

# ORBITS IN THREE DIMENSIONS

# 4

## 4.1 INTRODUCTION

The discussion of orbital mechanics up to now has been confined to two dimensions (i.e., to the plane of the orbits themselves). This chapter explores the means of describing orbits in three-dimensional space, which, of course, is the setting for real missions and orbital maneuvers. Our focus will be on the orbits of earth satellites, but the applications are to any two-body trajectories, including interplanetary missions to be discussed in [Chapters 8 and 9](#).

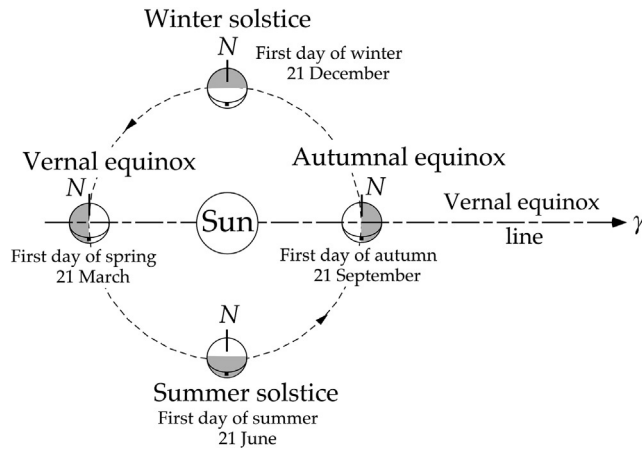
We begin with a discussion of the ancient concept of the celestial sphere and the use of right ascension and declination to define the location of stars, planets, and other celestial objects on the sphere. This leads to the establishment of the inertial geocentric equatorial frame of reference and the concept of state vector. The six components of this vector give the instantaneous position and velocity of an object relative to the inertial frame and define the characteristics of the orbit. Following this discussion is a presentation of the six classical orbital elements, which also uniquely define the shape and orientation of an orbit and the location of a body on it. We then show how to transform the state vector into orbital elements, and vice versa, taking advantage of the perifocal frame introduced in [Chapter 2](#).

We go on to summarize two of the major perturbations of earth orbits due to the earth's nonspherical shape. These perturbations are exploited to place satellites in sun-synchronous and Molniya orbits. The various perturbations of spacecraft trajectories are presented in more detail in [Chapter 10](#).

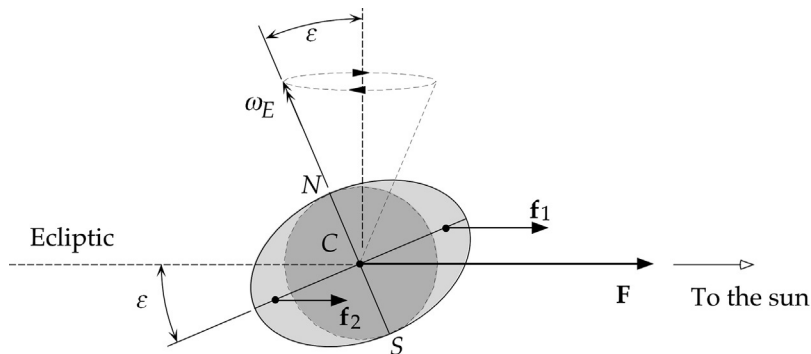
The chapter concludes with a discussion of ground tracks and how to compute them.

## 4.2 GEOCENTRIC RIGHT ASCENSION-DECLINATION FRAME

The coordinate system used to describe earth orbits in three dimensions is defined in terms of earth's equatorial plane, the ecliptic plane, and the earth's axis of rotation. The ecliptic is the plane of the earth's orbit around the sun, as illustrated in [Fig. 4.1](#). The earth's axis of rotation, which passes through the north and south poles, is not perpendicular to the ecliptic. It is tilted away by an angle known as the obliquity of the ecliptic,  $\epsilon$ . For the earth,  $\epsilon$  is approximately  $23.4^\circ$ . Therefore, the earth's equatorial plane and the ecliptic intersect along a line, which is known as the vernal equinox line. On the calendar, "vernal equinox" is the first day of spring in the northern hemisphere, when the noontime sun crosses the equator from south to north. The position of the sun at that instant defines the location of a point in the sky called the vernal equinox, for which the symbol  $\gamma$  is used. On the day of the vernal equinox, the

**FIG. 4.1**

The earth's orbit around the sun, viewed from above the ecliptic plane, showing the change of seasons in the northern hemisphere.

**FIG. 4.2**

Secondary (perturbing) gravitational forces on the earth.

number of hours of daylight and darkness are equal; hence, the word equinox. The other equinox occurs precisely half a year later, when the sun crosses back over the equator from north to south, thereby defining the first day of autumn. The vernal equinox lies today in the constellation Pisces, which is visible in the night sky during the fall. The direction of the vernal equinox line is from the earth toward  $\gamma$ , as shown in Fig. 4.1.

For many practical purposes, the vernal equinox line may be considered fixed in space. However, it actually rotates slowly because the earth's tilted spin axis precesses westward around the normal to the ecliptic at the rate of about  $1.4^\circ$  per century. This slow precession is due primarily to the action of the sun and the moon on the nonspherical distribution of mass within the earth. Due to the centrifugal force of rotation about its own axis, the earth bulges very slightly outward at its equator. This effect is shown highly exaggerated in Fig. 4.2. One of the bulging sides is closer to the sun than the other, so the force of

the sun's gravity  $\mathbf{f}_1$  on its mass is slightly larger than the force  $\mathbf{f}_2$  on the opposite side, farthest from the sun. The forces  $\mathbf{f}_1$  and  $\mathbf{f}_2$ , along with the dominant force  $\mathbf{F}$  on the spherical mass, comprise the total force of the sun on the earth, holding it in its solar orbit. Taken together,  $\mathbf{f}_1$  and  $\mathbf{f}_2$  produce a net clockwise moment (a vector into the page) about the center of the earth. That moment would rotate the earth's equator into alignment with the ecliptic if it were not for the fact that the earth has an angular momentum directed along its south-to-north polar axis due to its spin around that axis at an angular velocity  $\omega_E$  of about  $360^\circ$  per day. The effect of the moment is to rotate the angular momentum vector in the direction of the moment (into the page). The result is that the spin axis is forced to precess in a counterclockwise direction around the normal to the ecliptic, sweeping out a cone as illustrated in the figure. The moon exerts a torque on the earth for the same reason, and the combined effect of the sun and the moon is a precession of the spin axis, and hence  $\gamma$ , with a period of about 26,000 years. The moon's action also superimposes a small nutation on the precession. This causes the obliquity  $\epsilon$  to vary with a maximum amplitude of  $0.0025^\circ$  over a period of 18.6 years.

About 4000 years ago, when the first recorded astronomical observations were being made,  $\gamma$  was located in the constellation Aries, the ram. The Greek letter  $\gamma$  resembles the ancient symbol representing the head of a ram ( $\Upsilon$ ).

To the human eye, objects in the night sky appear as points on a celestial sphere surrounding the earth, as illustrated in Fig. 4.3. The north and south poles of this fixed sphere correspond to those of the earth rotating within it. Coordinates of latitude and longitude are used to locate points on the celestial

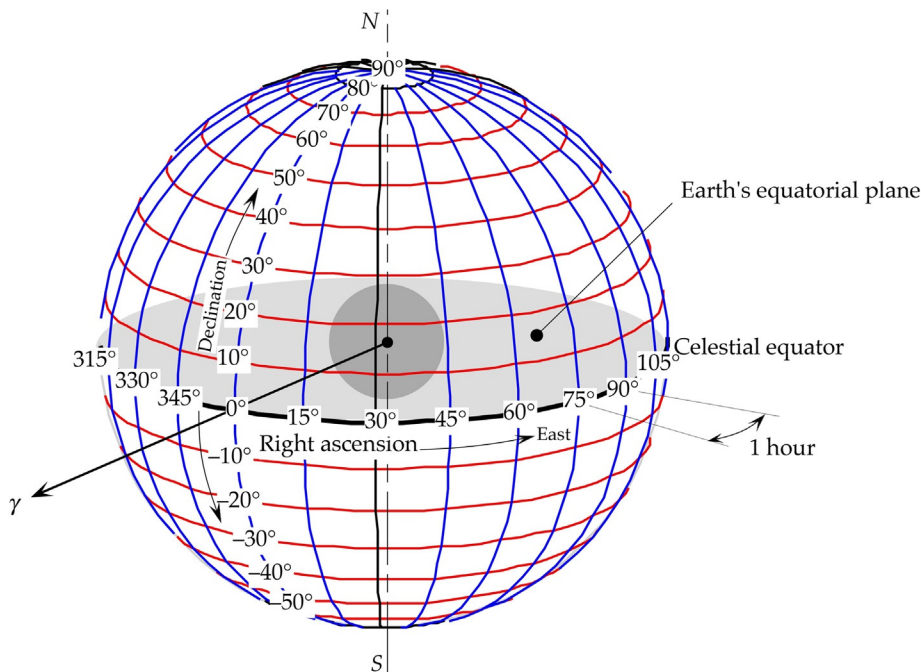


FIG. 4.3

The celestial sphere, with grid lines of right ascension and declination]

sphere in much the same way as on the surface of the earth. The projection of the earth's equatorial plane outward onto the celestial sphere defines the celestial equator. The vernal equinox  $\gamma$ , which lies on the celestial equator, is the origin for measurement of longitude, which in astronomical parlance is called right ascension. Right ascension (RA or  $\alpha$ ) is measured along the celestial equator in degrees east from the vernal equinox (astronomers measure right ascension in hours instead of degrees, where 24 h equals  $360^\circ$ ). Latitude on the celestial sphere is called declination. Declination (Dec or  $\delta$ ) is measured along a meridian in degrees, positive to the north of the equator and negative to the south. Fig. 4.4 is a sky chart showing how the heavenly grid appears from a given point on the earth. Notice that the sun is located at the intersection of the equatorial and ecliptic planes, so this must be the first day of spring.

Stars are so far away from the earth that their positions relative to each other appear stationary on the celestial sphere. Planets, comets, satellites, etc., move on the fixed backdrop of the stars. A table of the coordinates of celestial bodies as a function of time is called an ephemeris [e.g., the *Astronomical Almanac* (Department of the Navy, 2018)]. Table 4.1 is an abbreviated ephemeris for the moon and for Venus. An ephemeris depends on the location of the vernal equinox at a given time or epoch, for we know that even the positions of the stars relative to the equinox change slowly with time. For example, Table 4.2 shows the celestial coordinates of the star Regulus at five epochs since AD 1700. Currently, the position of the vernal equinox in the year 2000 is used to define the standard grid of the celestial sphere. In 2025, the position will be updated to that of the year 2050, and so on at 25-year intervals.

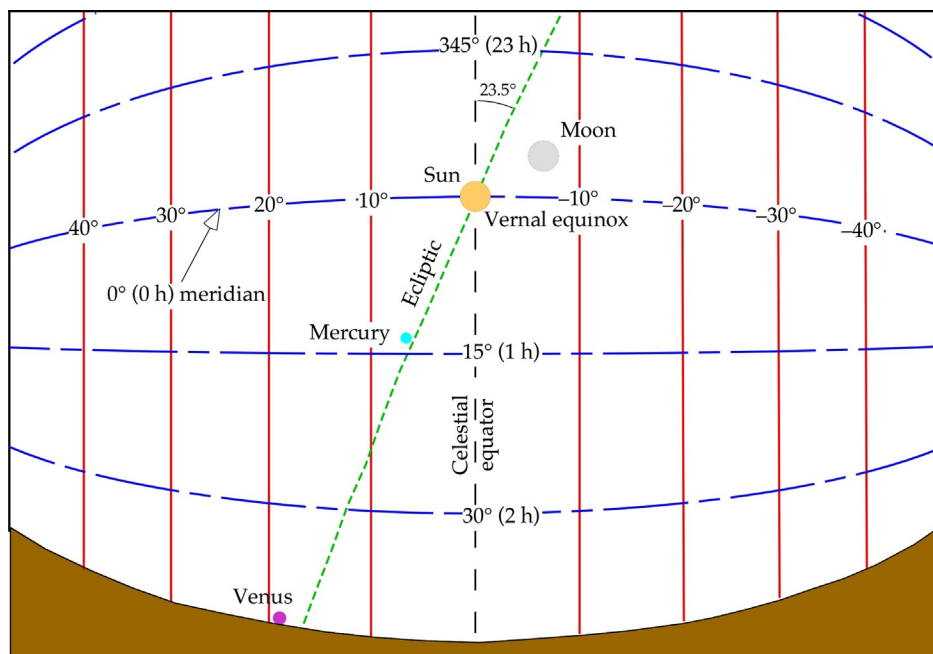


FIG. 4.4

A view of the sky above the eastern horizon from  $0^\circ$  longitude on the equator at 9 a.m. local time, March 20, 2004 (precession epoch AD 2000).

**Table 4.1 Venus and moon ephemeris for 0 h universal time (precession epoch: 2000 AD)**

Date	Venus		Moon	
	RA	Dec	RA	Dec
1 Jan 2004	21h 05.0min	−18° 36′	1h 44.9min	+8° 47′
1 Feb 2004	23h 28.0min	−04° 30′	4h 37.0min	+24° 11′
1 Mar 2004	01h 30.0min	+10° 26′	6h 04.0min	+08° 32′
1 Apr 2004	03h 37.6min	+22° 51′	9h 18.7min	+21° 08′
1 May 2004	05h 20.3min	+27° 44′	11h 28.8min	+07° 53′
1 Jun 2004	05h 25.9min	+24° 43′	14h 31.3min	−14° 48′
1 Jul 2004	04h 34.5min	+17° 48′	17h 09.0min	−26° 08′
1 Aug 2004	05h 37.4min	+19° 04′	21h 05.9min	−21° 49′
1 Sep 2004	07h 40.9min	+19° 16′	00h 17.0min	−00° 56′
1 Oct 2004	09h 56.5min	+12° 42′	02h 20.9min	+14° 35′
1 Nov 2004	12h 15.8min	+00° 01′	05h 26.7min	+27° 18′
1 Dec 2004	14h 34.3min	−13° 21′	07h 50.3min	+26° 14′
1 Jan 2005	17h 12.9min	−22° 15′	10h 49.4min	+11° 39′

**Table 4.2 Variation of the coordinates of the star Regulus due to precession of the equinox**

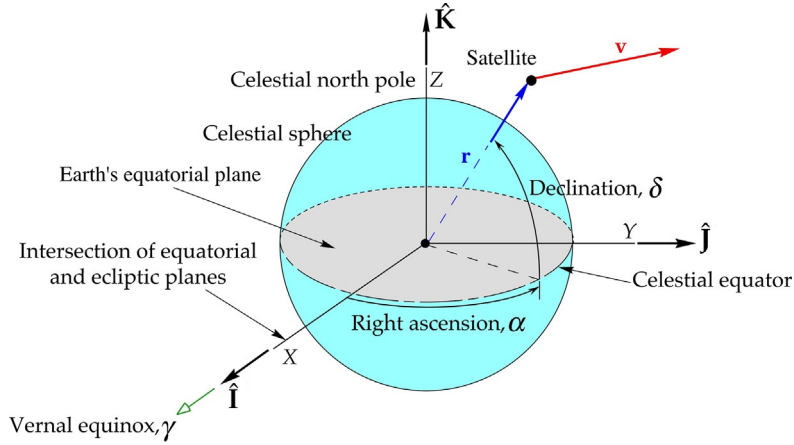
Precession epoch	RA	Dec
1700 AD	9h 52.2min (148.05°)	+13° 25′
1800 AD	9h 57.6min (149.40°)	+12° 56′
1900 AD	10h 3.0min (150.75°)	+12° 27′
1950 AD	10h 5.7min (151.42°)	+12° 13′
2000 AD	10h 8.4min (152.10°)	+11° 58′

Since observations are made relative to the actual orientation of the earth, these measurements must be transformed into the standardized celestial frame of reference. As [Table 4.2](#) suggests, the adjustments will be small if the current epoch is within 25 years of the standard precession epoch.

## 4.3 STATE VECTOR AND THE GEOCENTRIC EQUATORIAL FRAME

At any given time, the state vector of a satellite comprises its position  $\mathbf{r}$  and orbital velocity  $\mathbf{v}$ . Orbital mechanics is concerned with specifying or predicting state vectors over intervals of time. From [Chapter 2](#), we know that the equation governing the state vector of a satellite traveling around the earth is, under the familiar assumptions,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad (4.1)$$

**FIG. 4.5**

The geocentric equatorial frame.

where  $\mathbf{r}$  is the position vector of the satellite relative to the center of the earth. The components of  $\mathbf{r}$ , and especially, those of its time derivatives  $\dot{\mathbf{r}} = \mathbf{v}$  and  $\ddot{\mathbf{r}} = \mathbf{a}$ , must be measured in a nonrotating frame attached to the earth. A commonly used nonrotating right-handed Cartesian coordinate system is the geocentric equatorial frame shown in Fig. 4.5. The  $X$  axis points in the vernal equinox direction. The  $XY$  plane is the earth's equatorial plane, and the  $Z$  axis coincides with the earth's axis of rotation and points northward. The unit vectors  $\hat{\mathbf{I}}$ ,  $\hat{\mathbf{J}}$ , and  $\hat{\mathbf{K}}$  form a right-handed triad. The nonrotating geocentric equatorial frame serves as an inertial frame for the two-body earth satellite problem, as embodied in Eq. (4.1). It is not truly an inertial frame, however, since the center of the earth is always accelerating toward a third body, the sun (to say nothing of the moon), a fact that we ignore in the two-body formulation.

In the geocentric equatorial frame, the state vector is given in component form by

$$\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}} \quad (4.2)$$

$$\mathbf{v} = v_X\hat{\mathbf{I}} + v_Y\hat{\mathbf{J}} + v_Z\hat{\mathbf{K}} \quad (4.3)$$

If  $r$  is the magnitude of the position vector, then

$$\mathbf{r} = r\hat{\mathbf{u}}_r \quad (4.4)$$

Fig. 4.5 shows that the components of  $\hat{\mathbf{u}}_r$  (the direction cosines  $l$ ,  $m$ , and  $n$  of  $\hat{\mathbf{u}}_r$ ) are found in terms of the right ascension  $\alpha$  and declination  $\delta$  as follows:

$$\hat{\mathbf{u}}_r = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} = \cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}} \quad (4.5)$$

From this we see that the declination is obtained as  $\delta = \sin^{-1}n$ . There is no quadrant ambiguity since, by definition, the declination lies between  $-90^\circ$  and  $+90^\circ$ , which is precisely the range of the principal values of the arcsine function. It follows that  $\cos\delta$  cannot be negative. Eq. (4.5) also reveals that

$l = \cos \delta \cos \alpha$ . Hence, we find the right ascension from  $\alpha = \cos^{-1}(l/\cos \delta)$ , which yields two values of  $\alpha$  between  $0^\circ$  and  $360^\circ$ . To determine the correct quadrant for  $\alpha$ , we check the sign of the direction cosine,  $m = \cos \delta \sin \alpha$ . Since  $\cos \delta$  cannot be negative, the sign of  $m$  is the same as the sign of  $\sin \alpha$ . If  $\sin \alpha > 0$ , then  $\alpha$  lies in the range  $0^\circ$  to  $180^\circ$ , whereas  $\sin \alpha < 0$  means that  $\alpha$  lies between  $180^\circ$  and  $360^\circ$ .

