf{i}} = (p_e - p_a)A_e\Delta t\hat{\mathbf{i}} \tag{13.9}$$

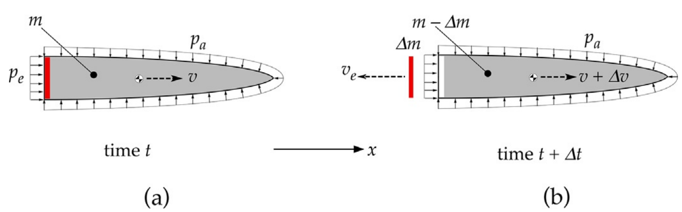


FIG. 13.2

(a) System of rocket and propellant at time t. (b) The system an instant later, after ejection of a small element Δm of combustion products.

Let $\dot{m}_e$ (a positive quantity) be the rate at which exhaust mass flows across the nozzle exit plane. The mass m of the rocket decreases at the rate $\mathrm{d}m/\mathrm{d}t$, and conservation of mass requires the decrease of mass to equal the mass flow rate out of the nozzle. Thus,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\dot{m}_e \tag{13.10}$$

Assuming $\dot{m}_e$ is constant, the vehicle mass as a function of time (from $t = 0$) may therefore be written

$$m(t) = m_0 - \dot{m}_e t \tag{13.11}$$

where m_0 is the initial mass of the vehicle. Since Δm is the mass that flows out in the time interval Δt, we have

$$\Delta m = \dot{m}_e \Delta t \tag{13.12}$$

Let us substitute this expression into Eq. (13.9) to obtain

$$\left[(m - \dot{m}_e\Delta t)(v + \Delta v)\hat{\mathbf{i}} + \dot{m}_e\Delta t\left(-v_e\hat{\mathbf{i}}\right)\right] - mv\hat{\mathbf{i}} = (p_e - p_a)A_e\Delta t\hat{\mathbf{i}}$$

Collecting terms, we get

$$m\Delta v\hat{\mathbf{i}} - \dot{m}_e\Delta t(v_e + v)\hat{\mathbf{i}} - \dot{m}_e\Delta t\Delta v\hat{\mathbf{i}} = (p_e - p_a)A_e\Delta t\hat{\mathbf{i}}$$

Dividing through by Δt, taking the limit as $\Delta t \rightarrow 0$, and canceling the common unit vector leads to

$$m\frac{\mathrm{d}v}{\mathrm{d}t} - \dot{m}_e c_a = (p_e - p_a)A_e \tag{13.13}$$

where c_a is the speed of the exhaust relative to the rocket,

$$c_a = v_e + v \tag{13.14}$$

Rearranging terms, Eq. (13.13) may be written

$$\dot{m}_e c_a + (p_e - p_a)A_e = m\frac{\mathrm{d}v}{\mathrm{d}t} \tag{13.15}$$

The left-hand side of this equation is the unbalanced force responsible for the acceleration $\mathrm{d}v/\mathrm{d}t$ of the system in Fig. 13.2. This unbalanced force is the thrust T,

$$T = \dot{m}_e c_a + (p_e - p_a)A_e \tag{13.16}$$

where $\dot{m}_e c_a$ is the jet thrust, and $(p_e - p_a)A_e$ is the pressure thrust. We can write Eq. (13.16) as

$$T = \dot{m}_e\left[c_a + \frac{(p_e - p_a)A_e}{\dot{m}_e}\right] \tag{13.17}$$

The term in brackets is called the effective exhaust velocity c,

$$c = c_a + \frac{(p_e - p_a)A_e}{\dot{m}_e} \tag{13.18}$$

In terms of the effective exhaust velocity, the thrust may be expressed simply as

$$T = \dot{m}_e c \tag{13.19}$$

The specific impulse I_{sp} is defined as the thrust per sea level weight rate (per second) of propellant consumption. That is,

$$I_{sp} = \frac{T}{\dot{m}_e g_0} \tag{13.20}$$

where g_0 is the standard sea level acceleration of gravity. The unit of specific impulse is force ÷ (force/s) or seconds. Together, Eqs. (13.19) and (13.20) imply that

$$c = I_{sp} g_0 \tag{13.21}$$

Obviously, we can infer the jet velocity directly from the specific impulse. Specific impulse is an important performance parameter for a given rocket engine and propellant combination. However, a large specific impulse equates to a large thrust only if the mass flow rate is large, which is true of chemical rocket engines. The specific impulse of chemical rockets typically lies in the range 200–300 s for solid fuels and 250–450 s for liquid fuels. Ion propulsion systems have very high specific impulse ($>10^4$ s), but their very low mass flow rates produce much smaller thrust than chemical rockets.

13.4 ROCKET PERFORMANCE

From Eqs. (13.10) and (13.20) we have

$$T = -I_{sp} g_0 \frac{dm}{dt} \tag{13.22}$$

or

$$\frac{dm}{dt} = -\frac{T}{I_{sp} g_0}$$

If the thrust and specific impulse are constant, then the integral of this expression over the burn time Δt is

$$\Delta m = -\frac{T}{I_{sp} g_0} \Delta t$$

from which we obtain.

$$\Delta t = \frac{I_{sp} g_0}{T}(m_0 - m_f) = \frac{I_{sp} g_0}{T} m_0 \left(1 - \frac{m_f}{m_0}\right) \tag{13.23}$$

where m_0 and m_f are the mass of the vehicle at the beginning and end of the burn, respectively. The mass ratio n is defined as the ratio of the initial mass to final mass,

$$n = \frac{m_0}{m_f} \tag{13.24}$$

Clearly, the mass ratio is always greater than unity. In terms of the mass ratio, Eq. (13.23) may be written.

$$\Delta t = \frac{n-1}{n} \frac{I_{sp}}{T/(m_0 g_0)} \tag{13.25}$$

$T/(mg_0)$ is the thrust-to-weight ratio. The thrust-to-weight ratio for a launch vehicle at liftoff is typically in the range 1.3 to 2.

Substituting Eq. (13.22) into Eq. (13.6), we get.

$$\frac{dv}{dt} = -I_{sp}g_0 \frac{dm/dt}{m} - \frac{D}{m} - g\sin\gamma$$

Integrating with respect to time, from t_0 to t_f, yields.

$$\Delta v = I_{sp}g_0 \ln\frac{m_0}{m_f} - \Delta v_D - \Delta v_G \tag{13.26}$$

where the drag loss Δv_D and the gravity loss Δv_G are given by the integrals.

$$\Delta v_D = \int_{t_0}^{t_f} \frac{D}{m} dt \qquad \Delta v_G = \int_{t_0}^{t_f} g\sin\gamma\, dt \tag{13.27}$$

Since the drag D, acceleration of gravity g, and flight path angle γ are unknown functions of time, these integrals cannot be computed. (Eqs. (13.6)–(13.8) must be integrated numerically to obtain $v(t)$ and $\gamma(t)$, and Δv would follow from those results.) Eq. (13.26) can be used for rough estimates where previous data and experience provide a basis for choosing conservative values of Δv_D and Δv_G. Obviously, if drag can be neglected, then $\Delta v_D = 0$. This would be a good approximation for the last stage of a satellite booster, for which it can also be said that $\Delta v_G = 0$, since $\gamma \approx 0°$ when the satellite is injected into orbit.

Sounding rockets are launched vertically and fly straight up to their maximum altitude before falling back to earth, usually by parachute. Their purpose is to measure remote portions of the earth's atmosphere. ("Sound" in this context means to measure or investigate.) If for a sounding rocket $\gamma = 90°$, then $\Delta v_G \approx g_0(t_f - t_0)$, since g is within 90% of g_0 up to 300 km altitude.

EXAMPLE 13.1

A sounding rocket of initial mass m_0 and mass m_f after all propellant is consumed is launched vertically ($\gamma = 90°$). The propellant mass flow rate $\dot{m}_e$ is constant.

(a) Neglecting drag and the variation of gravity with altitude, calculate the speed v_{bo}, the altitude h_{bo} at burnout, and the maximum height h_{max} attained by the rocket.

(b) For what flow rate is the greatest altitude reached?

