SOLUTIONS MANUAL

to accompany

ORBITAL MECHANICS FOR ENGINEERING STUDENTS

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Problem 1.1

(a)

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{A} &= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\\
&= A_x\hat{\mathbf{i}}\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + A_y\hat{\mathbf{j}}\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + A_z\hat{\mathbf{k}}\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\\
&= \left[A_x{}^2\left(\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}\right) + A_xA_y\left(\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}\right) + A_xA_z\left(\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}\right)\right] + \left[A_yA_x\left(\hat{\mathbf{j}}\cdot\hat{\mathbf{i}}\right) + A_y{}^2\left(\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}\right) + A_yA_z\left(\hat{\mathbf{j}}\cdot\hat{\mathbf{k}}\right)\right]\\
&\quad + \left[A_zA_x\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{i}}\right) + A_zA_y\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{j}}\right) + A_z{}^2\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}\right)\right]\\
&= \left[A_x{}^2(1) + A_xA_y(0) + A_xA_z(0)\right] + \left[A_yA_x(0) + A_y{}^2(1) + A_yA_z(0)\right] + \left[A_zA_x(0) + A_zA_y(0) + A_z{}^2(1)\right]\\
&= A_x{}^2 + A_y{}^2 + A_z{}^2
\end{aligned}$$

But, according to the Pythagorean Theorem, $A_x{}^2 + A_y{}^2 + A_z{}^2 = A^2$, where $A = \|\mathbf{A}\|$, the magnitude of the vector $\mathbf{A}$. Thus $\underline{\mathbf{A}\cdot\mathbf{A} = A^2}$.

(b)

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$\begin{aligned}
&= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\cdot\left[\hat{\mathbf{i}}\left(B_yC_z - B_zC_y\right) - \hat{\mathbf{j}}\left(B_xC_z - B_zC_x\right) + \hat{\mathbf{k}}\left(B_xC_y - B_yC_x\right)\right]\\
&= A_x\left(B_yC_z - B_zC_y\right) - A_y\left(B_xC_z - B_zC_x\right) + A_z\left(B_xC_y - B_yC_x\right)
\end{aligned}$$

or

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_xB_yC_z + A_yB_zC_x + A_zB_xC_y - A_xB_zC_y - A_yB_xC_z - A_zB_yC_x \tag{1}$$

Note that $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$, and according to (1)

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = C_xA_yB_z + C_yA_zB_x + C_zA_xB_y - C_xA_zB_y - C_yA_xB_z - C_zA_yB_x \tag{2}$$

The right hand sides of (1) and (2) are identical. Hence $\underline{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}}$.

(c)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_yC_z - B_zC_y & B_zC_x - B_xC_y & B_xC_y - B_yC_x\end{vmatrix}$$

$$\begin{aligned}
&= \left[A_y\left(B_xC_y - B_yC_x\right) - A_z\left(B_zC_x - B_xC_z\right)\right]\hat{\mathbf{i}} + \left[A_z\left(B_yC_z - B_zC_y\right) - A_x\left(B_xC_y - B_yC_x\right)\right]\hat{\mathbf{j}}\\
&\quad + \left[A_x\left(B_zC_x - B_xC_z\right) - A_y\left(B_yC_z - B_zC_y\right)\right]\hat{\mathbf{k}}\\
&= \left(A_yB_xC_y + A_zB_xC_z - A_yB_yC_x - A_zB_zC_x\right)\hat{\mathbf{i}} + \left(A_xB_yC_x + A_zB_yC_z - A_xB_xC_y - A_zB_zC_y\right)\hat{\mathbf{j}}\\
&\quad + \left(A_xB_zC_x + A_yB_zC_y - A_xB_xC_z - A_yB_yC_z\right)\hat{\mathbf{k}}\\
&= \left[B_x\left(A_yC_y + A_zC_z\right) - C_x\left(A_yB_y + A_zB_z\right)\right]\hat{\mathbf{i}} + \left[B_y\left(A_xC_x + A_zC_z\right) - C_y\left(A_xB_x + A_zB_z\right)\right]\hat{\mathbf{j}}\\
&\quad + \left[B_z\left(A_xC_x + A_yC_y\right) - C_z\left(A_xB_x + A_yB_y\right)\right]\hat{\mathbf{k}}
\end{aligned}$$

Add and subtract the underlined terms to get

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \left[B_x\left(A_yC_y + A_zC_z + \underline{A_xC_x}\right) - C_x\left(A_yB_y + A_zB_z + \underline{A_xB_x}\right)\right]\hat{\mathbf{i}}$$
$$+\left[B_y\left(A_xC_x + A_zC_z + \underline{A_yC_y}\right) - C_y\left(A_xB_x + A_zB_z + \underline{A_yB_y}\right)\right]\hat{\mathbf{j}}$$
$$+\left[B_z\left(A_xC_x + A_yC_y + \underline{A_zC_z}\right) - C_z\left(A_xB_x + A_yB_y + \underline{A_zB_z}\right)\right]\hat{\mathbf{k}}$$
$$= \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)\left(A_xC_x + A_yC_y + A_zC_z\right) - \left(C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}\right)\left(A_xB_x + A_yB_y + A_zB_z\right)$$

or

$$\underline{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})}$$

Problem 1.2 Using the interchange of Dot and Cross we get

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = [(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D}$$

But

$$[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D} = -[\mathbf{C}\times(\mathbf{A}\times\mathbf{B})]\cdot\mathbf{D} \tag{1}$$

Using the *bac – cab* rule on the right, yields

$$[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D} = -[\mathbf{A}(\mathbf{C}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{C}\cdot\mathbf{A})]\cdot\mathbf{D}$$

or

$$[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D} = -(\mathbf{A}\cdot\mathbf{D})(\mathbf{C}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{D})(\mathbf{C}\cdot\mathbf{A}) \tag{2}$$

Substituting (2) into (1) we get

$$\underline{[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\cdot\mathbf{D} = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})}$$

Problem 1.3

Velocity analysis

From Equation 1.38,

$$\mathbf{v} = \mathbf{v}_o + \boldsymbol{\Omega}\times\mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}\,. \tag{1}$$

From the given information we have

$$\mathbf{v}_o = -10\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 50\hat{\mathbf{K}} \tag{2}$$

$$\mathbf{r}_{\text{rel}} = \mathbf{r} - \mathbf{r}_o = \left(150\hat{\mathbf{I}} - 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}\right) - \left(300\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 100\hat{\mathbf{K}}\right) = -150\hat{\mathbf{I}} - 400\hat{\mathbf{J}} + 200\hat{\mathbf{K}} \tag{3}$$

$$\boldsymbol{\Omega}\times\mathbf{r}_{\text{rel}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.6 & -0.4 & 1.0 \\ -150 & -400 & 200 \end{vmatrix} = 320\hat{\mathbf{I}} - 270\hat{\mathbf{J}} - 300\hat{\mathbf{K}} \tag{4}$$

$$\begin{aligned}\mathbf{v}_{\text{rel}} &= -20\hat{\mathbf{i}} + 25\hat{\mathbf{j}} + 70\hat{\mathbf{k}}\\ &= -20\left(0.57735\hat{\mathbf{I}} + 0.57735\hat{\mathbf{J}} + 0.57735\hat{\mathbf{K}}\right)\\ &\quad + 25\left(-0.74296\hat{\mathbf{I}} + 0.66475\hat{\mathbf{J}} + 0.078206\hat{\mathbf{K}}\right)\\ &\quad + 70\left(-0.33864\hat{\mathbf{I}} - 0.47410\hat{\mathbf{J}} + 0.81274\hat{\mathbf{K}}\right)\end{aligned}$$

so that

$$\mathbf{v}_{\text{rel}} = -53.826\hat{\mathbf{I}} - 28.115\hat{\mathbf{J}} + 47.300\hat{\mathbf{K}}\,(\text{m/s}) \tag{5}$$

Substituting (2), (3), (4) and (5) into (1) yields

$$\mathbf{v} = \left(-10\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 50\hat{\mathbf{K}}\right) + \left(320\hat{\mathbf{I}} - 270\hat{\mathbf{J}} - 300\hat{\mathbf{K}}\right) + \left(-53.826\hat{\mathbf{I}} - 28.115\hat{\mathbf{J}} + 47.300\hat{\mathbf{K}}\right)$$

$$\begin{aligned}\mathbf{v} &= 256.17\hat{\mathbf{I}} - 268.12\hat{\mathbf{J}} - 302.7\hat{\mathbf{K}}\\ &= \underline{478.68\left(0.53516\hat{\mathbf{I}} - 0.56011\hat{\mathbf{J}} - 0.63236\mathbf{K}\right)(\text{m/s})}\end{aligned}$$

Acceleration analysis

From Equation 1.42,

$$\mathbf{a} = \mathbf{a}_O + \dot{\mathbf{\Omega}} \times \mathbf{r}_{\text{rel}} + \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \mathbf{r}_{\text{rel}}\right) + 2\mathbf{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{6}$$

Using the given data together with (4) and (5) we obtain

$$\mathbf{a}_o = 25\hat{\mathbf{I}} + 40\hat{\mathbf{J}} - 15\hat{\mathbf{K}} \tag{7}$$

$$\dot{\mathbf{\Omega}} \times \mathbf{r}_{\text{rel}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -0.4 & 0.3 & -1.0 \\ -150 & -400 & 200 \end{vmatrix} = -340\hat{\mathbf{I}} + 230\hat{\mathbf{J}} + 205\hat{\mathbf{K}} \tag{8}$$

$$\mathbf{\Omega} \times \left(\mathbf{\Omega} \times \mathbf{r}_{\text{rel}}\right) = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.6 & -0.4 & 1.0 \\ 320 & -270 & -300 \end{vmatrix} = 390\hat{\mathbf{I}} + 500\hat{\mathbf{J}} - 34\hat{\mathbf{K}} \tag{9}$$

$$2\mathbf{\Omega} \times \mathbf{v}_{\text{rel}} = 2\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.6 & -0.4 & 1.0 \\ -53.826 & -28.115 & 47.300 \end{vmatrix} = 2\left(9.151\hat{\mathbf{I}} - 82.206\hat{\mathbf{J}} - 38.399\hat{\mathbf{K}}\right) \tag{10}$$

$$\begin{aligned}\mathbf{a}_{\text{rel}} &= 7.5\hat{\mathbf{i}} - 8.5\hat{\mathbf{j}} + 6.0\hat{\mathbf{k}}\\ &= 7.5\left(0.57735\hat{\mathbf{I}} + 0.57735\hat{\mathbf{J}} + 0.57735\hat{\mathbf{K}}\right)\\ &\quad - 8.5\left(-0.74296\hat{\mathbf{I}} + 0.66475\hat{\mathbf{J}} + 0.078206\hat{\mathbf{K}}\right)\\ &\quad + 6.0\left(-0.33864\hat{\mathbf{I}} - 0.47410\hat{\mathbf{J}} + 0.81274\hat{\mathbf{K}}\right)\end{aligned}$$

$$\mathbf{a}_{\text{rel}} = 8.6134\hat{\mathbf{I}} - 4.1649\hat{\mathbf{J}} + 8.5418\hat{\mathbf{K}} \tag{11}$$

Substituting (7), (8), (9), (10) and (11) into (6) yields

$$\begin{aligned}\mathbf{a} &= \left(25\hat{\mathbf{I}} + 40\hat{\mathbf{J}} - 15\hat{\mathbf{K}}\right) + \left(-340\hat{\mathbf{I}} + 230\hat{\mathbf{J}} + 205\hat{\mathbf{K}}\right) + \left(390\hat{\mathbf{I}} + 500\hat{\mathbf{J}} - 34\hat{\mathbf{K}}\right)\\ &\quad + \left[2\left(9.151\hat{\mathbf{I}} - 82.206\hat{\mathbf{J}} - 38.399\hat{\mathbf{K}}\right)\right] + \left(8.6134\hat{\mathbf{I}} - 4.1649\hat{\mathbf{J}} + 8.5418\hat{\mathbf{K}}\right)\end{aligned}$$

$$\mathbf{a} = 102\hat{\mathbf{I}} + 601.42\hat{\mathbf{J}} + 87.743\hat{\mathbf{K}}$$
$$= \underline{616.29\left(0.16551\hat{\mathbf{I}} + 0.97588\hat{\mathbf{J}} + 0.14327\hat{\mathbf{K}}\right)\left(\text{m/s}^2\right)}$$

Problem 1.4 From Example 2.8, we have

$$\dddot{\mathbf{F}} = \ddot{\boldsymbol{\omega}} \times \mathbf{F} + 2\dot{\boldsymbol{\omega}} \times \left(\boldsymbol{\omega} \times \mathbf{F}\right) + \boldsymbol{\omega} \times \left[\dot{\boldsymbol{\omega}} \times \mathbf{F} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{F}\right)\right]$$

Substituting the given values for the quantities on the right hand side,

$$\ddot{\boldsymbol{\omega}} \times \mathbf{F} = \mathbf{0} \times 10\hat{\mathbf{i}} = \mathbf{0}$$
$$2\dot{\boldsymbol{\omega}} \times \left(\boldsymbol{\omega} \times \mathbf{F}\right) = 2\left(-2\hat{\mathbf{k}}\right) \times \left[\left(3\hat{\mathbf{k}}\right) \times \left(10\hat{\mathbf{i}}\right)\right] = 2\left(-2\hat{\mathbf{k}}\right) \times \left(30\hat{\mathbf{j}}\right) = 120\hat{\mathbf{i}}$$
$$\boldsymbol{\omega} \times \left(\dot{\boldsymbol{\omega}} \times \mathbf{F}\right) = \left(3\hat{\mathbf{k}}\right) \times \left[\left(-2\hat{\mathbf{k}}\right) \times \left(10\hat{\mathbf{i}}\right)\right] = \left(3\hat{\mathbf{k}}\right) \times \left(-20\hat{\mathbf{j}}\right) = 60\hat{\mathbf{i}}$$
$$\boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{F}\right)\right] = \left(3\hat{\mathbf{k}}\right) \times \left\{\left(3\hat{\mathbf{k}}\right) \times \left[\left(3\hat{\mathbf{k}}\right) \times \left(10\hat{\mathbf{i}}\right)\right]\right\} = \left(3\hat{\mathbf{k}}\right) \times \left[\left(3\hat{\mathbf{k}}\right) \times \left(30\hat{\mathbf{j}}\right)\right] = \left(3\hat{\mathbf{k}}\right) \times \left(-90\hat{\mathbf{i}}\right) = -270\hat{\mathbf{j}}$$

Thus, $\dddot{\mathbf{F}} = \mathbf{0} + 120\hat{\mathbf{i}} + 60\hat{\mathbf{i}} - 270\hat{\mathbf{j}} = \underline{120\hat{\mathbf{i}} - 270\hat{\mathbf{j}}}\left(\text{N/s}^3\right)$.

Problem 1.5

$$\hat{\mathbf{i}} = \sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}} \qquad \hat{\mathbf{j}} = -\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}} \qquad \hat{\mathbf{k}} = \hat{\mathbf{K}} \tag{1}$$

Velocity analysis

The absolute velocity of the airplane is

$$\mathbf{v} = v\hat{\mathbf{I}} \tag{2}$$

The absolute velocity of the origin of the moving frame is

$$\mathbf{v}_o = \mathbf{0} \tag{3}$$

The position of the airplane relative to the moving frame is

$$\mathbf{r}_{\text{rel}} = \frac{h}{\cos\theta}\hat{\mathbf{i}} = \frac{h}{\cos\theta}\left(\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}}\right) = h\frac{\sin\theta}{\cos\theta}\hat{\mathbf{I}} + h\hat{\mathbf{J}} \tag{4}$$

The angular velocity of the moving frame is

$$\boldsymbol{\Omega} = -\dot{\theta}\hat{\mathbf{K}} \tag{5}$$

The velocity of the airplane relative to the moving frame is, making use of (1)

$$\mathbf{v}_{\text{rel}} = v_{\text{rel}}\hat{\mathbf{i}} = v_{\text{rel}}\left(\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}}\right) = v_{\text{rel}}\sin\theta\hat{\mathbf{I}} + v_{\text{rel}}\cos\theta\hat{\mathbf{J}} \tag{6}$$

According to Equation 1.38, $\mathbf{v} = \mathbf{v}_o + \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}$. Substituting (2), (3), (4), (5) and (6) yields

$$v\hat{\mathbf{I}} = \mathbf{0} + \left(-\dot{\theta}\hat{\mathbf{K}}\right) \times \left(h\frac{\sin\theta}{\cos\theta}\hat{\mathbf{I}} + h\hat{\mathbf{J}}\right) + \left(v_{\text{rel}}\sin\theta\hat{\mathbf{I}} + v_{\text{rel}}\cos\theta\hat{\mathbf{J}}\right)$$

or

$$v\hat{\mathbf{I}} = \mathbf{0} + \left[(h\dot{\theta})\hat{\mathbf{I}} - \left(h\dot{\theta}\frac{\sin\theta}{\cos\theta}\right)\hat{\mathbf{J}}\right] + \left(v_{\text{rel}}\sin\theta\hat{\mathbf{I}} + v_{\text{rel}}\cos\theta\hat{\mathbf{J}}\right)$$

Collecting terms,

$$v\hat{\mathbf{I}} = \left(h\dot{\theta} + v_{\text{rel}}\sin\theta\right)\hat{\mathbf{I}} + \left(v_{\text{rel}}\cos\theta - h\dot{\theta}\frac{\sin\theta}{\cos\theta}\right)\hat{\mathbf{J}}$$

Equate the $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$ components on each side to obtain

$$h\dot{\theta} + v_{\text{rel}}\sin\theta = v$$
$$-h\dot{\theta}\frac{\sin\theta}{\cos\theta} + v_{\text{rel}}\cos\theta = 0$$

Solving these two equations for $\dot{\theta}$ and v_{rel} yields

$$\underline{\dot{\theta} = \frac{v}{h}\cos^2\theta} \tag{7}$$

$$\underline{v_{\text{rel}} = v\sin\theta} \tag{8}$$

Acceleration analysis

The absolute acceleration of the airplane, the absolute acceleration of the origin of the moving frame, and the angular acceleration of the moving frame are, respectively,

$$\mathbf{a} = \mathbf{0} \qquad \mathbf{a}_o = \mathbf{0} \qquad \dot{\mathbf{\Omega}} = -\ddot{\theta}\hat{\mathbf{K}} \tag{9}$$

The acceleration of the airplane relative to the moving frame is, making use of (1),

$$\mathbf{a}_{\text{rel}} = a_{\text{rel}}\hat{\mathbf{i}} = a_{\text{rel}}\left(\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}}\right) = a_{\text{rel}}\sin\theta\hat{\mathbf{I}} + a_{\text{rel}}\cos\theta\hat{\mathbf{J}} \tag{10}$$

Substituting (7) into (5), the angular velocity of the moving frame becomes

$$\mathbf{\Omega} = -\dot{\theta}\hat{\mathbf{K}} = -\frac{v}{h}\cos^2\theta\hat{\mathbf{K}} \tag{11}$$

Substituting (8) into (6) yields

$$\mathbf{v}_{\text{rel}} = v_{\text{rel}}\hat{\mathbf{i}} = v\sin\theta\left(\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}}\right) = v\sin^2\theta\hat{\mathbf{I}} + v\sin\theta\cos\theta\hat{\mathbf{J}} \tag{12}$$

From (4) and (9) we find

$$\dot{\mathbf{\Omega}} \times \mathbf{r}_{\text{rel}} = \left(-\ddot{\theta}\hat{\mathbf{K}}\right) \times \left(h\frac{\sin\theta}{\cos\theta}\hat{\mathbf{I}} + h\hat{\mathbf{J}}\right) = h\ddot{\theta}\hat{\mathbf{I}} - h\ddot{\theta}\frac{\sin\theta}{\cos\theta}\hat{\mathbf{J}} \tag{13}$$

Using (5) and (7) we get

$$\mathbf{\Omega} \times \mathbf{r}_{\text{rel}} = h\dot{\theta}\hat{\mathbf{I}} - h\dot{\theta}\frac{\sin\theta}{\cos\theta}\hat{\mathbf{J}} = h\left(\frac{v}{h}\cos^2\theta\right)\hat{\mathbf{I}} - h\left(\frac{v}{h}\cos^2\theta\right)\frac{\sin\theta}{\cos\theta}\hat{\mathbf{J}} = v\cos^2\theta\hat{\mathbf{I}} - v\sin\theta\cos\theta\hat{\mathbf{J}} \tag{14}$$

From (11) and (14) we have

$$\mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{r}_{\text{rel}}\right)=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 0 & 0 & -\dfrac{v}{h}\cos^2\theta\\ v\cos^2\theta & -v\sin\theta\cos\theta & 0\end{vmatrix}=-\frac{v^2}{h}\sin\theta\cos^3\theta\hat{\mathbf{I}}-\frac{v^2}{h}\cos^4\theta\hat{\mathbf{J}}\tag{15}$$

From (11) and (12),

$$2\mathbf{\Omega}\times\mathbf{v}_{\text{rel}}=2\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 0 & 0 & -\dfrac{v}{h}\cos^2\theta\\ v\sin^2\theta & v\sin\theta\cos\theta & 0\end{vmatrix}=2\frac{v^2}{h}\sin\theta\cos^3\theta\hat{\mathbf{I}}-2\frac{v^2}{h}\sin^2\theta\cos^2\theta\hat{\mathbf{J}}\tag{16}$$

According to Equation 1.42, $\mathbf{a}=\mathbf{a}_o+\dot{\mathbf{\Omega}}\times\mathbf{r}_{\text{rel}}+\mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{r}_{\text{rel}}\right)+2\mathbf{\Omega}\times\mathbf{v}_{\text{rel}}+\mathbf{a}_{\text{rel}}$. Substituting (9), (10), (13), (15) and (16) yields

$$\begin{aligned}\mathbf{0}=\mathbf{0}&+\left(h\ddot{\theta}\hat{\mathbf{I}}-h\ddot{\theta}\frac{\sin\theta}{\cos\theta}\hat{\mathbf{J}}\right)+\left(-\frac{v^2}{h}\sin\theta\cos^3\theta\hat{\mathbf{I}}-\frac{v^2}{h}\cos^4\theta\hat{\mathbf{J}}\right)\\&+\left(2\frac{v^2}{h}\sin\theta\cos^3\theta\hat{\mathbf{I}}-2\frac{v^2}{h}\sin^2\theta\cos^2\theta\hat{\mathbf{J}}\right)+\left(a_{\text{rel}}\sin\theta\hat{\mathbf{I}}+a_{\text{rel}}\cos\theta\hat{\mathbf{J}}\right)\end{aligned}$$

Collecting terms

$$\begin{aligned}\mathbf{0}&=\left(h\ddot{\theta}-\frac{v^2}{h}\sin\theta\cos^3\theta+2\frac{v^2}{h}\sin\theta\cos^3\theta+a_{\text{rel}}\sin\theta\right)\hat{\mathbf{I}}\\&+\left(-h\ddot{\theta}\frac{\sin\theta}{\cos\theta}-\frac{v^2}{h}\cos^4\theta-2\frac{v^2}{h}\sin^2\theta\cos^2\theta+a_{\text{rel}}\cos\theta\right)\hat{\mathbf{J}}\end{aligned}$$

or

$$\mathbf{0}=\left(h\ddot{\theta}+\frac{v^2}{h}\sin\theta\cos^3\theta+a_{\text{rel}}\sin\theta\right)\hat{\mathbf{I}}+\left(-h\ddot{\theta}\frac{\sin\theta}{\cos\theta}-\frac{v^2}{h}\cos^2\theta\left(1+\sin^2\theta\right)+a_{\text{rel}}\cos\theta\right)\hat{\mathbf{J}}$$

Equate the $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$ components on each side to obtain

$$\begin{aligned}h\ddot{\theta}+a_{\text{rel}}\sin\theta&=-\frac{v^2}{h}\sin\theta\cos^3\theta\\-h\ddot{\theta}\frac{\sin\theta}{\cos\theta}+a_{\text{rel}}\cos\theta&=\frac{v^2}{h}\cos^2\theta\left(1+\sin^2\theta\right)\end{aligned}$$

Solving these two equations for $\ddot{\theta}$ and a_{rel} yields

$$\underline{\ddot{\theta}=-2\frac{v^2}{h^2}\cos^3\theta\sin\theta}\qquad\underline{a_{rel}=\frac{v^2}{h}\cos^3\theta}$$

Problem 1.6 From Equation 2.58b with $z=0$ we have

$$\mathbf{a}=-2\Omega\dot{y}\sin\phi\hat{\mathbf{i}}+\Omega^2R_E\sin\phi\cos\phi\hat{\mathbf{j}}-\left[\frac{\dot{y}^2}{R_E}+\Omega^2R_E\cos^2\phi\right]\hat{\mathbf{k}}\tag{1}$$

where

$$R_E = 6378 \times 10^3 \text{ m}$$
$$\phi = 30°$$
$$\dot{y} = \frac{1000 \times 10^3}{3600} = 27.78 \text{ m/s}$$
$$\Omega = \frac{2\pi}{23.934 \times 3600} = 7.2921 \times 10^{-5} \text{ rad/s}$$

Substituting these numbers into (1), we find

$$\mathbf{a} = -0.0020256\hat{\mathbf{i}} + 0.014686\hat{\mathbf{j}} - 0.025557\hat{\mathbf{k}} \ (\text{m/s})$$