#### ALGORITHM 4.1

Given the position vector  $\mathbf{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}} + Z\hat{\mathbf{K}}$ , calculate the right ascension  $\alpha$  and declination  $\delta$ . This procedure is implemented in MATLAB as `ra_and_dec_from_r.m`, which appears in [Appendix D.17](#).

1. Calculate the magnitude of  $\mathbf{r}$ :  $r = \sqrt{X^2 + Y^2 + Z^2}$ .
2. Calculate the direction cosines of  $\mathbf{r}$ :  $l = X/r$   $m = Y/r$   $n = Z/r$
3. Calculate the declination:  $\delta = \sin^{-1}n$
4. Calculate the right ascension:  $\alpha = \begin{cases} \cos^{-1}(l/\cos \delta) & (m > 0) \\ 360^\circ - \cos^{-1}(l/\cos \delta) & (m \leq 0) \end{cases}$

Although the position vector furnishes the right ascension and declination, the right ascension and declination alone do not furnish  $\mathbf{r}$ . For that we need the distance  $r$  to obtain the position vector from Eq. (4.4).

#### EXAMPLE 4.1

If the position vector of the International Space Station in the geocentric equatorial frame is  $\mathbf{r} = -5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}}$  (km), what are its right ascension and declination?

##### Solution

We employ Algorithm 4.1.

Step 1:

$$r = \sqrt{(-5368)^2 + (-1784)^2 + 3691^2} = 6754 \text{ km}$$

Step 2:

$$l = \frac{-5368}{6754} = -0.7947 \quad m = \frac{-1784}{6754} = -0.2642 \quad n = \frac{3691}{6754} = 0.5462$$

Step 3:

$$\delta = \sin^{-1}0.5462 = \boxed{33.12^\circ}$$

Step 4:

Since the direction cosine  $m$  is negative,

$$\alpha = 360^\circ - \cos^{-1}\left(\frac{l}{\cos \delta}\right) = 360^\circ - \cos^{-1}\left(\frac{-0.7947}{\cos 33.12^\circ}\right) = 360^\circ - 161.6^\circ = \boxed{198.4^\circ}$$

From Section 2.11 we know that if we are provided the state vector  $(\mathbf{r}_0, \mathbf{v}_0)$  at a given instant, then we can determine the state vector at any other time in terms of the initial vector by means of the expressions

$$\begin{aligned}\mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0 \\ \mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0\end{aligned}\tag{4.6}$$

where the Lagrange coefficients  $f$  and  $g$  and their time derivatives are given in Eq. (3.69). Specifying the total of six components of  $\mathbf{r}_0$  and  $\mathbf{v}_0$  therefore completely determines the size, shape, and orientation of the orbit.

### EXAMPLE 4.2

At time  $t_0$ , the state vector of an earth satellite is

$$\mathbf{r}_0 = 1600\hat{\mathbf{i}} + 5310\hat{\mathbf{j}} + 3800\hat{\mathbf{k}} \text{ (km)} \tag{a}$$

$$\mathbf{v}_0 = -7.350\hat{\mathbf{i}} + 0.4600\hat{\mathbf{j}} + 2.470\hat{\mathbf{k}} \text{ (km/s)} \tag{b}$$

Determine the position and velocity vectors 3200 s later and plot the orbit in three dimensions.

#### Solution

We will use the universal variable formulation and Algorithm 3.4, which was illustrated in detail in Example 3.7. Therefore, only the results of each step are presented here.

Step 1: ( $\alpha$  here is not to be confused with right ascension)

$$\alpha = 1.4613(10^{-4})\text{km}^{-1}$$

Since this is positive, the orbit is an ellipse.

Step 2:

$$\chi = 294.42\text{km}^{1/2}$$

Step 3:

$$f = -0.94843 \quad \text{and} \quad g = -354.89\text{s}^{-1}$$

Step 4:

$$\boxed{\mathbf{r} = 1090.9\hat{\mathbf{i}} - 5199.4\hat{\mathbf{j}} - 4480.6\hat{\mathbf{k}} \text{ (km)}} \Rightarrow r = 6949.8\text{km}$$

Step 5:

$$\dot{f} = 0.00045324\text{s}^{-1} \quad \text{and} \quad \dot{g} = -0.88479$$

Step 6:

$$\boxed{\mathbf{v} = 7.2284\hat{\mathbf{i}} + 1.9997\hat{\mathbf{j}} - 0.46311\hat{\mathbf{k}} \text{ (km/s)}}$$

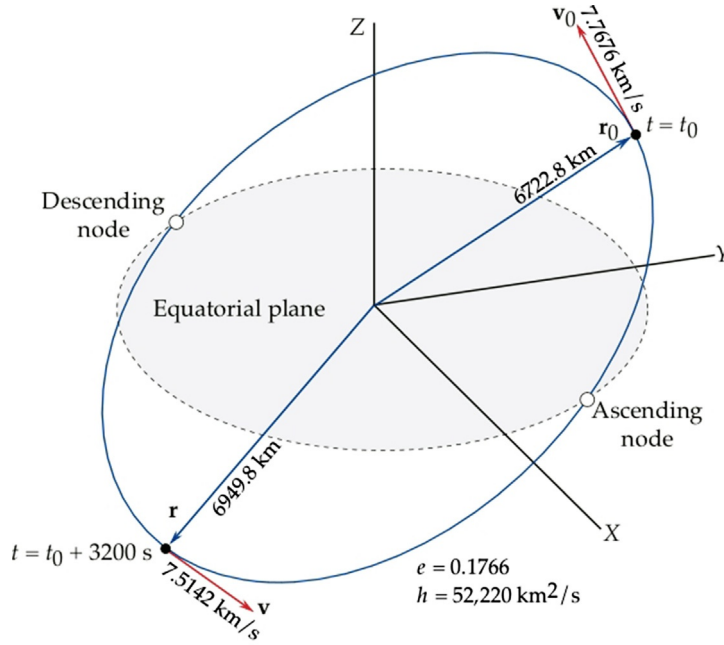
To plot the elliptical orbit, we observe that one complete revolution means a change in the eccentric anomaly  $E$  of  $2\pi$  radians. According to Eq. (3.57), the corresponding change in the universal anomaly is

$$\chi = \sqrt{a}E = \sqrt{\frac{1}{\alpha}}E = \sqrt{\frac{1}{0.00014613}} \cdot 2\pi = 519.77\text{km}^{1/2}$$

Letting  $\chi$  vary from 0 to 519.77 in small increments, we employ the Lagrange coefficient formulation (Eq. 3.67 plus Eqs. 3.69a and 3.69b) to compute

$$\mathbf{r} = \left[1 - \frac{\chi^2}{r_0}\right]C(\alpha\chi^2)\mathbf{r}_0 + \left[\Delta t - \frac{1}{\sqrt{\mu}}\chi^3S(\alpha\chi^2)\right]\mathbf{v}_0$$





**FIG. 4.6**

The orbit corresponding to the initial conditions given in Eqs. (a) and (b) of Example 4.2.

where  $\Delta t$  for a given value of  $\chi$  is given by Eq. (3.49). Using a computer to plot the points obtained in this fashion yields Fig. 4.6, which also shows the state vectors at  $t_0$  and  $t_0 + 3200$  s.

The previous example illustrates the fact that the six quantities or orbital elements comprising the state vector  $\mathbf{r}$  and  $\mathbf{v}$  completely determine the orbit. Other elements may be chosen. The classical orbital elements are introduced and related to the state vector in the next section.

## 4.4 ORBITAL ELEMENTS AND THE STATE VECTOR

To define an orbit in the plane requires two parameters: eccentricity and angular momentum. Other parameters, such as the semimajor axis, the specific energy, and (for an ellipse) the period are obtained from these two. To locate a point on the orbit requires a third parameter, the true anomaly, which leads us to the time since perigee. Describing the orientation of an orbit in three dimensions requires three additional parameters, called the Euler angles, which are illustrated in Fig. 4.7.

First, we locate the intersection of the orbital plane with the equatorial (XY) plane. This line is called the node line. The point on the node line where the orbit passes above the equatorial plane from below it is called the ascending node. The node line vector  $\mathbf{N}$  extends outward from the origin through the ascending node. At the other end of the node line, where the orbit dives below the equatorial plane, is the descending node. The angle between the positive X axis and the node line is the first Euler angle  $\Omega$ , the

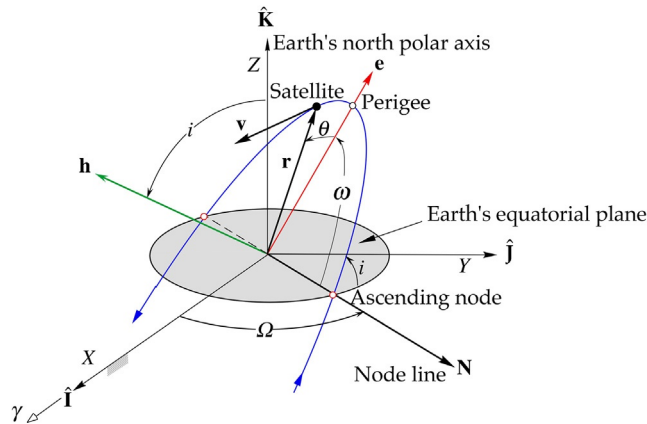


FIG. 4.7

Geocentric equatorial frame and the orbital elements.

right ascension of the ascending node. Recall from [Section 4.2](#) that right ascension is a positive number lying between  $0^\circ$  and  $360^\circ$ .

The dihedral angle between the orbital plane and the equatorial plane is the inclination  $i$ , measured according to the right-hand rule (i.e., counterclockwise around the node line vector from the equator to the orbit). The inclination is also the angle between the positive  $Z$  axis and the normal to the plane of the orbit. The two equivalent means of measuring  $i$  are indicated in [Fig. 4.7](#). Recall from [Chapter 2](#) that the angular momentum vector  $\mathbf{h}$  is normal to the plane of the orbit. Therefore, the inclination  $i$  is the angle between the positive  $Z$  axis and  $\mathbf{h}$ . The inclination is a positive number between  $0^\circ$  and  $180^\circ$ .

It remains to locate the perigee of the orbit. Recall that perigee lies at the intersection of the eccentricity vector  $\mathbf{e}$  with the orbital path. The third Euler angle  $\omega$ , the argument of perigee, is the angle between the node line vector  $\mathbf{N}$  and the eccentricity vector  $\mathbf{e}$ , measured in the plane of the orbit. The argument of perigee is a positive number between  $0^\circ$  and  $360^\circ$ .

In summary, the six orbital elements are

- $h$  specific angular momentum
- $i$  inclination
- $\Omega$  right ascension of the ascending node
- $e$  eccentricity
- $\omega$  argument of perigee
- $\theta$  true anomaly.

The angular momentum  $h$  and true anomaly  $\theta$  are frequently replaced by the semimajor axis  $a$  and the mean anomaly  $M$ , respectively.

Given the position  $\mathbf{r}$  and velocity  $\mathbf{v}$  of a spacecraft in the geocentric equatorial frame, how do we obtain the orbital elements? The step-by-step procedure is outlined next in [Algorithm 4.2](#). Note that each step incorporates results obtained in the previous steps. Several steps require resolving the quadrant ambiguity that arises in calculating the arccosine (recall [Fig. 3.4](#)).

**ALGORITHM 4.2**

Obtain orbital elements from the state vector. A MATLAB version of this procedure appears in [Appendix D.18](#). Applying this algorithm to orbits around other planets or the sun amounts to defining the frame of reference and substituting the appropriate gravitational parameter  $\mu$ .

1. Calculate the distance,  $r = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{X^2 + Y^2 + Z^2}$ .
2. Calculate the speed,  $v = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_X^2 + v_Y^2 + v_Z^2}$ .
3. Calculate the radial velocity,  $v_r = \mathbf{r} \cdot \mathbf{v} / r = (Xv_X + Yv_Y + Zv_Z) / r$ .

Note that if  $v_r > 0$ , the spacecraft is flying away from perigee. If  $v_r < 0$ , it is flying toward perigee.

4. Calculate the specific angular momentum,

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ X & Y & Z \\ v_X & v_Y & v_Z \end{vmatrix}$$

5. Calculate the magnitude of the specific angular momentum,  $h = \sqrt{\mathbf{h} \cdot \mathbf{h}}$ .  
This is the first orbital element.
6. Calculate the inclination,

$$i = \cos^{-1}(h_Z / h) \quad (4.7)$$

This is the second orbital element. Recall that  $i$  must lie between  $0^\circ$  and  $180^\circ$ , which is precisely the range (principal values) of the arccosine function. Hence, there is no quadrant ambiguity to contend with here. If  $90^\circ < i \leq 180^\circ$ , the angular momentum  $\mathbf{h}$  points in a southerly direction. In that case, the orbit is retrograde, which means that the motion of the satellite around the earth is opposite to earth's rotation.

7. Calculate

$$\mathbf{N} = \hat{\mathbf{k}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix} \quad (4.8)$$

This vector defines the node line.

8. Calculate the magnitude of  $\mathbf{N}$ ,  $N = \sqrt{\mathbf{N} \cdot \mathbf{N}}$ .
9. Calculate the right ascension of the ascending node,  $\Omega = \cos^{-1}(N_X / N)$ . This is the third orbital element. If  $N_X > 0$ , then  $\Omega$  lies in either the first or fourth quadrant. If  $N_X < 0$ , then  $\Omega$  lies in either the second or third quadrant. To place  $\Omega$  in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane ( $0 \leq \Omega < 180^\circ$ ) if  $N_Y > 0$ . On the other hand, the ascending node lies on the negative side of the XZ plane ( $180^\circ \leq \Omega < 360^\circ$ ) if  $N_Y < 0$ . Therefore,  $N_Y > 0$  implies that  $0 \leq \Omega < 180^\circ$ , whereas  $N_Y < 0$  implies that  $180^\circ \leq \Omega < 360^\circ$ . In summary,

$$\Omega = \begin{cases} \cos^{-1}\left(\frac{N_X}{N}\right) & (N_Y \geq 0) \\ 360^\circ - \cos^{-1}\left(\frac{N_X}{N}\right) & (N_Y < 0) \end{cases} \quad (4.9)$$

10. Calculate the eccentricity vector. Starting with Eq. (2.40),

$$\mathbf{e} = \frac{1}{\mu} \left[ \mathbf{v} \times \mathbf{h} - \mu \frac{\mathbf{r}}{r} \right] = \frac{1}{\mu} \left[ \mathbf{v} \times (\mathbf{r} \times \mathbf{v}) - \mu \frac{\mathbf{r}}{r} \right] = \frac{1}{\mu} \left[ \overbrace{\mathbf{r} \mathbf{v}^2 - \mathbf{v}(\mathbf{r} \cdot \mathbf{v})}^{\text{bac-cab rule}} - \mu \frac{\mathbf{r}}{r} \right]$$

so that

$$\mathbf{e} = \frac{1}{\mu} \left[ \left( v^2 - \frac{\mu}{r} \right) \mathbf{r} - r \mathbf{v}_r \mathbf{v} \right] \quad (4.10)$$

11. Calculate the eccentricity,  $e = \sqrt{\mathbf{e} \cdot \mathbf{e}}$ , which is the fourth orbital element. Substituting Eq. (4.10) leads to a form depending only on the scalars obtained thus far,

$$e = \sqrt{1 + \frac{h^2}{\mu^2} \left( v^2 - \frac{2\mu}{r} \right)} \quad (4.11)$$

12. Calculate the argument of perigee,

$$\omega = \cos^{-1} \left( \frac{\mathbf{N} \cdot \mathbf{e}}{N e} \right)$$

This is the fifth orbital element. If  $\mathbf{N} \cdot \mathbf{e} > 0$ , then  $\omega$  lies in either the first or fourth quadrant. If  $\mathbf{N} \cdot \mathbf{e} < 0$ , then  $\omega$  lies in either the second or third quadrant. To place  $\omega$  in the proper quadrant, observe that perigee lies above the equatorial plane ( $0^\circ \leq \omega < 180^\circ$ ) if  $\mathbf{e}$  points up (in the positive  $Z$  direction) and that perigee lies below the plane ( $180^\circ \leq \omega < 360^\circ$ ) if  $\mathbf{e}$  points down. Therefore,  $e_Z \geq 0$  implies that  $0^\circ < \omega < 180^\circ$ , whereas  $e_Z < 0$  implies that  $180^\circ < \omega < 360^\circ$ . To summarize,

$$\omega = \begin{cases} \cos^{-1} \left( \frac{\mathbf{N} \cdot \mathbf{e}}{N e} \right) & (e_Z \geq 0) \\ 360^\circ - \cos^{-1} \left( \frac{\mathbf{N} \cdot \mathbf{e}}{N e} \right) & (e_Z < 0) \end{cases} \quad (4.12)$$

13. Calculate the true anomaly,

$$\theta = \cos^{-1} \left( \frac{\mathbf{e} \cdot \mathbf{r}}{e r} \right)$$

This is the sixth and final orbital element. If  $\mathbf{e} \cdot \mathbf{r} > 0$ , then  $\theta$  lies in the first or fourth quadrant. If  $\mathbf{e} \cdot \mathbf{r} < 0$ , then  $\theta$  lies in the second or third quadrant. To place  $\theta$  in the proper quadrant, note that if the satellite is flying away from perigee ( $\mathbf{r} \cdot \mathbf{v} \geq 0$ ), then  $0 \leq \theta < 180^\circ$ , whereas if the satellite is flying toward perigee ( $\mathbf{r} \cdot \mathbf{v} < 0$ ), then  $180^\circ \leq \theta < 360^\circ$ . Therefore, using the results of Step 3 above

$$\theta = \begin{cases} \cos^{-1} \left( \frac{\mathbf{e} \cdot \mathbf{r}}{e \cdot r} \right) & (v_r \geq 0) \\ 360^\circ - \cos^{-1} \left( \frac{\mathbf{e} \cdot \mathbf{r}}{e \cdot r} \right) & (v_r < 0) \end{cases} \quad (4.13a)$$

Substituting Eq. (4.10) yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1} \left[ \frac{1}{e} \left( \frac{h^2}{\mu r} - 1 \right) \right] & (v_r \geq 0) \\ 360^\circ - \cos^{-1} \left[ \frac{1}{e} \left( \frac{h^2}{\mu r} - 1 \right) \right] & (v_r < 0) \end{cases} \quad (4.13b)$$

The procedure described above for calculating the orbital elements is not unique.

### EXAMPLE 4.3

Given the state vector,

$$\begin{aligned} \mathbf{r} &= -6045\hat{\mathbf{i}} - 3490\hat{\mathbf{j}} + 2500\hat{\mathbf{k}} \text{ (km)} \\ \mathbf{v} &= -3.457\hat{\mathbf{i}} + 6.618\hat{\mathbf{j}} + 2.533\hat{\mathbf{k}} \text{ (km/s)} \end{aligned}$$

find the orbital elements  $h$ ,  $i$ ,  $\Omega$ ,  $e$ ,  $\omega$ , and  $\theta$  using Algorithm 4.2.

#### Solution

Step 1:

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{(-6045)^2 + (-3490)^2 + 2500^2} = 7414 \text{ km}$$

Step 2:

$$v = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{(-3.457)^2 + 6.618^2 + 2.533^2} = 7.884 \text{ km/s}$$

Step 3:

$$v_r = \frac{\mathbf{v} \cdot \mathbf{r}}{r} = \frac{(-3.457) \cdot (-6045) + 6.618 \cdot (-3490) + 2.533 \cdot 2500}{7414} = 0.5575 \text{ km/s} \quad (a)$$

Since  $v_r > 0$ , the satellite is flying away from perigee.

