HW 4

written questions

Problem 1a) $\underline{A} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ $\underline{A}^{-1} = \begin{bmatrix} \frac{11}{9} & -\frac{1}{3} & -\frac{2}{9} \\ -2 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

$\underline{C} = \underline{A}\,\underline{A}^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$C_{11} = A_{11} A^{-1}_{11} + A_{12} A^{-1}_{21} + A_{13} A^{-1}_{31}$
$C_{12} = A_{11} A^{-1}_{12} + A_{12} A^{-1}_{22} + A_{13} A^{-1}_{32}$
$C_{13} = A_{11} A^{-1}_{13} + A_{12} A^{-1}_{23} + A_{13} A^{-1}_{33}$

$C_{21} = A_{21} A^{-1}_{11} + A_{22} A^{-1}_{21} + A_{23} A^{-1}_{31}$
$C_{22} = A_{21} A^{-1}_{12} + A_{22} A^{-1}_{22} + A_{23} A^{-1}_{32}$
$C_{23} = A_{21} A^{-1}_{13} + A_{22} A^{-1}_{23} + A_{23} A^{-1}_{33}$

$C_{31} = A_{31} A^{-1}_{11} + A_{32} A^{-1}_{21} + A_{33} A^{-1}_{31}$
$C_{32} = A_{31} A^{-1}_{12} + A_{32} A^{-1}_{22} + A_{33} A^{-1}_{32}$
$C_{33} = A_{31} A^{-1}_{13} + A_{32} A^{-1}_{23} + A_{33} A^{-1}_{33}$

$C_{11} = 3\left(\frac{11}{9}\right) + 1(-2) + 2\left(-\frac{1}{3}\right) = 1$
$C_{12} = 3\left(-\frac{1}{3}\right) + 1(1) + 2(0) = 0$
$C_{13} = 3\left(-\frac{2}{9}\right) + 1(0) + 2\left(\frac{1}{3}\right) = 0$
$C_{21} = 6\left(\frac{11}{9}\right) + 3(-2) + 4\left(-\frac{1}{3}\right) = 0$
$C_{22} = 6\left(-\frac{1}{3}\right) + 3(1) + 4(0) = 1$
$C_{23} = 6\left(-\frac{2}{9}\right) + 3(0) + 4\left(\frac{1}{3}\right) = 0$
$C_{31} = 3\left(\frac{11}{9}\right) + 1(-2) + 5\left(-\frac{1}{3}\right) = 0$
$C_{32} = 3\left(-\frac{1}{3}\right) + 1(1) + 5(0) = 0$
$C_{33} = 3\left(-\frac{2}{9}\right) + 1(0) + 5\left(\frac{1}{3}\right) = 1$

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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b) $\|\underline{A}\|_\infty = 13$, $\|\underline{A}^{-1}\|_\infty = 3$, $\text{cond}(\underline{A}) = \|\underline{A}\|_\infty * \|\underline{A}^{-1}\|_\infty = 39$

$6+3+4 = 13$ $\qquad 2+1 = 3$

c) $\underline{x}_c = (-.99, 1.01, .99)$
$\underline{x} = (-1, 1, 1)$

$$\underline{A} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \qquad \underline{b} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$\underline{r} = \underline{b} - \underline{A}\,\underline{x}_c$

$$\text{Error magnification factor} = \frac{\frac{\|\underline{x} - \underline{x}_c\|_\infty}{\|\underline{x}\|_\infty}}{\frac{\|\underline{r}\|_\infty}{\|\underline{b}\|_\infty}} = \frac{\frac{.01}{1}}{\frac{\|r\|_\infty}{3}}$$

$$\underline{A}\,\underline{x}_c = \begin{bmatrix} -.99(3) + -.99(1) + -.99(2) \\ 1.01(6) + 1.01(3) + 1.01(4) \\ .99(3) + .99(1) + .99(5) \end{bmatrix} = \begin{pmatrix} .02 \\ 1.05 \\ 2.99 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} .02 \\ 1.05 \\ 2.99 \end{pmatrix} = \begin{pmatrix} -.02 \\ -.05 \\ .01 \end{pmatrix}$$

$\|r\|_\infty = .05$

$$\boxed{EMF = \frac{.01}{\frac{.05}{3}} = .6}$$

This seems like a very good EMF, considering an EMF of 39 is possible for this system.