From $\mathbf{F} = m\mathbf{a}$, with $m = 1000$ kg, we obtain the net force on the car,

$$\mathbf{F}\text{=-}2.0256\hat{\mathbf{i}} + 14.686\hat{\mathbf{j}} - 25.557\hat{\mathbf{k}} \ (\text{N})$$

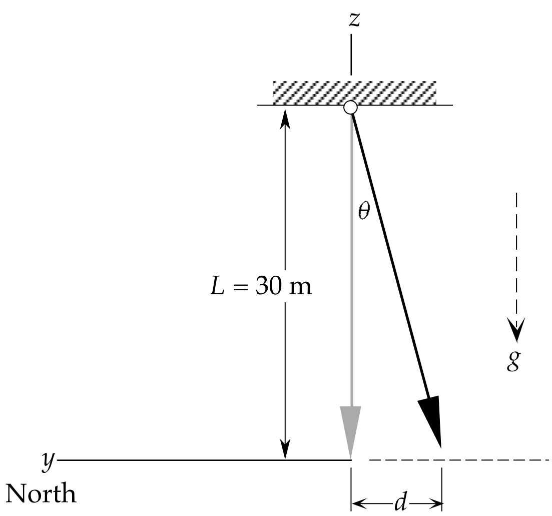
$F_{\text{lateral}} = F_x = -2.0256 \text{ N} = -0.4554 \text{ lb}$, that is

$\underline{F_{\text{lateral}} = 0.4554 \text{ lb} \quad \text{to the west}}$

The normal force N of the road on the car is given by $N = F_z + mg$, so that

$$N = -25.557 + 1000 \times 9.81 = \underline{9784 \text{ N}}$$

Problem 1.7 From Equation 1.61b, with $z = 0$,



$$\mathbf{a} = \Omega^2 R_E \sin l \cos l\hat{\mathbf{j}} - \Omega^2 R_E \cos^2 l\hat{\mathbf{k}}$$

From $\sum F_y = ma_y$ we get

$$T\sin\theta = m\Omega^2 R_E \sin\phi\cos\phi$$
$$\underline{T = \frac{m\Omega^2 R_E \sin\phi\cos\phi}{\sin\theta}}$$

From $\sum F_z = ma_z$ we obtain

$$T\cos\theta - mg = -m\Omega^2 R_E \cos^2\phi$$
$$\frac{m\Omega^2 R_E \sin\phi\cos\phi}{\sin\theta}\cos\theta - mg = -m\Omega^2 R_E \cos^2\phi$$
$$\tan\theta = \frac{\Omega^2 R_E \sin\phi\cos\phi}{g - \Omega^2 R_E \cos^2\phi}$$

Since $d = L\tan\theta$, we deduce

$$d = L\frac{\Omega^2 R_E \sin\phi\cos\phi}{g - \Omega^2 R_E \cos^2\phi}$$

Setting

$$L = 30 \text{ m}$$
$$R_E = 6378 \times 1000 = 6.378 \times 10^6 \text{ m}$$
$$\phi = 29°$$
$$g = 9.81 \text{ m/s}^2$$
$$\Omega = \frac{2\pi}{23.9344 \times 3600} = 7.2921 \times 10^{-5} \text{ rad/s}$$

yields

$\underline{d = 44.1 \text{ mm}}$ (to the south)

Problem 2.1

$$\mathbf{r} = 3t^4\hat{\mathbf{I}} + 2t^3\hat{\mathbf{J}} + 9t^2\hat{\mathbf{K}}$$
$$\|\mathbf{r}\| = \sqrt{\left(3t^4\hat{\mathbf{I}} + 2t^3\hat{\mathbf{J}} + 9t^2\hat{\mathbf{K}}\right)\cdot\left(3t^4\hat{\mathbf{I}} + 2t^3\hat{\mathbf{J}} + 9t^2\hat{\mathbf{K}}\right)} = \sqrt{9t^8 + 4t^6 + 81t^4}$$
$$\dot{r} = \frac{d\|\mathbf{r}\|}{dt} = \frac{36t^7 + 12t^5 + 162t^3}{\sqrt{9t^8 + 4t^6 + 81t^4}}$$

At $t = 2$ sec,

$$\dot{r} = \frac{4608 + 384 + 1296}{\sqrt{2304 + 256 + 1296}} = \underline{101.3\ \text{m/s}}$$

$$\dot{\mathbf{r}} = 12\mathrm{t}^3\hat{\mathbf{I}} + 6t^2\hat{\mathbf{J}} + 18t\hat{\mathbf{K}}$$
$$\|\dot{\mathbf{r}}\| = \sqrt{\left(12\mathrm{t}^3\hat{\mathbf{I}} + 6t^2\hat{\mathbf{J}} + 18t\hat{\mathbf{K}}\right)\cdot\left(12\mathrm{t}^3\hat{\mathbf{I}} + 6t^2\hat{\mathbf{J}} + 18t\hat{\mathbf{K}}\right)} = \sqrt{144t^6 + 36t^4 + 324t^2}$$

At $t = 2$ sec,

$$\|\dot{\mathbf{r}}\| = \sqrt{9216 + 576 + 1296} = \underline{105.3\ \text{m/s}}$$

Problem 2.2

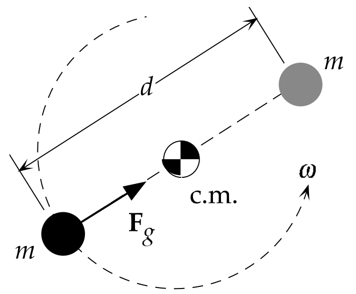
$$\hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_r = 1 \Rightarrow \frac{d}{dt}\left(\hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_r\right) = 0 \Rightarrow \frac{d\hat{\mathbf{u}}_r}{dt}\cdot\hat{\mathbf{u}}_r + \hat{\mathbf{u}}_r\cdot\frac{d\hat{\mathbf{u}}_r}{dt} = 0 \Rightarrow \underline{\hat{\mathbf{u}}_r \cdot \frac{d\hat{\mathbf{u}}_r}{dt} = 0}$$

Or,

$$\frac{d\hat{\mathbf{u}}_r}{dt} = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dfrac{d\mathbf{r}}{dt} - \mathbf{r}\dfrac{dr}{dt}}{r^2} = \frac{r\dot{\mathbf{r}} - \mathbf{r}\dot{r}}{r^2}$$
$$\hat{\mathbf{u}}_r \cdot \frac{d\hat{\mathbf{u}}_r}{dt} = \frac{\mathbf{r}}{r}\cdot\frac{r\dot{\mathbf{r}} - \mathbf{r}\dot{r}}{r^2} = \frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r^2} - \frac{r\dot{r}}{r^2}$$

But according to Equation 2.25, $\mathbf{r}\cdot\dot{\mathbf{r}} = r\dot{r}$. Hence $\underline{\hat{\mathbf{u}}_r \cdot \dfrac{d\hat{\mathbf{u}}_r}{dt} = 0}$

Problem 2.3 Both particles rotate with a constant angular velocity around the center of mass *c.m.*, which lies midway along the line joining the two masses. Let $\hat{\mathbf{u}}$ be the unit vector drawn from one of the masses to *c.m.*, which is the origin of an inertial frame. The only force on m is that of mutual gravitational attraction,



$$\mathbf{F}_g = G\frac{m^2}{d^2}\hat{\mathbf{u}}$$

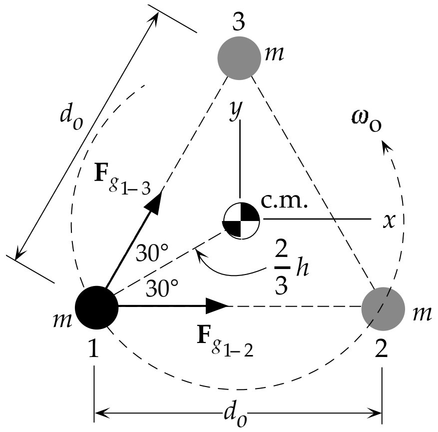
The absolute acceleration of m is normal to its circular path around *c.m.*,

$$\mathbf{a} = \omega^2\frac{d}{2}\hat{\mathbf{u}}$$

From Newton's second law, $\mathbf{F}_g = m\mathbf{a}$, so that $G\dfrac{m^2}{d^2}\hat{\mathbf{u}} = m\omega^2\dfrac{d}{2}\hat{\mathbf{u}}$, or

$$\underline{\omega = \sqrt{\frac{2Gm}{d^3}}}$$

Problem 2.4 The center of mass of the three equal masses lies at the centroid of the equilateral triangle, whose altitude h is given by $h = d_o \sin 60°$. The distance r of each mass from the center of mass is, therefore



$$r = \frac{2}{3}h = \frac{2}{3}d_o \sin 60°$$

Relative to an inertial frame with the center of mass as its origin, the acceleration of each particle is

$$a = \omega_o{}^2 r = \frac{2}{3}\omega_o{}^2 d_o \sin 60°$$

and this acceleration is directed toward the center of mass. The net force on each particle is the vector sum of the gravitational force of attraction of its two neighbors. This net force is directed towards the center of mass, so that its magnitude, focusing on particle 1 in the figure, is

$$F_{net} = F_{g_{1-2}} \cos 30° + F_{g_{1-3}} \cos 30° = G\frac{m \cdot m}{d_o{}^2}\cos 30° + G\frac{m \cdot m}{d_o{}^2}\cos 30° = 2\frac{Gm^2}{d_o{}^2}\cos 30°$$

Setting $F_{net} = ma$, we get

$$2\frac{Gm^2}{d_o{}^2}\cos 30° = m\frac{2}{3}\omega_o{}^2 d_o \sin 60°$$

$$\omega_o{}^2 = \frac{3Gm}{d_o{}^3}\frac{\cos 30°}{\sin 60°} = \frac{3Gm}{d_o{}^3}$$

$$\underline{\omega_o = \sqrt{\frac{3Gm}{d_o{}^3}}}$$

Problem 2.5

(a) $v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398{,}600}{6378 + 350}} = \underline{7.697 \text{ km/s}}$

(b) $T = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}}(6378 + 350)^{\frac{3}{2}} = 5492 \text{ sec} = \underline{91 \text{ min } 32 \text{ s}}$

Problem 2.6 The mass of the moon is 7.348×10^{22} kg. Therefore, for a satellite orbiting the moon,

$$\mu = Gm_{\text{moon}} = \left(6.67259 \times 10^{-20} \ \frac{\text{km}^3}{\text{kg} - \text{s}^2}\right)\left(7.348 \times 10^{22} \text{ kg}\right) = 4903 \ \frac{\text{km}^3}{\text{s}^2}$$

The radius of the moon as 1738 km. Hence

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{4903}{1738 + 80}} = \underline{1.642 \text{ km/s}}$$

$$T = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} = \frac{2\pi}{\sqrt{4903}}(1738 + 80)^{\frac{3}{2}} = 6956 \text{ sec} = \underline{1 \text{ hr } 56 \text{ min}}$$

Problem 2.7 The time between successive crossings of the equator equals the period of the orbit. That is

$$\frac{d}{\Omega_{\text{Earth}} R_{\text{Earth}}} = \frac{2\pi}{\sqrt{\mu}}\left(R_{\text{Earth}} + z\right)^{3/2}$$

where $d = 3000\text{ km}$ is the distance between ground tracks, z is the altitude of the orbit, $R_{\text{Earth}} = 6378\text{ km}$ and $\Omega_{\text{Earth}} = 2\pi/(23.934 \cdot 3600) = 7.2921\times 10^{-5}\text{ rad/s}$. Thus

$$\frac{3000}{\left(7.2921\times 10^{-5}\right)(6378)} = \frac{2\pi}{\sqrt{398\,600}}(6378 + z)^{3/2}$$

so that

$$\underline{z = 1440.7\text{ km}}$$

Problem 2.8 From Example 2.3 we know that $v_{\text{GEO}} = 3.0747\text{ km/s}$. From Equation 2.82 we know that

$v_{\text{esc}} = \sqrt{2}\,v_{\text{circular}}$. Hence $\Delta v = \left(\sqrt{2} - 1\right)v_{\text{GEO}} = 0.41421 \cdot 3.0747 = \underline{1.2736\text{ km/s}}$.

Problem 2.9

$$\mu_{\text{sun}} = 1.3271\times 10^{11}\text{ km}^3/\text{s}^2$$
$$r_{\text{earth}} = 149.6\times 10^{6}\text{ km}$$
$$v_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{r_{\text{earth}}}} = \sqrt{\frac{1.3271\times 10^{11}}{149.6\times 10^{6}}} = 29.784\text{ km/s}$$
$$v_{\text{esc}} = \sqrt{2}\cdot 29.784 = 42.121\text{ km/s}$$
$$v_{\text{relative}} = 42.121 - 29.784 = \underline{12.337\text{ km/s}}$$

Problem 2.10

$$\frac{A}{T/3} = \frac{\pi ab}{T}$$
$$A = \frac{\pi ab}{3} = \underline{1.0472ab}$$

Problem 2.11

$$v_r = \frac{\mu}{h}e\sin\theta$$
$$v_\perp = \frac{h}{r} = \frac{h}{\dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}}$$
$$v = \sqrt{{v_r}^2 + {v_\perp}^2}$$
$$= \frac{\mu}{h}\sqrt{e^2\left(\cos^2\theta + \sin^2\theta\right) + 2e\cos\theta + 1}$$
$$\underline{v = \frac{\mu}{h}\sqrt{e^2 + 2e\cos\theta + 1}}$$

Problem 2.12 For the ellipse, according to Problem 2.11,

$$v_{\text{ellipse}}{}^2 = \frac{\mu^2}{h^2}\left(e^2 + 2e\cos\theta + 1\right)$$

For the circle, at the point of intersection with the ellipse,

$$v_{\text{circle}}{}^2 = \frac{\mu}{r} = \frac{\mu}{\dfrac{h^2}{\mu}\dfrac{1}{1+e\cos\theta}} = \frac{\mu^2}{h^2}(1+e\cos\theta)$$

Setting $v_{\text{circle}}{}^2 = v_{\text{ellipse}}{}^2$,

$$\frac{\mu^2}{h^2}(1+e\cos\theta) = \frac{\mu^2}{h^2}\left(e^2 + 2e\cos\theta + 1\right)$$

yields $e\cos\theta = -e^2$, or $\underline{\theta = \cos^{-1}(-e)}$.

Problem 2.13 From Equation 3.42

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}$$

From Problem 2.12 $\theta = \cos^{-1}(-e)$. Hence

$$\tan\gamma = \frac{e\sin\left[\cos^{-1}(-e)\right]}{1-e^2}$$

But $\sin\left[\cos^{-1}(-e)\right] = \sqrt{1-e^2}$. Therefore,

$$\tan\gamma = \frac{e\sqrt{1-e^2}}{1-e^2} = \underline{\frac{e}{\sqrt{1-e^2}}}$$

Problem 2.14

(a)

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{70\,000 - 7000}{70\,000 + 7000} = \underline{0.81818} \ \ (\text{ellipse})$$

(b)

$$a = \frac{r_{\text{apogee}} + r_{\text{perigee}}}{2} = \frac{77\,000}{2} = \underline{38\,500 \text{ km}}$$

(c)

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398\,600}}(38\,500)^{3/2} = \underline{75\,180 \text{ s}} \ \ (20.88 \text{ h})$$

(d)

$$\varepsilon = -\frac{\mu}{2a} = -\frac{398\,600}{2\cdot 38\,500} = \underline{-5.1766 \ \ \text{km}^2/\text{s}^2}$$

(e) From Equation 3.62,

$$6378 + 1000 = \frac{38\,500\left(1 - 0.81818^2\right)}{1 + 0.81818\cos\theta}$$

$$\cos\theta = 0.88615 \ \Rightarrow \ \underline{\theta = 27.607°}$$

(f) From Equation 3.40

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398\,600 \cdot (1+0.81818) \cdot 7000} = \underline{71\,226 \ \text{km}^2/\text{s}}$$

Then

$$v_\perp = \frac{h}{r} = \frac{71\,226}{7378} = \underline{9.6538\ \text{km/s}}$$
$$v_r = \frac{\mu}{h} e \sin\theta = \frac{398\,600}{71\,226} \cdot 0.81818 \cdot \sin 27.607° = \underline{2.1218\ \text{km/s}}$$

(g)

$$v_{\text{perigee}} = \frac{h}{r_{\text{perigee}}} = \frac{71\,226}{7000} = \underline{10.175\ \text{km/s}}$$
$$v_{\text{apogee}} = \frac{h}{r_{\text{apogee}}} = \frac{71\,226}{70\,000} = \underline{1.0175\ \text{km/s}}$$

Problem 2.15

$$r_{\text{perigee}} = 6378 + 250 = 6628\ \text{km}$$
$$r_{\text{apogee}} = 6378 + 300 = 6678\ \text{km}$$
$$a = \frac{r_{\text{perigee}} + r_{\text{apogee}}}{2} = \frac{6628 + 6678}{2} = 6653\ \text{km}$$
$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398\,600}} \cdot 6653^{3/2} = 5400.5\ \text{s}\ (90.009\ \text{m})$$

$$t_{\text{perigee to apogee}} = \frac{T}{2} = \underline{45.005\ \text{m}}$$

Problem 2.16

(a)

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{(6378+1600)-(6378+600)}{(6378+1600)+(6378+600)} = \underline{0.066863}$$

(b)

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398\,600 \cdot (1+0.066863) \cdot (6378+600)} = \underline{54\,474\ \text{km}^2/s}$$

$$v_{\text{perigee}} = \frac{h}{r_{\text{perigee}}} = \frac{54\,474}{6378+600} = \underline{7.8065\ \text{km/s}}$$
$$v_{\text{apogee}} = \frac{h}{r_{\text{apogee}}} = \frac{54\,474}{6378+1600} = \underline{6.8280\ \text{km/s}}$$

(c)

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{54\,474}{\sqrt{1-0.066863^2}}\right)^3 = 6435.6\ \text{s} = \underline{107.26\ \text{m}}$$

Problem 2.17

$$h = r_{\text{perigee}} v_{\text{perigee}} = (6378 + 1270) \cdot 9 = 68\,832\ \text{km}^2/\text{s}$$
$$r_{\text{perigee}} = \frac{h^2}{\mu^2}\frac{1}{1+e}$$

$$6378 + 1270 = \frac{68\,832^2}{398\,600^2}\frac{1}{1+e} \Rightarrow e = 0.55416$$

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} = \frac{0.55416\cdot\sin 100°}{1+0.55416\cos 100°} = 0.60385 \Rightarrow \underline{\gamma = 31.13°}$$

$$z + R_{\text{earth}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$z + 6378 = \frac{68\,832^2}{398\,600}\frac{1}{1+0.55416\cos 100°} \Rightarrow \underline{z = 6773.8 \text{ km}}$$

$$h = r_{\text{perigee}}v_{\text{perigee}} = (6378+1270)\cdot 9 = 68\,832 \text{ km}^2/\text{s}$$

$$r_{\text{perigee}} = \frac{h^2}{\mu^2}\frac{1}{1+e}$$

$$6378 + 1270 = \frac{68\,832^2}{398\,600^2}\frac{1}{1+e} \Rightarrow e = 0.55416$$

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} = \frac{0.55416\cdot\sin 100°}{1+0.55416\cos 100°} = 0.60385 \Rightarrow \underline{\gamma = 31.13°}$$

$$z + R_{\text{earth}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$z + 6378 = \frac{68\,832^2}{398\,600}\frac{1}{1+0.55416\cos 100°} \Rightarrow \underline{z = 6773.8 \text{ km}}$$

Problem 2.18

$$v_r = v\sin\gamma = 9.2\cdot\sin 10° = 1.5976 \text{ km/s}$$

$$v_\perp = v\cos\gamma = 9.2\cdot\cos 10° = 9.0602 \text{ km/s}$$

$$\therefore h = rv_\perp = (6378+640)\cdot 9.0602 = 63\,585 \text{ km}^2/\text{s}$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$6378 + 640 = \frac{63\,585^2}{398\,600}\frac{1}{1+e\cos\theta}$$

$$e\cos\theta = 0.445\,29$$

$$v_r = \frac{\mu}{h}e\sin\theta$$

$$1.5976 = \frac{398\,600}{63\,585}e\sin\theta$$

$$e\sin\theta = 0.254\,84$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.254\,84}{0.445\,29} = 0.572\,31 \Rightarrow \underline{\theta = 29.783°}$$

$$e\sin 29.783° = 0.254\,84 \Rightarrow e = 0.51306$$

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{63\,585}{\sqrt{1-0.51306^2}}\right)^3 = 16\,075 \text{ s} = \underline{4.4654 \text{ h}}$$

Problem 2.19

$$a = \frac{r_{\text{perigee}} + r_{\text{apogee}}}{2} = \frac{(6378 + 250) + (6378 + 42\,000)}{2} = 27\ 503\ \text{km}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398\ 600}} 27\ 503^{3/2} = 45\,392\ \text{s} = \underline{12.61\ \text{h}}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} - r_{\text{perigee}}} = \frac{(6378 + 42\ 000) - (6378 + 250)}{(6378 + 42\ 000) + (6378 + 250)} = \underline{0.75901}$$

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398\ 600 \cdot (1 + 0.75901) \cdot (6378 + 250)} = 68\,170\ \ \text{km}^2/\text{s}$$

$$v_{\text{perigee}} = \frac{h}{r_{\text{perigee}}} = \frac{68\,170}{6378 + 250} = \underline{10.285\ \ \text{km/s}}$$

Problem 2.20

$$h = r_{\text{perigee}} v_{\text{perigee}} = (6378 + 640) \cdot 8 = 56\,144\ \ \text{km/s}$$

$$r_{\text{perigee}} = \frac{h^2}{\mu} \frac{1}{1+e}$$

$$7018 = \frac{56\,144^2}{398\ 600} \frac{1}{1+e} \quad \Rightarrow \quad e = 0.126\ 82$$

$$r_{\text{apogee}} = \frac{h^2}{\mu} \frac{1}{1-e} = \frac{56\,144^2}{398\ 600} \frac{1}{1 - 0.126\ 82} = 9056.6\ \text{km}$$

$$z_{\text{apogee}} = 9056.6 - 6378 = \underline{2678.6\ \text{km}}$$

$$a = \frac{r_{\text{perigee}} + r_{\text{apogee}}}{2} = \frac{(7018) + (9056.6)}{2} = 8037.3\ \text{km}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398\ 600}} 8037.3^{3/2} = 7171\ \text{s} = \underline{1.992\ \text{h}}$$

Problem 2.21

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$2 \cdot 3600 = \frac{2\pi}{\sqrt{398\ 600}} a^{3/2} \quad \Rightarrow \quad \underline{a = 8059\ \text{km}}$$

Using the energy equation

$$\frac{{v_{\text{perigee}}}^2}{2} - \frac{\mu}{r_{\text{perigee}}} = -\frac{\mu}{2a}$$

$$\frac{8^2}{2} - \frac{398\ 600}{r_{\text{perigee}}} = -\frac{398\ 600}{2 \cdot 8059} \quad \Rightarrow \quad r_{\text{perigee}} = 7026.2\ \text{km}$$

$$z_{\text{perigee}} = 7026.2 - 6378 = \underline{648.25\ \text{km}}$$

Problem 2.22

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$90 \cdot 60 = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \quad \Rightarrow \quad a = 6652.6 \text{ km}$$

$$r_{\text{perigee}} + r_{\text{apogee}} = 2a$$

$$(6378 + 150) + r_{\text{apogee}} = 2 \cdot 6652.6 \quad \Rightarrow \quad r_{\text{apogee}} = 6777.1 \text{ km}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{6777.1 - 6528}{6777.1 + 6528} = \underline{0.018\,723}$$

Problem 2.23

(a)

$$v_{\text{esc}} = \sqrt{2\frac{\mu}{r}} = \sqrt{2\frac{398\,600}{6378 + 300}} = 10.926 \text{ km/s}$$

$$v_{\infty} = \sqrt{v_{\text{perigee}}{}^2 - v_{\text{esc}}{}^2} = \sqrt{15^2 - 10.926^2} = \underline{10.277 \text{ km/s}}$$

(b)

$$h = r_{\text{perigee}} v_{\text{perigee}} = 6678 \cdot 15 = 100\,170 \text{ km}^2/\text{s}$$

$$r_{\text{perigee}} = \frac{h^2}{\mu} \frac{1}{1+e}$$

$$6678 = \frac{100\,170^2}{398\,600} \frac{1}{1+e} \quad \Rightarrow \quad e = 2.7696$$

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta} = \frac{100\,170^2}{398\,600} \frac{1}{1 + 2.7696\cos 100^\circ} = \underline{48\,497 \text{ km/s}}$$

(c)

$$v_r = \frac{\mu}{h} e \sin\theta = \frac{398\,600}{100\,170} \cdot 2.7696 \cdot \sin 100^\circ = \underline{10.853 \text{ km/s}}$$

$$v_{\perp} = \frac{h}{r} = \frac{100\,170}{48\,497} = \underline{2.0655 \text{ km/s}}$$

Problem 2.24

(a) From Equation 3.47

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{2.23^2}{2} - \frac{398\,600}{402\,000} = 1.4949 \text{ km}^2/\text{s}^2$$