Step 4:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -6045 & -3490 & 2500 \\ -3.457 & 6.618 & 2.533 \end{vmatrix} = -25,380\hat{\mathbf{i}} + 6670\hat{\mathbf{j}} - 52,070\hat{\mathbf{k}} \text{ (km}^2/\text{s)}$$

Step 5:

$$h = \sqrt{\mathbf{h} \cdot \mathbf{h}} = \sqrt{(-25,380)^2 + 6670^2 + (-52,070)^2} \Rightarrow \boxed{h = 58,310 \text{ km}^2/\text{s}}$$

Step 6:

$$i = \cos^{-1} \frac{h_z}{h} = \cos^{-1} \left( \frac{-52,070}{58,310} \right) \Rightarrow \boxed{i = 153.2^\circ}$$

Since  $i$  is greater than  $90^\circ$ , this is a retrograde orbit.

Step 7:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ -25,380 & 6670 & -52,070 \end{vmatrix} = -6670\hat{\mathbf{I}} - 25,380\hat{\mathbf{J}} \text{ (km}^2/\text{s)} \quad (\text{b})$$

Step 8:

$$N = \sqrt{\mathbf{N} \cdot \mathbf{N}} = \sqrt{(-6670)^2 + (-25,380)^2} = 26,250 \text{ km}^2/\text{s}$$

Step 9:

$$\Omega = \cos^{-1} \frac{N_X}{N} = \cos^{-1} \left( \frac{-6670}{26,250} \right) = 104.7^\circ \text{ or } 255.3^\circ$$

From Eq. (b) we know that  $N_Y < 0$ ; therefore,  $\Omega$  must lie in the third quadrant,

$$\boxed{\Omega = 255.3^\circ}$$

Step 10:

$$\begin{aligned} \mathbf{e} &= \frac{1}{\mu} \left[ \left( v^2 - \frac{\mu}{r} \right) \mathbf{r} - r v_r \mathbf{v} \right] \\ &= \frac{1}{398,600} \left[ \left( 7.884^2 - \frac{398,600}{7414} \right) (-6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}) \right. \\ &\quad \left. - (7414)(0.5575)(-3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}) \right] \\ \mathbf{e} &= -0.09160\hat{\mathbf{I}} - 0.1422\hat{\mathbf{J}} + 0.02644\hat{\mathbf{K}} \end{aligned} \quad (\text{c})$$

Step 11:

$$e = \sqrt{\mathbf{e} \cdot \mathbf{e}} = \sqrt{(-0.09160)^2 + (-0.1422)^2 + (0.02644)^2} \Rightarrow \boxed{e = 0.1712}$$

Clearly, the orbit is an ellipse.

Step 12:

$$\omega = \cos^{-1} \frac{\mathbf{N} \cdot \mathbf{e}}{Ne} = \cos^{-1} \left[ \frac{(-6670)(-0.09160) + (-25,380)(-0.1422) + (0)(0.02644)}{(26,250)(0.1712)} \right] = 20.07^\circ \text{ or } 339.9^\circ$$

 $\omega$  lies in the first quadrant if  $e_Z > 0$ , which is true in this case, as we see from Eq. (c). Therefore,

$$\boxed{\omega = 20.07^\circ}$$

Step 13:

$$\theta = \cos^{-1} \left( \frac{\mathbf{e} \cdot \mathbf{r}}{er} \right) = \cos^{-1} \left[ \frac{(-0.09160)(-6045) + (-0.1422) \cdot (-3490) + (0.02644)(2500)}{(0.1712)(7414)} \right] = 28.45^\circ \text{ or } 331.6^\circ$$

From Eq. (a) we know that  $v_r > 0$ , which means  $0^\circ \leq \theta < 180^\circ$ . Therefore,

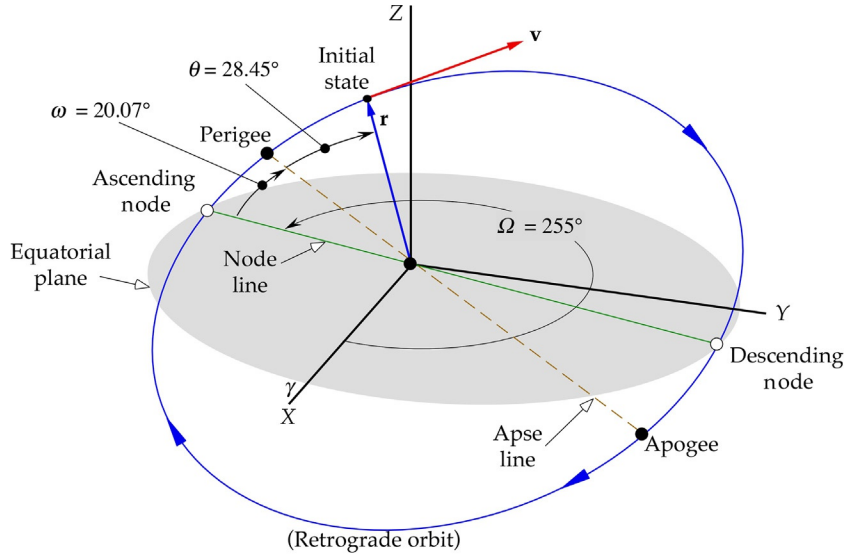
$$\boxed{\theta = 28.45^\circ}$$

Having found the six orbital elements, we can go on to compute other parameters. The perigee and apogee radii are

$$\begin{aligned} r_p &= \frac{h^2}{\mu} \frac{1}{1 + e \cos(0)} = \frac{58,310^2}{398,600} \frac{1}{1 + 0.1712} = 7284 \text{ km} \\ r_a &= \frac{h^2}{\mu} \frac{1}{1 + e \cos(180^\circ)} = \frac{58,310^2}{398,600} \frac{1}{1 - 0.1712} = 10,290 \text{ km} \end{aligned}$$

From these it follows that the semimajor axis of the ellipse is

$$a = \frac{1}{2}(r_p + r_a) = 8788 \text{ km}$$



**FIG. 4.8**

A plot of the orbit identified in Example 4.3.

This leads to the period,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = 2.278 \text{ h}$$

The orbit is illustrated in Fig. 4.8.

We have seen how to obtain the orbital elements from the state vector. To arrive at the state vector, given the orbital elements, requires performing coordinate transformations, which are discussed in the next section.

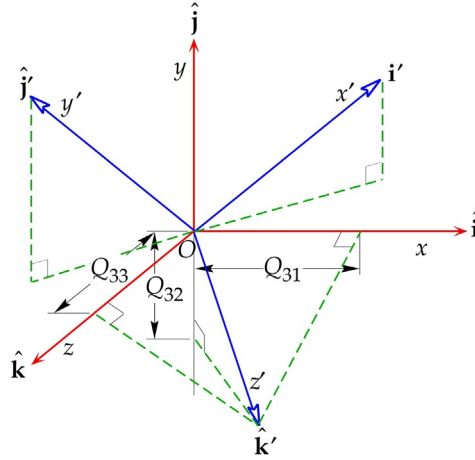
## 4.5 COORDINATE TRANSFORMATION

The Cartesian coordinate system was introduced in Section 1.2. Fig. 4.9 shows two such coordinate systems: the unprimed system with axes  $xyz$ , and the primed system with axes  $x'y'z'$ . The orthogonal unit basis vectors for the unprimed system are  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ . The fact they are unit vectors means

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \quad (4.14)$$

Since they are orthogonal,

$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0 \quad (4.15)$$

**FIG. 4.9**

Two sets of Cartesian reference axes,  $xyz$  and  $x'y'z'$ .

The orthonormal basis vectors  $\hat{\mathbf{i}}'$ ,  $\hat{\mathbf{j}}'$ , and  $\hat{\mathbf{k}}'$  of the primed system share these same properties. That is,

$$\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}' = \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}}' = \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}' = 1 \quad (4.16)$$

and

$$\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}}' = \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}}' = \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}}' = 0 \quad (4.17)$$

We can express the unit vectors of the primed system in terms of their components in the unprimed system as follows:

$$\begin{aligned} \hat{\mathbf{i}}' &= Q_{11}\hat{\mathbf{i}} + Q_{12}\hat{\mathbf{j}} + Q_{13}\hat{\mathbf{k}} \\ \hat{\mathbf{j}}' &= Q_{21}\hat{\mathbf{i}} + Q_{22}\hat{\mathbf{j}} + Q_{23}\hat{\mathbf{k}} \\ \hat{\mathbf{k}}' &= Q_{31}\hat{\mathbf{i}} + Q_{32}\hat{\mathbf{j}} + Q_{33}\hat{\mathbf{k}} \end{aligned} \quad (4.18)$$

The  $Q$ 's in these expressions are just the direction cosines of  $\hat{\mathbf{i}}'$ ,  $\hat{\mathbf{j}}'$ , and  $\hat{\mathbf{k}}'$ . Fig. 4.9 illustrates the components of  $\hat{\mathbf{k}}'$ , which are, of course, the projections of  $\hat{\mathbf{k}}'$  onto the  $x$ ,  $y$ , and  $z$  axes. The unprimed unit vectors may be resolved into components along the primed system to obtain a set of equations similar to Eq. (4.18).

$$\begin{aligned} \hat{\mathbf{i}} &= Q'_{11}\hat{\mathbf{i}}' + Q'_{12}\hat{\mathbf{j}}' + Q'_{13}\hat{\mathbf{k}}' \\ \hat{\mathbf{j}} &= Q'_{21}\hat{\mathbf{i}}' + Q'_{22}\hat{\mathbf{j}}' + Q'_{23}\hat{\mathbf{k}}' \\ \hat{\mathbf{k}} &= Q'_{31}\hat{\mathbf{i}}' + Q'_{32}\hat{\mathbf{j}}' + Q'_{33}\hat{\mathbf{k}}' \end{aligned} \quad (4.19)$$

However,  $\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}'$ , so that, from Eqs. (4.18) and (4.19), we find  $Q_{11} = Q_{11}'$ . Likewise,  $\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}'$ , which, according to Eqs. (4.18) and (4.19), means  $Q_{12} = Q_{21}'$ . Proceeding in this fashion, it is clear



that the direction cosines in Eq. (4.18) may be expressed in terms of those in Eq. (4.19). That is, Eq. (4.19) may be written

$$\begin{aligned}\hat{\mathbf{i}} &= Q_{11}\hat{\mathbf{i}}' + Q_{21}\hat{\mathbf{j}}' + Q_{31}\hat{\mathbf{k}}' \\ \hat{\mathbf{j}} &= Q_{12}\hat{\mathbf{i}}' + Q_{22}\hat{\mathbf{j}}' + Q_{32}\hat{\mathbf{k}}' \\ \hat{\mathbf{k}} &= Q_{13}\hat{\mathbf{i}}' + Q_{23}\hat{\mathbf{j}}' + Q_{33}\hat{\mathbf{k}}'\end{aligned}\quad (4.20)$$

Substituting Eq. (4.20) into Eq. (4.14) and making use of Eqs. (4.16) and (4.17), we get the three relations

$$\begin{aligned}\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} &= 1 \Rightarrow Q_{11}^2 + Q_{21}^2 + Q_{31}^2 = 1 \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} &= 1 \Rightarrow Q_{12}^2 + Q_{22}^2 + Q_{32}^2 = 1 \\ \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} &= 1 \Rightarrow Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1\end{aligned}\quad (4.21)$$

Substituting Eq. (4.20) into Eq. (4.15) and, again, making use of Eqs. (4.16) and (4.17), we obtain the three equations

$$\begin{aligned}\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} &= 0 \Rightarrow Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0 \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} &= 0 \Rightarrow Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} = 0 \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} &= 0 \Rightarrow Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} = 0\end{aligned}\quad (4.22)$$

Let  $[\mathbf{Q}]$  represent the matrix of direction cosines of  $\hat{\mathbf{i}}'$ ,  $\hat{\mathbf{j}}'$ , and  $\hat{\mathbf{k}}'$  relative to  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ , and  $\hat{\mathbf{k}}$ , as given by Eq. (4.18).  $[\mathbf{Q}]$  is referred to as the direction cosine matrix (DCM).

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} \end{bmatrix}\quad (4.23)$$

The transpose of the matrix  $[\mathbf{Q}]$ , denoted  $[\mathbf{Q}]^T$ , is obtained by interchanging the rows and columns of  $[\mathbf{Q}]$ . Thus,

$$[\mathbf{Q}]^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{k}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' \end{bmatrix}\quad (4.24)$$

Forming the product  $[\mathbf{Q}]^T[\mathbf{Q}]$ , we get

$$\begin{aligned}[\mathbf{Q}]^T[\mathbf{Q}] &= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \\ &= \begin{bmatrix} Q_{11}^2 + Q_{21}^2 + Q_{31}^2 & Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} & Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} \\ Q_{12}Q_{11} + Q_{22}Q_{21} + Q_{32}Q_{31} & Q_{12}^2 + Q_{22}^2 + Q_{32}^2 & Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} \\ Q_{13}Q_{11} + Q_{23}Q_{21} + Q_{33}Q_{31} & Q_{13}Q_{12} + Q_{23}Q_{22} + Q_{33}Q_{32} & Q_{13}^2 + Q_{23}^2 + Q_{33}^2 \end{bmatrix}\end{aligned}$$

From this we obtain, with the aid of Eqs. (4.21) and (4.22),

$$[\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{1}]\quad (4.25)$$

where

$$[\mathbf{1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[\mathbf{1}]$  stands for the identity matrix or unit matrix.

In a similar fashion, we can substitute Eq. (4.18) into Eqs. (4.16) and (4.17) and make use of Eqs. (4.14) and (4.15) to finally obtain

$$[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{1}] \quad (4.26)$$

Since  $[\mathbf{Q}]$  satisfies Eqs. (4.25) and (4.26), it is called an orthogonal matrix.

Let  $\mathbf{v}$  be a vector. It can be expressed in terms of its components along the unprimed system

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$$

or along the primed system

$$\mathbf{v} = v'_x \hat{\mathbf{i}}' + v'_y \hat{\mathbf{j}}' + v'_z \hat{\mathbf{k}}'$$

These two expressions for  $\mathbf{v}$  are equivalent ( $\mathbf{v} = \mathbf{v}$ ) since a vector is independent of the coordinate system used to describe it. Thus,

$$v'_x \hat{\mathbf{i}}' + v'_y \hat{\mathbf{j}}' + v'_z \hat{\mathbf{k}}' = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \quad (4.27)$$

Substituting Eq. (4.20) into the right-hand side of Eq. (4.27) yields

$$\begin{aligned} v'_x \hat{\mathbf{i}}' + v'_y \hat{\mathbf{j}}' + v'_z \hat{\mathbf{k}}' &= (Q_{11} \hat{\mathbf{i}} + Q_{21} \hat{\mathbf{j}} + Q_{31} \hat{\mathbf{k}}) v_x \\ &+ (Q_{12} \hat{\mathbf{i}} + Q_{22} \hat{\mathbf{j}} + Q_{32} \hat{\mathbf{k}}) v_y + (Q_{13} \hat{\mathbf{i}} + Q_{23} \hat{\mathbf{j}} + Q_{33} \hat{\mathbf{k}}) v_z \end{aligned}$$

On collecting terms on the right, we get

$$\begin{aligned} v'_x \hat{\mathbf{i}}' + v'_y \hat{\mathbf{j}}' + v'_z \hat{\mathbf{k}}' &= (Q_{11} v_x + Q_{12} v_y + Q_{13} v_z) \hat{\mathbf{i}}' \\ &+ (Q_{21} v_x + Q_{22} v_y + Q_{23} v_z) \hat{\mathbf{j}}' + (Q_{31} v_x + Q_{32} v_y + Q_{33} v_z) \hat{\mathbf{k}}' \end{aligned}$$

Equating the components of like unit vectors on each side of the equals sign yields

$$\begin{aligned} v'_x &= Q_{11} v_x + Q_{12} v_y + Q_{13} v_z \\ v'_y &= Q_{21} v_x + Q_{22} v_y + Q_{23} v_z \\ v'_z &= Q_{31} v_x + Q_{32} v_y + Q_{33} v_z \end{aligned} \quad (4.28)$$

In matrix notation, this may be written

$$\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\} \quad (4.29)$$

where

$$\{\mathbf{v}'\} = \begin{Bmatrix} v'_x \\ v'_y \\ v'_z \end{Bmatrix} \quad \{\mathbf{v}\} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \quad (4.30)$$

and  $[\mathbf{Q}]$  is given by Eq. (4.23). Eq. (4.28) (or Eq. 4.29) shows how to transform the components of the vector  $\mathbf{v}$  in the unprimed system into its components in the primed system. The inverse transformation, from primed to unprimed, is found by multiplying Eq. (4.29) throughout by  $[\mathbf{Q}]^T$ :

$$[\mathbf{Q}]^T \{\mathbf{v}'\} = [\mathbf{Q}]^T [\mathbf{Q}] \{\mathbf{v}\}$$

But, according to Eq. (4.25),  $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{1}]$ , so that

$$[\mathbf{Q}]^T \{\mathbf{v}'\} = [\mathbf{1}] \{\mathbf{v}\}$$

Since  $[\mathbf{1}]\{\mathbf{v}\} = \{\mathbf{v}\}$ , we obtain

$$\{\mathbf{v}\} = [\mathbf{Q}]^T \{\mathbf{v}'\} \quad (4.31)$$

Therefore, to go from the unprimed system to the primed system we use  $[\mathbf{Q}]$ , and in the reverse direction—from primed to unprimed—we use  $[\mathbf{Q}]^T$ .

#### EXAMPLE 4.4

In Fig. 4.10, the  $x'$  axis is defined by the line segment  $O'P$ . The  $x'y'$  plane is defined by the intersecting line segments  $O'P$  and  $O'Q$ . The  $z'$  axis is normal to the plane of  $O'P$  and  $O'Q$  and obtained by rotating  $O'P$  toward  $O'Q$  and using the right-hand rule.

- Find the direction cosine matrix  $[\mathbf{Q}]$ .
- If  $\{\mathbf{v}\} = [2 \ 4 \ 6]^T$ , find  $\{\mathbf{v}'\}$ .
- If  $\{\mathbf{v}'\} = [2 \ 4 \ 6]^T$ , find  $\{\mathbf{v}\}$ .

#### Solution

- Resolve the directed line segments  $\overrightarrow{O'P}$  and  $\overrightarrow{O'Q}$  into components along the unprimed system:

$$\begin{aligned} \overrightarrow{O'P} &= (-5-3)\hat{\mathbf{i}} + (5-1)\hat{\mathbf{j}} + (4-2)\hat{\mathbf{k}} = -8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}} \\ \overrightarrow{O'Q} &= (-6-3)\hat{\mathbf{i}} + (3-1)\hat{\mathbf{j}} + (5-2)\hat{\mathbf{k}} = -9\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} \end{aligned}$$

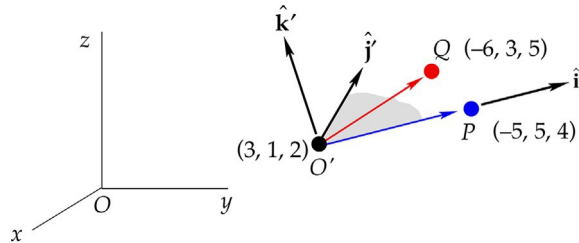


FIG. 4.10

Defining a unit triad from the coordinates of three noncollinear points,  $O'$ ,  $P$ , and  $Q$ .