Problem 2

a) Points: (0,0) (1,1) (2,2) (3,7) $x = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ $y = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 7 \end{pmatrix}$

$$L_k = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$k=1$ $L_1 = \underset{i=2}{\frac{x-1}{0-1}} \underset{i=3}{\frac{x-2}{0-2}} \underset{i=4}{\frac{x-3}{0-3}} =$

$k=2$ $L_2 = \underset{i=1}{\frac{x-0}{1-0}} \underset{i=3}{\frac{x-2}{1-2}} \underset{i=4}{\frac{x-3}{1-3}}$

$k=3$ $L_3 = \underset{i=1}{\frac{x-0}{2-0}} \underset{i=2}{\frac{x-1}{2-1}} \underset{i=4}{\frac{x-3}{2-3}}$

$k=4$ $L_4 = \underset{i=1}{\frac{x-0}{3-0}} \underset{i=2}{\frac{x-1}{3-1}} \underset{i=3}{\frac{x-2}{3-2}}$

$$P_3(x) = \sum_{k=1}^{n} y_k L_k(x)$$

$$P_3(x) = 0 + 1\left(\frac{x}{1} \frac{x-2}{-1} \frac{x-3}{-2}\right)$$
$$+ 2\left(\frac{x}{2} \frac{x-1}{1} \frac{x-3}{-1}\right)$$
$$+ 7\left(\frac{x}{3} \frac{x-1}{2} \frac{x-2}{1}\right)$$

$$P_3(x) = x(-x+2)\left(-\tfrac{1}{2}x + \tfrac{3}{2}\right) + x(x-1)(-x+3) + \tfrac{7}{3}x\left(\tfrac{x}{2} - \tfrac{1}{2}\right)(x-2)$$
$$= (-x^2+2x)\left(-\tfrac{1}{2}x + \tfrac{3}{2}\right) + (x^2 - x)(-x+3) + \left(\tfrac{7}{6}x^2 - \tfrac{7}{6}x\right)(x-2)$$
$$= \tfrac{1}{2}x^3 - \tfrac{3}{2}x^2 - x^2 + 3x + -x^3 + 3x^2 + x^2 - 3x + \tfrac{7}{6}x^3 - \tfrac{7}{6}x^2 - \tfrac{7}{3}x^2 + \tfrac{7}{3}x$$

$$= .66\bar{6}x^3 + -2x^2 + 2.33\bar{3}x + 0$$

b) no, there is no parabola that will go through these points.

c)

x	$f[x]$			
0	0			
		$1 = f[x_1, x_2]$		
1	1		$0 = f[x_1, x_2, x_3]$	
		$1 = f[x_2, x_3]$		$.6\bar{6} = f[x_1, x_2, x_3, x_4] = c$
2	2		$2 = f[x_2, x_3, x_4]$	
		$5 = f[x_3, x_4]$		
3	7			

$$P(x) = 0 + 1(x-0) + 0(x-0)(x-1) + .6\bar{6}(x-0)(x-1)(x-2)$$
$$= x + .66(x^2 - x)(x-2) = x + .66(x^3 - 2x^2 - x^2 + 2x)$$
$$x + .66x^3 - 1.3\bar{3}x^2 - .6\bar{6}x^2 + 1.3\bar{3}x = .6\bar{6}x^3 - 2x^2 + 2.3\bar{3}x$$

Problem 2

c) con't

yes, I got the same result as from lagrange method. This makes sense because there exists 1 unique interpolating polynomial of degree n-1 for these n data points. If the points all layed on a line or parabola this wouldn't be true.

d) add an extra point to the list of points to get a polynomial degree 4 that interpolates the orriginal 4 and the new 5th points, where $P_3(x) = .6\bar{6}x^3 + -2x^2 + 2.3\bar{3}x$, $P_4(x) = P_3(x) + f[x_1 x_2 x_3 x_4 x_5](x)(x-1)(x-2)(x-3)$ will interpolat the new list

x	f[x]				
0	0				
		1			
1	1		0		
		1		$.\bar{66}$	
2	2		2		$f[x_1\ x_2\ x_3\ x_4\ x_5] = \dfrac{\frac{y_5}{6} - \frac{8}{3} - \frac{2}{3}}{4} = \frac{y_5}{24} - \frac{10}{12}$
		5		$f[x_2 x_3 x_4 x_5] = \dfrac{\frac{y_5}{2} - 8}{3}$	
3	7		$f[x_3 x_4 x_5] = \dfrac{y_5 - 12}{2}$		
		$f[x_4 x_5] = y_5 - 7$			
4	y_5				

$$P_4(x) = P_3(x) + \left(\frac{y_5}{24} - \frac{10}{12}\right) x(x-1)(x-2)(x-3)$$

