CHAPTER

RIGID BODY DYNAMICS

11

11.1 INTRODUCTION

Just as Chapter 1 provides a foundation for development of the equations of orbital mechanics, this chapter serves as a basis for developing the equations of satellite attitude dynamics. Chapter 1 deals with particles, whereas here we are concerned with rigid bodies. Those familiar with rigid body dynamics can move on to the next chapter, perhaps returning from time to time to review concepts.

The kinematics of rigid bodies is presented first. The subject depends on a theorem of the French mathematician Michel Chasles (1793–1880). Chasles' theorem states that the motion of a rigid body can be described by the displacement of any point of the body (the base point) plus a rotation about a unique axis through that point. The magnitude of the rotation does not depend on the base point. Thus, at any instant, a rigid body in a general state of motion has an angular velocity vector whose direction is that of the instantaneous axis of rotation. Describing the rotational component of the motion of a rigid body in three dimensions requires taking advantage of the vector nature of angular velocity and knowing how to take the time derivative of moving vectors, which is explained in Chapter 1. Several examples in the current chapter illustrate how this is done.

We then move on to study the interaction between the motion of a rigid body and the forces acting on it. Describing the translational component of the motion requires simply concentrating all of the mass at a point known as the center of mass and applying the methods of particle mechanics to determine its motion. Indeed, our study of the two-body problem up to this point has focused on the motion of their centers of mass without regard to the rotational aspect. Analyzing the rotational dynamics requires computing the body's angular momentum, and that in turn requires accounting for how the mass is distributed throughout the body. The mass distribution is described by the six components of the moment of inertia tensor.

Writing the equations of rotational motion relative to coordinate axes embedded in the rigid body and aligned with the principal axes of inertia yields the nonlinear Euler equations of motion, which are applied to a study of the dynamics of a spinning top (or one-axis gyro).

The expression for the kinetic energy of a rigid body is derived because it will be needed in the following chapter.

The chapter next describes the two sets of three angles commonly employed to specify the orientation of a body in three-dimensional space. One of these comprises the Euler angles, which are the same as the right ascension of the node (Ω), the argument of periapsis (ω), and the inclination (i). These were introduced in Chapter 4 to orient orbits in space. The other set comprises the yaw, pitch, and roll

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angles, which are suitable for describing the orientation of an airplane. Both the Euler angles and the yaw, pitch, and roll angles will be employed in Chapter 12.

The chapter concludes with a brief discussion of quaternions and an example of how they are used to describe the evolution of the attitude of a rigid body.

11.2 KINEMATICS

Fig. 11.1 shows a moving rigid body and its instantaneous axis of rotation, which defines the direction of the absolute angular velocity vector $\boldsymbol{\omega}$. The XYZ axes are a fixed, inertial frame of reference. The position vectors $\mathbf{R}_A$ and $\mathbf{R}_B$ of two points on the rigid body are measured in the inertial frame. The vector $\mathbf{R}_{B/A}$ drawn from point A to point B is the position vector of B relative to A. Since the body is rigid, $\mathbf{R}_{B/A}$ has a constant magnitude even though its direction is continuously changing. Clearly,

$$\mathbf{R}_B = \mathbf{R}_A + \mathbf{R}_{B/A}$$

Differentiating this equation through with respect to time, we get

$$\dot{\mathbf{R}}_B = \dot{\mathbf{R}}_A + \frac{\mathrm{d}\mathbf{R}_{B/A}}{\mathrm{d}t} \tag{11.1}$$

where $\dot{\mathbf{R}}_A$ and $\dot{\mathbf{R}}_B$ are the absolute velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of points A and B. Because the magnitude of $\mathbf{R}_{B/A}$ does not change, its time derivative is given by Eq. (1.52). That is,

$$\frac{\mathrm{d}\mathbf{R}_{B/A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{R}_{B/A}$$

Thus, Eq. (11.1) becomes

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{R}_{B/A} \tag{11.2}$$

Taking the time derivative of Eq. (11.1) yields

$$\ddot{\mathbf{R}}_B = \ddot{\mathbf{R}}_A + \frac{\mathrm{d}^2\mathbf{R}_{B/A}}{\mathrm{d}t^2} \tag{11.3}$$

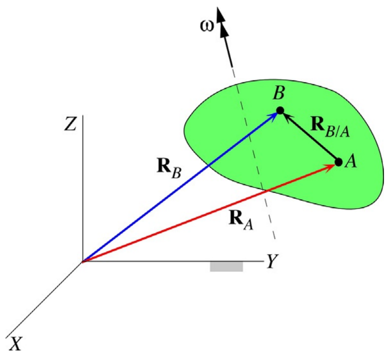


FIG. 11.1

Rigid body and its instantaneous axis of rotation.

where $\ddot{\mathbf{R}}_A$ and $\ddot{\mathbf{R}}_B$ are the absolute accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ of the two points of the rigid body, while from Eq. (1.53) we have

$$\frac{\mathrm{d}^2\mathbf{R}_{B/A}}{\mathrm{d}t^2} = \boldsymbol{\alpha} \times \mathbf{R}_{B/A} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{R}_{B/A}\right)$$

where $\boldsymbol{\alpha}$ is the angular acceleration, $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$. Therefore, Eq. (11.3) can be written

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{R}_{B/A} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{R}_{B/A}\right) \tag{11.4}$$

Eqs. (11.2) and (11.4) are the relative velocity and acceleration formulas. Note that all quantities in these expressions are measured in the same inertial frame of reference.

When the rigid body under consideration is connected to and moving relative to another rigid body, computation of its inertial angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ must be done with care. The key is to remember that angular velocity is a vector. It may be found as the vector sum of a sequence of angular velocities, each measured relative to another, starting with one measured relative to an absolute frame, as illustrated in Fig. 11.2. In that case, the absolute angular velocity $\boldsymbol{\omega}$ of body 4 is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_{2/1} + \boldsymbol{\omega}_{3/2} + \boldsymbol{\omega}_{4/3} \tag{11.5}$$

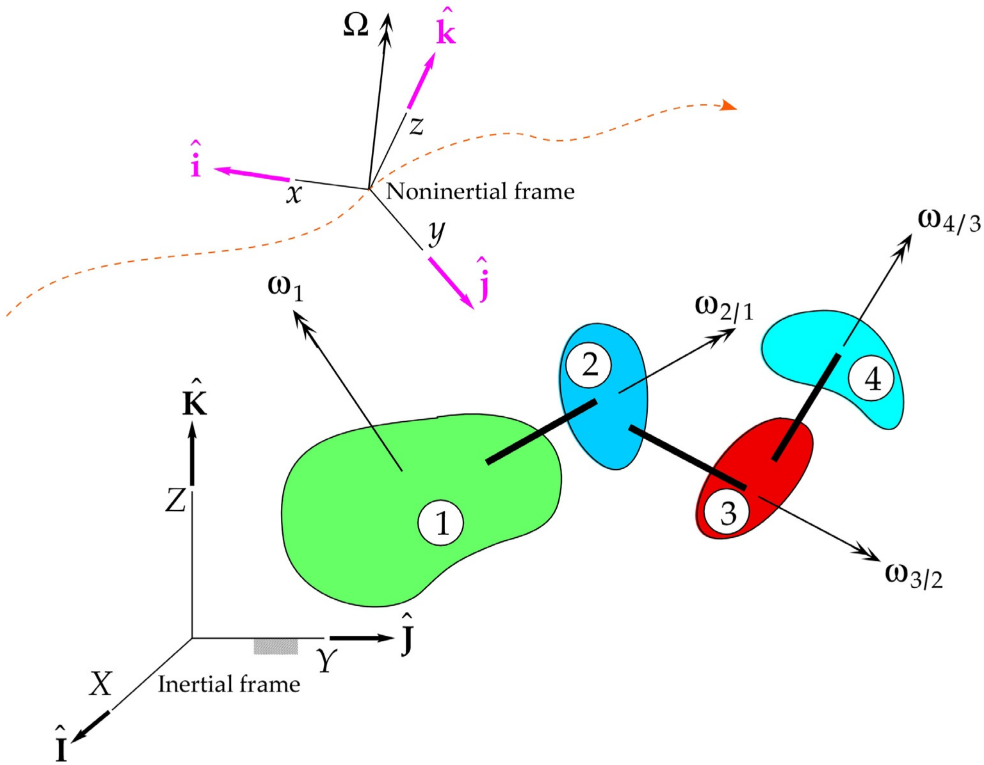


FIG. 11.2

Angular velocity is the vector sum of the relative angular velocities starting with $\boldsymbol{\omega}_1$, measured relative to the inertial frame.

Each of these angular velocities is resolved into components along the axes of the moving frame of reference xyz as shown in Fig. 11.2, so that the vector sum may be written as

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{11.6}$$

The moving frame is chosen for convenience of analysis, and its own inertial angular velocity vector is denoted as $\boldsymbol{\Omega}$, as discussed in Section 1.6. According to Eq. (1.56), the absolute angular acceleration $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$ is obtained from Eq. (11.6) by means of the following calculation:

$$\boldsymbol{\alpha} = \left.\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathrm{rel}} + \boldsymbol{\Omega}\times\boldsymbol{\omega} \tag{11.7}$$

where

$$\left.\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathrm{rel}} = \dot{\omega}_x\hat{\mathbf{i}} + \dot{\omega}_y\hat{\mathbf{j}} + \dot{\omega}_z\hat{\mathbf{k}} \tag{11.8}$$

and $\dot{\omega}_x = \mathrm{d}\omega_x/\mathrm{d}t$.

Being able to express the absolute angular velocity vector in an appropriately chosen moving reference frame, as in Eq. (11.6), is crucial to the analysis of rigid body motion. Once we have the components of $\boldsymbol{\omega}$, we simply differentiate each of them with respect to time to arrive at Eq. (11.8). Observe that the absolute angular acceleration $\boldsymbol{\alpha}$ and $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t)_{\mathrm{rel}}$, the angular acceleration relative to the moving frame, are the same if and only if $\boldsymbol{\Omega} = \boldsymbol{\omega}$. That occurs if the moving reference is a body-fixed frame (i.e., a set of xyz axes imbedded in the rigid body itself).

EXAMPLE 11.1

The airplane in Fig. 11.3 flies at a constant speed v while simultaneously undergoing a constant yaw rate ω_{yaw} about a vertical axis and describing a circular loop in the vertical plane with a radius ρ. The constant propeller spin rate is ω_{spin} relative to the airframe. Find the velocity and acceleration of the tip P of the propeller relative to the hub H, when P is directly above H. The propeller radius is ℓ.

Solution

The xyz axes are rigidly attached to the airplane. The x axis is aligned with the propeller's spin axis. The y axis is vertical, and the z axis is in the spanwise direction, so that xyz forms a right-handed triad. Although the xyz frame is not inertial, we can imagine it to instantaneously coincide with an inertial frame.

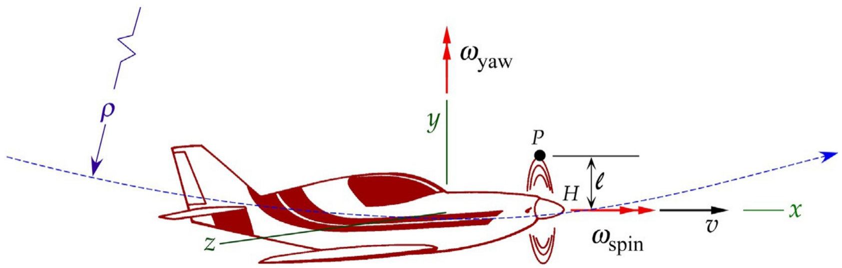


FIG. 11.3

Airplane with attached xyz body frame.

The absolute angular velocity of the airplane has two components, the yaw rate and the counterclockwise pitch angular velocity v/ρ of its rotation in the circular loop,

$$\boldsymbol{\omega}_{\text{airplane}} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \omega_{\text{pitch}}\hat{\mathbf{k}} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}$$

The angular velocity of the body-fixed moving frame is that of the airplane, $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{airplane}}$, so that

$$\boldsymbol{\Omega} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}} \tag{a}$$

The absolute angular velocity of the propeller is that of the airplane plus the angular velocity of the propeller relative to the airplane

$$\boldsymbol{\omega}_{\text{prop}} = \boldsymbol{\omega}_{\text{airplane}} + \omega_{\text{spin}}\hat{\mathbf{i}} = \boldsymbol{\Omega} + \omega_{\text{spin}}\hat{\mathbf{i}}$$

which means

$$\boldsymbol{\omega}_{\text{prop}} = \omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}} \tag{b}$$

From Eq. (11.2), the velocity of point P on the propeller relative to point H on the hub, $\mathbf{v}_{P/H}$, is given by

$$\mathbf{v}_{P/H} = \mathbf{v}_P - \mathbf{v}_H = \boldsymbol{\omega}_{\text{prop}} \times \mathbf{r}_{P/H}$$

where $\mathbf{r}_{P/H}$ is the position vector of P relative to H at this instant,

$$\mathbf{r}_{P/H} = \ell\hat{\mathbf{j}} \tag{c}$$

Thus, using Eqs. (b) and (c),

$$\mathbf{v}_{P/H} = \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\ell\hat{\mathbf{j}}\right)$$

from which

$$\boxed{\mathbf{v}_{P/H} = -\frac{v}{\rho}\ell\hat{\mathbf{i}} + \omega_{\text{spin}}\ell\hat{\mathbf{k}}}$$

The absolute angular acceleration of the propeller is found by substituting Eqs. (a) and (b) into Eq. (11.7),

$$\begin{aligned}
\boldsymbol{\alpha}_{\text{prop}} &= \left.\frac{d\boldsymbol{\omega}_{\text{prop}}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{prop}} \\
&= \left(\frac{d\omega_{\text{spin}}}{dt}\hat{\mathbf{i}} + \frac{d\omega_{\text{yaw}}}{dt}\hat{\mathbf{j}} + \frac{d(v/\rho)}{dt}\hat{\mathbf{k}}\right) + \left(\omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right)
\end{aligned}$$

Since ω_{spin}, ω_{yaw}, v, and ρ are all constant, this reduces to

$$\boldsymbol{\alpha}_{\text{prop}} = \left(\omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right)$$

Carrying out the cross product yields

$$\boldsymbol{\alpha}_{\text{prop}} = \frac{v}{\rho}\omega_{\text{spin}}\hat{\mathbf{j}} - \omega_{\text{yaw}}\omega_{\text{spin}}\hat{\mathbf{k}} \tag{d}$$

From Eq. (11.4), the acceleration of P relative to H, $\mathbf{a}_{P/H}$, is given by

$$\mathbf{a}_{P/H} = \mathbf{a}_P - \mathbf{a}_H = \boldsymbol{\alpha}_{\text{prop}} \times \mathbf{r}_{P/H} + \boldsymbol{\omega}_{\text{prop}} \times \left(\boldsymbol{\omega}_{\text{prop}} \times \mathbf{r}_{P/H}\right)$$

Substituting Eqs. (b), (c), and (d) into this expression yields

$$\mathbf{a}_{P/H} = \left(\frac{v}{\rho}\omega_{\text{spin}}\hat{\mathbf{j}} - \omega_{\text{yaw}}\omega_{\text{spin}}\hat{\mathbf{k}}\right) \times \left(\ell\hat{\mathbf{j}}\right) + \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left[\left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\ell\hat{\mathbf{j}}\right)\right]$$

From this we find that

$$\mathbf{a}_{P/H} = \left(\omega_{\text{yaw}}\omega_{\text{spin}}\ell\hat{\mathbf{i}}\right) + \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left[-\frac{v}{\rho}\ell\hat{\mathbf{i}} + \omega_{\text{spin}}\ell\hat{\mathbf{k}}\right]$$
$$= \left(\omega_{\text{yaw}}\omega_{\text{spin}}\ell\hat{\mathbf{i}}\right) + \left[\omega_{\text{yaw}}\omega_{\text{spin}}\ell\hat{\mathbf{i}} - \left(\frac{v^2}{\rho^2} + \omega_{\text{spin}}{}^2\right)\ell\hat{\mathbf{j}} + \omega_{\text{yaw}}\frac{v}{\rho}\ell\hat{\mathbf{k}}\right]$$

so that finally,

$$\boxed{\mathbf{a}_{P/H} = 2\omega_{\text{yaw}}\omega_{\text{spin}}\ell\hat{\mathbf{i}} - \left(\frac{v^2}{\rho^2} + \omega_{\text{spin}}{}^2\right)\ell\hat{\mathbf{j}} + \omega_{\text{yaw}}\frac{v}{\rho}\ell\hat{\mathbf{k}}}$$

EXAMPLE 11.2

The satellite in Fig. 11.4 is rotating about the z axis at a constant rate N. The xyz axes are attached to the spacecraft, and the z axis has a fixed orientation in inertial space. The solar panels rotate at a constant rate $\dot{\theta}$ in the direction shown. Relative to point O, which lies at the center of the spacecraft and on the centerline of the panels, calculate for point A on the panel

(a) its absolute velocity and
(b) its absolute acceleration.

Solution

(a) Since the moving xyz frame is attached to the body of the spacecraft, its angular velocity is

$$\boldsymbol{\Omega} = N\hat{\mathbf{k}} \tag{a}$$

The absolute angular velocity of the panel is the absolute angular velocity of the spacecraft plus the angular velocity of the panel relative to the spacecraft,

$$\boldsymbol{\omega}_{\text{panel}} = -\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}} \tag{b}$$

The position vector of A relative to O is

$$\mathbf{r}_{A/O} = -\frac{w}{2}\sin\theta\hat{\mathbf{i}} + d\hat{\mathbf{j}} + \frac{w}{2}\cos\theta\hat{\mathbf{k}} \tag{c}$$

According to Eq. (11.2), the velocity of A relative to O is

$$\mathbf{v}_{A/O} = \mathbf{v}_A - \mathbf{v}_O = \boldsymbol{\omega}_{\text{panel}} \times \mathbf{r}_{A/O} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\frac{w}{2}\sin\theta & d & \frac{w}{2}\cos\theta \end{vmatrix}$$

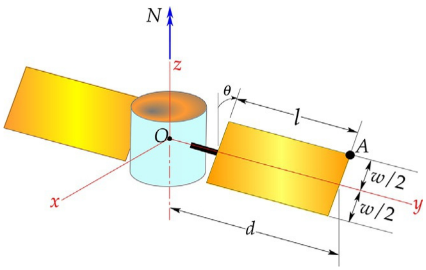


FIG. 11.4

Rotating solar panel on a rotating satellite.

from which we get

$$\boxed{\mathbf{v}_{A/O} = -\left(\frac{w}{2}\dot{\theta}\cos\theta + Nd\right)\hat{\mathbf{i}} - \frac{w}{2}N\sin\theta\hat{\mathbf{j}} - \frac{w}{2}\dot{\theta}\sin\theta\hat{\mathbf{k}}}$$

(b) The absolute angular acceleration of the panel is found by substituting Eqs. (a) and (b) into Eq. (11.7),

$$\begin{aligned}\boldsymbol{\alpha}_{\text{panel}} &= \left.\frac{\mathrm{d}\boldsymbol{\omega}_{\text{panel}}}{\mathrm{d}t}\right)_{\text{rel}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_{\text{panel}} \\ &= \left[\frac{\mathrm{d}(-\dot{\theta})}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}N}{\mathrm{d}t}\hat{\mathbf{k}}\right] + (N\hat{\mathbf{k}})\times\left(-\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}}\right)\end{aligned}$$

Since N and $\dot{\theta}$ are constants, this reduces to

$$\boldsymbol{\alpha}_{\text{panel}} = \dot{\theta}N\hat{\mathbf{i}} \tag{d}$$

To find the acceleration of A relative to O, we substitute Eqs. (b) through (d) into Eq. (11.4),

$$\begin{aligned}\mathbf{a}_{A/O} &= \mathbf{a}_A - \mathbf{a}_O = \boldsymbol{\alpha}_{\text{panel}}\times\mathbf{r}_{A/O} + \boldsymbol{\omega}_{\text{panel}}\times\left(\boldsymbol{\omega}_{\text{panel}}\times\mathbf{r}_{A/O}\right) \\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta}N & 0 & 0 \\ -\frac{w}{2}\sin\theta & d & \frac{w}{2}\cos\theta\end{vmatrix} + \left(-\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}}\right)\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\frac{w}{2}\sin\theta & d & \frac{w}{2}\cos\theta\end{vmatrix} \\ &= \left(-\frac{w}{2}N\dot{\theta}\cos\theta\hat{\mathbf{j}} + N\dot{\theta}d\hat{\mathbf{k}}\right) + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\frac{w}{2}\dot{\theta}\cos\theta - Nd & -N\frac{w}{2}\sin\theta & -\frac{w}{2}\dot{\theta}\sin\theta\end{vmatrix}\end{aligned}$$

which leads to

$$\boxed{\mathbf{a}_{A/O} = \frac{w}{2}\left(N^2 + \dot{\theta}^2\right)\sin\theta\hat{\mathbf{i}} - N\left(Nd + w\dot{\theta}\cos\theta\right)\hat{\mathbf{j}} - \frac{w}{2}\dot{\theta}^2\cos\theta\hat{\mathbf{k}}}$$

EXAMPLE 11.3

The gyro rotor illustrated in Fig. 11.5 has a constant spin rate ω_{spin} around axis b–a in the direction shown. The XYZ axes are fixed. The xyz axes are attached to the gimbal ring, whose angle θ with the vertical is increasing at a constant rate $\dot{\theta}$ in the direction shown. The assembly is forced to precess at a constant rate N around the vertical. For the rotor in the position shown, calculate

(a) the absolute angular velocity and

(b) the absolute angular acceleration.

Express the results in both the fixed XYZ frame and the moving xyz frame.

Solution

(a) We will need the instantaneous relationship between the unit vectors of the inertial XYZ axes and the comoving xyz frame, which on inspecting Fig. 11.6 can be seen to be

$$\begin{aligned}\hat{\mathbf{I}} &= -\cos\theta\hat{\mathbf{j}} + \sin\theta\hat{\mathbf{k}} \\ \hat{\mathbf{J}} &= \hat{\mathbf{i}} \\ \hat{\mathbf{K}} &= \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\end{aligned} \tag{a}$$

so that the matrix of the transformation from xyz to XYZ is (Section 4.5)

$$[\mathbf{Q}]_{xX} = \begin{bmatrix}0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta\end{bmatrix} \tag{b}$$

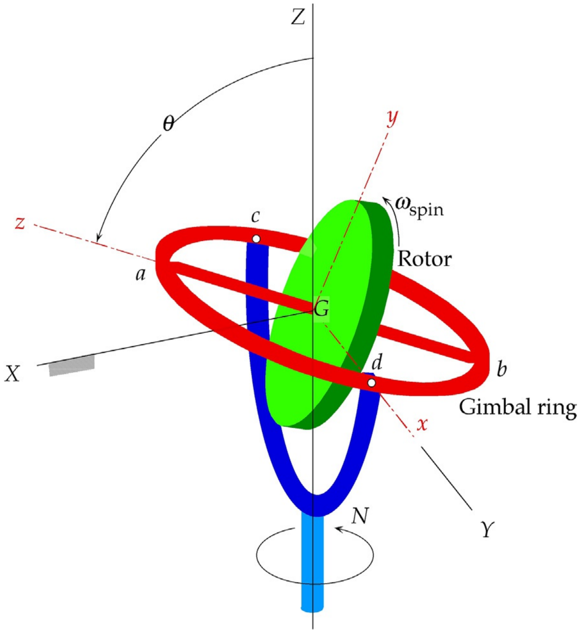


FIG. 11.5

Rotating, precessing, nutating gyro.

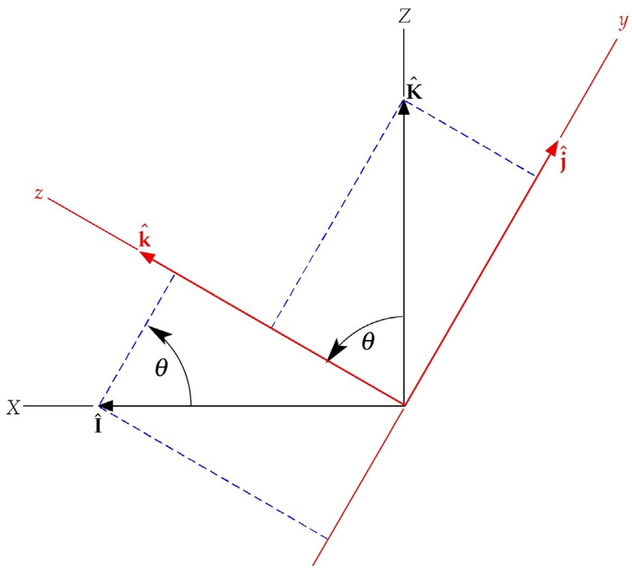


FIG. 11.6

Orientation of the fixed *XZ* axes relative to the rotating *xz* axes.

The absolute angular velocity of the gimbal ring is that of the base ($N\hat{\mathbf{K}}$) plus the angular velocity of the gimbal relative to the base ($\dot{\theta}\hat{\mathbf{i}}$), so that

$$\boldsymbol{\omega}_{\text{gimbal}} = N\hat{\mathbf{K}} + \dot{\theta}\hat{\mathbf{i}} = N\left(\sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\right) + \dot{\theta}\hat{\mathbf{i}} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + N\cos\theta\hat{\mathbf{k}} \tag{c}$$

where we made use of Eq. (a)$_3$ above. Since the moving xyz frame is attached to the gimbal, $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{gimbal}}$, so that

$$\boldsymbol{\Omega} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + N\cos\theta\hat{\mathbf{k}} \tag{d}$$

The absolute angular velocity of the rotor is its spin relative to the gimbal, plus the angular velocity of the gimbal,

$$\boldsymbol{\omega}_{\text{rotor}} = \boldsymbol{\omega}_{\text{gimbal}} + \omega_{\text{spin}}\hat{\mathbf{k}} \tag{e}$$

From Eq. (c), it follows that

$$\boxed{\boldsymbol{\omega}_{\text{rotor}} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + \left(N\cos\theta + \omega_{\text{spin}}\right)\hat{\mathbf{k}}} \tag{f}$$

Because $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ move with the gimbal, expression (f) is valid for any time, not just the instant shown in Fig. 11.5. Alternatively, applying the vector transformation

$$\{\boldsymbol{\omega}_{\text{rotor}}\}_{XYZ} = [\mathbf{Q}]_{xX}\{\boldsymbol{\omega}_{\text{rotor}}\}_{xyz} \tag{g}$$

we obtain the angular velocity of the rotor in the inertial frame, but only at the instant shown in the figure (i.e., when the x axis aligns with the Y axis):

$$\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ N\sin\theta \\ N\cos\theta + \omega_{\text{spin}} \end{Bmatrix} = \begin{Bmatrix} \omega_{\text{spin}}\sin\theta \\ \dot{\theta} \\ N + \omega_{\text{spin}}\cos\theta \end{Bmatrix}$$

or

$$\boxed{\boldsymbol{\omega}_{\text{rotor}} = \omega_{\text{spin}}\sin\theta\hat{\mathbf{I}} + \dot{\theta}\hat{\mathbf{J}} + \left(N + \omega_{\text{spin}}\cos\theta\right)\hat{\mathbf{K}}} \tag{h}$$

(b) The angular acceleration of the rotor can be found by substituting Eqs. (d) and (f) into Eq. (11.7), recalling that N, $\dot{\theta}$, and ω_{spin} are independent of time:

$$\boldsymbol{\alpha}_{\text{rotor}} = \left.\frac{\mathrm{d}\boldsymbol{\omega}_{\text{rotor}}}{\mathrm{d}t}\right)_{\text{rel}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_{\text{rotor}}$$

$$= \left[\frac{\mathrm{d}\left(\dot{\theta}\right)}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}(N\sin\theta)}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}\left(N\cos\theta + \omega_{\text{spin}}\right)}{\mathrm{d}t}\hat{\mathbf{k}}\right] + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta} & N\sin\theta & N\cos\theta \\ \dot{\theta} & N\sin\theta & N\cos\theta + \omega_{\text{spin}} \end{vmatrix}$$

$$= \left(N\dot{\theta}\cos\theta\hat{\mathbf{j}} - N\dot{\theta}\sin\theta\hat{\mathbf{k}}\right) + \left[N\omega_{\text{spin}}\sin\theta\hat{\mathbf{i}} - \omega_{\text{spin}}\dot{\theta}\hat{\mathbf{j}} + (0)\hat{\mathbf{k}}\right]$$

Upon collecting terms, we get

$$\boxed{\boldsymbol{\alpha}_{\text{rotor}} = N\omega_{\text{spin}}\sin\theta\hat{\mathbf{i}} + \dot{\theta}\left(N\cos\theta - \omega_{\text{spin}}\right)\hat{\mathbf{j}} - N\dot{\theta}\sin\theta\hat{\mathbf{k}}} \tag{i}$$

This expression, like Eq. (f), is valid at any time.

The components of $\boldsymbol{\alpha}_{\text{rotor}}$ along the XYZ axes are found in the same way as for $\boldsymbol{\omega}_{\text{rotor}}$,

$$\{\boldsymbol{\alpha}_{\text{rotor}}\}_{XYZ} = [\mathbf{Q}]_{xX}\{\boldsymbol{\alpha}_{\text{rotor}}\}_{xyz}$$

which means

$$\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} N\omega_{\text{spin}}\sin\theta \\ \dot{\theta}\left(N\cos\theta - \omega_{\text{spin}}\right) \\ -N\dot{\theta}\sin\theta \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}\omega_{\text{spin}}\cos\theta - N\dot{\theta} \\ N\omega_{\text{spin}}\sin\theta \\ -\dot{\theta}\omega_{\text{spin}}\sin\theta \end{Bmatrix}$$

or

$$\boxed{\boldsymbol{\alpha}_{\text{rotor}} = \dot{\theta}\left(\omega_{\text{spin}}\cos\theta - N\right)\hat{\mathbf{I}} + N\omega_{\text{spin}}\sin\theta\hat{\mathbf{J}} - \dot{\theta}\omega_{\text{spin}}\sin\theta\hat{\mathbf{K}}} \tag{j}$$

Note carefully that Eq. (j) is not simply the time derivative of Eq. (h). Eqs. (h) and (j) are valid only at the instant that the *xyz* and *XYZ* axes have the alignments shown in Fig. 11.5.

11.3 EQUATIONS OF TRANSLATIONAL MOTION

Fig. 11.7 again shows an arbitrary, continuous, three-dimensional body of mass m. "Continuous" means that as we zoom in on a point it remains surrounded by a continuous distribution of matter having the infinitesimal mass dm in the limit. The point never ends up in a void. In particular, we ignore the actual atomic and molecular microstructure in favor of this continuum hypothesis, as it is called. Molecular microstructure does bear on the overall dynamics of a finite body. We will use G to denote the center of mass. The position vectors of points relative to the origin of the inertial frame will be designated by capital letters. Thus, the position of the center of mass is $\mathbf{R}_G$, defined as

$$m\mathbf{R}_G = \int_m \mathbf{R}\,\mathrm{d}m \tag{11.9}$$

$\mathbf{R}$ is the position of a mass element dm within the continuum. Each element of mass is acted on by a net external force $\mathrm{d}\mathbf{F}_{\text{net}}$ and a net internal force $\mathrm{d}\mathbf{f}_{\text{net}}$. The external force comes from direct contact with other objects and from action at a distance, such as gravitational attraction. The internal forces are those exerted from within the body by neighboring particles. These are the forces that hold the body together. For each mass element, Newton's second law (Eq. 1.38) is written as

$$\mathrm{d}\mathbf{F}_{\text{net}} + \mathrm{d}\mathbf{f}_{\text{net}} = \mathrm{d}m\ddot{\mathbf{R}} \tag{11.10}$$

Writing this equation for the infinite number of mass elements of which the body is composed, and then summing them all together, leads, to the integral

$$\int_m \mathrm{d}\mathbf{F}_{\text{net}} + \int_m \mathrm{d}\mathbf{f}_{\text{net}} = \int_m \ddot{\mathbf{R}}\,\mathrm{d}m$$

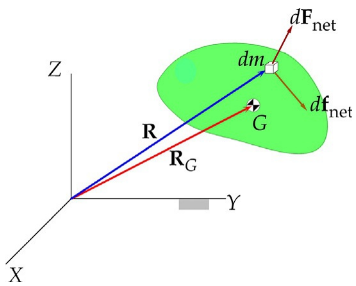


FIG. 11.7

Forces on the mass element dm of a continuous medium.

Because the internal forces occur in action–reaction pairs, $\int_m d\mathbf{f}_{\text{net}} = \mathbf{0}$. (External forces on the body are those without an internal reactant; the reactant lies outside the body and, hence, is outside our purview.) Thus,

$$\mathbf{F}_{\text{net}} = \int_m \ddot{\mathbf{R}} dm \tag{11.11}$$

where $\mathbf{F}_{\text{net}}$ is the resultant external force on the body, $\mathbf{F}_{\text{net}} = \int_m d\mathbf{F}_{\text{net}}$. From Eq. (11.9),

$$\int_m \ddot{\mathbf{R}} dm = m\ddot{\mathbf{R}}_G$$

where $\ddot{\mathbf{R}}_G = \mathbf{a}_G$, the absolute acceleration of the center of mass. Therefore, Eq. (11.11) can be written as

$$\mathbf{F}_{\text{net}} = m\ddot{\mathbf{R}}_G \tag{11.12}$$

We are therefore reminded that the motion of the center of mass of a body is determined solely by the resultant of the external forces acting on it. So far, our study of orbiting bodies has focused exclusively on the motion of their centers of mass. In this chapter, we turn our attention to rotational motion around the center of mass. To simplify things, we ultimately assume that the body is not only continuous but that it is also rigid. This means all points of the body remain at a fixed distance from each other and there is no flexing, bending, or twisting deformation.

11.4 EQUATIONS OF ROTATIONAL MOTION

Our development of the rotational dynamics equations does not require at the outset that the body under consideration be rigid. It may be a solid, fluid, or gas.

Point P in Fig. 11.8 is arbitrary; it need not be fixed in space nor attached to a point on the body. Then the moment about P of the forces on mass element dm (cf. Fig. 11.7) is

$$d\mathbf{M}_P = \mathbf{r} \times d\mathbf{F}_{\text{net}} + \mathbf{r} \times d\mathbf{f}_{\text{net}}$$

where $\mathbf{r}$ is the position vector of the mass element dm relative to the point P. Writing the right-hand side as $\mathbf{r} \times (d\mathbf{F}_{\text{net}} + d\mathbf{f}_{\text{net}})$, substituting Eq. (11.10), and integrating over all the mass elements of the body yields

$$\mathbf{M}_P)_{\text{net}} = \int_m \mathbf{r} \times \ddot{\mathbf{R}} dm \tag{11.13}$$

where $\ddot{\mathbf{R}}$ is the absolute acceleration of dm relative to the inertial frame and

$$\mathbf{M}_P)_{\text{net}} = \int_m \mathbf{r} \times d\mathbf{F}_{\text{net}} + \int_m \mathbf{r} \times d\mathbf{f}_{\text{net}}$$

But $\int_m \mathbf{r} \times d\mathbf{f}_{\text{net}} = \mathbf{0}$ because the internal forces occur in action–reaction pairs. Thus,

$$\mathbf{M}_P)_{\text{net}} = \int_m \mathbf{r} \times d\mathbf{F}_{\text{net}}$$

which means the net moment includes only the moment of all the external forces on the body.

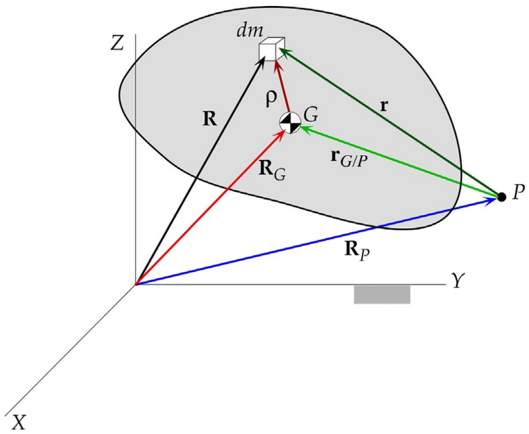


FIG. 11.8

Position vectors of a mass element in a continuum from several key reference points.