Taking the cross product of  $\vec{O'P}$  into  $\vec{O'Q}$  yields a vector  $\mathbf{Z}'$ , which lies in the direction of the desired positive  $z'$  axis:

$$\mathbf{Z}' = \vec{O'P} \times \vec{O'Q} = 8\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 20\hat{\mathbf{k}}$$

Taking the cross product of  $\mathbf{Z}'$  into  $\vec{O'P}$  then yields a vector  $\mathbf{Y}'$ , which points in the positive  $y'$  direction:

$$\mathbf{Y}' = \mathbf{Z}' \times \vec{O'P} = -68\hat{\mathbf{i}} + 176\hat{\mathbf{j}} + 80\hat{\mathbf{k}}$$

Normalizing the vectors  $\vec{O'P}$ ,  $\mathbf{Y}'$ , and  $\mathbf{Z}'$  produces the  $\hat{\mathbf{i}}'$ ,  $\hat{\mathbf{j}}'$ , and  $\hat{\mathbf{k}}'$  unit vectors, respectively. Thus

$$\hat{\mathbf{i}}' = \vec{O'P} / \|\vec{O'P}\| = -0.8729\hat{\mathbf{i}} + 0.4364\hat{\mathbf{j}} + 0.2182\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}}' = \mathbf{Y}' / \|\mathbf{Y}'\| = -0.3318\hat{\mathbf{i}} - 0.8588\hat{\mathbf{j}} + 0.3904\hat{\mathbf{k}}$$

$$\hat{\mathbf{k}}' = \mathbf{Z}' / \|\mathbf{Z}'\| = 0.3578\hat{\mathbf{i}} + 0.2683\hat{\mathbf{j}} + 0.8944\hat{\mathbf{k}}$$

The components of  $\hat{\mathbf{i}}'$ ,  $\hat{\mathbf{j}}'$ , and  $\hat{\mathbf{k}}'$  are the rows of the direction cosine matrix  $[\mathbf{Q}]$ . Thus,

$$[\mathbf{Q}] = \begin{bmatrix} -0.8729 & 0.4364 & 0.2182 \\ -0.3318 & -0.8588 & 0.3904 \\ 0.3578 & 0.2683 & 0.8944 \end{bmatrix}$$

(b)

$$\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\} = \begin{bmatrix} -0.8729 & 0.4364 & 0.2182 \\ -0.3318 & -0.8588 & 0.3904 \\ 0.3578 & 0.2683 & 0.8944 \end{bmatrix} \begin{Bmatrix} 2 \\ 4 \\ 6 \end{Bmatrix} \Rightarrow \{\mathbf{v}'\} = \begin{Bmatrix} 1.309 \\ -1.756 \\ 7.155 \end{Bmatrix}$$

(c)

$$\{\mathbf{v}\} = [\mathbf{Q}]^T \{\mathbf{v}'\} = \begin{bmatrix} -0.8729 & -0.3318 & 0.3578 \\ 0.4364 & -0.8588 & 0.2683 \\ 0.2182 & 0.3904 & 0.8944 \end{bmatrix} \begin{Bmatrix} 2 \\ 4 \\ 6 \end{Bmatrix} \Rightarrow \{\mathbf{v}\} = \begin{Bmatrix} -0.9263 \\ -0.9523 \\ 7.364 \end{Bmatrix}$$

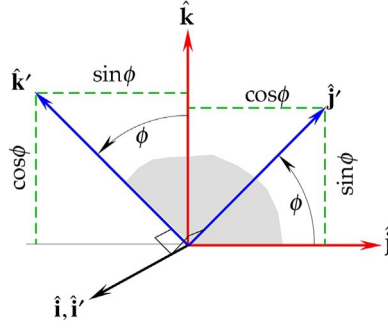
Let us consider the special case in which the coordinate transformation involves a rotation about only one of the coordinate axes, as shown in Fig. 4.11. If the rotation is about the  $x$  axis, then according to Eqs. (4.18) and (4.23),

$$\begin{aligned} \hat{\mathbf{i}}' &= \hat{\mathbf{i}} \\ \hat{\mathbf{j}}' &= (\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{j}} + \cos(90^\circ - \phi)\hat{\mathbf{k}} = \cos\phi\hat{\mathbf{j}} + \sin\phi\hat{\mathbf{k}} \\ \hat{\mathbf{k}}' &= (\hat{\mathbf{k}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} = \cos(90^\circ + \phi)\hat{\mathbf{j}} + \cos\phi\hat{\mathbf{k}} = -\sin\phi\hat{\mathbf{j}} + \cos\phi\hat{\mathbf{k}} \end{aligned}$$

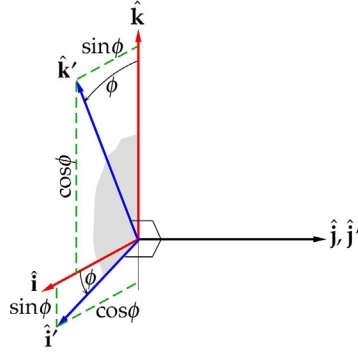
or

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

The transformation from the  $xyz$  coordinate system to the  $x'y'z'$  system having a common  $x$  axis is given by the direction cosine matrix on the right. Since this is a rotation through the angle  $\phi$  about the  $x$  axis, we denote this matrix by  $[\mathbf{R}_1(\phi)]$ , in which the subscript 1 stands for axis 1 (the  $x$  axis). Thus,


**FIG. 4.11**

Rotation about the x axis.


**FIG. 4.12**

Rotation about the y axis.

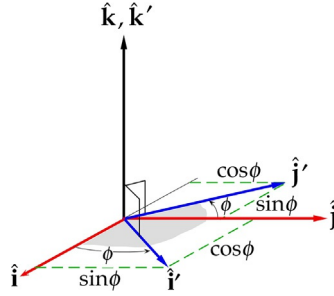
$$[\mathbf{R}_1(\phi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \quad (4.32)$$

If the rotation is about the y axis, as shown in Fig. 4.12, then Eq. (4.18) yields

$$\begin{aligned} \hat{\mathbf{i}}' &= (\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}) \hat{\mathbf{i}} + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} = \cos \phi \hat{\mathbf{i}} + \cos(\phi + 90^\circ) \hat{\mathbf{k}} = \cos \phi \hat{\mathbf{i}} - \sin \phi \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' &= \hat{\mathbf{j}} \\ \hat{\mathbf{k}}' &= (\hat{\mathbf{k}}' \cdot \hat{\mathbf{i}}) \hat{\mathbf{i}} + (\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} = \cos(90^\circ - \phi) \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{k}} = \sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{k}} \end{aligned}$$

or, more compactly,

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

**FIG. 4.13**

Rotation about the  $z$  axis.

We represent this transformation between two Cartesian coordinate systems having a common  $y$  axis (axis 2) as  $[\mathbf{R}_2(\phi)]$ . Therefore,

$$[\mathbf{R}_2(\phi)] = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix} \quad (4.33)$$

Finally, if the rotation is about the  $z$  axis, as shown in Fig. 4.13, then we have from Eq. (4.18) that

$$\begin{aligned} \hat{\mathbf{i}}' &= (\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}) \hat{\mathbf{i}} + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}}) \hat{\mathbf{j}} = \cos \phi \hat{\mathbf{i}} + \cos(90^\circ - \phi) \hat{\mathbf{j}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}} \\ \hat{\mathbf{j}}' &= (\hat{\mathbf{j}}' \cdot \hat{\mathbf{i}}) \hat{\mathbf{i}} + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}}) \hat{\mathbf{j}} = \cos(90^\circ + \phi) \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} \\ \hat{\mathbf{k}}' &= \hat{\mathbf{k}} \end{aligned}$$

or

$$\begin{Bmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{Bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

In this case, the rotation is around axis 3, the  $z$  axis, so

$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.34)$$

The single transformation between the  $xyz$  and  $x'y'z'$  Cartesian coordinate frames can be viewed as a sequence of three coordinate transformations, starting with  $xyz$ :

$$xyz \xrightarrow{\alpha} x_1 y_1 z_1 \xrightarrow{\beta} x_2 y_2 z_2 \xrightarrow{\gamma} x' y' z'$$

Each coordinate system is obtained from the previous one by means of a rotation about one of the axes of the previous frame. Two successive rotations cannot be about the same axis. The first rotation angle is  $\alpha$ , the second one is  $\beta$  and the final one is  $\gamma$ . In specific applications, the Greek letters that are traditionally used to represent the three rotations are not  $\alpha$ ,  $\beta$ , and  $\gamma$ . For those new to the subject, however, it might initially be easier to remember that the first, second, and third rotation angles are represented by the first, second, and third letters of the Greek alphabet ( $\alpha\beta\gamma$ ). Each one of the three transformations has the direction cosine matrix  $[\mathbf{R}_i(\phi)]$ , where  $i = 1, 2$ , or  $3$  and  $\phi = \alpha, \beta$  or  $\gamma$ . The sequence of three such elementary rotations relating two different Cartesian frames of reference is called an Euler angle sequence. Each of the 12 possible Euler angle sequences has a direction cosine matrix  $[\mathbf{Q}]$ , which is the product of three elementary rotation matrices. The six symmetric Euler sequences are those that begin and end with rotation about the same axis:

$$\begin{aligned} & [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] \quad [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] \\ & [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] \quad [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ & [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \quad [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \end{aligned} \quad (4.35)$$

The asymmetric Euler sequences involve rotations about all three axes:

$$\begin{aligned} & [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \quad [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ & [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] \quad [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \\ & [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] \quad [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] \end{aligned} \quad (4.36)$$

One of the symmetric sequences that has frequent application in space mechanics is the “classical” Euler angle sequence,

$$[\mathbf{Q}] = [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \quad (0 \leq \alpha < 360^\circ \quad 0 \leq \beta \leq 180^\circ \quad 0 \leq \gamma < 360^\circ) \quad (4.37)$$

which is underlined in Eq. (4.35) and illustrated in Fig. 4.14. The first rotation is around the  $z$  axis, through the angle  $\alpha$ . It rotates the  $x$  and  $y$  axes into the  $x_1$  and  $y_1$  directions. Viewed down the  $z$  axis, this rotation appears as shown in the insert at the top of the figure. The direction cosine matrix associated with this rotation is  $[\mathbf{R}_3(\alpha)]$ . The subscript means that the rotation is around the current (3) direction, which was the  $z$  axis (and is now the  $z_1$  axis). The second rotation, represented by  $[\mathbf{R}_1(\beta)]$ , is around the  $x_1$  axis through the angle  $\beta$  required to rotate the  $z_1$  axis into the  $z_2$  axis, which coincides with the target  $z'$  axis. Simultaneously,  $y_1$  rotates into  $y_2$ . The insert in the lower right of Fig. 4.14 shows how this rotation appears when viewed from the  $x_1$  direction.  $[\mathbf{R}_3(\gamma)]$  represents the third and final rotation, which rotates the  $x_2$  axis (formerly the  $x_1$  axis) and the  $y_2$  axis through the angle  $\gamma$  around the  $z'$  axis, so that they become aligned with the target  $x'$  and  $y'$  axes, respectively. This rotation appears from the  $z'$  direction as shown in the insert on the left of Fig. 4.14.

Applying the transformation in Eq. (4.37) to the  $xyz$  components  $\{\mathbf{b}\}_x$  of the vector  $\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$  yields the components of the same vector in the  $x'y'z'$  frame

$$\{\mathbf{b}\}_{x'} = [\mathbf{Q}]\{\mathbf{b}\}_x \quad (\mathbf{b} = b_{x'}\hat{\mathbf{i}}' + b_{y'}\hat{\mathbf{j}}' + b_{z'}\hat{\mathbf{k}}')$$

That is

$$[\mathbf{Q}]\{\mathbf{b}\}_x = [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)]\overbrace{[\mathbf{R}_3(\alpha)]\{\mathbf{b}\}_x}^{\{\mathbf{b}\}_{x_1}} = [\mathbf{R}_3(\gamma)]\overbrace{[\mathbf{R}_1(\beta)]\{\mathbf{b}\}_{x_1}}^{\{\mathbf{b}\}_{x_2}} = \overbrace{[\mathbf{R}_3(\gamma)]\{\mathbf{b}\}_{x_2}}^{\{\mathbf{b}\}_{x'}}$$

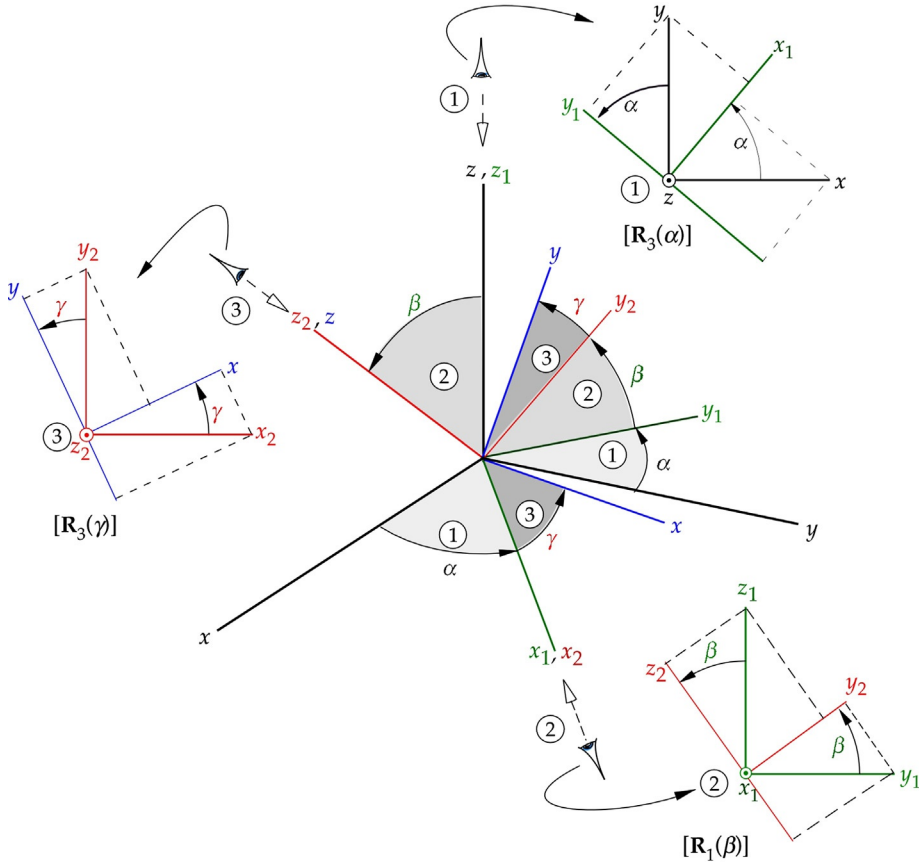


FIG. 4.14

Classical Euler sequence of three rotations transforming  $xyz$  into  $x'y'z'$ . The “eye” viewing down an axis sees the illustrated rotation about that axis.

The column vector  $\{\mathbf{b}\}_{x_1}$  contains the components of the vector  $\mathbf{b}$  ( $\mathbf{b} = b_{x_1}\hat{\mathbf{i}}_1 + b_{y_1}\hat{\mathbf{j}}_1 + b_{z_1}\hat{\mathbf{k}}_1$ ) in the first intermediate frame  $x_1y_1z_1$ . The column vector  $\{\mathbf{b}\}_{x_2}$  contains the components of the vector  $\mathbf{b}$  ( $\mathbf{b} = b_{x_2}\hat{\mathbf{i}}_2 + b_{y_2}\hat{\mathbf{j}}_2 + b_{z_2}\hat{\mathbf{k}}_2$ ) in the second intermediate frame  $x_2y_2z_2$ . Finally, the column vector  $\{\mathbf{b}\}_{x'}$  contains the components in the target  $x'y'z'$  frame.

Substituting Eqs. (4.32) and (4.34) into Eq. (4.37) yields the direction cosine matrix of the classical Euler sequence  $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$ ,

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\alpha\cos\beta\sin\gamma - \cos\alpha\cos\gamma & \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix} \quad (4.38)$$



From this we can see that, given a direction cosine matrix  $[\mathbf{Q}]$ , the angles of the classical Euler sequence may be found as follows:

$$\tan \alpha = \frac{Q_{31}}{-Q_{32}} \quad \cos \beta = Q_{33} \quad \tan \gamma = \frac{Q_{13}}{Q_{23}} \quad (\text{Classical Euler angle sequence}) \quad (4.39)$$

We see that  $\beta = \cos^{-1} Q_{33}$ . There is no quadrant uncertainty because the principal values of the arc-cosine function coincide with the range of the angle  $\beta$  given in Eq. (4.37) ( $0^\circ$  to  $180^\circ$ ). Finding  $\alpha$  and  $\gamma$  involves computing the inverse tangent (arctan), whose principal values lie in the range  $-90^\circ$  to  $+90^\circ$ , whereas the range of both  $\alpha$  and  $\gamma$  is  $0^\circ$  to  $360^\circ$ . Placing  $\tan^{-1}(y/x)$  in the correct quadrant is accomplished by taking into consideration the signs of  $x$  and  $y$ . The MATLAB function `atan2d_0_360.m` in [Appendix D.19](#) does just that.

### ALGORITHM 4.3

Given the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

find the angles  $\alpha\beta\gamma$  of the classical Euler rotation sequence. This algorithm is implemented by the MATLAB function `dcm_to_euler.m` in [Appendix D.20](#).

1.  $\alpha = \tan^{-1}(-Q_{31}/Q_{32}) \quad (0 \leq \alpha < 360^\circ)$
2.  $\beta = \cos^{-1} Q_{33} \quad (0 \leq \beta \leq 180^\circ)$
3.  $\gamma = \tan^{-1}(Q_{13}/Q_{23}) \quad (0 \leq \gamma < 360^\circ)$

### EXAMPLE 4.5

If the direction cosine matrix for the transformation from  $xyz$  to  $x'y'z'$  is

$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75309 & -0.15038 \\ 0.76737 & -0.63530 & 0.086823 \\ -0.30152 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  of the classical Euler sequence.

#### Solution

We should first verify that the matrix is indeed orthogonal by checking Eq. (4.25) (or Eq. 4.26).

$$[\mathbf{Q}]^T[\mathbf{Q}] = \begin{bmatrix} 1 & -3.5042(10^{-7}) & 4.2773(10^{-7}) \\ -3.5042(10^{-7}) & 1 & 3.4866(10^{-6}) \\ 4.2773(10^{-7}) & 3.4866(10^{-6}) & 1 \end{bmatrix}$$

The off-diagonal elements should all be zero, but they are not due to truncation of the numbers in the original matrix. Since their magnitudes are very much smaller than unity, we may deem  $[\mathbf{Q}]$  as very close to being orthogonal.

Use Algorithm 4.3.