$$x(x-1)(x-2)(x-3) = (x^2-x)(x-2)(x-3) = (x^3-2x^2-x^2+2x)(x-3)$$
$$= (x^3-3x^2+2x)(x-3) = x^4-3x^3+2x^2-3x^3+9x^2-6x =$$
$$= x^4 - 6x^3 + 11x^2 - 6x$$

choose y_5 to be 0, $P_4(x) = .6\bar{6}x^3 - 2x^2 + 2.3\bar{3}x - \frac{5}{6}x^4 + 5x^3 - 9.16\bar{6}x^2 + 5$

$$\doteq -.83\bar{3}x^4 + 5.6\bar{6}x^3 - 11.16\bar{6}x^2 + 7.3\bar{3}x$$

choose y_5 to be 8 $= P_4(x) = .6\bar{6}x^3 - 2x^2 + 2.3\bar{3}x - .5x^4 + 3x^3 - 5.5x^2 + 3x$

$$= .5x^4 + 3.6\bar{6}x^3 - 7.5x^2 + 5.3\bar{3}x$$

e) no, since $P_3(X) = .6\bar{6}X^3 - 2X^2 + 2.3\bar{3}X$ is the unique polynomial of degree 3 or less which interpolates $\underline{x}_1 = \begin{pmatrix}0\\1\\2\\3\end{pmatrix}$ $\underline{y}_1 = \begin{pmatrix}0\\1\\2\\7\end{pmatrix}$, then only if $(4,2)$ lies on $P_3(X)$, ie. if $P_3(4) = 2$, would there be a polynomial of degree 3 or less that interpolates

$\underline{x}_2 = \begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}$ $\underline{y}_2 = \begin{pmatrix}0\\1\\2\\7\\2\end{pmatrix}$. $P_3(4)$ doesn't equal 2, $P_3(4) = 20$, so no polynomial degree 3 or less interpolates $(\underline{x}_2, \underline{y}_2)$

Problem 3 a) $y = Be^{Ax}$, $\ln(y) = \ln(B) + \ln e^{Ax} = \ln B + Ax$

$\ln(y) = Y$, $\ln(B) = \beta$ $\Rightarrow Y = \beta + \alpha X$

$A = \alpha$, $X = x$

b) $\underline{x} = \begin{pmatrix}-1\\0\\1\\2\\3\end{pmatrix}$ $\underline{y} = \begin{pmatrix}6.62\\2.78\\1.51\\1.23\\0.89\end{pmatrix}$ $\underline{X} = \begin{pmatrix}-1\\0\\1\\2\\3\end{pmatrix}$ $\underline{Y} = \begin{pmatrix}1.890\\1.023\\.412\\.207\\-.117\end{pmatrix}$

c) $\underline{A}\underline{s} = \underline{Y}$ $\begin{bmatrix}1 & -1\\1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{pmatrix}\beta\\\alpha\end{pmatrix} = \begin{pmatrix}1.890\\1.023\\.412\\.207\\-.117\end{pmatrix}$

* to avoid confusion between data set and least squares solution, I name x^* s^*

$$\boxed{\text{Normal eq: } \underline{A}^T\underline{A}\,\underline{s}^* = \underline{A}^T\underline{Y}}$$

$\underline{A}^T\underline{A} = \begin{bmatrix}n & \Sigma x_i\\ \Sigma x_i & \Sigma x_i^2\end{bmatrix} =$ $\underline{A}^T\underline{Y} = \begin{pmatrix}\Sigma y_i\\ \Sigma x_i y_i\end{pmatrix}$

$$\begin{bmatrix}n & \Sigma x_i\\ \Sigma x_i & \Sigma x_i^2\end{bmatrix} s^* = \begin{pmatrix}\Sigma y_i\\ \Sigma x_i y_i\end{pmatrix} = \begin{bmatrix}5 & 5\\5 & 15\end{bmatrix} s^* = \begin{pmatrix}3.4151\\-1.4136\end{pmatrix}$$

Problem 3 c) cont

$$\begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 3.4151 \\ -1.4136 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 5 & 5 & 3.4151 \\ 0 & 10 & -4.8287 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 1 & .683 \\ 0 & 1 & -.4829 \end{array}\right]$$

$s_2 = -.4829$
$s_1 = 1.1659$ $\qquad s^* = \begin{pmatrix} 1.1659 \\ -.4829 \end{pmatrix}$

e) $\underline{r}^* = \underline{A}\,\underline{s}^* - \underline{Y}$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1.1659 \\ -.4829 \end{pmatrix} - \begin{pmatrix} 1.8901 \\ 1.0225 \\ .4121 \\ .2070 \\ -.1165 \end{pmatrix}$$

$$= \begin{pmatrix} 1.6488 \\ 1.1659 \\ .6830 \\ .2001 \\ -.2828 \end{pmatrix} - \begin{pmatrix} 1.8901 \\ 1.0225 \\ .4121 \\ .2070 \\ -.1165 \end{pmatrix} = \begin{pmatrix} -.2413 \\ .1434 \\ .2709 \\ -.0069 \\ -.1663 \end{pmatrix} = \underline{r}^*$$