From the product rule of calculus, we know that $\mathrm{d}(\mathbf{r}\times\dot{\mathbf{R}})/\mathrm{d}t=\mathbf{r}\times\ddot{\mathbf{R}}+\dot{\mathbf{r}}\times\dot{\mathbf{R}}$, so that the integrand in Eq. (11.13) may be written as

$$\mathbf{r}\times\ddot{\mathbf{R}}=\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\dot{\mathbf{R}})-\dot{\mathbf{r}}\times\dot{\mathbf{R}} \tag{11.14}$$

Furthermore, Fig. 11.8 shows that $\mathbf{r} = \mathbf{R} - \mathbf{R}_P$, where $\mathbf{R}_P$ is the absolute position vector of P. It follows that

$$\dot{\mathbf{r}}\times\dot{\mathbf{R}}=(\dot{\mathbf{R}}-\dot{\mathbf{R}}_P)\times\dot{\mathbf{R}}=-\dot{\mathbf{R}}_P\times\dot{\mathbf{R}} \tag{11.15}$$

Substituting Eq. (11.15) into Eq. (11.14) and then moving that result into Eq. (11.13), yields

$$\mathbf{M}_P)_{\text{net}}=\frac{\mathrm{d}}{\mathrm{d}t}\int_m \mathbf{r}\times\dot{\mathbf{R}}\mathrm{d}m+\dot{\mathbf{R}}_P\times\int_m \dot{\mathbf{R}}\mathrm{d}m \tag{11.16}$$

Now, $\mathbf{r}\times\dot{\mathbf{R}}\mathrm{d}m$ is the moment of the absolute linear momentum of mass element dm about P. The moment of momentum, or angular momentum, of the entire body is the integral of this cross product over all of its mass elements. That is, the absolute angular momentum of the body relative to point P is

$$\mathbf{H}_P=\int_m \mathbf{r}\times\dot{\mathbf{R}}\mathrm{d}m \tag{11.17}$$

Observing from Fig. 11.8 that $\mathbf{r} = \mathbf{r}_{G/P} + \boldsymbol{\rho}$, we can write Eq. (11.17) as

$$\mathbf{H}_P=\int_m (\mathbf{r}_{G/P}+\boldsymbol{\rho})\times\dot{\mathbf{R}}\mathrm{d}m=\mathbf{r}_{G/P}\times\int_m \dot{\mathbf{R}}\mathrm{d}m+\int_m \boldsymbol{\rho}\times\dot{\mathbf{R}}\mathrm{d}m \tag{11.18}$$

The last term is the absolute angular momentum relative to the center of mass G,

$$\mathbf{H}_G=\int_m \boldsymbol{\rho}\times\dot{\mathbf{R}}\mathrm{d}m \tag{11.19}$$

Furthermore, by the definition of center of mass (Eq. 11.9),

$$\int_m \dot{\mathbf{R}}\,\mathrm{d}m = m\dot{\mathbf{R}}_G \tag{11.20}$$

Eqs. (11.19) and (11.20) allow us to write Eq. (11.18) as

$$\mathbf{H}_P = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_G \tag{11.21}$$

This useful relationship shows how to obtain the absolute angular momentum about any point P once $\mathbf{H}_G$ is known.

For calculating the angular momentum about the center of mass, Eq. (11.19) can be cast in a much more useful form by making the substitution (cf. Fig. 11.8) $\mathbf{R} = \mathbf{R}_G + \boldsymbol{\rho}$, so that

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times \left(\dot{\mathbf{R}}_G + \dot{\boldsymbol{\rho}}\right)\mathrm{d}m = \int_m \boldsymbol{\rho} \times \dot{\mathbf{R}}_G \mathrm{d}m + \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,\mathrm{d}m$$

In the two integrals on the right, the variable is $\boldsymbol{\rho}$. $\dot{\mathbf{R}}_G$ is fixed and can therefore be factored out of the first integral to obtain

$$\mathbf{H}_G = \left(\int_m \boldsymbol{\rho}\,\mathrm{d}m\right) \times \dot{\mathbf{R}}_G + \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,\mathrm{d}m$$

By definition of the center of mass, $\int_m \boldsymbol{\rho}\,\mathrm{d}m = 0$ (the position vector of the center of mass relative to itself is zero), which means

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,\mathrm{d}m \tag{11.22}$$

Since $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ are the position and velocity vectors relative to the center of mass G, $\int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\,\mathrm{d}m$ is the total moment about the center of mass of the linear momentum relative to the center of mass, $\mathbf{H}_G)_{\mathrm{rel}}$. In other words,

$$\mathbf{H}_G = \mathbf{H}_G)_{\mathrm{rel}} \tag{11.23}$$

This is a rather surprising fact, hidden in Eq. (11.19), and is true in general for no other point of the body.

Another useful angular momentum formula, similar to Eq. (11.21), may be found by substituting $\mathbf{R} = \mathbf{R}_P + \mathbf{r}$ into Eq. (11.17),

$$\mathbf{H}_P = \int_m \mathbf{r} \times \left(\dot{\mathbf{R}}_P + \dot{\mathbf{r}}\right)\mathrm{d}m = \left(\int_m \mathbf{r}\,\mathrm{d}m\right) \times \dot{\mathbf{R}}_P + \int_m \mathbf{r} \times \dot{\mathbf{r}}\,\mathrm{dm} \tag{11.24}$$

The term on the far right is the net moment of relative linear momentum about P,

$$\mathbf{H}_P)_{\mathrm{rel}} = \int_m \mathbf{r} \times \dot{\mathbf{r}}\,\mathrm{d}m \tag{11.25}$$

Also, $\int_m \mathbf{r}\,\mathrm{d}m = m\mathbf{r}_{G/P}$, where $\mathbf{r}_{G/P}$ is the position vector of the center of mass relative to P. Thus, Eq. (11.24) can be written as

$$\mathbf{H}_P = \mathbf{H}_P)_{\mathrm{rel}} + \mathbf{r}_{G/P} \times m\mathbf{v}_P \tag{11.26}$$

Finally, substituting this into Eq. (11.21), solving for $\mathbf{H}_P)_{\text{rel}}$, and noting that $\mathbf{v}_G - \mathbf{v}_P = \mathbf{v}_{G/P}$, yields

$$\mathbf{H}_P)_{\text{rel}} = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_{G/P} \tag{11.27}$$

This expression is useful when the absolute velocity $\mathbf{v}_G$ of the center of mass, which is required in Eq. (11.21), is not available.

So far, we have written down some formulas for calculating the angular momentum about an arbitrary point in space and about the center of mass of the body itself. Let us now return to the problem of relating angular momentum to the applied torque. Substituting Eqs. (11.17) and (11.20) into Eq. (11.16), we obtain

$$\mathbf{M}_P)_{\text{net}} = \dot{\mathbf{H}}_P + \dot{\mathbf{R}}_P \times m\dot{\mathbf{R}}_G$$

Thus, for an arbitrary point P,

$$\mathbf{M}_P)_{\text{net}} = \dot{\mathbf{H}}_P + \mathbf{v}_P \times m\mathbf{v}_G \tag{11.28}$$

where $\mathbf{v}_P$ and $\mathbf{v}_G$ are the absolute velocities of points P and G, respectively. This expression is applicable to two important special cases.

If the point P is at rest in inertial space ($\mathbf{v}_P = \mathbf{0}$), then Eq. (11.28) reduces to

$$\mathbf{M}_P)_{\text{net}} = \dot{\mathbf{H}}_P \tag{11.29}$$

This equation holds as well if $\mathbf{v}_P$ and $\mathbf{v}_G$ are parallel (e.g., if P is the point of contact of a wheel rolling while slipping in the plane). Note that the validity of Eq. (11.29) depends neither on the body being rigid nor on it being in pure rotation about P. If point P is chosen to be the center of mass, then, since $\mathbf{v}_G \times \mathbf{v}_G = \mathbf{0}$, Eq. (11.28) becomes

$$\boxed{\mathbf{M}_G)_{\text{net}} = \dot{\mathbf{H}}_G} \tag{11.30}$$

This equation is valid for any state of motion.

If Eq. (11.30) is integrated over a time interval, then we obtain the angular impulse–momentum principle,

$$\int_{t_1}^{t_2} \mathbf{M}_G)_{\text{net}} dt = \mathbf{H}_G)_2 - \mathbf{H}_G)_1 \tag{11.31}$$

A similar expression follows from Eq. (11.29). $\int \mathbf{M} dt$ is the angular impulse. If the net angular impulse is zero, then $\Delta\mathbf{H} = \mathbf{0}$, which is a statement of the conservation of angular momentum. Keep in mind that the angular impulse–momentum principle is not valid for just any reference point.

Additional versions of Eqs. (11.29) and (11.30) can be obtained that may prove useful in special circumstances. For example, substituting the expression for $\mathbf{H}_P$ (Eq. 11.21) into Eq. (11.28) yields

$$\begin{aligned}
\mathbf{M}_P)_{\text{net}} &= \left[\dot{\mathbf{H}}_G + \frac{d}{dt}\left(\mathbf{r}_{G/P} \times m\mathbf{v}_G\right)\right] + \mathbf{v}_P \times m\mathbf{v}_G \\
&= \dot{\mathbf{H}}_G + \frac{d}{dt}[(\mathbf{r}_G - \mathbf{r}_P) \times m\mathbf{v}_G] + \mathbf{v}_P \times m\mathbf{v}_G \\
&= \dot{\mathbf{H}}_G + (\mathbf{v}_G - \mathbf{v}_P) \times m\mathbf{v}_G + \mathbf{r}_{G/P} \times m\mathbf{a}_G + \mathbf{v}_P \times m\mathbf{v}_G
\end{aligned}$$

or, finally,

$$\mathbf{M}_P)_{\text{net}} = \dot{\mathbf{H}}_G + \mathbf{r}_{G/P} \times m\mathbf{a}_G \tag{11.32}$$

This expression is useful when it is convenient to compute the net moment about a point other than the center of mass. Alternatively, by simply differentiating Eq. (11.27) we get

$$\dot{\mathbf{H}}_P)_{\text{rel}} = \dot{\mathbf{H}}_G + \overbrace{\mathbf{v}_{G/P} \times m\mathbf{v}_{G/P}}^{=\mathbf{0}} + \mathbf{r}_{G/P} \times m\mathbf{a}_{G/P}$$

Solving for $\dot{\mathbf{H}}_G$, invoking Eq. (11.30), and using the fact that $\mathbf{a}_{P/G} = -\mathbf{a}_{G/P}$ leads to

$$\mathbf{M}_G)_{\text{net}} = \dot{\mathbf{H}}_P)_{\text{rel}} + \mathbf{r}_{G/P} \times m\mathbf{a}_{P/G} \tag{11.33}$$

Finally, if the body is rigid, the magnitude of the position vector $\boldsymbol{\rho}$ of any point relative to the center of mass does not change with time. Therefore, Eq. (1.52) requires that $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$, leading us to conclude from Eq. (11.22) that

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m \quad \text{(Rigid body)} \tag{11.34}$$

Again, the absolute angular momentum about the center of mass depends only on the absolute angular velocity and not on the absolute translational velocity of any point of the body.

No such simplification of Eq. (11.17) exists for an arbitrary reference point P. However, if the point P is fixed in inertial space and the rigid body is rotating about P, then the position vector $\mathbf{r}$ from P to any point of the body is constant. It follows from Eq. (1.52) that $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. According to Fig. 11.8,

$$\mathbf{R} = \mathbf{R}_P + \mathbf{r}$$

Differentiating with respect to time gives

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_P + \dot{\mathbf{r}} = \mathbf{0} + \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}$$

Substituting this into Eq. (11.17) yields the formula for angular momentum in this special case as

$$\mathbf{H}_P = \int_m \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\mathrm{d}m \quad \text{(Rigid body rotating about fixed point } P\text{)} \tag{11.35}$$

Although Eqs. (11.34) and (11.35) are mathematically identical, we must keep in mind the notation of Fig. 11.8. Eq. (11.35) applies only if the rigid body is in pure rotation about a stationary point in inertial space, whereas Eq. (11.34) applies unconditionally to any situation.

11.5 MOMENTS OF INERTIA

To use Eq. (11.29) or Eq. (11.30) to solve problems, the vectors within them have to be resolved into components. To find the components of angular momentum, we must appeal to its definition. We focus on the formula for angular momentum of a rigid body about its center of mass (Eq. (11.34)) because the expression for fixed point rotation (Eq. (11.35)) is mathematically the same. The integrand of Eq. (11.34) can be rewritten using the bac–cab vector identity presented in Eq. (2.33),

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \boldsymbol{\omega}\rho^2 - \boldsymbol{\rho}(\boldsymbol{\omega} \cdot \boldsymbol{\rho}) \tag{11.36}$$

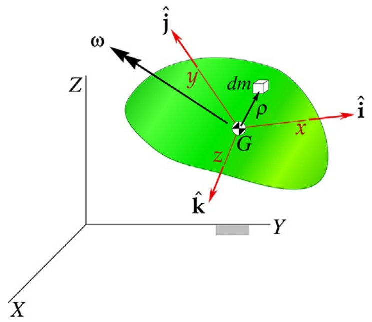


FIG. 11.9

Comoving *xyz* frame used to compute the moments of inertia.

Let the origin of a comoving *xyz* coordinate system be attached to the center of mass G, as shown in Fig. 11.9. The unit vectors of this frame are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. The vectors $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ can be resolved into components in the *xyz* directions to get $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. Substituting these vector expressions into the right-hand side of Eq. (11.36) yields

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \left(\omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}\right)\left(x^2 + y^2 + z^2\right) - \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\left(\omega_x x + \omega_y y + \omega_z z\right)$$

Expanding the right-hand side and collecting the terms having the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in common, we get

$$\begin{aligned}\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = &\left[\left(y^2 + z^2\right)\omega_x - xy\omega_y - xz\omega_z\right]\hat{\mathbf{i}} + \left[-yx\omega_x + \left(x^2 + z^2\right)\omega_y - yz\omega_z\right]\hat{\mathbf{j}} \\ &+ \left[-zx\omega_x - zy\omega_y + \left(x^2 + y^2\right)\omega_z\right]\hat{\mathbf{k}}\end{aligned} \tag{11.37}$$

We put this result into the integrand of Eq. (11.34) to obtain

$$\mathbf{H}_G = H_x\hat{\mathbf{i}} + H_y\hat{\mathbf{j}} + H_z\hat{\mathbf{k}} \tag{11.38}$$

where

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{11.39a}$$

or, in matrix notation,

$$\{\mathbf{H}\} = [\mathbf{I}]\{\boldsymbol{\omega}\} \tag{11.39b}$$

The nine components of the moment of inertia matrix $[\mathbf{I}]$ about the center of mass are

$$\begin{aligned} I_x &= \int_m (y^2 + z^2)\mathrm{d}m & I_{xy} &= -\int_m xy\mathrm{d}m & I_{xz} &= -\int_m xz\mathrm{d}m \\ I_{yx} &= -\int_m yx\mathrm{d}m & I_y &= \int_m (x^2 + z^2)\mathrm{d}m & I_{yz} &= -\int_m yz\mathrm{d}m \\ I_{zx} &= -\int_m zx\mathrm{d}m & I_{zy} &= -\int_m zy\mathrm{d}m & I_z &= \int_m (x^2 + y^2)\mathrm{d}m \end{aligned} \tag{11.40}$$

Since $I_{yx}=I_{xy}$, $I_{zx}=I_{xz}$, and $I_{zy}=I_{yz}$, it follows that $[\mathbf{I}]$ is a symmetric matrix (i.e., $[\mathbf{I}]^T=[\mathbf{I}]$). Therefore, $[\mathbf{I}]$ has just six independent components instead of nine. Observe that, whereas the products of inertia I_{xy}, I_{xz}, and I_{yz} can be positive, negative, or zero, the moments of inertia I_x, I_y, and I_z are always positive (never zero or negative) for bodies of finite dimensions. For this reason, $[\mathbf{I}]$ is a symmetric positive definite matrix. Keep in mind that Eqs. (11.38) and (11.39) are valid as well for axes attached to a fixed point P about which the body is rotating.

The moments of inertia reflect how the mass of a rigid body is distributed. They manifest a body's rotational inertia (i.e., its resistance to being set into rotary motion or stopped once rotation is under way). It is not an object's mass alone but how that mass is distributed that determines how the body will respond to applied torques.

If the xy plane is a plane of symmetry, then for any x and y within the body there are identical mass elements located at $+z$ and $-z$. This means the products of inertia with z in the integrand vanish. Similar statements are true if xz or yz are symmetry planes. In summary, we conclude

If the xy plane is a plane of symmetry of the body, then $I_{xz}=I_{yz}=0$.
If the xz plane is a plane of symmetry of the body, then $I_{xy}=I_{yz}=0$.
If the yz plane is a plane of symmetry of the body, then $I_{xy}=I_{xz}=0$.

It follows that if the body has two planes of symmetry relative to the xyz frame of reference, then all three products of inertia vanish, and $[\mathbf{I}]$ becomes a diagonal matrix such that,

$$[\mathbf{I}]=\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{11.41}$$

where A, B, and C are the principal moments of inertia (all positive), and the xyz axes are the body's principal axes of inertia or principal directions. In this case, relative to either the center of mass or a fixed point of rotation, as appropriate, we have

$$H_x=A\omega_x \quad H_y=B\omega_y \quad H_z=C\omega_z \tag{11.42}$$

In general, the angular velocity vector $\boldsymbol{\omega}$ and the angular momentum vector $\mathbf{H}$ are not parallel ($\boldsymbol{\omega}\times\mathbf{H}\neq 0$). However, if, for example, $\boldsymbol{\omega}=\omega\hat{\mathbf{i}}$, then according to Eq. (11.42), $\mathbf{H}=A\boldsymbol{\omega}$. In other words, if the angular velocity is aligned with a principal direction, so is the angular momentum. In that case, the two vectors $\boldsymbol{\omega}$ and $\mathbf{H}$ are indeed parallel.

Each of the three principal moments of inertia can be expressed as follows:

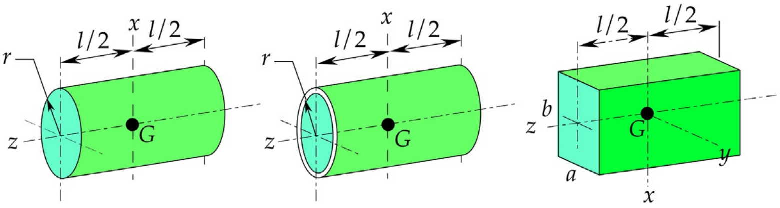
$$A=mk_x{}^2 \quad B=mk_y{}^2 \quad C=mk_z{}^2 \tag{11.43}$$

where m is the mass of the body and k_x, k_y, and k_z are the three radii of gyration. One may imagine the mass of a body to be concentrated around a principal axis at a distance equal to the radius of gyration.

The moments of inertia for several common shapes are listed in Fig. 11.10. By symmetry, their products of inertia vanish for the coordinate axes used. Formulas for other solid geometries can be found in engineering handbooks and in dynamics textbooks.

For a mass concentrated at a point, the moments of inertia in Eq. (11.40) are just the mass times the integrand evaluated at the point. That is, the moment of inertia matrix $[\mathbf{I}^{(m)}]$ of a point mass m is given by

$$\left[\mathbf{I}^{(m)}\right]=\begin{bmatrix} m(y^2+z^2) & -mxy & -mxz \\ -mxy & m(x^2+z^2) & -myz \\ -mxz & -myz & m(x^2+y^2) \end{bmatrix} \tag{11.44}$$



$$I_x = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$$
$$I_z = \frac{1}{2}mr^2$$

(a)

$$I_x = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$$
$$I_z = mr^2$$

(b)

$$I_x = \frac{1}{12}m(a^2 + l^2)$$
$$I_y = \frac{1}{12}m(b^2 + l^2)$$
$$I_z = \frac{1}{12}m(a^2 + b^2)$$

(c)

FIG. 11.10

Moments of inertia for three common homogeneous solids of mass *m*. (a) Solid circular cylinder. (b) Circular cylindrical shell. (c) Rectangular parallelepiped.

EXAMPLE 11.4

The following table lists the mass and coordinates of seven point masses. Find the center of mass of the system and the moments of inertia about the origin.

Point, i	Mass, m_i (kg)	x_i (m)	y_i (m)	z_i (m)
1	3	−0.5	0.2	0.3
2	7	0.2	0.75	−0.4
3	5	1	−0.8	0.9
4	6	1.2	−1.3	1.25
5	2	−1.3	1.4	−0.8
6	4	−0.3	1.35	0.75
7	1	1.5	−1.7	0.85

Solution

The total mass of this system is

$$m = \sum_{i=1}^{7} m_i = 28 \text{ kg}$$

For concentrated masses, the integral in Eq. (11.9) is replaced by the mass times its position vector. Therefore, in this case, the three components of the position vector of the center of mass are $x_G = (1/m)\sum_{i=1}^{7} m_i x_i$, $y_G = (1/m)\sum_{i=1}^{7} m_i y_i$, and $z_G = (1/m)\sum_{i=1}^{7} m_i z_i$, so that

$$\boxed{x_G = 0.35 \text{ m} \quad y_G = 0.01964 \text{ m} \quad z_G = 0.4411 \text{ m}}$$

The total moment of inertia is the sum over all the particles of Eq. (11.44) evaluated at each point. Thus,

$$[\mathbf{I}] = \overbrace{\begin{bmatrix} 0.39 & 0.3 & 0.45 \\ 0.3 & 1.02 & -0.18 \\ 0.45 & -0.18 & 0.87 \end{bmatrix}}^{(1)} + \overbrace{\begin{bmatrix} 5.0575 & -1.05 & 0.56 \\ -1.05 & 1.4 & 2.1 \\ 0.56 & 2.1 & 4.2175 \end{bmatrix}}^{(2)} + \overbrace{\begin{bmatrix} 7.25 & 4 & -4.5 \\ 4 & 9.05 & 3.6 \\ -4.5 & 3.6 & 8.2 \end{bmatrix}}^{(3)}$$

$$+ \overbrace{\begin{bmatrix} 19.515 & 9.36 & -9 \\ 9.36 & 18.015 & 9.75 \\ -9 & 9.75 & 18.78 \end{bmatrix}}^{(4)} + \overbrace{\begin{bmatrix} 5.2 & 3.64 & -2.08 \\ 3.64 & 4.66 & 2.24 \\ -2.08 & 2.24 & 7.3 \end{bmatrix}}^{(5)} + \overbrace{\begin{bmatrix} 9.54 & 1.62 & 0.9 \\ 1.62 & 2.61 & -4.05 \\ 0.9 & -4.05 & 7.65 \end{bmatrix}}^{(6)}$$

$$+ \overbrace{\begin{bmatrix} 3.6125 & 2.55 & -1.275 \\ 2.55 & 2.9725 & 1.445 \\ -1.275 & 1.445 & 5.14 \end{bmatrix}}^{(7)}$$

or

$$\boxed{[\mathbf{I}] = \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} (\mathrm{kg \cdot m^2})}$$

EXAMPLE 11.5

Calculate the moments of inertia of a slender, homogeneous straight rod of length ℓ and mass m shown in Fig. 11.11. One end of the rod is at the origin and the other has coordinates (a, b, c).

Solution

A slender rod is one whose cross-sectional dimensions are negligible compared with its length. The mass is concentrated along its centerline. Since the rod is homogeneous, the mass per unit length ρ is uniform and given by

$$\rho = \frac{m}{\ell} \tag{a}$$

The length of the rod is

$$\ell = \sqrt{a^2 + b^2 + c^2}$$

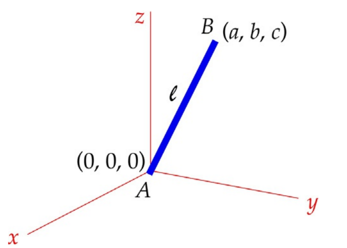


FIG. 11.11

Uniform slender bar of mass m and length ℓ.

Starting with I_x we have from Eq. (11.40),

$$I_x = \int_0^{\ell} (y^2 + z^2)\rho \mathrm{d}s$$

in which we replaced the element of mass dm by ρds, where ds is the element of length along the rod. The distance s is measured from end A of the rod, so that the x, y, and z coordinates of any point along it are found in terms of s by the following relations:

$$x = \frac{s}{\ell}a \quad y = \frac{s}{\ell}b \quad z = \frac{s}{\ell}c$$

Thus,

$$I_x = \int_0^{\ell} \left(\frac{s}{\ell^2}b^2 + \frac{s}{\ell^2}c^2\right)\rho \mathrm{d}s = \rho\frac{b^2 + c^2}{\ell^2}\int_0^{\ell} s^2 \mathrm{d}s = \frac{1}{3}\rho\left(b^2 + c^2\right)\ell$$

Substituting Eq. (a) yields

$$\boxed{I_x = \frac{1}{3}m\left(b^2 + c^2\right)}$$

In precisely the same way, we find

$$\boxed{I_y = \frac{1}{3}m\left(a^2 + c^2\right)} \quad \boxed{I_z = \frac{1}{3}m\left(a^2 + b^2\right)}$$

For I_{xy} we have

$$I_{xy} = -\int_0^{\ell} xy\rho \mathrm{d}s = -\int_0^{\ell} \left(\frac{s}{\ell}a\right)\left(\frac{s}{\ell}b\right)\rho \mathrm{d}s = -\rho\frac{ab}{\ell^2}\int_0^{\ell} s^2 \mathrm{d}s = -\frac{1}{3}\rho ab\ell$$

Once again using Eq. (a),

$$\boxed{I_{xy} = -\frac{1}{3}mab}$$

Likewise,

$$\boxed{I_{xz} = -\frac{1}{3}mac} \quad \boxed{I_{yz} = -\frac{1}{3}mbc}$$

EXAMPLE 11.6

The gyro rotor (Fig. 11.12) in Example 11.3 has a mass m of 5 kg, radius r of 0.08 m, and thickness t of 0.025 m. If $N = 2.1$ rad/s, $\dot{\theta} = 4$ rad/s, $\omega = 10.5$ rad/s, and $\theta = 60°$, calculate
(a) the angular momentum of the rotor about its center of mass G in the body-fixed xyz frame and
(b) the angle between the rotor's angular velocity vector and its angular momentum vector.

Solution

Eq. (f) from Example 11.3 gives the components of the absolute angular velocity of the rotor in the moving xyz frame.

$$\begin{aligned}
\omega_x &= \dot{\theta} = 4 \text{ rad/s} \\
\omega_y &= N\sin\theta = 2.1 . \sin 60° = 1.819 \text{ rad/s} \\
\omega_z &= \omega_{\text{spin}} + N\cos\theta = 10.5 + 2.1 . \cos 60° = 11.55 \text{ rad/s}
\end{aligned} \tag{a}$$

Therefore,

$$\boldsymbol{\omega} = 4\hat{\mathbf{i}} + 1.819\hat{\mathbf{j}} + 11.55\hat{\mathbf{k}} \text{ (rad/s)} \tag{b}$$

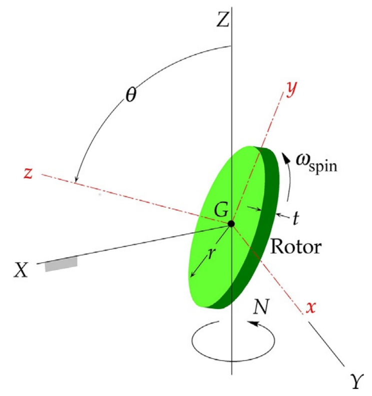


FIG. 11.12

Rotor of the gyroscope in Fig. 11.4.

All three coordinate planes of the body-fixed xyz frame contain the center of mass G and all are planes of symmetry of the circular cylindrical rotor. Therefore, $I_{xy} = I_{xz} = I_{yz} = 0$.

From Fig. 11.10a, we see that the nonzero diagonal entries in the moment of inertia tensor are

$$\begin{aligned} A = B &= \frac{1}{12}mt^2 + \frac{1}{4}mr^2 = \frac{1}{12}(5)(0.025)^2 + \frac{1}{4}(5)(0.08)^2 = 0.008260 \text{ kg}\cdot\text{m}^2 \\ C &= \frac{1}{2}mr^2 = \frac{1}{2}(5)(0.08)^2 = 0.0160 \text{ kg}\cdot\text{m}^2 \end{aligned} \tag{c}$$

We can use Eq. (11.42) to calculate the angular momentum, because the origin of the xyz frame is the rotor's center of mass (which in this case also happens to be a fixed point of rotation, which is another reason why we can use Eq. 11.42). Substituting Eqs. (a) and (c) into Eq. (11.42) yields

$$\begin{aligned} H_x &= A\omega_x = (0.008260)(4) = 0.03304 \text{ kg}\cdot\text{m}^2/\text{s} \\ H_y &= B\omega_y = (0.008260)(1.819) = 0.0150 \text{ kg}\cdot\text{m}^2/\text{s} \\ H_z &= C\omega_z = (0.0160)(11.55) = 0.1848 \text{ kg}\cdot\text{m}^2/\text{s} \end{aligned} \tag{d}$$

so that

$$\boxed{\mathbf{H} = 0.03304\hat{\mathbf{i}} + 0.0150\hat{\mathbf{j}} + 0.1848\hat{\mathbf{k}} \text{ (kg}\cdot\text{m}^2/\text{s)}} \tag{e}$$

The angle ϕ between $\mathbf{H}$ and $\boldsymbol{\omega}$ is found by taking the dot product of the two vectors,

$$\phi = \cos^{-1}\left(\frac{\mathbf{H}\cdot\boldsymbol{\omega}}{H\omega}\right) = \cos^{-1}\left(\frac{2.294}{0.1883\cdot 12.36}\right) = \boxed{9.717^\circ} \tag{f}$$

As this problem illustrates, the angular momentum and the angular velocity are in general not collinear.

Consider a Cartesian coordinate system $x'y'z'$ with the same origin as xyz but a different orientation. Let $[\mathbf{Q}]$ be the orthogonal matrix ($[\mathbf{Q}]^{-1} = [\mathbf{Q}]^T$) that transforms the components of a vector from the xyz system to the $x'y'z'$ frame. Recall from Section 4.5 that the rows of $[\mathbf{Q}]$ are the direction cosines of the $x'y'z'$ axes relative to xyz. If $\{\mathbf{H}'\}$ comprises the components of the angular momentum vector along the $x'y'z'$ axes, then $\{\mathbf{H}'\}$ is obtained from its components $\{\mathbf{H}\}$ in the xyz frame by the relation

$$\{\mathbf{H}'\} = [\mathbf{Q}]\{\mathbf{H}\}$$

From Eq. (11.39), we can write this as

$$\{\mathbf{H}'\} = [\mathbf{Q}][\mathbf{I}]\{\boldsymbol{\omega}\} \tag{11.45}$$

where $[\mathbf{I}]$ is the moment of inertia matrix (Eq. 11.39) in xyz coordinates. Like the angular momentum vector, the components $\{\boldsymbol{\omega}\}$ of the angular velocity vector in the xyz system are related to those in the primed system ($\{\boldsymbol{\omega}'\}$) by the expression

$$\{\boldsymbol{\omega}'\} = [\mathbf{Q}]\{\boldsymbol{\omega}\}$$

The inverse relation is simply

$$\{\boldsymbol{\omega}\} = [\mathbf{Q}]^{-1}\{\boldsymbol{\omega}'\} = [\mathbf{Q}]^T\{\boldsymbol{\omega}'\} \tag{11.46}$$

Substituting this into Eq. (11.45), we get

$$\{\mathbf{H}'\} = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T\{\boldsymbol{\omega}'\} \tag{11.47}$$

But the components of angular momentum and angular velocity in the $x'y'z'$ frame are related by an equation of the same form as Eq. (11.39), so that

$$\{\mathbf{H}'\} = [\mathbf{I}']\{\boldsymbol{\omega}'\} \tag{11.48}$$

where $[\mathbf{I}']$ comprises the components of the inertia matrix in the primed system. Comparing the right-hand sides of Eqs. (11.47) and (11.48), we conclude that

$$[\mathbf{I}'] = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T \tag{11.49a}$$

That is,

$$\begin{bmatrix} I_{x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \tag{11.49b}$$

This shows how to transform the components of the inertia matrix from the xyz coordinate system to any other orthogonal system with a common origin. Thus, for example,

$$\begin{aligned} I_{x'} &= \overbrace{\lfloor Q_{11}\ Q_{12}\ Q_{13} \rfloor}^{\lfloor \text{Row } 1 \rfloor} \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \overbrace{\begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{13} \end{Bmatrix}}^{\lfloor \text{Row } 1 \rfloor^T} \\ I_{y'z'} &= \overbrace{\lfloor Q_{21}\ \ Q_{22}\ \ Q_{23} \rfloor}^{\lfloor \text{Row } 2 \rfloor} \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \overbrace{\begin{Bmatrix} Q_{31} \\ Q_{32} \\ Q_{33} \end{Bmatrix}}^{\lfloor \text{Row } 3 \rfloor^T} \end{aligned} \tag{11.50}$$

etc.

Any object represented by a square matrix whose components transform according to Eq. (11.49) is called a second-order tensor. We may therefore refer to $[\mathbf{I}]$ as the inertia tensor.

EXAMPLE 11.7

Find the mass moment of inertia of the system of point masses in Example 11.4 about an axis from the origin through the point with coordinates (2 m, −3 m, 4 m).

Solution

From Example 11.4, the moment of inertia tensor for the system of point masses is

$$[\mathbf{I}] = \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} \left(\mathrm{kg}\cdot\mathrm{m}^2\right)$$

The vector $\mathbf{V}$ connecting the origin with (2 m, −3 m, 4 m) is

$$\mathbf{V} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

The unit vector in the direction of $\mathbf{V}$ is

$$\hat{\mathbf{u}}_V = \frac{\mathbf{V}}{\|\mathbf{V}\|} = 0.3714\hat{\mathbf{i}} - 0.5571\hat{\mathbf{j}} + 0.7428\hat{\mathbf{k}}$$

We may consider $\hat{\mathbf{u}}_V$ as the unit vector along the x' axis of a rotated Cartesian coordinate system. Then, from Eq. (11.50) we get

$$I_{V'} = \lfloor 0.3714 \quad -0.5571 \quad 0.7428 \rfloor \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} \begin{bmatrix} 0.3714 \\ -0.5571 \\ 0.7428 \end{bmatrix}$$

$$= \lfloor 0.3714 \quad -0.5571 \quad 0.7428 \rfloor \begin{Bmatrix} -3.695 \\ -3.482 \\ 24.90 \end{Bmatrix} = \boxed{19.06\ \mathrm{kg}\cdot\mathrm{m}^2}$$

EXAMPLE 11.8

For the satellite of Example 11.2, which is reproduced in Fig. 11.13, the data are as follows: $N = 0.1$ rad/s and $\dot{\theta} = 0.01\ \mathrm{rad/s}$, in the directions shown. $\theta = 40°$ and $d_0 = 1.5$ m. The length, width, and thickness of the panel are $\ell = 6$ m, $w = 2$ m, and $t = 0.025$ m. The uniformly distributed mass of the panel is 50 kg. Find the angular momentum of the panel relative to the center of mass O of the satellite.

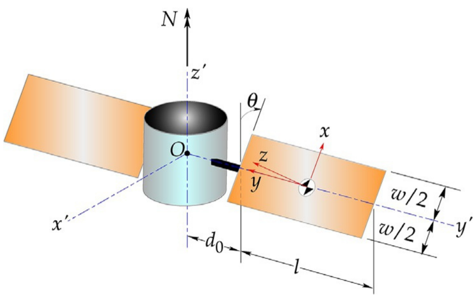


FIG. 11.13

Satellite and solar panel.