Solution

The vehicle mass as a function of time, up to burnout, is

$$m = m_0 - \dot{m}_e t \tag{a}$$

At burnout, $m = m_f$, so the burnout time t_{bo} is

$$t_{bo} = \frac{m_0 - m_f}{\dot{m}_e} \tag{b}$$

The drag loss Δv_D is assumed to be zero, and the gravity loss for $g = g_0 =$ constant is

$$\Delta v_G = \int_0^{t_{bo}} g_0 \sin(90°) dt = g_0 t_{bo}$$

Recalling that $I_{sp}g_0 = c$ and using Eq. (a), it follows from Eq. (13.26) that, up to burnout, the velocity as a function of time is

$$v = c\ \ln\frac{m_0}{m_0 - \dot{m}_e t} - g_0 t - \overbrace{\int_{t_0}^{t}\frac{D}{m}\mathrm{d}t}^{=0} \tag{c}$$

Since $\mathrm{d}h/\mathrm{d}t = v$, the altitude as a function of time is.

$$h = \int_0^t v\mathrm{d}t = \int_0^t\left(c\ \ln\frac{m_0}{m_0 - \dot{m}_e t} - g_0 t\right)\mathrm{d}t = \frac{c}{\dot{m}_e}\left[(m_0 - \dot{m}_e t)\ \ln\frac{m_0 - \dot{m}_e t}{m_0} + \dot{m}_e t\right] - \frac{1}{2}g_0 t^2 \tag{d}$$

The height at burnout h_{bo} is found by substituting Eq. (b) into this expression,

$$\boxed{h_{bo} = \frac{c}{\dot{m}_e}\left(m_f\ \ln\frac{m_f}{m_0} + m_0 - m_f\right) - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)^2 g_0} \tag{e}$$

Likewise, the burnout velocity is obtained by substituting t_{bo} from Eq. (b) into Eq. (c),

$$\boxed{v_{bo} = c\ \ln\frac{m_0}{m_f} - \frac{g_0}{\dot{m}_e}(m_0 - m_f)} \tag{f}$$

After burnout, the rocket coasts upward with the constant downward acceleration of gravity, so that

$$v = v_{bo} - g_0(t - t_{bo})$$
$$h = h_{bo} + v_{bo}(t - t_{bo}) - \frac{1}{2}g_0(t - t_{bo})^2$$

Substituting Eqs. (b), (e), and (f) into these two expressions yields, for $t > t_{bo}$,

$$v = c\ \ln\frac{m_0}{m_f} - g_0 t$$
$$h = \frac{c}{\dot{m}_e}\left(m_0\ \ln\frac{m_f}{m_0} + m_0 - m_f\right) + ct\ \ln\frac{m_0}{m_f} - \frac{1}{2}g_0 t^2 \tag{g}$$

The maximum height h_{max} is reached when $v = 0$,

$$c\ \ln\frac{m_0}{m_f} - g_0 t_{max} = 0 \quad \Rightarrow \quad t_{max} = \frac{c}{g_0}\ \ln\frac{m_0}{m_f} \tag{h}$$

Substituting $t = t_{max}$ into Eq. (g) leads to our result,

$$\boxed{h_{max} = \frac{1}{2}\frac{c^2}{g_0}\ \ln^2 n - \frac{cm_0}{\dot{m}_e}\frac{n\ \ln n - (n-1)}{n}} \tag{i}$$

where n is the mass ratio ($n > 1$). Since n ln n is greater than $n - 1$, it follows that the second term in this expression is positive. Hence, h_{max} can be increased by increasing the mass flow rate $\dot{m}_e$. In fact,

$$\boxed{\text{The greatest height is achieved when } \dot{m}_e \to \infty}$$

In that extreme, all the propellant is expended at once, like a mortar shell.

Since we neglected both drag and the variation of gravity with altitude, our results (Eqs. e, f, and i) are not accurate, but only estimates.

EXAMPLE 13.2

The data for a single-stage rocket are as follows:

Launch mass: $m_0 = 68{,}000$ kg
Mass ratio: $n = 7$
Specific impulse: $I_{sp} = 390$ s
Thrust: $T = 933.91$ kN

It is launched into a vertical trajectory, like a sounding rocket. Neglecting drag and assuming that the gravitational acceleration is constant at its sea level value $g_0 = 9.81$ m/s^2, estimate.

(a) the time until burnout;
(b) the burnout altitude;
(c) the burnout velocity; and
(d) the maximum altitude reached.

Solution

(a) From Eq. (b) of Example 13.1, the burnout time t_{bo} is

$$t_{bo} = \frac{m_0 - m_f}{\dot{m}_e} \tag{a}$$

The burnout mass m_f is obtained from Eq. (13.24),

$$m_f = \frac{m_0}{n} = \frac{68{,}000}{7} = 9714.3\,\text{kg} \tag{b}$$

The propellant mass flow rate $\dot{m}_e$ is given by Eq. (13.20),

$$\dot{m}_e = \frac{T}{I_{sp} g_0} = \frac{933{,}910}{390 \cdot 9.81} = 244.10\,\text{kg/s} \tag{c}$$

Substituting Eqs. (b) and (c), and $m_0 = 68,000$ kg into Eq. (a) yields the burnout time,

$$t_{bo} = \frac{68{,}000 - 9714.3}{244.10} = \boxed{238.8\ \text{s}}$$

(b) The burnout altitude is given by Eq. (e) of Example 13.1,

$$h_{bo} = \frac{c}{\dot{m}_e}\left(m_f \ln\frac{m_f}{m_0} + m_0 - m_f\right) - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)^2 g_0 \tag{d}$$

The exhaust velocity c is found in Eq. (13.21),

$$c = I_{sp} g_0 = 390 \cdot 9.81 = 3825.9\,\text{m/s} \tag{e}$$

Substituting Eqs. (b), (c), and (e), along with $m_0 = 68{,}000$ kg and $g_0 = 9.81$ m/s^2, into Eq. (d), we get

$$h_{bo} = \frac{3825.9}{244.1}\left(9714.3 \ln\frac{9714.3}{68{,}000} + 68,000 - 9714.3\right) - \frac{1}{2}\left(\frac{68{,}000 - 9714.3}{244.1}\right)^2 \cdot 9.81$$

$$\boxed{h_{bo} = 337.6\text{km}}$$

(c) From Eq. (f) of Example 13.1, we find

$$v_{bo} = c \ln\frac{m_0}{m_f} - \frac{g_o}{\dot{m}_e}(m_0 - m_f) = 3825.9 \ln\frac{68{,}000}{9714.3} - \frac{9.81}{244.1}(68,000 - 9714.3)$$

$$\boxed{v_{bo} = 5.102\text{km/s}}$$

(d) To find $h_{\max}$, where the speed of the rocket falls to zero, we use Eq. (i) of Example 13.1,

$$h_{\max} = \frac{1}{2}\frac{c^2}{g_0}\ln^2 n - \frac{c m_0}{\dot{m}_e}\frac{n \ln n - (n-1)}{n} = \frac{1}{2}\frac{3825.9^2}{9.81}\ln^2 7 - \frac{3825.9 \cdot 68{,}000}{244.1} \cdot \frac{7 \ln 7 - (7-1)}{7}$$

$$\boxed{h_{\max} = 1664.6\text{km}}$$

Note that the rocket coasts to a height nearly five times the burnout altitude.

We can employ the integration schemes introduced in Section 1.8 to solve Eqs. (13.6)–(13.8) numerically. This permits a more accurate accounting of the effects of gravity and drag. It also yields the trajectory.

EXAMPLE 13.3

The rocket in Example 13.2 has a diameter of 5 m. It is to be launched on a gravity turn trajectory. Pitchover begins at an altitude of 130 m with an initial flight path angle γ_0 of 89.85°. What are the altitude h and speed v of the rocket at burnout ($t_{bo} = 260$ s)? What are the velocity losses due to drag and gravity (cf. Eq. 13.27)?

Solution

The MATLAB program *Example_13_03.m* in Appendix D.53 finds the speed v, the flight path angle γ, the altitude h, and the downrange distance x as a function of time. It does so by using the ordinary differential equation solver *rkf_45.m* (Appendix D.4) to numerically integrate Eqs. (13.6)–(13.8), namely

$$\frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma \tag{a}$$

$$\frac{d\gamma}{dt} = -\frac{1}{v}\left(g - \frac{v^2}{R_E + h}\right)\cos\gamma \tag{b}$$

$$\frac{dh}{dt} = v\sin\gamma \tag{c}$$

$$\frac{dx}{dt} = \frac{R_E}{R_E + h} v\cos\gamma \tag{d}$$

The variable mass m is given in terms of the initial mass $m_0 = 68{,}000$ kg and the constant mass flow rate $\dot{m}_e$ by Eq. (13.11),

$$m = m_0 - \dot{m}_e t \tag{e}$$

The thrust $T = 933.913$ kN is assumed constant, and $\dot{m}_e$ is obtained from T and the specific impulse $I_{sp} = 390$ s by means of Eq. (13.20),

$$\dot{m}_e = \frac{T}{I_{sp} g_0} \tag{f}$$

The drag force D in Eq. (a) is given by Eq. (13.1),

$$D = \frac{1}{2}\rho v^2 A C_D \tag{g}$$

The drag coefficient is assumed to have the constant value $C_D = 0.5$. The frontal area $A = \pi d^2/4$ is found from the rocket diameter $d = 5$ m. The atmospheric density profile is assumed to be exponential,

$$\rho = \rho_0 e^{-h/h_0} \tag{h}$$

where $\rho_0 = 1.225$ kg/m^3 is the sea level atmospheric density, and $h_0 = 7.5$ km is the scale height of the atmosphere. (The scale height is the altitude at which the density of the atmosphere is about 37% of its sea level value.)

Finally, the acceleration of gravity varies with altitude h according to Eq. (1.36),

$$g = \frac{g_0}{(1 + h/R_E)^2} \qquad (R_E = 6378\text{ km},\ g_0 = 9.81\text{ m/s}^2) \tag{i}$$

The drag loss and gravity loss are found by numerically integrating Eqs. (13.27).

Between liftoff and pitchover, the flight path angle γ is held at 90°. Pitchover begins at the altitude $h_p = 130$ m with the flight path angle set at $\gamma_0 = 89.85°$.

We write the above system of equations in the standard form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{j}$$

where

$$\mathbf{y} = [v \ \ \gamma \ \ x \ \ h \ \ v_D \ \ v_G]^T$$

$$\mathbf{f} = \left\lfloor \overbrace{\frac{T}{m} - \frac{D}{m} - g\sin\gamma}^{\dot{v}} \quad \overbrace{-\frac{1}{v}\left(g - \frac{v^2}{R_E + h}\right)\cos\gamma}^{\dot{\gamma}} \quad \overbrace{\frac{R_E}{R_E + h} v\cos\gamma}^{\dot{x}} \quad \overbrace{v\sin\gamma}^{\dot{h}} \quad \overbrace{-\frac{D}{m}}^{\dot{v}_D} \quad \overbrace{-g\sin\gamma}^{\dot{v}_G} \right\rfloor^T$$

For the input data described above, the output of *Example_13_03.m* is listed below. The solution is very sensitive to the choice of h_p and γ_0.