From Equation 3.50

$$h^2 = -\frac{1}{2}\frac{\mu^2}{\varepsilon}\left(1 - e^2\right) = -\frac{1}{2}\frac{398\,600^2}{1.4949}\left(1 - e^2\right)$$

$$h^2 = \left(5.3141e^2 - 1\right) \times 10^{10}$$

From the orbit equation

$$h^2 = \mu r(1 + e\cos\theta) = 398\ 600 \cdot 402\ 000 \cdot (1 + e\cos 150°)$$
$$h^2 = (16.024 - 13.877e) \times 10^{10}$$

Equating the two expressions for h^2,

$$\left(5.3141e^2 - 1\right) \times 10^{10} = (16.024 - 13.877e) \times 10^{10}$$

yields

$$e^2 + 22.6113e - 4.0153 = 0$$

which has the positive root

$$\underline{e = 1.086}$$

(b) Using this value of the eccentricity we find

$$h^2 = (16.024 - 13.877 \cdot 1.086) \times 10^{10} = 9.5334 \times 10^9 \ \text{km}^4/\text{s}^2$$

so that

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{9.5334 \times 10^9}{398\ 600}\frac{1}{1 + 1.086} = 11\ 466 \ \text{km}$$
$$z_{\text{perigee}} = 11\ 466 - 6378 = \underline{5087.6 \ \text{km}}$$

(c)

$$v_{\text{perigee}} = \frac{h}{r_{\text{perigee}}} = \frac{\sqrt{9.5334 \times 10^9}}{11\ 466} = \underline{8.5158 \ \text{km/s}}$$

Problem 2.25 From the energy equation

$$v_\infty{}^2 + v_{\text{esc}}{}^2 = v^2$$
$$v_\infty{}^2 + \frac{2\mu}{r} = \left(1.1v_\infty\right)^2$$
$$r = 9.5238\frac{\mu}{v_\infty{}^2}$$

Substituting Equation 3.105

$$r = 9.5238\frac{\mu}{\left(\frac{\mu}{h}\sqrt{e^2 - 1}\right)^2} = 9.5238\frac{h^2}{\mu}\frac{1}{e^2 - 1}$$

Using Equation 3.40

$$r = 9.5238\left[r_{\text{perigee}}(1 + e)\right]\frac{1}{e^2 - 1} = \underline{9.5238\frac{r_{\text{perigee}}}{e - 1}}$$

Problem 2.26

$$v_\infty = \frac{\mu}{h}\sqrt{e^2 - 1} = \frac{398\ 600}{105\ 000}\sqrt{3^2 - 1} = \underline{10.737 \ \text{km/s}}$$

Problem 2.27

(a)

$$r_1 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_1}$$

$$6378 + 1700 = \frac{h^2}{398\ 600}\frac{1}{1+e\cos 130°}$$

$$h^2 = (3.2199 - 2.0697e)\times 10^9$$

$$r_2 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_2}$$

$$6378 + 500 = \frac{h^2}{398\ 600}\frac{1}{1+e\cos 50°}$$

$$h^2 = (2.7416 + 1.7622e)\times 10^9$$

$$(3.2199 - 2.0697e)\times 10^9 = (2.7416 + 1.7622e)\times 10^9$$

$$3.832e = 0.478\ 32$$

$$\underline{e = 0.124\ 82}$$

(b)

$$h^2 = (2.7416 + 1.7622\cdot 0.124\ 82)\times 10^9 = 2.9615\times 10^9 \ \ \text{km}^4/\text{s}^2$$

$$r_{perigee} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{2.9615\times 10^9}{398\ 600}\frac{1}{1+0.124\ 82} = 6605.4 \text{ km}$$

$$z_{perigee} = 6605.4 - 6378 = \underline{227.35 \text{ km}}$$

(c) From Equation 3.63,

$$a = \frac{r_{perigee}}{1-e} = \frac{6605.4}{1-0.124\ 82} = \underline{7547.5 \text{ km}}$$

Problem 2.28

(a) $\dfrac{v_r}{v_\perp} = \tan\gamma = \tan 15° = 0.26795 \ \Rightarrow\ v_r = 0.26795 v_\perp$

$$v^2 = {v_r}^2 + {v_\perp}^2$$

$$7^2 = (0.26795 v_\perp)^2 + {v_\perp}^2$$

$$\therefore v_\perp = 6.7615 \ \text{km/s}$$

$$v_r = 1.8117 \ \text{km/s}$$

$$h = r v_\perp = 9000\cdot 6.7615 = 60\ 853 \ \ \text{km}^2/\text{s}$$

$$v_r = \frac{\mu}{h} e\sin\theta$$

$$1.8117 = \frac{398\ 600}{60\ 853} e\sin\theta$$

$$e\sin\theta = 0.27659$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$9000 = \frac{60\,853^2}{398\,600}\frac{1}{1+e\cos\theta}$$

$$e\cos\theta = 0.032259$$

$$\tan\theta = \frac{e\sin\theta}{e\cos\theta} = \frac{0.27659}{0.032259} = 8.574 \Rightarrow \underline{\theta = 83.348°}$$

(b)

$$e\cos 83.348° = 0.032259 \Rightarrow \underline{e = 0.27847}$$

Problem 2.29 From Equation 3.50,

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right)$$

$$-20 = -\frac{1}{2}\frac{398\,600^2}{60\,000^2}\left(1-e^2\right) \Rightarrow e = 0.30605$$

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{60\,000^2}{398\,600}\frac{1}{1+0.30605} = 6915.2 \text{ km}$$

$$z_{\text{perigee}} = 6915.2 - 6378 = \underline{537.21 \text{ km}}$$

$$r_{\text{apogee}} = \frac{h^2}{\mu}\frac{1}{1-e} = \frac{60\,000^2}{398\,600}\frac{1}{1-0.30605} = 13\,015 \text{ km}$$

$$z_{\text{apogee}} = 13\,015 - 6378 = \underline{6636.8 \text{ km}}$$

Problem 2.30

(a)

$$v_\perp = v\cos\gamma = 8.85\cdot\cos 6° = 8.8015 \text{ km/s}$$

$$h = rv_\perp = (6378 + 550)\cdot 8.8015 = 60\,977 \text{ km}^2/\text{s}$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$6928 = \frac{60\,977^2}{398\,600}\frac{1}{1+e\cos\theta} \Rightarrow e\cos\theta = 0.34644$$

$$v_r = v\sin\gamma = 8.85\cdot\sin 6° = 0.92508 \text{ km/s}$$

$$v_r = \frac{\mu}{h}e\sin\theta$$

$$0.92508 = \frac{398\,600}{60\,977}e\sin\theta \Rightarrow e\sin\theta = 0.14152$$

$$\therefore \tan\theta = \frac{e\sin\theta}{e\cos\theta} = 0.40849 \Rightarrow \theta = 22.22°$$

$$e\sin 22.22° = 0.14152 \Rightarrow \underline{e = 0.37423}$$

(b)

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{60\,977}{\sqrt{1-0.37423^2}}\right)^3 = 11\,243 \text{ s} = \underline{187.39 \text{ m}}$$

Problem 2.31

$$v_\perp = v\cos\gamma = 10\cdot\cos 30° = 8.6603 \text{ km/s}$$
$$h = rv_\perp = (10\,000)\cdot 8.6603 = 86\,603 \text{ km}^2/\text{s}$$
$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$
$$10\,000 = \frac{86\,603^2}{398\,600}\frac{1}{1+e\cos\theta} \Rightarrow e\cos\theta = 0.88159$$

$$v_r = v\sin\gamma = 10\cdot\sin 30° = 5 \text{ km/s}$$
$$v_r = \frac{\mu}{h}e\sin\theta$$
$$5 = \frac{398\,600}{86\,603}e\sin\theta \Rightarrow e\sin\theta = 1.0863$$

$$\therefore \tan\theta = \frac{e\sin\theta}{e\cos\theta} = 1.2323 \Rightarrow \underline{\theta = 50.94°}$$

Problem 2.32

$$v_\perp = v\cos\gamma = 10\cdot\cos 20° = 9.3969 \text{ km/s}$$
$$h = rv_\perp = (15\,000)\cdot 9.3969 = 140\,950 \text{ km}^2/\text{s}$$
$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$
$$15\,000 = \frac{140\,950^2}{398\,600}\frac{1}{1+e\cos\theta} \Rightarrow e\cos\theta = 2.323$$

$$v_r = v\sin\gamma = 10\cdot\sin 20° = 3.4202 \text{ km/s}$$
$$v_r = \frac{\mu}{h}e\sin\theta$$
$$3.4202 = \frac{398\,600}{140\,950}e\sin\theta \Rightarrow e\sin\theta = 1.2095$$

$$\therefore \tan\theta = \frac{e\sin\theta}{e\cos\theta} = 0.52065 \Rightarrow \underline{\theta = 27.504°}$$

Problem 2.33

Using the orbit equation $r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$ we find

$$\frac{h^2}{\mu} = 10\,000(1+e\cos 30°) = 10\,000 + 8660.3e$$
$$\frac{h^2}{\mu} = 30\,000(1+e\cos 105°) = 30\,000 - 7764.6e$$
$$\therefore 10\,000 + 8660.3e = 30\,000 - 7764.6e$$
$$16\,425e = 20\,000 \Rightarrow \underline{e = 1.2177}$$

Problem 2.34

$$v_1 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6378+500}} = 7.6127 \text{ km/s}$$

$$v_2 = v_1 + \frac{v_1}{2} = 11.419 \text{ km/s}$$

$$\frac{v_\infty{}^2}{2} - \frac{\mu}{\infty} = \frac{v_2{}^2}{2} - \frac{\mu}{r}$$

$$\frac{v_\infty{}^2}{2} = \frac{11.419^2}{2} - \frac{398\ 600}{6878} = 7.2441 \Rightarrow \underline{v_\infty = 3.8062 \text{ km/s}}$$

Problem 2.35

$$v_1 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\ 600}{6378 + 320}} = 7.7143 \text{ km/s}$$

$$v_{\text{perigee}} = v_1 + 0.5 = 8.2143 \text{ km/s}$$

$$h = r_{\text{perigee}} v_{\text{perigee}} = 6698 \cdot 8.2143 = 55\ 019$$

$$r_{\text{perigee}} = \frac{h^2}{\mu} \frac{1}{1+e}$$

$$6698 = \frac{55\ 019^2}{398\ 600} \frac{1}{1+e} \Rightarrow e = 0.13383$$

$$r_{\text{apogee}} = \frac{h^2}{\mu} \frac{1}{1-e} = \frac{55\ 019^2}{398\ 600} \frac{1}{1 - 0.13383} = 8767.8 \text{ km}$$

$$z_{\text{apogee}} = 8767.8 - 6378 = \underline{2389.8 \text{ km}}$$

Problem 2.36

$$v = \sqrt{\frac{\mu}{r_{\text{perigee}}}}$$

$$v_{\text{perigee}} = v + \alpha v = (1+\alpha)\sqrt{\frac{\mu}{r_{\text{perigee}}}}$$

$$h = r_{\text{perigee}} v_{\text{perigee}} = r_{\text{perigee}}(1+\alpha)\sqrt{\frac{\mu}{r_{\text{perigee}}}} = (1+\alpha)\sqrt{\mu r_{\text{perigee}}}$$

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{(1+\alpha)^2 \mu r_{\text{perigee}}}{\mu}\frac{1}{1+e}$$

$$\therefore 1 + e = (1+\alpha)^2$$

$$1 + e = 1 + 2\alpha + \alpha^2 \Rightarrow \underline{e = \alpha(\alpha + 2)}$$

Problem 2.37

(a)

$$v_1 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\ 600}{6378 + 400}} = 7.6686 \text{ km/s}$$

$$v_{\perp 2} = v_1 + 0.24 = 7.9086 \text{ km/s}$$

$$h_2 = r v_{\perp 2} = 6778 \cdot 7.9086 = 53\ 605 \text{ km}^2/\text{s}$$

$$r = \frac{h_2{}^2}{\mu}\frac{1}{1+e}$$

$$6778 = \frac{53\ 605^2}{398\ 600}\frac{1}{1+e} \Rightarrow e = 0.063572$$

$$z_{\text{perigee}_2} = \underline{400 \text{ km}}$$

$$6378 + z_{\text{apogee}_2} = \frac{h_2{}^2}{\mu}\frac{1}{1-e} = \frac{53\,605^2}{398\,600}\frac{1}{1-0.063572} \Rightarrow \underline{z_{\text{apogee}_2} = 1320.3 \text{ km}}$$

(b)

$$v_{\perp 2} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6378+400}} = 7.6686 \text{ km/s}$$

$$h_2 = rv_{\perp 2} = 6778 \cdot 7.6686 = 51\,978 \text{ km}^2/\text{s}$$

$$v_{r2} = \frac{\mu}{h_2}e_2 \sin\theta$$

$$0.24 = \frac{398\,600}{51\,978}e_2 \sin\theta \Rightarrow e_2 \sin\theta = 0.031296$$

$$r = \frac{h_2{}^2}{\mu}\frac{1}{1+e_2\cos\theta}$$

$$6778 = \frac{51\,978^2}{398\,600}\frac{1}{1+e_2\cos\theta} \Rightarrow e_2\cos\theta = 0 \Rightarrow \theta = 90^\circ \ (\text{since } e \text{ cannot be zero if } v_r \neq 0)$$

$$e_2 \sin 90^\circ = 0.031296 \Rightarrow e_2 = 0.031296$$

$$6378 + z_{\text{perigee}_2} = \frac{h_2{}^2}{\mu}\frac{1}{1+e} = \frac{51\,978^2}{398\,600}\frac{1}{1+0.031296} \Rightarrow \underline{z_{\text{perigee}_2} = 196.49 \text{ km}}$$

$$6378 + z_{\text{apogee}_2} = \frac{h_2{}^2}{\mu}\frac{1}{1-e} = \frac{51\,978^2}{398\,600}\frac{1}{1-0.031296} \Rightarrow \underline{z_{\text{apogee}_2} = 631.3 \text{ km}}$$

Problem 2.38 In Figure 3.30

$$m_1 = m_{\text{sun}} = 1.989 \times 10^{30} \text{ kg}$$
$$m_2 = m_{\text{earth}} = 5.974 \times 10^{24} \text{ kg}$$
$$r_{12} = 149.6 \times 10^6 \text{ km}$$

From Equation 3.169

$$\pi_2 = \frac{m_2}{m_1 + m_2} = 3.0035 \times 10^{-6}$$

Substitute π_2 into Equation 3.195,

$$f(\xi) = \frac{1-\pi_2}{|\xi+\pi_2|^3}(\xi+\pi_2) + \frac{\pi_2}{|\xi+\pi_2-1|^3}(\xi+\pi_2-1) - \xi$$

The graph of $f(\xi)$ is similar to Figure 3.33, with the two crossings on the right much more closely spaced. Zeroing in on the regions where $f(\xi) = 0$, with the aid of a computer, reveals

$$\xi_1 = 0.990\,026\,6$$
$$\xi_2 = 1.010\,034$$
$$\xi_3 = -1.000\,001$$

Then,

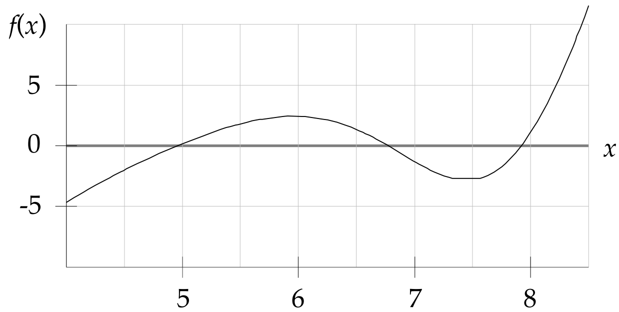
$$x_1 = \xi_1 r_{12} = \underline{148.108 \times 10^6 \text{ km}}$$
$$x_2 = \xi_2 r_{12} = \underline{151.101 \times 10^6 \text{ km}}$$
$$x_3 = \xi_3 r_{12} = \underline{-149.600 \times 10^6 \text{ km}}$$

These are the locations of L_1, L_2 and L_3 relative to the center of mass of the sun-earth system (essentially the center of the sun).

Problem 3.1 Graph the function

$$f = x^2 - 5x + 4 - 10e^{\sin x}$$

to get an idea where the roots lie.



$$f' = 2x - 5 - 10\cos x e^{\sin x}$$

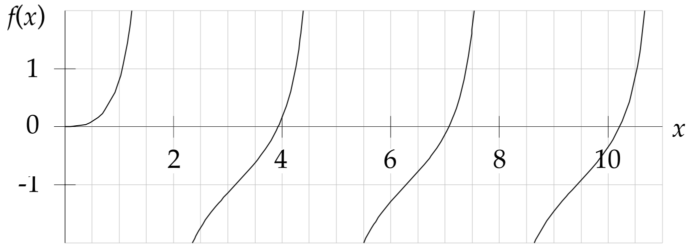
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{x_i^2 - 5x_i + 4 - 10e^{\sin x_i}}{2x_i - 5 - 10\cos x_i e^{\sin x_i}}$$

First root:	Second root:	Third root:
$x_0 = 4$ (Estimate)	$x_0 = 7$ (Estimate)	$x_0 = 8$ (Estimate)
$x_1 = 4.7733482$	$x_1 = 6.7673080$	$x_1 = 7.9259101$
$x_2 = 4.9509264$	$x_2 = 6.7732223$	$x_2 = 7.9198260$
$x_3 = 4.9577657$	$x_3 = 6.7732128$	$x_3 = 7.9197836$
$x_4 = 4.9577768$	$x_5 = \underline{6.7732128}$	$x_4 = \underline{7.9197836}$
$x_5 = \underline{4.9577768}$		

Problem 3.2 Graph the function

$$f = \tan x - \tanh x$$



to get an idea where the roots lie.
Clearly, the first root is $x = 0$.

$$f' = \sec^2 x - \operatorname{sech}^2 x$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{\tan x_i - \tanh x_i}{\sec^2 x_i - \operatorname{sech}^2 x_i}$$

Second root:	Third root:	Fourth root:
$x_0 = 4$ (Estimate)	$x_0 = 7$ (Estimate)	$x_0 = 10$ (Estimate)
$x_1 = 3.9322455$	$x_1 = 7.0730641$	$x_1 = 10.247568$
$x_2 = 3.9266343$	$x_2 = 7.0686029$	$x_2 = 10.211608$
$x_3 = 3.9266023$	$x_3 = 7.0685827$	$x_3 = 10.210178$
$x_4 = \underline{3.9266023}$	$x_5 = \underline{7.0685827}$	$x_4 = 10.210176$
		$x_5 = \underline{10.210176}$

Problem 3.3

$$a = \frac{1}{2}\left(r_{\text{apogee}} + r_{\text{perigee}}\right) = \frac{1}{2}(6978 + 6578) = 6778 \text{ km}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = \frac{2\pi}{\sqrt{398\,600}} 6778^{3/2} = 5553.5 \text{ s}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{6978 - 6578}{6978 + 6578} = 0.029\,507$$

Let B denote the point where the satellite flies through 400 km altitude on the way to apogee.

$$r_B = \frac{a(1-e^2)}{1+e\cos\theta_B}$$

$$6378+400 = \frac{6778(1-0.029\ 507^2)}{1+0.029\ 507\cos\theta_B} \Rightarrow \theta_B = 91.691°$$

$$\tan\frac{E_B}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_B}{2} = \sqrt{\frac{1-0.029\ 507}{1+0.029\ 507}}\tan\frac{91.691°}{2} \Rightarrow E_B = 1.5708 \text{ rad}$$

$$M_B = E_B - e\sin E_B = 1.5708 - 0.029\ 507\sin 1.5708 = 1.5413 \text{ rad}$$

$$t_B = \frac{M_B T}{2\pi} = \frac{1.5413\cdot 5553.5}{2\pi} = 1362.3 \text{ s}$$

t_B is the time after perigee at which the spacecraft goes above 400 km. Let C denote the point at which the satellite flies downward through 400 km altitude on its way to perigee. The time of flight t_{BC} from B to C is

$$t_{BC} = T - 2t_B = 5553.5 - 2\cdot 1362.3 = 2828.9 \text{ s} = \underline{47.148 \text{ m}}$$

Problem 3.4

(a)

$$a = \frac{1}{2}\left(r_{\text{apogee}} + r_{\text{perigee}}\right) = \frac{1}{2}(10\,000 + 7000) = 8500 \text{ km}$$

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398\ 600}}8500^{3/2} = 7799 \text{ s}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{10\,000 - 7000}{10\,000 + 7000} = 0.176\,47$$

$$t_1 = 0.5\cdot 3600 = 1800 \text{ s}$$

$$M_1 = \frac{2\pi t_1}{T} = \frac{2\pi\cdot 1800}{7799} = 1.4501 \text{ rad}$$

$$E_1 - e\sin E_1 = M_1$$

$$E_1 - 0.176\,47\sin E_1 = 1.4501$$

$$E_1 = 1.6263 \text{ rad} \quad (\text{Algorithm 3.1})$$

$$\therefore \tan\frac{\theta_1}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_1}{2} = \sqrt{\frac{1+0.176\,47}{1-0.176\,47}}\tan\frac{1.6263}{2} \Rightarrow \theta_1 = 103.28°$$

$$t_2 = 1.5\cdot 3600 = 5400 \text{ s}$$

$$M_2 = \frac{2\pi t_2}{T} = \frac{2\pi\cdot 5400}{7799} = 4.3504 \text{ rad}$$

$$E_2 - e\sin E_2 = M_2$$

$$E_2 - 0.176\,47\sin E_2 = 4.3504$$

$$E_2 = 4.1969 \text{ rad} \quad (\text{Algorithm 3.1})$$

$$\tan\frac{\theta_2}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_2}{2} = \sqrt{\frac{1+0.176\,47}{1-0.176\,47}}\tan\frac{4.1969}{2} \Rightarrow \theta_2 = 231.99°$$

$$\Delta\theta = \theta_2 - \theta_1 = \underline{128.7°}$$

(b)

$$h = \sqrt{\mu r_{\text{perigee}}(1+e)} = \sqrt{398\,600\cdot 7000\cdot(1+0.176\,47)} = 57\,294 \text{ km}^2/s$$

$$\frac{\Delta A}{\Delta t} = \frac{h}{2}$$

$$\Delta t = 3600 \text{ s}$$

$$\therefore \Delta A = \frac{1}{2} h \Delta t = \frac{1}{2} 57\,294 \cdot 3600 = \underline{103.13 \times 10^6 \text{ km}^2}$$

Problem 3.5

(a)

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$15.743 \cdot 3600 = \frac{2\pi}{\sqrt{398\,600}} a^{3/2} \Rightarrow a = 31\,890 \text{ km}$$

$$a = \frac{1}{2}\left(r_{\text{perigee}} + r_{\text{apogee}}\right)$$

$$31\,890 = \frac{1}{2}\left(12\,756 + r_{\text{apogee}}\right) \Rightarrow r_{\text{apogee}} = 51\,024 \text{ km}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{51\,024 - 12\,756}{51\,024 + 12\,756} = 0.6000$$

$$M = 2\pi \frac{t}{T} = 2\pi \frac{10}{15.743} = 3.9911 \text{ rad}$$

$$E - e \sin E = M$$

$$E - 0.6 \sin E = 3.9911$$

$$E = 3.6823 \text{ rad (Algorithm 3.1)}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan\frac{E}{2} = \sqrt{\frac{1+0.6}{1-0.6}} \tan\frac{3.6823}{2} \Rightarrow \theta = 195.78°$$

$$r = \frac{a\left(1-e^2\right)}{1 + e\cos\theta} = \frac{31\,890\left(1 - 0.6^2\right)}{1 + 0.6\cos 195.78°} = \underline{48\,924 \text{ km}}$$

(b)

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$\frac{v^2}{2} - \frac{398\,600}{48\,924} = -\frac{398\,600}{2 \cdot 31\,890} \Rightarrow \underline{v = 2.0019 \text{ km/s}}$$

(c)

$$h = \sqrt{\mu a\left(1-e^2\right)} = \sqrt{398\,600 \cdot 31\,890 \cdot \left(1 - 0.6^2\right)} = 90\,196 \text{ km}^2/\text{s}$$

$$v_r = \frac{\mu}{h} e \sin\theta = \frac{398\,600}{90\,196} \cdot 0.6 \cdot \sin 195.78° = \underline{-0.720\,97 \text{ km/s}}$$

Problem 3.6

(a)

$$M_{PB} = E - e\sin E$$

$$\frac{2\pi t_{PB}}{T} = \frac{\pi}{2} - e\sin\frac{\pi}{2}$$

$$t_{PB} = \left(\frac{\pi}{2} - e\right)\frac{T}{2\pi}$$

$$t_{DPB} = 2t_{PB} = \underline{\left(\frac{1}{2} - \frac{e}{\pi}\right)T}$$

(b)

$$t_{BA} = \frac{T}{2} - t_{PB}$$

$$t_{BA} = \frac{T}{2} - \left(\frac{\pi}{2} - e\right)\frac{T}{2\pi} = \left(\frac{1}{4} + \frac{e}{2\pi}\right)T$$

$$t_{BAD} = 2t_{BA} = \underline{\left(\frac{1}{2} + \frac{e}{\pi}\right)T}$$

Problem 3.7

$$\tan\frac{E_B}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_B}{2} = \sqrt{\frac{1-0.3}{1+0.3}}\tan\frac{\pi}{4} = 0.733\,80 \;\Rightarrow\; E_B = 1.2661 \text{ rad}$$

$$M_B = E_B - e\sin E_B = 1.2661 - 0.6\sin 1.2661 = 0.97992 \text{ rad}$$

$$t_B = \frac{M_B}{2\pi}T = \frac{0.97992}{2\pi}T = \underline{0.48996T}$$

Problem 3.8

$$a = \frac{r_{\text{apogee}} + r_{\text{perigee}}}{2} = \frac{14\,000 + 7000}{2} = 10\,500 \text{ km}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{14\,000 - 7000}{14\,000 + 7000} = 0.33333$$