$\|\underline{r}^*\|_2 = \sqrt{.2413^2 + .1434^2 + .2709^2 + .0069^2 + 1.663^2} = .4241$

f) 1) $s = (0,0)$ $\quad r = -Y$, $\|r\|_2 = \|Y\|_2 = \sqrt{1.8901^2 + 1.0225^2 + .4121^2 + .2070^2 + .1165^2}$
$= 2.2009$

f) 2) $s = (1,0)$ $\quad As = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$ $\quad r = As - Y$

$$r = \begin{pmatrix} -.8901 \\ -.0225 \\ .5879 \\ .7930 \\ 1.1165 \end{pmatrix}, \|r\|_2 = \sqrt{.8901^2 + .0225^2 + .5879^2 + .793^2 + 1.1165^2} = 1.7361$$

f) 3) $s = (1, -.5)$

$$r = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ -.5 \end{pmatrix} - \begin{pmatrix} 1.8901 \\ 1.0225 \\ .4121 \\ .2070 \\ -.1165 \end{pmatrix} = \begin{pmatrix} -.2413 \\ .1434 \\ .2709 \\ -.0069 \\ -.1662 \end{pmatrix}, \quad \|r\|_2 = \sqrt{.2413^2 + .1434^2 + .2709^2 + .0069^2 + .1662^2}$$