The Mach number is the vehicle speed v divided by the speed of sound a, which as a function of altitude h is found from the International Standard Atmosphere model that MATLAB provides as the function *atmosisa*.

```
Initial flight path angle  =   89.850 deg
Pitchover altitude         =  130.000 m
Burn time                  =  238.776 s
Maximum dynamic pressure   =    0.156 atm
   Time                    =    1.110 min
   Speed                   =    0.302 km/s
   Altitude                =    9.424 km
   Mach number             =    0.999
At burnout:
   Speed                   =    5.737 km/s
   Flight path angle       =    9.154 deg
   Altitude                =  110.324 km
   Downrange distance      =  318.364 km
   Drag loss               =    0.298 km/s
   Gravity loss            =    1.410 km/s
```

Thus, at burnout

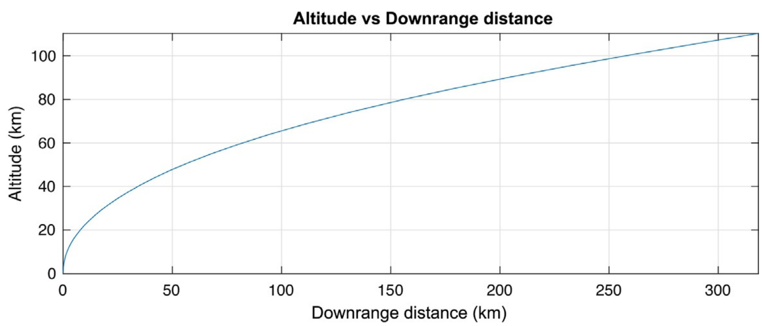
$$\boxed{\begin{array}{r}\text{Altitude} = 110.3\,\text{km}\\ \text{Speed} = 5.737\,\text{km/s}\end{array}}$$

The speed losses are

$$\boxed{\begin{array}{r}\text{Due to drag: } 0.298\,\text{km/s}\\ \text{Due to gravity: } 1.410\,\text{km/s}\end{array}}$$

Note that the drag loss is much less than the gravity loss.

Fig. 13.3 shows the computed gravity turn trajectory and the dynamic pressure variation. The maximum dynamic pressure, $q_{max} = 15.8$ kPa (0.156 atm), occurs 66.6 s into the flight at an altitude of 9.42 km and a speed of 0.302 km/s (Mach 1). By comparison, the dynamic pressure of the wind on your hand sticking out of a car window traveling 80 km/h is about 0.003 atm.

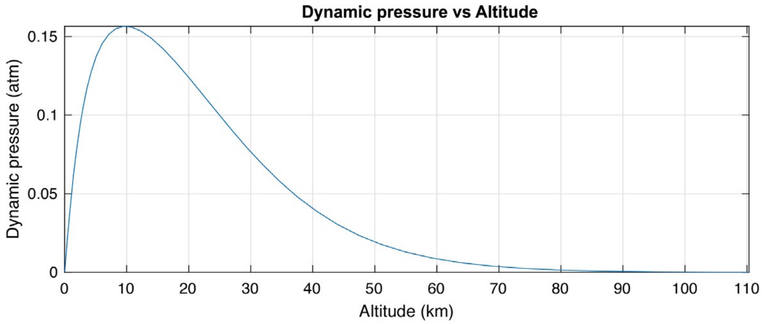


(a)

FIG. 13.3

(a) Gravity turn trajectory for the data given in Examples 13.2 and 13.3.

(Continued)



(b)

FIG. 13.3, cont'd

(b) Dynamic pressure variation with altitude (1 atm = 101.3 kPa).

13.5 RESTRICTED STAGING IN FIELD-FREE SPACE

In field-free space, we neglect drag and gravitational attraction. In that case, Eq. (13.26) becomes

$$\Delta v = I_{sp} g_0 \ln \frac{m_0}{m_f} \tag{13.28}$$

This is at best a poor approximation for high-thrust rockets, but it will suffice to shed some light on the rocket-staging problem. Observe that we can solve this equation for the mass ratio to obtain

$$\frac{m_0}{m_f} = e^{\Delta v / \left(I_{sp} g_0\right)} \tag{13.29}$$

The amount of propellant expended to produce the velocity increment Δv is $m_0 - m_f$. If we let $\Delta m = m_0 - m$, then Eq. (13.29) can be written as.

$$\frac{\Delta m}{m_0} = 1 - e^{-\Delta v / \left(I_{sp} g_0\right)} \tag{13.30}$$

This relation is used to compute the propellant required to produce a given delta-v.

The gross mass m_0 of a launch vehicle consists of the empty mass m_E, the propellant mass m_p, and the payload mass m_{PL},

$$m_0 = m_E + m_p + m_{PL} \tag{13.31}$$

The empty mass comprises the mass of the structure, the engines, fuel tanks, control systems, etc. m_E is also called the structural mass, although it embodies much more than just structure. Dividing Eq. (13.31) through by m_0, we obtain

$$\pi_E + \pi_p + \pi_{PL} = 1 \tag{13.32}$$

where the mass ratios $\pi_E = m_E/m_0$, $\pi_p = m_p/m_0$, and $\pi_{PL} = m_{PL}/m_0$ are the structural fraction, propellant fraction, and payload fraction, respectively. It is convenient to define the payload ratio λ,

$$\lambda = \frac{m_{PL}}{m_E + m_p} = \frac{m_{PL}}{m_0 - m_{PL}} \tag{13.33}$$

and the structural ratio ε,

$$\varepsilon = \frac{m_E}{m_E + m_p} = \frac{m_E}{m_0 - m_{PL}} \tag{13.34}$$

The mass ratio n was introduced in Eq. (13.24). Assuming that all of the propellant is consumed, that may now be written

$$n = \frac{m_E + m_p + m_{PL}}{m_E + m_{PL}} \tag{13.35}$$

λ, ε, and n are not independent. From Eq. (13.34) we have

$$m_E = \frac{\varepsilon}{1-\varepsilon} m_p \tag{13.36}$$

whereas Eq. (13.33) gives

$$m_{PL} = \lambda(m_E + m_p) = \lambda\left(\frac{\varepsilon}{1-\varepsilon} m_p + m_p\right) = \frac{\lambda}{1-\varepsilon} m_p \tag{13.37}$$

Substituting Eqs. (13.36) and (13.37) into Eq. (13.35) leads to

$$n = \frac{1+\lambda}{\varepsilon+\lambda} \tag{13.38}$$

Thus, given any two of the ratios λ, ε, and n, we obtain the third from Eq. (13.38). Using this relation in Eq. (13.28) and setting Δv equal to the burnout speed v_{bo}, when the propellants have been used up, yields

$$v_{bo} = I_{sp} g_0 \ln n = I_{sp} g_0 \ln \frac{1+\lambda}{\varepsilon+\lambda} \tag{13.39}$$

This equation is plotted in Fig. 13.4 for a range of structural ratios. Clearly, for a given empty mass, the greatest possible Δv occurs when the payload is zero. However, what we want to do is maximize the amount of payload while keeping the structural weight to a minimum. Of course, the mass of load-bearing structure, rocket motors, pumps, piping, etc. cannot be made arbitrarily small. Current materials technology places a lower limit on ε of about 0.1. For this value of the structural ratio and $\lambda = 0.05$, Eq. (13.39) yields.

$$v_{bo} = 1.94 I_{sp} g_0 = 0.019 I_{sp}\,(\text{km/s})$$

The specific impulse of a typical chemical rocket is about 300 s, which in this case would provide $\Delta v = 5.7$ km/s. However, the circular orbital velocity at the earth's surface is 7.905 km/s. Therefore, this booster by itself could not orbit the payload. The minimum specific impulse required for a single stage to orbit would be 416 s. Only today's most advanced liquid hydrogen/liquid oxygen engines (e.g., the Space Shuttle main engines), have had this kind of performance. Practicality and economics would likely dictate going the route of a multistage booster.

Fig. 13.5 shows a series or tandem two-stage rocket configuration, with one stage sitting on top of the other. Each stage has its own engines and propellant tanks. The dividing lines between the stages are

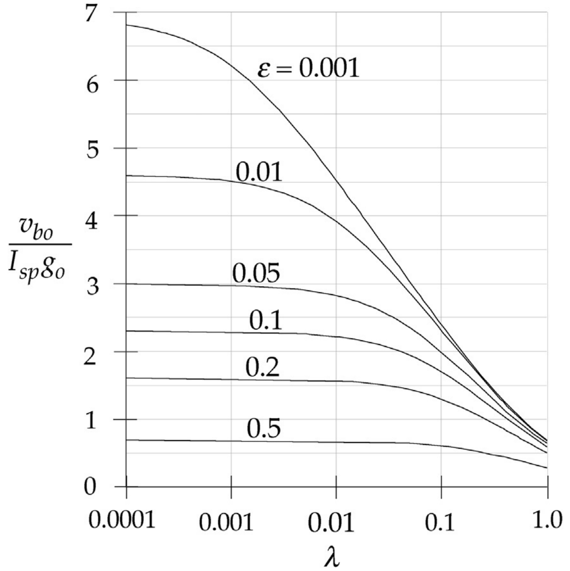


FIG. 13.4

Dimensionless burnout speed vs. payload ratio.