Step 1:

$$\alpha = \tan^{-1}\left(\frac{Q_{31}}{-Q_{32}}\right) = \tan^{-1}\left(\frac{-0.030152}{-[-0.17101]}\right)$$

Since the numerator is negative and the denominator is positive,  $\alpha$  must lie in the fourth quadrant. Thus

$$\tan^{-1}\left(\frac{-0.030152}{-[-0.17101]}\right) = \tan^{-1}(-0.17632) = -10^\circ \Rightarrow \boxed{\alpha = 350^\circ}$$

Step 2:

$$\beta = \cos^{-1}Q_{33} = \cos^{-1}(-0.98481) = \boxed{170.0^\circ}$$

Step 3:

$$\gamma = \tan^{-1}\frac{Q_{13}}{Q_{23}} = \tan^{-1}\left(\frac{-0.15038}{0.086823}\right)$$

The numerator is negative and the denominator is positive, so  $\gamma$  lies in the fourth quadrant,

$$\tan^{-1}\left(\frac{-0.15038}{0.086824}\right) = \tan^{-1}(-0.17320) = -60^\circ \Rightarrow \boxed{\gamma = 300^\circ}$$

Another commonly used set of Euler angles for rotating  $xyz$  into alignment with  $x'y'z'$  is the asymmetric “yaw, pitch, and roll” sequence underlined in Eq. 4.36,

$$[\mathbf{Q}] = [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \quad (0 \leq \alpha < 360^\circ \quad -90^\circ < \beta < 90^\circ \quad 0 \leq \gamma < 360^\circ) \quad (4.40)$$

It is illustrated in Fig. 4.15.

The first rotation  $[\mathbf{R}_3(\alpha)]$  is about the  $z$  axis through the angle  $\alpha$ . It carries the  $y$  axis into the  $y_1$  axis normal to the plane of  $z$  and  $x'$ , while rotating the  $x$  axis into  $x_1$ . This rotation appears as shown in the insert at the top right of Fig. 4.15. The second rotation  $[\mathbf{R}_2(\beta)]$ , shown in the auxiliary view at the bottom right of the figure, is a pitch around  $y_1$  through the angle  $\beta$ . This carries the  $x_1$  axis into  $x_2$ , lined up with the target  $x'$  direction, and rotates the original  $z$  axis (now  $z_1$ ) into  $z_2$ . The final rotation  $[\mathbf{R}_1(\gamma)]$  is a roll through the angle  $\gamma$  around the  $x_2$  axis so as to carry  $y_2$  (originally  $y_1$ ) and  $z_2$  into alignment with the target  $y'$  and  $z'$  axes.

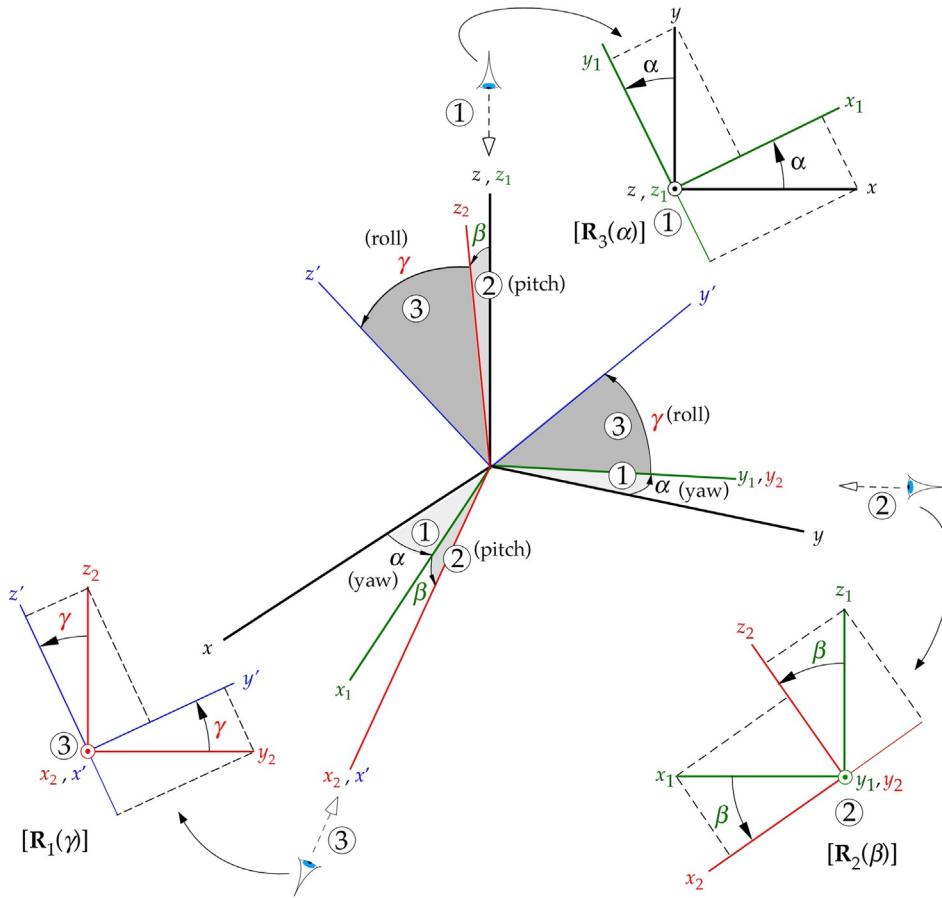
Substituting Eqs. (4.32)–(4.34) into Eq. (4.40) yields the direction cosine matrix for the yaw, pitch, and roll sequence,

$$[\mathbf{Q}] = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\beta & \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{bmatrix} \quad (4.41)$$

From this it is apparent that

$$\tan\alpha = \frac{Q_{12}}{Q_{11}} \quad \sin\beta = -Q_{13} \quad \tan\gamma = \frac{Q_{23}}{Q_{33}} \quad (\text{Yaw, pitch and roll sequence}) \quad (4.42)$$

For  $\beta$  we simply compute  $\sin^{-1}(-Q_{13})$ . There is no quadrant uncertainty because the principal values of the arcsine function coincide with the range of the pitch angle ( $-90^\circ < \beta < 90^\circ$ ). Finding  $\alpha$  and  $\gamma$  involves computing the inverse tangent, so we must once again be careful to place the results of these calculations in the range  $0^\circ$ – $360^\circ$ . As pointed out previously, the MATLAB function `atan2d_0_360.m` in Appendix D.19 takes care of that.



**FIG. 4.15**

Yaw, pitch, and roll sequence transforming  $xyz$  into  $x'y'z'$ .

#### ALGORITHM 4.4

Given the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

Find the angles  $\alpha\beta\gamma$  of the yaw, pitch, and roll sequence. This algorithm is implemented by the MATLAB function *dcm\_to\_ypr.m* in [Appendix D.21](#).

1.  $\alpha = \tan^{-1}(Q_{12}/Q_{11}) \quad (0 \leq \alpha < 360^\circ)$
2.  $\beta = \sin^{-1}(-Q_{13}) \quad (-90^\circ < \beta < 90^\circ)$
3.  $\gamma = \tan^{-1}(Q_{23}/Q_{33}) \quad (0 \leq \gamma < 360^\circ)$

### EXAMPLE 4.6

If the direction cosine matrix for the transformation from  $xyz$  to  $x'y'z'$  is the same as it was in Example 4.5,

$$[Q] = \begin{bmatrix} 0.64050 & 0.75309 & -0.15038 \\ 0.76737 & -0.63530 & 0.086823 \\ -0.30152 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  of the yaw, pitch, and roll sequence.

#### Solution

Use Algorithm 4.4.

Step 1:

$$\alpha = \tan^{-1} \frac{Q_{12}}{Q_{11}} = \tan^{-1} \left( \frac{0.75309}{0.64050} \right)$$

Since both the numerator and the denominator are positive,  $\alpha$  must lie in the first quadrant. Thus,

$$\tan^{-1} \left( \frac{0.75309}{0.64050} \right) = \tan^{-1} 1.1758 = \boxed{49.62^\circ}$$

Step 2:

$$\beta = \sin^{-1}(-Q_{13}) = \sin^{-1}[-(-0.15038)] = \sin^{-1}(0.15038) = \boxed{8.649^\circ}$$

Step 3:

$$\gamma = \tan^{-1} \frac{Q_{23}}{Q_{33}} = \tan^{-1} \left( \frac{0.086823}{-0.98481} \right)$$

The numerator is positive and the denominator is negative, so  $\gamma$  lies in the second quadrant,

$$\tan^{-1} \left( \frac{0.086823}{-0.98481} \right) = \tan^{-1}(-0.088162) = -5.0383^\circ \Rightarrow \boxed{\gamma = 174.96^\circ}$$

## 4.6 TRANSFORMATION BETWEEN GEOCENTRIC EQUATORIAL AND PERIFOCAL FRAMES

The perifocal frame of reference for a given orbit was introduced in Section 2.10. Fig. 4.16 illustrates the relationship between the perifocal and geocentric equatorial frames. Since the orbit lies in the  $\bar{x}\bar{y}$  plane, the components of the state vector of a body relative to its perifocal reference are, according to Eqs. (2.119) and (2.125),

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \frac{h^2}{\mu} \frac{1}{1 + e \cos \theta} (\cos \theta \hat{\mathbf{p}} + \sin \theta \hat{\mathbf{q}}) \quad (4.43)$$

$$\mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \frac{\mu}{h} [-\sin \theta \hat{\mathbf{p}} + (e + \cos \theta) \hat{\mathbf{q}}] \quad (4.44)$$

In matrix notation, these may be written

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu} \frac{1}{1 + e \cos \theta} \begin{Bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{Bmatrix} \quad (4.45)$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h} \begin{Bmatrix} -\sin \theta \\ e + \cos \theta \\ 0 \end{Bmatrix} \quad (4.46)$$

The subscript  $\bar{x}$  is shorthand for “the  $\bar{x}\bar{y}\bar{z}$  coordinate system” and is used to indicate that the components of these vectors are given in the perifocal frame, as opposed to, say, the geocentric equatorial frame (Eqs. 4.2 and 4.3).

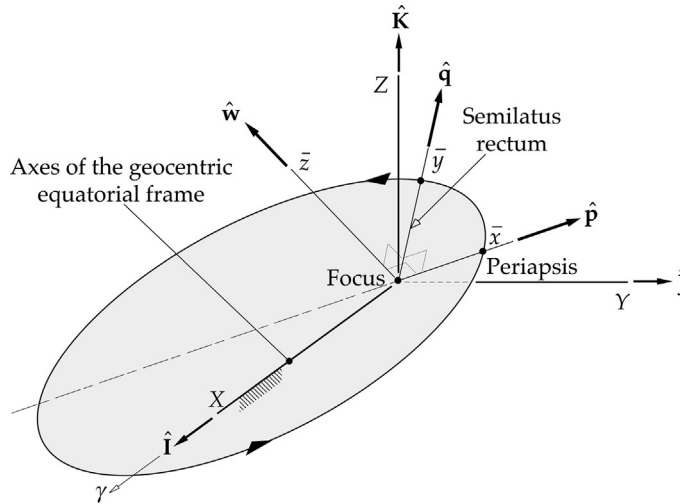


FIG. 4.16

Perifocal ( $\bar{x}\bar{y}\bar{z}$ ) and geocentric equatorial ( $XYZ$ ) frames.

The transformation from the geocentric equatorial frame into the perifocal frame may be accomplished by the classical Euler angle sequence  $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$  in Eq. (4.37) (see Fig. 4.7). In this case, the first rotation angle is  $\Omega$ , the right ascension of the ascending node. The second rotation is  $i$ , the orbital inclination angle, and the third rotation angle is  $\omega$ , the argument of perigee.  $\Omega$  is measured around the  $Z$  axis of the geocentric equatorial frame,  $i$  is measured around the node line, and  $\omega$  is measured around the  $\bar{z}$  axis of the perifocal frame. Therefore, the direct cosine matrix  $[\mathbf{Q}]_{X\bar{x}}$  of the transformation from  $XYZ$  to  $\bar{x}\bar{y}\bar{z}$  is

$$[\mathbf{Q}]_{X\bar{x}} = [\mathbf{R}_3(\omega)][\mathbf{R}_1(i)][\mathbf{R}_3(\Omega)] \quad (4.47)$$

From Eq. (4.38) we get

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \quad (4.48)$$

Remember that this is an orthogonal matrix, which means that the inverse transformation  $[\mathbf{Q}]_{\bar{x}X}$  from  $\bar{x}\bar{y}\bar{z}$  to  $XYZ$ , is given by  $[\mathbf{Q}]_{\bar{x}X} = ([\mathbf{Q}]_{X\bar{x}})^T$ , or

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \sin\Omega\sin i \\ \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & -\cos\Omega\sin i \\ \sin\Omega\sin i & \sin i\cos\omega & \cos i \end{bmatrix} \quad (4.49)$$

If the components of the state vector are given in the geocentric equatorial frame,

$$\{\mathbf{r}\}_X = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad \{\mathbf{v}\}_X = \begin{Bmatrix} v_X \\ v_Y \\ v_Z \end{Bmatrix}$$

then the components in the perifocal frame are found by carrying out the matrix multiplications

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} \bar{x} \\ \bar{y} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{r}\}_X \quad \{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{v}\}_X \quad (4.50)$$

Likewise, the transformation from perifocal to geocentric equatorial components is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} \quad \{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} \quad (4.51)$$

#### ALGORITHM 4.5

Given the orbital elements  $h$ ,  $e$ ,  $i$ ,  $\Omega$ ,  $\omega$ , and  $\theta$ , compute the state vectors  $\mathbf{r}$  and  $\mathbf{v}$  in the geocentric equatorial frame of reference. A MATLAB implementation of this procedure is listed in [Appendix D.22](#). This algorithm can be applied to orbits around other planets or the sun.

1. Calculate position vector  $\{\mathbf{r}\}_{\bar{x}}$  in perifocal coordinates using Eq. (4.45).
2. Calculate velocity vector  $\{\mathbf{v}\}_{\bar{x}}$  in perifocal coordinates using Eq. (4.46).
3. Calculate the matrix  $[\mathbf{Q}]_{\bar{x}X}$  of the transformation from perifocal to geocentric equatorial coordinates using Eq. (4.49).
4. Transform  $\{\mathbf{r}\}_{\bar{x}}$  and  $\{\mathbf{v}\}_{\bar{x}}$  into the geocentric frame by means of Eq. (4.51).

### EXAMPLE 4.7

For a given earth orbit, the elements are  $h = 80,000 \text{ km}^2/\text{s}$ ,  $e = 1.4$ ,  $i = 30^\circ$ ,  $\Omega = 40^\circ$ ,  $\omega = 60^\circ$ , and  $\theta = 30^\circ$ . Using Algorithm 4.5, find the state vectors  $\mathbf{r}$  and  $\mathbf{v}$  in the geocentric equatorial frame.

#### Solution

Step 1:

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu} \frac{1}{1 + e \cos \theta} \begin{Bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{Bmatrix} = \frac{80,000^2}{398,600} \frac{1}{1 + 1.4 \cos 30^\circ} \begin{Bmatrix} \cos 30^\circ \\ \sin 30^\circ \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6285.0 \\ 3628.6 \\ 0 \end{Bmatrix} \text{ (km)}$$

Step 2:

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h} \begin{Bmatrix} -\sin \theta \\ e + \cos \theta \\ 0 \end{Bmatrix} = \frac{398,600}{80,000} \begin{Bmatrix} -\sin 30^\circ \\ 1.4 + \cos 30^\circ \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.4913 \\ 11.290 \\ 0 \end{Bmatrix} \text{ (km/s)}$$

Step 3:

$$\begin{aligned} [\mathbf{Q}]_{\bar{x}\bar{x}} &= \begin{bmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos 60^\circ & \sin 60^\circ & 0 \\ -\sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & \sin 30^\circ \\ 0 & -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} \cos 40^\circ & \sin 40^\circ & 0 \\ -\sin 40^\circ & \cos 40^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.099068 & 0.89593 & 0.43301 \\ -0.94175 & -0.22496 & 0.25000 \\ 0.32139 & -0.38302 & 0.86603 \end{bmatrix} \end{aligned}$$

This is the direction cosine matrix for  $XYZ \rightarrow \bar{x} \bar{y} \bar{z}$ . The transformation matrix for  $\bar{x} \bar{y} \bar{z} \rightarrow XYZ$  is the transpose,

$$[\mathbf{Q}]_{\bar{x}\bar{x}} = \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25000 & 0.86603 \end{bmatrix}$$

Step 4:

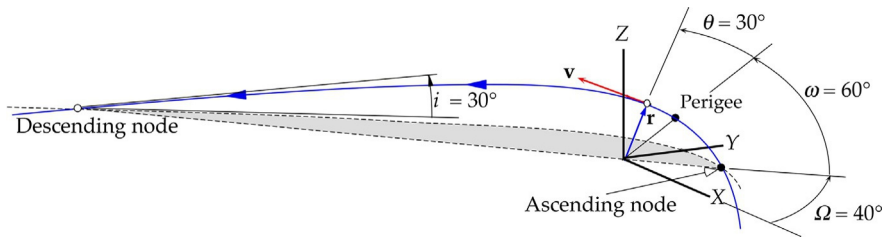
The geocentric equatorial position vector is

$$\begin{aligned} \{\mathbf{r}\}_X &= [\mathbf{Q}]_{\bar{x}\bar{x}} \{\mathbf{r}\}_{\bar{x}} \\ &= \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25 & 0.86603 \end{bmatrix} \begin{Bmatrix} 6285.0 \\ 3628.6 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -4040 \\ 4815 \\ 3629 \end{Bmatrix} \text{ (km)} \end{aligned} \quad (a)$$

whereas the geocentric equatorial velocity vector is

$$\begin{aligned} \{\mathbf{v}\}_X &= [\mathbf{Q}]_{\bar{x}\bar{x}} \{\mathbf{v}\}_{\bar{x}} \\ &= \begin{bmatrix} -0.099068 & -0.94175 & 0.32139 \\ 0.89593 & -0.22496 & -0.38302 \\ 0.43301 & 0.25 & 0.86603 \end{bmatrix} \begin{Bmatrix} -2.4913 \\ 11.290 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -10.39 \\ -4.772 \\ 1.744 \end{Bmatrix} \text{ (km/s)} \end{aligned}$$

The state vectors  $\mathbf{r}$  and  $\mathbf{v}$  are shown in Fig. 4.17. By holding all the orbital parameters except the true anomaly fixed and allowing  $\theta$  to take on a range of values, we generate a sequence of position vectors  $\{\mathbf{r}\}_{\bar{x}}$  from Eq. (4.45). Each of these is



**FIG. 4.17**  
A portion of the hyperbolic trajectory of Example 4.7.

projected into the geocentric equatorial frame as in Eq. (a), using repeatedly the same transformation matrix  $[Q]_{I_{\infty}X}$ . By connecting the end points of all the position vectors  $\{r\}_X$ , we trace out the trajectory illustrated in Fig. 4.17.

### 4.7 EFFECTS OF THE EARTH'S OBLATENESS

The earth, like all planets with comparable or higher rotational rates, bulges out at the equator because of centrifugal force. The earth's equatorial radius is 21 km (13 miles) larger than the polar radius. This flattening at the poles is called oblateness, which is defined as follows:

$$\text{Oblateness} = \frac{\text{Equatorial radius} - \text{Polar radius}}{\text{Equatorial radius}}$$

The earth is an oblate spheroid, lacking the perfect symmetry of a sphere (a basketball can be made an oblate spheroid by sitting on it). This lack of symmetry means that the force of gravity on an orbiting body is not directed toward the center of the earth. Although the gravitational field of a perfectly spherical planet depends only on the distance from its center, oblateness causes a variation also with latitude (i.e., the angular distance from the equator (or pole)). This is called a zonal variation. The dimensionless parameter that quantifies the major effects of oblateness on orbits is  $J_2$ , the second zonal harmonic.  $J_2$  is not a universal constant. Each planet has its own value, as illustrated in Table 4.3, which lists variations of  $J_2$  as well as oblateness.