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398\,600}}10\,500^{3/2} = 10708 \text{ s}$$

$$\tan\frac{E_{\theta=60^\circ}}{2} = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{60^\circ}{2}\right) = \sqrt{\frac{1-0.33333}{1+0.33333}}\tan(30^\circ) = 0.40825 \;\Rightarrow E_{\theta=60^\circ} = 0.77519 \text{ rad}$$

$$M_{\theta=60^\circ} = 0.77519 - 0.33333\sin(0.77519) = 0.54191 \text{ rad}$$

$$t_{\theta=60^\circ} = \frac{M_{\theta=60^\circ}}{2\pi}T = \frac{0.54191}{2\pi}10708 = 923.51 \text{ s}$$

$$t_\theta = t_{\theta=60^\circ} + 30\cdot 60 = 2723.5 \text{ s}$$

$$M_\theta = 2\pi\frac{t_\theta}{T} = 2\pi\frac{2723.5}{10708} = 1.5981 \text{ rad}$$

$$E_\theta - e\sin E_\theta = M_\theta$$

$$E_\theta - 0.33333\sin E_\theta = 1.5981 \;\Rightarrow\; E_\theta = 1.9122 \text{ rad (Algorithm 3.1)}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_\theta}{2} = \sqrt{\frac{1+0.33333}{1-0.33333}}\tan\frac{1.9122}{2} \;\Rightarrow\; \underline{\theta = 126.95^\circ}$$

Problem 3.9

$$a\cos\theta + b\sin\theta = c$$

$$\cos\theta + \frac{b}{a}\sin\theta = \frac{c}{a}$$

$$\frac{b}{a} \equiv \tan\phi = \frac{\sin\phi}{\cos\phi}$$

$$\cos\theta + \frac{\sin\phi}{\cos\phi}\sin\theta = \frac{c}{a}$$

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \frac{c}{a}\cos\phi$$

$$\cos(\theta - \phi) = \frac{c}{a}\cos\phi$$

$$\theta - \phi = \pm\cos^{-1}\left(\frac{c}{a}\cos\phi\right)$$

$$\underline{\theta = \phi \pm \cos^{-1}\left(\frac{c}{a}\cos\phi\right)}$$

Problem 3.10

$$M_B = E_B - e\sin E_B$$
$$2\pi\frac{t_B}{T} = \frac{\pi}{2} - e\sin\frac{\pi}{2}$$
$$t_B = \frac{\frac{\pi}{2} - e\sin\frac{\pi}{2}}{2\pi}T = \underline{(0.25 - 0.15915e)T}$$

Problem 3.11

$$\left.\begin{aligned} r &= \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \\ r_{\text{perigee}} &= \frac{h^2}{\mu}\frac{1}{1+e} \end{aligned}\right\} \Rightarrow \ r = \frac{r_{\text{perigee}}(1+e)}{1+e\cos\theta}$$
$$2r_{\text{perigee}} = \frac{r_{\text{perigee}}(1+0.5)}{1+0.5\cos\theta_B}$$
$$\cos\theta_B = -0.5 \ \Rightarrow \ \theta_B = 120°$$

$$\tan\frac{E_B}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_B}{2} = \sqrt{\frac{1-0.5}{1+.5}}\tan\frac{120°}{2} \ \Rightarrow \ E_B = \frac{\pi}{2}\text{ rad}$$
$$M_B = E_B - e\sin E_B = \frac{\pi}{2} - 0.5\sin\frac{\pi}{2} = 1.0708\text{ rad}$$
$$t_B = \frac{M_B}{2\pi}T = \frac{1.0708}{2\pi}T = \underline{0.17042T}$$

Problem 3.12 From Example 3.3 we have

$$e = 0.24649$$
$$T = 8679.1\text{ s}$$
$$\theta_c = 143.36$$

Thus

$$\tan\frac{E_c}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_c}{2} = \sqrt{\frac{1-0.24649}{1+0.24649}}\tan\frac{143.36°}{2} \ \Rightarrow \ \underline{E_c = 2.3364\text{ rad}}$$
$$M_c = E_c - e\sin E_c = 2.3364 - 0.24649\cdot\sin 2.3364 = \underline{2.1587\text{ rad}}$$
$$t_c = \frac{M_c}{2\pi}T = \frac{2.1587}{2\pi}\cdot 8679.1 = \underline{2981.8\text{ s}}$$

Problem 3.13

$$r_{\text{SOI}} = 925\,000\text{ km}$$
$$r = \frac{r_{\text{perigee}}(1+e)}{1+e\cos\theta}$$
$$925\,000 = \frac{(6378+500)(1+1)}{1+1\cdot\cos\theta} \ \Rightarrow \ \theta = 170.11°$$
$$M_p = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} = \frac{1}{2}\tan\frac{170.11°}{2} + \frac{1}{6}\tan^3\frac{170.11°}{2} = 262.82$$
$$r_{\text{perigee}} = \frac{h^2}{2\mu} \ \Rightarrow \ h = \sqrt{2\mu r_{\text{perigee}}} = \sqrt{2\cdot 398\,600\cdot 6878} = 74\,048\text{ km}^2/\text{s}$$

$$t = \frac{h^3}{\mu}M_p = \frac{74\,048^3}{398\,600}\cdot 262.82 = 671\,630\text{ s} = \underline{7\text{d }18\text{h }34\text{m}}$$

Problem 3.14

(a)

$$h = \sqrt{\mu r_{\text{perigee}}(1+e)} = \sqrt{398\,600\cdot 7500\cdot(1+1)} = 77\,324\ \text{km}^2/\text{s}$$

$$M_p\Big)_{\theta=90°} = \frac{1}{2}\tan\frac{90°}{2} + \frac{1}{6}\tan^3\frac{90°}{2} = 0.66667$$

$$t_{\theta=90°} = \frac{h^3}{\mu^2}M_p\Big)_{\theta=90°} = \frac{77\,324^3}{389\,600^2}0.66667 = 1939.9\text{ s}$$

$$t_{-90°\text{ to }+90°} = 2\cdot 1939.9 = 3879.8\text{ s} = \underline{1.0777\text{ h}}$$

(b)

$$M_p = \frac{\mu^2 t}{h^3} = \frac{398\,600^3\cdot(24\cdot 3600)}{77\,324^3} = 29.692$$

$$\tan\frac{\theta}{2} = \left[3M_p + \sqrt{(3M_p)^2+1}\right]^{1/3} - \left[3M_p + \sqrt{(3M_p)^2+1}\right]^{-1/3}$$

$$\tan\frac{\theta}{2} = \left[3\cdot 29.692 + \sqrt{(3\cdot 29.692)^2+1}\right]^{1/3} - \left[3\cdot 29.692 + \sqrt{(3\cdot 29.692)^2+1}\right]^{-1/3} = 5.4492$$

$$\theta = 159.2°$$

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} = \frac{77\,324^2}{398\,600}\frac{1}{1+\cos 159.2°} = \underline{230\,200\text{ km}}$$

Problem 3.15

(a)

$$v_{\text{perigee}} = 1.1\sqrt{\frac{2\mu}{r_{\text{perigee}}}} = 1.1\sqrt{\frac{2\cdot 398\,600}{7500}} = 11.341\ \text{km/s}$$

$$h = r_{\text{perigee}}v_{\text{perigee}} = 7500\cdot 11.341 = 85\,056\ \text{km}^2/\text{s}$$

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e}$$

$$7500 = \frac{85\,056^2}{398\,600}\frac{1}{1+e} \quad\Rightarrow\quad e = 1.4200$$

$$\tanh\frac{F_{90°}}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{90°}{2} = \sqrt{\frac{1.42-1}{1.42+1}}\tan\frac{90°}{2} \quad\Rightarrow\quad F_{90°} = 0.887\,14$$

$$M_h\big)_{90°} = e\sinh F_{90°} - F_{90°} = 1.42\cdot\sinh 0.88714 - 0.88714 = 0.544\,46$$

$$M_h\big)_{90°} = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t_{90°}$$

$$0.544\,46 = \frac{398\,600^2}{85\,056^3}\left(1.42^2-1\right)^{3/2}t_{90°} \quad\Rightarrow\quad t_{90°} = 2057.9\text{ s}$$

$$t_{-90°\text{ to }90°} = 2t_{90°} = \underline{4115.7\text{ s}} = \underline{1.1433\text{ h}}$$

(b)

$$M_h = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t = \frac{398\,600^2}{85\,056^3}\left(1.42^2-1\right)^{3/2}\cdot 24\cdot 3600 = 22.859$$

$$e\sinh F - F = M_h$$

$$1.42\sinh F - F = 22.859 \quad\Rightarrow\quad F = 3.6196\ \ \text{(Algorithm 3.2)}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{1.42+1}{1.42-1}}\tanh\frac{3.6196}{2} \;\Rightarrow\; \theta = 132.55°$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{85056^2}{398\,600}\frac{1}{1+1.42\cdot\cos 132.55°} = \underline{455\,660 \text{ km}}$$

Problem 3.16

$$h = r_{\text{perigee}}v_{\text{perigee}} = (6378+300)\cdot 11.5 = 76\,797 \;\; \text{km}^2/\text{s}$$

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e}$$

$$e = \frac{h^2}{\mu r_{\text{perigee}}} - 1 = \frac{76\,797^2}{398\,600\cdot 6878} - 1 = 1.2157 \;\; (\text{hyperbola})$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1} = \frac{76\,797^2}{398\,600}\frac{1}{1.2157^2-1} = 30\,964 \text{ km}$$

At 6 AM:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

$$\frac{10^2}{2} - \frac{398\,600}{r} = \frac{398\,600}{2\cdot 30\,964} \;\Rightarrow\; r = 9149.9 \text{ km}$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$9149.9 = \frac{76\,797^2}{398\,600}\frac{1}{1+1.2157\cos\theta} \;\Rightarrow\; \theta = -59.494° \;\; (\text{flying } \textit{towards} \text{ earth})$$

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2} = \sqrt{\frac{1.2157-1}{1.2157+1}}\tan\frac{-59.494°}{2} = -0.10384 \;\Rightarrow\; F = -0.360\,45$$

$$M_h = e\sinh F - F = 1.2157\sinh(-0.360\,45) - (-0.360\,45) = -0.087\,287$$

$$M_h = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t$$

$$-0.087\,287 = \frac{398\,600^2}{76\,797^3}\left(1.2157^2-1\right)^{3/2}t \;\Rightarrow\; t = -753.3 \text{ s} \;\; (\text{negative means time } \textit{until} \text{ perigee})$$

At 11 AM:

$$t = 5\cdot 3600 - |-753.3| = 17\,247 \text{ s} \;\; (\text{time } \textit{since} \text{ perigee at 11 AM})$$

$$M_h = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t_2 = \frac{398\,600^2}{76\,797^3}\left(1.2157^2-1\right)^{3/2}\cdot 17\,247 = 1.9984$$

$$e\sinh F - F = M_h$$

$$1.2157\sinh F - F = 1.9984 \;\Rightarrow\; F = 1.8760 \;\; (\text{Algorithm 3.2})$$

$$\tanh\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tan\frac{F}{2} = \sqrt{\frac{1.2157+1}{1.2157-1}}\tan\frac{1.8760}{2} \;\Rightarrow\; \theta = 133.96°$$

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{76\,797^2}{398\,600}\frac{1}{1+1.2157\cdot\cos 133.96°} = 94771 \text{ km}$$

$$z = r - 6378 = \underline{88\,393 \text{ km}}$$

Problem 3.17

$$v_\perp = v\cos\gamma = 8\cdot\cos(-65°) = 3.3809 \text{ km/s}$$
$$h = rv_\perp = (37\,000 + 6378)\cdot 3.3809 = 146\,660 \text{ km}^2/\text{s}$$
$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$
$$43\,378 = \frac{146\,660^2}{398\,600}\frac{1}{1+e\cos\theta} \Rightarrow e\cos\theta = 0.24397$$
$$v_r = v\sin\gamma = 8\cdot\sin(-65°) = -7.2505 \text{ km/s}$$
$$v_r = \frac{\mu}{h}e\sin\theta$$
$$-7.2505 = \frac{398\,600}{146\,660}e\sin\theta \Rightarrow e\sin\theta = -2.6677$$
$$\tan\theta = \frac{e\sin\theta}{e\cos\theta} = \frac{-2.6677}{0.24397} = -10.935 \Rightarrow \theta = -84.775°$$
$$e\sin(-84.775°) = -2.6677 \Rightarrow e = 2.6788 \text{ (hyperbola)}$$
$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{146\,660^2}{398\,600}\frac{1}{1+2.6788} = 14\,668 \text{ km (no impact)}$$

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2} = \sqrt{\frac{2.6788-1}{2.6788+1}}\tan\left(\frac{-84.775°}{2}\right) \Rightarrow F = -1.4389$$
$$M_h = e\sinh F - F = 2.6788\sinh(-1.4389) - (-1.4389) = -3.8906$$
$$M_h = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t$$
$$-3.8906 = \frac{398\,600^2}{46\,660^3}\left(2.6788^2-1\right)^{3/2}t \Rightarrow t = -5032.5 \text{ s} = -1.3979 \text{ h}$$

1.3979 hours until perigee passage.

Problem 3.18 Write the following MATLAB script to use M-function `kepler_U` to implement *Algorithm 3.3*.

```
clear
global mu
mu  = 398600;

ro  = 7200;
vro = 1;
a   = 10000;
dt  = 3600;

x   = kepler_U(dt, ro, vro, 1/a);

fprintf('\n\n-----------------------------------------------------\n')
fprintf('\n Initial radial coordinate = %g',ro)
fprintf('\n Initial radial velocity   = %g',vro)
fprintf('\n Elapsed time              = %g',dt)
fprintf('\n Semimajor axis            = %g\n',a)
fprintf('\n Universal anomaly         = %g',x)
fprintf('\n\n-----------------------------------------------------\n\n')
```

Running this program produces the following output in the MATLAB Command Window:

```
-----------------------------------------------------

 Initial radial coordinate = 7200
 Initial radial velocity   = 1
 Elapsed time              = 3600
 Semimajor axis            = 10000

 Universal anomaly         = 229.341

-----------------------------------------------------
```

That is, $\underline{\chi = 229.34\ \text{km}^{1/2}}$. To check this with Equation 3.55, proceed as follows.

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

$$7200 = \frac{h^2}{398\,600}\frac{1}{1+e\cos\theta}$$

$$\therefore \cos\theta = \frac{3.844\times 10^{-10}h^2 - 1}{e} \tag{1}$$

$$v_r = \frac{\mu}{h}e\sin\theta$$

$$1 = \frac{398\,600}{h}e\sin\theta$$

$$\therefore \sin\theta = \frac{1}{398\,600}\frac{h}{e} \tag{2}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Substitute (1) and (2):

$$\left(\frac{1}{398\,600}\frac{h}{e}\right)^2 + \left(\frac{3.844\times 10^{-10}h^2 - 1}{e}\right)^2 = 1 \tag{3}$$

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2}$$

$$10\,000 = \frac{h^2}{398\,600}\frac{1}{1-e^2}$$

$$\therefore h = \sqrt{3.986\times 10^9\left(1-e^2\right)} \tag{4}$$

Substitute (4) into (3):

$$\left(\frac{1}{398\,600}\frac{\sqrt{3.986\times 10^9\left(1-e^2\right)}}{e}\right)^2 + \left\{\frac{3.844\times 10^{-10}\left[3.986\times 10^9\left(1-e^2\right)\right]-1}{e}\right\}^2 = 1$$

Expanding and collecting terms yields

$$\frac{1}{1993e^2}\left(3844.5e^4 - 4195.9e^2 + 351.41\right) = 0$$

or

$$3844.5e^4 - 4195.9e^2 + 351.41 = 0$$

The positive roorts of this equation are $e = 1.0000$ and $e = 0.30233$. Obviously, we choose the latter.

$$e = 0.30233 \tag{5}$$

Substituting (5) into (4), we get

$$h = 60\,180 \text{ km}^2/\text{s} \tag{6}$$

Substituting (5) and (6) into (1) or (2) yields

$$\theta_1 = 29.959° \tag{7}$$

Compute the time at this initial true anomaly as follows:

$$\tan\frac{E_1}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_1}{2} = \sqrt{\frac{1-0.30233}{1+0.30233}}\tan\frac{29.959°}{2} = 0.19584$$

$$\therefore E_1 = 0.38678 \text{ rad} \tag{8}$$

$$M_1 = E_1 - e\sin E_1 = 0.38678 - 0.30233 \cdot \sin 0.38678 = 0.27273 \text{ rad}$$
$$t_1 = \frac{M_1}{2\pi}T = \frac{0.27273}{2\pi} \cdot 9952.0 = 431.99 \text{ s}$$

Obtain E one hour later.

$$t_2 = t_1 + 3600 = 4032 \text{ s}$$
$$M_2 = 2\pi\frac{t_2}{T} = 2\pi\frac{4032}{9952.0} = 2.5456 \text{ rad}$$
$$E_2 - e\sin E_2 = M_2$$
$$E_2 - 0.30233\sin E_2 = 2.5456$$

Using *Algorithm 3.1*,

$$E_2 = 2.6802 \text{ rad} \tag{9}$$

According to Equation 3.55,

$$\chi = \sqrt{a}(E_2 - E_1) = \sqrt{10\,000}(2.6802 - 0.38678) = \underline{229.34 \text{ km}^{1/2}}$$

This is the same as the value obtained via *Algorithm 3.3*

Problem 3.19 Write the following MATLAB script to use the M-function `rv_from_r0v0` to execute *Algorithm 3.4.*

```
clear
global mu
mu = 398600;

R0  = [20000 -105000 -19000];
V0  = [  0.9    -3.4   -1.5];
t   =2*3600;

[R V] = rv_from_r0v0(R0, V0, t);

fprintf('-----------------------------------------------------')
fprintf('\n Initial position vector (km):')
fprintf('\n   r0 = (%g, %g, %g)\n', R0(1), R0(2), R0(3))
fprintf('\n Initial velocity vector (km/s):')
```

```
fprintf('\n   v0 = (%g, %g, %g)', V0(1), V0(2), V0(3))
fprintf('\n\n Elapsed time = %g s\n',t)
fprintf('\n Final position vector (km):')
fprintf('\n   r = (%g, %g, %g)\n', R(1), R(2), R(3))
fprintf('\n Final velocity vector (km/s):')
fprintf('\n   v = (%g, %g, %g)', V(1), V(2), V(3))
fprintf('\n-----------------------------------------------------\n')
```

The output to the MATLAB Command Window is as follows:

```
-----------------------------------------------------
 Initial position vector (km):
   r0 = (20000, -105000, -19000)

 Initial velocity vector (km/s):
   v0 = (0.9, -3.4, -1.5)

 Elapsed time = 7200 s

 Final position vector (km):
   r = (26337.8, -128752, -29655.9)

 Final velocity vector (km/s):
   v = (0.862796, -3.2116, -1.46129)
-----------------------------------------------------
```

Problem 4.1 *Algorithm 4.1* (MATLAB M-function `coe_from_sv` in Appendix D.8):

$$(1)\ r = \|\mathbf{r}\| = 16\,850 \text{ km}$$

$$(2)\ v = \|\mathbf{v}\| = 5.7415 \text{ km/s}$$

$$(3)\ v_r = \frac{\mathbf{r}\cdot\mathbf{v}}{r} = 0.001\,885\,6 \text{ km/s} \quad (>0)$$

$$(4)\ \mathbf{h} = \mathbf{r}\times\mathbf{v} = 82\,234\hat{\mathbf{I}} - 23\,035\hat{\mathbf{J}} + 41\,876\hat{\mathbf{K}}\ \left(\text{km}^2/\text{s}\right)$$

$$(5)\ h = \|\mathbf{h}\| = \underline{95\,360 \text{ km}^2/\text{s}}$$

$$(6)\ i = \cos^{-1}\left(\frac{h_Z}{h}\right) = \cos^{-1}\left(\frac{41\,876}{96\,350}\right) = \underline{63.952°}$$

$$(7)\ \mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = 24\,035\hat{\mathbf{I}} + 82\,234\hat{\mathbf{J}}\ \left(\text{km}^2/\text{s}\right) \quad (N_Y > 0)$$

$$(8)\ N = \|\mathbf{N}\| = 85\,674 \text{ km}^2/\text{s}$$

$$(9)\ \Omega = \cos^{-1}\left(\frac{N_X}{N}\right) = \cos^{-1}\left(\frac{24\,035}{85\,674}\right) = \underline{73.707°}$$

$$(10)\ \mathbf{e} = \frac{1}{398\,600}\Big[\left(5.7415^2 - 398\,600/16\,850\right)\left(2615\hat{\mathbf{I}} + 15\,881\hat{\mathbf{J}} + 3980\hat{\mathbf{K}}\right)$$
$$-\left(16\,580\right)\left(0.001\,855\,6\right)\left(-2.767\hat{\mathbf{I}} - 0.7905\hat{\mathbf{J}} + 4.98\hat{\mathbf{K}}\right)\Big]$$
$$= 0.059\,521\hat{\mathbf{I}} + 0.360\,32\hat{\mathbf{J}} + 0.08988\hat{\mathbf{K}} \quad (e_Z > 0)$$

$$(11)\ e = \|\mathbf{e}\| = \underline{0.37602}$$

$$(12)\ \omega = \cos^{-1}\left(\frac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) = \underline{15.43°}$$

$$(13)\ \theta = \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = \underline{0.067\,42°}$$

Problem 4.2 *Algorithm 4.1* (MATLAB M-function `coe_from_sv` in Appendix D.8):

$$(1)\ r = \|\mathbf{r}\| = 12\,670 \text{ km}$$

$$(2)\ v = \|\mathbf{v}\| = 3.9538 \text{ km/s}$$

$$(3)\ v_r = \frac{\mathbf{r}\cdot\mathbf{v}}{r} = -0.7905 \text{ km/s} \quad (<0)$$

$$(4)\ \mathbf{h} = \mathbf{r}\times\mathbf{v} = 49\,084\hat{\mathbf{I}}\ \left(\text{km}^2/\text{s}\right)$$

$$(5)\ h = \|\mathbf{h}\| = \underline{49\,084 \text{ km}^2/\text{s}}$$

$$(6)\ i = \cos^{-1}\left(\frac{h_Z}{h}\right) = \cos^{-1}\left(\frac{0}{49\,084}\right) = \underline{90°}$$

$$(7)\ \mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = 49\,084\hat{\mathbf{J}}\ \left(\text{km}^2/\text{s}\right) \quad (N_Y > 0)$$

$$(8)\ N = \|\mathbf{N}\| = 49\,084 \text{ km}^2/\text{s}$$

$$(9)\ \Omega = \cos^{-1}\left(\frac{N_X}{N}\right) = \cos^{-1}\left(\frac{0}{49\,084}\right) = \underline{90°}$$

$$(10)\ \mathbf{e} = \frac{1}{398\,600}\Big[\left(3.9538^2 - 398\,600/12\,670\right)\left(12\,670\hat{\mathbf{K}}\right)$$
$$-\left(12\,670\right)\left(-0.7905\right)\left(-3.874\hat{\mathbf{J}} - 0.7905\hat{\mathbf{K}}\right)\Big]$$
$$= -0.097342\hat{\mathbf{J}} - 0.52296\hat{\mathbf{K}} \quad (e_Z < 0)$$

$$(11)\ e = \|\mathbf{e}\| = \underline{0.53194}$$

(12) $\omega = 360° - \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) = 360° - 100.54° = \underline{259.46°}$

(13) $\theta = 360° - \cos^{-1}\left(\dfrac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = 360° - 169.46° = \underline{190.54°}$

Problem 4.3 *Algorithm 4.1* (MATLAB M-function `coe_from_sv` in Appendix D.8):

(1) $\|\mathbf{r}\| = 10189$ km

(2) $\|\mathbf{v}\| = 5.8805$ km/s

(3) $v_r = \dfrac{\mathbf{r}\cdot\mathbf{v}}{r} = 1.2874$ km/s (>0)

(4) $\mathbf{h} = \mathbf{r}\times\mathbf{v} = 31\,509\hat{\mathbf{I}} + 11\,468\hat{\mathbf{J}} + 47\,888\hat{\mathbf{K}}\ \left(\text{km}^2/\text{s}\right)$

(5) $h = \|\mathbf{h}\| = \underline{58\,461\ \text{km}^2/\text{s}}$

(6) $i = \cos^{-1}\left(\dfrac{h_Z}{h}\right) = \cos^{-1}\left(\dfrac{47\,888}{58\,461}\right) = \underline{35°}$

(7) $\mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = -11\,468\hat{\mathbf{I}} + 31\,509\hat{\mathbf{J}}\ \left(\text{km}^2/\text{s}\right)\ \ (N_Y > 0)$