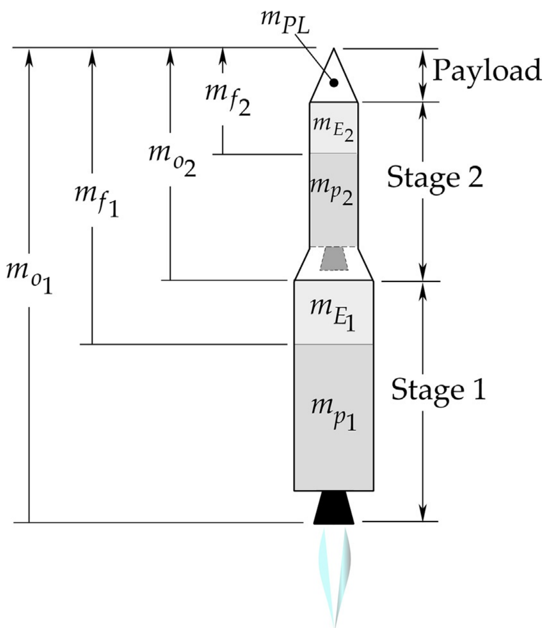


FIG. 13.5

Tandem two-stage booster.

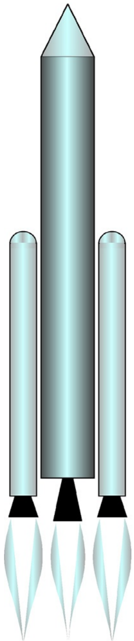


FIG. 13.6

Parallel staging.

where they separate during flight. The first stage drops off first, the second stage next, etc. The payload of an N-stage rocket is actually stage $N+1$. Indeed, satellites commonly carry their own propulsion systems into orbit. The payload of a given stage is everything above it. Therefore, as illustrated in Fig. 13.5, the initial mass m_0 of stage 1 is that of the entire vehicle. After stage 1 expels all its fuel, the mass m_f that remains is stage 1's empty mass m_E plus the mass of stage 2 and the payload. After separation of stage 1, the process continues likewise for stage 2, with m_0 being its initial mass.

Titan II, the launch vehicle for the Gemini program, had the two-stage, tandem configuration. So did the Saturn 1B, used to launch earth orbital flights early in the Apollo program, as well as to send crews to Skylab and an Apollo spacecraft to dock with a Russian Soyuz in 1975.

Fig. 13.6 illustrates the concept of parallel staging. Two or more solid or liquid rockets are attached ("strapped on") to a core vehicle carrying the payload. In the tandem arrangement, the motors in a given stage cannot ignite until separation of the previous stage, whereas all the rockets ignite at once in the parallel-staged vehicle. The strap-on boosters fall away after they burn out early in the ascent. The Space Shuttle was a most obvious example of parallel staging. Its two solid rocket boosters were mounted on the external tank, which fueled the three "main" engines built into the orbiter. The solid rocket boosters and the external tank were cast off after they were depleted. In more common use is the combination of parallel and tandem staging, in which solid or liquid propellant boosters are strapped to the first stage of a multistage stack. A few of the many examples from several countries include:

China:	Long March series
Europe:	Later versions in the Ariane series
India:	Geosynchronous and Polar Satellite launch vehicles
Japan:	H-2 and H-3 series
Russia:	Soyuz and Proton series
United States:	Later versions of the Atlas, Delta, Titan, and Falcon

The venerable Atlas, used in many variants to, among other things, launch the orbital flights of the Mercury program, had three main liquid-fuel engines at its base. They all fired simultaneously at launch, but several minutes into the flight, the outer two "boosters" dropped away, leaving the central sustainer engine to burn the rest of the way to orbit. Since the booster engines shared the sustainer's propellant tanks, the Atlas exhibited partial staging and is sometimes referred to as a one-and-a-half-stage rocket.

We will for simplicity focus on tandem staging, although parallel-staged systems are handled in a similar way (Wiesel, 2010). Restricted staging involves the simple but unrealistic assumption that all stages are similar. That is, each stage has the same specific impulse I_{sp}, the same structural ratio ε, and the same payload ratio λ. From Eq. (13.38), it follows that the mass ratios n are identical too. Let us investigate the effect of restricted staging on the final burnout speed v_{bo} for a given payload mass m_{PL} and overall payload fraction

$$\pi_{PL} = \frac{m_{PL}}{m_0} \tag{13.40}$$

where m_0 is the total mass of the tandem-stacked vehicle.

For a single-stage vehicle, the payload ratio is

$$\lambda = \frac{m_{PL}}{m_0 - m_{PL}} = \frac{1}{\dfrac{m_0}{m_{PL}} - 1} = \frac{\pi_{PL}}{1 - \pi_{PL}} \tag{13.41}$$

so that, from Eq. (13.38), the mass ratio is

$$n = \frac{1}{\pi_{PL}(1-\varepsilon) + \varepsilon} \tag{13.42}$$

According to Eq. (13.39), the burnout speed is.

$$v_{bo} = I_{sp} g_0 \ln \frac{1}{\pi_{PL}(1-\varepsilon) + \varepsilon} \tag{13.43}$$

Let m_0 be the total mass of the two-stage rocket of Fig. 13.5. That is,

$$m_0 = m_0)_1 \tag{13.44}$$

The payload of stage 1 is the entire mass m_0 of stage 2. Thus, for stage 1 the payload ratio is

$$\lambda_1 = \frac{m_0)_2}{m_0)_1 - m_0)_2} = \frac{m_0)_2}{m_0 - m_0)_2} \tag{13.45}$$

The payload ratio of stage 2 is

$$\lambda_2 = \frac{m_{PL}}{m_0)_2 - m_{PL}} \tag{13.46}$$

By virtue of the two stages being similar, $\lambda_1 = \lambda_2$, or

$$\frac{m_0)_2}{m_0 - m_0)_2} = \frac{m_{PL}}{m_0)_2 - m_{PL}}$$

Solving this equation for m_0 yields

$$m_0)_2 = \sqrt{m_0}\sqrt{m_{PL}}$$

But $m_0 = m_{PL}/\pi_{PL}$, so the gross mass of the second stage is

$$m_0)_2 = \sqrt{\frac{1}{\pi_{PL}}} m_{PL} \tag{13.47}$$

Putting this back into Eq. (13.45) or Eq. (13.46), we obtain the common two-stage payload ratio $\lambda = \lambda_1 = \lambda_2$,

$$\lambda_{2\text{-stage}} = \frac{\pi_{PL}{}^{1/2}}{1 - \pi_{PL}{}^{1/2}} \tag{13.48}$$

This together with Eq. (13.38) and the assumption that $\varepsilon_1 = \varepsilon_2 = \varepsilon$ leads to the common mass ratio for each stage,

$$n_{2\text{-stage}} = \frac{1}{\pi_{PL}{}^{1/2}(1-\varepsilon) + \varepsilon} \tag{13.49}$$

If stage 2 ignites immediately after burnout of stage 1, the final velocity of the two-stage vehicle is the sum of the burnout velocities of the individual stages,

$$v_{bo} = v_{bo})_1 + v_{bo})_2$$

or

$$v_{bo_{2\text{-stage}}} = I_{sp}g_0 \ln n_{2\text{-stage}} + I_{sp}g_0 \ln n_{2\text{-stage}} = 2I_{sp}g_0 \ln n_{2\text{-stage}}$$

so that, with Eq. (13.49), we get

$$v_{bo_{2\text{-stage}}} = I_{sp}g_0 \ln\left[\frac{1}{\pi_{PC}{}^{1/2}(1-\varepsilon) + \varepsilon}\right]^2 \tag{13.50}$$

The empty mass of each stage can be found in terms of the payload mass using the common structural ratio ε,

$$\frac{m_E)_1}{m_0)_1 - m_0)_2} = \varepsilon \qquad \frac{m_E)_2}{m_0)_2 - m_{PL}} = \varepsilon$$

Substituting Eqs. (13.40) and (13.44) together with Eq. (13.47) yields.

$$m_E)_1 = \frac{\left(1 - \pi_{PL}{}^{1/2}\right)\varepsilon}{\pi_{PL}} m_{PL} \qquad m_E)_2 = \frac{\left(1 - \pi_{PL}{}^{1/2}\right)\varepsilon}{\pi_{PL}{}^{1/2}} m_{PL} \tag{13.51}$$

Likewise, we can find the propellant mass for each stage from the expressions

$$m_{\mathrm{p}})_1 = m_0)_1 - \left[m_{\mathrm{E}})_1 + m_0)_2\right] \qquad m_{\mathrm{p}})_2 = m_0)_2 - \left[m_{\mathrm{E}})_2 + m_{\mathrm{PL}}\right] \tag{13.52}$$

Substituting Eqs. (13.40) and (13.4), together with Eqs. (13.47) and (13.51), we get

$$m_{\mathrm{p}})_1 = \frac{\left(1-\pi_{\mathrm{PL}}{}^{1/2}\right)(1-\varepsilon)}{\pi_{\mathrm{PL}}} m_{\mathrm{PL}} \qquad m_{\mathrm{p}})_2 = \frac{\left(1-\pi_{\mathrm{PL}}{}^{1/2}\right)(1-\varepsilon)}{\pi_{\mathrm{PL}}{}^{1/2}} m_{\mathrm{PL}} \tag{13.53}$$

EXAMPLE 13.4

The following data are given:

$$m_{\mathrm{PL}} = 10{,}000\,\mathrm{kg} \quad \pi_{\mathrm{PL}} = 0.05 \quad \varepsilon = 0.15 \quad I_{\mathrm{sp}} = 350\ \mathrm{s} \quad g_0 = 0.00981\,\mathrm{km/s^2}$$

Calculate the payload velocity v_{bo} at burnout, the empty mass of the launch vehicle, and the propellant mass for.
(a) a single-stage vehicle; and.
(b) a restricted, two-stage vehicle.