Table 4.3 Oblateness and second zonal harmonics		
Planet	Oblateness	$J_2$
Mercury	0.000	$60(10^{-6})$
Venus	0.000	$4.458(10^{-6})$
Earth	0.003353	$1.08263(10^{-3})$
Mars	0.00648	$1.96045(10^{-3})$
Jupiter	0.06487	$14.736(10^{-3})$
Saturn	0.09796	$16.298(10^{-3})$
Uranus	0.02293	$3.34343(10^{-3})$
Neptune	0.01708	$3.411(10^{-3})$
(Moon)	0.0012	$202.7(10^{-6})$



Oblateness causes the right ascension  $\Omega$  and the argument of periapsis  $\omega$  to vary significantly with time. In Chapter 10, we will show that the average rates of change of these two angles are

$$\dot{\Omega} = - \left[ \frac{3}{2} \frac{\sqrt{\mu} J_2 R^2}{(1-e^2)^2 a^{7/2}} \right] \cos i \quad (4.52)$$

and

$$\dot{\omega} = - \left[ \frac{3}{2} \frac{\sqrt{\mu} J_2 R^2}{(1-e^2)^2 a^{7/2}} \right] \left( \frac{5}{2} \sin^2 i - 2 \right) \quad (4.53)$$

where  $R$  and  $\mu$  are the radius and gravitational parameter of the planet, respectively;  $a$  and  $e$  are the semimajor axis and eccentricity of the orbit, respectively; and  $i$  is the orbit's inclination.

In Eq. (4.52), observe that if  $0^\circ \leq i < 90^\circ$ , then  $\dot{\Omega} < 0$ . That is, for prograde orbits, the node line drifts westward. Since the right ascension of the node continuously decreases, this phenomenon is called regression of the nodes. If  $90^\circ < i \leq 180^\circ$ , we see that  $\dot{\Omega} > 0$ . The node line of retrograde orbits therefore advances eastward. For polar orbits ( $i = 90^\circ$ ), the node line is stationary.

In Eq. (4.53), we see that if  $0^\circ \leq i < 63.4^\circ$  or  $116.6^\circ < i \leq 180^\circ$ , then  $\dot{\omega}$  is positive, which means the perigee advances in the direction of the motion of the satellite (hence, the name “advance of perigee” for this phenomenon). If  $63.4^\circ < i \leq 116.6^\circ$ , the perigee regresses, moving opposite to the direction of motion.  $i = 63.4^\circ$  and  $i = 116.6^\circ$  are the critical inclinations at which the apse line does not move.

Observe that the coefficient of the trigonometric terms in Eqs. (4.52) and (4.53) are identical, so that

$$\dot{\omega} = \dot{\Omega} \frac{(5/2) \sin^2 i - 2}{\cos i} \quad (4.54)$$

Fig. 4.18 is a plot of Eqs. (4.52) and (4.53) for several circular low earth orbits. The effect of oblateness on both  $\dot{\Omega}$  and  $\dot{\omega}$  is greatest at low inclinations, for which the orbit is near the equatorial bulge for longer portions of each revolution. The effect decreases with increasing semimajor axis because the satellite

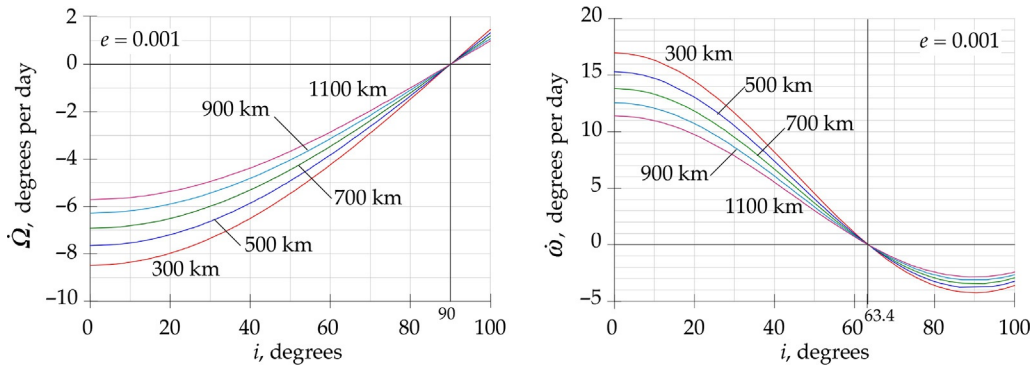


FIG. 4.18

Regression of the node and advance of perigee for nearly circular orbits of altitudes 300–1100 km.

becomes farther from the bulge and its gravitational influence. Obviously,  $\dot{\Omega} = \dot{\omega} = 0$  if  $J_2 = 0$  (no equatorial bulge).

It turns out (Chapter 10) that the  $J_2$  effect produces zero time-averaged variations of the inclination, eccentricity, angular momentum, and semimajor axis.

### EXAMPLE 4.8

A spacecraft is in a 280 km by 400 km orbit with an inclination of  $51.43^\circ$ . Find the rates of node regression and perigee advance.

#### Solution

The perigee and apogee radii are

$$r_p = 6378 + 280 = 6658 \text{ km} \quad r_a = 6378 + 400 = 6778 \text{ km}$$

Therefore, the eccentricity and semimajor axis are

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.008931 \quad a = \frac{r_a + r_p}{2} = 6718 \text{ km}$$

From Eq. (4.52), we obtain the rate of node line regression.

$$\dot{\Omega} = - \left[ \frac{3 \sqrt{398,600} \times 0.0010826 \times 6378^2}{2 (1 - 0.0089312^2)^2 \times 6718^{7/2}} \right] \cos 51.43^\circ = -1.0465 (10^{-6}) \text{ rad/s}$$

or

$$\dot{\Omega} = 5.181 \text{ degrees per day to the west}$$

From Eq. 4.54,

$$\dot{\omega} = -1.0465 (10^{-6}) \cdot \frac{(5/2) \sin^2 51.43^\circ - 2}{\cos 51.43^\circ} = +7.9193 (10^{-7}) \text{ rad/s}$$

or

$$\dot{\omega} = 3.920 \text{ degrees per day in the flight direction}$$

The effect of orbit inclination on node regression and advance of perigee is taken advantage of for two very important types of orbits. Sun-synchronous orbits are those whose orbital plane makes a constant angle  $\alpha$  with the radial from the sun, as illustrated in Fig. 4.19. For that to occur, the orbital plane must rotate in inertial space with the angular velocity of the earth in its orbit around the sun, which is  $360^\circ$  per 365.26 days, or  $0.9856^\circ$  per day. With the orbital plane precessing eastward at this rate, the ascending node will lie at a fixed local time. In the illustration, it happens to be 3 p.m. During every orbit, the satellite sees any given swath of the planet under nearly the same conditions of daylight or darkness day after day. The satellite also has a constant perspective on the sun. Sun-synchronous satellites, like the NOAA Polar-orbiting Operational Environmental Satellites and those of the Defense Meteorological Satellite Program, are used for global weather coverage, while Landsat and the French SPOT series are intended for high-resolution earth observation.

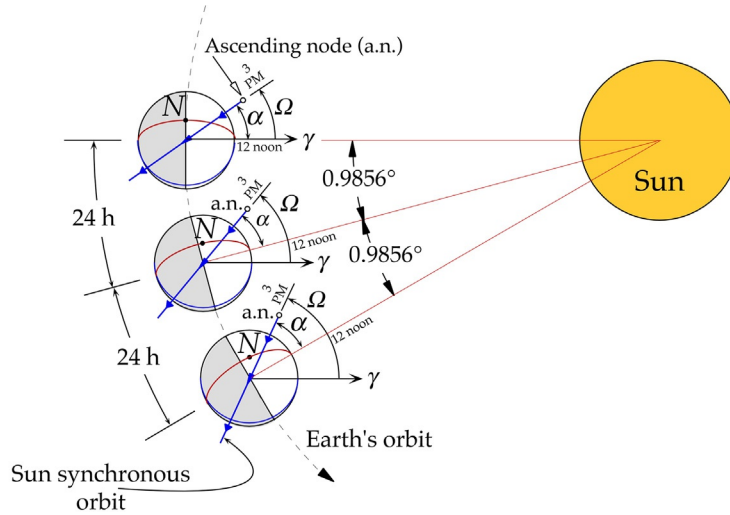


FIG. 4.19

Sun-synchronous orbit.

### EXAMPLE 4.9

A satellite is to be launched into a sun-synchronous circular orbit with a period of 100 min. Determine the required altitude and inclination of its orbit.

#### Solution

We find the altitude  $z$  from the period relation for a circular orbit, Eq. (2.64):

$$T = \frac{2\pi}{\sqrt{\mu}} (R_E + z)^{3/2} \Rightarrow 100 \times 60 = \frac{2\pi}{\sqrt{398,600}} (6378 + z)^{3/2} \Rightarrow \boxed{z = 758.63 \text{ km}}$$

For a sun-synchronous orbit, the ascending node must advance at the rate

$$\dot{\Omega} = \frac{2\pi \text{ rad}}{365.26 \times 24 \times 3600 \text{ s}} = 1.991 (10^{-7}) \text{ rad/s}$$

Substituting this and the altitude into Eq. (4.47), we obtain,

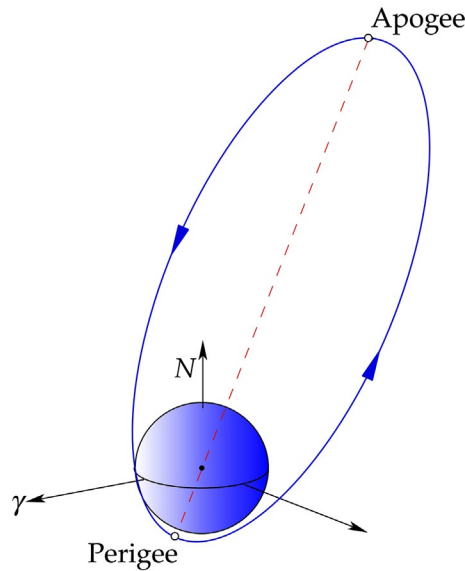
$$1.991 \times 10^{-7} = - \left[ \frac{3 \sqrt{398,600} \times 0.00108263 \times 6378^2}{2 (1 - 0^2)^2 (6378 + 758.63)^{7/2}} \right] \cos i \Rightarrow \cos i = -0.14658$$

Thus, the inclination of the orbit is

$$\boxed{i = 98.43^\circ}$$

This illustrates the fact that sun-synchronous orbits are very nearly polar orbits ( $i = 90^\circ$ ).

If a satellite is launched into an orbit with an inclination of  $63.4^\circ$  (prograde) or  $116.6^\circ$  (retrograde), then Eq. (4.53) shows that the apse line will remain stationary. The Russian space program made this a

**FIG. 4.20**

A typical Molniya orbit (to scale).

key element in the design of the system of Molniya (“lightning”) communications satellites. All the Russian launch sites are above  $45^\circ$  latitude, with the northernmost, Plesetsk, being located at  $62.8^\circ\text{N}$ . As we shall see in [Chapter 6](#), launching a satellite into a geostationary orbit would involve a costly plane change maneuver. Furthermore, recall from Example 2.4 that a geostationary satellite cannot view effectively the far northern latitudes into which Russian territory extends.

The Molniya telecommunications satellites are launched from Plesetsk into  $63^\circ$  inclination orbits having a period of 12 h. From Eq. (2.83), we conclude that the major axis of these orbits is 53,000 km long. Perigee (typically 500 km altitude) lies in the southern hemisphere, while apogee is at an altitude of 40,000 km (25,000 miles) above the northern latitudes, farther out than the geostationary satellites. [Fig. 4.20](#) illustrates a typical Molniya orbit. A Molniya “constellation” consists of eight satellites in planes separated by  $45^\circ$ . Each satellite is above  $30^\circ\text{N}$  latitude for over 8 h, coasting toward and away from apogee.

### EXAMPLE 4.10

Determine the perigee and apogee for an earth satellite whose orbit satisfies all the following conditions: it is sun synchronous, its argument of perigee is constant, and its period is 3 h.

#### Solution

The period determines the semimajor axis,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \Rightarrow 3 \cdot 3600 = \frac{2\pi}{\sqrt{398,600}} a^{3/2} \Rightarrow a = 10,560 \text{ km}$$

For the apse line to be stationary, we know from Eq. (4.53) that  $i = 64.435^\circ$  or  $i = 116.57^\circ$ . However, an inclination of less than  $90^\circ$  causes a westward regression of the node, whereas a sun-synchronous orbit requires an eastward advance, which  $i = 116.57^\circ$  provides. Substituting this, the semimajor axis and the  $\Omega$  in radians per second for a sun-synchronous orbit (cf. Example 4.9) into Eq. (4.52), we get

$$1.991 \times 10^{-7} = -\frac{3\sqrt{398,600} \times 0.0010826 \times 6378^2}{(1-e^2)^2 \times 10,560^{7/2}} \cos 116.57^\circ \Rightarrow e = 0.3466$$

Now we can find the angular momentum from the period expression (Eq. 2.82)

$$T = \frac{2\pi}{\mu^2} \left( \frac{h}{\sqrt{1-e^2}} \right)^3 \Rightarrow 3 \cdot 3600 = \frac{2\pi}{398,600^2} \left( \frac{h}{\sqrt{1-0.34655^2}} \right)^3 \Rightarrow h = 60,850 \text{ km}^2/\text{s}$$

Finally, to obtain the perigee and apogee radii, we use the orbit formula:

$$z_p + 6378 = \frac{h^2}{\mu} \frac{1}{1+e} = \frac{60,860^2}{398,600} \frac{1}{1+0.34655} \Rightarrow z_p = \boxed{522.6 \text{ km}}$$

$$z_a + 6378 = \frac{h^2}{\mu} \frac{1}{1-e} \Rightarrow z_a = \boxed{7842 \text{ km}}$$

### EXAMPLE 4.11

Given the following state vector of a satellite in geocentric equatorial coordinates,

$$\mathbf{r} = -3670\hat{\mathbf{i}} - 3870\hat{\mathbf{j}} + 4400\hat{\mathbf{k}} \text{ km}$$

$$\mathbf{v} = 4.7\hat{\mathbf{i}} - 7.4\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \text{ km/s}$$

find the state vector after 4 days (96 h) of coasting flight, assuming that there are no perturbations other than the influence of the earth's oblateness on  $\Omega$  and  $\omega$ .

#### Solution

A time interval of 4 days is long enough for us to take into consideration not only the change in true anomaly, but also the regression of the ascending node and the advance of perigee. First, we must determine the orbital elements at the initial time using Algorithm 4.2, which yields

$$h = 58,930 \text{ km}^2/\text{s}$$

$$i = 39.687^\circ$$

$$e = 0.42607 \text{ (The orbit is an ellipse)}$$

$$\Omega_0 = 130.32^\circ$$

$$\omega_0 = 42.373^\circ$$

$$\theta_0 = 52.404^\circ$$

We use Eq. (2.71) to determine the semimajor axis,

$$a = \frac{h^2}{\mu} \frac{1}{1-e^2} = \frac{58,930^2}{398,600} \frac{1}{1-0.4261^2} = 10,640 \text{ km}$$

so that, according to Eq. (2.83), the period is

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = 10,928 \text{ s}$$

From this we obtain the mean motion

$$n = \frac{2\pi}{T} = 0.00057495 \text{ rad/s}$$

The initial value  $E_0$  of eccentric anomaly is found from the true anomaly  $\theta_0$  using Eq. (3.13a),

$$\tan \frac{E_0}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta_0}{2} = \sqrt{\frac{1-0.42607}{1+0.42607}} \tan \frac{52.404^\circ}{2} \Rightarrow E_0 = 0.60520 \text{ rad}$$

With  $E_0$ , we use Kepler's equation to calculate the time  $t_0$  since perigee at the initial epoch,

$$nt_0 = E_0 - e \sin E_0 \Rightarrow 0.00057495 t_0 = 0.60520 - 0.42607 \sin 0.60520 \Rightarrow t_0 = 631.00 \text{ s}$$

Now we advance the time to  $t_f$ , that of the final epoch, given as 96 h later. That is,  $\Delta t = 345,600$  s, so that

$$t_f = t_0 + \Delta t = 631.00 + 345,600 = 346,230 \text{ s}$$

The number of periods  $n_p$  since passing perigee in the first orbit is

$$n_p = \frac{t_f}{T} = \frac{346,230}{10,928} = 31.682$$

From this we see that the final epoch occurs in the 32nd orbit, whereas  $t_0$  was in orbit 1. Time since passing perigee in the 32nd orbit, which we will denote  $t_{32}$ , is

$$t_{32} = (31.682 - 31)T \Rightarrow t_{32} = 7455.7 \text{ s}$$

The mean anomaly corresponding to that time in the 32nd orbit is

$$M_{32} = nt_{32} = 0.00057495 \times 7455.7 = 4.2866 \text{ rad}$$

Kepler's equation yields the eccentric anomaly:

$$\begin{aligned} E_{32} - e \sin E_{32} &= M_{32} \\ E_{32} - 0.42607 \sin E_{32} &= 4.2866 \\ E_{32} &= 3.9721 \text{ rad (Algorithm 3.1)} \end{aligned}$$

The true anomaly follows in the usual way,

$$\tan \frac{\theta_{32}}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E_{32}}{2} \Rightarrow \theta_{32} = 211.25^\circ$$

At this point, we use the newly found true anomaly to calculate the state vector of the satellite in perifocal coordinates. Thus, from Eq. (4.43)

$$\mathbf{r} = r \cos \theta_{32} \hat{\mathbf{p}} + r \sin \theta_{32} \hat{\mathbf{q}} = -11,714 \hat{\mathbf{p}} - 7108.8 \hat{\mathbf{q}} \text{ (km)}$$

or, in matrix notation,

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} -11,714 \\ -7108.8 \\ 0 \end{Bmatrix} \text{ (km)}$$

Likewise, from Eq. (4.44),

$$\mathbf{v} = \frac{\mu}{h} \sin \theta_{32} \hat{\mathbf{p}} + \frac{\mu}{h} (e + \cos \theta_{32}) \hat{\mathbf{q}} = 3.5093 \hat{\mathbf{p}} - 2.9007 \hat{\mathbf{q}} \text{ (km/s)}$$

or

$$\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} 3.5093 \\ -2.9007 \\ 0 \end{Bmatrix} \text{ (km/s)}$$

Before we can project  $\mathbf{r}$  and  $\mathbf{v}$  into the geocentric equatorial frame, we must update the right ascension of the node and the argument of perigee. The regression rate of the ascending node is

$$\begin{aligned}\dot{\Omega} &= -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2a^{7/2}}\right]\cos i = -\frac{3\sqrt{398,600}\times 0.00108263\times 6378^2}{2(1-0.42607^2)^2\times 10,644^{7/2}}\cos 39.69^\circ \\ &= -3.8514\times 10^{-7}\text{ rad/s}\end{aligned}$$

or

$$\dot{\Omega} = -2.2067\times 10^{-5}\text{ deg/s}$$

Therefore, right ascension at epoch in the 32nd orbit is

$$\Omega_{32} = \Omega_0 + \dot{\Omega}\Delta t = 130.32 + (-2.2067\times 10^{-5})\times 345,600 = 122.70^\circ$$

Likewise, the perigee advance rate is

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R^2}{(1-e^2)^2a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right) = 4.9072\times 10^{-7}\text{ rad/s} = 2.8116\times 10^{-5}\text{ deg/s}$$

which means the argument of perigee at epoch in the 32nd orbit is

$$\omega_{32} = \omega_0 + \dot{\omega}\Delta t = 42.373 + 2.8116\times 10^{-5}\times 345,600 = 52.090^\circ$$