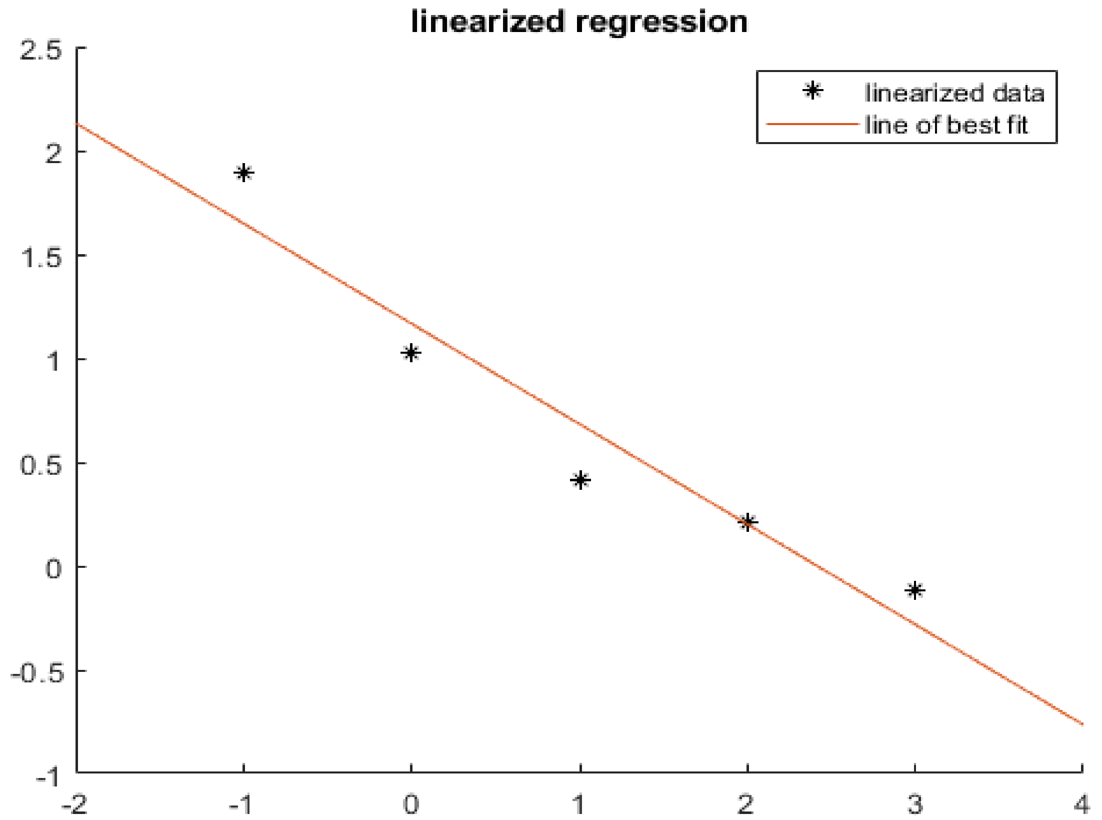
$$= .5914$$

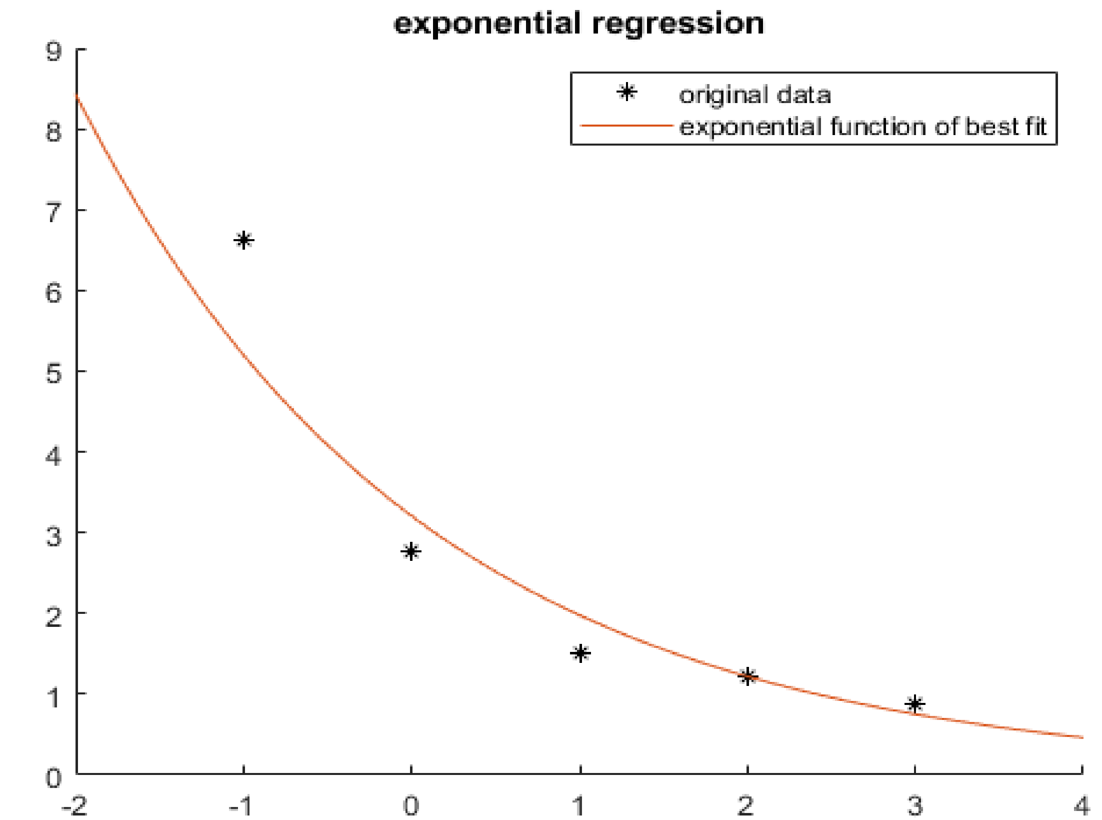
f) as expected the values of s I chose are less good than s^*. This makes sense since s^* by definition reduces r as low as possible

g) $A = \alpha = s_2$ $\qquad$ $A = s_2 = -.4829$

$\ln(B) = \beta = s_1$ $\qquad$ $B = e^{s_1} = 3.2088$

```
close all
x=[-1;0;1;2;3]; % data
y=[6.62;2.78;1.51;1.23;.89];
Y = log(y);

A = [ones(5,1),x]; % A matrix

a = -.4829; % a and alpha
bet = 1.1659; % beta
B = exp(bet); % B

lin = @(x) a*x + bet;
Xlin = linspace(-2,4);
Ylin = lin(Xlin);

figure
hold on
plot(x,Y, '* k')
plot(Xlin,Ylin)
legend("linearized data","line of best fit")
title("linearized regression")

lin = @(x) B*exp(a*x);
Xlin = linspace(-2,4);
Ylin = lin(Xlin);
figure

hold on
plot(x,y, '* k')
plot(Xlin,Ylin)
legend("original data","exponential function of best fit")
title("exponential regression")
```


linearized regression
linearized data
line of best fit
2.5
2
1.5
1
0.5
0
-0.5
-1
-2
-1
0
1
2
3
4


exponential regression
original data
exponential function of best fit
9
8
7
6
5
4
3
2
1
0
-2
-1
0
1
2
3
4