Solution

(a) From Eq. (13.43), we find

$$v_{bo} = 350 \cdot 0.00981 \cdot \ \ln \frac{1}{0.05(1-0.15)+0.15} = \boxed{5.657\ \mathrm{km/s}}$$

Eq. (13.40) yields the gross mass

$$m_0 = \frac{10{,}000}{0.05} = 200{,}000\,\mathrm{kg}$$

from which we obtain the empty mass using Eq. (13.34),

$$m_{\mathrm{E}} = \varepsilon(m_0 - m_{\mathrm{PL}}) = 0.15(200{,}000 - 10{,}000) = \boxed{28{,}500\ \mathrm{kg}}$$

The mass of propellant is

$$m_{\mathrm{p}} = m_0 - m_{\mathrm{E}} - m_{\mathrm{PL}} = 200{,}000 - 28{,}500 - 10{,}000 = \boxed{161{,}500\ \mathrm{kg}}$$

(b) For a restricted two-stage vehicle, the burnout speed is given by Eq. (13.50),

$$v_{bo})_{\text{2-stage}} = 350 \cdot 0.00981 \ln \left[\frac{1}{0.05^{1/2}(1-0.15)+0.15}\right]^2 = \boxed{7.407\,\mathrm{km/s}}$$

The empty mass of each stage is found using Eq. (13.51),

$$m_{\mathrm{E}})_1 = \frac{\left(1-0.05^{1/2}\right)\cdot 0.15}{0.05} \cdot 10{,}000 = \boxed{23{,}292\,\mathrm{kg}}$$

$$m_{\mathrm{E}})_2 = \frac{\left(1-0.05^{1/2}\right)\cdot 0.15}{0.05^{1/2}} \cdot 10{,}000 = \boxed{5208\,\mathrm{kg}}$$

For the propellant masses, we turn to Eq. (13.53),

$$m_{\mathrm{p}})_1 = \frac{\left(1-0.05^{1/2}\right)\cdot(1-0.15)}{0.05} \cdot 10{,}000 = \boxed{131{,}990\,\mathrm{kg}}$$

$$m_{\mathrm{p}})_2 = \frac{\left(1-0.05^{1/2}\right)\cdot(1-0.15)}{0.05^{1/2}} \cdot 10{,}000 = \boxed{29{,}513\,\mathrm{kg}}$$

The total empty mass, $m_{\mathrm{E}} = m_{\mathrm{E}})_1 + m_{\mathrm{E}})_2$, and the total propellant mass, $m_{\mathrm{p}} = m_{\mathrm{p}})_1 + m_{\mathrm{p}})_2$, are the same as for the single-stage rocket. The mass of the second stage, including the payload, is 22.4% of the total vehicle mass.

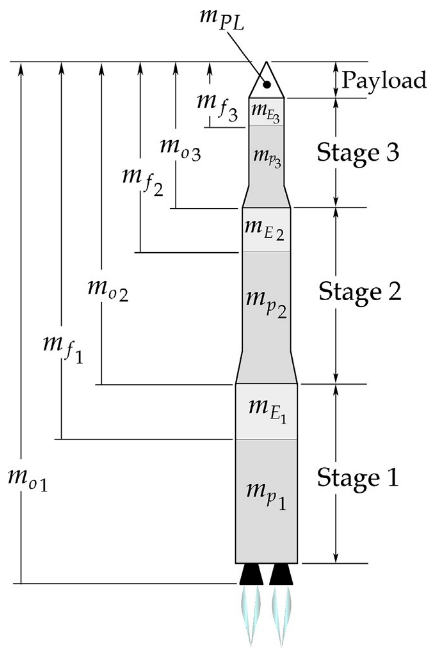


FIG. 13.7

Tandem three-stage launch vehicle.

Observe in the previous example that, although the total vehicle mass was unchanged, the burnout velocity increased 31% for the two-stage arrangement. The reason is that the second stage is lighter and can therefore be accelerated to a higher speed. Let us determine the velocity gain associated with adding another stage, as illustrated in Fig. 13.7.

The payload ratios of the three stages are

$$\lambda_1 = \frac{m_0)_2}{m_0)_1 - m_0)_2} \qquad \lambda_2 = \frac{m_0)_3}{m_0)_2 - m_0)_3} \qquad \lambda_3 = \frac{m_{PL}}{m_0)_3 - m_{PL}}$$

Since the stages are similar, these payload ratios are all the same. Setting $\lambda_1 = \lambda_2$ and recalling that $m_0)_1 = m_0$, we find

$$m_0)_2{}^2 - m_0)_3 m_0 = 0$$

Similarly, $\lambda_1 = \lambda_3$ yields

$$m_0)_2 m_0)_3 - m_0 m_{PL} = 0$$

These two equations imply that.

$$m_0)_2 = \frac{m_{PL}}{\pi_{PL}{}^{2/3}} \qquad m_0)_3 = \frac{m_{PL}}{\pi_{PL}{}^{1/3}} \tag{13.54}$$

Substituting these results back into any one of the above expressions for λ_1, λ_2, or λ_3 yields the common payload ratio for the restricted three-stage rocket,

$$\lambda_{\text{3-stage}} = \frac{{\pi_{PL}}^{1/3}}{1 - {\pi_{PL}}^{1/3}}$$

With this result and Eq. (13.38), we find the common mass ratio,

$$n_{\text{three-stage}} = \frac{1}{{\pi_{PL}}^{1/3}(1-\varepsilon)+\varepsilon} \tag{13.55}$$

Since the payload burnout velocity is $v_{bo} = v_{bo})_1 + v_{bo})_2 + v_{bo})_3$, we have

$$v_{bo})_{\text{3-stage}} = 3I_{sp}g_0 \ln n_{\text{3-stage}} = I_{sp}g_0 \ln\left(\frac{1}{{\pi_{PL}}^{1/3}(1-\varepsilon)+\varepsilon}\right)^3 \tag{13.56}$$

Because of the common structural ratio across each stage,

$$\frac{m_E)_1}{m_0)_1 - m_0)_2} = \varepsilon \qquad \frac{m_E)_2}{m_0)_2 - m_0)_3} = \varepsilon \qquad \frac{m_E)_3}{m_0)_3 - m_{PL}} = \varepsilon$$

Substituting Eqs. (13.40) and (13.54) and solving the resultant expressions for the empty stage masses yields

$$m_E)_1 = \frac{\left(1-{\pi_{PL}}^{1/3}\right)\varepsilon}{\pi_{PL}}m_{PL} \qquad m_E)_2 = \frac{\left(1-{\pi_{PL}}^{1/3}\right)\varepsilon}{{\pi_{PL}}^{2/3}}m_{PL} \qquad m_E)_3 = \frac{\left(1-{\pi_{PL}}^{1/3}\right)\varepsilon}{{\pi_{PL}}^{1/3}}m_{PL} \tag{13.57}$$

The stage propellant masses are

$$\begin{aligned} m_p\big)_1 &= m_0)_1 - \left[m_E)_1 + m_0)_2\right] \\ m_p\big)_2 &= m_0)_2 - \left[m_E)_2 + m_0)_3\right] \\ m_p\big)_3 &= m_0)_3 - \left[m_E)_3 + m_{PL}\right] \end{aligned}$$

Substituting Eqs. (13.40), (13.54), and (13.57) leads to

$$\begin{aligned} m_p\big)_1 &= \frac{\left(1-{\pi_{PL}}^{1/3}\right)(1-\varepsilon)}{\pi_{PL}}m_{PL} \\ m_p\big)_2 &= \frac{\left(1-{\pi_{PL}}^{1/3}\right)(1-\varepsilon)}{{\pi_{PL}}^{2/3}}m_{PL} \\ m_p\big)_2 &= \frac{\left(1-{\pi_{PL}}^{1/3}\right)(1-\varepsilon)}{{\pi_{PL}}^{1/3}}m_{PL} \end{aligned} \tag{13.58}$$

EXAMPLE 13.5

Repeat Example 13.4 for the restricted three-stage launch vehicle.

Solution

Eq. (13.56) gives the burnout velocity for three stages.

$$v_{bo} = 350 \cdot 0.00981 \cdot \ln\left[\frac{1}{0.05^{1/3}(1-0.15)+0.15}\right]^3 = \boxed{7.928 \text{ km/s}}$$

Substituting $m_{\text{PL}} = 10,000$ kg, $\pi_{\text{PL}} = 0.05$, and $\varepsilon = 0.15$ into Eqs. (13.57) and (13.58) yields

$m_E)_1 = 18{,}948$ kg	$m_E)_2 = 6980$ kg	$m_E)_3 = 2572$ kg
$m_{\text{p}})_1 = 107{,}370$ kg	$m_{\text{p}})_2 = 39{,}556$ kg	$m_{\text{p}})_3 = 14{,}573$ kg

Again, the total empty mass and total propellant mass are the same as for the single- and two-stage vehicles. Note that the velocity increase over the two-stage rocket is just 7%, which is much less than the advantage the two-stage vehicle had over the single-stage vehicle.