Substituting the updated values of  $\Omega$  and  $\omega$ , together with the inclination  $i$ , into Eq. (4.47) yields the updated direction cosine matrix from geocentric equatorial to the perifocal frame,

$$\begin{aligned}[\mathbf{Q}]_{\bar{x}\bar{y}} &= \begin{bmatrix} \cos\omega_{32} & \sin\omega_{32} & 0 \\ -\sin\omega_{32} & \cos\omega_{32} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega_{32} & \sin\Omega_{32} & 0 \\ -\sin\Omega_{32} & \cos\Omega_{32} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos 52.09^\circ & \sin 52.09^\circ & 0 \\ -\sin 52.09^\circ & \cos 52.09^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 39.687^\circ & \sin 39.687^\circ \\ 0 & -\sin 39.687^\circ & \cos 39.687^\circ \end{bmatrix} \cdot \begin{bmatrix} \cos 122.70^\circ & \sin 122.70^\circ & 0 \\ -\sin 122.70^\circ & \cos 122.70^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

or

$$[\mathbf{Q}]_{\bar{x}\bar{y}} = \begin{bmatrix} -0.84285 & 0.18910 & 0.50383 \\ 0.028276 & -0.91937 & 0.39237 \\ 0.53741 & 0.34495 & 0.76955 \end{bmatrix}$$

For the inverse transformation, from perifocal to geocentric equatorial, we need the transpose of this matrix,

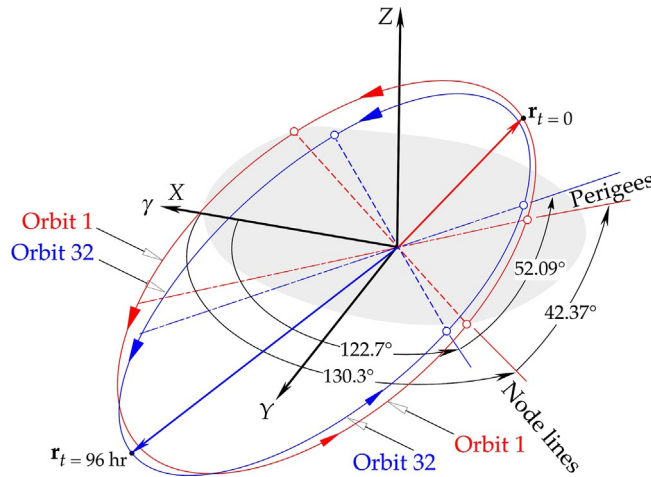
$$[\mathbf{Q}]_{\bar{y}\bar{x}} = \begin{bmatrix} -0.84285 & 0.18910 & 0.50383 \\ 0.028276 & -0.91937 & 0.39237 \\ 0.53741 & 0.34495 & 0.76955 \end{bmatrix}^T = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix}$$

Thus, according to Eq. 4.51, the final state vector in the geocentric equatorial frame is

$$\begin{aligned}\{\mathbf{r}\}_X &= [\mathbf{Q}]_{\bar{y}\bar{x}}\{\mathbf{r}\}_{\bar{x}} = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix} \begin{Bmatrix} -11,714 \\ -7108.8 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9672 \\ 4320 \\ -8691 \end{Bmatrix} \text{ (km)} \\ \{\mathbf{v}\}_X &= [\mathbf{Q}]_{\bar{y}\bar{x}}\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} -0.84285 & 0.028276 & 0.53741 \\ 0.18910 & -0.91937 & 0.34495 \\ 0.50383 & 0.39237 & 0.76955 \end{bmatrix} \begin{Bmatrix} 3.5093 \\ -2.9007 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -3.040 \\ 3.330 \\ 0.6299 \end{Bmatrix} \text{ (km/s)}\end{aligned}$$

or, in vector notation,

$$\boxed{\mathbf{r} = 9672\hat{\mathbf{i}} + 4320\hat{\mathbf{j}} - 8691\hat{\mathbf{k}} \text{ (km)} \quad \mathbf{v} = -3.040\hat{\mathbf{i}} + 3.330\hat{\mathbf{j}} + 0.6299\hat{\mathbf{k}} \text{ (km/s)}}$$

**FIG. 4.21**

The initial and final position vectors.

The two orbits are plotted in Fig. 4.21.

### 4.7.1 GROUND TRACKS

The projection of a satellite's orbit onto the earth's surface is called its ground track. At a given instant, we can imagine a radial line drawn outward from the center of the earth to the satellite. Where this line pierces the earth's spherical surface is a point on the ground track. We locate this point by giving its latitude and longitude relative to the earth. As the satellite moves around the earth, the trace of these points is its ground track.

Because the satellite reaches a maximum and minimum latitude ("amplitude") during each orbit, while passing over the equator twice, on a Mercator projection, the ground track of a satellite in low earth orbit often resembles a sine curve. If the earth did not rotate, there would be just one sinusoid-like track, traced repeatedly as the satellite orbits the earth. However, the earth rotates eastward beneath the satellite orbit at 15.04 deg/h, so the ground track advances westward at that rate. Fig. 4.22 shows about two and a half orbits of a satellite, with the beginning and end of this portion of the ground track labeled. The distance between two successive crossings of the equator is measured to be 23.2°, which is the amount of earth rotation in one orbit of the spacecraft. Therefore, the ground track reveals that the period of the satellite is

$$T = \frac{23.2^\circ}{15.04^\circ/\text{h}} = 1.54\text{h} = 92.6\text{ min}$$

This is a typical low earth orbital period.

Given a satellite's position vector  $\mathbf{r}$ , we can use Algorithm 4.1 to find its right ascension and declination relative to the geocentric equatorial XYZ frame, which is fixed in space. The earth rotates at an angular velocity  $\omega_E$  relative to this system. Let us attach an  $x'y'z'$  Cartesian coordinate system to the



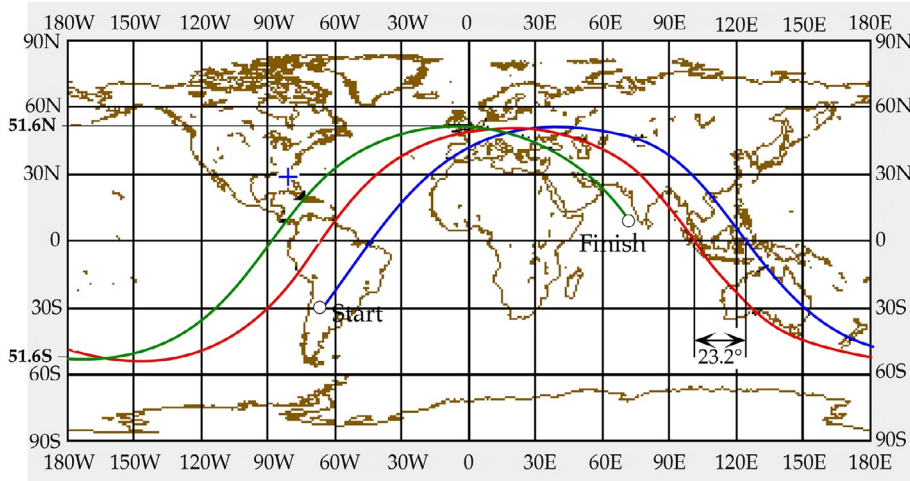


FIG. 4.22

Ground track of a satellite.

earth with its origin located at the earth's center, as illustrated in Fig. 1.18. The  $x'y'$  axes lie in the equatorial plane and the  $z'$  axis points north. (In Fig. 1.18 the  $x'$  axis is directed toward the prime meridian, which passes through Greenwich, England.) The  $XYZ$  and  $x'y'z'$  axes differ only by the angle  $\theta$  between the stationary  $X$  axis and the rotating  $x'$  axis. If the  $X$  and  $x'$  axes line up at time  $t_0$ , then at any time  $t$  thereafter the angle  $\theta$  will be given by  $\omega_E(t - t_0)$ . The transformation from  $XYZ$  to  $x'y'z'$  is represented by the elementary rotation matrix (recall Eq. 4.34),

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \theta = \omega_E(t - t_0) \quad (4.55)$$

Thus, if the components of the position vector  $\mathbf{r}$  in the inertial  $XYZ$  frame are given by  $\{\mathbf{r}\}_X$ , its components  $\{\mathbf{r}\}_{x'}$  in the rotating, earth-fixed  $x'y'z'$  frame are

$$\{\mathbf{r}\}_{x'} = [\mathbf{R}_3(\theta)]\{\mathbf{r}\}_X \quad (4.56)$$

Knowing  $\{\mathbf{r}\}_{x'}$ , we use Algorithm 4.1 to determine the right ascension (longitude east of  $x'$ ) and declination (latitude) in the earth-fixed system. These points are usually plotted on a rectangular Mercator projection of the earth's surface, as in Fig. 4.22.

#### ALGORITHM 4.6

Given the initial orbital elements ( $h$ ,  $e$ ,  $a$ ,  $T$ ,  $i$ ,  $\omega_0$ ,  $\Omega_0$ , and  $\theta_0$ ) of a satellite relative to the geocentric equatorial frame, compute the right ascension  $\alpha$  and declination  $\delta$  relative to the rotating earth after a time interval  $\Delta t$ . This algorithm is implemented in MATLAB as the script *ground\_track.m* in Appendix D.23.

1. Compute  $\dot{\Omega}$  and  $\dot{\omega}$  from Eqs. (4.52) and (4.53).

2. Calculate the initial time  $t_0$  (time since perigee passage):
  - a. Find the eccentric anomaly  $E_0$  from Eq. (3.13b).
  - b. Find the mean anomaly  $M_0$  from Eq. (3.14).
  - c. Find  $t_0$  from Eq. (3.15).
3. At time  $t = t_0 + \Delta t$ , calculate  $\alpha$  and  $\delta$ .
  - a. Calculate the true anomaly:
    - i. Find  $M$  from Eq. (3.8).
    - ii. Find  $E$  from Eq. (3.14) using Algorithm 3.1.
    - iii. Find  $\theta$  from Eq. (3.13a).
  - b. Update  $\Omega$  and  $\omega$ :  $\Omega = \Omega_0 + \dot{\Omega}\Delta t$   $\omega = \omega_0 + \dot{\omega}\Delta t$ .
  - c. Find  $\{\mathbf{r}\}_X$  using Algorithm 4.5.
  - d. Find  $\{\mathbf{r}\}_{X'}$  using Eqs. (4.55) and (4.56).
  - e. Use Algorithm 4.1 to compute  $\alpha$  and  $\delta$  from  $\{\mathbf{r}\}_{X'}$ .
4. Repeat Step 3 for additional times ( $t = t_0 + 2\Delta t$ ,  $t = t_0 + 3\Delta t$ , etc.).

### EXAMPLE 4.12

An earth satellite has the following orbital parameters:

$r_p = 6700\text{km}$	Perigee
$r_a = 10,000\text{km}$	Apogee
$\theta_0 = 230^\circ$	True anomaly
$\Omega_0 = 270^\circ$	Right ascension of the ascending node
$i_0 = 60^\circ$	Inclination
$\omega_0 = 45^\circ$	Argument of perigee

Calculate the right ascension (longitude east of  $x'$ ) and declination (latitude) relative to the rotating earth 45 min later.

#### Solution

First, we compute the semimajor axis  $a$ , eccentricity  $e$ , the angular momentum  $h$ , the semimajor axis  $a$ , and the period  $T$ . For the semimajor axis, we recall that

$$a = \frac{r_p + r_a}{2} = \frac{6700 + 10,000}{2} = 8350\text{km}$$

From Eq. (2.84) we get the eccentricity,

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{10,000 - 6700}{10,000 + 6700} = 0.19760$$

Eq. (2.50) yields the angular momentum

$$h = \sqrt{\mu r_p (1 + e)} = \sqrt{398,600 \cdot 6700 \cdot (1 + 0.19760)} = 56,554\text{km}^2/\text{s}$$

Finally, we obtain the period from Eq. (2.83),

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398,600}} 8350^{3/2} = 7593.5\text{s}$$

Now we can proceed with Algorithm 4.6.

Step 1:

$$\dot{\Omega} = - \left[ \frac{3}{2} \frac{\sqrt{\mu} J_2 R^2}{(1-e^2)^2 a^{7/2}} \right] \cos i_0 = - \left[ \frac{3 \sqrt{398,600} \cdot 0.0010836 \cdot 6378^2}{2 (1-0.19760^2)^2 \cdot 8350^{7/2}} \right] \cos 60^\circ = -2.3394 (10^{-5})^\circ/\text{s}$$

$$\dot{\omega} = \dot{\Omega} \frac{(5/2) \sin^2 i_0 - 2}{\cos i_0} = -2.3394 \times 10^{-5} \left[ \frac{(5/2) \sin^2 60^\circ - 2}{\cos 60^\circ} \right] = 5.8484 (10^{-6})^\circ/\text{s}$$

Step 2:

(a)  $E = 2 \tan^{-1} \left( \tan \frac{\theta}{2} \sqrt{\frac{1-e}{1+e}} \right) = 2 \tan^{-1} \left( \tan \frac{230^\circ}{2} \sqrt{\frac{1-0.19760}{1+0.19760}} \right) = -2.1059 \text{ rad}$

(b)  $M = E - e \sin E = -2.1059 - 0.19760 \sin(-2.1059) = -1.9360 \text{ rad}$

(c)  $t_0 = \frac{M}{2\pi} T = \frac{-1.9360}{2\pi} \cdot 7593.5 = -2339.7 \text{ s} (\therefore 2339.7 \text{ seconds until perigee})$

Step 3:

(a)  $t = t_0 + 45 \text{ min} = -2339.7 + (45 \times 60) = 360.33 \text{ s} (360.33 \text{ s after perigee})$

$$M = 2\pi \frac{t}{T} = 2\pi \frac{360.33}{7593.5} = 0.29815 \text{ rad}$$

Algorithm 3.1

$$E - 0.19760 \sin E = 0.29815 \quad \Rightarrow \quad E = 0.36952 \text{ rad}$$

$$\theta = 2 \tan^{-1} \left( \tan \frac{E}{2} \sqrt{\frac{1+e}{1-e}} \right) = 2 \tan^{-1} \left( \tan \frac{0.36952}{2} \sqrt{\frac{1+0.19760}{1-0.19760}} \right) = 25.723^\circ$$

(b)  $\Omega = \Omega_0 + \dot{\Omega} \Delta t = 270^\circ + (-2.3394 \times 10^{-5})(2700 \text{ s})$

$$\omega = \omega_0 + \dot{\omega} \Delta t = 45^\circ + (5.8484 \times 10^{-6} / \text{s})(2700 \text{ s}) = 45.016^\circ$$

(c)

Algorithm 4.5

$$\{\mathbf{r}\}_X \quad \Rightarrow \quad \begin{Bmatrix} 3212.6 \\ -2250.5 \\ 5568.6 \end{Bmatrix} (\text{km})$$

(d)

$$\theta = \omega_E \Delta t = \frac{360^\circ \left( 1 + \frac{1}{365.26} \right)}{24 \cdot 3600 \text{ s}} \cdot 2700 \text{ s} = 11.28^\circ$$

$$[\mathbf{R}_3(\theta)] = \begin{bmatrix} \cos 11.281^\circ & \sin 11.281^\circ & 0 \\ -\sin 11.281^\circ & \cos 11.281^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.98068 & 0.19562 & 0 \\ -0.19562 & 0.98068 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

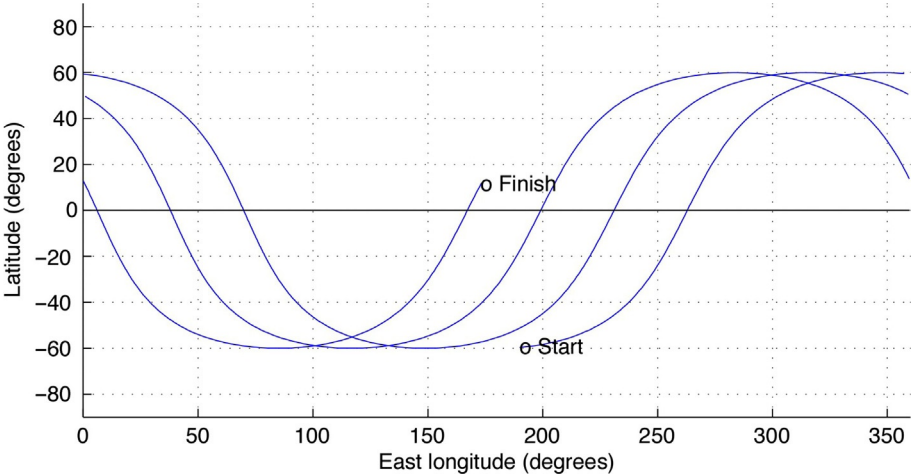
$$\{\mathbf{r}\}_{X'} = [\mathbf{R}_3(\theta)] \{\mathbf{r}\}_X = \begin{bmatrix} 0.98068 & 0.19562 & 0 \\ -0.19562 & 0.98068 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 3212.6 \\ -2250.5 \\ 5568.6 \end{Bmatrix} = \begin{Bmatrix} 2710.3 \\ -2835.4 \\ 5568.6 \end{Bmatrix} (\text{km})$$

(e)

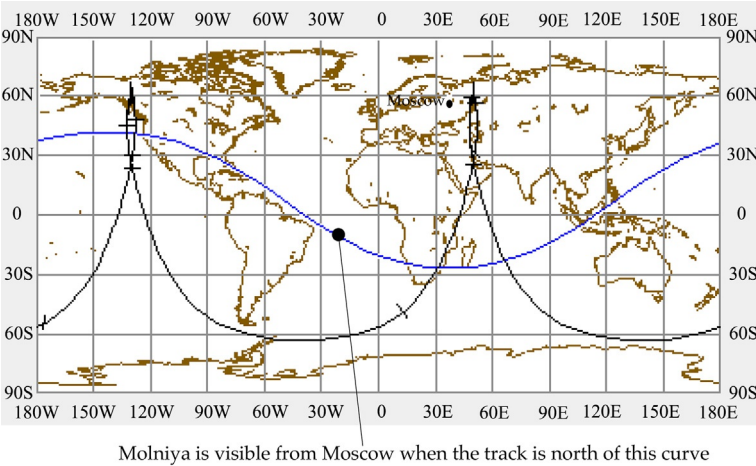
Algorithm 4.1

$$\mathbf{r} = 2710.3 \hat{\mathbf{i}}' - 2835.4 \hat{\mathbf{j}}' + 5568.6 \hat{\mathbf{k}}' \quad \Rightarrow \quad \boxed{\alpha = 313.7^\circ \quad \delta = 54.84^\circ}$$

The script *ground\_track.m* in [Appendix D.23](#) can be used to plot ground tracks. For the data of Example 4.12, the ground track for 3.25 periods appears in [Fig. 4.23](#). The ground track for one orbit of a Molniya satellite is featured more elegantly in [Fig. 4.24](#).