Solution

We can treat the panel as a thin parallelepiped. The panel's xyz axes have their origin at the center of mass G of the panel and are parallel to its three edge directions. According to Fig. 11.10c, the moments of inertia of the panel relative to the xyz coordinate system are

$$\begin{aligned}
I_G)_x &= \frac{1}{12}m\left(\ell^2+t^2\right) = \frac{1}{12}\cdot 50\cdot\left(6^2+0.025^2\right) = 150.0\ \mathrm{kg\cdot m^2}\\
I_G)_y &= \frac{1}{12}m\left(w^2+t^2\right) = \frac{1}{12}\cdot 50\cdot\left(2^2+0.025^2\right) = 16.67\ \mathrm{kg\cdot m^2}\\
I_G)_z &= \frac{1}{12}m\left(w^2+\ell^2\right) = \frac{1}{12}\cdot 50\cdot\left(2^2+6^2\right) = 166.7\ \mathrm{kg\cdot m^2}\\
I_G)_{xy} &= I_G)_{xz} = I_G)_{yz} = 0
\end{aligned} \tag{a}$$

In matrix notation,

$$[\mathbf{I}_G] = \begin{bmatrix} 150.0 & 0 & 0\\ 0 & 16.67 & 0\\ 0 & 0 & 166.7\end{bmatrix}\left(\mathrm{kg\cdot m^2}\right) \tag{b}$$

The unit vectors of the satellite's $x'y'z'$ system are related to those of the panel's xyz frame by inspection.

$$\begin{aligned}
\hat{\mathbf{i}}' &= -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{k}} = -0.6428\hat{\mathbf{i}} + 0.7660\hat{\mathbf{k}}\\
\hat{\mathbf{j}}' &= -\hat{\mathbf{j}}\\
\hat{\mathbf{k}}' &= \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{k}} = 0.7660\hat{\mathbf{i}} + 0.6428\hat{\mathbf{k}}
\end{aligned} \tag{c}$$

The matrix $[\mathbf{Q}]$ of the transformation from xyz to $x'y'z'$ comprises the direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$, which we infer from Eqs. (c),

$$[\mathbf{Q}] = \begin{bmatrix} -0.6428 & 0 & 0.7660\\ 0 & -1 & 0\\ 0.7660 & 0 & 0.6428\end{bmatrix} \tag{d}$$

In Example 11.2, we found that the absolute angular velocity of the panel, in the satellite's $x'y'z'$ frame of reference, is

$$\boldsymbol{\omega} = -\dot{\theta}\hat{\mathbf{j}}' + N\hat{\mathbf{k}}' = -0.01\hat{\mathbf{j}}' + 0.1\hat{\mathbf{k}}'\ \left(\mathrm{rad/s}\right)$$

That is,

$$\{\boldsymbol{\omega}'\} = \begin{Bmatrix} 0\\ -0.01\\ 0.1\end{Bmatrix}\left(\mathrm{rad/s}\right) \tag{e}$$

To find the absolute angular momentum $\{\mathbf{H}'_G\}$ of the panel in the satellite system requires the use of Eq. (11.39),

$$\left\{\mathbf{H}'_G\right\} = \left[\mathbf{I}'_G\right]\{\boldsymbol{\omega}'\} \tag{f}$$

Before doing so, we must transform the components of the moments of the inertia tensor in Eq. (b) from the unprimed (panel) system to the primed (satellite) system, by means of Eq. (11.49),

$$\begin{aligned}
\left[\mathbf{I}'_G\right] &= [\mathbf{Q}][\mathbf{I}_G][\mathbf{Q}]^T\\
&= \begin{bmatrix} -0.6428 & 0 & 0.7660\\ 0 & -1 & 0\\ 0.7660 & 0 & 0.6428\end{bmatrix}\begin{bmatrix} 150 & 0 & 0\\ 0 & 16.67 & 0\\ 0 & 0 & 166.7\end{bmatrix}\begin{bmatrix} -0.6428 & 0 & 0.7660\\ 0 & -1 & 0\\ 0.7660 & 0 & 0.6428\end{bmatrix}
\end{aligned}$$

so that

$$\left[\mathbf{I}'_G\right] = \begin{bmatrix} 159.8 & 0 & 8.205\\ 0 & 16.67 & 0\\ 8.205 & 0 & 156.9\end{bmatrix}\left(\mathrm{kg\cdot m^2}\right) \tag{g}$$

Then Eq. (f) yields

$$\{\mathbf{H}'_G\} = \begin{bmatrix} 159.8 & 0 & 8.205 \\ 0 & 16.67 & 0 \\ 8.205 & 0 & 156.9 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.01 \\ 0.1 \end{Bmatrix} = \begin{Bmatrix} 0.8205 \\ -0.1667 \\ 15.69 \end{Bmatrix} (\mathrm{kg \cdot m^2/s})$$

or, in vector notation,

$$\mathbf{H}_G = 0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 15.69\mathbf{k}' \ (\mathrm{kg \cdot m^2/s}) \tag{h}$$

This is the absolute angular momentum of the panel about its own center of mass G, and it is used in Eq. (11.27) to calculate the angular momentum $\mathbf{H}_O)_{\text{rel}}$ relative to the satellite's center of mass O,

$$\mathbf{H}_O)_{\text{rel}} = \mathbf{H}_G + \mathbf{r}_{G/O} \times m\mathbf{v}_{G/O} \tag{i}$$

$\mathbf{r}_{G/O}$ is the position vector from O to G,

$$\mathbf{r}_{G/O} = \left(d_O + \frac{\ell}{2}\right)\hat{\mathbf{j}}' = \left(1.5 + \frac{6}{2}\right)\hat{\mathbf{j}}' = 4.5\hat{\mathbf{j}}' \ (\mathrm{m}) \tag{j}$$

The velocity of G relative to O, $\mathbf{v}_{G/O}$, is found from Eq. (11.2),

$$\mathbf{v}_{G/O} = \boldsymbol{\omega}_{\text{satellite}} \times \mathbf{r}_{G/O} = N\hat{\mathbf{k}}' \times \mathbf{r}_{G/O} = 0.1\hat{\mathbf{k}}' \times 4.5\hat{\mathbf{j}}' = -0.45\hat{\mathbf{i}}' \ (\mathrm{m/s}) \tag{k}$$

Substituting Eqs. (h), (j), and (k) into Eq. (i) finally yields

$$\begin{aligned} \mathbf{H}_O)_{\text{rel}} &= \left(0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 15.69\hat{\mathbf{k}}'\right) + 4.5\hat{\mathbf{j}}' \times \left[50\left(-0.45\hat{\mathbf{i}}'\right)\right] \\ &= \boxed{0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 116.9\hat{\mathbf{k}}' \ (\mathrm{kg \cdot m^2/s})} \end{aligned} \tag{l}$$

Note that we were unable to use Eq. (11.21) to find the absolute angular momentum $\mathbf{H}_O$ because that requires knowing the absolute velocity $\mathbf{v}_G$, which in turn depends on the absolute velocity of O, which was not provided.

How can we find the direction cosine matrix $[\mathbf{Q}]$ such that Eq. (11.49) will yield a moment of inertia matrix $[\mathbf{I}']$ that is diagonal (i.e., of the form given by Eq. (11.41))? In other words, how do we find the principal directions (eigenvectors) and the corresponding principal values (eigenvalues) of the moment of inertia tensor?

Let the angular velocity vector $\boldsymbol{\omega}$ be parallel to the principal direction defined by the vector $\mathbf{e}$, so that $\boldsymbol{\omega} = \beta\mathbf{e}$, where β is a scalar. Since $\boldsymbol{\omega}$ points in the principal direction of the inertia tensor, so must $\mathbf{H}$, which means $\mathbf{H}$ is also parallel to $\mathbf{e}$. Therefore, $\mathbf{H} = \alpha\mathbf{e}$, where α is a scalar. From Eq. (11.39), it follows that

$$\alpha\{\mathbf{e}\} = \{\mathbf{I}\}(\beta\{\mathbf{e}\})$$

or

$$[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$$

where $\lambda = \alpha/\beta$ (a scalar). That is,

$$\begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \lambda \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix}$$

This can be written

$$\begin{bmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{11.51}$$

The trivial solution of Eq. (11.51) is $\mathbf{e} = \mathbf{0}$, which is of no interest. The only way that Eq. (11.51) will not yield the trivial solution is if the coefficient matrix on the left is singular. That will occur if its determinant vanishes. That is, if

$$\begin{vmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{vmatrix} = 0 \tag{11.52}$$

Expanding the determinant, we find

$$\begin{vmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{vmatrix} = -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 \tag{11.53}$$

where

$$J_1 = I_x + I_y + I_z \qquad J_2 = \begin{vmatrix} I_x & I_{xy} \\ I_{xy} & I_y \end{vmatrix} + \begin{vmatrix} I_x & I_{xz} \\ I_{xz} & I_z \end{vmatrix} + \begin{vmatrix} I_y & I_{yz} \\ I_{yz} & I_z \end{vmatrix} \qquad J_3 = \begin{vmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{vmatrix} \tag{11.54}$$

J_1, J_2, and J_3 are invariants (i.e., they have the same value in every Cartesian coordinate system).

Eqs. (11.52) and (11.53) yield the characteristic equation of the tensor [**I**],

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0 \tag{11.55}$$

The three roots λ_p ($p = 1, 2, 3$) of this cubic equation are real, since [**I**] is symmetric; furthermore, they are all positive, since [**I**] is a positive definite matrix. We substitute each root, or eigenvalue, λ_p back into Eq. (11.51) to obtain

$$\begin{bmatrix} I_x - \lambda_p & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda_p & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda_p \end{bmatrix} \begin{Bmatrix} e_x^{(p)} \\ e_y^{(p)} \\ e_z^{(p)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (p = 1, 2, 3) \tag{11.56}$$

Solving this system yields the three eigenvectors $\mathbf{e}^{(p)}$ corresponding to each of the three eigenvalues λ_p. The three eigenvectors are orthogonal, also due to the symmetry of matrix [**I**]. Each eigenvalue is a principal moment of inertia and its corresponding eigenvector is a principal direction.

EXAMPLE 11.9

Find the principal moments of inertia and the principal axes of inertia of the inertia tensor

$$[\mathbf{I}] = \begin{bmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{bmatrix} \text{kg} \cdot \text{m}^2$$

Solution

We seek the nontrivial solutions of the system $[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$. That is,

$$\begin{bmatrix} 100-\lambda & -20 & -100 \\ -20 & 300-\lambda & -50 \\ -100 & -50 & 500-\lambda \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{a}$$

From Eq. (11.54),

$$\begin{aligned} J_1 &= 100+300+500 = 900 \\ J_2 &= \begin{vmatrix} 100 & -20 \\ -20 & 300 \end{vmatrix} + \begin{vmatrix} 100 & -100 \\ -100 & 500 \end{vmatrix} + \begin{vmatrix} 300 & -50 \\ -50 & 500 \end{vmatrix} = 217{,}100 \\ J_3 &= \begin{vmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{vmatrix} = 11{,}350{,}000 \end{aligned} \tag{b}$$

Thus, the characteristic equation is

$$\lambda^3 - 900\lambda^2 + 217{,}100\lambda - 11{,}350{,}000 = 0 \tag{c}$$

The three roots are the principal moments of inertia, which are found to be

$$\boxed{\lambda_1 = 532.052 \quad \lambda_2 = 295.840 \quad \lambda_3 = 72.1083}\,(\mathrm{kg\cdot m^2}) \tag{d}$$

We substitute each of these, in turn, back into Eq. (a) to find its corresponding principal direction.

Substituting $\lambda_1 = 532.052\ \mathrm{kg\cdot m^2}$ into Eq. (a) we obtain

$$\begin{bmatrix} -432.052 & -20.0000 & -100.0000 \\ -20.0000 & -232.052 & -50.0000 \\ -100.0000 & -50.0000 & -32.0519 \end{bmatrix} \begin{Bmatrix} e_x^{(1)} \\ e_y^{(1)} \\ e_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{e}$$

Since the determinant of the coefficient matrix is zero, at most two of the three equations in Eq. (e) are independent. Thus, at most, two of the three components of the vector $\mathbf{e}^{(1)}$ can be found in terms of the third. We can therefore arbitrarily set $e_x^{(1)} = 1$ and solve for $e_y^{(1)}$ and $e_z^{(1)}$ using any two of the independent equations in Eq. (e). With $e_x^{(1)} = 1$, the first two of Eq. (e) become

$$\begin{aligned} -20.0000e_y^{(1)} - 100.000e_z^{(1)} &= 432.052 \\ -232.052e_y^{(1)} - 50.000e_z^{(1)} &= 20.0000 \end{aligned} \tag{f}$$

Solving these two equations for $e_y^{(1)}$ and $e_z^{(1)}$ yields, together with the assumption that $e_x^{(1)} = 1$,

$$e_x^{(1)} = 1.00000 \quad e_y^{(1)} = 0.882793 \quad e_z^{(1)} = -4.49708 \tag{g}$$

The unit vector in the direction of $\mathbf{e}^{(1)}$ is

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}^{(1)}}{\|\mathbf{e}^{(1)}\|} = \frac{1.00000\hat{\mathbf{i}} + 0.882793\hat{\mathbf{j}} - 4.49708\hat{\mathbf{k}}}{\sqrt{1.00000^2 + 0.882793^2 + (-4.49708)^2}}$$

or

$$\boxed{\hat{\mathbf{e}}_1 = 0.213186\hat{\mathbf{i}} + 0.188199\hat{\mathbf{j}} - 0.958714\hat{\mathbf{k}} \quad (\lambda_1 = 532.052\ \mathrm{kg\cdot m^2})} \tag{h}$$

Substituting $\lambda_2 = 295.840\ \ \mathrm{kg\cdot m^2}$ into Eq. (a) and proceeding as above we find that

$$\boxed{\hat{\mathbf{e}}_2 = 0.17632\hat{\mathbf{i}} - 0.972512\hat{\mathbf{j}} - 0.151609\hat{\mathbf{k}} \quad (\lambda_2 = 295.840\ \mathrm{kg\cdot m^2})} \tag{i}$$

The two unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ define two of the three principal directions of the inertia tensor. Observe that $\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0$, as must be the case for symmetric matrices.

To obtain the third principal direction $\hat{\mathbf{e}}_3$, we can substitute $\lambda_3 = 72.1083\ \text{kg}\cdot\text{m}^2$ into Eq. (a) and proceed as above. However, since the inertia tensor is symmetric, we know that the three principal directions are mutually orthogonal, which means $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$. Substituting Eqs. (h) and (i) into this cross product, we find that

$$\boxed{\hat{\mathbf{e}}_3 = -0.960894\hat{\mathbf{i}} - 0.137114\hat{\mathbf{j}} - 0.240587\hat{\mathbf{k}}\quad (\lambda_3 = 72.1083\ \text{kg}\cdot\text{m}^2)} \tag{j}$$

We can check our work by substituting λ_3 and $\hat{\mathbf{e}}_3$ into Eq. (a) and verify that it is indeed satisfied:

$$\begin{bmatrix} 100-72.1083 & -20 & -100 \\ -20 & 300-72.1083 & -50 \\ -100 & -50 & 500-72.1083 \end{bmatrix} \begin{Bmatrix} -0.960894 \\ -0.137114 \\ -0.240587 \end{Bmatrix} \overset{\text{verify}}{=} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{k}$$

The components of the vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ define the three rows of the orthogonal transformation $[\mathbf{Q}]$ from the xyz system into the $x'y'z'$ system that is aligned along the three principal directions:

$$[\mathbf{Q}] = \begin{bmatrix} 0.213186 & 0.188199 & -0.958714 \\ 0.176732 & -0.972512 & -0.151609 \\ -0.960894 & -0.137114 & -0.240587 \end{bmatrix} \tag{l}$$

Indeed, if we apply the transformation in Eq. (11.49), $[\mathbf{I}'] = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T$, we find

$$\begin{aligned} [\mathbf{I}'] &= \begin{bmatrix} 0.213186 & 0.188199 & -0.958714 \\ 0.176732 & -0.972512 & -0.151609 \\ -0.960894 & -0.137114 & -0.240587 \end{bmatrix} \begin{bmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0.213186 & 0.176732 & -0.960894 \\ 0.188199 & -0.972512 & -0.137114 \\ -0.958714 & -0.151609 & -0.240587 \end{bmatrix} \\ &= \begin{bmatrix} 532.052 & 0 & 0 \\ 0 & 295.840 & 0 \\ 0 & 0 & 72.1083 \end{bmatrix} (\text{kg}\cdot\text{m}^2) \end{aligned}$$

An alternative to the above hand calculations in Example 11.9 is to type the following lines in the MATLAB Command Window:

```
I = [ 100  -20  -100
       -20  300   -50
      -100  -50   500];
[eigenVectors, eigenValues] = eig(I)
```

Hitting the Enter (or Return) key yields the following output to the Command Window:

```
eigenVectors =
      0.9609   0.1767  -0.2132
      0.1371  -0.9725  -0.1882
      0.2406  -0.1516   0.9587
eigenValues =
      72.1083         0         0
            0  295.8398         0
            0         0  532.0519
```

Two of the eigenvectors delivered by MATLAB are opposite in direction to those calculated in Example 11.9. This illustrates the fact that we can determine an eigenvector only to within an arbitrary scalar factor. To show this, suppose $\mathbf{e}$ is an eigenvector of the tensor $[\mathbf{I}]$ so that $[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$. Multiplying this equation through by an arbitrary scalar a yields $([\mathbf{I}]\{\mathbf{e}\})a = (\lambda\{\mathbf{e}\})a$, or $[\mathbf{I}]\{a\mathbf{e}\} = \lambda\{a\mathbf{e}\}$, which means that $\{a\mathbf{e}\}$ is an eigenvector corresponding to the same eigenvalue λ.

11.5.1 PARALLEL AXIS THEOREM

Suppose the rigid body in Fig. 11.14 is in pure rotation about point P. Then, according to Eq. (11.39),

$$\{\mathbf{H}_P)_{\text{rel}}\} = [\mathbf{I}_P]\{\boldsymbol{\omega}\} \tag{11.57}$$

where $[\mathbf{I}_P]$ is the moment of inertia tensor about P, given by Eq. (11.40) with

$$x = x_{G/P} + \xi \quad y = y_{G/P} + \eta \quad z = z_{G/P} + \zeta$$

On the other hand, we have from Eq. (11.27) that

$$\mathbf{H}_P)_{\text{rel}} = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_{G/P} \tag{11.58}$$

The vector $\mathbf{r}_{G/P} \times m\mathbf{v}_{G/P}$ is the angular momentum about P of the concentrated mass m located at the center of mass G. Using matrix notation, it is computed as follows:

$$\{\mathbf{r}_{G/P} \times m\mathbf{v}_{G/P}\} \equiv \mathbf{H}_P^{(m)}\Big)_{\text{rel}} = \left[\mathbf{I}_P^{(m)}\right]\{\boldsymbol{\omega}\} \tag{11.59}$$

where $[\mathbf{I}_P^{(m)}]$, the moment of inertia of the point mass m about P, is obtained from Eq. (11.44), with $x = x_{G/P}$, $y = y_{G/P}$, and $z = z_{G/P}$. That is,

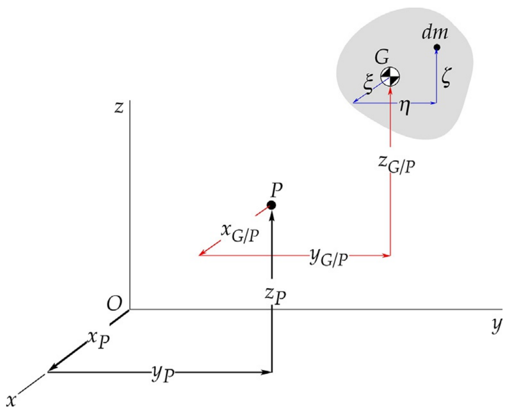


FIG. 11.14

The moments of inertia are to be computed at P, given their values at G.

$$\left[\mathbf{I}_P^{(m)}\right] = \begin{bmatrix} m\left(y_{G/P}^2 + z_{G/P}^2\right) & -mx_{G/P}y_{G/P} & -mx_{G/P}z_{G/P} \\ -mx_{G/P}y_{G/P} & m\left(x_{G/P}^2 + z_{G/P}^2\right) & -my_{G/P}z_{G/P} \\ -mx_{G/P}z_{G/P} & -my_{G/P}z_{G/P} & m\left(x_{G/P}^2 + y_{G/P}^2\right) \end{bmatrix} \tag{11.60}$$

Of course, Eq. (11.39) requires

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\}$$

Substituting this together with Eqs. (11.57) and (11.59) into Eq. (11.58) yields

$$[\mathbf{I}_P]\{\boldsymbol{\omega}\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_P^{(m)}\right]\{\boldsymbol{\omega}\} = \left[\mathbf{I}_G + \mathbf{I}_P^{(m)}\right]\{\boldsymbol{\omega}\}$$

From this, we may infer the parallel axis theorem,

$$\mathbf{I}_P = \mathbf{I}_G + \mathbf{I}_P^{(m)} \tag{11.61}$$

The moment of inertia about P is the moment of inertia about the parallel axes through the center of mass plus the moment of inertia of the center of mass about P. That is,

$$\begin{array}{lll} I_{P_x} = I_{G_x} + m\left(y_{G/P}^2 + z_{G/P}^2\right) & I_{P_y} = I_{G_y} + m\left(y_{G/P}^2 + x_{G/P}^2\right) & I_{P_z} = I_{G_z} + m\left(x_{G/P}^2 + y_{G/P}^2\right) \\ I_{P_{xy}} = I_{G_{xy}} - mx_{G/P}y_{G/P} & I_{P_{xz}} = I_{G_{xz}} - mx_{G/P}z_{G/P} & I_{P_{yz}} = I_{G_{yz}} - my_{G/P}z_{G/P} \end{array} \tag{11.62}$$

EXAMPLE 11.10

Find the moments of inertia of the rod in Example 11.5 (Fig. 11.15) about its center of mass G.

Solution

From Example 11.5,

$$[\mathbf{I}_A] = \begin{bmatrix} \frac{1}{3}m\left(b^2+c^2\right) & -\frac{1}{3}mab & -\frac{1}{3}mac \\ -\frac{1}{3}mab & \frac{1}{3}m\left(a^2+c^2\right) & -\frac{1}{3}mbc \\ -\frac{1}{3}mac & -\frac{1}{3}mbc & \frac{1}{3}m\left(a^2+b^2\right) \end{bmatrix}$$

Using Eq. $(11.62)_1$ and noting the coordinates of the center of mass in Fig. 11.15,

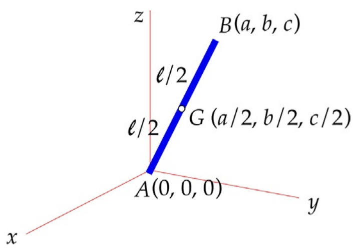


FIG. 11.15

Uniform slender rod.

$$I_{G_x} = I_{A_x} - m\left[(y_G - 0)^2 + (z_G - 0)^2\right] = \frac{1}{3}m(b^2 + c^2) - m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2\right] = \frac{1}{12}m(b^2 + c^2)$$

Eq. (11.62)$_4$ yields

$$I_{G_{xy}} = I_{A_{xy}} + m(x_G - 0)(y_G - 0) = -\frac{1}{3}mab + m \cdot \frac{a}{2} \cdot \frac{b}{2} = -\frac{1}{12}mab$$

The remaining four moments of inertia are found in a similar fashion, so that

$$\boxed{[\mathbf{I}_G] = \begin{bmatrix} \frac{1}{12}m(b^2+c^2) & -\frac{1}{2}mab & -\frac{1}{12}mac \\ -\frac{1}{12}mab & \frac{1}{12}m(a^2+c^2) & -\frac{1}{12}mbc \\ -\frac{1}{12}mac & -\frac{1}{12}mbc & \frac{1}{12}m(a^2+b^2) \end{bmatrix}} \tag{11.63}$$

EXAMPLE 11.11

Calculate the principal moments of inertia about the center of mass and the corresponding principal directions for the bent rod in Fig. 11.16. Its mass is uniformly distributed at 2 kg/m.

Solution

The mass of each of the four slender rod segments is

$$m_1 = 2 \cdot 0.4 = 0.8\text{kg} \quad m_2 = 2 \cdot 0.5 = 1\text{kg} \quad m_3 = 2 \cdot 0.3 = 0.6\text{kg} \quad m_4 = 2 \cdot 0.2 = 0.4\text{kg} \tag{a}$$

The total mass of the system is

$$m = \sum_{i=1}^{4} m_i = 2.8\text{kg} \tag{b}$$

The coordinates of each segment's center of mass are

$$\begin{array}{lll} x_{G_1} = 0 & y_{G_1} = 0 & z_{G_1} = 0.2\text{m} \\ x_{G_2} = 0 & y_{G_2} = 0.25\text{m} & z_{G_2} = 0.2\text{m} \\ x_{G_3} = 0.15\text{m} & y_{G_3} = 0.5\text{m} & z_{G_3} = 0 \\ x_{G_4} = 0.3\text{m} & y_{G_4} = 0.4\text{m} & z_{G_4} = 0 \end{array} \tag{c}$$

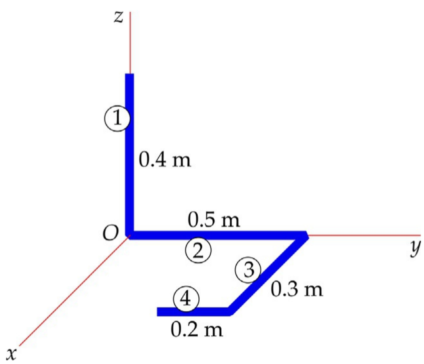


FIG. 11.16

Bent rod for which the principal moments of inertia are to be determined.

If the slender rod in Fig. 11.15 is aligned with, say, the x axis, then $a = \ell$ and $b = c = 0$, so that according to Eq. (11.63),

$$[\mathbf{I}_G] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m\ell^2 & 0 \\ 0 & 0 & \frac{1}{12}m\ell^2 \end{bmatrix}$$

That is, the moment of inertia of a slender rod about the axes normal to the rod at its center of mass is $m\ell^2/12$, where m and ℓ are the mass and length of the rod, respectively. Since the mass of a slender bar is assumed to be concentrated along the axis of the bar (its cross-sectional dimensions are infinitesimal), the moment of inertia about the centerline is zero. By symmetry, the products of inertia about the axes through the center of mass are all zero. Using this information and the parallel axis theorem, we find the moments and products of inertia of each rod segment about the origin O of the xyz system as follows:

Rod 1:

$$I_x^{(1)} = I_{G_1}^{(1)}\Big)_x + m_1\left(y_{G_1}^2 + z_{G_1}^2\right) = \frac{1}{12}\cdot 0.8\cdot 0.4^2 + 0.8\left(0 + 0.2^2\right) = 0.04267\ \text{kg}\cdot\text{m}^2$$

$$I_y^{(1)} = I_{G_1}^{(1)}\Big)_y + m_1\left(x_{G_1}^2 + z_{G_1}^2\right) = \frac{1}{12}\cdot 0.8\cdot 0.4^2 + 0.8\left(0 + 0.2^2\right) = 0.04267\ \text{kg}\cdot\text{m}^2$$

$$I_z^{(1)} = I_{G_1}^{(1)}\Big)_z + m_1\left(x_{G_1}^2 + y_{G_1}^2\right) = 0 + 0.8(0 + 0) = 0$$

$$I_{xy}^{(1)} = I_{G_1}^{(1)}\Big)_{xy} - m_1 x_{G_1} y_{G_1} = 0 - 0.8(0)(0) = 0$$

$$I_{xz}^{(1)} = I_{G_1}^{(1)}\Big)_{xz} - m_1 x_{G_1} z_{G_1} = 0 - 0.8(0)(0.2) = 0$$

$$I_{xz}^{(1)} = I_{G_1}^{(1)}\Big)_{yz} - m_1 y_{G_1} z_{G_1} = 0 - 0.8(0)(0) = 0$$

Rod 2:

$$I_x^{(2)} = I_{G_2}^{(2)}\Big)_x + m_2\left(y_{G_2}^2 + z_{G_2}^2\right) = \frac{1}{12}\cdot 1.0\cdot 0.5^2 + 1.0\left(0 + 0.25^2\right) = 0.08333\ \text{kg}\cdot\text{m}^2$$

$$I_y^{(2)} = I_{G_2}^{(2)}\Big)_y + m_2\left(x_{G_2}^2 + z_{G_2}^2\right) = 0 + 1.0(0 + 0) = 0$$

$$I_z^{(2)} = I_{G_2}^{(2)}\Big)_z + m_2\left(x_{G_2}^2 + y_{G_2}^2\right) = \frac{1}{12}\cdot 1.0\cdot 0.5^2 + 1.0\left(0 + 0.5^2\right) = 0.08333\ \text{kg}\cdot\text{m}^2$$

$$I_{xy}^{(2)} = I_{G_2}^{(2)}\Big)_{xy} - m_2 x_{G_2} y_{G_2} = 0 - 1.0(0)(0.5) = 0$$

$$I_{xz}^{(2)} = I_{G_2}^{(2)}\Big)_{xz} - m_2 x_{G_2} z_{G_2} = 0 - 1.0(0)(0) = 0$$

$$I_{yz}^{(2)} = I_{G_2}^{(2)}\Big)_{yz} - m_2 y_{G_2} z_{G_2} = 0 - 1.0(0.5)(0) = 0$$

Rod 3:

$$I_x^{(3)} = I_{G_3}^{(3)}\Big)_x + m_3\left(y_{G_3}^2 + z_{G_3}^2\right) = 0 + 0.6\left(0.5^2 + 0\right) = 0.15\ \text{kg}\cdot\text{m}^2$$

$$I_y^{(3)} = I_{G_3}^{(2)}\Big)_y + m_3\left(x_{G_3}^2 + z_{G_3}^2\right) = \frac{1}{12}\cdot 0.6\cdot 0.3^2 + 0.6\left(0.15^2 + 0\right) = 0.018\ \text{kg}\cdot\text{m}^2$$

$$I_z^{(3)} = I_{G_3}^{(3)}\Big)_z + m_3\left(x_{G_3}^2 + y_{G_3}^2\right) = \frac{1}{2}\cdot 0.6\cdot 0.3^2 + 0.6\left(0.15^2 + 0.5^2\right) = 0.1680\ \text{kg}\cdot\text{m}^2$$

$$I_{xy}^{(3)} = I_{G_3}^{(3)}\Big)_{xy} - m_3 x_{G_3} y_{G3} = 0 - 0.6(0.15)(0.5) = -0.045\ \text{kg}\cdot\text{m}^2$$

$$I_{xz}^{(3)} = I_{G_3}^{(3)}\Big)_{xz} - m_3 x_{G_3} z_{G_3} = 0 - 0.6(0.15)(0) = 0$$

$$I_{yz}^{(3)} = I_{G_3}^{(3)}\Big)_{yz} - m_3 y_{G_3} z_{G_3} = 0 - 0.6(0.5)(0) = 0$$

Rod 4:

$$I_x^{(4)} = \left(I_{G_4}^{(4)}\right)_x + m_4\left(y_{G_4}^2 + z_{G_4}^2\right) = \frac{1}{12}\cdot 0.4\cdot 0.2^2 + 0.4\left(0.4^2+0\right) = 0.06533\ \text{kg}\cdot\text{m}^2$$
$$I_y^{(4)} = \left(I_{G_4}^{(4)}\right)_y + m_4\left(x_{G_4}^2 + z_{G4}^2\right) = 0 + 0.4\left(0.3^2+0\right) = 0.0360\ \text{kg}\cdot\text{m}^2$$
$$I_z^{(4)} = \left(I_{G_4}^{(4)}\right)_z + m_4\left(x_{G_4}^2 + y_{G_4}^2\right) = \frac{1}{12}\cdot 0.4\cdot 0.2^2 + 0.4\left(0.3^2+0.4^2\right) = 0.1013\ \text{kg}\cdot\text{m}^2$$
$$I_{xy}^{(4)} = \left(I_{G_4}^{(4)}\right)_{xy} - m_4 x_{G4} y_{G4} = 0 - 0.4(0.3)(0.4) = -0.0480\ \text{kg}\cdot\text{m}^2$$
$$I_{xz}^{(4)} = \left(I_{G_4}^{(4)}\right)_{xz} - m_4 x_{G_4} z_{G_4} = 0 - 0.4(0.3)(0) = 0$$
$$I_{yz}^{(4)} = \left(I_{G_4}^{(4)}\right)_{yz} - m_4 y_{G_4} z_{G_4} = 0 - 0.4(0.4)(0) = 0$$

The total moments of inertia for all the four rods about O are

$$\begin{array}{lll} I_x = \sum_{i=1}^{4} I_x^{(i)} = 0.3413\ \text{kg}\cdot\text{m}^2 & I_y = \sum_{i=1}^{4} I_y^{(i)} = 0.09667\ \text{kg}\cdot\text{m}^2 & I_z = \sum_{i=1}^{4} I_z^{(i)} = 0.3527\ \text{kg}\cdot\text{m}^2 \\ I_{xy} = \sum_{i=1}^{4} I_{xy}^{(i)} = 0.0930\ \text{kg}\cdot\text{m}^2 & I_{xz} = \sum_{i=1}^{4} I_{xz}^{(i)} = 0 & I_{yz} = \sum_{i=1}^{4} I_{yz}^{(i)} = 0 \end{array} \tag{d}$$

The coordinates of the center of mass of the system of four rods are, from Eqs. (a) through (c),

$$\begin{aligned} x_G &= \frac{1}{m}\sum_{i=1}^{4} m_i x_{G_i} = \frac{1}{2.8}\cdot 0.21 = 0.075\text{m} \\ y_G &= \frac{1}{m}\sum_{i=1}^{4} m_i y_{G_i} = \frac{1}{2.8}\cdot 0.71 = 0.2536\text{m} \\ z_G &= \frac{1}{m}\sum_{i=1}^{4} m_i z_{G_i} = \frac{1}{2.8}\cdot 0.16 = 0.05714\text{m} \end{aligned} \tag{e}$$

We use the parallel axis theorems to shift the moments of inertia in Eq. (d) to the center of mass G of the system

$$I_{G_x} = I_x - m\left(y_G^2 + z_G^2\right) = 0.3413 - 0.1892 = 0.1522\ \text{kg}\cdot\text{m}^2$$
$$I_{G_y} = I_y - m\left(x_G^2 + z_G^2\right) = 0.09667 - 0.02489 = 0.17177\ \text{kg}\cdot\text{m}^2$$
$$I_{G_z} = I_z - m\left(x_G^2 + y_G^2\right) = -0.3527 - 0.1958 = 0.1569\ \text{kg}\cdot\text{m}^2$$
$$I_{G_{xy}} = I_{xy} + m x_G y_G = -0.093 + 0.05325 = -0.03975\ \text{kg}\cdot\text{m}^2$$
$$I_{G_{xz}} = I_{xz} + m x_G z_G = 0 + 0.012 = 0.012\ \text{kg}\cdot\text{m}^2$$
$$I_{G_{yz}} = I_{yz} + m y_G z_G = 0 + 0.04057 = 0.04057\ \text{kg}\cdot\text{m}^2$$

Therefore, the inertia tensor, relative to the center of mass, is

$$[\mathbf{I}] = \begin{bmatrix} I_{G_x} & I_{G_{xy}} & I_{G_{xz}} \\ I_{G_{xy}} & I_{G_y} & I_{G_{yz}} \\ I_{G_{xz}} & I_{G_{yz}} & I_{G_z} \end{bmatrix} = \begin{bmatrix} 0.1522 & -0.03975 & 0.012 \\ -0.03975 & 0.07177 & 0.04057 \\ 0.012 & 0.04057 & 0.1569 \end{bmatrix} \left(\text{kg}\cdot\text{m}^2\right) \tag{f}$$

To find the three principal moments of inertia, we may proceed as in Example 11.9, or simply enter the following lines in the MATLAB Command Window:

```
IG = [ 0.1522  -0.03975  0.012
      -0.03975  0.07177  0.04057
       0.012    0.04057  0.1569];
[eigenVectors, eigenValues] = eig(IG)
```

to obtain

```
eigenVectors =
     0.3469  -0.8482  -0.4003
     0.8742   0.1378   0.4656
    -0.3397  -0.5115   0.7893
eigenValues =
     0.0402        0        0
          0   0.1658        0
          0        0   0.1747
```

Hence, the three principal moments of inertia and their principal directions are

$$\boxed{\begin{array}{ll} \lambda_1 = 0.04023\ \mathrm{kg\cdot m^2} & \mathbf{e}^{(1)} = 0.3469\hat{\mathbf{i}} + 0.8742\hat{\mathbf{j}} - 0.3397\hat{\mathbf{k}} \\ \lambda_2 = 0.1658\ \mathrm{kg\cdot m^2} & \mathbf{e}^{(2)} = -0.8482\hat{\mathbf{i}} + 0.1378\hat{\mathbf{j}} - 0.5115\hat{\mathbf{k}} \\ \lambda_3 = 0.1747\ \mathrm{kg\cdot m^2} & \mathbf{e}^{(3)} = -0.4003\hat{\mathbf{i}} + 0.4656\hat{\mathbf{j}} + 0.7893\hat{\mathbf{k}} \end{array}}$$

11.6 EULER EQUATIONS

For either the center of mass G or for a fixed point P about which the body is in pure rotation, we know from Eqs. (11.29) and (11.30) that

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}} \tag{11.64}$$

Using a comoving coordinate system, with angular velocity $\boldsymbol{\Omega}$ and its origin located at the point (G or P), the angular momentum has the analytical expression

$$\mathbf{H} = H_x\hat{\mathbf{i}} + H_y\hat{\mathbf{j}} + H_z\hat{\mathbf{k}} \tag{11.65}$$

For simplicity, we shall henceforth assume

(a) the moving xyz axes are the principal axes of inertia; (11.66a)

(b) the moments of inertia relative to xyz are constant in time. (11.66b)

Eqs. (11.42) and (11.66a) imply that

$$\mathbf{H} = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{11.67}$$

where A, B, and C are the principal moments of inertia.