Looking back over the velocity formulas for one-, two-, and three-stage vehicles (Eqs. 13.43, 13.50, and 13.56), we can induce that for an N-stage rocket,

$$v_{bo})_{\text{N-stage}} = I_{\text{sp}} g_0 \ln\left(\frac{1}{\pi_{\text{PL}}{}^{1/N}(1-\varepsilon)+\varepsilon}\right)^N = I_{\text{sp}} g_0 N \ln\left(\frac{1}{\pi_{\text{PL}}{}^{1/N}(1-\varepsilon)+\varepsilon}\right) \tag{13.59}$$

What happens as we let N become very large? First of all, it can be shown using Taylor series expansion that, for large N,

$$\pi_{\text{PL}}{}^{1/N} \approx 1 + \frac{1}{N}\ln \pi_{\text{PL}} \tag{13.60}$$

Substituting this into Eq. (13.59), we find that

$$v_{bo})_{\text{N-stage}} \approx I_{\text{sp}} g_0 N \ln\left[\frac{1}{1+\dfrac{1}{N}(1-\varepsilon)\ln \pi_{\text{PL}}}\right]$$

Since the term $\frac{1}{N}(1-\varepsilon)\ln \pi_{\text{PL}}$ is arbitrarily small, we can use the fact that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

to write

$$\frac{1}{1+\dfrac{1}{N}(1-\varepsilon)\ln \pi_{\text{PL}}} \approx 1 - \frac{1}{N}(1-\varepsilon)\ln \pi_{\text{PL}}$$

which means

$$v_{bo})_{\text{N-stage}} \approx I_{\text{sp}} g_0 N \ln\left[1 - \frac{1}{N}(1-\varepsilon)\ln \pi_{\text{PL}}\right]$$

Finally, since $\ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4 - \cdots$, we can write this as

$$v_{bo})_{\text{N-stage}} \approx I_{\text{sp}} g_0 N \left[-\frac{1}{N}(1-\varepsilon)\ln \pi_{\text{PL}}\right]$$

Therefore, as N, the number of stages, tends toward infinity, the burnout velocity approaches

$$v_{bo})_\infty = I_{\text{sp}} g_0 (1-\varepsilon)\ln\frac{1}{\pi_{\text{PL}}} \tag{13.61}$$

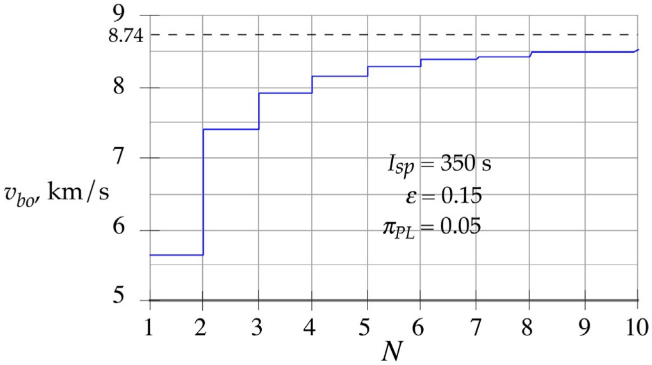


FIG. 13.8

Burnout velocity vs. number of stages (Eq. 13.59).

Thus, no matter how many similar stages we use, for a given specific impulse, payload fraction, and structural ratio, we cannot exceed this burnout speed. For example, using $I_{sp} = 350$ s, $\pi_{PL} = 0.05$, and $\varepsilon = 0.15$ from the previous two examples yields $v_{bo})_\infty = 8.743$ km/s, which is only 10% greater than v_{bo} of a three-stage vehicle. The trend of v_{bo} toward this limiting value is illustrated by Fig. 13.8.

Our simplified analysis does not take into account the added weight and complexity accompanying additional stages. Practical reality has limited the number of stages of actual launch vehicles to rarely more than three.

13.6 OPTIMAL STAGING

Let us now abandon the restrictive assumption that all stages of a tandem-stacked vehicle are similar. Instead, we will specify the specific impulse I_{sp_i} and structural ratio ε_i of each stage i and then seek the minimum-mass N-stage vehicle that will carry a given payload m_{PL} to a specified burnout velocity v_{bo}. To optimize the mass requires using the Lagrange multiplier method, which we shall briefly review.

13.6.1 LAGRANGE MULTIPLIER

Consider a bivariate function f on the xy plane. Then $z = f(x, y)$ is a surface lying above or below the plane, or both. $f(x, y)$ is stationary at a given point if it takes on a local maximum or a local minimum (i.e., an extremum) at that point. For f to be stationary means d$f = 0$. That is,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{13.62}$$

where dx and dy are independent and not necessarily zero. It follows that for an extremum to exist,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{13.63}$$

Now let $g(x, y) = 0$ be a curve in the xy plane. Let us find the points on the curve $g = 0$ at which f is stationary. That is, rather than searching the entire xy plane for extreme values of f, we confine our attention to the curve $g = 0$, which is therefore a constraint. Since $g = 0$, it follows that $\mathrm{d}g = 0$, or

$$\frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y = 0 \tag{13.64}$$

If Eqs. (13.62) and (13.64) are both valid at a given point, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial g/\partial x}{\partial g/\partial y}$$

That is,

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} = -\eta$$

From this we obtain

$$\frac{\partial f}{\partial x} + \eta\frac{\partial g}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} + \eta\frac{\partial g}{\partial y} = 0$$

But these, together with the constraint $g(x, y) = 0$, are the very conditions required for the function

$$h(x, y, \eta) = f(x, y) + \eta g(x, y) \tag{13.65}$$

to have an extremum, namely,

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial x} + \eta\frac{\partial g}{\partial x} = 0 \\ \frac{\partial h}{\partial y} &= \frac{\partial f}{\partial y} + \eta\frac{\partial g}{\partial y} = 0 \\ \frac{\partial h}{\partial \eta} &= g = 0 \end{aligned} \tag{13.66}$$

where η is the Lagrange multiplier. The procedure generalizes to functions of any number of variables.

One can determine mathematically whether the extremum is a maximum or a minimum by checking the sign of the second differential d^2h of the function h in Eq. (13.65),

$$\mathrm{d}^2h = \frac{\partial^2 h}{\partial x^2}\mathrm{d}x^2 + 2\frac{\partial^2 h}{\partial x \partial y}\mathrm{d}x\,\mathrm{d}y + \frac{\partial^2 h}{\partial y^2}\mathrm{d}y^2 \tag{13.67}$$

If $\mathrm{d}^2h < 0$ at the extremum for all $\mathrm{d}x$ and $\mathrm{d}y$ satisfying the constraint condition (Eq. 13.64), then the extremum is a local maximum. Likewise, if $\mathrm{d}^2h > 0$, then the extremum is a local minimum.

EXAMPLE 13.6

(a) Find the extrema of the function $z = -x^2 - y^2$. (b) Find the extrema of the same function under the constraint $y = 2x + 3$.

Solution

(a) To find the extrema we must use Eq. (13.63). Since $\partial z/\partial x = -2x$ and $\partial z/\partial y = -2y$, it follows that $\partial z/\partial x = \partial z/\partial y = 0$ at $x = y = 0$, at which point $z = 0$. Since z is negative everywhere else (Fig. 13.9), it is clear that the extreme value is the maximum value.

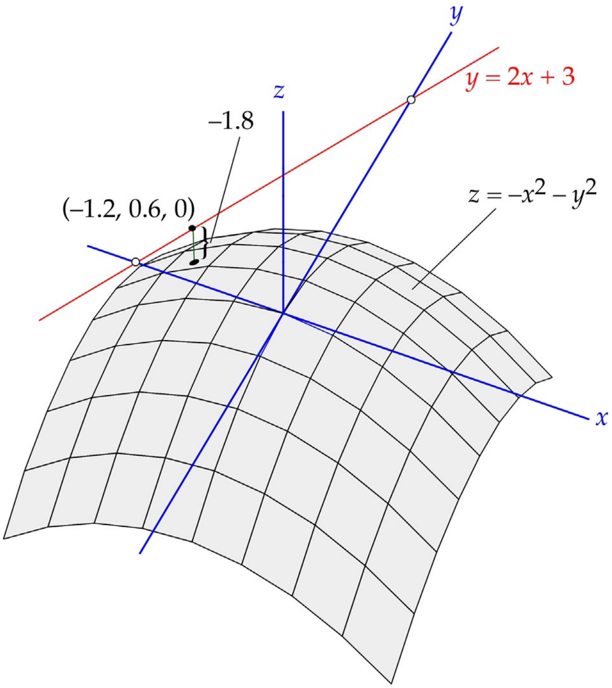


FIG. 13.9

Location of the point on the line $y = 2 \times + 3$ at which the surface $z = -x^2 - y^2$ is closest to the xy plane.