(8) $N = \|\mathbf{N}\| = 33\,532\ \text{km}^2/\text{s}$

(9) $\Omega = \cos^{-1}\left(\dfrac{N_X}{N}\right) = \cos^{-1}\left(\dfrac{-11\,468}{33\,532}\right) = \underline{110°}$

(10) $$\mathbf{e} = \frac{1}{398\,600}\Big[\left(5.8805^2 - 398\,600/10189\right)\left(6472.7\hat{\mathbf{I}} - 7470.8\hat{\mathbf{J}} - 2469.8\hat{\mathbf{K}}\right)$$
$$-\left(10189\right)\left(1.2874\right)\left(3.9914\hat{\mathbf{I}} + 2.7916\hat{\mathbf{J}} - 3.2948\hat{\mathbf{K}}\right)\Big]$$
$$= -0.2051\hat{\mathbf{I}} - 0.006\,738\,2\hat{\mathbf{J}} + 0.13657\hat{\mathbf{K}}\ \ (e_Z > 0)$$

(11) $e = \|\mathbf{e}\| = \underline{0.2465}$

(12) $\omega = \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) = \underline{74.996°}$

(13) $\theta = \cos^{-1}\left(\dfrac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = \underline{130°}$

Problem 4.4

$v_r = \dfrac{\mathbf{v}\cdot\mathbf{r}}{r} = -2.30454$ km/s (Flying *towards* perigee)

$\theta = 360° - \cos^{-1}\left(\dfrac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = 360° - 30° = \underline{330°}$

Problem 4.5

$$\hat{\mathbf{w}} = \frac{\mathbf{r}\times\mathbf{e}}{\|\mathbf{r}\times\mathbf{e}\|} = \frac{-3353.1\hat{\mathbf{I}} + 6361.8\hat{\mathbf{J}} + 2718.1\hat{\mathbf{K}}}{7687.9}$$

$$\hat{\mathbf{w}} = -0.436\,16\hat{\mathbf{I}} + 0.827\,51\hat{\mathbf{J}} + 0.353\,55\hat{\mathbf{K}}$$

$$i = \cos^{-1}\left(\hat{\mathbf{w}}\cdot\hat{\mathbf{K}}\right) = \cos^{-1}\left(0.353\,55\right) = \underline{69.295°}$$

Problem 4.6

(a)

$$\overrightarrow{AB} = (4-1)\hat{\mathbf{i}} + (6-2)\hat{\mathbf{j}} + (5-3)\hat{\mathbf{k}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\overrightarrow{AC} = (3-1)\hat{\mathbf{i}} + (9-2)\hat{\mathbf{j}} + (-2-3)\hat{\mathbf{k}} = 2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

$$\mathbf{X}' = \overrightarrow{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\mathbf{Z}' = \mathbf{X}' \times \overrightarrow{AC} = -34\hat{\mathbf{i}} + 19\hat{\mathbf{j}} + 13\hat{\mathbf{k}}$$

$$\mathbf{Y}' = \mathbf{Z}' \times \mathbf{X}' = -14\hat{\mathbf{i}} + 107\hat{\mathbf{j}} - 193\hat{\mathbf{k}}$$

$$\hat{\mathbf{i}}' = \frac{\mathbf{X}'}{\|\mathbf{X}'\|} = 0.557\,09\hat{\mathbf{i}} + 0.742\,78\hat{\mathbf{j}} + 0.371\,39\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}}' = \frac{\mathbf{Y}'}{\|\mathbf{Y}'\|} = -0.063\,314\hat{\mathbf{i}} + 0.483\,90\hat{\mathbf{j}} - 0.872\,83\hat{\mathbf{k}}$$

$$\hat{\mathbf{k}}' = \frac{\mathbf{Z}'}{\|\mathbf{Z}'\|} = -0.828\,04\hat{\mathbf{i}} + 0.462\,73\hat{\mathbf{j}} + 0.316\,60\hat{\mathbf{k}}$$

$$[\mathbf{Q}] = \begin{Bmatrix} \lfloor \hat{\mathbf{i}}' \rfloor \\ \lfloor \hat{\mathbf{j}}' \rfloor \\ \lfloor \hat{\mathbf{k}}' \rfloor \end{Bmatrix} = \underline{\begin{bmatrix} 0.557\,09 & 0.742\,78 & 0.371\,39 \\ -0.063\,314 & 0.483\,90 & -0.872\,83 \\ -0.828\,04 & 0.462\,73 & 0.316\,60 \end{bmatrix}}$$

(b)

$$\{\mathbf{v}\} = [\mathbf{Q}]^T\{\mathbf{v}'\} = \begin{bmatrix} 0.557\,09 & -0.063\,314 & -0.828\,04 \\ 0.742\,78 & 0.483\,90 & 0.462\,73 \\ 0.371\,39 & -0.872\,83 & 0.316\,60 \end{bmatrix} \begin{Bmatrix} 2 \\ -1 \\ 3 \end{Bmatrix}$$

$$\{\mathbf{v}\} = \underline{\begin{Bmatrix} -1.3066 \\ 2.3898 \\ 2.5654 \end{Bmatrix}} \quad \left(\mathbf{v} = -1.3066\hat{\mathbf{i}} + 2.3898\hat{\mathbf{j}} + 2.5654\hat{\mathbf{k}}\right)$$

Problem 4.7

$$\{\mathbf{V}\}_u = [\mathbf{Q}]_{xu}\{\mathbf{V}\}_x = \begin{bmatrix} 0.267\,26 & 0.534\,52 & 0.801\,78 \\ -0.443\,76 & 0.806\,84 & -0.389\,97 \\ -0.855\,36 & -0.251\,58 & 0.452\,84 \end{bmatrix} \begin{Bmatrix} -50 \\ 100 \\ 75 \end{Bmatrix}$$

$$\{\mathbf{V}\}_u = \underline{\begin{Bmatrix} 100.22 \\ 73.624 \\ 51.573 \end{Bmatrix}} \quad (\mathbf{V} = 100.22\hat{\mathbf{u}} + 73.624\hat{\mathbf{v}} + 51.573\hat{\mathbf{w}})$$

Problem 4.8

$$[\mathbf{R}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 40° & \sin 40° \\ 0 & -\sin 40° & \cos 40° \end{bmatrix} \qquad [\mathbf{R}_2] = \begin{bmatrix} \cos 25° & 0 & -\sin 25° \\ 0 & 1 & 0 \\ \sin 25° & 0 & \cos 25° \end{bmatrix}$$

$$[\mathbf{Q}] = [\mathbf{R}_2][\mathbf{R}_1] = \begin{bmatrix} \cos 25° & \sin 40° \sin 25° & -\cos 40° \sin 25° \\ 0 & \cos 40° & \sin 40° \\ \sin 25° & -\sin 40° \cos 25° & \cos 40° \cos 25° \end{bmatrix}$$

$$[\mathbf{Q}] = \underline{\begin{bmatrix} 0.906\,31 & 0.271\,65 & -0.323\,74 \\ 0 & 0.766\,04 & 0.642\,79 \\ 0.422\,62 & -0.582\,56 & 0.694\,27 \end{bmatrix}}$$

Problem 4.9 $\mathbf{r}_o = -5102\hat{\mathbf{I}} - 8228\hat{\mathbf{J}} - 2106\hat{\mathbf{K}}$ (km) $\mathbf{v}_o = -4.348\hat{\mathbf{I}} + 3.478\hat{\mathbf{J}} - 2.846\hat{\mathbf{K}}$ (km/s)

Method 1

Use *Algorithm 3.4* (MATLAB M-function rv_from_r0v0 in Appendix D.7):

$$(1a)\quad r_o = 9907.6 \text{ km} \qquad v_o = 6.2531 \text{ km/s}$$
$$(1b)\quad v_{ro} = -0.044\,678 \text{ km/s}$$
$$(1c)\quad \alpha = 103.77\times 10^{-6} \text{ km}^{-1}$$
$$(2)\quad \chi = 195 \text{ km}^{1/2}$$
$$(3)\quad f = -0.365\,39 \qquad g = 1394.4 \text{ s}^{-1}$$
$$(4)\quad \mathbf{r} = (-0.36\,539)\left(-5102\hat{\mathbf{I}} - 8228\hat{\mathbf{J}} - 2106\hat{\mathbf{K}}\right) + 1394.4\left(-4.348\hat{\mathbf{I}} + 3.478\hat{\mathbf{J}} - 2.846\hat{\mathbf{K}}\right)$$
$$= \underline{-4198.4\hat{\mathbf{I}} + 7856.1\hat{\mathbf{J}} - 3199.2\hat{\mathbf{K}} \text{ (km)}}$$
$$(5)\quad \dot{f} = -6.0467\times 10^{-4} \text{ s}^{-1} \qquad \dot{g} = -0.429\,31$$
$$(6)\quad \mathbf{v} = \left(-6.0467\times 10^{-4}\right)\left(-5102\hat{\mathbf{I}} - 8228\hat{\mathbf{J}} - 2106\hat{\mathbf{K}}\right) + (-0.429\,31)\left(-4.348\hat{\mathbf{I}} + 3.478\hat{\mathbf{J}} - 2.846\hat{\mathbf{K}}\right)$$
$$= \underline{4.9517\hat{\mathbf{I}} + 3.4821\hat{\mathbf{J}} + 2.4946\hat{\mathbf{K}} \text{ (km/s)}}$$

Method 2

Compute the orbital elements using *Algorithm 4.1* (MATLAB M-function `coe_from_sv` in Appendix D.8):

$$(1)\quad r = \|\mathbf{r}\| = 9907.6 \text{ km}$$
$$(2)\quad v = \|\mathbf{v}\| = 6.2531 \text{ km/s}$$
$$(3)\quad v_r = \frac{\mathbf{r}\cdot\mathbf{v}}{r} = -0.044\,678 \text{ km/s} \quad (<0)$$
$$(4)\quad \mathbf{h} = \mathbf{r}\times\mathbf{v} = 30\,738\hat{\mathbf{I}} - 5367.8\hat{\mathbf{J}} - 53\,520\hat{\mathbf{K}} \left(\text{km}^2/\text{s}\right)$$
$$(5)\quad h = \|\mathbf{h}\| = \underline{61\,952 \text{ km}^2/\text{s}}$$
$$(6)\quad i = \cos^{-1}\left(\frac{h_Z}{h}\right) = \cos^{-1}\left(\frac{-53\,520}{61\,952}\right) = \underline{149.76^\circ}$$
$$(7)\quad \mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = 5367.8\hat{\mathbf{I}} + 30\,738\hat{\mathbf{J}} \left(\text{km}^2/\text{s}\right) \quad (N_Y > 0)$$
$$(8)\quad N = \|\mathbf{N}\| = 31\,203 \text{ km}^2/\text{s}$$
$$(9)\quad \Omega = \cos^{-1}\left(\frac{N_X}{N}\right) = \cos^{-1}\left(\frac{5367.8}{31\,203}\right) = \underline{80.094^\circ}$$
$$(10)\quad \mathbf{e} = \frac{1}{398\,600}\Big[\left(5.7415^2 - 398\,600/9907.6\right)\left(-5102\hat{\mathbf{I}} - 8228\hat{\mathbf{J}} - 2106\hat{\mathbf{K}}\right)$$
$$-(9907.6)(-0.044\,678)\left(-4.348\hat{\mathbf{I}} + 3.478\hat{\mathbf{J}} - 2.846\hat{\mathbf{K}}\right)\Big]$$
$$= 0.009\,638\,51\hat{\mathbf{I}} + 0.027\,193\hat{\mathbf{J}} + 0.002\,808\,3\hat{\mathbf{K}} \quad (e_Z > 0)$$
$$(11)\quad e = \|\mathbf{e}\| = \underline{0.028\,987}$$
$$(12)\quad \omega = \cos^{-1}\left(\frac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) = \underline{11.09^\circ}$$
$$(13)\quad \theta = 360^\circ - \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = 360^\circ - 166.14^\circ = \underline{193.86^\circ}$$
$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = 9414.9 \text{ s}$$

Determine the time since perigee passage at true anomaly $\theta_o = 193.86^\circ$:

$$\tan\frac{E_0}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_0}{2} = \sqrt{\frac{1-0.028\,987}{1+0.028\,987}}\tan\frac{193.86°}{2} \Rightarrow E_0 = -2.8926 \text{ rad}$$

$$M_0 = E_0 - e\sin E_0 = -2.8926 - 0.028\,987\cdot\sin(-2.8926) = -2.8855 \text{ rad}$$

$$t_0 = \frac{M_0}{2\pi}T = \frac{-2.8855}{2\pi}9414.9 = -4323.7 \text{ s} \quad (\text{minus means time } \textit{until} \text{ perigee passage})$$

Update the true anomaly of the spacecraft. $t = t_0 + 50\cdot 60 = -1323.7$ s.

$$M = 2\pi\frac{t}{T} = 2\pi\frac{-1323.7}{9414.9} = -0.883\,41 \text{ rad}$$

$$E - e\sin E = M$$

$$E - 0.028\,987\sin E = -0.883\,41 \quad \Rightarrow E = -0.90623 \text{ rad (Algorithm 3.1)}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \sqrt{\frac{1+0.028\,987}{1-0.028\,987}}\tan\frac{-0.90623}{2} \Rightarrow \theta = -53.242°$$

Algorithm 4.2 (MATLAB M-function `sv_from_coe` in Appendix D.9):

(1) $$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{61\,952^2}{398\,600}\frac{1}{1+0.028\,987\cos(-53.242°)}\begin{Bmatrix}\cos(-53.242°)\\ \sin(-53.242°)\\ 0\end{Bmatrix}$$

$$= \begin{Bmatrix}5663.9\\ -7582.8\\ 0\end{Bmatrix} (\text{km})$$

(2) $$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398\,600}{61\,952}\begin{Bmatrix}-\sin(-53.242°)\\ 0.028\,987+\cos(-53.242°)\\ 0\end{Bmatrix} = \begin{Bmatrix}5.1548\\ 4.0368\\ 0\end{Bmatrix} (\text{km/s})$$

(3) $$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i\\ \sin i\sin\omega & \sin i\cos\omega & \cos i\end{bmatrix}$$

$$= \begin{bmatrix}0.33251 & 0.80204 & 0.49616\\ 0.93811 & -0.33532 & -0.086644\\ 0.09688 & 0.49426 & -0.8639\end{bmatrix}$$

(4) $$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix}-4198.4\\ 7856.1\\ -3199.2\end{Bmatrix} (\text{km}) \quad \textit{or} \quad \underline{\mathbf{r} = -4198.4\hat{\mathbf{I}} + 7856.1\hat{\mathbf{J}} - 3199.2\hat{\mathbf{K}} \ (\text{km})}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix}4.9517\\ 3.4821\\ 2.4946\end{Bmatrix} (\text{km/s}) \quad \textit{or} \quad \underline{\mathbf{v} = 4.9517\hat{\mathbf{I}} + 3.4821\hat{\mathbf{J}} + 2.4946\hat{\mathbf{K}} \ (\text{km/s})}$$

Problem 4.10 $e = 1.5 \quad \Omega = 130° \quad i = 35° \quad \omega = 115° \quad \theta = 0° \quad r_{\text{perigee}} = 6678$ km

$$h = \sqrt{\mu(1+e)r_{\text{perigee}}} = \sqrt{398\,600\cdot(1+1.5)\cdot 6678} = 81\,576 \text{ km}^2/\text{s}$$

Algorithm 4.2 (MATLAB M-function `sv_from_coe` in Appendix D.9):

$$(1)\ \{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{81\,576^2}{398\,600}\frac{1}{1+1.5\cos(0)}\begin{Bmatrix}\cos(0)\\ \sin(0)\\ 0\end{Bmatrix} = \begin{Bmatrix}6678\\ 0\\ 0\end{Bmatrix}\ (\text{km})$$

$$\underline{\mathbf{r} = 6678\hat{\mathbf{p}}\ (\text{km})}$$

$$(2)\ \{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398\,600}{81\,576}\begin{Bmatrix}-\sin(0)\\ 1.5+\cos(0)\\ 0\end{Bmatrix} = \begin{Bmatrix}0\\ 12.216\\ 0\end{Bmatrix}\ (\text{km/s})$$

$$\underline{\mathbf{v} = 12.216\hat{\mathbf{q}}\ (\text{km/s})}$$

$$(3)\quad [\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i\\ \sin i\sin\omega & \sin i\cos\omega & \cos i\end{bmatrix}$$

$$= \begin{bmatrix}-0.297\,06 & 0.847\,76 & 0.439\,39\\ -0.800\,95 & -0.471\,75 & 0.368\,69\\ 0.519\,84 & -0.2424 & 0.819\,15\end{bmatrix}$$

$$(4)\ \{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix}-1983.8\\ -5348.8\\ 3471.5\end{Bmatrix}\ (\text{km}) \quad \textit{or} \quad \underline{\mathbf{r} = -1983.8\hat{\mathbf{I}} - 5348.8\hat{\mathbf{J}} + 3471.5\hat{\mathbf{K}}\ (\text{km})}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix}10.356\\ -5.7627\\ -2.9611\end{Bmatrix}\ (\text{km/s}) \quad \textit{or} \quad \underline{\mathbf{v} = 10.356\hat{\mathbf{I}} - 5.7627\hat{\mathbf{J}} - 2.9611\hat{\mathbf{K}}\ (\text{km/s})}$$

Problem 4.11 $h = 81\,576\ \text{km}^2/\text{s}$ $e = 1.5$ $\Omega = 130°$ $i = 35°$ $\omega = 115°$.

$t = 7200\ \text{s}$

$$M_h = \frac{\mu^2}{h^3}\left(e^2-1\right)^{3/2}t = \frac{398\,600^2}{81\,576^3}\left(1.5^2-1\right)^{3/2}\cdot 7200 = 2.945$$

$$e\sinh F - F = M_h$$

$$1.5\sinh F - F = 2.945 \Rightarrow F = 1.886\ (\textit{Algorithm}\ 3.2)$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{1.5+1}{1.5-1}}\tan\frac{1.886}{2} \Rightarrow \theta = 117.47°$$

Algorithm 4.2 (MATLAB M-function `sv_from_coe` in Appendix D.9):

$$(1)\ \{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{81\,576^2}{398\,600}\frac{1}{1+1.5\cos(117.47°)}\begin{Bmatrix}\cos(117.47°)\\ \sin(117.47°)\\ 0\end{Bmatrix} = \begin{Bmatrix}-25\,007\\ 48\,093\\ 0\end{Bmatrix}\ (\text{km})$$

$$\underline{\mathbf{r} = -25\,007\hat{\mathbf{p}} + 48\,093\hat{\mathbf{q}}\ (\text{km})}$$

$$(2)\ \{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398\,600}{81\,576}\begin{Bmatrix}-\sin(117.47°)\\ 1.5+\cos(117.47°)\\ 0\end{Bmatrix} = \begin{Bmatrix}-4.3352\\ 5.0752\\ 0\end{Bmatrix}\ (\text{km/s})$$

$$\underline{\mathbf{v} = -4.3352\hat{\mathbf{p}} + 5.0752\hat{\mathbf{q}}\ (\text{km/s})}$$

$$(3)\quad [\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -0.29706 & 0.84776 & 0.43939 \\ -0.80095 & -0.47175 & 0.36869 \\ 0.51984 & -0.2424 & 0.81915 \end{bmatrix} \quad \text{(Problem 4.10)}$$

$$(4)\quad \{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} 48200 \\ -2658 \\ -24658 \end{Bmatrix} \text{ (km)} \quad \textit{or} \quad \underline{\mathbf{r} = 48200\hat{\mathbf{I}} - 2658\hat{\mathbf{J}} - 24658\hat{\mathbf{K}} \text{ (km)}}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} 5.5903 \\ 1.0781 \\ -3.4838 \end{Bmatrix} \text{ (km/s)} \quad \textit{or} \quad \underline{\mathbf{v} = 5.5903\hat{\mathbf{I}} + 1.0781\hat{\mathbf{J}} - 3.4838\hat{\mathbf{K}} \text{ (km/s)}}$$

Problem 4.12 $\mathbf{r}_o = 6472.7\hat{\mathbf{I}} - 7470.8\hat{\mathbf{J}} - 2469.8\hat{\mathbf{K}}$ (km) $\mathbf{v}_o = 3.9914\hat{\mathbf{I}} + 2.7916\hat{\mathbf{J}} - 3.2948\hat{\mathbf{K}}$ $\left(\text{km/s}^2\right)$

Method 1

Use *Algorithm 3.4* (MATLAB M-function rv_from_r0v0 in Appendix D.7):

$(1a)$ $r_o = 10189$ km $\quad v_o = 5.805$ km/s

$(1b)$ $v_{ro} = 1.2874$ km/s

$(1c)$ $\alpha = 109.54 \times 10^{-6}$ km^{-1}

(2) $\chi = 171.31$ km$^{1/2}$

(3) $f = -0.093379 \quad g = 1870.6$ s^{-1}

$$(4)\quad \mathbf{r} = (-0.093379)\left(6472.7\hat{\mathbf{I}} - 7470.8\hat{\mathbf{J}} - 2469.8\hat{\mathbf{K}}\right) + 1870.6\left(3.9914\hat{\mathbf{I}} + 2.7916\hat{\mathbf{J}} - 3.2948\hat{\mathbf{K}}\right)$$
$$= \underline{6861.9\hat{\mathbf{I}} + 5919.6\hat{\mathbf{J}} - 5932.7\hat{\mathbf{K}} \text{ (km)}}$$

(5) $\dot{f} = -5.3316 \times 10^{-4}$ s^{-1} $\quad \dot{g} = -0.028475$

$$(6)\quad \mathbf{v} = \left(-5.3316 \times 10^{-4}\right)\left(6472.7\hat{\mathbf{I}} - 7470.8\hat{\mathbf{J}} - 2469.8\hat{\mathbf{K}}\right) + (-0.028475)\left(3.9914\hat{\mathbf{I}} + 2.7916\hat{\mathbf{J}} - 3.2948\hat{\mathbf{K}}\right)$$
$$= \underline{-3.5647\hat{\mathbf{I}} + 3.9037\hat{\mathbf{J}} + 1.4106\hat{\mathbf{K}} \text{ (km/s)}}$$

Method 2

From Problem 4.3 $h = 58461$ km^2/s, $e = 0.2465$, $i = 35°$, $\Omega = 110°$, $\omega = 74.996°$, $\theta = 130°$.