According to Eq. (1.56), the time derivative of $\mathbf{H}$ is $\dot{\mathbf{H}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{H}$, so that Eq. (11.64) can be written as

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{H} \tag{11.68}$$

Keep in mind that, whereas $\boldsymbol{\Omega}$ (the angular velocity of the moving xyz coordinate system) and $\boldsymbol{\omega}$ (the angular velocity of the rigid body itself) are both absolute kinematic quantities, Eq. (11.68) contains their components as projected onto the axes of the noninertial xyz frame given by

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$
$$\boldsymbol{\Omega} = \Omega_x\hat{\mathbf{i}} + \Omega_y\hat{\mathbf{j}} + \Omega_z\hat{\mathbf{k}}$$

The absolute angular acceleration $\boldsymbol{\alpha}$ is obtained using Eq. (1.56) as

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \overbrace{\dot{\omega}_x\hat{\mathbf{i}} + \dot{\omega}_y\hat{\mathbf{j}} + \dot{\omega}_z\hat{\mathbf{k}}}^{\boldsymbol{\alpha}_{\text{rel}}} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

That is,

$$\boldsymbol{\alpha} = \left(\dot{\omega}_x + \Omega_y\dot{\omega}_z - \Omega_z\omega_y\right)\hat{\mathbf{i}} + \left(\dot{\omega}_y + \Omega_z\omega_x - \Omega_x\omega_z\right)\hat{\mathbf{j}} + \left(\dot{\omega}_z + \Omega_x\omega_y - \Omega_y\omega_x\right)\hat{\mathbf{k}} \tag{11.69}$$

Clearly, it is generally true that

$$\alpha_x \neq \dot{\omega}_x \quad \alpha_y \neq \dot{\omega}_y \quad \alpha_z \neq \dot{\omega}_z$$

From Eq. (1.57) and Eq. (11.67),

$$\dot{\mathbf{H}})_{\text{rel}} = \frac{\mathrm{d}(A\omega_x)}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}\left(B\omega_y\right)}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}(C\omega_z)}{\mathrm{d}t}\hat{\mathbf{k}}$$

Since A, B, and C are constant, this becomes

$$\dot{\mathbf{H}})_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{11.70}$$

Substituting Eqs. (11.67) and (11.70) into Eq. (11.68) yields

$$\mathbf{M}_{\text{net}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Omega_x & \Omega_y & \Omega_z \\ A\omega_x & B\omega_y & C\omega_z \end{vmatrix}$$

Expanding the cross product and collecting the terms leads to

$$\begin{aligned} M_x)_{\text{net}} &= A\dot{\omega}_x + C\Omega_y\omega_z - B\Omega_z\omega_y \\ M_y)_{\text{net}} &= B\dot{\omega}_y + A\Omega_z\omega_x - C\Omega_z\omega_z \\ M_z)_{\text{net}} &= C\dot{\omega}_z + B\Omega_x\omega_y - A\Omega_y\omega_x \end{aligned} \tag{11.71}$$

If the comoving frame is a body-fixed frame, then its angular velocity vector is the same as that of the body (i.e., $\boldsymbol{\Omega} = \boldsymbol{\omega}$). In that case, Eq. (11.68) reduces to the classical Euler equation of motion, namely,

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H} \tag{11.72a}$$

the three components of which are obtained from Eq. (11.71) as

$$\begin{aligned} M_x)_{\text{net}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\ M_y)_{\text{net}} &= B\dot{\omega}_y + (A - C)\omega_z\omega_x \\ M_z)_{\text{net}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y \end{aligned} \tag{11.72b}$$

Eq. (11.68) is sometimes referred to as the modified Euler equation.

When $\boldsymbol{\Omega} = \boldsymbol{\omega}$, it follows from Eq. (11.69) that

$$\dot{\omega}_x = \alpha_x \quad \dot{\omega}_y = \alpha_y \quad \dot{\omega}_z = \alpha_z \tag{11.73}$$

That is, the relative angular acceleration equals the absolute angular acceleration when $\boldsymbol{\Omega} = \boldsymbol{\omega}$. Rather than calculating the time derivatives $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ for use in Eq. (11.72), we may in this case first compute $\boldsymbol{\alpha}$ in the absolute XYZ frame

$$\boldsymbol{\alpha} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}\omega_X}{\mathrm{d}t}\hat{\mathbf{I}} + \frac{\mathrm{d}\omega_Y}{\mathrm{d}t}\hat{\mathbf{J}} + \frac{\mathrm{d}\omega_z}{\mathrm{d}t}\hat{\mathbf{K}}$$

and then project these components onto the xyz body frame, so that

$$\begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} = [\mathbf{Q}]_{Xx} \begin{Bmatrix} \mathrm{d}\omega_X/\mathrm{d}t \\ \mathrm{d}\omega_Y/\mathrm{d}t \\ \mathrm{d}\omega_Z/\mathrm{d}t \end{Bmatrix} \tag{11.74}$$

where $[\mathbf{Q}]_{Xx}$ is the time-dependent orthogonal transformation from the inertial XYZ frame to the non-inertial xyz frame.

EXAMPLE 11.12

Calculate the net moment on the solar panel of Examples 11.2 and 11.8 (Fig. 11.17).

Solution

Since the comoving frame is rigidly attached to the panel, the Euler equation (Eq. 11.72a) applies to this problem. That is

$$\mathbf{M}_{G})_{\text{net}} = \dot{\mathbf{H}}_{G})_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_G \tag{a}$$

where

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{b}$$

and

$$\dot{\mathbf{H}}_{G}\big)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{c}$$

In Example 11.2, the angular velocity of the panel in the satellite's $x'y'z'$ frame was found to be

$$\boldsymbol{\omega} = -\dot{\theta}\hat{\mathbf{j}}' + N\hat{\mathbf{k}}' \tag{d}$$

In Example 11.8, we showed that the direction cosine matrix for the transformation from the panel's xyz frame to that of the satellite is

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \tag{e}$$

We use the transpose of $[\mathbf{Q}]$ to transform the components of $\boldsymbol{\omega}$ into the panel frame of reference,

$$\{\boldsymbol{\omega}\}_{xyz} = [\mathbf{Q}]^T\{\boldsymbol{\omega}\}_{x'y'z'} = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} 0 \\ -\dot{\theta} \\ N \end{Bmatrix} = \begin{Bmatrix} N\cos\theta \\ \dot{\theta} \\ N\sin\theta \end{Bmatrix}$$

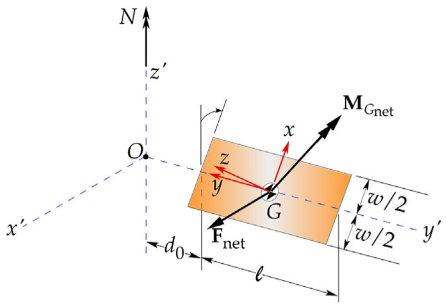


FIG. 11.17

Free body diagram of the solar panel in Examples 11.2 and 11.8.

or

$$\omega_x = N\cos\theta \quad \omega_y = \dot{\theta} \quad \omega_z = N\sin\theta \tag{f}$$

In Example 11.2, N and $\dot{\theta}$ were said to be constant. Therefore, the time derivatives of Eq. (f) are

$$\dot{\omega}_x = \frac{\mathrm{d}(N\cos\theta)}{\mathrm{d}t} = -N\dot{\theta}\sin\theta \quad \dot{\omega}_x = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 0 \quad \dot{\omega}_z = \frac{\mathrm{d}(N\sin\theta)}{\mathrm{d}t} = N\dot{\theta}\cos\theta \tag{g}$$

In Example 11.8, the moments of inertia in the panel frame of reference were listed as

$$A = \frac{1}{12}m\left(\ell^2 + t^2\right) \quad B = \frac{1}{12}m\left(w^2 + t^2\right) \quad C = \frac{1}{12}m\left(w^2 + \ell^2\right) \quad I_G)_{xy} = I_G)_{xz} = I_G)_{yz} = 0 \tag{h}$$

Substituting Eqs. (b), (c), (f), (g), and (h) into Eq. (a) yields

$$\mathbf{M}_G)_{\text{net}} = \frac{1}{12}m\left(\ell^2 + t^2\right)\left(-N\dot{\theta}\sin\theta\right)\hat{\mathbf{i}} + \frac{1}{12}m\left(w^2 + t^2\right)(0)\hat{\mathbf{j}} + \frac{1}{12}m\left(w^2 + \ell^2\right)\left(N\dot{\theta}\cos\theta\right)\hat{\mathbf{k}}$$
$$+ \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ N\cos\theta & \dot{\theta} & N\sin\theta \\ \frac{1}{12}m\left(\ell^2 + t^2\right)(N\cos\theta) & \frac{1}{12}m\left(w^2 + t^2\right)\dot{\theta} & \frac{1}{12}m\left(w^2 + \ell^2\right)(N\sin\theta) \end{vmatrix}$$

Upon expanding the cross product determinant and collecting terms, this reduces to

$$\mathbf{M}_G)_{\text{net}} = -\frac{1}{6}mt^2N\dot{\theta}\sin\theta\hat{\mathbf{i}} + \frac{1}{24}m\left(t^2 - w^2\right)N^2\sin 2\theta\hat{\mathbf{j}} + \frac{1}{6}mw^2N\dot{\theta}\cos\theta\hat{\mathbf{k}}$$

Using the numerical data of Example 11.8 ($m = 50$ kg, $N = 0.1$ rad/s, $\theta = 40°$, $\dot{\theta} = 0.01$ rad/s, $\ell = 6$ m, $w = 2$ m, and $t = 0.025$ m), we find

$$\boxed{\mathbf{M}_G)_{\text{net}} = -3.348\left(10^{-6}\right)\hat{\mathbf{i}} - 0.08205\hat{\mathbf{j}} + 0.02554\hat{\mathbf{k}}\,(\mathrm{N\cdot m})}$$

EXAMPLE 11.13

Calculate the net moment on the gyro rotor of Examples 11.3 and 11.6.

Solution

Fig. 11.18 is a free body diagram of the rotor. Since in this case the comoving frame is not rigidly attached to the rotor, we must use Eq. (11.68) to find the net moment about G. That is

$$\mathbf{M}_G)_{\text{net}} = \dot{\mathbf{H}}_G)_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{H}_G \tag{a}$$

where

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{b}$$

and

$$\dot{\mathbf{H}}_G)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{c}$$

From Eq. (f) of Example 11.3, we know that the components of the angular velocity of the rotor in the moving reference frame are

$$\omega_x = \dot{\theta} \quad \omega_y = N\sin\theta \quad \omega_z = \omega_{\text{spin}} + N\cos\theta \tag{d}$$

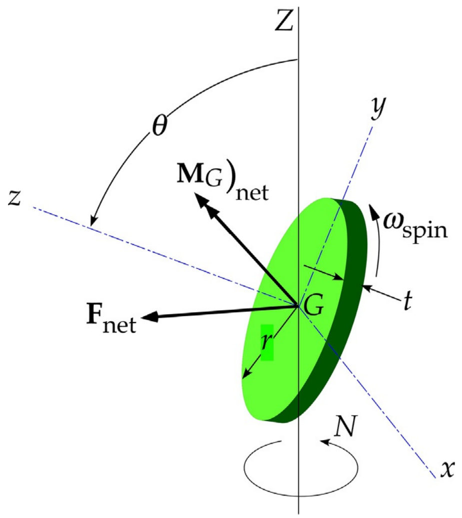


FIG. 11.18

Free body diagram of the gyro rotor of Examples 11.6 and 11.3.

Since, as specified in Example 11.3, $\dot{\theta}$, N, and ω_{spin} are all constant, it follows that

$$\begin{aligned}\dot{\omega}_x &= \frac{d\dot{\theta}}{dt} = 0\\ \dot{\omega}_y &= \frac{d(N\sin\theta)}{dt} = N\dot{\theta}\cos\theta\\ \dot{\omega}_z &= \frac{d\left(\omega_{spin} + N\cos\theta\right)}{dt} = -N\dot{\theta}\sin\theta\end{aligned} \tag{e}$$

The angular velocity $\mathbf{\Omega}$ of the comoving xyz frame is that of the gimbal ring, which equals the angular velocity of the rotor minus its spin. Therefore,

$$\Omega_x = \dot{\theta} \quad \Omega_y = N\sin\theta \quad \Omega_z = N\cos\theta \tag{f}$$

In Example 11.6, we found that

$$A = B = \frac{1}{12}mt^2 + \frac{1}{4}mr^2 \quad C = \frac{1}{2}mr^2 \tag{g}$$

Substituting Eqs. (b) through (g) into Eq. (a), we get

$$\begin{aligned}\mathbf{M}_G)_{net} = &\left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)(0)\hat{\mathbf{i}} + \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)(N\dot{\theta}\cos\theta)\hat{\mathbf{j}} + \frac{1}{2}mr^2(-N\dot{\theta}\sin\theta)\hat{\mathbf{k}}\\ &+ \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dot{\theta} & N\sin\theta & N\cos\theta\\ \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)\dot{\theta} & \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)N\sin\theta & \frac{1}{2}mr^2\left(\omega_{spin} + N\cos\theta\right)\end{vmatrix}\end{aligned}$$

Expanding the cross product, collecting terms, and simplifying leads to

$$\begin{aligned}\mathbf{M}_G)_{net} = &\left[\frac{1}{2}\omega_{spin} + \frac{1}{12}\left(3 - \frac{t^2}{r^2}\right)N\cos\theta\right]mr^2 N\sin\theta\hat{\mathbf{i}}\\ &+ \left(\frac{1}{6}\frac{t^2}{r^2}N\cos\theta - \frac{1}{2}\omega_{spin}\right)mr^2\dot{\theta}\hat{\mathbf{j}} - \frac{1}{2}N\dot{\theta}\sin\theta mr^2\hat{\mathbf{k}}\end{aligned} \tag{h}$$

In Example 11.3, the following numerical data were provided: $m = 5$ kg, $r = 0.08$ m, $t = 0.025$ m, $N = 2.1$ rad/s, $\theta = 60°$, $\dot{\theta} = 4\,\text{rad/s}$, and $\omega_{\text{spin}} = 105\,\text{rad/s}$. For this set of numbers, Eq. (h) becomes

$$\boxed{\mathbf{M}_G)_{\text{net}} = 0.3203\hat{\mathbf{i}} - 0.6698\hat{\mathbf{j}} - 0.1164\hat{\mathbf{k}}\,(\text{N}\cdot\text{m})}$$

11.7 KINETIC ENERGY

The kinetic energy T of a rigid body is the integral of the kinetic energy $(1/2)v^2\mathrm{d}m$ of its individual mass elements,

$$T = \int_m \frac{1}{2}v^2\mathrm{d}m = \int_m \frac{1}{2}\mathbf{v}.\mathbf{v}\mathrm{d}m \tag{11.75}$$

where $\mathbf{v}$ is the absolute velocity $\dot{\mathbf{R}}$ of the element of mass dm. From Fig. 11.8, we infer that $\dot{\mathbf{R}} = \dot{\mathbf{R}}_G + \dot{\boldsymbol{\rho}}$. Furthermore, Eq. (1.52) requires that $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$. Thus, $\mathbf{v} = \mathbf{v}_G + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}$, which means

$$\mathbf{v}\cdot\mathbf{v} = (\mathbf{v}_G + \boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\mathbf{v}_G + \boldsymbol{\omega}\times\boldsymbol{\rho}) = v_G{}^2 + 2\mathbf{v}_G\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}) + (\boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho})$$

We can apply the vector identity introduced in Eq. (1.21), namely

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) \tag{11.76}$$

to the last term to get

$$\mathbf{v}\cdot\mathbf{v} = v_G{}^2 + 2\mathbf{v}_G\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}) + \boldsymbol{\omega}\cdot[\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})]$$

Therefore, Eq. (11.75) becomes

$$T = \int_m \frac{1}{2}v_G{}^2\mathrm{d}m + \mathbf{v}_G\cdot\left(\boldsymbol{\omega}\times\int_m \boldsymbol{\rho}\mathrm{d}m\right) + \frac{1}{2}\boldsymbol{\omega}\cdot\int_m \boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})\mathrm{d}m$$

Since the position vector $\boldsymbol{\rho}$ is measured from the center of mass, $\int_m \boldsymbol{\rho}\mathrm{d}m = \mathbf{0}$. Recall that, according to Eq. (11.34),

$$\int_m \boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})\mathrm{d}m = \mathbf{H}_G$$

It follows that the kinetic energy may be written as

$$T = \frac{1}{2}mv_G{}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G \tag{11.77}$$

The second term is the rotational kinetic energy T_R,

$$T_R = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G \tag{11.78}$$

If the body is rotating about a point P that is at rest in inertial space, we have from Eq. (11.2) and Fig. 11.8 that

$$\mathbf{v}_G = \mathbf{v}_P + \boldsymbol{\omega}\times\mathbf{r}_{G/P} = \mathbf{0} + \boldsymbol{\omega}\times\mathbf{r}_{G/P} = \boldsymbol{\omega}\times\mathbf{r}_{G/P}$$

It follows that

$$v_G{}^2 = \mathbf{v}_G\cdot\mathbf{v}_G = \left(\boldsymbol{\omega}\times\mathbf{r}_{G/P}\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_{G/P}\right)$$

Making use once again of the vector identity in Eq. (11.76), we find

$$v_G{}^2 = \boldsymbol{\omega} \cdot \left[\mathbf{r}_{G/P} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{G/P}\right)\right] = \boldsymbol{\omega} \cdot \left(\mathbf{r}_{G/P} \times \mathbf{v}_G\right)$$

Substituting this into Eq. (11.77) yields

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \left[\mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_G\right]$$

Eq. (11.21) shows that this can be written as

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_P \tag{11.79}$$

In this case, of course, all the kinetic energy is rotational.

In terms of the components of $\boldsymbol{\omega}$ and $\mathbf{H}$, whether it is $\mathbf{H}_P$ or $\mathbf{H}_G$, the rotational kinetic energy expression becomes, with the aid of Eq. (11.39),

$$T_R = \frac{1}{2}\left(\omega_x H_x + \omega_y H_y + \omega_y H_z\right) = \frac{1}{2}\lfloor \omega_x \quad \omega_y \quad \omega_z \rfloor \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Expanding, we obtain

$$T_R = \frac{1}{2}I_x\omega_x{}^2 + \frac{1}{2}I_y\omega_y{}^2 + \frac{1}{2}I_z\omega^2{}_z + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z \tag{11.80}$$

Obviously, if the xyz axes are principal axes of inertia, then Eq. (11.80) simplifies considerably,

$$T_R = \frac{1}{2}A\omega_x{}^2 + \frac{1}{2}B\omega_y{}^2 + \frac{1}{2}C\omega_z{}^2 \tag{11.81}$$

EXAMPLE 11.14

A satellite in a circular geocentric orbit of 300 km altitude has a mass of 1500 kg and the moments of inertia relative to a body frame with origin at the center of mass G are

$$[\mathbf{I}] = \begin{bmatrix} 2000 & -1000 & 2500 \\ -1500 & 3000 & -1500 \\ 2500 & -1500 & 4000 \end{bmatrix} \left(\mathrm{kg} \cdot \mathrm{m}^2\right)$$

If at a given instant the components of angular velocity in this frame of reference are

$$\boldsymbol{\omega} = 1\mathbf{i} - 0.9\hat{\mathbf{j}} + 1.5\hat{\mathbf{k}}\,(\mathrm{rad/s})$$

calculate the total kinetic energy of the satellite.

Solution

The speed of the satellite in its circular orbit is

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398{,}600}{6378 + 300}} = 7.7258\,\mathrm{km/s}$$

The angular momentum of the satellite is

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} = \begin{bmatrix} 2000 & -1000 & 2500 \\ -1500 & 3000 & -1500 \\ 2500 & -1500 & 4000 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.9 \\ 1.5 \end{Bmatrix} = \begin{Bmatrix} 6650 \\ -5950 \\ 9850 \end{Bmatrix} (\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s})$$

Therefore, the total kinetic energy is

$$T = \frac{1}{2}mv_G{}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G = \overbrace{\frac{1}{2}(1500)\left(7.7258\times 10^3\right)^2}^{44.766\left(10^9\right)\mathrm{J}} + \overbrace{\frac{1}{2}\lfloor 1 \quad -0.9 \quad 1.5 \rfloor \begin{Bmatrix} 6650 \\ -5950 \\ 9850 \end{Bmatrix}}^{13,390\mathrm{J}}$$

$$\boxed{T = 44.766\,\mathrm{GJ}}$$

Obviously, the kinetic energy is dominated by that due to the orbital motion.

11.8 THE SPINNING TOP

Let us analyze the motion of the simple axisymmetric top in Fig. 11.19. It is constrained to rotate about point O, which is fixed in space.

The moving xyz coordinate system is chosen to have its origin at O. The z axis is aligned with the spin axis of the top (the axis of rotational symmetry). The x axis is the node line, which passes through O and is perpendicular to the plane defined by the inertial Z axis and the spin axis of the top. The y axis is then perpendicular to x and z, such that $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$. By symmetry, the moment of inertia matrix of the top relative to the xyz frame is diagonal, with $I_x = I_y = A$ and $I_z = C$. From Eqs. (11.68) and (11.70), we have

$$\mathbf{M}_O)_{\text{net}} = A\dot{\omega}_x\hat{\mathbf{i}} + A\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Omega_x & \Omega_y & \Omega_z \\ A\omega_x & A\omega_y & C\omega_z \end{vmatrix} \tag{11.82}$$

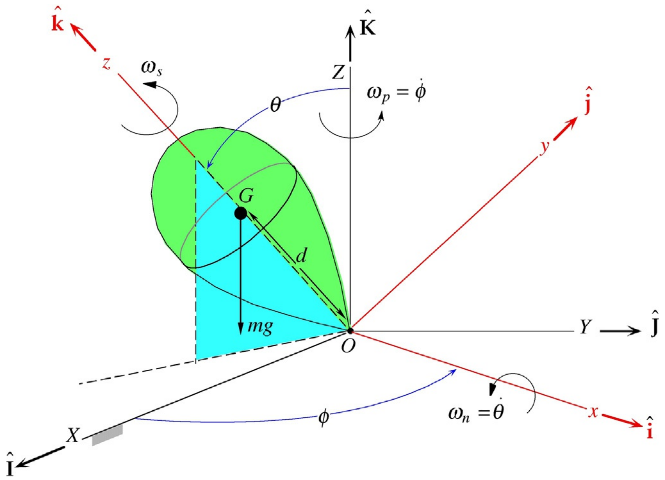


FIG. 11.19

Simple top rotating about the fixed point O.

The angular velocity $\boldsymbol{\omega}$ of the top is the vector sum of the spin rate ω_s and the rates of precession ω_p and nutation ω_n, where

$$\omega_p = \dot{\phi} \qquad \omega_n = \dot{\theta} \tag{11.83}$$

Thus,

$$\boldsymbol{\omega} = \omega_n \hat{\mathbf{i}} + \omega_p \hat{\mathbf{K}} + \omega_s \hat{\mathbf{k}}$$

From the geometry, we see that

$$\hat{\mathbf{K}} = \sin\theta \hat{\mathbf{j}} + \cos\theta \hat{\mathbf{k}} \tag{11.84}$$

Therefore, relative to the comoving system,

$$\boldsymbol{\omega} = \omega_n \hat{\mathbf{i}} + \omega_p \sin\theta \hat{\mathbf{j}} + \left(\omega_s + \omega_p \cos\theta\right) \hat{\mathbf{k}} \tag{11.85}$$

From Eq. (11.85), we see that

$$\omega_x = \omega_n \qquad \omega_y = \omega_p \sin\theta \qquad \omega_z = \omega_s + \omega_p \cos\theta \tag{11.86}$$

Computing the time rates of these three expressions yields the components of angular acceleration relative to the xyz frame, given by

$$\dot{\omega}_x = \dot{\omega}_n \qquad \dot{\omega}_y = \dot{\omega}_p \sin\theta + \omega_p \omega_n \cos\theta \qquad \dot{\omega}_z = \dot{\omega}_s + \dot{\omega}_p \cos\theta - \omega_p \omega_n \sin\theta \tag{11.87}$$

The angular velocity $\boldsymbol{\Omega}$ of the xyz system is $\boldsymbol{\Omega} = \omega_p \hat{\mathbf{K}} + \omega_n \hat{\mathbf{i}}$, so that, using Eq. (11.84),

$$\boldsymbol{\Omega} = \omega_n \hat{\mathbf{i}} + \omega_p \sin\theta \hat{\mathbf{j}} + \omega_p \cos\theta \hat{\mathbf{k}} \tag{11.88}$$

From Eq. (11.88), we obtain

$$\Omega_x = \omega_n \qquad \Omega_y = \omega_p \sin\theta \qquad \Omega_z = \omega_p \cos\theta \tag{11.89}$$

In Fig. 11.19, the moment about O is that of the weight vector acting through the center of mass G:

$$\mathbf{M}_O)_{\text{net}} = \left(d\hat{\mathbf{k}}\right) \times \left(-mg\hat{\mathbf{K}}\right) = -mgd\hat{\mathbf{k}} \times \left(\sin\theta \hat{\mathbf{j}} + \cos\theta \hat{\mathbf{k}}\right)$$

or

$$\mathbf{M}_O)_{\text{net}} = mgd \sin\theta \hat{\mathbf{i}} \tag{11.90}$$

Substituting Eqs. (11.86) through (11.90) into Eq. (11.82), we get

$$\begin{aligned} mgd \sin\theta \hat{\mathbf{i}} = {} & A\dot{\omega}_n \hat{\mathbf{i}} + A\left(\dot{\omega}_p \sin\theta + \dot{\omega}_p \dot{\omega}_n \cos\theta\right)\hat{\mathbf{j}} + C\left(\dot{\omega}_s + \dot{\omega}_p \cos\theta - \dot{\omega}_p \dot{\omega}_n \sin\theta\right)\hat{\mathbf{k}} \\ & + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_n & \omega_p \sin\theta & \omega_p \cos\theta \\ A\omega_n & A\omega_p \sin\theta & C\left(\omega_s + \omega_p \cos\theta\right) \end{vmatrix} \end{aligned} \tag{11.91}$$

Let us consider the special case in which θ is constant (i.e., there is no nutation), so that $\omega_n = \dot{\omega}_n = 0$. Then, Eq. (11.91) reduces to

$$mgd \sin\theta \hat{\mathbf{i}} = A\dot{\omega}_p \sin\theta \hat{\mathbf{j}} + C\left(\dot{\omega}_s + \dot{\omega}_p \cos\theta\right)\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \omega_p \sin\theta & \omega_p \cos\theta \\ 0 & A\omega_p \sin\theta & C\left(\omega_s + \omega_p \cos\theta\right) \end{vmatrix} \tag{11.92}$$

Expanding the determinant yields

$$mgd\sin\theta\hat{\mathbf{i}} = A\dot{\omega}_p\sin\theta\hat{\mathbf{j}} + C\left(\dot{\omega}_s + \dot{\omega}_p\cos\theta\right)\hat{\mathbf{k}} + \left[C\omega_p\omega_s\sin\theta + (C-A)\omega_p{}^2\cos\theta\sin\theta\right]\hat{\mathbf{i}}$$

Equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ on each side of this equation and assuming that $0° < \theta < 180°$ leads to

$$mgd = C\omega_p\omega_s + (C-A)\omega_p{}^2\cos\theta \tag{11.93a}$$

$$A\dot{\omega}_p = 0 \tag{11.93b}$$

$$C\left(\dot{\omega}_s + \dot{\omega}_p\cos\theta\right) = 0 \tag{11.93c}$$

Eq. (11.93b) implies $\dot{\omega}_p = 0$, and from Eq. (11.93c) it follows that $\dot{\omega}_s = 0$. Therefore, the rates of spin and precession are both constant. From Eq. (11.93a), we find

$$(A-C)\cos\theta\omega_p{}^2 - C\omega_s\omega_p + mgd = 0 \tag{11.94}$$

If the spin rate is zero, Eq. (11.94) yields

$$\omega_p\big)_{\omega_s=0} = \pm\sqrt{\frac{mgd}{(C-A)\cos\theta}} \quad \text{if } (C-A)\cos\theta > 0 \tag{11.95}$$

In this case, the top rotates about O at this rate, without spinning. If $A > C$ (prolate), its symmetry axis must make an angle between 90° and 180° to the vertical; otherwise, ω_p is imaginary. On the other hand, if $A < C$ (oblate), the angle lies between 0° and 90°. Thus, in steady rotation without spin, the top's axis sweeps out a cone that lies either below the horizontal plane ($A > C$) or above the plane ($A < C$).

In the special case where $(A - C)\cos\theta = 0$, Eq. (11.94) yields a steady precession rate that is inversely proportional to the spin rate,

$$\omega_p = \frac{mgd}{C\omega_s} \quad \text{if } (A-C)\cos\theta = 0 \tag{11.96}$$

If $A = C$, this precession apparently occurs irrespective of tilt angle θ. If $A \neq C$, this rate of precession occurs at $\theta = 90°$ (i.e., the spin axis is perpendicular to the precession axis).

In general, Eq. (11.94) is a quadratic equation in ω_p, so we can use the quadratic formula to find

$$\omega_p = \frac{C}{2(A-C)\cos\theta}\left(\omega_s \pm \sqrt{\omega_s{}^2 - \frac{4mgd(A-C)\cos\theta}{C^2}}\right) \tag{11.97}$$

Thus, for a given spin rate and tilt angle θ ($\theta \neq 90°$), there are two rates of precession $\dot{\phi}$.

Observe that if $(A - C)\cos\theta > 0$, then ω_p is imaginary when $\omega_s{}^2 < 4mgd(A-C)\cos\theta/C^2$. Therefore, the minimum spin rate required for steady precession at a constant inclination θ is

$$\omega_s\big)_{\min} = \frac{2}{C}\sqrt{mgd(A-C)\cos\theta} \quad \text{if } (A-C)\cos\theta > 0 \tag{11.98}$$

If $(A - C)\cos\theta < 0$, the radical in Eq. (11.97) is real for all ω_s. In this case, as $\omega_s \to 0$, ω_p approaches the value given above in Eq. (11.95).

EXAMPLE 11.15

Calculate the precession rate ω_p for a toy top like that in Fig. 11.19 if $m = 0.5$ kg, $A(=I_x = I_y) = 12(10^{-4})\text{kg}\cdot\text{m}^2$, $C\ (=I_z) = 4.5(10^{-4})\,\text{kg}\cdot\text{m}^2$, and $d = 0.05$ m.

Solution

For an inclination of, say, 60°, $(A - C)\cos\theta > 0$, so that Eq. (11.98) requires $\omega_s)_{\text{min}} = 407.01$ rpm. Let us choose the spin rate to be $\omega_s = 1000\ \text{rpm} = 104.7$ rad/s. Then, from Eq. (11.97), the steady precession rate as a function of the inclination θ is given by either one of the following formulas:

$$\boxed{\omega_p = 31.42\frac{1+\sqrt{1-0.3312\cos\theta}}{\cos\theta} \quad \text{and} \quad \omega_p = 31.42\frac{1-\sqrt{1-0.3312\cos\theta}}{\cos\theta}} \tag{a}$$

These are plotted in Fig. 11.20. For $\theta = 60°$, the high-energy precession rate is 1148.1 rpm, which exceeds the spin rate, whereas the low-energy precession rate is a leisurely 51.93 rpm.

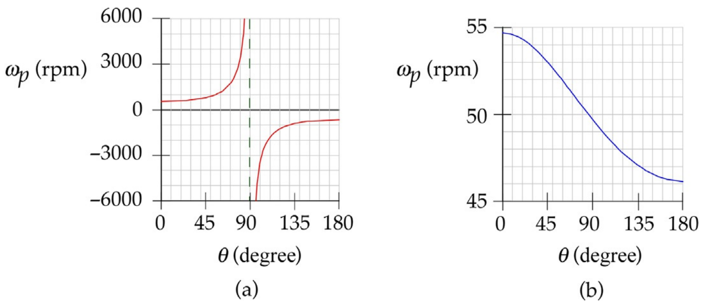


FIG. 11.20

(a) High-energy precession rate (unlikely to be observed). (b) Low-energy precession rate (the one almost always seen).

Fig. 11.21 shows an axisymmetric rotor mounted so that its spin axis (z) remains perpendicular to the precession axis (y). In that case, Eq. (11.85) with $\theta = 90°$ yields

$$\boldsymbol{\omega} = \omega_p\hat{\mathbf{j}} + \omega_s\hat{\mathbf{k}} \tag{11.99}$$

Likewise, from Eq. (11.88), the angular velocity of the comoving xyz system is $\mathbf{\Omega} = \omega_p\hat{\mathbf{j}}$. If we assume that the spin rate and precession rate are constant ($d\omega_p/dt = d\omega_s/dt = 0$), then Eq. (11.68), written for the center of mass G, becomes

$$\mathbf{M}_G)_{\text{net}} = \mathbf{\Omega}\times\mathbf{H} = \left(\omega_p\hat{\mathbf{j}}\right)\times\left(A\omega_p\hat{\mathbf{j}} + C\omega_s\hat{\mathbf{k}}\right) \tag{11.100}$$

where A and C are the moments of inertia of the rotor about the x and z axes, respectively. Setting $C\omega_s\hat{\mathbf{k}} = \mathbf{H}_s$, the spin angular momentum, and $\omega_p\hat{\mathbf{j}} = \boldsymbol{\omega}_p$, we obtain

$$\mathbf{M}_G)_{\text{net}} = \boldsymbol{\omega}_p \times \mathbf{H}_s \qquad (\mathbf{H}_s = C\omega_s\mathbf{k}) \tag{11.101}$$

Since the center of mass is the reference point, there is no restriction on the motion G for which Eq. (11.101) is valid. Observe that the net gyroscopic moment $\mathbf{M}_G)_{\text{net}}$ exerted on the rotor by its supports is perpendicular to the plane of the spin and the precession vectors. If a spinning rotor is forced to

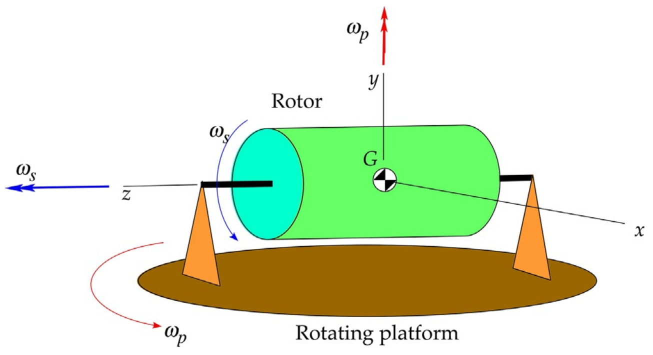


FIG. 11.21

A spinning rotor on a rotating platform.

precess, the gyroscopic moment $\mathbf{M}_G)_{\text{net}}$ develops. Or, if a moment is applied normal to the spin axis of a rotor, it will precess so as to cause the spin axis to turn toward the moment axis.

EXAMPLE 11.16

A uniform cylinder of radius r, length L, and mass m spins at a constant angular velocity ω_s. It rests on simple supports (which cannot exert couples), mounted on a platform that rotates at an angular velocity of ω_p. Find the reactions at A and B. Neglect the weight (i.e., calculate the reactions due just to the gyroscopic effects).

Solution

The net vertical force on the cylinder is zero, so the reactions at each end must be equal and opposite in direction, as shown on the free body diagram insert in Fig. 11.22. Noting that the moment of inertia of a uniform cylinder about its axis of rotational symmetry is $mr^2/2$, Eq. (11.101) yields

$$RL\hat{\mathbf{i}} = \left(\omega_p\hat{\mathbf{j}}\right) \times \left(\frac{mr^2}{2}\omega_s\hat{\mathbf{k}}\right) = \frac{1}{2}mr^2\omega_\rho\omega_s\hat{\mathbf{i}}$$

so that

$$\boxed{R = \frac{mr^2\omega_\rho\omega_s}{2L}}$$

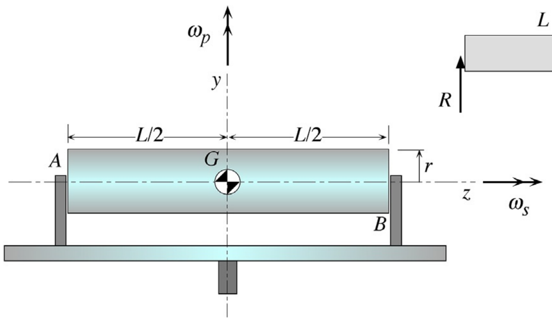


FIG. 11.22

Illustration of the gyroscopic effect.

11.9 EULER ANGLES

Three angles are required to specify the orientation of a rigid body relative to an inertial frame. The choice is not unique, but there are two sets in common use: Euler angles and yaw, pitch, and roll angles. We will discuss each of them in turn. The reader is urged to review Section 4.5 on orthogonal coordinate transformations and, in particular, the discussion of Euler angle sequences.

The three Euler angles ϕ, θ, and ψ shown in Fig. 11.23 give the orientation of a body-fixed xyz frame of reference relative to the XYZ inertial frame of reference. The xyz frame is obtained from the XYZ frame by a sequence of rotations through each of the Euler angles in turn. The first rotation is around the $Z\,(=z_1)$ axis through the precession angle ϕ. This takes X into x_1 and Y into y_1. The second rotation is around the $x_2\,(=x_1)$ axis through the nutation angle θ. This carries y_1 and z_1 into y_2 and z_2, respectively. The third and final rotation is around the $z\,(=z_2)$ axis through the spin angle ψ, which takes x_2 into x and y_2 into y.

The matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the inertial frame to the body-fixed frame is given by the classical Euler angle sequence (Eq. 4.37):

$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_3(\psi)][\mathbf{R}_1(\theta)][\mathbf{R}_3(\phi)] \tag{11.102}$$

From Eqs. (4.32) and (4.34), we have

$$[\mathbf{R}_3(\psi)] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\mathbf{R}_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad [\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.103}$$

According to Eq. (4.38), the direction cosine matrix is

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \sin\theta\sin\psi \\ -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\psi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix} \tag{11.104}$$

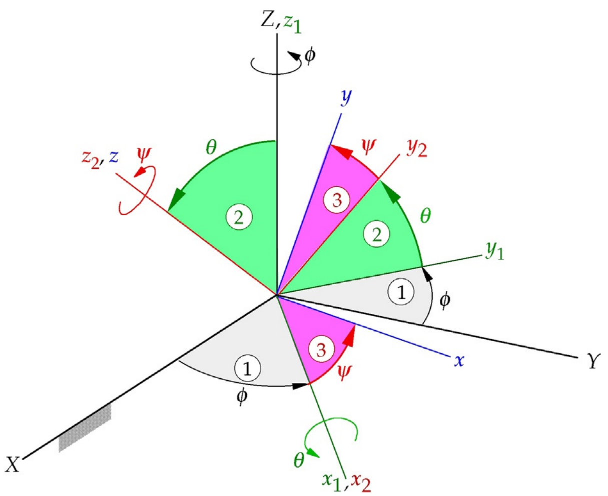


FIG. 11.23

Classical Euler angle sequence (see also fig. 4.14).

Since this is an orthogonal matrix, the inverse transformation from xyz to XYZ is $[\mathbf{Q}]_{xX} = [\mathbf{Q}]_{Xx}^T$,

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\psi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \tag{11.105}$$

Algorithm 4.3 is used to find the three Euler angles θ, ϕ, and ψ from a given direction cosine matrix $[\mathbf{Q}]_{Xx}$.

EXAMPLE 11.17

The direction cosine matrix of an orthogonal transformation from XYZ to xyz is

$$[\mathbf{Q}] = \begin{bmatrix} -0.32175 & 0.89930 & -0.29620 \\ 0.57791 & -0.061275 & -0.81380 \\ -0.75000 & -0.43301 & -0.5000 \end{bmatrix}$$

Use Algorithm 4.3 to find the Euler angles ϕ, θ, and ψ for this transformation.

Solution

Step 1 (precession angle, ϕ):

$$\phi = \tan^{-1}\left(\frac{Q_{31}}{-Q_{32}}\right) = \tan^{-1}\left(\frac{-0.75000}{0.43301}\right) \quad (0 \le \phi < 360°)$$

Since the numerator is negative and the denominator is positive, the angle ϕ lies in the fourth quadrant:

$$\boxed{\phi = \tan^{-1}(-1.7320) = 300°}$$

Step 2 (nutation angle, θ):

$$\theta = \cos^{-1} Q_{33} = \cos^{-1}(-0.5000) \quad (0 \le \theta \le 180°)$$

$$\boxed{\theta = 120°}$$

Step 3 (spin angle, ψ):

$$\psi = \tan^{-1}\frac{Q_{13}}{Q_{23}} = \tan^{-1}\left(\frac{-0.29620}{-0.81380}\right) \quad (0 \le \psi < 360°)$$

Since both the numerator and denominator are negative, the angle ψ lies in the third quadrant:

$$\boxed{\psi = \tan^{-1}(0.36397) = 200°}$$

The time rates of change of the Euler angles ϕ, θ, and ψ are, respectively, the precession rate ω_p, the nutation rate ω_n, and the spin ω_s. That is,

$$\omega_p = \dot{\phi} \qquad \omega_n = \dot{\theta} \qquad \omega_s = \dot{\psi} \tag{11.106}$$

The absolute angular velocity $\boldsymbol{\omega}$ of a rigid body can be resolved into components ω_x, ω_y, and ω_z along the body-fixed xyz axes, so that

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{11.107}$$

Fig. 11.23 shows that precession is measured around the inertial Z axis (unit vector $\hat{\mathbf{K}}$), nutation is measured around the intermediate x_1 axis (node line) with unit vector $\hat{\mathbf{i}}_1$, and spin is measured around the body-fixed z axis (unit vector $\hat{\mathbf{k}}$). Therefore, the absolute angular velocity can alternatively be written in terms of the nonorthogonal Euler angle rates as

$$\boldsymbol{\omega} = \omega_p \hat{\mathbf{K}} + \omega_n \hat{\mathbf{i}}_1 + \omega_s \hat{\mathbf{k}} \tag{11.108}$$

To find the relationship between the body rates ω_x, ω_y, and ω_z and the Euler angle rates ω_p, ω_n, and ω_s, we must express $\hat{\mathbf{K}}$ and $\hat{\mathbf{i}}_1$ in terms of the unit vectors $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ of the body-fixed frame. To accomplish that, we proceed as follows.

The first rotation $[\mathbf{R}_3(\phi)]$ in Eq. (11.102) rotates the unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ of the inertial frame into the unit vectors $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ of the intermediate $x_1y_1z_1$ axes in Fig. 11.23. Hence $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ are rotated into $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ by the inverse transformation given by

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = [\mathbf{R}_3(\phi)]^T \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} \tag{11.109}$$

The second rotation $[\mathbf{R}_1(\theta)]$ rotates $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ into the unit vectors $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ of the second intermediate frame $x_2y_2z_2$ in Fig. 11.23. The inverse transformation rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ back into $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = [\mathbf{R}_1(\theta)]^T \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} \tag{11.110}$$

Finally, the third rotation $[\mathbf{R}_3(\psi)]$ rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, the target unit vectors of the body-fixed xyz frame. $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ are obtained from $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ by the reverse rotation,

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = [\mathbf{R}_3(\psi)]^T \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} \tag{11.111}$$

From Eqs. (11.109) through (11.111), we observe that

$$\hat{\mathbf{K}} \overbrace{=}^{11.109} \hat{\mathbf{k}}_1 \overbrace{=}^{11.110} \sin\theta\hat{\mathbf{j}}_2 + \cos\theta\hat{\mathbf{k}}_2 \overbrace{=}^{11.111} \sin\theta\left(\sin\psi\hat{\mathbf{i}} + \cos\psi\hat{\mathbf{j}}\right) + \cos\theta\hat{\mathbf{k}}$$

or

$$\hat{\mathbf{K}} = \sin\theta\sin\psi\hat{\mathbf{i}} + \sin\theta\cos\psi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{11.112}$$

Similarly, Eqs. (11.110) and (11.111) imply that

$$\hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2 = \cos\psi\hat{\mathbf{i}} - \sin\psi\hat{\mathbf{j}} \tag{11.113}$$

Substituting Eqs. (11.112) and (11.113) into Eq. (11.108) yields

$$\boldsymbol{\omega} = \omega_p\left(\sin\theta\sin\psi\hat{\mathbf{i}} + \sin\theta\cos\psi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\right) + \omega_n\left(\cos\psi\hat{\mathbf{i}} - \sin\psi\hat{\mathbf{j}}\right) + \omega_s\hat{\mathbf{k}}$$

or

$$\boldsymbol{\omega} = \left(\omega_p\sin\theta\sin\psi + \omega_n\cos\psi\right)\hat{\mathbf{i}} + \left(\omega_p\sin\theta\cos\psi - \omega_n\sin\psi\right)\hat{\mathbf{j}} + \left(\omega_s + \omega_p\cos\theta\right)\hat{\mathbf{k}} \tag{11.114}$$

Comparing the coefficients of $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ in this equation with those in Eqs. (11.107), we see that

$$\begin{aligned}
\omega_x &= \omega_p \sin\theta \sin\psi + \omega_n \cos\psi \\
\omega_y &= \omega_p \sin\theta \cos\psi - \omega_n \sin\psi \\
\omega_z &= \omega_s + \omega_p \cos\theta
\end{aligned} \tag{11.115a}$$

or

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\sin\psi & -\sin\psi & 0 \\ \cos\theta & & 1 \end{bmatrix} \begin{Bmatrix} \omega_p \\ \omega_n \\ \omega_s \end{Bmatrix} \tag{11.115b}$$

(Note that the precession angle ϕ does not appear.) We solve these three equations to obtain the Euler rates in terms of ω_x, ω_y, and ω_z:

$$\begin{Bmatrix} \omega_p \\ \omega_n \\ \omega_s \end{Bmatrix} = \begin{bmatrix} \sin\psi/\sin\theta & \cos\psi/\sin\theta & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\sin\psi/\tan\theta & -\cos\psi/\tan\theta & 1 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{11.116a}$$

or

$$\begin{aligned}
\omega_p &= \dot{\phi} = \frac{1}{\sin\theta}(\omega_x \sin\psi + \omega_y \cos\psi) \\
\omega_n &= \dot{\theta} = \omega_x \cos\psi - \omega_y \sin\psi \\
\omega_s &= \dot{\psi} = -\frac{1}{\tan\theta}(\omega_x \sin\psi + \omega_y \cos\psi) + \omega_z
\end{aligned} \tag{11.116b}$$

Observe that if ω_x, ω_y, and ω_z are given functions of time, found by solving the Euler equations of motion (Eq. 11.72), then Eqs. (11.116b) are three coupled differential equations that may be solved to obtain the three time-dependent Euler angles, namely

$$\phi = \phi(t) \quad \theta = \theta(t) \quad \psi = \psi(t)$$

With this solution, the orientation of the xyz frame, and hence the body to which it is attached, is known for any given time t. Note, however, that Eqs. (11.116) "blow up" when $\theta = 0$ (i.e., when the xy plane is parallel to the XY plane).

EXAMPLE 11.18

At a given instant, the unit vectors of a body frame are

$$\begin{aligned}
\hat{\mathbf{i}} &= 0.40825\hat{\mathbf{I}} - 0.40825\hat{\mathbf{J}} + 0.81649\hat{\mathbf{K}} \\
\hat{\mathbf{j}} &= -0.10102\hat{\mathbf{I}} - 0.90914\hat{\mathbf{J}} - 0.40405\hat{\mathbf{K}} \\
\hat{\mathbf{k}} &= 0.90726\hat{\mathbf{I}} + 0.082479\hat{\mathbf{J}} - 0.41240\hat{\mathbf{K}}
\end{aligned} \tag{a}$$

and the angular velocity is

$$\boldsymbol{\omega} = -3.1\hat{\mathbf{I}} + 2.5\hat{\mathbf{J}} + 1.7\hat{\mathbf{K}}\,(\text{rad/s}) \tag{b}$$

Calculate ω_p, ω_n, and ω_s (the precession, nutation, and spin rates) at this instant.

Solution

We will ultimately use Eq. (11.116) to find ω_p, ω_n, and ω_s. To do so we must first obtain the Euler angles ϕ, θ, and ψ as well as the components of the angular velocity in the body frame.

The three rows of the direction cosine matrix $[\mathbf{Q}]_{Xx}$ comprise the components of the given unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \tag{c}$$

Therefore, the components of the angular velocity in the body frame are

$$\{\boldsymbol{\omega}\}_x = [\mathbf{Q}]_{Xx}\{\boldsymbol{\omega}\}_X = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \begin{Bmatrix} -3.1 \\ 2.5 \\ 1.7 \end{Bmatrix} = \begin{Bmatrix} -0.89817 \\ -2.6466 \\ -3.3074 \end{Bmatrix}$$

or

$$\omega_x = -0.89817\,\text{rad/s} \qquad \omega_y = -2.6466\,\text{rad/s} \qquad \omega_z = -3.3074\,\text{rad/s} \tag{d}$$

To obtain the Euler angles ϕ, θ, and ψ from the direction cosine matrix in Eq. (c), we use Algorithm 4.3, as was illustrated in Example 11.17. That algorithm is implemented as the MATLAB function *dcm_to_Euler.m* in Appendix D.20. Typing the following lines in the MATLAB Command Window:

```
Q = [ .40825  -.40825    .81649
     -.10102  -.90914   -.40405
      .90726   .082479  -.41240];
[phi theta psi] = dcm_to_euler(Q)
```

produces the following output:

```
phi =
     95.1945
theta =
      114.3557
psi =
      116.3291
```

Substituting $\theta = 114.36°$ and $\psi = 116.33°$ together with the angular velocities of Eq. (d) into Eqs. (11.116) yields

$$\begin{aligned}\omega_p &= \frac{1}{\sin 114.36°}[-0.89817 \cdot \sin 116.33° + (-2.6466) \cdot \cos 116.33°] \\ &= \boxed{0.40492\,\text{rad/s}}\end{aligned}$$

$$\begin{aligned}\omega_n &= -0.89817 \cdot \cos 116.33° - (-2.6466) \cdot \sin 116.33° \\ &= \boxed{2.7704\,\text{rad/s}}\end{aligned}$$

$$\begin{aligned}\omega_s &= -\frac{1}{\tan 114.36°}[-0.89817 \cdot \sin 116.33° + (-2.6466) \cdot \cos 116.33°] + (-3.3074) \\ &= \boxed{-3.1404\,\text{rad/s}}\end{aligned}$$

EXAMPLE 11.19

The mass moments of inertia of a body about the principal body frame axes with origin at the center of mass G are

$$A = 1000\ \mathrm{kg \cdot m^2} \quad B = 2000\ \mathrm{kg \cdot m^2} \quad C = 3000\ \mathrm{kg \cdot m^2} \tag{a}$$

The Euler angles in radians are given as functions of time in seconds as follows:

$$\begin{aligned} \phi &= 2te^{-0.05t} \\ \theta &= 0.02 + 0.3\sin 0.25t \\ \psi &= 0.6t \end{aligned} \tag{b}$$

At $t = 10$ s, find

(a) the net moment about G and

(b) the components α_X, α_Y, and α_Z of the absolute angular acceleration in the inertial frame.

Solution

(a) We must use Euler equations (Eq. 11.72) to calculate the net moment, which means we must first obtain $\omega_x, \omega_y, \omega_z, \dot{\omega}_x, \dot{\omega}_y$, and $\dot{\omega}_z$. Since we are given the Euler angles as functions of time, we can compute their time derivatives and then use Eq. (11.115) to find the body frame angular velocity components and their derivatives. Starting with the first of Eqs. (b), we get

$$\begin{aligned} \omega_p &= \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2te^{-0.05t}\right) = 2e^{-0.05t} - 0.1te^{-0.05t} \\ \dot{\omega}_p &= \frac{\mathrm{d}\omega_p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2e^{-0.05t} - 0.1e^{-0.05t}\right) = -0.2e^{-0.05t} + 0.005te^{-0.05t} \end{aligned}$$

Proceeding to the remaining two Euler angles leads to

$$\begin{aligned} \omega_n &= \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(0.02 + 0.3\sin 0.25t) = 0.075\cos 0.25t \\ \dot{\omega}_n &= \frac{\mathrm{d}\omega_n}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(0.075\cos 0.25t) = -0.01875\sin 0.25t \\ \omega_s &= \frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(0.6t) = 0.6 \\ \dot{\omega}_s &= \frac{\mathrm{d}\omega_s}{\mathrm{d}t} = 0 \end{aligned}$$

Evaluating all these quantities, including those in Eqs. (b), at $t = 10$ s yields

$$\begin{array}{lll} \phi = 335.03^\circ & \omega_p = 0.60653\,\mathrm{rad/s} & \dot{\omega}_p = -0.09098\,\mathrm{rad/s^2} \\ \theta = 11.433^\circ & \omega_n = -0.060086\,\mathrm{rad/s} & \dot{\omega} = -0.011221\,\mathrm{rad/s^2} \\ \psi = 343.77 & \omega_s = 0.6\,\mathrm{rad/s} & \dot{\omega}_s = 0 \end{array} \tag{c}$$

Eq. (11.115) relates the Euler angle rates to the angular velocity components,

$$\begin{aligned} \omega_x &= \omega_p \sin\theta \sin\psi + \omega_n \cos\psi \\ \omega_y &= \omega_p \sin\theta \cos\psi - \omega_n \sin\psi \\ \omega_z &= \omega_s + \omega_p \cos\theta \end{aligned} \tag{d}$$

Taking the time derivative of each of these equations in turn leads to the following three equations:

$$\begin{aligned} \dot{\omega}_x &= \omega_p\omega_n \cos\theta \sin\psi + \omega_p\omega_s \sin\theta \cos\psi - \omega_n\omega_s \sin\psi + \dot{\omega}_p \sin\theta \sin\psi + \dot{\omega}_n \cos\psi \\ \dot{\omega}_y &= \omega_p\omega_n \cos\theta \cos\psi - \omega_p\omega_s \sin\theta \sin\psi - \omega_n\omega_s \cos\psi + \dot{\omega}_p \sin\theta \cos\psi - \dot{\omega}_n \sin\psi \\ \dot{\omega}_z &= -\omega_p\omega_n \sin\theta + \dot{\omega}_p \cos\theta + \dot{\omega}_s \end{aligned} \tag{e}$$

Substituting the data in Eqs. (c) into Eqs. (d) and (e) yields

$$\begin{array}{lll}\omega_x = -0.091286\,\text{rad/s} & \omega_y = 0.098649\,\text{rad/s} & \omega_z = 1.1945\,\text{rad/s} \\ \dot{\omega}_x = 0.063435\,\text{rad/s}^2 & \dot{\omega}_y = 2.2346\left(10^{-5}\right)\text{rad/s}^2 & \dot{\omega}_z = -0.08195\,\text{rad/s}^2\end{array} \tag{f}$$

With Eqs. (a) and (f) we have everything we need for the Euler equations, namely,

$$\begin{aligned} M_x)_{\text{net}} &= A\dot{\omega}_x + (C-B)\omega_y\omega_z \\ M_y)_{\text{net}} &= B\dot{\omega}_y + (A-C)\omega_z\omega_x \\ M_z)_{\text{net}} &= C\dot{\omega}_z + (B-A)\omega_x\omega_y \end{aligned}$$

from which we find

$$\boxed{\begin{aligned} M_x)_{\text{net}} &= 181.27\ \text{N}\cdot\text{m} \\ M_y)_{\text{net}} &= 218.12\ \text{N}\cdot\text{m} \\ M_z)_{\text{net}} &= -254.86\ \text{N}\cdot\text{m} \end{aligned}}$$

(b) Since the comoving xyz frame is a body frame, rigidly attached to the solid, we know from Eq. (11.74) that

$$\begin{Bmatrix} a_X \\ a_Y \\ a_Z \end{Bmatrix} = [\mathbf{Q}]_{xX} \begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} \tag{g}$$

In other words, the absolute angular acceleration and the relative angular acceleration of the body are the same. All we have to do is project the components of relative acceleration in Eqs. (f), onto the axes of the inertial frame. The required orthogonal transformation matrix is given in Eq. (11.105),

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\phi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

Upon substituting the numerical values of the Euler angles from Eqs. (c), this becomes

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} 0.75484 & 0.65055 & -0.083668 \\ -0.65356 & 0.73523 & -0.17970 \\ -0.055386 & 0.19033 & 0.98016 \end{bmatrix}$$

Substituting this and the relative angular velocity rates from Eqs. (f) into Eq. (g) yields

$$\begin{aligned} \begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} &= \begin{bmatrix} 0.75484 & 0.65055 & -0.083668 \\ -0.65356 & 0.73523 & -0.17970 \\ -0.055386 & 0.19033 & 0.98016 \end{bmatrix} \begin{Bmatrix} 0.063435 \\ 2.2345\left(10^{-5}\right) \\ -0.08195 \end{Bmatrix} \\ &= \boxed{\begin{Bmatrix} 0.054755 \\ -0.026716 \\ -0.083833 \end{Bmatrix} \left(\text{rad/s}^2\right)} \end{aligned}$$

EXAMPLE 11.20

Fig. 11.24 shows a rotating platform on which is mounted a rectangular parallelepiped shaft (with dimensions b, h, and ℓ) spinning about the inclined axis DE. If the mass of the shaft is m, and the angular velocities ω_p and ω_s are constant, calculate the bearing forces at D and E as a function of ϕ and ψ. Neglect gravity, since we are interested only in the gyroscopic forces. (The small extensions shown at each end of the parallelepiped are just for clarity; the distance between the bearings at D and E is ℓ.)

Solution

The inertial XYZ frame is centered at O on the platform, and it is right handed $\left(\hat{\mathbf{I}} \times \hat{\mathbf{J}} = \hat{\mathbf{K}}\right)$. The origin of the right-handed comoving body frame xyz is at the shaft's center of mass G, and it is aligned with the symmetry axes of the parallelepiped.

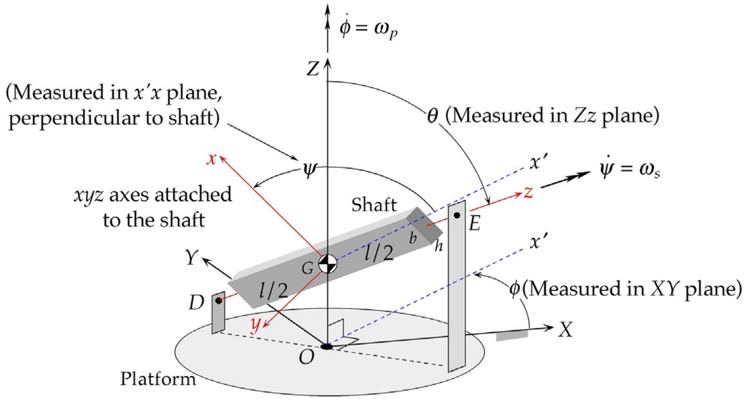


FIG. 11.24

Spinning block mounted on rotating platform.

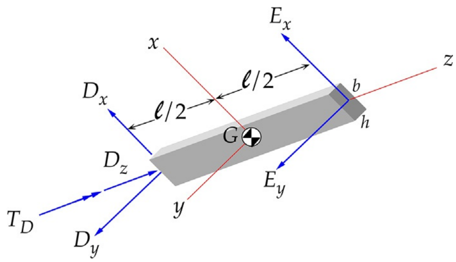


FIG. 11.25

Free body diagram of the block in Fig. 11.24.

The three Euler angles ϕ, θ, and ψ are shown in Fig. 11.24. Since θ is constant, the nutation rate is zero ($\omega_n = 0$). Thus, Eqs. (11.115) reduce to

$$\omega_x = \omega_p \sin\theta \sin\psi \quad \omega_y = \omega_p \sin\theta \cos\psi \quad \omega_z = \omega_p \cos\theta + \omega_s \tag{a}$$

Since ω_p, ω_s, and θ are constant, it follows (recalling Eq. 11.106) that

$$\dot{\omega}_x = \omega_p \omega_s \sin\theta \cos\psi \quad \dot{\omega}_y = -\omega_p \omega_s \sin\theta \sin\psi \quad \dot{\omega}_z = 0 \tag{b}$$

The principal moments of inertia of the parallelepiped are (Fig. 11.10c)

$$A = I_x = \frac{1}{12}m\left(h^2 + \ell^2\right) \quad B = I_y = \frac{1}{12}m\left(b^2 + l^2\right) \quad C = I_z = \frac{1}{12}m\left(b^2 + h^2\right) \tag{c}$$

Fig. 11.25 is a free body diagram of the shaft. Let us assume that the bearings at D and E are such as to exert just the six body frame components of force shown. Thus, D is a thrust bearing to which the axial torque T_D is applied from, say, a motor of some kind. At E, there is a simple journal bearing.

From Newton's laws of motion, we have $\mathbf{F}_{\text{net}} = m\mathbf{a}_G$. But G is fixed in inertial space, so $\mathbf{a}_G = \mathbf{0}$. Thus,

$$\left(D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}} + D_z\hat{\mathbf{k}}\right) + \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}\right) = 0$$

It follows that

$$E_x = -D_x \quad E_y = -D_y \quad D_z = 0 \tag{d}$$

Summing moments about G we get

$$\begin{aligned}\mathbf{M}_G)_{\text{net}} &= \frac{\ell}{2}\hat{\mathbf{k}} \times \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}\right) + \left(-\frac{\ell}{2}\hat{\mathbf{k}}\right) \times \left(D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}}\right) + T_D\hat{\mathbf{k}} \\ &= \left(D_y\frac{\ell}{2} - E_y\frac{\ell}{2}\right)\hat{\mathbf{i}} + \left(-D_x\frac{\ell}{2} + E_x\frac{\ell}{2}\right)\hat{\mathbf{j}} + T_D\hat{\mathbf{k}} \\ &= D_y\ell\hat{\mathbf{i}} - D_x\ell\hat{\mathbf{j}} + T_D\hat{\mathbf{k}}\end{aligned}$$

where we made use of Eq. (d)$_2$. Thus,

$$M_x)_{\text{net}} = D_y\ell \quad M_y)_{\text{net}} = -D_x\ell \quad M_z)_{\text{net}} = T_D \tag{e}$$

We substitute Eqs. (a) through (c) and (e) into the Euler equations (Eqs. 11.72):

$$\begin{aligned}M_x)_{\text{net}} &= A\dot{\omega}_x + (C-B)\omega_y\omega_z \\ M_y)_{\text{net}} &= B\dot{\omega}_y + (A-C)\omega_x\omega_z \\ M_z)_{\text{net}} &= C\dot{\omega}_z + (B-A)\omega_x\omega_y\end{aligned} \tag{f}$$

After making the substitutions and simplifying, the first Euler equation, Eq. (f)$_1$, becomes

$$D_y = \frac{1}{12}\frac{m}{\ell}\left[\left(h^2 - l^2\right)\omega_p\cos\theta + 2h^2\omega_s\right]\omega_p\sin\theta\cos\psi \tag{g}$$

Likewise, from Eq. (f)$_2$ we obtain

$$D_x = \frac{1}{12}\frac{m}{\ell}\left[\left(b^2 - l^2\right)\omega_p\cos\theta + 2b^2\omega_s\right]\omega_p\sin\theta\sin\psi \tag{h}$$

Finally, Eq. (f)$_3$ yields

$$T_D = \frac{1}{24}m\left(b^2 - h^2\right){\omega_p}^2\sin^2\theta\sin 2\psi \tag{i}$$

This completes the solution, since $E_y = -D_y$ and $E_z = -D_z$. Note that the resultant transverse bearing load V at D (and E) is

$$V = \sqrt{{D_x}^2 + {D_y}^2} \tag{j}$$

As a numerical example, let

$$\ell = 1\,\text{m} \quad h = 0.1\,\text{m} \quad b = 0.025\,\text{m} \quad \theta = 30° \quad m = 10\,\text{kg}$$

and

$$\omega_p = 100\,\text{rpm} = 10.47\,\text{rad/s} \qquad \omega_s = 2000\,\text{rpm} = 209.4\,\text{rad/s}$$

For these numbers, the variation of V and T_D with ψ are as shown in Fig. 11.26.

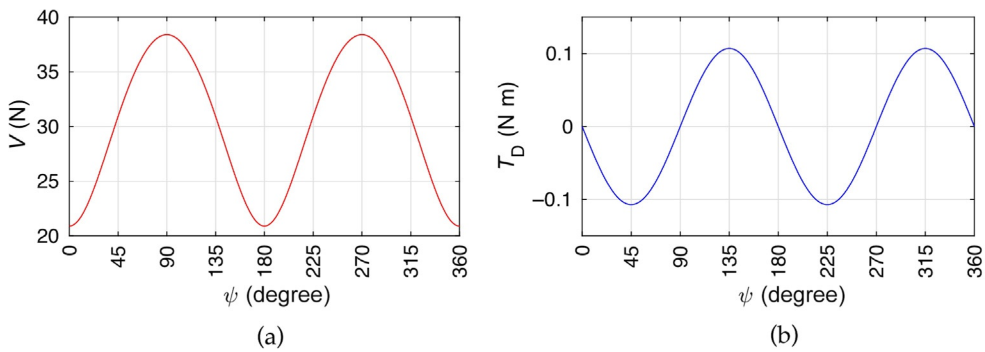


FIG. 11.26

(A) Transverse bearing load. (B) Axial torque at D.

11.10 YAW, PITCH, AND ROLL ANGLES

The problem of the Euler angle relations (Eq. 11.116) becoming singular when the nutation angle θ is zero can be alleviated by using the yaw, pitch, and roll angles discussed in Section 4.5. As in the classical Euler sequence, the yaw–pitch–roll sequence rotates the inertial XYZ axes into the triad of body-fixed xyz axes by means of a series of three elementary rotations, as illustrated in Fig. 11.27. Like the classical Euler sequence, the first rotation is around the Z $(=z_1)$ axis through the yaw angle ϕ. This takes X into x_1 and Y into y_1. The second rotation is around the y_2 $(=y_1)$ axis through the pitch angle θ. This carries x_1 and z_1 into x_2 and z_2, respectively. The third and final rotation is around the x $(=x_2)$ axis through the roll angle ψ, which takes y_2 into y and z_2 into z.

Eq. (4.40) gives the matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the inertial frame into the body-fixed frame,

$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_1(\psi)][\mathbf{R}_2(\theta)][\mathbf{R}_3(\phi)] \tag{11.117}$$

From Eqs. (4.32) through (4.34), the elementary rotation matrices are

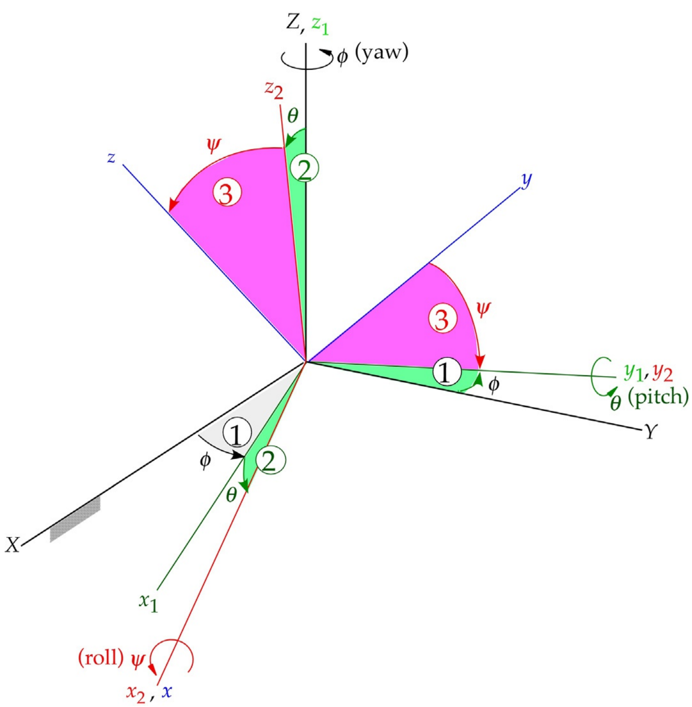


FIG. 11.27

Yaw, pitch, and roll sequence (see also fig. 4.15).

$$[\mathbf{R}_1(\psi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \quad [\mathbf{R}_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11.118}$$

According to Eq. (4.41), the multiplication on the right of Eq. (11.117) yields the following direction cosine matrix for the yaw, pitch, and roll sequence:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\psi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\theta\cos\psi \end{bmatrix} \tag{11.119}$$

The inverse matrix $[\mathbf{Q}]_{xX}$, which transforms xyz into XYZ, is just the transpose

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\sin\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{bmatrix} \tag{11.120}$$

Algorithm 4.4 (*dcm_to_ypr.m* in Appendix D.21) is used to determine the yaw, pitch, and roll angles for a given direction cosine matrix. The following brief MATLAB session reveals that the yaw, pitch, and roll angles for the direction cosine matrix in Example 11.17 are $\phi = 109.69°$, $\theta = 17.230°$, and $\psi = 238.43°$.

```
Q = [-0.32175   0.89930   -0.29620
      0.57791  -0.061275  -0.81380
     -0.75000  -0.43301   -0.5000];
[yaw pitch roll] = dcm_to_ypr(Q)


yaw =
     109.6861
pitch =
     17.2295
roll =
     238.4334
```

Fig. 11.27 shows that yaw ϕ is measured around the inertial Z axis (unit vector $\hat{\mathbf{K}}$), pitch θ is measured around the intermediate y_1 axis (unit vector $\hat{\mathbf{j}}_1$), and roll ψ is measured around the body-fixed x axis (unit vector $\hat{\mathbf{i}}$). The angular velocity $\boldsymbol{\omega}$, expressed in terms of the rates of yaw, pitch, and roll, is

$$\boldsymbol{\omega} = \omega_{yaw}\hat{\mathbf{K}} + \omega_{pitch}\hat{\mathbf{j}}_2 + \omega_{roll}\hat{\mathbf{i}} \tag{11.121}$$

in which

$$\omega_{yaw} = \dot{\phi} \qquad \omega_{pitch} = \dot{\theta} \qquad \omega_{roll} = \dot{\psi} \tag{11.122}$$

The first rotation $[\mathbf{R}_3(\phi)]$ in Eq. (11.117) rotates the unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ of the inertial frame into the unit vectors $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ of the intermediate $x_1y_1z_1$ axes in Fig. 11.27. Thus, $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ are rotated into $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ by the inverse transformation

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} \tag{11.123}$$

The second rotation $[\mathbf{R}_2(\theta)]$ rotates $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ into the unit vectors $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ of the second intermediate frame $x_2y_2z_2$ in Fig. 11.27. The inverse transformation rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ back into $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} \tag{11.124}$$

Lastly, the third rotation $[\mathbf{R}_1(\psi)]$ rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, the unit vectors of the body-fixed xyz frame. $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ are obtained from $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ by the reverse transformation,

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} \tag{11.125}$$

From Eqs. (11.123) through (11.125), we see that

$$\hat{\mathbf{K}} \overset{11.123}{\overbrace{=}} \hat{\mathbf{k}}_1 \overset{11.124}{\overbrace{=}} -\sin\theta\hat{\mathbf{i}}_2 + \cos\theta\hat{\mathbf{k}}_2 \overset{11.125}{\overbrace{=}} -\sin\theta\hat{\mathbf{i}} + \cos\theta\left(\sin\psi\hat{\mathbf{j}} + \cos\psi\hat{\mathbf{k}}\right)$$

or

$$\hat{\mathbf{K}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}} \tag{11.126}$$

From Eq. (11.125),

$$\hat{\mathbf{j}}_2 = \cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}} \tag{11.127}$$

Substituting Eqs. (11.126) and (11.127) into Eq. (11.121) yields

$$\boldsymbol{\omega} = \omega_{\text{yaw}}\left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}}\right) + \omega_{\text{pitch}}\left(\cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}}\right) + \omega_{\text{roll}}\hat{\mathbf{i}}$$

or

$$\begin{aligned} \boldsymbol{\omega} = &\left(-\omega_{\text{yaw}}\sin\theta + \omega_{\text{roll}}\right)\hat{\mathbf{i}} + \left(\omega_{\text{yaw}}\cos\theta\sin\psi + \omega_{\text{pitch}}\cos\psi\right)\hat{\mathbf{j}} \\ &+ \left(\omega_{\text{yaw}}\cos\theta\cos\psi - \omega_{\text{pitch}}\sin\psi\right)\hat{\mathbf{k}} \end{aligned} \tag{11.128}$$

Comparing the coefficients of $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ in Eqs. (11.107) and (11.128), we conclude that the body angular velocities are related to the yaw, pitch, and roll rates as follows:

$$\begin{aligned} \omega_x &= \omega_{\text{roll}} - \omega_{\text{yaw}}\sin\theta_{\text{pitch}} \\ \omega_y &= \omega_{\text{yaw}}\cos\theta_{\text{pitch}}\sin\psi_{\text{roll}} + \omega_{\text{pitch}}\cos\psi_{\text{roll}} \\ \omega_z &= \omega_{\text{yaw}}\cos\theta_{\text{pitch}}\cos\psi_{\text{roll}} - \omega_{\text{pitch}}\sin\psi_{\text{roll}} \end{aligned} \tag{11.129a}$$

or

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} -\sin\theta_{\text{pitch}} & 0 & 1 \\ \cos\theta_{\text{pitch}}\sin\psi_{\text{roll}} & \cos\psi_{\text{roll}} & 0 \\ \cos\theta_{\text{pitch}}\cos\psi_{\text{roll}} & -\sin\psi_{\text{roll}} & 0 \end{bmatrix} \begin{Bmatrix} \omega_{\text{yaw}} \\ \omega_{\text{pitch}} \\ \omega_{\text{roll}} \end{Bmatrix} \tag{11.129b}$$

wherein the subscript on each symbol helps us remember the physical rotation it describes. Note that ϕ_{yaw} does not appear. The inverse of these equations is

$$\begin{Bmatrix} \omega_{\text{yaw}} \\ \omega_{\text{pitch}} \\ \omega_{\text{roll}} \end{Bmatrix} = \begin{bmatrix} 0 & \sin\psi_{\text{roll}}/\cos\theta_{\text{pitch}} & \cos\psi_{\text{roll}}/\cos\theta_{\text{pitch}} \\ 0 & \cos\psi_{\text{roll}} & -\sin\psi_{\text{roll}} \\ 1 & \sin\psi_{\text{roll}}\tan\theta_{\text{pitch}} & \cos\psi_{\text{roll}}\tan\theta_{\text{pitch}} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{11.130a}$$

or

$$\begin{aligned} \omega_{\text{yaw}} &= \frac{1}{\cos\theta_{\text{pitch}}}(\omega_y\sin\psi_{\text{roll}} + \omega_z\cos\psi_{\text{roll}}) \\ \omega_{\text{pitch}} &= \omega_y\cos\psi_{\text{roll}} - \omega_z\sin\psi_{\text{roll}} \\ \omega_{\text{roll}} &= \omega_x + \omega_y\tan\theta_{\text{pitch}}\sin\psi_{\text{roll}} + \omega_z\tan\theta_{\text{pitch}}\cos\psi_{\text{roll}} \end{aligned} \tag{11.130b}$$

Note that this system becomes singular ($\cos\theta_{\text{pitch}} = 0$) when the pitch angle is $\pm 90°$.