(b) The constraint may be written $g = y - 2x - 3$. Clearly, $g = 0$. Multiply the constraint by the Lagrange multiplier η and add the result (zero!) to the function $-(x^2 + y^2)$ to obtain

$$h = -\left(x^2 + y^2\right) + \eta(y - 2x - 3)$$

This is a function of the three variables x, y, and η. For it to be stationary, the partial derivatives with respect to all three of these variables must vanish. First, we have

$$\frac{\partial h}{\partial x} = -2x - 2\eta$$

Setting this equal to zero yields

$$x = -\eta \tag{a}$$

Next,

$$\frac{\partial h}{\partial y} = -2y + \eta$$

For this to be zero means

$$y = \frac{\eta}{2} \tag{b}$$

Finally,

$$\frac{\partial h}{\partial \eta} = y - 2x - 3$$

Setting this equal to zero gives us back the original constraint condition,

$$y - 2x - 3 = 0 \tag{c}$$

Substituting Eqs. (a) and (b) into Eq. (c) yields $\eta = 1.2$, from which Eqs. (a) and (b) imply,

$$\boxed{x = -1.2 \quad y = 0.6} \tag{d}$$

These are the coordinates of the point on the line $y = 2x + 3$ at which $z = -x^2 - y^2$ is stationary. Using Eqs. (d), we find that $z = -1.8$ at this point.

Fig. 13.9 is an illustration of this problem, and shows that the computed extremum (a maximum, in the sense that small negative numbers exceed large negative numbers) is where the surface $z = -x^2 - y^2$ is closest to the line $y = 2x + 3$, as measured in the z direction. Note that in this case, Eq. (13.67) yields $d^2h = -2dx^2 - 2dy^2$, which is negative, confirming our conclusion that the extremum is a maximum.

Now let us return to the optimal staging problem. It is convenient to introduce the step mass m_i of the ith stage. The step mass is the empty mass plus the propellant mass of the stage, exclusive of all the other stages,

$$m_i = m_{\mathrm{E}})_i + m_{\mathrm{p}})_i \tag{13.68}$$

The empty mass of stage i can be expressed in terms of its step mass and its structural ratio ε_i as follows:

$$m_{\mathrm{E}})_i = \varepsilon_i\left[m_{\mathrm{E}})_i + m_{\mathrm{p}})_i\right] = \varepsilon_i m_i \tag{13.69}$$

The total mass of the rocket excluding the payload is M, which is the sum of all the step masses,

$$M = \sum_{i=1}^{N} m_i \tag{13.70}$$

Thus, recalling that m_0 is the total mass of the vehicle, we have

$$m_0 = M + m_{\mathrm{PL}} \tag{13.71}$$

Our goal is to minimize m_0.

For simplicity, we will deal first with a two-stage rocket, and then generalize our results to N stages. For a two-stage vehicle, $m_0 = m_1 + m_2 + m_{PL}$, so we can write,

$$\frac{m_0}{m_{\mathrm{PL}}} = \frac{m_1 + m_2 + m_{\mathrm{PL}}}{m_2 + m_{\mathrm{PL}}} \cdot \frac{m_2 + m_{\mathrm{PL}}}{m_{\mathrm{PL}}} \tag{13.72}$$

The mass ratio of stage 1 is

$$n_1 = \frac{m_0)_1}{m_E)_1 + m_2 + m_{\mathrm{PL}}} = \frac{m_1 + m_2 + m_{\mathrm{PL}}}{\varepsilon_1 m_1 + m_2 + m_{\mathrm{PL}}} \tag{13.73}$$

where Eq. (13.69) was used. Likewise, the mass ratio of stage 2 is

$$n_2 = \frac{m_0)_2}{\varepsilon_2 m_2 + m_{\mathrm{PL}}} = \frac{m_2 + m_{\mathrm{PL}}}{\varepsilon_2 m_2 + m_{\mathrm{PL}}} \tag{13.74}$$

We can solve Eqs. (13.73) and (13.74) to obtain the step masses from the mass ratios,

$$\begin{aligned} m_2 &= \frac{n_2 - 1}{1 - n_2\varepsilon_2} m_{PL} \\ m_1 &= \frac{n_1 - 1}{1 - n_1\varepsilon_1}(m_2 + m_{PL}) \end{aligned} \tag{13.75}$$

Now,

$$\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{1 - \varepsilon_1}{1 - \varepsilon_1} \cdot \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL} + (\varepsilon_1 m_1 - \varepsilon_1 m_1)} \cdot \frac{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}}$$

These manipulations leave the right-hand side unchanged. Carrying out the multiplications proceeds as follows:

$$\begin{aligned} \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} &= \frac{(1 - \varepsilon_1)(m_1 + m_2 + m_{PL})}{\varepsilon_1 m_1 + m_2 + m_{PL} - \varepsilon_1(m_1 + m_2 + m_{PL})} \cdot \frac{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{\dfrac{1}{\varepsilon_1 m_1 + m_2 + m_{PL}}} \\ &= \frac{(1 - \varepsilon_1)\dfrac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{\dfrac{\varepsilon_1 m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}} - \varepsilon_1 \dfrac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}}} \end{aligned}$$

Finally, with the aid of Eq. (13.73), this algebraic trickery reduces to

$$\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{(1 - \varepsilon_1)n_1}{1 - \varepsilon_1 n_1} \tag{13.76}$$

Likewise,

$$\frac{m_2 + m_{PL}}{m_{PL}} = \frac{(1 - \varepsilon_2)n_2}{1 - \varepsilon_2 n_2} \tag{13.77}$$

so that Eq. (13.72) may be written in terms of the stage mass ratios instead of the step masses as

$$\frac{m_o}{m_{PL}} = \frac{(1 - \varepsilon_1)n_1}{1 - \varepsilon_1 n_1} \cdot \frac{(1 - \varepsilon_2)n_2}{1 - \varepsilon_2 n_2} \tag{13.78}$$

Taking the natural logarithm of both sides of this equation, we get

$$\ln\frac{m_0}{m_{PL}} = \ln\frac{(1 - \varepsilon_1)n_1}{1 - \varepsilon_1 n_1} + \ln\frac{(1 - \varepsilon_2)n_2}{1 - \varepsilon_2 n_2}$$

Expanding the logarithms on the right-hand side leads to

$$\ln\frac{m_0}{m_{PL}} = [\ln(1 - \varepsilon_1) + \ln n_1 - \ln(1 - \varepsilon_1 n_1)] + [\ln(1 - \varepsilon_2) + \ln n_2 - \ln(1 - \varepsilon_2 n_2)] \tag{13.79}$$

Observe that for m_{PL} fixed, $\ln(m_0/m_{PL})$ is a monotonically increasing function of m_0,

$$\frac{d}{dm_0}\left(\ln\frac{m_0}{m_{PL}}\right) = \frac{1}{m_0} > 0$$

Therefore, $\ln(m_0/m_{PL})$ is stationary when m_0 is.

From Eqs. (13.21) and (13.39), the burnout velocity of the two-stage rocket is.

$$v_{bo} = v_{bo})_1 + v_{bo})_2 = c_1 \ln n_1 + c_2 \ln n_2 \tag{13.80}$$

which means that, given v_{bo}, our constraint equation is

$$v_{bo} - c_1 \ln n_1 - c_2 \ln n_2 = 0 \tag{13.81}$$

Introducing the Lagrange multiplier η, we combine Eqs. (13.79) and (13.81) to obtain

$$h = [\ln(1-\varepsilon_1) + \ln n_1 - \ln(1-\varepsilon_1 n_1)] + [\ln(1-\varepsilon_2) + \ln n_2 - \ln(1-\varepsilon_2 n_2)] + \eta(v_{bo} - c_1 \ln n_1 - c_2 \ln n_2) \tag{13.82}$$

Finding the values of n_1 and n_2 for which h is stationary will extremize ln (m_0/m_{PL}) (and, hence, m_0) for the prescribed burnout velocity v_{bo}. h is stationary when $\partial h/\partial n_1 = \partial h/\partial n_2 = \partial h/\partial \eta = 0$. Thus,

$$\frac{\partial h}{\partial n_1} = \frac{1}{n_1} + \frac{\varepsilon_1}{1-\varepsilon_1 n_1} - \eta\frac{c_1}{n_1} = 0$$
$$\frac{\partial h}{\partial n_2} = \frac{1}{n_2} + \frac{\varepsilon_2}{1-\varepsilon_2 n_2} - \eta\frac{c_2}{n_2} = 0$$
$$\frac{\partial h}{\partial \eta} = v_{bo} - c_1 \ln n_1 - c_2 \ln n_2 = 0$$

These three equations yield, respectively,

$$n_1 = \frac{c_1\eta - 1}{c_1\varepsilon_1\eta} \qquad n_2 = \frac{c_2\eta - 1}{c_2\varepsilon_2\eta} \qquad v_{bo} = c_1 \ln n_1 + c_2 \ln n_2 \tag{13.83}$$

Substituting n_1 and n_2 into the expression for v_{bo}, we get

$$c_1 \ln\left(\frac{c_1\eta - 1}{c_1\varepsilon_1\eta}\right) + c_2 \ln\left(\frac{c_2\eta - 1}{c_2\varepsilon_2\eta}\right) = v_{bo} \tag{13.84}$$

This equation must be solved iteratively for η, after which η is substituted into Eq. (13.83) to obtain the stage mass ratios n_1 and n_2. These mass ratios are then used in Eq. (13.75) together with the assumed structural ratios, exhaust velocities, and payload mass to obtain the step masses of each stage.