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3 = \frac{2\pi}{398600^2}\left(\frac{58461}{\sqrt{1-0.2465^2}}\right)^3 = 8680.3 \text{ s}$$

$$\tan\frac{E_o}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_o}{2} = \sqrt{\frac{1-0.2465}{1+0.2465}}\tan\frac{130°}{2} \Rightarrow E_o = 2.0612 \text{ rad}$$

$$M_o = E_o - e\sin E_o = 2.0612 - 0.2465\cdot\sin(2.0612) = 1.8437 \text{ rad}$$

$$t_o = \frac{M_o}{2\pi}T = \frac{1.8437}{2\pi}8680.3 = 2547.1 \text{ s} \quad \text{(minus means time \textit{until} perigee passage)}$$

Update the true anomaly: $t = t_o + 50\cdot 60 = 5547.1$ s

$$M = 2\pi\frac{t}{T} = 2\pi\frac{5547.1}{8680.3} = 4.0153 \text{ rad}$$

$$E - e\sin E = M$$

$$E - 0.2465\sin E = 4.0153 \quad \Rightarrow E = 3.8541 \text{ rad (Algorithm 3.1)}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \sqrt{\frac{1+0.2465}{1-0.2465}}\tan\frac{3.8541}{2} \quad \Rightarrow \quad \theta = -147.73°$$

Algorithm 4.2 (MATLAB M-function `sv_from_coe` in Appendix D.9):

$$(1) \quad \{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{58\,461^2}{398\,600}\frac{1}{1+0.2465\cos(-147.73°)}\begin{Bmatrix}\cos(-147.73°)\\ \sin(-147.73°)\\ 0\end{Bmatrix}$$

$$= \begin{Bmatrix}-9158.1\\ -5783.8\\ 0\end{Bmatrix} \text{ (km)}$$

$$(2) \quad \{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398\,600}{8\,461}\begin{Bmatrix}-\sin(-147.73°)\\ 0.2465+\cos(-147.73°)\\ 0\end{Bmatrix} = \begin{Bmatrix}3.6408\\ -4.0841\\ 0\end{Bmatrix} \text{ (km/s)}$$

$$(3) \quad [\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i\\ \sin i\sin\omega & \sin i\cos\omega & \cos i\end{bmatrix}$$

$$= \begin{bmatrix}-0.8321 & 0.13078 & 0.53899\\ -0.026991 & -0.9802 & 0.19617\\ 0.55397 & 0.14869 & 0.81915\end{bmatrix}$$

$$(4) \quad \{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix}6864\\ 5916.5\\ -5933.3\end{Bmatrix} \text{ (km)} \quad \textit{or} \quad \underline{\mathbf{r} = 6864\hat{\mathbf{I}} + 5916.5\hat{\mathbf{J}} - 5933.3\hat{\mathbf{K}} \text{ (km)}}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix}-3.5636\\ 3.905\\ 1.4096\end{Bmatrix} \text{ (km/s)} \quad \textit{or} \quad \underline{\mathbf{v} = -3.5636\hat{\mathbf{I}} + 3.905\hat{\mathbf{J}} + 1.4096\hat{\mathbf{K}} \text{ (km/s)}}$$

Problem 4.13 $e = 1.2 \quad \Omega = 75° \quad i = 50° \quad \omega = 80° \quad \theta = 0° \quad r_{perigee} = 6578 \text{ km}$

$$h = \sqrt{\mu(1+e)r_{perigee}} = \sqrt{398\,600\cdot(1+1.2)\cdot 6578} = 79\,950 \text{ km}^2/\text{s}$$

Algorithm 4.2 (MATLAB M-function `sv_from_coe` in Appendix D.9):

$$(1) \quad \{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{79\,950^2}{398\,600}\frac{1}{1+1.2\cos(0)}\begin{Bmatrix}\cos(0)\\ \sin(0)\\ 0\end{Bmatrix} = \begin{Bmatrix}6578\\ 0\\ 0\end{Bmatrix} \text{ (km)}$$

$$\underline{\mathbf{r} = 6578\hat{\mathbf{p}} \text{ (km)}}$$

$$(2) \quad \{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398\,600}{79\,950}\begin{Bmatrix}-\sin(0)\\ 1.2+\cos(0)\\ 0\end{Bmatrix} = \begin{Bmatrix}0\\ 11.546\\ 0\end{Bmatrix} \text{ (km/s)}$$

$$\underline{\mathbf{v} = 11.546\hat{\mathbf{q}} \text{ (km/s)}}$$

$$(3)\quad [\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix}$$

$$= \begin{bmatrix} -0.56\,561 & -0.3627 & 0.739\,94 \\ 0.331\,57 & -0.92236 & -0.198\,27 \\ 0.754\,41 & 0.133\,02 & 0.642\,79 \end{bmatrix}$$

$$(4)\quad \{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} -3726.5 \\ 2181.1 \\ 4962.5 \end{Bmatrix} \text{ (km)} \quad \textit{or} \quad \underline{\mathbf{r} = -3726.5\hat{\mathbf{I}} + 2181.1\hat{\mathbf{J}} + 4962.5\hat{\mathbf{K}} \text{ (km)}}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} -4.1878 \\ -10.65 \\ 1.5359 \end{Bmatrix} \text{ (km/s)} \quad \textit{or} \quad \underline{\mathbf{v} = -4.1878\hat{\mathbf{I}} + 10.65\hat{\mathbf{J}} + 1.5359\hat{\mathbf{K}} \text{ (km/s)}}$$

Problem 4.14 $h = 75950 \text{ km}^2/\text{s} \quad e = 1.2 \quad \Omega = 130° \quad i = 50° \quad \omega = 80°$

$$t = 7200 \text{ s}$$

$$M_h = \frac{\mu^2}{h^3}\left(e^2 - 1\right)^{3/2} t = \frac{398\,600^2}{75950^3}\left(1.2^2 - 1\right)^{3/2} \cdot 7200 = 0.762\,09$$

$$e\sinh F - F = M_h$$

$$1.2\sinh F - F = 0.762\,09 \;\Rightarrow\; F = 1.3174 \;\left(\textit{Algorithm } 3.2\right)$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} = \sqrt{\frac{1.2+1}{1.2-1}}\tan\frac{0.762\,09}{2} \Rightarrow \theta = 124.86°$$

Algorithm 4.2 (MATLAB M-function `sv_from_coe` in Appendix D.9):

$$(1)\quad \{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} = \frac{75950^2}{398\,600}\frac{1}{1+1.2\cos(124.86°)}\begin{Bmatrix} \cos(124.86°) \\ \sin(124.86°) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -26\,336 \\ 37\,806 \\ 0 \end{Bmatrix} \text{ (km)}$$

$$\underline{\mathbf{r} = -26\,336\hat{\mathbf{p}} + 37\,806\hat{\mathbf{q}} \text{ (km)}}$$

$$(2)\quad \{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix} -\sin\theta \\ e+\cos\theta \\ 0 \end{Bmatrix} = \frac{398\,600}{81\,576}\begin{Bmatrix} -\sin(124.86°) \\ 1.5+\cos(124.86°) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -4.3064 \\ 3.298 \\ 0 \end{Bmatrix} \text{ (km/s)}$$

$$\underline{\mathbf{v} = -4.3064\hat{\mathbf{p}} + 3.298\hat{\mathbf{q}} \text{ (km/s)}}$$

$$(3)\quad [\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -0.56\,561 & -0.3627 & 0.739\,94 \\ 0.331\,57 & -0.92236 & -0.198\,27 \\ 0.754\,41 & 0.133\,02 & 0.642\,79 \end{bmatrix} \quad \text{(Problem 4.13)}$$

$$(4)\quad \{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} 1207.2 \\ -43\,603 \\ -14\,839 \end{Bmatrix} \text{ (km)} \quad \textit{or} \quad \underline{\mathbf{r} = 1207.2\hat{\mathbf{I}} - 43\,603\hat{\mathbf{J}} - 14\,839\hat{\mathbf{K}} \text{ (km)}}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} 1.2434 \\ -4.4698 \\ -2.8100 \end{Bmatrix} \text{ (km/s)} \quad \textit{or} \quad \underline{\mathbf{v} = 1.2434\hat{\mathbf{I}} - 4.4698\hat{\mathbf{J}} - 2.8100\hat{\mathbf{K}} \text{ (km/s)}}$$

Problem 4.15 $h = 75000 \text{ km}^2/\text{s} \quad e = 0.7 \quad \theta = 25°$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix} -\sin\theta \\ e + \cos\theta \\ 0 \end{Bmatrix} = \frac{398\,600}{75000}\begin{Bmatrix} -\sin(25°) \\ 0.7 + \cos(25°) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.2461 \\ 8.537 \\ 0 \end{Bmatrix} \text{ (km/s)}$$

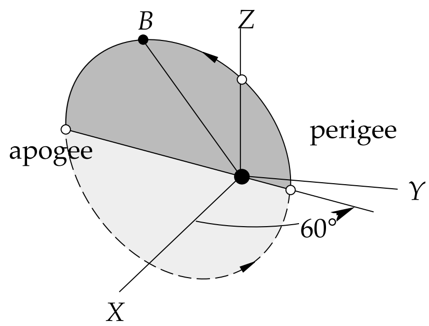
$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{bmatrix} -0.83204 & 0.02741 & 0.55403 \\ -0.13114 & -0.98019 & -0.14845 \\ 0.53899 & -0.19617 & 0.81915 \end{bmatrix}\begin{Bmatrix} -2.2461 \\ 8.537 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2.1028 \\ -8.0733 \\ -2.8853 \end{Bmatrix} \text{ (km/s)}$$

or

$$\underline{\mathbf{v} = 2.1028\hat{\mathbf{I}} - 8.0733\hat{\mathbf{J}} - 2.8853\hat{\mathbf{K}} \text{ (km/s)}}$$

Problem 4.16

$\Omega = 60° \quad \omega = 0 \quad i = 90°$



$$\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} -3.208 \\ -0.8288 \\ 0 \end{Bmatrix} \text{ (km/s)}$$

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0.86603 \\ 0.86603 & 0 & -0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} -1.604 \\ -2.7782 \\ -0.8288 \end{Bmatrix} \text{ (km/s)}$$

or

$$\underline{\mathbf{v} = -1.604\hat{\mathbf{I}} - 2.7782\hat{\mathbf{J}} - 0.8288\hat{\mathbf{K}} \text{ (km/s)}}$$

Problem 4.17 $a = 7016 \text{ km} \quad e = 0.05 \quad i = 45° \quad \Omega = 0 \quad \omega = 20° \quad \theta = 10°$

$$\{\mathbf{r}\}_{\bar{x}} = \frac{a(1-e^2)}{1 + e\cos\theta}\begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} = \frac{7016(1 - 05^2)}{1 + 0.05\cos 10°}\begin{Bmatrix} \cos 10° \\ \sin 10° \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6568.7 \\ 1158.2 \\ 0 \end{Bmatrix} \text{ (km)}$$

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix}$$

$$= \begin{bmatrix} 0.93969 & -0.34202 & 0 \\ 0.24184 & 0.66446 & -0.70711 \\ 0.24184 & 0.66446 & 0.70711 \end{bmatrix}$$

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} 5776.4 \\ 2358.2 \\ 2358.2 \end{Bmatrix} \text{ (km)}$$

or

$$\underline{\mathbf{r} = 5776.4\hat{\mathbf{I}} + 2358.2\hat{\mathbf{J}} + 2358.2\hat{\mathbf{K}} \text{ (km)}}$$

Problem 4.18

$$\dot{\Omega} = -\frac{3}{2}\frac{\sqrt{\mu}J_2R_{\text{earth}}^{\ 2}}{\left(1-e^2\right)^2 a^{7/2}}\cos i$$

$$a = \frac{1}{2}\left(r_{\text{perigee}} + r_{\text{apogee}}\right) = \frac{1}{2}(6878 + 7378) = 7128 \text{ km}$$

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{7378 - 6878}{7378 + 6878} = 0.035073$$

$$\dot{\Omega} = \frac{2\pi}{365.26\cdot 24\cdot 3600} = 1.991\times 10^{-7} \text{ rad/s}$$

$$1.991\times 10^{-7} = -\frac{3}{2}\frac{\sqrt{398600}\cdot 0.0010826\cdot 6378^2}{\left(1-0.035073^2\right)^2 7128^{7/2}}\cos i$$

$$1.991\times 10^{-7} = -1.367\times 10^{-6}\cos i \Rightarrow \underline{i = 98.372°}$$

Problem 4.19

$$T = \frac{2\pi}{\sqrt{\mu}}r^{3/2} = \frac{2\pi}{\sqrt{398600}}(6378 + 180)^{3/2} = 5285.3 \text{ s}$$

$$\dot{\Omega} = -\frac{3}{2}\frac{\sqrt{\mu}J_2R_{\text{earth}}^{\ 2}}{r^{7/2}}\cos i = -\frac{3}{2}\frac{\sqrt{398600}\cdot 0.0010826\cdot 6378^2}{(6378+180)^{7/2}}\cos 30° = -1.5814\times 10^{-6} \text{ rad/s}$$

$$\omega_{\text{earth}} = \frac{2\pi + \dfrac{2\pi}{365.26}}{24\cdot 3600} = 7.2921\times 10^{-5} \text{ rad/s}$$

Change in east longitude of the ascending node after 1 orbit of the satellite:

$$\Delta\lambda = \left(\omega_{\text{earth}} - \dot{\Omega}\right)T = \left[7.2921\times 10^{-5} - \left(-1.5814\times 10^{-6}\right)\right]\cdot 5285.3 = 0.39377 \text{ rad}$$

Spacing s,

$$s = R_{\text{earth}}\Delta\lambda = 6378\cdot 0.39377 = \underline{2511.4 \text{ km}}$$

Problem 4.20

The change in east longitude λ of the ascending node of a satellite after n_s orbits is

$$\Delta\lambda = \left(\omega_{\text{earth}} - \dot{\Omega}\right)n_s T$$

If $\Delta\lambda$ is an integral multiple n_e of earth rotations (2π) then the ground track will close on itself,

$$2\pi n_e = \left(\omega_{\text{earth}} - \dot{\Omega}\right)n_s T$$

Let $\nu = n_s/n_e$. Then $2\pi = \left(\omega_{\text{earth}} - \dot{\Omega}\right)\nu T$, or

$$T = \frac{1}{\nu}\frac{2\pi}{\omega_{\text{earth}} - \dot{\Omega}} \tag{1}$$

But $T = 2\pi r^{3/2}/\sqrt{\mu}$, where r is the radius of the orbit. Thus

$$\frac{2\pi}{\sqrt{\mu}}r^{3/2} = \frac{1}{\nu}\frac{2\pi}{\omega_{\text{earth}} - \dot{\Omega}}$$

or

$$r = \left[\frac{\sqrt{\mu}}{\nu\left(\omega_{\text{earth}} - \dot{\Omega}\right)}\right]^{2/3} \tag{2}$$

For a circular orbit

$$\dot{\Omega} = -\frac{3}{2}\frac{\sqrt{\mu}J_2 R_{\text{earth}}{}^2}{r^{7/2}}\cos i$$

or

$$\cos i = -\frac{2}{3}\frac{r^{7/2}}{\sqrt{\mu}J_2 R_{\text{earth}}{}^2}\dot{\Omega}$$

Substituting (2) we get

$$\cos i = -\frac{2}{3}\frac{\left[\dfrac{\sqrt{\mu}}{\nu\left(\omega_{\text{earth}} - \dot{\Omega}\right)}\right]^{7/3}}{\sqrt{\mu}J_2 R_{\text{earth}}{}^2}\dot{\Omega} \tag{3}$$

Substituting

$$\omega_{\text{earth}} = 7.2921 \times 10^{-5} \text{ rad} \qquad \dot{\Omega} = 1.997 \times 10^{-7} \text{ rad/s}$$
$$J_2 = 0.0010826 \qquad R_{\text{earth}} = 6378 \text{ km} \qquad \mu = 398\,600 \text{ km}^3/\text{s}^2$$

into (1), (2) and (3) we get

$$i = \cos^{-1}\left(-\frac{73.948}{\nu^{7/3}}\right) \qquad T = \frac{86\,400}{\nu} \qquad z = \frac{42\,241}{\nu^{2/3}} - 6378$$

From these we obtain the following table of scenarios:

ν	z (km)	i (deg)	T (h)
17	11.042	95.711	1.4118
16	274.55	96.583	1.5000
15	567.03	97.658	1.6000
14	893.93	99.006	1.7143
13	1262.2	100.72	2.000
12	1681.0	102.96	2.1818
11	2162.6	105.95	2.4000
10	2722.6	110.07	2.6667
9	3384.8	116.03	3.0000
8	4182.3	125.29	3.4286

Problem 4.21

$$\dot{\omega} = -f\left(\frac{5}{2}\sin^2 i - 2\right)$$

$$f = -\frac{\dot{\omega}}{\frac{5}{2}\sin^2 i - 2} = -\frac{7}{\frac{5}{2}\sin^2 40° - 2} = 7.2384 \text{ deg/day}$$

$$\dot{\Omega} = -f\cos i = -7.2384 \cdot \cos 40° = \underline{-5.545 \text{ deg/day}}$$

Problem 4.22 From

$$\mathbf{r}_o = -2429.1\hat{\mathbf{I}} + 4555.1\hat{\mathbf{J}} + 4577.0\hat{\mathbf{K}} \text{ (km)} \quad \mathbf{v}_o = -4.7689\hat{\mathbf{I}} - 5.6113\hat{\mathbf{J}} + 3.0535\hat{\mathbf{K}} \text{ (km/s)}$$

we obtain the orbital elements by means of *Algorithm 4.1*:

$$h = 55\,000 \left(\text{km}^2/\text{s}\right) \quad e = 0.1 \quad \Omega_o = 70° \quad i = 50° \quad \omega_o = 60° \quad \theta_o = 0$$

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} = \frac{55\,000^2}{398\,600}\frac{1}{1-0.1^2} = 7665.8 \text{ km}$$

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398\,600}}7665.8^{3/2} = 6679.5 \text{ s}$$

The satellite is at perigee ($\theta_o = 0$) so $t_o = 0$. After 72 hours, $t_f = t_o + 72 \cdot 3600 = 259\,000$ s. $t_f/T = 38.805$, so t_f is in orbit 39. The time since perigee in orbit 39 is

$$t_{39} = (38.805 - 38)T = 5378.3 \text{ s}$$

$$\therefore M_{39} = \frac{2\pi}{T}t_{39} = \frac{2\pi}{6679.5}5378.3 = 5.0952 \text{ rad}$$

$$E_{39} - e\sin E_{39} = M_{39}$$

$$E_{39} - 0.1\sin E_{39} = 5.0952 \Rightarrow E_{39} = 4.9623 \text{ rad } (\textit{Algorithm 3.1})$$

$$\tan\frac{\theta_{39}}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_{39}}{2} = \sqrt{\frac{1+0.1}{1-0.1}}\tan\frac{4.9623}{2} \Rightarrow \theta_{39} = 278.68°$$

The perifocal state vector is

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{55\,000^2}{398\,600}\frac{1}{1+0.1\cos(278.68°)}\begin{Bmatrix}\cos(278.68°)\\ \sin(278.68°)\\ 0\end{Bmatrix} = \begin{Bmatrix}1128.9\\ -7390.5\\ 0\end{Bmatrix}\ (\text{km})$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398\,600}{55\,000}\begin{Bmatrix}-\sin(278.68°)\\ 0.1+\cos(278.68°)\\ 0\end{Bmatrix} = \begin{Bmatrix}7.1642\\ 1.8191\\ 0\end{Bmatrix}\ (\text{km/s})$$

Update Ω and ω:

$$\dot{\Omega} = -\frac{3}{2}\frac{\sqrt{\mu}J_2R_{\text{earth}}{}^2}{\left(1-e^2\right)^2 a^{7/2}}\cos i = -\frac{3}{2}\frac{\sqrt{3998\,600}\cdot 0.0010826\cdot 6378^2}{\left(1-0.1^2\right)^2 7665.8^{7/2}}\cos 50° = -1.0789\times 10^{-6}\cos 50°$$
$$= -6.9352\times 10^{-7}\ \text{rad/s} = -3.9736\times 10^{-5}\ \text{deg/s}$$
$$\therefore \Omega_{39} = \Omega_1 + \dot{\Omega}\Delta t = 70° - 3.9736\times 10^{-5}\cdot 259\,200 = 59.701°$$

$$\dot{\omega} = -\frac{3}{2}\frac{\sqrt{\mu}J_2R_{\text{earth}}{}^2}{\left(1-e^2\right)^2 a^{7/2}}\left(\frac{5}{2}\sin^2 i - 2\right) = -1.0789\times 10^{-6}\left(\frac{5}{2}\sin^2 50° - 2\right)$$
$$= 5.7599\times 10^{-7}\ \text{rad/s} = 3.2945\times 10^{-5}\ \text{deg/s}$$
$$\therefore \omega_{39} = \omega_1 + \dot{\omega}\Delta t = 60° + 3.2945\times 10^{-5}\cdot 259\,200 = 68.539°$$

Update the transformation matrix between perifocal and geocentric equatorial coordinates:

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix}\cos\Omega_{39}\cos\omega_{39} - \sin\Omega_{39}\sin\omega_{39}\cos i & -\cos\Omega_{39}\sin\omega_{39} - \sin\Omega_{39}\cos i\cos\omega_{39} & \sin\Omega_{39}\sin i\\ \sin\Omega_{39}\cos\omega_{39} + \cos\Omega_{39}\cos i\sin\omega_{39} & -\sin\Omega_{39}\sin\omega_{39} + \cos\Omega_{39}\cos i\cos\omega_{39} & -\cos\Omega_{39}\sin i\\ \sin i\sin\omega & \sin i\cos\omega_{39} & \cos i\end{bmatrix}$$
$$= \begin{bmatrix}-0.331\,92 & -0.672\,58 & 0.661\,41\\ 0.6177 & -0.684\,89 & -0.386\,48\\ 0.712\,94 & 0.280\,27 & 0.642\,78\end{bmatrix}$$

Compute the geocentric equatorial state vector at t_f:

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix}4596\\ 5759\\ -1266.5\end{Bmatrix}\ (\text{km}) \quad \textit{or} \quad \underline{\mathbf{r} = 4596\hat{\mathbf{I}} + 5759\hat{\mathbf{J}} - 1266.5\hat{\mathbf{K}}\ (\text{km})}$$

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix}-3.6014\\ 3.1794\\ 5.6174\end{Bmatrix}\ (\text{km/s}) \quad \textit{or} \quad \underline{\mathbf{v} = -3.6014\hat{\mathbf{I}} + 3.1794\hat{\mathbf{J}} + 5.6174\hat{\mathbf{K}}\ (\text{km/s})}$$

Problem 5.1

The following MATLAB script uses the given data to compute $\mathbf{v}_2$ by means of *Algorithm 5.1,* which is implemented as the M-function gibbs in Appendix D.10. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_01
% ~~~~~~~~~~~~
%
% This program uses Algorithm 5.1 (Gibbs method) to obtain the state
% vector from the three coplanar position vectors provided in
% Problem 5.1.
%
% mu         - gravitational parameter (km^3/s^2)
% r1, r2, r3 - three coplanar geocentric position vectors (km)
% ierr       - 0 if r1, r2, r3 are found to be coplanar
%              1 otherwise
% v2         - the velocity corresponding to r2 (km/s)
%
% User M-function required: gibbs
% ----------------------------------------------------------------------
clear
global mu
mu = 398600;

r1 = [5887 -3520 -1204];
r2 = [5572 -3457 -2376];
r3 = [5088 -3289 -3480];

%...Echo the input data to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.1: Gibbs Method\n')
fprintf('\n Input data:\n')
fprintf('\n  Gravitational parameter (km^3/s^2)  = %g\n', mu)
fprintf('\n  r1 (km) = [%g  %g  %g]', r1(1), r1(2), r1(3))
fprintf('\n  r2 (km) = [%g  %g  %g]', r2(1), r2(2), r2(3))
fprintf('\n  r3 (km) = [%g  %g  %g]', r3(1), r3(2), r3(3))
fprintf('\n\n');

%...Algorithm 5.1:
[v2, ierr] = gibbs(r1, r2, r3);

%...If the vectors r1, r2, r3, are not coplanar, abort:
if ierr == 1
    fprintf('\n  These vectors are not coplanar.\n\n')
    return
end

%...Output the results to the command window:
fprintf(' Solution:')
fprintf('\n');
fprintf('\n  v2 (km/s) = [%g  %g  %g]', v2(1), v2(2), v2(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.1: Gibbs Method

 Input data:

  Gravitational parameter (km^3/s^2)  = 398600
```

```
  r1 (km) = [5887  -3520  -1204]
  r2 (km) = [5572  -3457  -2376]
  r3 (km) = [5088  -3289  -3480]

 Solution:

  v2 (km/s) = [-2.50254  0.723248  -7.13125]
-----------------------------------------------------
```

$\underline{\mathbf{v}_2 = -2.5025\hat{\mathbf{I}} + 0.72325\hat{\mathbf{J}} - 7.1312\hat{\mathbf{K}}\ (\text{ km/s })}$

Problem 5.2

The following MATLAB script uses $\mathbf{r}_2$ and $\mathbf{v}_2$ from Problem 5.1 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_02
% ~~~~~~~~~~~~
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.1.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ----------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.2:
r = [    5572     -3457     -2376];
v = [-2.50254  0.723248  -7.13125];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.2: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                    = [%g  %g  %g]', ...
```

```
                                          r(1), r(2), r(3))
fprintf('\n v (km/s)                    = [%g  %g  %g]', ...
                                          v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                   = %g', T)
    fprintf('\n   Minutes                   = %g', T/60)
    fprintf('\n   Hours                     = %g', T/3600)
    fprintf('\n   Days                      = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.2: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                    = [5572  -3457  -2376]
 v (km/s)                  = [-2.50254  0.723248  -7.13125]

 Angular momentum (km^2/s) = 52948.9
 Eccentricity              = 0.0127382
 Right ascension (deg)     = 150.003
 Inclination (deg)         = 95.0071
 Argument of perigee (deg) = 151.688
 True anomaly (deg)        = 48.3093
 Semimajor axis (km):      = 7034.71
 Period:
   Seconds                 = 5871.93
   Minutes                 = 97.8655
   Hours                   = 1.63109
   Days                    = 0.0679622
-----------------------------------------------------
```

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{52\,949^2}{398\,600}\frac{1}{1+0.012738} = 6945.1 \text{ km}$$

$$z_{\text{perigee}} = 6945.1 - 6378 = \underline{567.11 \text{ km}}$$

Problem 5.3

As in Example 5.3, we set $\mathbf{r}_1 = r_1\hat{\mathbf{i}}$ and $\mathbf{r}_2 = r_2\left(\cos\Delta\theta\hat{\mathbf{i}} + \sin\Delta\theta\hat{\mathbf{j}}\right)$, where $r_1 = 6978$ km, $r_2 = 6678$ km and $\Delta\theta = 60°$. The following MATLAB script uses this data to compute $\mathbf{v}_1$ and $\mathbf{v}_2$ by means of *Algorithm 5.2,* which is implemented as the M-function `lambert` in Appendix D.11. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem 5_03a
```

```
% ~~~~~~~~~~~~~
%
% This program uses Algorithm 5.2 to solve Lambert's problem for the
% data provided in Problem 5.3.