11.11 QUATERNIONS

In Chapter 4, we showed that the transformation from any Cartesian coordinate frame to another having the same origin can be accomplished by three Euler angle sequences, each being an elementary rotation about one of the three coordinate axes. We have focused on the commonly used classical Euler angle sequence $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$ and the yaw–pitch–roll sequence $[\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)]$.

Another of Euler's theorems, which we used in Section 1.6, states that any two Cartesian coordinate frames are related by a unique rotation about a single line through their common origin. This line is called the Euler axis and the angle is referred to as the principal angle.

Let $\hat{\mathbf{u}}$ be the unit vector along the Euler axis. A vector $\mathbf{v}$ can be resolved into orthogonal components $\mathbf{v}_\perp$ normal to $\hat{\mathbf{u}}$ and $\mathbf{v}_{||}$ parallel to $\hat{\mathbf{u}}$, so that we may write

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_\perp \tag{11.131}$$

The component of $\mathbf{v}$ along $\hat{\mathbf{u}}$ is given by $\mathbf{v}\cdot\hat{\mathbf{u}}$. That is,

$$\mathbf{v}_{||} = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{11.132}$$

From Eqs. (11.131) and (11.132), we have

$$\mathbf{v}_\perp = \mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{11.133}$$

Let $\mathbf{v}'$ be the vector obtained by rotating $\mathbf{v}$ through an angle θ around $\hat{\mathbf{u}}$, as illustrated in Fig. 11.28. This rotation leaves the magnitude of $\mathbf{v}_\perp$ and its component along $\hat{\mathbf{u}}$ unchanged. That is

$$\|\mathbf{v}'_\perp\| = \|\mathbf{v}_\perp\| \tag{11.134}$$

$$\mathbf{v}'_{||} = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{11.135}$$

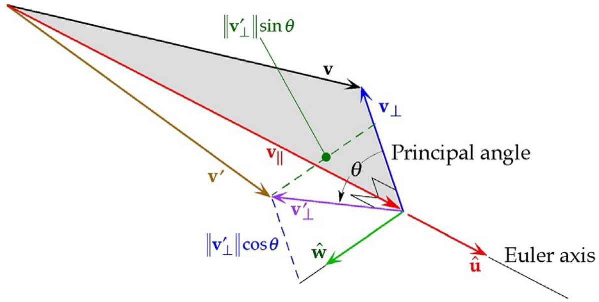


FIG. 11.28

Rotation of a vector through an angle θ about an axis with unit vector $\hat{\mathbf{u}}$.

$\mathbf{v}'_\perp$, having been rotated about $\hat{\mathbf{u}}$, has the component $\left\|\mathbf{v}'_\perp\right\| \cos\theta$ along the original vector $\mathbf{v}_\perp$ and the component $\left\|\mathbf{v}'_\perp\right\| \sin\theta$ along the vector normal to the plane of $\hat{\mathbf{u}}$ and $\mathbf{v}$. Let $\hat{\mathbf{w}}$ be the unit vector normal to that plane. Then,

$$\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \frac{\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|} \tag{11.136}$$

Thus,

$$\mathbf{v}'_\perp = \left\|\mathbf{v}'_\perp\right\| \cos\theta \frac{\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|} + \left\|\mathbf{v}'_\perp\right\| \sin\theta \frac{\hat{\mathbf{u}} \times \mathbf{v}_\perp}{\|\mathbf{v}_\perp\|}$$

According to Eq. (11.134), this reduces to

$$\mathbf{v}'_\perp = \cos\theta \mathbf{v}_\perp + \sin\theta \hat{\mathbf{u}} \times \mathbf{v}_\perp \tag{11.137}$$

Observe that

$$\hat{\mathbf{u}} \times \mathbf{v}_\perp = \hat{\mathbf{u}} \times \left(\mathbf{v} - \mathbf{v}_\|\right) = \hat{\mathbf{u}} \times \mathbf{v}$$

since $\mathbf{v}_\|$ is parallel to $\hat{\mathbf{u}}$. This, together with Eq. (11.133), means we can write Eq. (11.137) as

$$\mathbf{v}'_\perp = \cos\theta[\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}] + \sin\theta(\hat{\mathbf{u}} \times \mathbf{v}) \tag{11.138}$$

Since $\mathbf{v}' = \mathbf{v}'_\perp + \mathbf{v}'_\|$, we find, upon substituting Eqs. (11.135) and (11.138) and collecting terms, that

$$\mathbf{v}' = \cos\theta\mathbf{v} + (1 - \cos\theta)(\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}} + \sin\theta(\hat{\mathbf{u}} \times \mathbf{v}) \tag{11.139}$$

This is known as Rodrigues' *rotation* formula, named for the same French mathematician who gave us the Rodrigues' formula for Legendre polynomials (Eq. 10.22). Eq. (11.139) is useful for determining the result of rotating a vector about a line.

We can obtain the body-fixed xyz Cartesian frame from the inertial XYZ frame by a single rotation through the principal angle θ about the Euler axis $\hat{\mathbf{u}}$. The unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ are thereby rotated into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$. The two sets of unit vectors are related by Eq. (11.139). Thus,

$$\begin{aligned}
\hat{\mathbf{i}} &= \cos\theta\hat{\mathbf{I}} + (1 - \cos\theta)\left(\hat{\mathbf{u}} \cdot \hat{\mathbf{I}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}} \times \hat{\mathbf{I}} \\
\hat{\mathbf{j}} &= \cos\theta\hat{\mathbf{J}} + (1 - \cos\theta)\left(\hat{\mathbf{u}} \cdot \hat{\mathbf{J}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}} \times \hat{\mathbf{J}} \\
\hat{\mathbf{k}} &= \cos\theta\hat{\mathbf{K}} + (1 - \cos\theta)\left(\hat{\mathbf{u}} \cdot \hat{\mathbf{K}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}} \times \hat{\mathbf{K}}
\end{aligned} \tag{11.140}$$

Let us express the unit vector $\hat{\mathbf{u}}$ in terms of its direction cosines l, m, and n along the original XYZ axes. That is

$$\hat{\mathbf{u}} = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} \qquad l^2 + m^2 + n^2 = 1 \tag{11.141}$$

Substituting these into Eq. (11.140), carrying out the vector operations, and collecting the terms yields

$$\begin{aligned}
\hat{\mathbf{i}} &= \left[l^2(1-\cos\theta)+\cos\theta\right]\hat{\mathbf{I}} + [lm(1-\cos\theta)+n\sin\theta]\hat{\mathbf{J}} + [ln(1-\cos\theta)-m\sin\theta]\hat{\mathbf{K}} \\
\hat{\mathbf{j}} &= [lm(1-\cos\theta)-\sin\theta]\hat{\mathbf{I}} + \left[m^2(1-\cos\theta)+\cos\theta\right]\hat{\mathbf{J}} + [mn(1-\cos\theta)+l\sin\theta]\hat{\mathbf{K}} \\
\hat{\mathbf{k}} &= [ln(1-\cos\theta)+m\sin\theta]\hat{\mathbf{I}} + [mn(1-\cos\theta)-l\sin\theta]\hat{\mathbf{J}} + \left[n^2(1-\cos\theta)+\cos\theta\right]\hat{\mathbf{K}}
\end{aligned} \tag{11.142}$$

Recall that the rows of the matrix $[\mathbf{Q}]_{Xx}$ of the transformation from XYZ to xyz comprise the direction cosines of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively. That is,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)+n\sin\theta & ln(1-\cos\theta)-m\sin\theta \\ lm(1-\cos\theta)-n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)+l\sin\theta \\ ln(1-\cos\theta)+m\sin\theta & mn(1-\cos\theta)-l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{11.143}$$

The direction cosine matrix is thus expressed in terms of the Euler axis direction cosines and the principal angle.

Quaternions (also known as Euler symmetric parameters) were introduced in 1843 by the Irish mathematician Sir William R. Hamilton (1805–65). They provide an alternative to the use of direction cosine matrices for describing the orientation of a body frame in three-dimensional space. Quaternions can be used to avoid encountering the singularities we observed for the classical Euler angle sequence when the nutation angle θ becomes zero (Eqs. 11.116) or for the yaw, pitch, and roll sequence when the pitch angle θ approaches 90° (Eqs. (11.130)).

As the name implies, a quaternion $\widehat{\mathbf{q}}$ comprises four numbers:

$$\widehat{\mathbf{q}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \hdashline q_4 \end{Bmatrix} = \begin{Bmatrix} \mathbf{q} \\ \hdashline q_4 \end{Bmatrix} \tag{11.144}$$

where $\mathbf{q}$ is called the vector part $\left(\mathbf{q} = q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}}\right)$, and q_4 is the scalar part. (It is common to see the scalar part denoted q_0 and listed first, in which case $\widehat{\mathbf{q}} = \begin{Bmatrix} q_0 \\ \hdashline \mathbf{q} \end{Bmatrix}$.) Regardless, a quaternion whose scalar part is zero is called a pure quaternion.

The norm $\|\widehat{\mathbf{q}}\|$ of the quaternion $\widehat{\mathbf{q}}$ is defined as

$$\|\widehat{\mathbf{q}}\| = \sqrt{\|\mathbf{q}\|^2 + q_4^2} = \sqrt{\mathbf{q}\cdot\mathbf{q} + q_4^2} \tag{11.145}$$

Obviously, the norm of a pure quaternion ($q_4 = 0$) is just the norm of its vector part. A unit quaternion is one whose norm is unity ($\|\widehat{\mathbf{q}}\| = 1$).

Quaternions obey the familiar vector rules of addition and scalar multiplication. That is,

$$\widehat{\mathbf{p}} + \widehat{\mathbf{q}} = \begin{Bmatrix} \mathbf{p}+\mathbf{q} \\ \hdashline p_4 + q_4 \end{Bmatrix} \qquad a\widehat{\mathbf{p}} = \begin{Bmatrix} a\mathbf{p} \\ \hdashline ap_4 \end{Bmatrix}$$

Addition is both associative and commutative, so that

$$(\widehat{\mathbf{p}}+\widehat{\mathbf{q}})+\widehat{\mathbf{r}}=\widehat{\mathbf{p}}+(\widehat{\mathbf{q}}+\widehat{\mathbf{r}}) \qquad \widehat{\mathbf{p}}+\widehat{\mathbf{q}}=\widehat{\mathbf{q}}+\widehat{\mathbf{p}}$$

We use the special symbol $\otimes$ to denote the product or "composition" of two quaternions. The somewhat complicated rule for quaternion multiplication involves ordinary scalar multiplication as well as the familiar vector dot product and cross product operations,

$$\widehat{\mathbf{p}}\otimes\widehat{\mathbf{q}}=\left\{\begin{array}{c} p_4\mathbf{q}+q_4\mathbf{p}+\mathbf{p}\times\mathbf{q} \\ \hdashline p_4q_4-\mathbf{p}\cdot\mathbf{q} \end{array}\right\} \tag{11.146}$$

Switching the order of multiplication yields

$$\widehat{\mathbf{q}}\otimes\widehat{\mathbf{p}}=\left\{\begin{array}{c} q_4\mathbf{p}+p_4\mathbf{q}+\mathbf{q}\times\mathbf{p} \\ \hdashline q_4p_4-\mathbf{q}\cdot\mathbf{p} \end{array}\right\}$$

We are familiar with the fact that $\mathbf{q}\times\mathbf{p}=-(\mathbf{p}\times\mathbf{q})$, which means that quaternion multiplication is generally not commutative,

$$\widehat{\mathbf{p}}\otimes\widehat{\mathbf{q}}\neq\widehat{\mathbf{q}}\otimes\widehat{\mathbf{p}}$$

EXAMPLE 11.21

Find the product of the quaternions

$$\widehat{\mathbf{p}}=\left\{\begin{array}{c}\mathbf{p}\\ \hdashline p_4\end{array}\right\}=\left\{\begin{array}{c}\hat{\mathbf{j}}\\ \hdashline 1\end{array}\right\} \qquad \widehat{\mathbf{q}}=\left\{\begin{array}{c}\mathbf{q}\\ \hdashline q_4\end{array}\right\}=\left\{\begin{array}{c}0.5\hat{\mathbf{i}}+0.5\hat{\mathbf{j}}+0.75\hat{\mathbf{k}}\\ \hdashline 1\end{array}\right\} \tag{a}$$

Solution

$$\begin{aligned}
\widehat{\mathbf{p}}\otimes\widehat{\mathbf{q}} &= \left\{\begin{array}{c} p_4\mathbf{q}+q_4\mathbf{p}+\mathbf{p}\times\mathbf{q} \\ \hdashline p_4q_4-\mathbf{p}\cdot\mathbf{q} \end{array}\right\} \\
&= \left\{\begin{array}{c} \overbrace{1\cdot\left(0.5\hat{\mathbf{i}}+0.5\hat{\mathbf{j}}+0.75\hat{\mathbf{k}}\right)}^{0.5\hat{\mathbf{i}}+0.5\hat{\mathbf{j}}+0.75\hat{\mathbf{k}}}+\overbrace{1\cdot\hat{\mathbf{j}}}^{\hat{\mathbf{j}}}+\overbrace{\hat{\mathbf{j}}\times\left(0.5\hat{\mathbf{i}}+0.5\hat{\mathbf{j}}+0.75\hat{\mathbf{k}}\right)}^{-0.5\hat{\mathbf{k}}+0.75\hat{\mathbf{i}}} \\ \hdashline \underbrace{1\cdot 1}_{1}-\underbrace{\hat{\mathbf{j}}\cdot\left(0.5\hat{\mathbf{i}}+0.5\hat{\mathbf{j}}+0.75\hat{\mathbf{k}}\right)}_{0.5} \end{array}\right\} \\
&= \left\{\begin{array}{c} (0.5+0.75)\hat{\mathbf{i}}+(0.5+1.0)\hat{\mathbf{j}}+(0.75-0.5)\hat{\mathbf{k}} \\ \hdashline 0.5 \end{array}\right\}
\end{aligned}$$

or

$$\boxed{\widehat{\mathbf{p}}\otimes\widehat{\mathbf{q}}=\left\{\begin{array}{c}1.25\hat{\mathbf{i}}+1.5\hat{\mathbf{j}}+0.25\hat{\mathbf{k}}\\ \hdashline 0.5\end{array}\right\}}$$

The conjugate $\widehat{\mathbf{q}}^*$ of a quaternion $\widehat{\mathbf{q}}$ is found by simply multiplying its vector part by -1, thereby changing the signs of its vector components:

$$\widehat{\mathbf{q}}^*=\left\{\begin{array}{c}-\mathbf{q}\\ \hdashline q_4\end{array}\right\} \tag{11.147}$$

The identity quaternion $\widehat{\mathbf{1}}$ has zero for its vector part and 1 for its scalar part,

$$\widehat{\mathbf{1}}=\left\{\begin{array}{c}\mathbf{0}\\ \hdashline 1\end{array}\right\} \tag{11.148}$$

The product of any quaternion with $\widehat{\mathbf{1}}$ is commutative and yields the original quaternion,

$$\widehat{\mathbf{q}}\otimes\widehat{\mathbf{1}}=\widehat{\mathbf{1}}\otimes\widehat{\mathbf{q}}=\left\{\begin{array}{c}q_4\cdot\mathbf{0}+1\cdot\mathbf{q}+\mathbf{q}\times\mathbf{0}\\ \hdashline q_4\cdot 1-\mathbf{q}\cdot\mathbf{0}\end{array}\right\}=\left\{\begin{array}{c}\mathbf{q}\\ \hdashline q_4\end{array}\right\}=\widehat{\mathbf{q}}$$

Multiplication of a quaternion by its conjugate is also a commutative operation that yields a quaternion proportional to $\widehat{\mathbf{1}}$,

$$\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^*=\widehat{\mathbf{q}}^*\otimes\widehat{\mathbf{q}}=\left\{\begin{array}{c}q_4(-\mathbf{q})+q_4\mathbf{q}+\mathbf{q}\times(-\mathbf{q})\\ \hdashline q_4q_4-\mathbf{q}\cdot(-\mathbf{q})\end{array}\right\}=\left\{\begin{array}{c}\mathbf{0}\\ \hdashline \|\widehat{\mathbf{q}}\|^2\end{array}\right\}=\|\widehat{\mathbf{q}}\|^2\,\widehat{\mathbf{1}} \tag{11.149}$$

The inverse $\widehat{\mathbf{q}}^{-1}$ of a quaternion is defined as

$$\widehat{\mathbf{q}}^{-1}=\frac{\widehat{\mathbf{q}}^*}{\|\widehat{\mathbf{q}}\|^2} \tag{11.150}$$

Substituting $\widehat{\mathbf{q}}^*=\|\widehat{\mathbf{q}}\|^2\widehat{\mathbf{q}}^{-1}$ into Eq. (11.149) yields

$$\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^{-1}=\widehat{\mathbf{q}}^{-1}\otimes\widehat{\mathbf{q}}=\widehat{\mathbf{1}} \tag{11.151}$$

Clearly, for unit quaternions the inverse and the conjugate are the same, $\widehat{\mathbf{q}}^*=\widehat{\mathbf{q}}^{-1}$, and

$$\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^*=\widehat{\mathbf{q}}^*\otimes\widehat{\mathbf{q}}=\widehat{\mathbf{1}}\quad\left(\text{if }\|\widehat{\mathbf{q}}\|=1\right) \tag{11.152}$$

Let us restrict our attention to unit quaternions, in which case

$$\widehat{\mathbf{q}} = \left\{ \begin{array}{c} \sin(\theta/2)\hat{\mathbf{u}} \\ \hdashline \cos(\theta/2) \end{array} \right\} \tag{11.153}$$

where $\hat{\mathbf{u}}$ is the unit vector along the Euler axis around which the inertial reference frame is rotated into the body-fixed frame, and θ is the Euler principal rotation angle. Recalling Eq. (11.141), we observe that

$$q_1 = l\sin(\theta/2) \quad q_2 = m\sin(\theta/2) \quad q_3 = n\sin(\theta/2) \quad q_4 = \cos(\theta/2) \tag{11.154}$$

The conjugate quaternion $\widehat{\mathbf{q}}^*$ is found by reversing the sign of the vector part of $\widehat{\mathbf{q}}$, so that

$$\widehat{\mathbf{q}}^* = \left\{ \begin{array}{c} -\sin(\theta/2)\hat{\mathbf{u}} \\ \hdashline \cos(\theta/2) \end{array} \right\} \tag{11.155}$$

Employing these and the trigonometric identities

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) \quad \sin\theta = 2\cos(\theta/2)\sin(\theta/2) \tag{11.156}$$

we show in Appendix G that the direction cosine matrix $[\mathbf{Q}]_{Xx}$ of the body frame in Eq. (11.143) is obtained from the quaternion $\widehat{\mathbf{q}}$ by means of the following algorithm.

ALGORITHM 11.1

Obtain the direction cosine matrix $[\mathbf{Q}]_{Xx}$ from the unit quaternion $\widehat{\mathbf{q}}$. This procedure is implemented in the MATLAB function *dcm_from_q.m* in Appendix D.49.

1. Write the quaternion as

$$\widehat{\mathbf{q}} = \left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \hdashline q_4 \end{array} \right\}$$

where $\lfloor q_1 \ \ q_2 \ \ q_3 \rfloor^T$ is the vector part, q_4 is the scalar part, and $\|\widehat{\mathbf{q}}\| = 1$.

2. Compute the direction cosine matrix of the transformation from XYZ to xyz as follows:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_2q_1 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_3q_1 + q_2q_4) & 2(q_3q_2 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \tag{11.157}$$

Note that every element of the matrix $[\mathbf{Q}]_{Xx}$ in Eq. (11.157) contains products of two components of $\widehat{\mathbf{q}}$. Since $\widehat{\mathbf{q}}$ and $-\widehat{\mathbf{q}}$ therefore yield the same direction cosine matrix, they represent the same rotation. We can verify by carrying out the matrix multiplication and using Eq. (11.145) that $[\mathbf{Q}]_{Xx}$ in Eq. (11.157) exhibits the required orthogonality property,

$$[\mathbf{Q}]_{Xx}[\mathbf{Q}]_{Xx}{}^{T} = [\mathbf{Q}]_{Xx}{}^{T}[\mathbf{Q}]_{Xx} = [\mathbf{1}]$$

To find the unit quaternion ($q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$) for a given direction cosine matrix, we observe from Eq. (11.157) that

$$\begin{aligned} q_4 &= \frac{1}{2}\sqrt{1 + Q_{11} + Q_{22} + Q_{33}} \\ q_1 &= \frac{Q_{23} - Q_{32}}{4q_4} \qquad q_2 = \frac{Q_{31} - Q_{13}}{4q_4} \qquad q_3 = \frac{Q_{12} - Q_{21}}{4q_4} \end{aligned} \tag{11.158}$$

This procedure obviously fails for pure quaternions ($q_4 = 0$). The following algorithm (Bar-Itzhack, 2000) avoids having to deal with this situation.

ALGORITHM 11.2

Obtain the (unit) quaternion from the direction cosine matrix $[\mathbf{Q}]_{Xx}$. This procedure is implemented as the MATLAB function *q_from_dcm.m* in Appendix D.50.

1. Form the 4-by-4 *symmetric* matrix

$$[\mathbf{K}] = \frac{1}{3}\begin{bmatrix} Q_{11} - Q_{22} - Q_{33} & Q_{21} + Q_{12} & Q_{31} + Q_{13} & Q_{23} - Q_{32} \\ Q_{21} + Q_{12} & -Q_{11} + Q_{22} - Q_{33} & Q_{32} + Q_{23} & Q_{31} - Q_{13} \\ Q_{31} + Q_{13} & Q_{32} + Q_{23} & -Q_{11} - Q_{22} + Q_{33} & Q_{12} - Q_{21} \\ Q_{23} - Q_{32} & Q_{31} - Q_{13} & Q_{12} - Q_{21} & Q_{11} + Q_{22} + Q_{33} \end{bmatrix} \tag{11.159}$$

2. Solve the eigenvalue problem $[\mathbf{K}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$ for the largest eigenvalue $\lambda_{\max}$. The corresponding eigenvector is the quaternion, $\left\{\widehat{\mathbf{q}}\right\} = \{\mathbf{e}\}$. Since we are interested in only the dominant eigenvalue of $[\mathbf{K}]$, we can use the iterative power method (Jennings, 1977), which converges to $\lambda_{\max}$. Thus, starting with an estimate $\{\mathbf{e}_0\}$ of the eigenvector, we normalize it,

$$\{\hat{\mathbf{e}}_0\} = \frac{\{\mathbf{e}_0\}}{\|\mathbf{e}_0\|}$$

and use $\{\hat{\mathbf{e}}_0\}$ to compute an updated normalized estimate

$$\{\hat{\mathbf{e}}_1\} = \frac{[\mathbf{K}]\{\hat{\mathbf{e}}_0\}}{\|[\mathbf{K}]\{\hat{\mathbf{e}}_0\}\|} \quad (\|\hat{\mathbf{e}}_1\| = 1)$$

We estimate the corresponding eigenvalue by using the Rayleigh quotient,

$$r_1 = \frac{\{\hat{\mathbf{e}}_1\}^T[\mathbf{K}]\{\hat{\mathbf{e}}_1\}}{\|\hat{\mathbf{e}}_1\|^2} = \{\hat{\mathbf{e}}_1\}^T[\mathbf{K}]\{\hat{\mathbf{e}}_1\}$$

We repeat this process, using $\hat{\mathbf{e}}_1$ to compute an updated normalized estimate $\hat{\mathbf{e}}_2$ followed by using the Rayleigh quotient to find r_2, and so on, over and over again. After n steps we have $\hat{\mathbf{e}}_n$ and r_n. As n increases, r_n converges toward $\lambda_{\max}$, the *maximum* eigenvalue of $[\mathbf{K}]$. When $|(r_n - r_{n-1})/r_{n-1}| < \varepsilon$, where ε is our chosen tolerance, we terminate the iteration and declare that $\lambda_{\max} = r_n$ and that $\hat{\mathbf{e}}_n$ is the corresponding eigenvector.

Of course, instead of the power method we can take advantage of commercial software, such as MATLAB's eigenvalue extraction program `eig`.

EXAMPLE 11.22

(a) Write down the unit quaternion for a rotation about the x axis through an angle θ.
(b) Obtain the corresponding direction cosine matrix.

Solution

(a) According to Eq. (11.151),

$$\mathbf{q} = \sin(\theta/2)\hat{\mathbf{i}} \quad q_4 = \cos(\theta/2) \tag{a}$$

so that

$$\boxed{\widehat{\mathbf{q}} = \begin{Bmatrix} \sin(\theta/2) \\ 0 \\ 0 \\ \hdashline \cos(\theta/2) \end{Bmatrix}} \tag{b}$$

(b) Substituting $q_1 = \sin(\theta/2)$, $q_2 = q_3 = 0$, and $q_4 = \cos(\theta/2)$ into Eq. (11.152) yields

$$[\mathbf{Q}] = \begin{bmatrix} \sin^2(\theta/2) + \cos^2(\theta/2) & 0 & 0 \\ 0 & -\sin^2(\theta/2) + \cos^2(\theta/2) & 2\sin(\theta/2)\cos(\theta/2) \\ 0 & -2\sin(\theta/2)\cos(\theta/2) & -\sin^2(\theta/2) + \cos^2(\theta/2) \end{bmatrix} \tag{c}$$

From trigonometry, we have

$$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1 \qquad 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta \qquad \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$

Therefore, Eq. (c) becomes

$$\boxed{[\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}} \tag{d}$$

We recognize this as the direction cosine matrix $[\mathbf{R}_1(\theta)]$ for a rotation θ around the x axis (Eq. 4.33).

EXAMPLE 11.23

For the yaw–pitch–roll sequence $\phi_{\text{yaw}} = 50°$, $\theta_{\text{pitch}} = 90°$, and $\psi_{\text{roll}} = 120°$, calculate
(a) the quaternion and
(b) the rotation angle and the axis of rotation.

Solution

(a) Substituting the given angles into Eq. (11.119) yields the direction cosine matrix

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0 & 0 & -1 \\ 0.93969 & 0.34202 & 0 \\ 0.34202 & -0.93969 & 0 \end{bmatrix} \tag{a}$$

Substituting the components of $[\mathbf{Q}]_{Xx}$ into Eq. (11.159), we get

$$[\mathbf{K}] = \begin{bmatrix} -0.11401 & 0.31323 & -0.21933 & 0.31323 \\ 0.31323 & 0.11401 & -0.31323 & 0.44734 \\ -0.21933 & -0.31323 & -0.11401 & -0.31323 \\ 0.31323 & 0.44734 & -0.31323 & 0.11401 \end{bmatrix} \tag{b}$$

The following is a MATLAB script that implements the power method described in Algorithm 11.2.

```
K                  = [-0.11401  0.31323 -0.21933  0.31323
                        0.31323  0.11401 -0.31323  0.44734
                       -0.21933 -0.31323 -0.11401 -0.31323
                        0.31323  0.44734 -0.31323  0.11401];

v0                 = [1 1 1 1]';   %Initial estimate of the eigenvector.
v0                 = v0/norm(v0);  %Normalize it.
lamda_new          = v0'*K*v0;     %Rayleigh quotient (norm(v0) = 1)
                                   % estimate of the eigenvalue.
lamda_old          = 10*lamda_new; %Just to begin the iteration.
no_iterations      = 0;            %Count the number of iterations.
tolerance          = 1.e-10;

while abs((lamda_new - lamda_old)/lamda_old) > tolerance
    no_iterations = no_iterations + 1;
    lamda_old     = lamda_new;
    v             = v0;
    vnew          = K*v/norm(K*v);
    lamda_new     = vnew'*K*vnew; %Rayleigh quotient (norm(vnew) = 1).
    v0            = vnew;
end

no_iterations      = no_iterations
disp(' ')
lamda_max          = lamda_new
eigenvector        = vnew
```

The output of this program to the Command Window is as follows:

```
no_iterations =
    12
lamda_max =
             1
eigenvector =
       0.40558
       0.57923
      -0.40558
       0.57923
```

The quaternion is the eigenvector associated with $\lambda_{\max}$, so that

$$\boxed{\widehat{\mathbf{q}} = \left\{ \begin{array}{r} 0.40558 \\ 0.57923 \\ -0.40558 \\ \hdashline 0.57923 \end{array} \right\}}$$

Observe that $\|\widehat{\mathbf{q}}\| = 1$. $\widehat{\mathbf{q}}$ must be a unit quaternion.

(b) From Eq. (11.146), we find that the principal angle is

$$\theta = 2\cos^{-1}(q_4) = 2\cos^{-1}(0.57923) = \boxed{54.604°}$$

and the Euler axis is

$$\hat{\mathbf{u}} = \frac{0.40558\hat{\mathbf{I}} + 0.57923\hat{\mathbf{J}} - 0.40558\hat{\mathbf{K}}}{\sin(54.604°/2)} = \boxed{0.4975\hat{\mathbf{I}} + 0.71056\hat{\mathbf{J}} - 0.49754\hat{\mathbf{K}}}$$

We have seen that a unit quaternion of Eq. (11.153) represents a rotation about the unit vector $\hat{\mathbf{u}}$ through the angle θ. Let us show that the Rodrigues' formula (Eq. 11.139) for rotating the vector $\mathbf{v}$ into the vector $\mathbf{v}'$ may be written in terms of quaternions as follows:

$$\widehat{\mathbf{v}}' = \widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}} \otimes \widehat{\mathbf{q}}^* \tag{11.160}$$

where the conjugate quaternion $\widehat{\mathbf{q}}^*$ is given by Eq. (11.155), and $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{v}}'$ are the pure quaternions having $\mathbf{v}$ and $\mathbf{v}'$ as their vector parts,

$$\widehat{\mathbf{v}} = \left\{ \begin{array}{c} \mathbf{v} \\ \hdashline 0 \end{array} \right\} \qquad \widehat{\mathbf{v}}' = \left\{ \begin{array}{c} \mathbf{v}' \\ \hdashline 0 \end{array} \right\} \tag{11.161}$$

Eq. (11.160) is implemented in MATLAB as the function *quat_rotate.m* in Appendix D.51.

Using Eq. (11.146) we first calculate the product of $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{v}}$,

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}} = \left\{ \begin{array}{c} \cos(\theta/2)\mathbf{v} + \sin(\theta/2)(\hat{\mathbf{u}} \times \mathbf{v}) \\ \hdashline -\sin(\theta/2)(\hat{\mathbf{u}} \cdot \mathbf{v}) \end{array} \right\}$$

Multiply this quaternion on the right by $\widehat{\mathbf{q}}^*$ to get

$$\widehat{\mathbf{v}}' = (\widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}}) \otimes \widehat{\mathbf{q}}^* = \left\{ \begin{array}{c} \mathbf{v}' \\ \hdashline v_4' \end{array} \right\} \tag{11.162}$$

where $\mathbf{v}'$ and v_4' are, respectively, the vector and scalar parts of the quaternion $\widehat{\mathbf{v}}'$.

Once again we employ Eq. (11.146) to obtain

$$\begin{aligned}
\mathbf{v}' &= \overbrace{[-\sin(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v})]}^{(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}})_4}\overbrace{(-\sin(\theta/2)\hat{\mathbf{u}})}^{\mathbf{q}^*} + \overbrace{\cos(\theta/2)}^{q^*{}_4}\overbrace{[\cos(\theta/2)\mathbf{v}+\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})]}^{\mathbf{q}\otimes\mathbf{v}} \\
&\quad + \underbrace{[\cos(\theta/2)\mathbf{v}+\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})]}_{\mathbf{q}\otimes\mathbf{v}}\times\underbrace{(-\sin(\theta/2)\hat{\mathbf{u}})}_{\mathbf{q}^*} \\
&= \sin^2(\theta/2)\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v}) + \left[\cos^2(\theta/2)\mathbf{v}+\cos(\theta/2)\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})\right] \\
&\quad + \cos(\theta/2)\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v}) - \sin^2(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}}
\end{aligned}$$

According to the bac–cab rule (Eq. 1.20), $(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}} = \mathbf{v}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})$. Substituting this into the above equation and collecting terms we get

$$\mathbf{v}' = \mathbf{v}\left[\cos^2(\theta/2)-\sin^2(\theta/2)\right]+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})\left[2\sin^2(\theta/2)\right]+(\hat{\mathbf{u}}\times\mathbf{v})[2\cos(\theta/2)\sin(\theta/2)]$$

But

$$\cos^2(\theta/2)-\sin^2(\theta/2)=\cos\theta \qquad 2\sin^2(\theta/2)=1-\cos\theta \qquad 2\cos(\theta/2)\sin(\theta/2)=\sin\theta$$

so that finally

$$\mathbf{v}' = \mathbf{v}\cos\theta+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})(1-\cos\theta)+(\hat{\mathbf{u}}\times\mathbf{v})\sin\theta \tag{11.163}$$

According to Eq. (11.146), the scalar part v_4 of the quaternion product in Eq. (11.162) is

$$\begin{aligned}
v_4' &= \overbrace{[-\sin(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v})]}^{(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}})_4}\overbrace{[\cos(\theta/2)]}^{q^*{}_4} - \overbrace{[\cos(\theta/2)\mathbf{v}+\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})]}^{\mathbf{q}\otimes\mathbf{v}}\cdot\overbrace{[-\sin(\theta/2)\hat{\mathbf{u}}]}^{\mathbf{q}^*} \\
&= -\sin(\theta/2)\cos(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v})+\cos(\theta/2)\sin(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v})+\sin^2(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})\cdot\hat{\mathbf{u}} \\
&= 0
\end{aligned}$$

Thus, the scalar part of $\widehat{\mathbf{v}}'$ vanishes, which means that $\widehat{\mathbf{v}}'$ is a pure quaternion whose vector part $\mathbf{v}'$ is identical to Eq. (11.139). We have therefore shown that the quaternion operation $\widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}^*$ indeed rotates the vector $\mathbf{v}$ around the axis of the quaternion (the Euler axis) through the angle θ. In the same way we can show that the operation $\widehat{\mathbf{q}}^*\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}$ rotates the vector $\mathbf{v}$ through the angle $-\theta$. In fact, if we follow the operation $\widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}^*$ (rotation through $+\theta$) with the operation $\widehat{\mathbf{q}}^*\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}$ (rotation through $-\theta$) we end up where we started (namely, with the pure quaternion $\widehat{\mathbf{v}}$):

$$\widehat{\mathbf{q}}\otimes\left(\widehat{\mathbf{q}}^*\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}\right)\otimes\widehat{\mathbf{q}}^* = \left(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^*\right)\otimes\widehat{\mathbf{v}}\otimes\left(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^*\right) = \widehat{\mathbf{1}}\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{1}} = \left(\widehat{\mathbf{1}}\otimes\widehat{\mathbf{v}}\right)\otimes\widehat{\mathbf{1}} = \widehat{\mathbf{v}}\otimes\widehat{\mathbf{1}} = \widehat{\mathbf{v}}$$

The operation in Eq. (11.160) is a vector rotation. The frame of reference remains fixed while the vector $\mathbf{v}$ is rotated into the vector $\mathbf{v}'$. On the other hand, the familiar operation $\{\mathbf{v}\}_{x'} = [\mathbf{Q}]_{xx'}\{\mathbf{v}\}$ is a frame rotation (a coordinate transformation), in which the vector $\mathbf{v}$ remains fixed while the reference frame is rotated. We can easily illustrate this by revisiting Example 11.22.