We can now generalize the optimization procedure to an N-stage vehicle, for which Eq. (13.82) becomes

$$h = \sum_{i=1}^{N} [\ln(1-\varepsilon_i) + \ln n_i - \ln(1-\varepsilon_i n_i)] - \eta\left(v_{bo} - \sum_{i=1}^{N} c_i \ln n_i\right) \tag{13.85}$$

At the outset, we know the required burnout velocity v_{bo} and the payload mass m_{PL}, and for every stage we have the structural ratio ε_i and the exhaust velocity c_i (i.e., the specific impulse). The first step is to solve for the Lagrange parameter η using Eq. (13.84), which, for N stages is written

$$\sum_{i=1}^{N} c_i \ln\frac{c_i\eta - 1}{c_i\varepsilon_i\eta} = v_{bo}$$

Expanding the logarithm, this can be written

$$\sum_{i=1}^{N} c_i \ln(c_i\eta - 1) - \ln\eta \sum_{i=1}^{N} c_i - \sum_{i=1}^{N} c_i \ln c_i\varepsilon_i = v_{bo} \tag{13.86}$$

After solving this equation iteratively for η, we use that result to calculate the optimum mass ratio for each stage (cf. Eq. 13.83),

$$n_i = \frac{c_i\eta - 1}{c_i\varepsilon_i\eta} \quad i = 1, 2, \ldots, N \tag{13.87}$$

Of course, each n_i must be greater than 1.

Referring to Eq. (13.75), we next obtain the step masses of each stage, beginning with stage N and working our way down the stack to stage 1,

$$\begin{aligned} m_N &= \frac{n_N - 1}{1 - n_N\varepsilon_N} m_{\mathrm{PL}} \\ m_{N-1} &= \frac{n_{N-1} - 1}{1 - n_{N-1}\varepsilon_{N-1}}(m_N + m_{\mathrm{PL}}) \\ m_{N-2} &= \frac{n_{N-2} - 1}{1 - n_{N-2}\varepsilon_{N-2}}(m_{N-1} + m_N + m_{\mathrm{PL}}) \\ &\vdots \\ m_1 &= \frac{n_1 - 1}{1 - n_1\varepsilon_1}(m_2 + m_3 + \cdots m_{\mathrm{PL}}) \end{aligned} \tag{13.88}$$

Having found each step mass, each empty stage mass is

$$\left.m_{\mathrm{E}}\right)_i = \varepsilon_i m_i \tag{13.89}$$

and each stage propellant mass is

$$\left.m_{\mathrm{p}}\right)_i = m_i - \left.m_{\mathrm{E}}\right)_i \tag{13.90}$$

For the function h in Eq. (13.85), it is easily shown that

$$\frac{\partial^2 h}{\partial n_i \partial n_j} = 0 \quad i, j = 1, \ldots, N \ (i \neq j)$$

It follows that the second differential of h is

$$\mathrm{d}^2 h = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 h}{\partial n_i \partial n_j} \mathrm{d}n_i \mathrm{d}n_j = \sum_{i=1}^{N} \frac{\partial^2 h}{\partial {n_i}^2} (\mathrm{d}n_i)^2 \tag{13.91}$$

where it can be shown, again using Eq. (13.85), that

$$\frac{\partial^2 h}{\partial {n_i}^2} = \frac{\eta c_i(\varepsilon_i n_i - 1)^2 + 2\varepsilon_i n_i - 1}{(\varepsilon_i n_i - 1)^2 {n_i}^2} \tag{13.92}$$

For h to be minimum at the mass ratios n_i given by Eq. (13.87), it must be true that $d^2h > 0$. Eqs. (13.91) and (13.92) indicate that this will be the case if

$$\eta c_i(\varepsilon_i n_i - 1)^2 + 2\varepsilon_i n_i - 1 > 0 \quad i = 1, \ldots, N \tag{13.93}$$

EXAMPLE 13.7

Find the optimal mass for a three-stage launch vehicle that is required to lift a 5000-kg payload to a speed of 10 km/s. For each stage, we are given that

Stage 1: $I_{sp})_1 = 400$ s ($c_1 = 3.924$ km/s) $\varepsilon_1 = 0.10$
Stage 2: $I_{sp})_2 = 350$ s ($c_2 = 3.434$ km/s) $\varepsilon_2 = 0.15$
Stage 3: $I_{sp})_3 = 300$ s ($c_3 = 2.943$ km/s) $\varepsilon_2 = 0.20$

Solution

Substituting these data into Eq. (13.86), we get

$$3.924\ \ln(3.924\eta - 1) + 3.434\ \ln(3.434\eta - 1) + 2.943\ \ln(2.943\eta - 1) - 10.30\ \ln\eta + 7.5089 = 10$$

As can be checked by substitution, the iterative solution of this equation is

$$\eta = 0.4668$$

Substituting η into Eq. (13.87) yields the optimum mass ratios,

$$n_1 = 4.541 \quad n_2 = 2.507 \quad n_3 = 1.361$$

For the step masses, we appeal to Eq. (13.88) to obtain

$$m_1 = 165{,}700\,\text{kg} \quad m_2 = 18{,}070\,\text{kg} \quad m_3 = 2477\,\text{kg}$$

The total mass of the vehicle is

$$m_0 = m_1 + m_2 + m_3 + m_{PL} = \boxed{191{,}200\ \text{kg}}$$

Using Eqs. (13.89) and (13.90), the empty masses and propellant masses are found to be

$$\begin{array}{lll} m_E)_1 = 16{,}570\,\text{kg} & m_E)_2 = 2710\,\text{kg} & m_E)_3 = 495.4\,\text{kg} \\ m_p)_1 = 149{,}100\,\text{kg} & m_p)_2 = 15{,}360\,\text{kg} & m_p)_3 = 1982\,\text{kg} \end{array}$$

The payload ratios for each stage are

$$\lambda_1 = \frac{m_2 + m_3 + m_{PL}}{m_1} = 0.1542$$
$$\lambda_2 = \frac{m_3 + m_{PL}}{m_2} = 0.4139$$
$$\lambda_3 = \frac{m_{PL}}{m_3} = 2.018$$

The overall payload fraction is

$$\pi_{PL} = \frac{m_{PL}}{m_0} = \frac{5000}{191{,}200} = 0.0262$$

Finally, let us check Eq. (13.93),

$$\eta c_1(\varepsilon_1 n_1 - 1)^2 + 2\varepsilon_1 n_1 - 1 = 0.4541$$
$$\eta c_2(\varepsilon_2 n_2 - 1)^2 + 2\varepsilon_2 n_2 - 1 = 0.3761$$
$$\eta c_3(\varepsilon_3 n_3 - 1)^2 + 2\varepsilon_3 n_3 - 1 = 0.2721$$

A positive number in every instance means that we have indeed found a local minimum of the function in Eq. (13.85).

PROBLEMS

Section 13.4

13.1 A two-stage, solid-propellant sounding rocket has the following properties:

First stage: $m_0 = 249.5$ kg $m_f = 170.1$ kg $\dot{m}_e = 10.61\,\text{kg/s}$ $I_{sp} = 235$ s
Second stage: $m_0 = 113.4$ kg $m_f = 58.97$ kg $\dot{m}_e = 4.053\,\text{kg/s}$ $I_{sp} = 235$ s

The delay time between burnout of first stage and ignition of second stage is 3 s. As a preliminary estimate, neglect drag and the variation of earth's gravity with altitude to calculate the maximum height reached by the second stage after burnout.
{Ans.: 322 km}

13.2 A two-stage launch vehicle has the following properties.

First stage: Two solid-propellant rockets, each with a total mass of 525,000 kg, 450,000 kg of which is propellant, and $I_{sp} = 290$ s.

Second stage: Two liquid rockets with $I_{sp} = 450$ s, dry mass = 30,000 kg, and propellant mass = 600,000 kg.

Calculate the payload mass to a 300-km orbit if launched due east from Kennedy Space Center. Let the total gravity and drag loss be 2 km/s.
{Ans.: 114,000 kg}

Section 13.5

13.3 Suppose a spacecraft in permanent orbit around the earth is to be used for delivering payloads from low earth orbit (LEO) to geostationary equatorial orbit (GEO). Before each flight from LEO, the spacecraft is refueled with propellant, which it uses up in its round trip to GEO. The outbound leg requires four times as much propellant as the inbound return leg. The delta-v for transfer from LEO to GEO is 4.22 km/s. The specific impulse of the propulsion system is 450 s. If the payload mass is 3500 kg, calculate the empty mass of the vehicle.
{Ans.: 2733 kg}

13.4 Consider a rocket comprising three similar stages (i.e., each stage has the same specific impulse, structural ratio, and payload ratio). The common specific impulse is 310 s. The total mass of the vehicle is 150,000 kg, the total structural mass (empty mass) is 20,000 kg, and the payload mass is 10,000 kg. Calculate

(a) the mass ratio n and the total Δv for the three-stage rocket.
{Ans.: $n = 2.04$, $\Delta v = 6.50$ km/s}

(b) $m_p)_1$, $m_p)_2$, and $m_p)_3$.

(c) $m_E)_1$, $m_E)_2$ and $m_E)_3$.

(d) $m_0)_1$, $m_0)_2$ and $m_0)_3$.

Section 13.6

13.5 Find the extrema of the function $z = (x + y)^2$ subject to y and z lying on the circle $(x - 1)^2 + y^2 = 1$.
{Ans.: $z = 0.1716$ at $(x, y) = (0.2929, -0.7071)$;
$z = 5.828$ at $(x, y) = (1.707, 0.7071)$; and
$z = 0$ at $(x, y) = (0, 0)$ and $(x, y) = (1, -1)$}

13.6 A small two-stage vehicle is to propel a 10-kg payload to a speed of 6.2 km/s. The properties of the stages are as follows. For the first stage, $I_{sp} = 300$ s and $\varepsilon = 0.2$. For the second stage, $I_{sp} = 235$ s and $\varepsilon = 0.3$. Estimate the optimum mass of the vehicle.
{Ans.: 1125 kg}

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