% deg    - factor for converting between degrees and radians
% pi     - 3.1415926...
% mu     - gravitational parameter (km^3/s^2)
% r1, r2 - initial and final radii (km)
% dt     - time between r1 and r2 (s)
% dtheta - change in true anomaly during dt (degrees)
% R1, R2 - initial and final position vectors (km)
% string - = 'pro' if the orbit is prograde
%          = 'retro if the orbit is retrograde
% V1, V2 - initial and final velocity vectors (km/s)

% User M-function required: lambert
% -------------------------------------------------------------
clear
global  mu
mu     = 398600;       %km^3/s^2
deg    = pi/180;

r1     = 6378 + 600;   %km
r2     = 6378 + 300;   %km
dt     = 15*60;        %sec
dtheta = 60;           %degrees

R1 = [r1  0  0];
R2 = [r2*cos(dtheta*deg)  r2*sin(dtheta*deg)   0];

%...Algorithm 5.2:
string = 'pro';
[V1 V2] = lambert(R1, R2, dt, string);

%...Echo the input data and output results to the command window:
fprintf('\n-----------------------------------------------------')
fprintf('\n Problem 5.3: Lambert''s Problem\n')

fprintf('\n Input data:\n');
fprintf('\n   Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n   Radius 1 (km)                      = %g', r1)
fprintf('\n   Position vector R1 (km)            = [%g  %g  %g]\n',...
                                           R1(1), R1(2), R1(3))
fprintf('\n   Radius 2 (km)                      = %g', r2)
fprintf('\n   Position vector R2 (km)            = [%g  %g  %g]\n',...
                                           R2(1), R2(2), R2(3))
fprintf('\n   Elapsed time (s)                   = %g', dt)
fprintf('\n   Change in true anomaly (deg) = %g', dtheta)

fprintf('\n\n Solution:\n')
fprintf('\n  Velocity vector V1 (km/s) = [%g  %g  %g]',...
                                       V1(1), V1(2), V1(3))
fprintf('\n  Velocity vector V2 (km/s) = [%g  %g  %g]',...
                                       V2(1), V2(2), V2(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.3: Lambert's Problem

 Input data:
```

```
   Gravitational parameter (km^3/s^2) = 398600

   Radius 1 (km)                      = 6978
   Position vector R1 (km)            = [6978  0  0]

   Radius 2 (km)                      = 6678
   Position vector R2 (km)            = [3339  5783.32  0]

   Elapsed time (s)                   = 900
   Change in true anomaly (deg) = 60

 Solution:

  Velocity vector V1 (km/s) = [-0.544135  7.68498  0]
  Velocity vector V2 (km/s) = [-6.98129  3.96849  0]
-----------------------------------------------------
```

To find the perigee altitude, we need the orbital elements. The following MATLAB script uses $\mathbf{r}_1 = 6978\hat{\mathbf{i}}\ (\text{km})$ and $\mathbf{v}_1 = -0.544135\hat{\mathbf{i}} + 7.68498\hat{\mathbf{j}}\ (\text{km/s})$ from above to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_03b
% ~~~~~~~~~~~~
%
% This program employs Algorithm 4.1 to obtain the orbital
% elements from the state vector found from the solution
% of Lambert's problem using the data given in Problem 5.3.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ---------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.3:
r = [6978  0  0];
v = [-0.544135  7.68498  0];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
```

```
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.3: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                        = [%g  %g  %g]', ...
                                           r(1), r(2),  r(3))
fprintf('\n v (km/s)                      = [%g  %g  %g]', ...
                                           v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                     = %g', T)
    fprintf('\n   Minutes                     = %g', T/60)
    fprintf('\n   Hours                       = %g', T/3600)
    fprintf('\n   Days                        = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.3: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                    = [6978  0  0]
 v (km/s)                  = [-0.544135  7.68498  0]

 Angular momentum (km^2/s) = 53625.8
 Eccentricity              = 0.0806743
 Right ascension (deg)     = 0
 Inclination (deg)         = 0
 Argument of perigee (deg) = 0
 True anomaly (deg)        = 294.849
 Semimajor axis (km):      = 7261.83
 Period:
   Seconds                 = 6158.57
   Minutes                 = 102.643
   Hours                   = 1.71071
   Days                    = 0.0712798
-----------------------------------------------------
```

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{53\,626^2}{398\,600}\frac{1}{1+0.0806743} = 6676 \text{ km}$$

$$z_{\text{perigee}} = 6676 - 6378 = \underline{298 \text{ km}}$$

Problem 5.4 The following MATLAB script uses $\mathbf{r}_1$, $\mathbf{r}_2$ and Δt to compute $\mathbf{v}_1$ and $\mathbf{v}_2$ by means of *Algorithm 5.2,* which is implemented as the M-function `lambert` in Appendix D.11. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_04
% ~~~~~~~~~~~~
%
% This program uses Algorithm 5.2 to solve Lambert's problem for the
% data provided in Problem 5.4.
%
% mu     - gravitational parameter (km^3/s^2)
% r1, r2 - initial and final position vectors (km)
% dt     - time between r1 and r2 (s)
% string - = 'pro' if the orbit is prograde
%          = 'retro if the orbit is retrograde
% v1, v2 - initial and final velocity vectors (km/s)
% coe    - orbital elements [h e RA incl w TA a]
%
% User M-function required: lambert
% -----------------------------------------------------------

clear
global mu
deg = pi/180;

%...Data declaration for Problem 5.4:
mu     = 398600;
r1     = [-3600   3600   5100];
r2     = [-5500  -6240   -520];
dt     = 30*60;
string = 'pro';
%...

%...Algorithm 5.2:
[v1, v2] = lambert(r1, r2, dt, string);

%...Echo the input data and output the results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.4: Lambert''s Problem\n')
fprintf('\n Input data:\n');
fprintf('\n   Gravitational parameter (km^3/s^2) = %g\n', mu);
fprintf('\n   r1 (km)          = [%g  %g  %g]', ...
                                  r1(1), r1(2), r1(3))
fprintf('\n   r2 (km)          = [%g  %g  %g]', ...
                                  r2(1), r2(2), r2(3))
fprintf('\n   Elapsed time (s) = %g', dt);
fprintf('\n\n Solution:\n')

fprintf('\n   v1 (km/s)        = [%g  %g  %g]', ...
                                  v1(1), v1(2), v1(3))
fprintf('\n   v2 (km/s)        = [%g  %g  %g]', ...
                                  v2(1), v2(2), v2(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.4: Lambert's Problem

 Input data:

   Gravitational parameter (km^3/s^2) = 398600

   r1 (km)          = [-3600  3600  5100]
   r2 (km)          = [-5500  -6240  -520]
   Elapsed time (s) = 1800
```

```
 Solution:

   v1 (km/s)        = [-5.68521  -5.19833  0.348733]
   v2 (km/s)        = [3.42204  -3.24131  -4.71994]
-----------------------------------------------------
```

$\underline{\mathbf{v}_1 = -5.6852\hat{\mathbf{I}} - 5.1983\hat{\mathbf{J}} + 0.34873\hat{\mathbf{K}}\,(\text{km/s})}$ $\underline{\mathbf{v}_2 = 3.4220\hat{\mathbf{I}} - 3.2413\hat{\mathbf{J}} - 4.7199\hat{\mathbf{K}}\,(\text{km/s})}$

Problem 5.5 The following MATLAB script uses $\mathbf{r}_1 = -3600\hat{\mathbf{I}} + 3600\hat{\mathbf{J}} + 5100\hat{\mathbf{K}}\,(\text{km})$ and $\mathbf{v}_1 = -5.6852\hat{\mathbf{I}} - 5.1983\hat{\mathbf{J}} + 0.34873\hat{\mathbf{K}}\,(\text{km/s})$ from Problem 5.4 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_05
% ~~~~~~~~~~~~
%
% This program employs Algorithm 4.1 to obtain the orbital
% elements from the state vector found in Problem 5.4.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ----------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.5:
r = [   -3600      3600      5100];
v = [-5.68521  -5.19833  0.348733];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.5: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                   = [%g  %g  %g]', ...
                                      r(1), r(2),  r(3))
fprintf('\n v (km/s)                 = [%g  %g  %g]', ...
```

```
                                         v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                   = %g', T)
    fprintf('\n   Minutes                   = %g', T/60)
    fprintf('\n   Hours                     = %g', T/3600)
    fprintf('\n   Days                      = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.5: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                    = [-3600  3600  5100]
 v (km/s)                  = [-5.68521  -5.19833  0.348733]

 Angular momentum (km^2/s) = 55458
 Eccentricity              = 0.0982445
 Right ascension (deg)     = 45.0287
 Inclination (deg)         = 45.0497
 Argument of perigee (deg) = 46.034
 True anomaly (deg)        = 43.9458
 Semimajor axis (km):      = 7791.19
 Period:
   Seconds                 = 6844.1
   Minutes                 = 114.068
   Hours                   = 1.90114
   Days                    = 0.0792142
-----------------------------------------------------
```

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{55458^2}{398\,600}\frac{1}{1+0.098\,244} = 7025.7 \text{ km}$$

$$z_{\text{perigee}} = 7025.7 - 6378 = \underline{647.74 \text{ km}}$$

Problem 5.6 The following MATLAB script uses $\mathbf{r}_1$, $\mathbf{r}_2$ and Δt to compute $\mathbf{v}_1$ and $\mathbf{v}_2$ by means of *Algorithm 5.2,* which is implemented as the M-function `lambert` in Appendix D.11. The output to the MATLAB Command Window is listed afterwards.

(a)

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_06
% ~~~~~~~~~~~~
%
% This program uses Algorithm 5.2 to solve Lambert's problem for the
% data provided in Problem 5.6.
%
```

```
% mu     - gravitational parameter (km^3/s^2)
% r1, r2 - initial and final position vectors (km)
% dt     - time between r1 and r2 (s)
% string - = 'pro' if the orbit is prograde
%          = 'retro if the orbit is retrograde
% v1, v2 - initial and final velocity vectors (km/s)
% coe    - orbital elements [h e RA incl w TA a]
%
% User M-function required: lambert
% ------------------------------------------------------------

clear
global mu
deg = pi/180;

%...Data declaration for Problem 5.6:
mu     = 398600;
r1     = [ 5644  -2830  -4170];
r2     = [-2240   7320  -4980];
dt     = 20*60;
string = 'pro';
%...

%...Algorithm 5.2:
[v1, v2] = lambert(r1, r2, dt, string);

%...Echo the input data and output the results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.6: Lambert''s Problem\n')
fprintf('\n Input data:\n');
fprintf('\n   Gravitational parameter (km^3/s^2) = %g\n', mu);
fprintf('\n   r1 (km)          = [%g  %g  %g]', ...
                                 r1(1), r1(2), r1(3))
fprintf('\n   r2 (km)          = [%g  %g  %g]', ...
                                 r2(1), r2(2), r2(3))
fprintf('\n   Elapsed time (s) = %g', dt);
fprintf('\n\n Solution:\n')

fprintf('\n   v1 (km/s)        = [%g  %g  %g]', ...
                                 v1(1), v1(2), v1(3))
fprintf('\n   v2 (km/s)        = [%g  %g  %g]', ...
                                 v2(1), v2(2), v2(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.6: Lambert's Problem

 Input data:

   Gravitational parameter (km^3/s^2) = 398600

   r1 (km)          = [5644  -2830  -4170]
   r2 (km)          = [-2240  7320  -4980]
   Elapsed time (s) = 1200

 Solution:

   v1 (km/s)        = [-4.13223  9.01237  -4.3781]
   v2 (km/s)        = [-7.28524  6.31978  2.5272]
-----------------------------------------------------
```

$\underline{\mathbf{v}_1 = -4.1322\hat{\mathbf{I}} + 9.0124\hat{\mathbf{J}} - 4.3781\hat{\mathbf{K}}\,(\text{km/s})}$ $\underline{\mathbf{v}_2 = -7.2852\hat{\mathbf{I}} + 6.3198\hat{\mathbf{J}} + 2.5272\hat{\mathbf{K}}\,(\text{km/s})}$

Problem 5.7 The following MATLAB script uses $\mathbf{r}_1 = 5644\hat{\mathbf{I}} - 2830\hat{\mathbf{J}} - 4170\hat{\mathbf{K}}\,(\text{km})$ and $\mathbf{v}_1 = -4.1322\hat{\mathbf{I}} + 9.0124\hat{\mathbf{J}} - 4.3781\hat{\mathbf{K}}\,(\text{km/s})$ from Problem 5.4 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_07
% ~~~~~~~~~~~~
%
% This program employs Algorithm 4.1 to obtain the orbital
% elements from the state vector found in Problem 5.6.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ----------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.7:
r = [    5644    -2830    -4170];
v = [-4.13223  9.01237  -4.3781];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.7: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                          = [%g  %g  %g]', ...
                                            r(1), r(2), r(3))
fprintf('\n v (km/s)                        = [%g  %g  %g]', ...
                                            v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
```

```
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                  = %g', T)
    fprintf('\n   Minutes                  = %g', T/60)
    fprintf('\n   Hours                    = %g', T/3600)
    fprintf('\n   Days                     = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.7: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                       = [5644  -2830  -4170]
 v (km/s)                     = [-4.13223  9.01237  -4.3781]

 Angular momentum (km^2/s) = 76096.4
 Eccentricity              = 1.20053
 Right ascension (deg)     = 130.007
 Inclination (deg)         = 59.0184
 Argument of perigee (deg) = 259.98
 True anomaly (deg)        = 320.023
 Semimajor axis (km):      = -32922.3
-----------------------------------------------------
```

$$r_{\text{perigee}} = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{76\,096^2}{398\,600}\frac{1}{1+1.2005} = 6601.8 \text{ km}$$

$$z_{\text{perigee}} = 6601.8 - 6378 = \underline{223.82 \text{ km}}$$

Problem 5.8 The following MATLAB script uses the M-function `J0` in Appendix D.12 to compute the Julian day for the date given in part (a) of Problem 5.8. The output to the MATLAB Command Window is listed afterwards, as are the results for the dates (b) through (c).

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_08a
% ~~~~~~~~~~~~~
%
% This program computes J0 and the Julian day number using the data
% in Problem 5.8.
%
% year   - range: 1901 - 2099
% month  - range: 1 - 12
% day    - range: 1 - 31
% hour   - range: 0 - 23 (Universal Time)
% minute - rage: 0 - 60
% second - range: 0 - 60
% ut     - universal time (hr)
% j0     - Julian day number at 0 hr UT
% jd     - Julian day number at specified UT
%
% User M-function required: J0
% ---------------------------------------------------------------------
```

```
clear

%...Data declaration for Problem 5.8a:
year   = 1914;
month  = 8;
day    = 14;

hour   = 5;
minute = 30;
second = 00;
%...

ut = hour + minute/60 + second/3600;

%...Equation 5.46:
j0 = J0(year, month, day);

%...Equation 5.47:
jd = j0 + ut/24;

%...Echo the input data and output the results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Example 5.8a: Julian day calculation\n')
fprintf('\n Input data:\n');
fprintf('\n   Year            = %g',   year)
fprintf('\n   Month           = %g',   month)
fprintf('\n   Day             = %g',   day)
fprintf('\n   Hour            = %g',   hour)
fprintf('\n   Minute          = %g',   minute)
fprintf('\n   Second          = %g\n', second)

fprintf('\n Julian day number = %11.3f', jd);
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Example 5.8a: Julian day calculation

 Input data:

   Year            = 1914
   Month           = 8
   Day             = 14
   Hour            = 5
   Minute          = 30
   Second          = 0

 Julian day number = 2420358.729
-----------------------------------------------------
Example 5.8b: Julian day calculation

 Input data:

   Year            = 1946
   Month           = 4
   Day             = 18
   Hour            = 14
   Minute          = 0
   Second          = 0

 Julian day number = 2431929.083
-----------------------------------------------------
Example 5.8c: Julian day calculation
```

```
 Input data:

   Year           = 2010
   Month          = 9
   Day            = 1
   Hour           = 0
   Minute         = 0
   Second         = 0

 Julian day number = 2455440.500
-----------------------------------------------------
Example 5.8d: Julian day calculation

 Input data:

   Year           = 2007
   Month          = 10
   Day            = 16
   Hour           = 12
   Minute         = 0
   Second         = 0

 Julian day number = 2454390.000
-----------------------------------------------------
```

Problem 5.9 This is similar to Example 5.5. The MATLAB script listed in the solution to Problem 5.8 can be used to obtain the Julian day numbers.

Problem 5.10 The following MATLAB script uses *Algorithm 5.3*, which is implemented in MATLAB by the M-function `LST` in Appendix D.13, to compute the local sidereal time for the data given in part (a) of Problem 5.10. The output to the MATLAB Command Window is listed afterwards, as are the results for the data in (b) through (e).

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_10a
% ~~~~~~~~~~~~~
%
% This program uses Algorithm 5.3 to obtain the local sidereal
% time from the data provided in Problem 5.10.
%
% lst   - local sidereal time (degrees)
% EL    - east longitude of the site (west longitude is negative):
%            degrees (0 - 360)
%            minutes (0 - 60)
%            seconds (0 - 60)
% WL    - west longitude
% year  - range: 1901 - 2099
% month - range: 1 - 12
% day   - range: 1 - 31
% ut    - universal time
%            hour (0 - 23)
%            minute (0 - 60)
%            second (0 - 60)
%
% User M-function required: LST
% --------------------------------------------------------------------

clear

%...Data declaration for Problem 5.10a:
```

```
% East longitude:
degrees = 18;
minutes = 3;
seconds = 0;

% Date:
year    = 2008;
month   = 1;
day     = 1;

% Universal time:
hour    = 12;
minute  = 0;
second  = 0;

%...

%...Convert negative (west) longitude to east longitude:
if degrees < 0
    degrees = degrees + 360;
end

%...Express the longitudes as decimal numbers:
EL = degrees + minutes/60 + seconds/3600;
WL = 360 - EL;

%...Express universal time as a decimal number:
ut = hour + minute/60 + second/3600;

%...Algorithm 5.3:
lst = LST(year, month, day, ut, EL);

%...Echo the input data and output the results to the command window:
fprintf('---------------------------------------------------')
fprintf('\n Problem 5.10a: Local sidereal time calculation\n')
fprintf('\n Input data:\n');
fprintf('\n   Year                     = %g', year)
fprintf('\n   Month                    = %g', month)
fprintf('\n   Day                      = %g', day)
fprintf('\n   UT (hr)                  = %g', ut)
fprintf('\n   West Longitude (deg)     = %g', WL)
fprintf('\n   East Longitude (deg)     = %g', EL)
fprintf('\n\n');

fprintf(' Solution:')
fprintf('\n');
fprintf('\n  Local Sidereal Time (deg) = %g', lst)
fprintf('\n  Local Sidereal Time (hr)  = %g', lst/15)
fprintf('\n---------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.10a: Local sidereal time calculation

 Input data:

   Year                     = 2008
   Month                    = 1
   Day                      = 1
   UT (hr)                  = 12
   West Longitude (deg)     = 341.95
   East Longitude (deg)     = 18.05
```

```
Solution:

 Local Sidereal Time (deg) = 298.572
 Local Sidereal Time (hr)  = 19.9048
-----------------------------------------------------
Problem 5.10b: Local sidereal time calculation

 Input data:

   Year                       = 2007
   Month                      = 12
   Day                        = 21
   UT (hr)                    = 10
   West Longitude (deg)       = 215.033
   East Longitude (deg)       = 144.967

 Solution:

 Local Sidereal Time (deg) = 24.5646
 Local Sidereal Time (hr)  = 1.63764
-----------------------------------------------------
Problem 5.10c: Local sidereal time calculation

 Input data:

   Year                       = 2005
   Month                      = 7
   Day                        = 4
   UT (hr)                    = 20
   West Longitude (deg)       = 118.25
   East Longitude (deg)       = 241.75

 Solution:

 Local Sidereal Time (deg) = 104.676
 Local Sidereal Time (hr)  = 6.9784
-----------------------------------------------------
Problem 5.10d: Local sidereal time calculation

 Input data:

   Year                       = 2006
   Month                      = 2
   Day                        = 15
   UT (hr)                    = 3
   West Longitude (deg)       = 43.1
   East Longitude (deg)       = 316.9

 Solution:

 Local Sidereal Time (deg) = 146.884
 Local Sidereal Time (hr)  = 9.79228
-----------------------------------------------------
Problem 5.10e: Local sidereal time calculation

 Input data:

   Year                       = 2006
   Month                      = 3
   Day                        = 21
   UT (hr)                    = 8
   West Longitude (deg)       = 228.067
   East Longitude (deg)       = 131.933
```

```
Solution:

 Local Sidereal Time (deg) = 70.6348
 Local Sidereal Time (hr)  = 4.70899
-----------------------------------------------------
```

Problem 5.11. $\theta = 117°$ $\phi = 51°$ $A = 28°$ $a = 68°$.

From Equation 5.83a,

$$\delta = \sin^{-1}(\cos\phi\cos A\cos a + \sin\phi\sin a) = \sin^{-1}(\cos 51°\cos 28°\cos 68° + \sin 117°\sin 68°)$$

$\underline{\delta = 68.235°}$

From Equation 5.83b, since $A < 180°$,

$$\begin{aligned} h &= 360° - \cos^{-1}\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) \\ &= 360° - \cos^{-1}\left(\frac{\cos 51°\sin 68° - \sin 51°\cos 28°\cos 68°}{\cos 68.235°}\right) \\ &= 331.69° \end{aligned}$$

From Equation 5.83c

$$\alpha = \theta - h = 117° - 331.69° = -214.69$$

Placing this within the range $0 \le \theta \le 360°$,

$\underline{\alpha = 145.31°}$

Problem 5.12 The following MATLAB script uses *Algorithm 5.4*, which is implemented in MATLAB by the M-function `rv_from_observe` in Appendix D.14, to compute the state vector of a space object from the data given in Problem 5.12. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_12
% ~~~~~~~~~~~~
%
% This program uses Algorithm 5.4 to obtain the state
% vector from the observational data provided in Problem 5.12.
%
% deg    - conversion factor between degrees and radians
% pi     - 3.1415926...
% mu     - gravitational parameter (km^3/s^2)

% Re     - equatorial radius of the earth (km)
% f      - earth's flattening factor
% wE     - angular velocity of the earth (rad/s)
% omega  - earth's angular velocity vector (rad/s) in the
%          geocentric equatorial frame

% rho    - slant range of object (km)
% rhodot - range rate (km/s)
% A      - azimuth (deg) of object relative to observation site
% Adot   - time rate of change of azimuth (deg/s)
% a      - elevation angle (deg) of object relative to observation
site
```

```
% adot   - time rate of change of elevation angle (degrees/s)

% theta  - local sidereal time (deg) of tracking site
% phi    - geodetic latitude (deg) of site
% H      - elevation of site (km)

% r      - geocentric equatorial position vector of object (km)
% v      - geocentric equatorial velocity vector of object (km)

% coe    - orbital elements [h e RA incl w TA a]
%          where
%              h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = inclination of the orbit (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% rp    - perigee radius (km)
% T     - period of elliptical orbit (s)
%
% User M-function required: rv_from_observe
% --------------------------------------------------------------------

clear
global  f Re wE mu

deg    = pi/180;
f      = 0.0033528;
Re     = 6378;
wE     = 7.2921e-5;
mu     = 398600;

%...Data declaration for Problem 5.12:
rho    = 988;
rhodot = 4.86;
A      = 36;
Adot   = 0.59;
a      = 36.6;
adot   = -0.263;
theta  = 40;
phi    = 35;
H      = 0;
%...

%...Algorithm 5.4:
[r,v] = rv_from_observe(rho, rhodot, A, Adot, a, adot, theta, phi, H);

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.12')
fprintf('\n\n Input data:\n');
fprintf('\n Slant range (km)              = %g', rho);
fprintf('\n Slant range rate (km/s)       = %g', rhodot);
fprintf('\n Azimuth (deg)                 = %g', A);
fprintf('\n Azimuth rate (deg/s)          = %g', Adot);
fprintf('\n Elevation (deg)               = %g', a);
fprintf('\n Elevation rate (deg/s)        = %g', adot);
fprintf('\n Local sidereal time (deg)     = %g', theta);
fprintf('\n Latitude (deg)                = %g', phi);
fprintf('\n Altitude above sea level (km) = %g', H);
fprintf('\n\n');
```

```
fprintf(' Solution:')

fprintf('\n\n State vector:\n');
fprintf('\n r (km)                          = [%g, %g, %g]', ...
                                       r(1), r(2), r(3));
fprintf('\n v (km/s)                        = [%g, %g, %g]', ...
                                       v(1), v(2), v(3));
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.12

 Input data:

 Slant range (km)             = 988
 Slant range rate (km/s)      = 4.86
 Azimuth (deg)                = 36
 Azimuth rate (deg/s)         = 0.59
 Elevation (deg)              = 36.6
 Elevation rate (deg/s)       = -0.263
 Local sidereal time (deg)    = 40
 Latitude (deg)               = 35
 Altitude above sea level (km) = 0

 Solution:

 State vector:

 r (km)                       = [3794.66, 3792.71, 4501.31]
 v (km/s)                     = [-7.72483, 7.72134, 0.0186586]
-----------------------------------------------------
```

$\underline{\mathbf{r} = 3794.7\hat{\mathbf{I}} + 3792.7\hat{\mathbf{J}} + 4501.3\hat{\mathbf{K}}\,(\text{km})}$ $\underline{\mathbf{v} = -7.7248\hat{\mathbf{I}} + 7.72134\hat{\mathbf{J}} + 0.018\,659\hat{\mathbf{K}}\,(\text{km/s})}$

Problem 5.13 The following MATLAB script uses the state vector found in Problem 5.12 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function coe_from_sv in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_13
% ~~~~~~~~~~~~
%
% This program employs Algorithm 4.1 to obtain the orbital
% elements from the state vector found in Problem 5.12.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
```

```
% User M-function required: coe_from_sv
% ---------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.13:
r = [ 3794.66, 3792.71,   4501.31];
v = [-7.72483, 7.72134, 0.0186586];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.13: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                       = [%g  %g  %g]', ...
                                           r(1), r(2), r(3))
fprintf('\n v (km/s)                     = [%g  %g  %g]', ...
                                           v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                 = %g', T)
    fprintf('\n   Minutes                 = %g', T/60)
    fprintf('\n   Hours                   = %g', T/3600)
    fprintf('\n   Days                    = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.13: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                    = [3794.66  3792.71  4501.31]
 v (km/s)                  = [-7.72483  7.72134  0.0186586]

 Angular momentum (km^2/s) = 76490.5
 Eccentricity              = 1.09593
 Right ascension (deg)     = 315.13
 Inclination (deg)         = 39.9968
 Argument of perigee (deg) = 89.8097
 True anomaly (deg)        = 0.0797759
 Semimajor axis (km):      = -73003.5
-----------------------------------------------------
```

Problem 5.14 The local sidereal time θ, azimuth A, angular elevation a and slant range ρ are provided at three observation times. The rates are not provided, but we can still use *Algorithm 5.4*, implemented in MATLAB as `rv_from_observe` in Appendix D.14, to find just the position vectors at each of the times. The following MATLAB script carries out this procedure, passing zeros to `rv_from_observe` as values for the rates. The output to the MATLAB Command Window follows.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_10a
% ~~~~~~~~~~~~~
%
% This program uses Algorithm 5.4 to find the geocentric position
% vectors corresponding to the three sets of azimuth, elevation
% and slant range data given in Problem 5.14

% deg    - conversion factor between degrees and radians
% pi     - 3.1415926...