EXAMPLE 11.24

Consider the vector $\mathbf{v} = v\hat{\mathbf{j}}$. Using the quaternion and corresponding direction cosine matrix in Example 11.22, carry out the following operations and interpret the results geometrically:

(i) $\widehat{\mathbf{v}}' = \widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}^*$

(ii) $\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\}$
where

$$\widehat{\mathbf{v}} = \left\{ \begin{array}{c} v\hat{\mathbf{j}} \\ \hdashline 0 \end{array} \right\} \qquad \widehat{\mathbf{q}} = \left\{ \begin{array}{c} \sin(\theta/2)\hat{\mathbf{i}} \\ \hdashline \cos(\theta/2) \end{array} \right\} \qquad [\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Solution

(i) We do the two quaternion products one after the other using Eq. (11.146). The first product is

$$\begin{aligned} \widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}} &= \left\{ \begin{array}{c} \cos(\theta/2)(v\hat{\mathbf{j}}) + 0\cdot\sin(\theta/2)\hat{\mathbf{i}} + \sin(\theta/2)\hat{\mathbf{i}} \times v\hat{\mathbf{j}} \\ \hdashline \cos(\theta/2)\cdot 0 - \sin(\theta/2)\hat{\mathbf{i}}\cdot v\hat{\mathbf{j}} \end{array} \right\} \\ &= \left\{ \begin{array}{c} v\cos(\theta/2)\hat{\mathbf{j}} + v\sin(\theta/2)\hat{\mathbf{k}} \\ \hdashline 0 \end{array} \right\} \end{aligned}$$

Following this by the second product, we get

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}} \otimes \widehat{\mathbf{q}}^* = \left\{ \begin{array}{c} v\cos(\theta/2)\hat{\mathbf{j}} + v\sin(\theta/2)\hat{\mathbf{k}} \\ \hdashline 0 \end{array} \right\} \otimes \left\{ \begin{array}{c} -\sin(\theta/2)\hat{\mathbf{i}} \\ \hdashline \cos(\theta/2) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 0\cdot[-\sin(\theta/2)\hat{\mathbf{i}}] + \cos(\theta/2)\cdot[v\cos(\theta/2)\hat{\mathbf{j}} + v\sin(\theta/2)\hat{\mathbf{k}}] + [v\cos(\theta/2)\hat{\mathbf{j}} + v\sin(\theta/2)\hat{\mathbf{k}}] \times [-\sin(\theta/2)\hat{\mathbf{i}}] \\ \hdashline 0\cdot[\cos(\theta/2)] - [v\cos(\theta/2)\hat{\mathbf{j}} + v\sin(\theta/2)\hat{\mathbf{k}}]\cdot[-\sin(\theta/2)\hat{\mathbf{i}}] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \overbrace{[v\cos^2(\theta/2) - v\sin^2(\theta/2)]}^{\cos\theta}\hat{\mathbf{j}} + \overbrace{[2v\sin(\theta/2)\cos(\theta/2)]}^{\sin\theta}\hat{\mathbf{k}} \\ \hdashline \mathbf{0} \end{array} \right\} = \left\{ \begin{array}{c} v\cos\theta\hat{\mathbf{j}} + v\sin\theta\hat{\mathbf{k}} \\ \hdashline 0 \end{array} \right\}$$

Finally, therefore,

$$\boxed{\mathbf{v}' = v\cos\theta\hat{\mathbf{j}} + v\sin\theta\hat{\mathbf{k}}} \tag{a}$$

(ii)

$$\{\mathbf{v}'\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} 0 \\ v \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ v\cos\theta \\ -v\sin\theta \end{Bmatrix}$$

or

$$\boxed{\mathbf{v}' = v\cos\theta\hat{\mathbf{j}} - v\sin\theta\hat{\mathbf{k}}} \tag{b}$$

These two results are illustrated in Fig. 11.29.

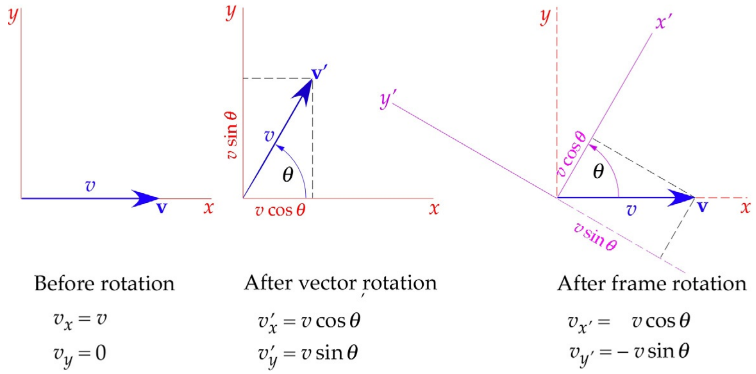


FIG. 11.29

Vector rotation vs frame rotation.

The Euler equations of motion for a rigid body (Eqs. 11.72) provide the angular velocity rates $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ as functions of time. We can integrate those equations to find the time history of the angular velocities ω_x, ω_y, and ω_z. In addition, to obtain the orientation history of the body, we need the time history of the Euler angles ϕ, θ, and ψ. These are found by integrating Eq. (11.116),

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} \dfrac{\sin\psi}{\sin\theta} & \dfrac{\cos\psi}{\sin\theta} & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\dfrac{\sin\psi}{\tan\theta} & -\dfrac{\cos\psi}{\tan\theta} & 1 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

which provide the Euler angle rates (precession, nutation, and spin) in terms of the angular velocities. If we elect to use quaternions instead of Euler angles to describe the attitude of the body, then we need a formula for the rate of change of $\widehat{\mathbf{q}}$ in terms of the angular velocities.

To find the time derivative of a pure quaternion $\widehat{\mathbf{q}}$, we simply differentiate Eq. (11.153) to get

$$\dot{\widehat{\mathbf{q}}} = \begin{Bmatrix} \dfrac{d}{dt}\left[\hat{\mathbf{u}}\sin(\theta/2)\right] \\ \hdashline \dfrac{d}{dt}\cos(\theta/2) \end{Bmatrix} = \begin{Bmatrix} \dot{\hat{\mathbf{u}}}\sin(\theta/2) + \hat{\mathbf{u}}(\dot{\theta}/2)\cos(\theta/2) \\ \hdashline -(\dot{\theta}/2)\sin(\theta/2) \end{Bmatrix}$$

The Euler axis unit vector $\hat{\mathbf{u}}$ is constant in magnitude, but not in direction. According to Gelman (1971) and Hughes (2004), its time derivative is

$$\dot{\hat{\mathbf{u}}} = \frac{1}{2}[\hat{\mathbf{u}} \times \boldsymbol{\omega} - \cot(\theta/2)\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\omega})] \tag{11.164}$$

where $\boldsymbol{\omega}$ is the angular velocity vector. Clearly, if the instantaneous axis of rotation and the Euler axis happen to coincide (i.e., if $\boldsymbol{\omega} = \omega\hat{\mathbf{u}}$), then $\dot{\hat{\mathbf{u}}} = \mathbf{0}$, because in that case $\hat{\mathbf{u}} \times \boldsymbol{\omega} = \mathbf{0}$. However, in general

$\dot{\hat{\mathbf{u}}}$ does not vanish. Substituting Eq. (11.164) into the expression for $\dot{\widehat{\mathbf{q}}}$, we find after expanding and collecting terms that

$$\dot{\widehat{\mathbf{q}}} = \frac{1}{2}\left\{\begin{array}{c} \sin(\theta/2)\dot{\hat{\mathbf{u}}}\times\boldsymbol{\omega}+\cos(\theta/2)\boldsymbol{\omega} \\ \hdashline -\dot{\theta}\sin(\theta/2) \end{array}\right\} \tag{11.165}$$

From Eq. (11.153) we know that $\hat{\mathbf{u}}\sin(\theta/2) = \mathbf{q}$ and $\cos(\theta/2) = q_4$. Observing furthermore that $\dot{\theta}$ is the component of the angular velocity $\boldsymbol{\omega}$ along the Euler axis ($\dot{\theta} = \boldsymbol{\omega}\cdot\hat{\mathbf{u}}$), the expression for $\dot{\widehat{\mathbf{q}}}$ becomes

$$\dot{\widehat{\mathbf{q}}} = \frac{1}{2}\left\{\begin{array}{c} \mathbf{q}\times\boldsymbol{\omega}+q_4\boldsymbol{\omega} \\ \hdashline -\boldsymbol{\omega}\cdot\mathbf{q} \end{array}\right\} \tag{11.166}$$

According to the quaternion composition rule (Eq. 11.146), this may be written

$$\dot{\widehat{\mathbf{q}}} = \frac{1}{2}\widehat{\mathbf{q}}\otimes\widehat{\boldsymbol{\omega}}$$

where $\widehat{\boldsymbol{\omega}} = \left\{\begin{array}{c}\boldsymbol{\omega}\\ \hdashline 0\end{array}\right\}$ is the pure quaternion version of the angular velocity vector. Substituting $\mathbf{q} = q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}}$ and $\boldsymbol{\omega} = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} + \omega_3\hat{\mathbf{k}}$ into Eq. (11.166) and expanding the vector and scalar products leads to

$$\dot{\widehat{\mathbf{q}}} = \frac{1}{2}\left\{\begin{array}{c} (q_2\omega_3 - q_3\omega_2) + q_4\omega_1 \\ (q_3\omega_1 - q_1\omega_3) + q_4\omega_2 \\ (q_1\omega_2 - q_2\omega_1) + q_4\omega_3 \\ \hdashline -\omega_1 q_1 - \omega_2 q_2 - \omega_3 q_3 \end{array}\right\} = \frac{1}{2}\left[\begin{array}{ccc:c} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ \hdashline -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{array}\right]\left\{\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \hdashline q_4 \end{array}\right\}$$

That is,

$$\frac{d}{dt}\left\{\widehat{\mathbf{q}}\right\} = \frac{1}{2}[\boldsymbol{\Omega}]\left\{\widehat{\mathbf{q}}\right\} \tag{11.167}$$

where

$$[\boldsymbol{\Omega}] = \left[\begin{array}{ccc:c} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ \hdashline -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{array}\right] \tag{11.168}$$

ω_1, ω_2, and ω_3 are the x, y, and z components of angular velocity in the body-fixed frame.

If the components of the angular velocity are constant, then the matrix $[\boldsymbol{\Omega}]$ is constant and we can readily integrate Eq. (11.167) to obtain

$$\left\{\widehat{\mathbf{q}}\right\} = \exp\left(\frac{[\boldsymbol{\Omega}]}{2}t\right)\left\{\widehat{\mathbf{q}}_0\right\} \tag{11.169}$$

where $\left\{\widehat{\mathbf{q}}_0\right\}$ is the value of the quaternion at time $t = 0$. This expression may be inferred directly from scalar calculus, in which we know that if c is a constant, then the solution of the differential equation

$dx/dt = cx$ is simply $x = x_0 e^{ct}$. In linear algebra we learn that a 4-by-4 matrix $[\mathbf{A}]$ has four eigenvalues λ_i and four corresponding eigenvectors $\{\mathbf{e}_i\}$, satisfying the equation

$$[\mathbf{A}]\{\mathbf{e}_i\} = \lambda_i\{\mathbf{e}_i\} \quad i = 1,\ldots,4$$

It turns out that

$$\exp([\mathbf{A}]) = [\mathbf{V}]\exp([\mathbf{\Lambda}])[\mathbf{V}]^{-1} \tag{11.170}$$

where $[\mathbf{\Lambda}]$ is the 4-by-4 diagonal matrix of eigenvalues,

$$[\mathbf{\Lambda}] = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

and $[\mathbf{V}]$ is the 4-by-4 matix whose columns comprise the four distinct eigenvectors $\{\mathbf{e}_i\}$,

$$[\mathbf{V}] = [\{\mathbf{e}_1\} \;\; \{\mathbf{e}_2\} \;\; \{\mathbf{e}_3\} \;\; \{\mathbf{e}_4\}]$$

Using, for example, MATLAB's symbolic math feature, we find that the eigenvalues and corresponding eigenvectors of the matrix $[\mathbf{\Omega}]$ in Eq. (11.168) are

$$\lambda_1 = \lambda_2 = \omega i: \quad \mathbf{e}_1 = \begin{Bmatrix} (\omega_y \omega i - \omega_x \omega_z)/\omega_{xy}{}^2 \\ -(\omega_x \omega i + \omega_y \omega_z)/\omega_{xy}{}^2 \\ 1 \\ 0 \end{Bmatrix} \quad \mathbf{e}_2 = \begin{Bmatrix} -(\omega_x \omega i + \omega_y \omega_z)/\omega_{xy}{}^2 \\ (-\omega_y \omega i + \omega_x \omega_z)/\omega_{xy}{}^2 \\ 0 \\ 1 \end{Bmatrix}$$

$$\lambda_3 = \lambda_4 = -\omega i: \quad \mathbf{e}_3 = \begin{Bmatrix} -(\omega_y \omega i + \omega_x \omega_z)/\omega_{xy}{}^2 \\ (\omega_x \omega i - \omega_y \omega_z)/\omega_{xy}{}^2 \\ 1 \\ 0 \end{Bmatrix} \quad \mathbf{e}_4 = \begin{Bmatrix} (\omega_x \omega i - \omega_y \omega_z)/\omega_{xy}{}^2 \\ (\omega_y \omega i + \omega_x \omega_z)/\omega_{xy}{}^2 \\ 0 \\ 1 \end{Bmatrix}$$

where $\omega = \sqrt{\omega_x{}^2 + \omega_y{}^2 + \omega_z{}^2}$ (the magnitude of the angular velocity vector), $\omega_{xy} = \sqrt{\omega_x{}^2 + \omega_y{}^2}$, and $i = \sqrt{-1}$. Substituting these results into Eq. (11.170) yields, again with the considerable aid of MATLAB,

$$\exp\left(\frac{[\mathbf{\Omega}]}{2}t\right) = \begin{bmatrix} \cos\dfrac{\omega t}{2} & \dfrac{\omega_z}{\omega}\sin\dfrac{\omega t}{2} & -\dfrac{\omega_y}{\omega}\sin\dfrac{\omega t}{2} & \dfrac{\omega_x}{\omega}\sin\dfrac{\omega t}{2} \\ -\dfrac{\omega_z}{\omega}\sin\dfrac{\omega t}{2} & \cos\dfrac{\omega t}{2} & \dfrac{\omega_x}{\omega}\sin\dfrac{\omega t}{2} & \dfrac{\omega_y}{\omega}\sin\dfrac{\omega t}{2} \\ \dfrac{\omega_y}{\omega}\sin\dfrac{\omega t}{2} & -\dfrac{\omega_x}{\omega}\sin\dfrac{\omega t}{2} & \cos\dfrac{\omega t}{2} & \dfrac{\omega_z}{\omega}\sin\dfrac{\omega t}{2} \\ -\dfrac{\omega_x}{\omega}\sin\dfrac{\omega t}{2} & -\dfrac{\omega_y}{\omega}\sin\dfrac{\omega t}{2} & -\dfrac{\omega_z}{\omega}\sin\dfrac{\omega t}{2} & \cos\dfrac{\omega t}{2} \end{bmatrix} \tag{11.171}$$

We know that for Eq. (11.171) to be valid, the angular velocity components must all be constant. The rigid body equations of motion (Eq. 11.72) show that $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$ if the net torque on the body is zero *and* the principal moments of inertia are all the same. Whereas torque-free motion (Chapter 12) is quite common for space vehicles, spherical symmetry ($A = B = C$) is not. Thus, we cannot make much practical use of Eqs. (11.169) and (11.171). In general, we must instead use numerical integration to obtain the angular velocities from the Euler equations and the quaternions from Eq. (11.167).

EXAMPLE 11.25

At time $t = 0$ the body-fixed axes and inertial angular velocity of a rigid body are those given in Example 11.18, namely

$$\begin{aligned}\hat{\mathbf{i}}_0 &= 0.40825\hat{\mathbf{I}} - 0.40825\hat{\mathbf{J}} + 0.81649\hat{\mathbf{K}} \\ \hat{\mathbf{j}}_0 &= -0.10102\hat{\mathbf{I}} - 0.90914\hat{\mathbf{J}} - 0.40405\hat{\mathbf{K}} \\ \hat{\mathbf{k}}_0 &= 0.90726\hat{\mathbf{I}} + 0.082479\hat{\mathbf{J}} - 0.41240\hat{\mathbf{K}}\end{aligned} \tag{a}$$

and

$$\boldsymbol{\omega}_X = -3.1\hat{\mathbf{I}} + 2.5\hat{\mathbf{J}} + 1.7\hat{\mathbf{K}}\,(\mathrm{rad/s}) \tag{b}$$

If the angular velocity is constant, find the time histories of the Euler angles and the quaternion.

Solution

Because the angular velocity is constant, the motion of the body will be pure rotation about the fixed axis of rotation defined by the angular velocity vector. Once we find the direction cosine matrix as a function of time, we can use Algorithm 4.3 to obtain the Euler angles at each time.

Step 1:

As in Example 11.18, we find that the direction cosine matrix at time $t = 0$ is

$$[\mathbf{Q}_0]_{Xx} = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \tag{c}$$

Step 2:

As in Example 11.18, use $[\mathbf{Q}_0]_{Xx}$ to project the angular velocity $\boldsymbol{\omega}_X$ onto the axes of the body-fixed frame

$$\{\boldsymbol{\omega}\}_x = [\mathbf{Q}_0]_{Xx}\{\boldsymbol{\omega}\}_X = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \begin{Bmatrix} -3.1 \\ 2.5 \\ 1.7 \end{Bmatrix}$$

so that

$$\omega_x = -0.89817\,\mathrm{rad/s} \quad \omega_y = -2.6466\,\mathrm{rad/s} \quad \omega_z = -3.3074\,\mathrm{rad/s} \tag{d}$$

The magnitude of the constant angular velocity is

$$\omega = \sqrt{{\omega_x}^2 + {\omega_y}^2 + {\omega_z}^2} = 4.3301\,\mathrm{rad/s} \tag{e}$$

The period of the rotation is $T = 2\pi/\omega = 1.451$ s.

Step 3:

Use the angular velocities in (d) to form the matrix $[\boldsymbol{\Omega}]$ in Eq. (11.168),

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & -3.3074 & 2.6466 & -0.89817 \\ 3.3074 & 0 & -0.89817 & -2.6466 \\ -2.6466 & 0.89817 & 0 & -3.3074 \\ 0.89817 & 2.6466 & 3.3074 & 0 \end{bmatrix} \tag{f}$$

$[\boldsymbol{\Omega}]$ remains constant.

Step 4:

Use Algorithm 11.2 to obtain the quaternion at $t=0$ from the direction cosine matrix in (c).

$$\widehat{\mathbf{q}}_0 = \left\{ \begin{array}{c} -0.82610 \\ 0.15412 \\ -0.52165 \\ \hdashline 0.14724 \end{array} \right\} \tag{g}$$

Step 5
At each time through t_{final}:
Compute the quaternion $\widehat{\mathbf{q}}(t)$ from Eqs. (11.169) and (11.171).
Use $\widehat{\mathbf{q}}(t)$ to update the direction cosine matrix $[\mathbf{Q}(t)]_{Xx}$ using Algorithm 11.1.
Use $[\mathbf{Q}(t)]_{Xx}$ to calculate the Euler angles $\phi(t)$, $\theta(t)$, and $\psi(t)$by means of Algorithm 4.3.

Fig. 11.30 shows the time variation of the four components of $\widehat{\mathbf{q}}$ during one rotation of the body. The variation of the three Euler angles is shown in Fig. 11.31. Observe that their values at $t=0$ agree with those found in Example 11.18. Fig. 11.32 shows the initial orientation of the orthonormal body-fixed xyz axes given in Eq. (a). The dotted lines trace out the subsequent motion of their end points as they rotate at 4.33 rad/s about the fixed angular velocity vector $\boldsymbol{\omega}$. Finally, Fig. 11.33 shows the initial position of the Euler axis $\hat{\mathbf{u}}$ and its subsequent motion during rotation of the body. The unit vector $\hat{\mathbf{u}}$ is obtained from the unit quaternion $\widehat{\mathbf{q}}(t)$ at any instant by means of Eq. (11.153),

$$\hat{\mathbf{u}}(t) = \frac{\mathbf{q}(t)}{\sin\{\cos^{-1}[q_4(t)]\}}$$

where $\mathbf{q}(t)$ is the vector part of $\widehat{\mathbf{q}}(t)$. Fig. 11.33 amply illustrates the fact that the unit vectors $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{u}}$ are not the same.

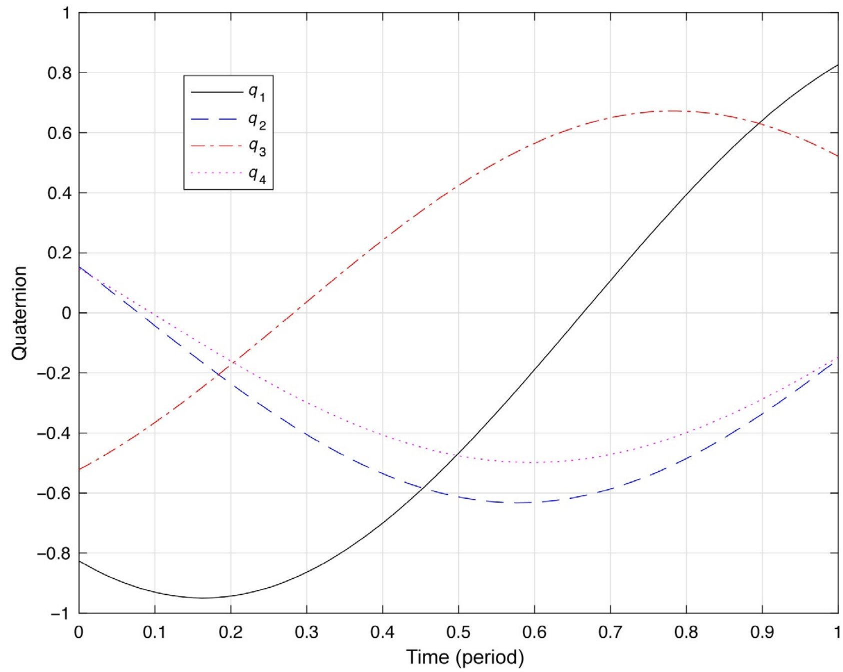


FIG. 11.30

History of the components of $\widehat{\mathbf{q}}$ for one rotation of the body.

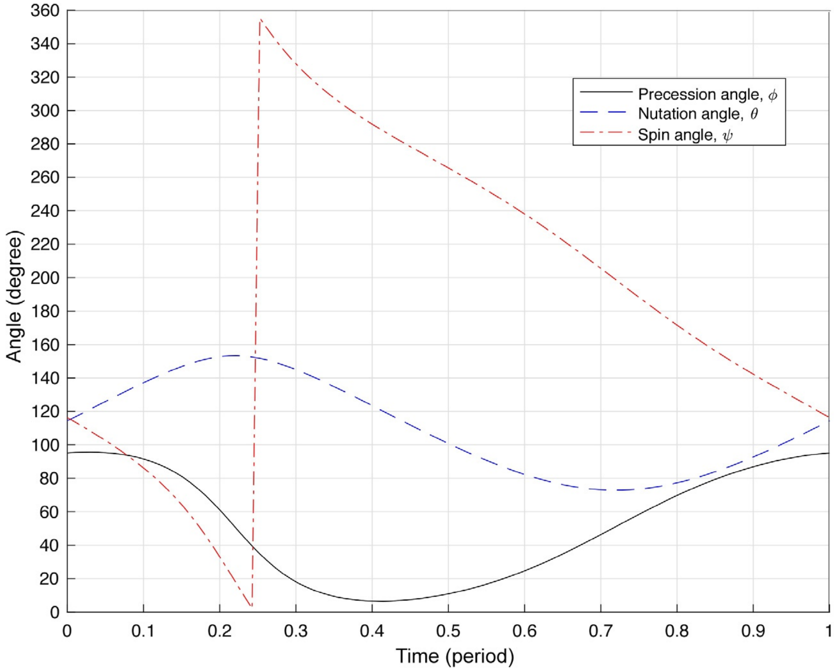


FIG. 11.31

History of the three Euler angles for one rotation of the body.

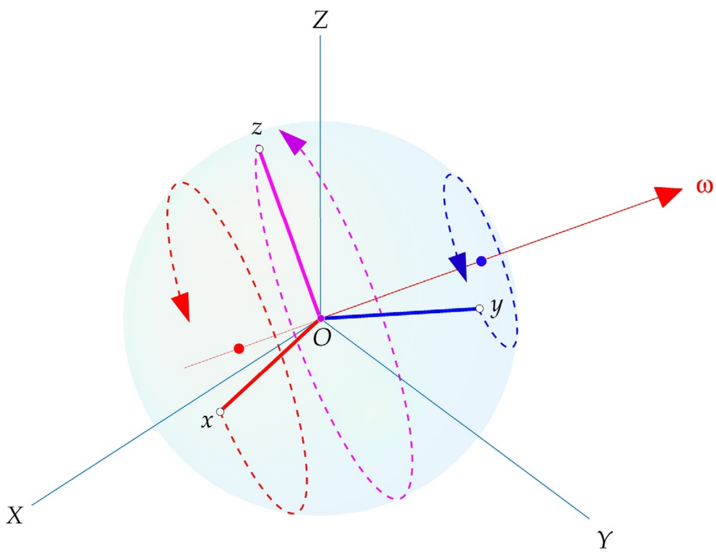


FIG. 11.32

The motion of three orthogonal lines during rotation of the body.

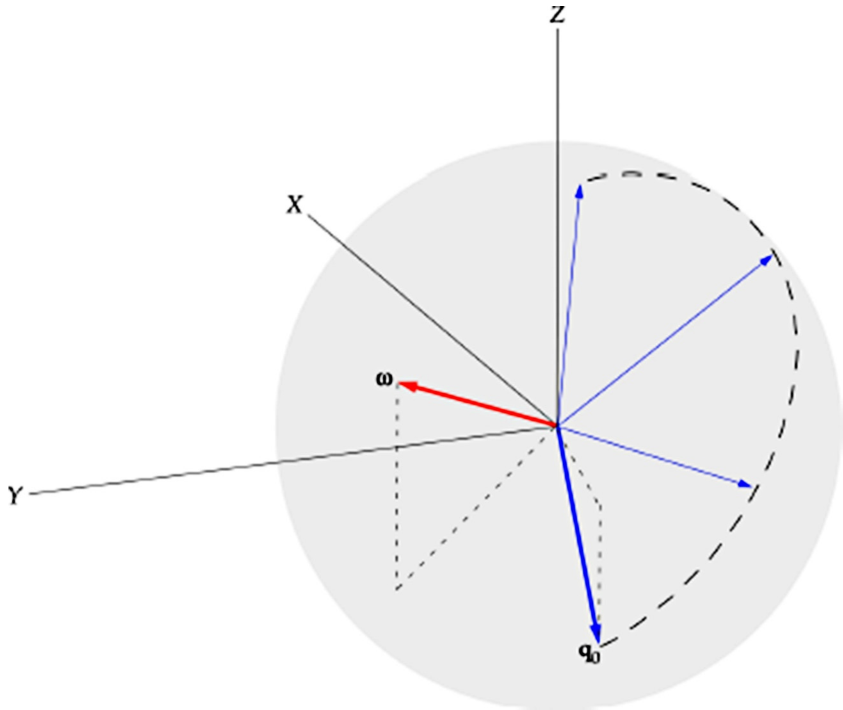


FIG. 11.33

Motion of the Euler axis during one rotation of the body.

EXAMPLE 11.26

Solve the spinning top problem of Example 11.15 numerically, using quaternions. Use the low-energy precession rate, $\omega_p = 51.93\,\text{rpm}$.

Solution

We will use Eq. (11.72) (the Euler equations) to compute the body frame angular velocity derivatives:

$$\begin{aligned}\frac{d\omega_x}{dt} &= \frac{M_x}{A} - \frac{C-B}{A}\omega_y\omega_z \\ \frac{d\omega_y}{dt} &= \frac{M_y}{B} - \frac{A-C}{B}\omega_z\omega_x \\ \frac{d\omega_z}{dt} &= \frac{M_z}{C} - \frac{B-A}{C}\omega_x\omega_y\end{aligned} \tag{a}$$

These require that the moving xyz axes are all rigidly attached to the top. In Example 11.15 only the x axis was fixed to the top along its spin axis; the y and z axes did not rotate with the top.

The moments in Eq. (a) must be expressed in components along the body-fixed axes. From Fig. 11.19, the moment of the weight vector about O is

$$\mathbf{M} = d\hat{\mathbf{k}} \times \left(-mg\hat{\mathbf{K}}\right) = -mgd\left(\hat{\mathbf{k}} \times \hat{\mathbf{K}}\right) \tag{b}$$

where, recalling Eq. (4.18)$_3$

$$\hat{\mathbf{k}} = Q_{31}\hat{\mathbf{I}} + Q_{32}\hat{\mathbf{J}} + Q_{33}\hat{\mathbf{K}} \tag{c}$$

The Qs are the time-dependent components of the direction cosine matrix $[\mathbf{Q}]_{Xx}$ in Eq. (11.157). Carrying out the cross product in Eq. (b) yields the components of $\mathbf{M}$ along the XYZ axes of the fixed space frame,

$$\{\mathbf{M}\}_X = \begin{Bmatrix} -mgdQ_{32} \\ mgdQ_{31} \\ 0 \end{Bmatrix}$$

To obtain the components of $\mathbf{M}$ in the body-fixed frame, we perform the transformation

$$\{\mathbf{M}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{M}\}_X = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{Bmatrix} -mgdQ_{32} \\ mgdQ_{31} \\ 0 \end{Bmatrix} = mgd\begin{Bmatrix} Q_{12}Q_{31} - Q_{32}Q_{11} \\ Q_{22}Q_{31} - Q_{32}Q_{21} \\ 0 \end{Bmatrix} \tag{d}$$

It can be shown that (Problem 11.27)

$$Q_{12}Q_{31} - Q_{32}Q_{11} = Q_{23} \qquad Q_{22}Q_{31} - Q_{32}Q_{21} = -Q_{13}$$

Therefore, at any instant the moments in Eq. (a) are

$$M_x = mgdQ_{23} \qquad M_y = -mgdQ_{13} \qquad M_z = 0 \tag{e}$$

The MATLAB implementation of the following procedure is listed in Appendix D.51.
Step 1:
Specify the initial orientation of the xyz axes of the body frame, thereby defining the initial value of the direction cosine matrix $[\mathbf{Q}]_{Xx}$.

According to Fig. 11.19, the body z axis is the top's spin axis, and we shall assume here that it initially lies in the global YZ plane, tilted 60° away from the Z axis, as it is in Example 11.15. Let the body x axis be initially aligned with the global X axis. The body y axis is then found from the cross product $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$. Thus,

$$\begin{aligned}
&\hat{\mathbf{k}} = -\sin 60°\hat{\mathbf{J}} + \cos 60°\hat{\mathbf{K}} \\
&\hat{\mathbf{i}} = \hat{\mathbf{I}} \\
&\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \cos 60°\hat{\mathbf{J}} + \sin 60°\hat{\mathbf{K}}
\end{aligned}$$

It follows that the direction cosine matrix relating XYZ to xyz at the start of the simulation is

$$[\mathbf{Q}_0]_{Xx} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60° & \sin 60° \\ 0 & -\sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \tag{f}$$

Step 2:
Compute the initial quaternion $\widehat{\mathbf{q}}_0$ using Algorithm 11.2. Substituting Eq. (f) into Eq. (11.150) yields

$$[\mathbf{K}] = \begin{bmatrix} 0 & 0 & 0 & 0.57735 \\ 0 & -0.33333 & 0 & 0 \\ 0 & 0 & -0.33333 & 0 \\ 0.57735 & 0 & 0 & 0.66667 \end{bmatrix}$$

Rather than finding the dominant eigenvector by means of the power method, as we did in Example 11.23, we shall here use MATLAB's `eig` function, which obtains all of the eigenpairs. The snippet of MATLAB code for doing so is:

```
[eigvectors, eigvalues] = eig(K);
%Find the dominant eigenvalue and the column of 'eigvectors' that
% contains its eigenvector:
[dominant_eigvalue,column] = max(max(abs(eigvalues)));
dominant_eigenvector = eigvectors(:,column)
```

The output of this code is,

```
dominant_eigenvector =
     0.5
     0
     0
     0.86603
```

Therefore, the initial value of the quaternion is

$$\widehat{\mathbf{q}}_0 = \left\{ \begin{array}{c} 0.5 \\ 0 \\ 0 \\ \hline 0.86603 \end{array} \right\} \tag{g}$$

Step 3:
Specify the initial values of the body frame components of angular velocity $\boldsymbol{\omega}_0 = \lfloor \omega_x)_0 \quad \omega_y)_0 \quad \omega_z)_0 \rfloor^T$.

Recall that Eqs. (11.115) relate these body frame angular velocities to the initial values of the top's Euler angles and their rates,

$$\begin{aligned} \omega_x)_0 &= \omega_p)_0 \sin\theta_0 \sin\psi_0 + \omega_n)_0 \cos\psi_0 \\ \omega_y)_0 &= \omega_p)_0 \sin\theta_0 \cos\psi_0 - \omega_n)_0 \sin\psi_0 \\ \omega_z)_0 &= \omega_s)_0 + \omega_p)_0 \cos\theta_0 \end{aligned} \tag{h}$$

The top is released from rest with a given tilt angle θ_0 and spin rate $\omega_s)_0$. According to Example 11.15,

$$\begin{aligned} \theta_0 &= 60^\circ \\ \psi_0 &= 0 \\ \omega_s)_0 &= 1000\,\text{rpm} = 104.72\ \text{rad/s} \\ \omega_p)_0 &= 51.93\,\text{rpm} = 5.438\ \text{rad/s}\ (\text{low energy precession rate}) \\ \omega_n)_0 &= 0 \end{aligned} \tag{i}$$

Substituting these into Eq. (h) we find

$$\boldsymbol{\omega}_0 = \lfloor 0 \quad 4.7095 \quad 107.44 \rfloor^T\ (\text{rad/s}) \tag{j}$$

Step 4:
Supply $\boldsymbol{\omega}_0$ and $\widehat{\mathbf{q}}_0$ as initial conditions to, say, the Runge–Kutta–Fehlberg 4(5) numerical integration procedure (Algorithm 1.3) to solve the system $\{\dot{\mathbf{y}}\} = \{\mathbf{f}\}$, where

$$\{\mathbf{y}\} = \lfloor \omega_x \;\; \omega_y \;\; \omega_z \;\; q_1 \;\; q_2 \;\; q_3 \;\; q_4 \rfloor^T$$
$$\{\mathbf{f}\} = \lfloor \mathrm{d}\omega_x/\mathrm{d}t \;\; \mathrm{d}\omega_y/\mathrm{d}t \;\; \mathrm{d}\omega_z/\mathrm{d}t \;\; \mathrm{d}q_1/\mathrm{d}t \;\; \mathrm{d}q_2/\mathrm{d}t \;\; \mathrm{d}q_3/\mathrm{d}t \;\; \mathrm{d}q_4/\mathrm{d}t \rfloor^T \tag{k}$$

thereby obtaining the angular velocity $\boldsymbol{\omega}$ and quaternion $\widehat{\mathbf{q}}$ as functions of time. At each step of the numerical integration process:

(i) Use the current value of $\widehat{\mathbf{q}}$ to compute $[\mathbf{Q}]_{Xx}$ from Algorithm 11.1.
(ii) Use the current value of $[\mathbf{Q}]_{Xx}$ and $\boldsymbol{\omega}$ to compute $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$ from (a), (d), and (e).
(iii) Use the current value of $\widehat{\mathbf{q}}$ and $\boldsymbol{\omega}$ to compute $\mathrm{d}\widehat{\mathbf{q}}/\mathrm{d}t$ from Eqs. (11.164).

Step 5:
At each solution time:
(i) Use Algorithm 11.1 to compute the direction cosine matrix $[\mathbf{Q}]_{Xx}$.
(ii) Use Algorithm 4.3 to compute the Euler angles ϕ (precession), θ (nutation), and ψ(spin).
(iii) Use Eq. (11.116) to compute the Euler angle rates $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$.

Step 6:
Plot the time histories of the Euler angles and their rates.

Fig. 11.34 shows the numerical solution for the precession, nutation, and spin angles as well as their rates as functions of time. We see that the constant precession rate (51.93 rpm) and spin rate (1000 rpm) are in agreement with Example 11.15, as is the nutation angle, which is fixed at 60°. The sawtooth appearance of the spin angle $\psi(t)$ reflects the fact that it is confined to the range 0 to 360°.