**FIG. 4.23**  
Ground track for 3.25 orbits of the satellite in Example 4.12.



**FIG. 4.24**  
Ground track for two orbits of a Molniya satellite with a 12-h period. Tick marks are 1 h apart.

## PROBLEMS

### Section 4.3

- 4.1 For each of the following geocentric equatorial position vectors (in kilometers) find the right ascension and declination.

$$\mathbf{r} = -3000\hat{\mathbf{i}} - 6000\hat{\mathbf{j}} - 9000\hat{\mathbf{k}} \quad (\text{a})$$

$$\mathbf{r} = -3000\hat{\mathbf{i}} - 6000\hat{\mathbf{j}} - 9000\hat{\mathbf{k}} \quad (\text{b})$$

$$\mathbf{r} = -9000\hat{\mathbf{i}} - 3000\hat{\mathbf{j}} - 6000\hat{\mathbf{k}} \quad (\text{c})$$

$$\mathbf{r} = 6000\hat{\mathbf{i}} - 9000\hat{\mathbf{j}} - 3000\hat{\mathbf{k}} \quad (\text{d})$$

{Partial Ans.: (b)  $\alpha = 243.4^\circ$ ,  $\delta = -53.30^\circ$ }

- 4.2 At a given instant, a spacecraft is 500 km above the earth, with a right ascension of  $300^\circ$  and a declination of  $-20^\circ$  relative to the geocentric equatorial frame. Its velocity is 10 km/s directly north, normal to the equatorial plane. Find  $\alpha$  and  $\delta$  30 min later.  
{Ans.:  $\alpha = 120^\circ$ ,  $\delta = -29.98^\circ$ }

### Section 4.4

- 4.3 Find the orbital elements of a geocentric satellite whose inertial position and velocity vectors in a geocentric equatorial frame are

$$\begin{aligned} \mathbf{r} &= 2500\hat{\mathbf{i}} + 16,000\hat{\mathbf{j}} + 4000\hat{\mathbf{k}} \text{ (km)} \\ \mathbf{v} &= -3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}} \text{ (km/s)} \end{aligned}$$

{Ans.:  $e = 0.4658$ ,  $h = 98,623 \text{ km}^2/\text{s}$ ,  $i = 62.52^\circ$ ,  $\Omega = 73.74^\circ$ ,  $\omega = 22.08^\circ$ ,  $\theta = 353.6^\circ$ }

- 4.4 At a given instant, the position  $\mathbf{r}$  and velocity  $\mathbf{v}$  of a satellite in the geocentric equatorial frame are

$$\begin{aligned} \mathbf{r} &= -13,000\hat{\mathbf{k}} \text{ (km)} \\ \mathbf{v} &= 4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 6\hat{\mathbf{k}} \text{ (km/s)} \end{aligned}$$

Find the orbital elements.

{Ans.:  $e = 1.298$ ,  $h = 83,240 \text{ km}^2/\text{s}$ ,  $i = 90^\circ$ ,  $\Omega = 51.34^\circ$ ,  $\omega = 344.9^\circ$ ,  $\theta = 285.1^\circ$ }

- 4.5 At time  $t_0$  (relative to perigee passage) the position  $\mathbf{r}$  and velocity  $\mathbf{v}$  of a satellite in the geocentric equatorial frame are

$$\begin{aligned} \mathbf{r} &= 6500\hat{\mathbf{i}} - 7500\hat{\mathbf{j}} - 2500\hat{\mathbf{k}} \text{ (km)} \\ \mathbf{v} &= 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}} \text{ (km/s)} \end{aligned}$$

Find the orbital elements.

{Ans.:  $e = 0.2226$ ,  $h = 58,656 \text{ km}^2/\text{s}$ ,  $i = 32.44^\circ$ ,  $\Omega = 107.6^\circ$ ,  $\omega = 72.36^\circ$ ,  $\theta = 134.7^\circ$ }

- 4.6 With respect to the geocentric equatorial frame, the position vector of a spacecraft is  $\mathbf{r} = -6000\hat{\mathbf{i}} - 1000\hat{\mathbf{j}} - 5000\hat{\mathbf{k}}$  (km) and the orbit's eccentricity vector is  $\mathbf{e} = 0.4\hat{\mathbf{i}} + 0.5\hat{\mathbf{j}} + 0.6\hat{\mathbf{k}}$ . Calculate the true anomaly  $\theta$  if the satellite is approaching perigee.  
{Ans.:  $328.6^\circ$ }

- 4.7 Given that, relative to the geocentric equatorial frame,  $\mathbf{r} = -6600\hat{\mathbf{i}} - 1300\hat{\mathbf{j}} - 5200\hat{\mathbf{k}}$  (km), the eccentricity vector is  $\mathbf{e} = -0.4\hat{\mathbf{i}} - 0.5\hat{\mathbf{j}} - 0.6\hat{\mathbf{k}}$ , and the satellite is flying toward perigee, calculate the inclination of the orbit.  
{Ans.:  $43.3^\circ$ }

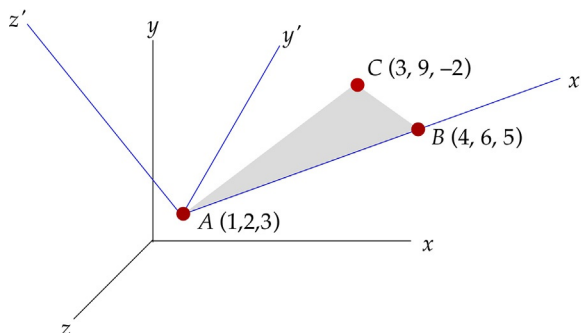
## Section 4.5

**4.8** The right-handed, Cartesian coordinate system  $x'y'z'$  is defined by the three points  $A$ ,  $B$ , and  $C$ . The  $x'y'$  plane is defined by the plane  $ABC$ . The  $x'$  axis runs from  $A$  through  $B$ . The  $z'$  axis is defined by the cross product of the vector  $\vec{AB}$  into the vector  $\vec{AC}$ , so that the  $+y'$  axis lies on the same side of the  $x'$  axis as point  $C$ .

(a) Find the direction cosine matrix  $[Q]$  relating the two coordinate bases.

(b) If the components of a vector  $\mathbf{v}$  in the primed system are  $[2 \ -1 \ 3]^T$ , find the components of  $\mathbf{v}$  in the unprimed system.

{Partial Ans.: (b)  $[-1.307 \ 2.390 \ 2.565]^T$ }



**4.9** The unit vectors in a  $uvw$  Cartesian coordinate frame have the following components in the  $xyz$  frame:

$$\hat{\mathbf{u}} = 0.26726\hat{\mathbf{i}} + 0.53452\hat{\mathbf{j}} + 0.80178\hat{\mathbf{k}}$$

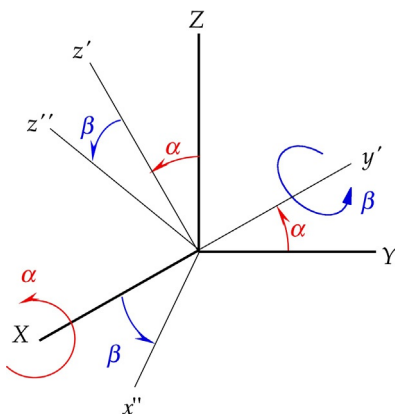
$$\hat{\mathbf{v}} = -0.44376\hat{\mathbf{i}} + 0.80684\hat{\mathbf{j}} + 0.38997\hat{\mathbf{k}}$$

$$\hat{\mathbf{w}} = -0.85536\hat{\mathbf{i}} + 0.25158\hat{\mathbf{j}} + 0.45284\hat{\mathbf{k}}$$

If, in the  $xyz$  frame  $\mathbf{V} = -50\hat{\mathbf{i}} + 100\hat{\mathbf{j}} + 75\hat{\mathbf{k}}$ , find the components of the vector  $\mathbf{V}$  in the  $uvw$  frame.  
{Ans.:  $\mathbf{V} = 100.2\hat{\mathbf{u}} + 73.62\hat{\mathbf{v}} + 51.57\hat{\mathbf{w}}$ }

**4.10** Calculate the direction cosine matrix  $[Q]$  for the sequence of two rotations:  $\alpha = 40^\circ$  about the positive  $X$  axis, followed by  $\beta = 25^\circ$  about the positive  $y'$  axis. The result is that the  $XYZ$  axes are rotated into the  $x''y''z''$  axes.

{Partial Ans.:  $Q_{11} = 0.9063 \quad Q_{12} = 0.2716 \quad Q_{13} = -0.3237$ }



4.11 For the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} 0.086824 & -0.77768 & 0.62264 \\ -0.49240 & -0.57682 & -0.65178 \\ 0.86603 & -0.25000 & -0.43301 \end{bmatrix}$$

calculate:

(a) The classical Euler angle sequence.

(b) The yaw, pitch, and roll angle sequence.

{Ans.: (a)  $\alpha = 73.90^\circ$   $\beta = 115.7^\circ$   $\gamma = 136.31^\circ$

(b)  $\alpha = 276.37^\circ$   $\beta = -38.51^\circ$   $\gamma = 236.40^\circ$ }

4.12 What yaw, pitch, and roll sequence yields the same direction cosine matrix as the classical Euler sequence  $\alpha = 350^\circ$ ,  $\beta = 170^\circ$ ,  $\gamma = 300^\circ$ ?

{Ans.:  $\alpha = 49.62^\circ$ ,  $\beta = 8.649^\circ$ ,  $\gamma = 175.0^\circ$ }

4.13 What classical Euler angle sequence yields the same direction cosine matrix as the yaw, pitch, and roll sequence  $\alpha = 300^\circ$ ,  $\beta = -80^\circ$ ,  $\gamma = 30^\circ$ ?

{Ans.:  $\alpha = 240.4^\circ$ ,  $\beta = 81.35^\circ$ ,  $\gamma = 84.96^\circ$ }

#### Section 4.6

4.14 At time  $t_0$  (relative to perigee passage), the position  $\mathbf{r}$  and velocity  $\mathbf{v}$  of a satellite in the geocentric equatorial frame are

$$\begin{aligned} \mathbf{r} &= -5000\hat{\mathbf{i}} - 8000\hat{\mathbf{j}} - 2100\hat{\mathbf{k}} \text{ (km)} \\ \mathbf{v} &= -4\hat{\mathbf{i}} + 3.5\hat{\mathbf{j}} - 3\hat{\mathbf{k}} \text{ (km/s)} \end{aligned}$$

Find  $\mathbf{r}$  and  $\mathbf{v}$  at time  $t_0 + 50$  min.

{Ans.:  $\mathbf{r} = -1717\hat{\mathbf{i}} + 7604\hat{\mathbf{j}} - 2101\hat{\mathbf{k}}$  (km);  $\mathbf{v} = 6.075\hat{\mathbf{i}} + 1.925\hat{\mathbf{j}} + 3.591\hat{\mathbf{k}}$  (km/s)}

4.15 At time  $t_0$  (relative to perigee passage), a spacecraft has the following orbital parameters:  $e = 1.5$ ; perigee altitude = 300 km;  $i = 35^\circ$ ;  $\Omega = 130^\circ$ ; and  $\omega = 115^\circ$ . Calculate  $\mathbf{r}$  and  $\mathbf{v}$  at perigee relative to

(a) The perifocal reference frame.

(b) The geocentric equatorial frame.

{Ans.: (a)  $\mathbf{r} = 6678\hat{\mathbf{p}}$  (km),  $\mathbf{v} = 12.22\hat{\mathbf{q}}$  (km/s)

(b)  $\mathbf{r} = -1984\hat{\mathbf{i}} - 5348\hat{\mathbf{j}} + 3471\hat{\mathbf{k}}$  (km),  $\mathbf{v} = 10.36\hat{\mathbf{i}} - 5.763\hat{\mathbf{j}} - 2.961\hat{\mathbf{k}}$  (km/s) }

4.16 For the spacecraft of Problem 4.15, calculate  $\mathbf{r}$  and  $\mathbf{v}$  at two hours past perigee relative to

(a) The perifocal reference frame.

(b) The geocentric equatorial frame.

{Ans.: (a)  $\mathbf{r} = -25,010\hat{\mathbf{p}} + 48,090\hat{\mathbf{q}}$  (km),  $\mathbf{v} = -4.335\hat{\mathbf{p}} + 5.075\hat{\mathbf{q}}$  (km/s)

(b)  $\mathbf{r} = 48,200\hat{\mathbf{i}} - 2658\hat{\mathbf{j}} - 24,660\hat{\mathbf{k}}$  (km),  $\mathbf{v} = 5.590\hat{\mathbf{i}} + 1.078\hat{\mathbf{j}} - 3.484\hat{\mathbf{k}}$  (km/s) }

4.17 Calculate  $\mathbf{r}$  and  $\mathbf{v}$  relative to the geocentric equatorial frame for the satellite in Problem 4.15 at time  $t_0 + 50$  min.

{Ans.:  $\mathbf{r} = 23,047\hat{\mathbf{i}} - 6972.4\hat{\mathbf{j}} - 9219.6\hat{\mathbf{k}}$  (km)

$\mathbf{v} = 6.6563\hat{\mathbf{i}} + 0.88638\hat{\mathbf{j}} - 3.9680\hat{\mathbf{k}}$  (km/s)}

4.18 For a spacecraft, the following orbital parameters are given:  $e = 1.2$ ; perigee altitude = 200 km;  $i = 50^\circ$ ;  $\Omega = 75^\circ$ ; and  $\omega = 80^\circ$ . Calculate  $\mathbf{r}$  and  $\mathbf{v}$  at perigee relative to

(a) The perifocal reference frame.

(b) The geocentric equatorial frame.

{Ans.: (a)  $\mathbf{r} = 6578\hat{\mathbf{p}}$  (km),  $\mathbf{v} = 11.55\hat{\mathbf{q}}$  (km/s)

(b)  $\mathbf{r} = -3726\hat{\mathbf{I}} + 2181\hat{\mathbf{J}} + 4962\hat{\mathbf{K}}$  (km)  
 $\mathbf{v} = -4.188\hat{\mathbf{I}} - 10.65\hat{\mathbf{J}} + 1.536\hat{\mathbf{K}}$  (km/s)}

4.19 For the spacecraft of Problem 4.18, calculate  $\mathbf{r}$  and  $\mathbf{v}$  at 2 h past perigee relative to

(a) The perifocal reference frame.

(b) The geocentric equatorial frame.

{Ans.: (a)  $\mathbf{r} = -26,340\hat{\mathbf{p}} + 37,810\hat{\mathbf{q}}$  (km),  $\mathbf{v} = -4.306\hat{\mathbf{p}} + 3.298\hat{\mathbf{q}}$  (km/s)

(b)  $\mathbf{r} = 1207\hat{\mathbf{I}} - 43,600\hat{\mathbf{J}} - 14,840\hat{\mathbf{K}}$  (km)  
 $\mathbf{v} = 1.243\hat{\mathbf{I}} - 4.470\hat{\mathbf{J}} - 2.810\hat{\mathbf{K}}$  (km/s)}

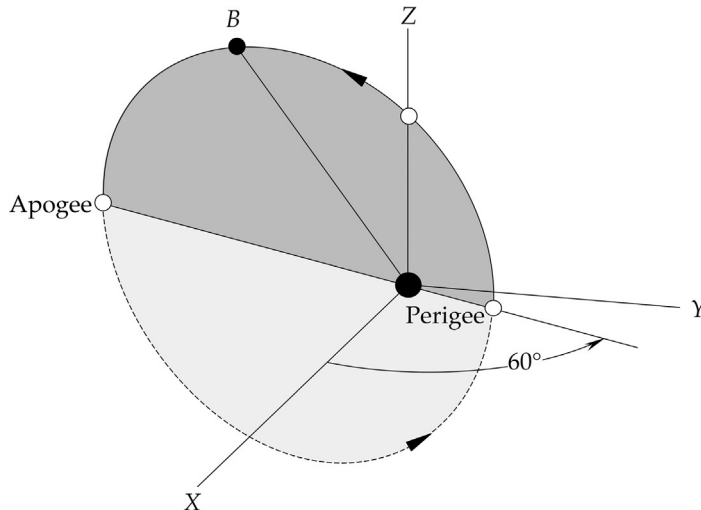
4.20 Given that  $e = 0.7$ ,  $h = 75,000 \text{ km}^2/\text{s}$ , and  $\theta = 25^\circ$ , calculate the components of velocity in the geocentric equatorial frame if

$$[\mathbf{Q}]_{\bar{X}\bar{Y}} = \begin{bmatrix} -0.83204 & -0.13114 & 0.53899 \\ 0.02741 & -0.98019 & -0.19617 \\ 0.55403 & -0.14845 & 0.81915 \end{bmatrix}$$

{Ans.:  $\mathbf{v} = 2.103\hat{\mathbf{I}} - 8.073\hat{\mathbf{J}} - 2.885\hat{\mathbf{K}}$  (km/s)}

4.21 The apse line of the elliptical orbit lies in the  $XY$  plane of the geocentric equatorial frame, whose  $Z$  axis lies in the plane of the orbit. At  $B$  (for which  $\theta = 140^\circ$ ), the perifocal velocity vector is  $\{\mathbf{v}\}_{\bar{X}} = [-3.208 \ -0.8288 \ 0]^T$  (km/s). Calculate the geocentric equatorial components of the velocity at  $B$ .

{Ans.:  $\{\mathbf{v}\}_X = [-1.604 \ -2.778 \ -0.8288]^T$  (km/s)}



4.22 A satellite in earth orbit has the following orbital parameters:  $a = 7016$  km,  $e = 0.05$ ,  $i = 45^\circ$ ,  $\Omega = 0^\circ$ ,  $\omega = 20^\circ$ , and  $\theta = 10^\circ$ . Find the position vector in the geocentric equatorial frame.

{Ans.:  $\mathbf{r} = 5776.4\hat{\mathbf{I}} + 2358.2\hat{\mathbf{J}} + 2358.2\hat{\mathbf{K}}$  (km)}



**Section 4.7**

- 4.23** Calculate the orbital inclination required to place an earth satellite in a 300 km by 600 km sun-synchronous orbit.  
{Ans.:  $97.21^\circ$ .}
- 4.24** A satellite in a circular, sun-synchronous low earth orbit passes over the same point on the equator once each day, at 12 o'clock noon. Calculate the inclination, altitude, and period of the orbit.  
{Ans.: This problem has more than one solution.}
- 4.25** The orbit of a satellite around an unspecified planet has an inclination of  $45^\circ$ , and its perigee advances at the rate of  $6^\circ$  per day. At what rate does the node line regress?  
{Ans.:  $\dot{\Omega} = 5.656^\circ/\text{day}$ }
- 4.26** At a given time, the position and velocity of an earth satellite in the geocentric equatorial frame are  $\mathbf{r} = -2429.1\hat{\mathbf{i}} + 4555.1\hat{\mathbf{j}} + 4577.0\hat{\mathbf{k}}$  (km) and  $\mathbf{v} = -4.7689\hat{\mathbf{i}} - 5.6113\hat{\mathbf{j}} + 3.0535\hat{\mathbf{k}}$  (km/s). Find  $\mathbf{r}$  and  $\mathbf{v}$  precisely 72 h later, taking into consideration the node line regression and the advance of perigee.  
{Ans.:  $\mathbf{r} = 4596\hat{\mathbf{i}} + 5759\hat{\mathbf{j}} - 1266\hat{\mathbf{k}}$  (km),  $\mathbf{v} = -3.601\hat{\mathbf{i}} + 3.179\hat{\mathbf{j}} + 5.617\hat{\mathbf{k}}$  (km/s)}

**Section 4.8**

- 4.27** A spacecraft is in a circular orbit of 180 km altitude and inclination  $30^\circ$ . What is the spacing, in kilometers, between successive ground tracks at the equator, including the effect of earth's oblateness?  
{Ans.: 2511 km}

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**REFERENCE**

Department of the Navy, 2018. Nautical Almanac Office: The Astronomical Almanac for the Year 2019. GPO.