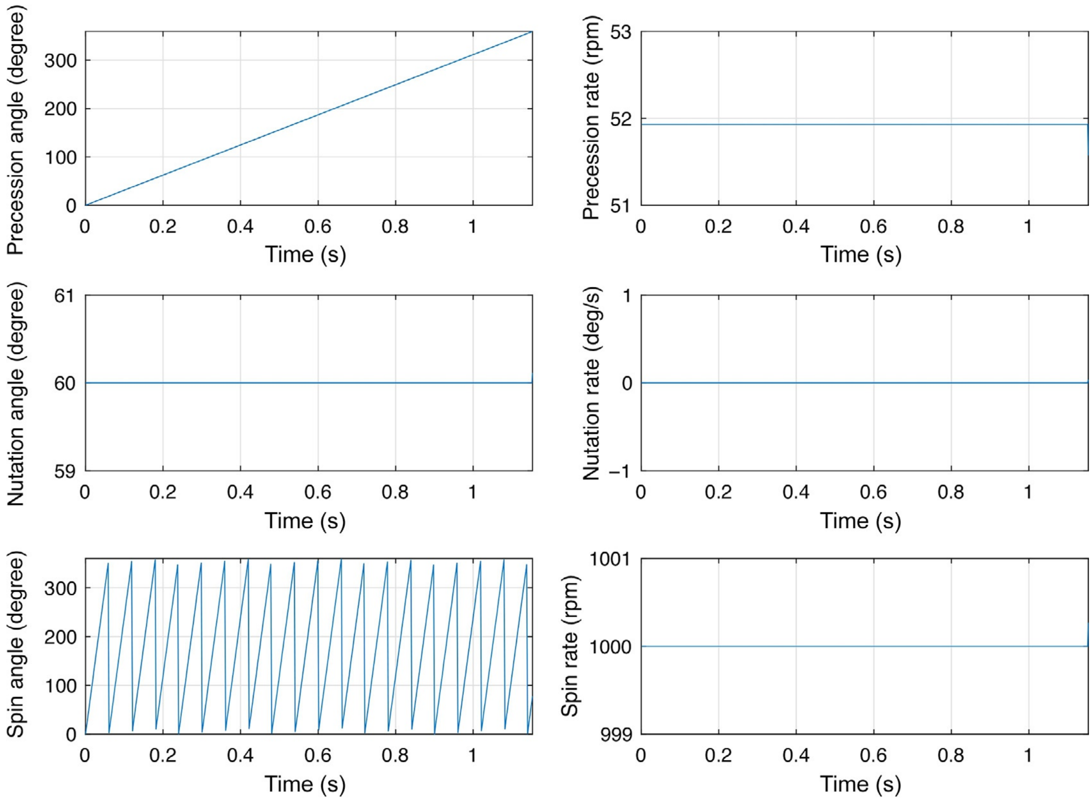


FIG. 11.34

Precession (ϕ), nutation (θ), and spin (ψ) angles and their rates for the top in Example 11.15. $A = B = 0.0012\ \mathrm{kg\cdot m^2}$, $C = 0.00045\ \mathrm{kg\cdot m^2}$.

Clearly , the solution of the steady-state spinning top problem by numerical integration yields no new insight into the top's motion and may be deemed a waste of effort. However, suppose we solve the same problem, but release the top from rest with *zero* precession rate, so that instead of Eqs. (i), the initial conditions are

$$\begin{aligned}\theta_0 &= 60^\circ\\ \psi_0 &= 0\\ \omega_s)_0 &= 1000\text{rpm} = 104.72 \text{ rad/s}\\ \omega_p)_0 &= 0\\ \omega_n)_0 &= 0\end{aligned} \tag{l}$$

The initial orientation of the top is unchanged, so the initial quaternion $\widehat{\mathbf{q}}_0$ remains as shown in Eq. (g). On the other hand, the initially zero precession rate yields a different initial angular velocity vector, namely,

$$\boldsymbol{\omega}_0 = \lfloor 0 \;\; 0 \;\; 104.72 \rfloor^T (\text{rad/s}) \tag{m}$$

With only this change, the above numerical integration procedure yields the results shown in Fig. 11.35.
This is an example of unsteady precession, in which we see that the spin axis, instead of making a constant angle of 60° to the vertical, nutates between 60° and 75.4° at a rate of about 5.7 Hz, while the spin rate itself varies between 975 and

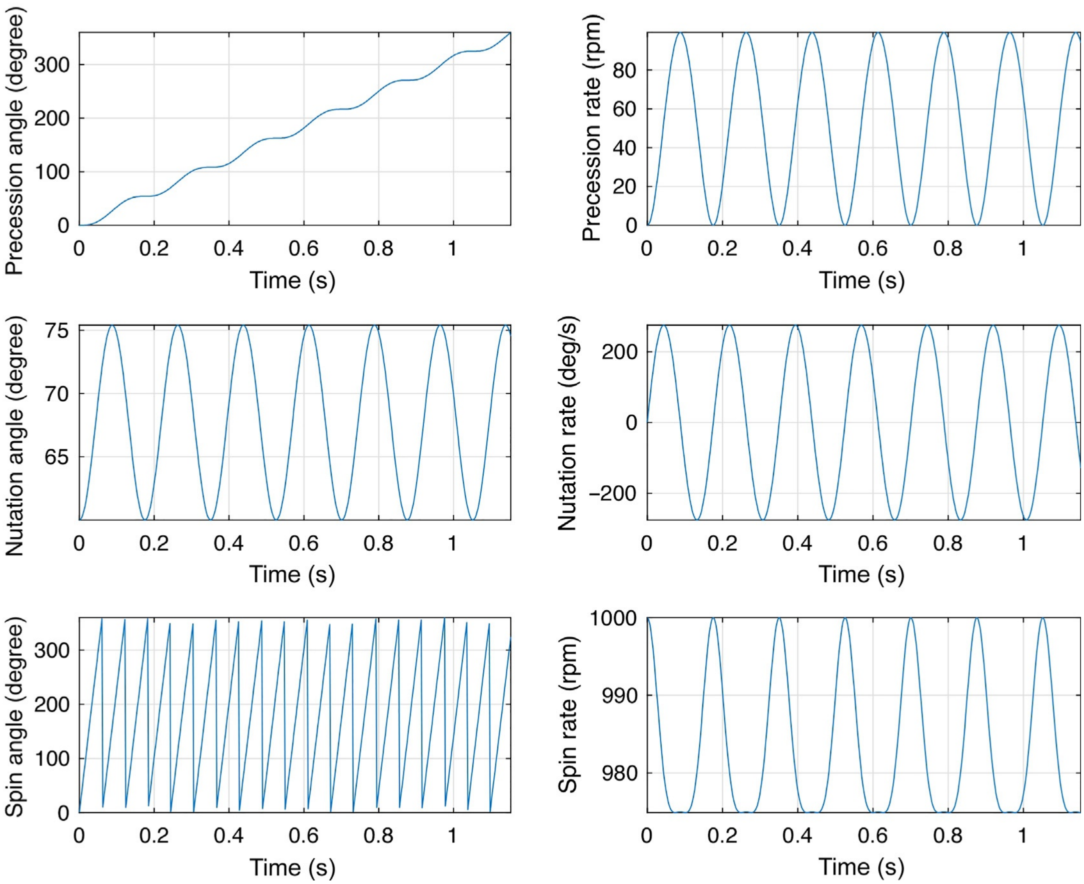


FIG. 11.35

Precession (ϕ), nutation (θ), and spin (ψ) angles and their rates for the top in Fig. 11.19, released from rest with initially zero precession. $A = B = 0.0012 \text{ kg}\cdot\text{m}^2$, $C = 0.00045 \text{ kg}\cdot\text{m}^2$.

1000 rpm at the same frequency. The precession rate oscillates between 0 and 99.4 rpm, also at a frequency of 5.7 Hz, with an average rate of 51.9 rpm, which happens to be the steady-state precession rate (Fig. 11.34).

These numerical results can be compared with formulas from the classical analysis of tops in unsteady precession. For example, it can be shown (Greenwood, 1988) that the relationship between the minimum and maximum nutation angles is

$$\cos\theta_{max} = \lambda - \sqrt{\lambda^2 - 2\lambda\cos\theta_{min} + 1}$$

where $\lambda = C^2\omega_z{}^2/(4Amgd)$. For the data of this problem, $\lambda = 1.887$, so that

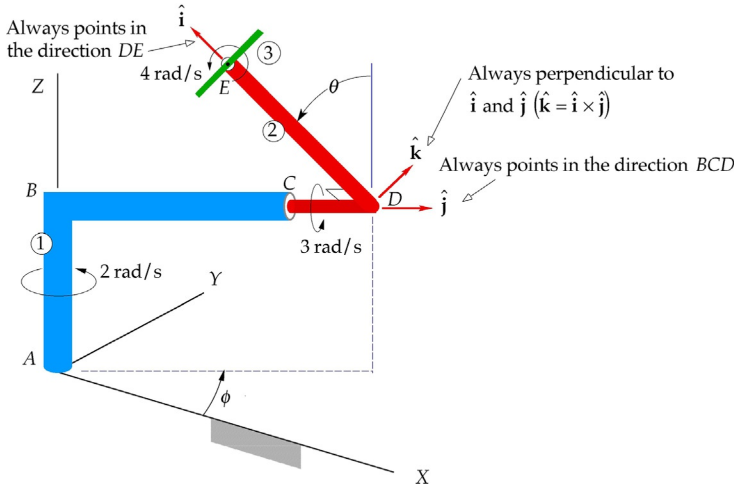
$$\begin{aligned}\cos\theta_{max} &= 1.887 - \sqrt{1.887^2 - 2\cdot 1.887\cdot\cos 60° + 1} = 0.2518\\ \theta_{max} &= 75.41°\end{aligned}$$

This is precisely what we observe for the nutation angle in Fig. 11.35. By the way, ω_z remains constant at its initial value of 104.72 rad/s because the top is axisymmetric ($A = B$) and $M_z = 0$ (Eq. (e)$_3$), so that $d\omega_z/dt = 0$ (Eq. (a)$_3$).

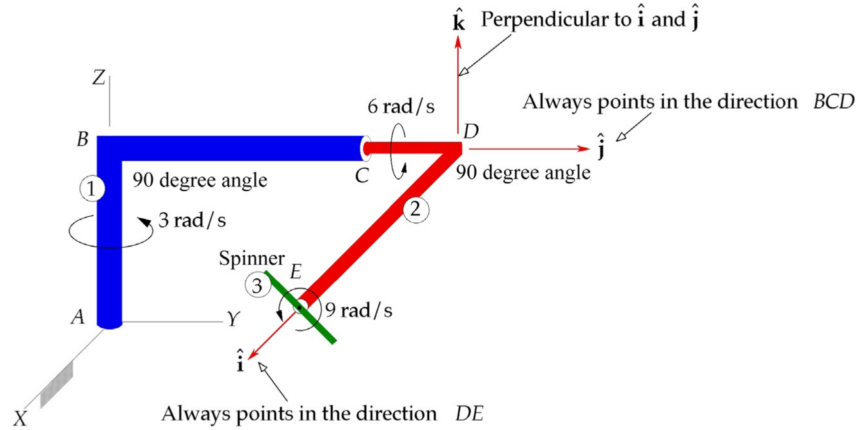
PROBLEMS

Section 2

11.1 Rigid, bent shaft 1 (ABC) rotates at a constant angular velocity of $2\hat{\mathbf{K}}$ rad/s around the positive Z axis of the inertial frame. Bent shaft 2 (CDE) rotates around BC with a constant angular velocity of $3\hat{\mathbf{j}}$ rad/s, relative to BC. Spinner 3 at E rotates around DE with a constant angular velocity of $4\hat{\mathbf{j}}$ rad/s relative to DE. Calculate the magnitude of the absolute angular acceleration vector $\boldsymbol{\alpha}_3$ of the spinner at the instant shown.
{Ans.: $\|\boldsymbol{\alpha}_3\| = \sqrt{180 + 64\sin^2\theta - 144\cos\theta}\,(\text{rad/s}^2)$}



11.2 All the spin rates shown are constant. Calculate the magnitude of the absolute angular acceleration vector $\boldsymbol{\alpha}_3$ of the spinner at the instant shown (i.e., at the instant when the unit vector $\hat{\mathbf{i}}$ is parallel to the X axis and the unit vector $\hat{\mathbf{j}}$ is parallel to the Y axis).
{Ans.: $\|\boldsymbol{\alpha}_3\| = 63$ rad/s^2}

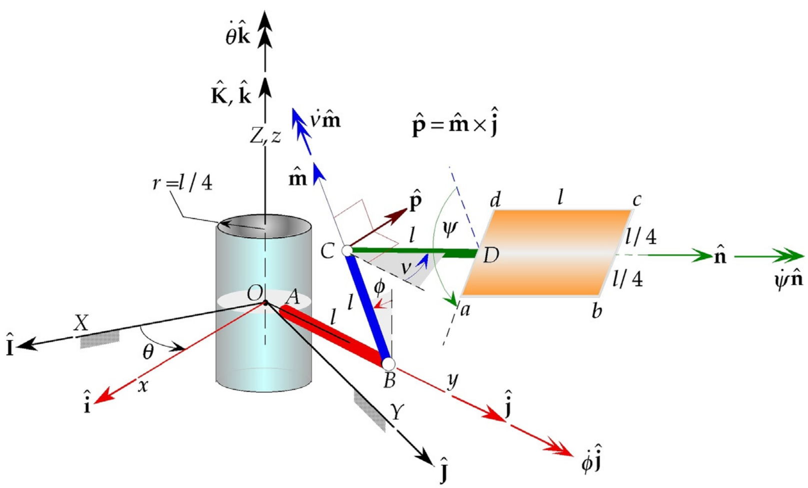


11.3 The body-fixed xyz frame is attached to the cylinder as shown. The cylinder rotates around the inertial Z axis, which is collinear with the z axis, with a constant absolute angular velocity $\dot{\theta}\hat{\mathbf{k}}$. Rod AB is attached to the cylinder and aligned with the y axis. Rod BC is perpendicular to AB and rotates around AB with the constant angular velocity $\dot{\phi}\hat{\mathbf{j}}$ relative to the cylinder. Rod CD is perpendicular to BC and rotates around BC with the constant angular velocity $\dot{\nu}\hat{\mathbf{m}}$ relative to BC, where $\hat{\mathbf{m}}$ is the unit vector in the direction of BC. The plate $abcd$ rotates around CD with a constant angular velocity $\dot{\psi}\hat{\mathbf{n}}$ relative to CD, where the unit vector $\hat{\mathbf{n}}$ points in the direction of CD. Thus, the absolute angular velocity of the plate is $\boldsymbol{\omega}_{\text{plate}} = \dot{\theta}\hat{\mathbf{k}} + \dot{\phi}\hat{\mathbf{j}} + \dot{\nu}\hat{\mathbf{m}} + \dot{\psi}\hat{\mathbf{n}}$. Show that

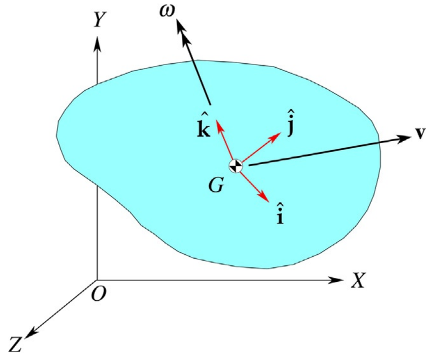
(a) $$\boldsymbol{\omega}_{\text{plate}} = (\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu)\hat{\mathbf{i}} + (\dot{\phi} + \dot{\psi}\cos\nu)\hat{\mathbf{j}} + (\dot{\theta} + \dot{\nu}\cos\phi + \dot{\psi}\sin\phi\sin\nu)\hat{\mathbf{k}}$$

(b) $$\begin{aligned}\boldsymbol{\alpha}_{\text{plate}} = \frac{d\boldsymbol{\omega}_{\text{plate}}}{dt} &= \left[\dot{\nu}\left(\dot{\phi}\cos\phi - \dot{\psi}\cos\phi\cos\nu\right) + \dot{\psi}\dot{\phi}\sin\phi\sin\nu - \dot{\psi}\dot{\theta}\cos\nu - \dot{\phi}\dot{\theta}\right]\hat{\mathbf{i}} \\ &+ \left[\dot{\nu}\left(\dot{\theta}\sin\phi - \dot{\psi}\sin\nu\right) - \dot{\psi}\dot{\theta}\cos\phi\sin\nu\right]\hat{\mathbf{j}} \\ &+ \left(\dot{\psi}\dot{\nu}\cos\nu\sin\phi + \dot{\psi}\dot{\phi}\cos\phi\sin\nu - \dot{\phi}\dot{\nu}\sin\dot{\varphi}\right)\hat{\mathbf{k}}\end{aligned}$$

(c) $$\mathbf{a}_C = -l\left(\dot{\phi}^2 + \dot{\theta}^2\right)\sin\phi\hat{\mathbf{i}} + \left(2l\dot{\phi}\dot{\theta}\cos\phi - \frac{5}{4}l\dot{\theta}^2\right)\hat{\mathbf{j}} - l\dot{\phi}^2\cos\phi\hat{\mathbf{k}}$$



11.4 The mass center G of a rigid body has a velocity $\mathbf{v} = t^3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ (m/s) and an angular velocity $\boldsymbol{\omega} = 2t^2\hat{\mathbf{k}}$ (rad/s), where t is time in seconds. The $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ unit vectors are attached to and rotate with the rigid body. Calculate the magnitude of the acceleration $\mathbf{a}_G$ of the center of mass at $t = 2$ s.
{Ans.: $\mathbf{a}_G = -20\hat{\mathbf{i}} + 64\hat{\mathbf{j}}$ $(\mathrm{m/s^2})$}



11.5 A rigid body is in pure rotation with angular velocity $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$ about the origin of the inertial xyz frame. If point A with position vector $\mathbf{r}_A = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (m) has velocity $\mathbf{v}_A = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (m/s), what is the magnitude of the velocity of the point B with position vector $\mathbf{r}_B = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ (m)?
{Ans.: 1.871 m/s}

11.6 The inertial angular velocity of a rigid body is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors of a comoving frame whose inertial angular velocity is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}}$. Calculate the components of angular acceleration of the rigid body in the moving frame, assuming that ω_x, ω_y, and ω_z are all constant.
{Ans.: $\boldsymbol{\alpha} = \omega_y\omega_z\hat{\mathbf{i}} - \omega_x\omega_z\hat{\mathbf{j}}$}

Section 5

11.7 Find the moments of inertia about the center of mass of the system of six point masses listed in the table.

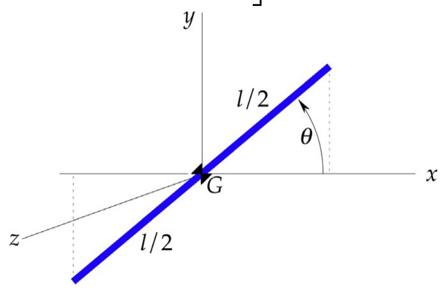
Point, i	Mass, m_i (kg)	x_i (m)	y_i (m)	z_i (m)
1	10	1	1	1
2	10	−1	−1	−1
3	8	4	−4	4
4	8	−2	2	−2
5	12	3	−3	−3
6	12	−3	3	3

$$\{\text{Ans.: } [\mathbf{I}_G] = \begin{bmatrix} 783.5 & 351.7 & 40.27 \\ 351.7 & 783.5 & -80.27 \\ 40.27 & -80.27 & 783.5 \end{bmatrix} (\mathrm{kg \cdot m^2})\}$$

11.8 Find the mass moment of inertia of the configuration of Problem 11.7 about an axis through the origin and the point with coordinates (1, 2, 2 m).
{Ans.: 898.7 $\mathrm{kg \cdot m^2}$}

11.9 A uniform slender rod of mass m and length l lies in the xy plane inclined to the x axis by an angle θ. Use the results of Example 11.10 to find the mass moments of inertia about the xyz axes passing through the center of mass G.

$$\{\text{Ans.: } [\mathbf{I}_G] = \tfrac{1}{12}ml^2 \begin{bmatrix} \sin^2\theta & -\frac{1}{2}\sin 2\theta & 0 \\ -\frac{1}{2}\sin 2\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$$



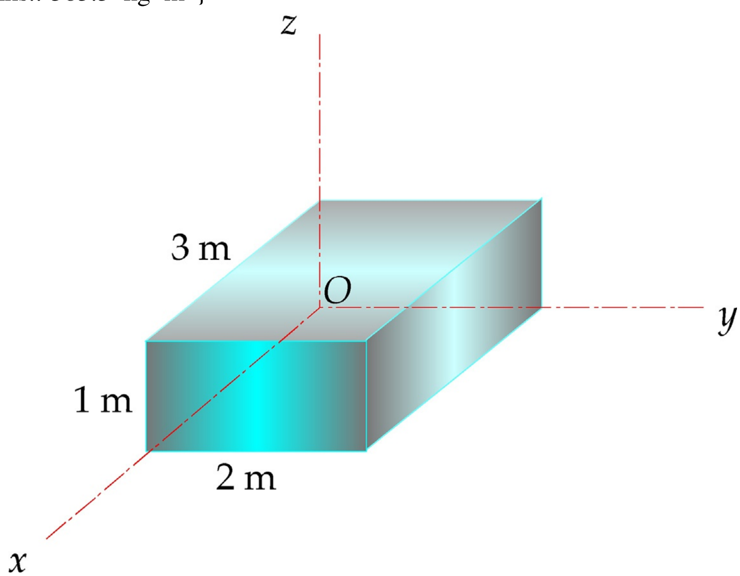
11.10 The uniform rectangular box has a mass of 1000 kg. The dimensions of its edges are shown.

(a) Find the mass moments of inertia about the xyz axes.

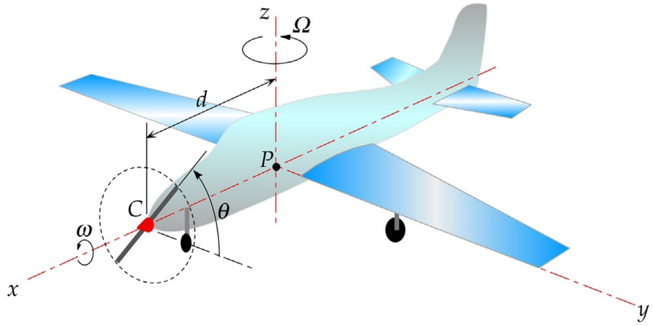
$$\{\text{Ans.: } [\mathbf{I}_O] = \begin{bmatrix} 1666.7 & -1500 & -750 \\ -1500 & 3333.3 & -500 \\ -750 & -500 & 4333.3 \end{bmatrix} (\text{kg}\cdot\text{m}^2)\}$$

(b) Find the principal moments of inertia and the principal directions about the xyz axes through O.
{Partial Ans.: $I_1 = 568.9\ \text{kg}\cdot\text{m}^2$, $\hat{\mathbf{e}}_1 = 0.8366\hat{\mathbf{i}} + 0.4960\hat{\mathbf{j}} + 0.2326\hat{\mathbf{k}}$}

(c) Find the moment of inertia about the line through O and the point with coordinates (3 m, 2 m, 1 m).
{Ans.: $583.3\ \text{kg}\cdot\text{m}^2$}



11.11 A taxiing airplane turns about its vertical axis with an angular velocity Ω while its propeller spins at an angular velocity $\omega = \dot{\theta}$. Determine the components of the angular momentum of the propeller about the body-fixed xyz axes centered at P. Treat the propeller as a uniform slender rod of mass m and length l.
{Ans.: $\mathbf{H}_p = \frac{1}{12}m\omega l^2\hat{\mathbf{i}} - \frac{1}{24}m\Omega l^2 \sin 2\theta\hat{\mathbf{j}} + \left(\frac{1}{12}ml^2\cos^2\theta + md^2\right)\Omega\hat{\mathbf{k}}$}



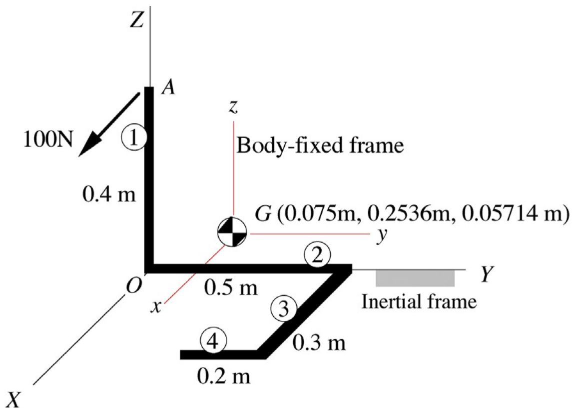
11.12 Relative to an xyz frame of reference the components of angular momentum $\mathbf{H}$ are given by

$$\{\mathbf{H}\} = \begin{bmatrix} 1000 & 0 & -300 \\ 0 & 1000 & 500 \\ -300 & 500 & 1000 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} (\text{kg}\cdot\text{m}^2/\text{s})$$

where ω_x, ω_y, and ω_z are the components of the angular velocity vector $\boldsymbol{\omega}$. Find the components $\boldsymbol{\omega}$ such that $\{\mathbf{H}\} = 1000\{\boldsymbol{\omega}\}$, where the magnitude of $\boldsymbol{\omega}$ is 20 rad/s.
{Ans.: $\boldsymbol{\omega} = 174.15\hat{\mathbf{i}} + 10.29\hat{\mathbf{j}}$ (rad/s)}

11.13 Relative to a body-fixed xyz frame $[\mathbf{I}_G] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} (\text{kg}\cdot\text{m}^2)$ and $\boldsymbol{\omega} = 2t^2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$ (rad/s), where t is the time in seconds. Calculate the magnitude of the net moment about the center of mass G at $t = 3$ s.
{Ans.: 3374 N m}

11.14 In Example 11.11, the system is at rest when a 100-N force is applied to point A as shown. Calculate the inertial components of angular acceleration at that instant.
{Ans.: $\alpha_X = 143.9$ rad/s^2, $\alpha_Y = 553.1$ rad/s^2, $\alpha_Z = 7.61$ rad/s^2}

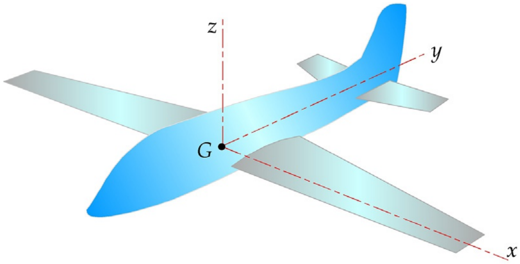


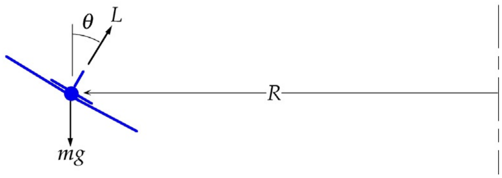
11.15 The body-fixed xyz axes pass through the center of mass G of the airplane and are the principal axes of inertia. The moments of inertia about these axes are A, B, and C, respectively. The airplane is in a level turn of radius R with a speed v.

(a) Calculate the bank angle θ.

(b) Use the Euler equations to calculate the rolling moment M_y that must be applied by the aerodynamic surfaces.

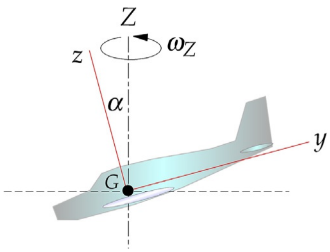
{Ans.: (a) $\theta = \tan^{-1} v^2/Rg$; (b) $M_y = v^2 \sin 2\theta (C - A)/2R^2$}





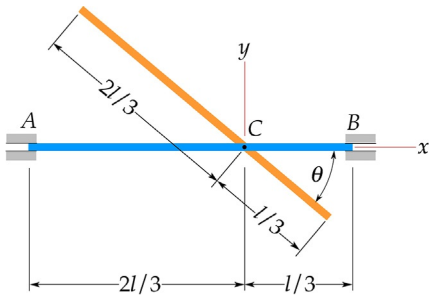
11.16 The airplane in Problem 11.15 is spinning with an angular velocity ω_Z about the vertical Z axis. The nose is pitched down at the angle α. What external moments must accompany this maneuver?

{Ans.: $M_y = M_z = 0$, $M_x = \omega_Z{}^2 \sin 2\alpha (C - B)/2$}

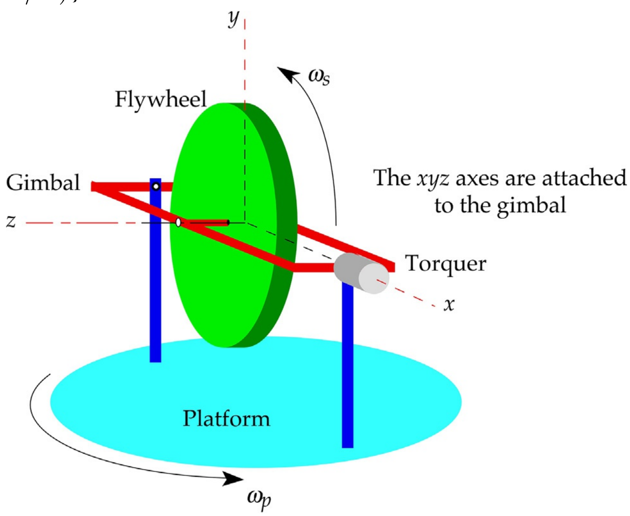


11.17 Two identical slender rods of mass m and length l are rigidly joined together at an angle θ at point C, their 2/3 point. Determine the bearing reactions at A and B if the shaft rotates at a constant angular velocity ω. Neglect gravity and assume that the only bearing forces are normal to rod AB.

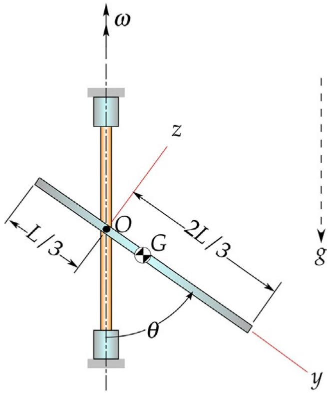
{Ans.: $\|\mathbf{F}_A\| = m\omega^2 l \sin \theta (1 + 2 \cos \theta)/18$, $\|\mathbf{F}_B\| = m\omega^2 l \sin \theta (1 - \cos \theta)/9$}



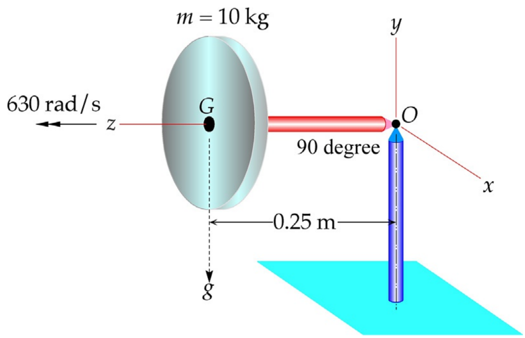
11.18 The flywheel ($A = B = 5\ \text{kg}\cdot\text{m}^2$, $C = 10\ \text{kg}\cdot\text{m}^2$) spins at a constant angular velocity of $\boldsymbol{\omega}_s = 100\hat{\mathbf{k}}$ (rad/s). It is supported by a massless gimbal that is mounted on the platform as shown. The gimbal is initially stationary relative to the platform, which rotates with a constant angular velocity of $\boldsymbol{\omega}_p = 0.5\hat{\mathbf{j}}$ (rad/s). What will be the gimbal's angular acceleration when the torquer applies a torque of $600\hat{\mathbf{i}}$ (N m) to the flywheel?
{Ans.: $70\hat{\mathbf{i}}$ (rad/s^2)}



11.19 A uniform slender rod of length L and mass m is attached by a smooth pin at O to a vertical shaft that rotates at constant angular velocity ω. Use the Euler equations and the body frame shown to calculate ω at the instant shown.
{Ans.: $\omega = \sqrt{3g/(2L\cos\theta)}$}



11.20 A uniform, thin circular disk of mass 10 kg spins at a constant angular velocity of 630 rad/s about axis OG, which is normal to the disk and pivots about the frictionless ball joint at O. Neglecting the mass of the shaft OG, determine the rate of precession if OG remains horizontal as shown. Gravity acts down, as shown. G is the center of mass and the y axis remains fixed in space. The moments of inertia about G are $I_G)_z = 0.02812\ \text{kg}\cdot\text{m}^2$ and $I_G)_x = I_G)_y = 0.01406\ \text{kg}\cdot\text{m}^2$.
{Ans.: 1.38 rad/s}



Section 7

11.21 Consider a rigid body experiencing rotational motion associated with an angular velocity vector $\boldsymbol{\omega}$. The inertia tensor (relative to body-fixed axes through the center of mass G) is

$$\begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} (\text{kg}\cdot\text{m}^2)$$

and $\boldsymbol{\omega} = 10\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 30\hat{\mathbf{k}}$ (rad/s). Calculate

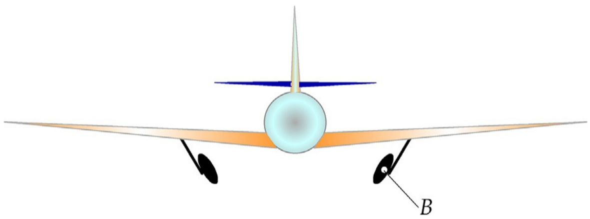
(a) the angular momentum $\mathbf{H}_G$ and

(b) the rotational kinetic energy (about G).

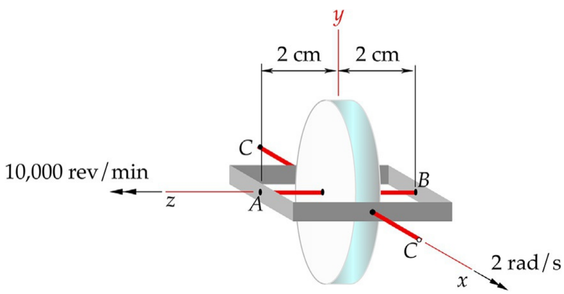
{Partial Ans.: (b) $T_R = 23,000$ J}

Section 8

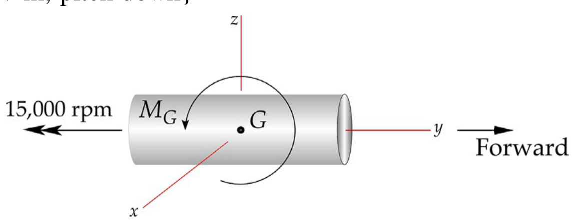
11.22 At the end of its takeoff run, an airplane with retractable landing gear leaves the runway with a speed of 130 km/h. The gear rotates into the wing with an angular velocity of 0.8 rad/s with the wheels still spinning. Calculate the gyroscopic bending moment in the wheel bearing B. The wheels have a diameter of 0.6 m, a mass of 25 kg, and a radius of gyration of 0.2 m.
{Ans.: 96.3 N m}



11.23 The gyro rotor, including shaft AB, has a mass of 4 kg and a radius of gyration 7 cm around AB. The rotor spins at 10,000 rpm while also being forced to rotate around the gimbal axis CC at 2 rad/s. What are the transverse forces exerted on the shaft at A and B? Neglect gravity.
{Ans.: 1.03 kN}

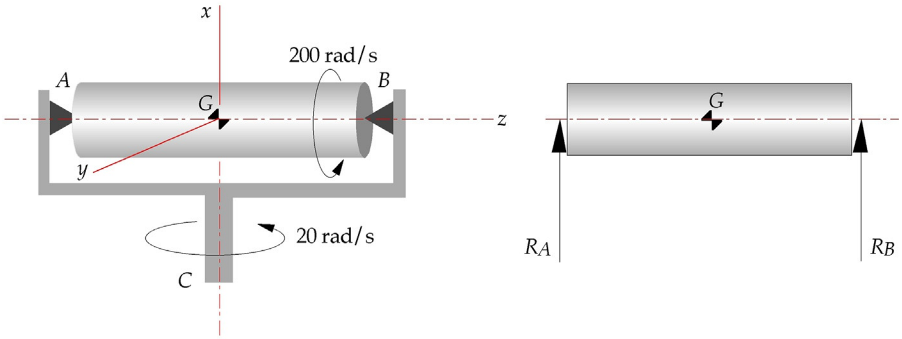


11.24 A jet aircraft is making a level, 2.5-km radius turn to the left at a speed of 650 km/h. The rotor of the turbojet engine has a mass of 200 kg, a radius of gyration of 0.25 m, and rotates at 15,000 rpm clockwise as viewed from the front of the airplane. Calculate the gyroscopic moment that the engine exerts on the airframe and explain why it tends to pitch the nose up or down.
{Ans.: 1.418 kN m; pitch down}



11.25 A cylindrical rotor of mass 10 kg, radius 0.05 m, and length 0.60 m is simply supported at each end in a cradle that rotates at a constant 20 rad/s counterclockwise as viewed from above. Relative to the cradle, the rotor spins at 200 rad/s counterclockwise as viewed from the right

(from B toward A). Assuming that there is no gravity, calculate the bearing reactions R_A and R_B. Use the comoving xyz frame shown, which is attached to the cradle but not to the rotor.
{Ans.: $R_A = -R_B = 83.3$ N}



Section 9

11.26 The Euler angles of a rigid body are $\phi = 50°$, $\theta = 25°$, and $\psi = 70°$. Calculate the angle (a positive number) between the body-fixed x axis and the inertial X axis.
{Ans.: 115.6°}

Section 11

11.27 Let $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ and $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ be two right-handed triads of orthogonal unit vectors related as in Eq. (4.18) by the direction cosine matrix $[\mathbf{Q}]$, so that

$$\begin{aligned}\hat{\mathbf{i}} &= Q_{11}\hat{\mathbf{I}} + Q_{12}\hat{\mathbf{J}} + Q_{13}\hat{\mathbf{K}}\\ \hat{\mathbf{j}} &= Q_{21}\hat{\mathbf{I}} + Q_{22}\hat{\mathbf{J}} + Q_{23}\hat{\mathbf{K}}\\ \hat{\mathbf{k}} &= Q_{31}\hat{\mathbf{I}} + Q_{32}\hat{\mathbf{J}} + Q_{33}\hat{\mathbf{K}}\end{aligned}$$

Show that $\hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{k}}$ implies that $Q_{13} = Q_{32}Q_{21} - Q_{22}Q_{31}$, whereas $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$ implies that

$$Q_{23} = Q_{12}Q_{31} - Q_{32}Q_{11}.$$

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