% Re     - equatorial radius of the earth (km)
% f      - earth's flattening factor
% wE     - angular velocity of the earth (rad/s) (not required in
%                                 this problem)

% t      - vector of three observation times (min)
% rho    - vector of slant ranges (km) of the object at the three
%          observation times
% az     - vector of azimuths (deg) of the object relative to the
%          observation site at the three observation times
% el     - vector of elevation angles (deg) of the object relative to
%          the observation site at the three observation times

% theta  - vector of local sidereal times (deg) of the tracking site
at
%          the three observation times
% phi    - geodetic latitude (deg) of site
% H      - elevation of site (km)

% r      - geocentric equatorial position vector of object (km)
% v      - geocentric equatorial velocity vector of object (km)
%          (not computed since the rates of rho, az and el are not
given)

% User M-function required: rv_from_observe
% ----------------------------------------------------------------------
clear

global f Re wE

deg   = pi/180;
Re    = 6378;
f     = 0.0033528;
wE    = 7.292115e-5;

%...Data declaration for Problem 5.14:
phi   = -20;
H     = 0.5;
t     = [0 2 4];
theta = [60.0    60.5014  61.0027];
az    = [165.931  145.967  2.40962];
el    = [9.53549  45.7711  21.8825];
rho   = [1214.89  421.441  732.079];
%...
```

```
%...Echo the input data to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.14')
fprintf('\n\n Input data (angles in degrees):\n');
fprintf('\n  Time')
fprintf('\n  (min)      Azimuth      Elevation     Slant range\n')

for i = 1:3
fprintf('\n %5.1f%15.5e%15.5e%15.5e',t(i), az(i), el(i), rho(i))
end

%...Output the solution to the command window:
fprintf('\n\n Solution:')
fprintf('\n\n  Time')
fprintf('\n  (min)    Geocentric position vector (km)\n')

for i = 1:3
%...Algorithm 5.4:
    [r,v] = rv_from_observe(rho(i), 0, az(i), 0, el(i), ...
                                0, theta(i), phi, H);
  fprintf('\n %5.1f      [%g  %g  %g]',t(i), r(1), r(2), r(3))
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.14

 Input data (angles in degrees):

  Time
  (min)      Azimuth      Elevation     Slant range

   0.0    1.65931e+02    9.53549e+00    1.21489e+03
   2.0    1.45967e+02    4.57711e+01    4.21441e+02
   4.0    2.40962e+00    2.18825e+01    7.32079e+02

 Solution:

  Time
  (min)    Geocentric position vector (km)

   0.0      [2641.68  5158.02  -3328.73]
   2.0      [2908.04  5474.36  -2500.03]
   4.0      [3118.6  5685.65  -1623.34]
-----------------------------------------------------
```

$$\mathbf{r}_1 = 2641.7\hat{\mathbf{I}} + 5158.0\hat{\mathbf{J}} - 3328.7\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{r}_2 = 2908.0\hat{\mathbf{I}} + 5474.4\hat{\mathbf{J}} - 2500.0\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{r}_3 = 3118.6\hat{\mathbf{I}} + 5685.6\hat{\mathbf{J}} - 1623.3\hat{\mathbf{K}}\,(\text{km})$$

Using these three position vectors we employ Gibbs' method, Algrorithm 5.1, which is implemented in MATLAB as the M-function `gibbs` in Appendix D.10. The following MATLAB script calls upon `gibbs` to find the velocity vector $\mathbf{v}_2$ corresponding to the position vector $\mathbf{r}_2$. The output is listed afterward.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_14b
% ~~~~~~~~~~~~~
%
% This program uses Algorithm 5.1 (Gibbs method) to obtain the state
```

```
% vector from the three coplanar position vectors found in the first
% part of Problem 5.14.
%
% mu          - gravitational parameter (km^3/s^2)
% r1, r2, r3 - three coplanar geocentric position vectors (km)
% ierr        - 0 if r1, r2, r3 are found to be coplanar
%               1 otherwise
% v2          - the velocity corresponding to r2 (km/s)
%
% User M-function required: gibbs
% ------------------------------------------------------------------------
clear
global mu
mu = 398600;

r1 = [2641.68  5158.02  -3328.73];
r2 = [2908.04  5474.36  -2500.03];
r3 = [3118.6  5685.65  -1623.34];

%...Echo the input data to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.14: Gibbs Method\n')
fprintf('\n Input data:\n')
fprintf('\n  Gravitational parameter (km^3/s^2)  = %g\n', mu)
fprintf('\n  r1 (km) = [%g  %g  %g]', r1(1), r1(2), r1(3))
fprintf('\n  r2 (km) = [%g  %g  %g]', r2(1), r2(2), r2(3))
fprintf('\n  r3 (km) = [%g  %g  %g]', r3(1), r3(2), r3(3))
fprintf('\n\n');

%...Algorithm 5.1:
[v2, ierr] = gibbs(r1, r2, r3);

%...If the vectors r1, r2, r3, are not coplanar, abort:
if ierr == 1
    fprintf('\n  These vectors are not coplanar.\n\n')
    return
end

%...Output the results to the command window:
fprintf(' Solution:')
fprintf('\n');
fprintf('\n  v2 (km/s) = [%g  %g  %g]', v2(1), v2(2), v2(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.14: Gibbs Method

 Input data:

  Gravitational parameter (km^3/s^2)  = 398600

  r1 (km) = [2641.68  5158.02  -3328.73]
  r2 (km) = [2908.04  5474.36  -2500.03]
  r3 (km) = [3118.6  5685.65  -1623.34]

 Solution:

  v2 (km/s) = [1.99357  2.20552  7.12881]
-----------------------------------------------------
```

$$\underline{\mathbf{v}_2 = 1.9936\hat{\mathbf{I}} + 2.2055\hat{\mathbf{J}} + 7.1288\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.15 The following MATLAB script uses $\mathbf{r}_2$ and $\mathbf{v}_2$ from Problem 5.14 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_15
% ~~~~~~~~~~~~
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.14.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ----------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.15:
r = [2908.04  5474.36  -2500.03];
v = [1.99357  2.20552   7.12881];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.15: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                       = [%g  %g  %g]', ...
                                            r(1), r(2), r(3))
fprintf('\n v (km/s)                     = [%g  %g  %g]', ...
                                            v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity                 = %g', coe(2))
fprintf('\n Right ascension (deg)        = %g', coe(3)/deg)
fprintf('\n Inclination (deg)            = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)           = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):         = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
```

```
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                  = %g', T)
    fprintf('\n   Minutes                  = %g', T/60)
    fprintf('\n   Hours                    = %g', T/3600)
    fprintf('\n   Days                     = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.15: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                     = [2908.04  5474.36  -2500.03]
 v (km/s)                   = [1.99357  2.20552  7.12881]

 Angular momentum (km^2/s) = 51626.3
 Eccentricity              = 0.00102595
 Right ascension (deg)     = 60.0001
 Inclination (deg)         = 95.0003
 Argument of perigee (deg) = 270.34
 True anomaly (deg)        = 67.6075
 Semimajor axis (km):      = 6686.59
 Period:
   Seconds                 = 5441.5
   Minutes                 = 90.6916
   Hours                   = 1.51153
   Days                    = 0.0629803
-----------------------------------------------------
```

Problems 5.16 and 5.17 The following MATLAB script uses Equations 5.55 and 5.56 to convert the data given in Problem 5.16 into three tracking site position vectors ($\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$) and three space object direction cosine vectors ($\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2, \hat{\boldsymbol{\rho}}_3$). These vectors together with the three observation times are then handed off to the M-function gauss (Appendix D.15). gauss implements both the Gauss *Algorithm 5.5* to compute an approximation of the state vector $(\mathbf{r}, \mathbf{v})$ and *Algorithm 5.6*. which iteratively improves it. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_16
% ~~~~~~~~~~~~
%
% This program uses Algorithms 5.5 and 5.6 (Gauss's method) to compute
% the state vector from the angles only data provided in Problem 5.16.
%
% deg          - factor for converting between degrees and radians
% pi           - 3.1415926...
% mu           - gravitational parameter (km^3/s^2)
% Re           - earth's equatorial radius (km)
% f            - earth's flattening factor
% H            - elevation of observation site (km)
% phi          - latitude of site (deg)
% t            - vector of observation times t1, t2, t3 (s)
% ra           - vector of topocentric equatorial right ascensions
%                at t1, t2, t3 (deg)
% dec          - vector of topocentric equatorial right declinations
%                at t1, t2, t3 (deg)
% theta        - vector of local sidereal times for t1, t2, t3 (deg)
% R            - matrix of site position vectors at t1, t2, t3 (km)
```

```
% rho          - matrix of direction cosine vectors at t1, t2, t3
% fac1, fac2   - common factors
% r_old, v_old - the state vector without iterative improvement (km,
km/s)
% r, v         - the state vector with iterative improvement (km,
km/s)
%
% User M-function required: gauss
% --------------------------------------------------------------------

clear

global mu

deg = pi/180;
mu  = 398600;
Re  = 6378;
f   = 1/298.26;

%...Data declaration for Problen 5.16:
H     = 0;
phi   = 29*deg;
t     = [      0          60             120    ];
ra    = [      0      6.59279e+01    7.98500e+01]*deg;
dec   = [5.15110e+01  2.79911e+01    1.46609e+01]*deg;
theta = [      0      2.50684e-01    5.01369e-01]*deg;
%...

%...Equations 5.56, 5.57:
fac1 = Re/sqrt(1-(2*f - f*f)*sin(phi)^2);
fac2 = (Re*(1-f)^2/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*sin(phi);
for i = 1:3
    R(i,1) = (fac1 + H)*cos(phi)*cos(theta(i));
    R(i,2) = (fac1 + H)*cos(phi)*sin(theta(i));
    R(i,3) = fac2;
    rho(i,1) = cos(dec(i))*cos(ra(i));
    rho(i,2) = cos(dec(i))*sin(ra(i));
    rho(i,3) = sin(dec(i));
end

%...Algorithms 5.5 and 5.6:
[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                               R(1,:),    R(2,:),  R(3,:), ...
                               t(1),      t(2),    t(3));

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problems 5.16 and 5.17: Orbit determination')
fprintf('\n                          by the Gauss method\n')
fprintf('\n Radius of earth (km)                = %g', Re)
fprintf('\n Flattening factor                   = %g', f)
fprintf('\n Gravitational parameter (km^3/s^2) = %g', mu)
fprintf('\n\n Input data:\n');
fprintf('\n Latitude (deg) of tracking site = %g', phi/deg);
fprintf('\n Altitude (km) above sea level   = %g', H);
fprintf('\n\n Observations:')
fprintf('\n                Right')
fprintf('                                   Local')
fprintf('\n   Time (s)   Ascension (deg)   Declination (deg)')
fprintf('   Sidereal time (deg)')
for i = 1:3
    fprintf('\n %9.4g %11.4f %19.4f %20.4f', ...
```

```
                 t(i), ra(i)/deg, dec(i)/deg, theta(i)/deg)
end

fprintf('\n\n Solution:\n')

fprintf('\n Without iterative improvement (Problem 5.16)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n');
fprintf('\n\n With iterative improvement (Problem 5.17)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problems 5.16 and 5.17: Orbit determination
                         by the Gauss method

 Radius of earth (km)               = 6378
 Flattening factor                  = 0.00335278
 Gravitational parameter (km^3/s^2) = 398600

 Input data:

 Latitude (deg) of tracking site = 29
 Altitude (km) above sea level   = 0

 Observations:
               Right                                    Local
   Time (s)   Ascension (deg)   Declination (deg)   Sidereal time
(deg)
         0      0.0000              51.5110              0.0000
        60     65.9279              27.9911              0.2507
       120     79.8500              14.6609              0.5014

 Solution:

 Without iterative improvement (Problem 5.16)...

  r (km)   = [5788.09, 484.257, 3341.52]
  v (km/s) = [-0.460072, 8.05816, -0.265618]


 With iterative improvement (Problem 5.17)...

  r (km)   = [5788.42, 485.007, 3341.96]
  v (km/s) = [-0.460926, 8.0706, -0.266112]
-----------------------------------------------------
```

Problem 5.18 The following MATLAB script uses $\mathbf{r}_2$ and $\mathbf{v}_2$ from Problem 5.17 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_18
% ~~~~~~~~~~~~
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.17.
%
% pi   - 3.1415926...
```

```
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ---------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.18:
r = [  5788.42, 485.007,   3341.96];
v = [-0.460926,  8.0706, -0.266112];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.18: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                            = [%g  %g  %g]', ...
                                            r(1), r(2), r(3))
fprintf('\n v (km/s)                          = [%g  %g  %g]', ...
                                            v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                   = %g', T)
    fprintf('\n   Minutes                   = %g', T/60)
    fprintf('\n   Hours                     = %g', T/3600)
    fprintf('\n   Days                      = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
Problem 5.18: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600
```

```
State vector:

r (km)                       = [5788.42  485.007  3341.96]
v (km/s)                     = [-0.460926  8.0706  -0.266112]

Angular momentum (km^2/s) = 54201.2
Eccentricity                 = 0.100054
Right ascension (deg)        = 270
Inclination (deg)            = 30.0001
Argument of perigee (deg)    = 89.9993
True anomaly (deg)           = 4.15098
Semimajor axis (km):         = 7444.75
Period:
  Seconds                    = 6392.73
  Minutes                    = 106.546
  Hours                      = 1.77576
  Days                       = 0.07399
-----------------------------------------------------
```

Problems 5.19 The following MATLAB script uses Equations 5.55 and 5.56 to convert the data given in Problem 5.19 into three tracking site position vectors $(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3)$ and three space object direction cosine vectors $(\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2, \hat{\boldsymbol{\rho}}_3)$. These vectors together with the three observation times are then handed off to the M-function `gauss` (Appendix D.15). `gauss` implements both the Gauss *Algorithm 5.5* to compute an approximation of the state vector $(\mathbf{r}, \mathbf{v})$ and *Algorithm 5.6*. which iteratively improves it. The output to the MATLAB Command Window is listed afterwards.

```
%
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_19
% ~~~~~~~~~~~~
%
% This program uses Algorithms 5.5 and 5.6 (Gauss's method) to
compute
% the state vector from the angles only data provided in Problem
5.16.
%
% deg          - factor for converting between degrees and radians
% pi           - 3.1415926...
% mu           - gravitational parameter (km^3/s^2)
% Re           - earth's equatorial radius (km)
% f            - earth's flattening factor
% H            - elevation of observation site (km)
% phi          - latitude of site (deg)
% t            - vector of observation times t1, t2, t3 (s)
% ra           - vector of topocentric equatorial right ascensions
%                at t1, t2, t3 (deg)
% dec          - vector of topocentric equatorial right declinations
%                at t1, t2, t3 (deg)
% theta        - vector of local sidereal times for t1, t2, t3 (deg)
% R            - matrix of site position vectors at t1, t2, t3 (km)
% rho          - matrix of direction cosine vectors at t1, t2, t3
% fac1, fac2   - common factors
% r_old, v_old - the state vector without iterative improvement (km,
km/s)
% r, v         - the state vector with iterative improvement (km,
km/s)
%
% User M-function required: gauss
% --------------------------------------------------------------------
-
```

```
clear

global mu

deg = pi/180;
mu  = 398600;
Re  = 6378;
f   = 1/298.26;

%...Data declaration for Problem 5.19:
H     = 0;
phi   = 29*deg;
t     = [      0       60       120];
ra    = [15.0394  25.7539   48.6055]*deg;
dec   = [20.7487  30.1410   43.8910]*deg;
theta = [     90  90.2507   90.5014]*deg;
%...

%...Equations 5.56, 5.57:
fac1 = Re/sqrt(1-(2*f - f*f)*sin(phi)^2);
fac2 = (Re*(1-f)^2/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*sin(phi);
for i = 1:3
    R(i,1) = (fac1 + H)*cos(phi)*cos(theta(i));
    R(i,2) = (fac1 + H)*cos(phi)*sin(theta(i));
    R(i,3) = fac2;
    rho(i,1) = cos(dec(i))*cos(ra(i));
    rho(i,2) = cos(dec(i))*sin(ra(i));
    rho(i,3) = sin(dec(i));
end

%...Algorithms 5.5 and 5.6:
[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                              R(1,:),    R(2,:),  R(3,:), ...
                              t(1),      t(2),    t(3));

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problems 5.19 and 5.20: Orbit determination')
fprintf('\n                         by the Gauss method\n')
fprintf('\n Radius of earth (km)               = %g', Re)
fprintf('\n Flattening factor                  = %g', f)
fprintf('\n Gravitational parameter (km^3/s^2) = %g', mu)
fprintf('\n\n Input data:\n');
fprintf('\n Latitude (deg) of tracking site = %g', phi/deg);
fprintf('\n Altitude (km) above sea level   = %g', H);
fprintf('\n\n Observations:')
fprintf('\n               Right')
fprintf('                                Local')
fprintf('\n   Time (s)   Ascension (deg)   Declination (deg)')
fprintf('   Sidereal time (deg)')
for i = 1:3
    fprintf('\n %9.4g %11.4f %19.4f %20.4f', ...
                 t(i), ra(i)/deg, dec(i)/deg, theta(i)/deg)
end

fprintf('\n\n Solution:\n')

fprintf('\n Without iterative improvement (Problem 5.19)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n');
```

```
fprintf('\n\n With iterative improvement (Problem 5.20)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
fprintf('\n-----------------------------------------------------\n')
```

```
Problems 5.19 and 5.20: Orbit determination
                         by the Gauss method

 Radius of earth (km)               = 6378
 Flattening factor                  = 0.00335278
 Gravitational parameter (km^3/s^2) = 398600

 Input data:

 Latitude (deg) of tracking site = 29
 Altitude (km) above sea level   = 0

 Observations:
               Right                                     Local
   Time (s)   Ascension (deg)   Declination (deg)   Sidereal time
(deg)
        0      15.0394              20.7487              90.0000
       60      25.7539              30.1410              90.2507
      120      48.6055              43.8910              90.5014

 Solution:

 Without iterative improvement (Problem 5.19)...

  r (km)   = [765.19, 5963.6, 3582.88]
  v (km/s) = [-7.50919, 0.705265, 0.423729]


 With iterative improvement (Problem 5.20)...

  r (km)   = [766.265, 5964.12, 3583.57]
  v (km/s) = [-7.51882, 0.706233, 0.42431]
-----------------------------------------------------
```

Approximate state vector:

$$\underline{\mathbf{r} = 765.19\hat{\mathbf{I}} + 5963.60\hat{\mathbf{J}} + 3582.88\hat{\mathbf{K}}\,(\text{km})} \quad \underline{\mathbf{v} = -7.50919\hat{\mathbf{I}} + 0.705265\hat{\mathbf{J}} + 0.423729\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.20 From the MATLAB output in the previous problem solution, the refined state vector is

$$\underline{\mathbf{r} = 766.265\hat{\mathbf{I}} + 5964.12\hat{\mathbf{J}} + 3583.57\hat{\mathbf{K}}\,(\text{km})} \quad \underline{\mathbf{v} = -7.518\,82\hat{\mathbf{I}} + 0.706\,233\hat{\mathbf{J}} + 0.424\,31\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.21 The following MATLAB script uses $\mathbf{r}$ and $\mathbf{v}$ from Problem 5.20 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_21
% ~~~~~~~~~~~~
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.20.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
```

```
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ---------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.21:
r = [ 766.265,   5964.12,  3583.57];
v = [-7.51882,  0.706233,  0.42431];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.21: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                       = [%g  %g  %g]', ...
                                          r(1), r(2), r(3))
fprintf('\n v (km/s)                     = [%g  %g  %g]', ...
                                          v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                 = %g', T)
    fprintf('\n   Minutes                 = %g', T/60)
    fprintf('\n   Hours                   = %g', T/3600)
    fprintf('\n   Days                    = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.21: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600
```

```
 State vector:

 r (km)                      = [766.265  5964.12  3583.57]
 v (km/s)                    = [-7.51882  0.706233  0.42431]

 Angular momentum (km^2/s) = 52946.7
 Eccentricity              = 0.00474691
 Right ascension (deg)     = 360
 Inclination (deg)         = 30.9997
 Argument of perigee (deg) = 90.3282
 True anomaly (deg)        = 353.388
 Semimajor axis (km):      = 7033.16
 Period:
   Seconds                 = 5869.98
   Minutes                 = 97.833
   Hours                   = 1.63055
   Days                    = 0.0679396
-----------------------------------------------------
```

Problem 5.22 The following MATLAB script uses the given three tracking site position vectors ($\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$) and three space object direction cosine vectors $(\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2, \hat{\boldsymbol{\rho}}_3)$ together with the three observation times to find the state vector $(\mathbf{r}, \mathbf{v})$ by means of *Algorithm 5.5* and then iteratively improve it using *Algorithm 5.6*. Both algorithms are implemented in the MATLAB M-function `gauss` in Appendix D.15. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_22
% ~~~~~~~~~~~~
%
% This program uses Algorithms 5.5 and 5.6 (Gauss's method) to compute
% the state vector from the angles only data provided in Problem 5.16.
%
% deg          - factor for converting between degrees and radians
% pi           - 3.1415926...
% mu           - gravitational parameter (km^3/s^2)
% t            - vector of observation times t1, t2, t3 (s)
% theta        - vector of local sidereal times for t1, t2, t3 (deg)
% R            - matrix of site position vectors at t1, t2, t3 (km)
% rho          - matrix of direction cosine vectors at t1, t2, t3
% r_old, v_old - the state vector without iterative improvement (km,
km/s)
% r, v         - the state vector with iterative improvement (km,
km/s)
%
% User M-function required: gauss
% ----------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.22:
t     = [        0         60        120];

R     = [ -1825.96    3583.66    4933.54
          -1841.63    3575.63    4933.54
          -1857.25    3567.54    4933.54];

rho   = [-0.301687   0.200673   0.932049
         -0.793090  -0.210324   0.571640
```

```
         -0.873085  -0.362969   0.325539];
%...

%...Algorithms 5.5 and 5.6:
[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                               R(1,:),   R(2,:),   R(3,:), ...
                                 t(1),     t(2),     t(3));

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problems 5.22 and 5.23: Orbit determination')
fprintf('\n                         by the Gauss method\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g', mu)
fprintf('\n\n Input data:\n');
fprintf('\n Site position vector (R) and space object')
fprintf('\n direction cosine vector (rho) at three times:\n')
for i = 1:3
    fprintf('\n  t = %g s:\n',t(i))
   fprintf('\n   R   = [%g  %g  %g]', R(i,1), R(i,2), R(i,3))
fprintf('\n   rho = [%g  %g  %g]', rho(i,1), rho(i,2), rho(i,3))
                           disp(' ')
end
fprintf('\n\n Solution:\n')
fprintf('\n Without iterative improvement (Problem 5.22)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n');
fprintf('\n\n With iterative improvement (Problem 5.23)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problems 5.22 and 5.23: Orbit determination
                         by the Gauss method

 Gravitational parameter (km^3/s^2) = 398600

 Input data:

 Site position vector (R) and space object
 direction cosine vector (rho) at three times:

  t = 0 s:

   R   = [-1825.96  3583.66  4933.54]
   rho = [-0.301687  0.200673  0.932049]

  t = 60 s:

   R   = [-1841.63  3575.63  4933.54]
   rho = [-0.79309  -0.210324  0.57164]

  t = 120 s:

   R   = [-1857.25  3567.54  4933.54]
   rho = [-0.873085  -0.362969  0.325539]


 Solution:
```

```
Without iterative improvement (Problem 5.22)...

 r (km)   = [-2350.74, 3440.62, 5300.49]
 v (km/s) = [-6.61345, -3.88226, -0.413321]


With iterative improvement (Problem 5.23)...

 r (km)   = [-2351.59, 3440.39, 5301.1]
 v (km/s) = [-6.62403, -3.8885, -0.414013]
-----------------------------------------------------
```

Approximate state vector:

$$\underline{\mathbf{r} = -2350.74\hat{\mathbf{I}} + 3440.62\hat{\mathbf{J}} + 5300.49\hat{\mathbf{K}}\,(\text{km})} \quad \underline{\mathbf{v} = -6.613\,45\hat{\mathbf{I}} - 3.882\,26\hat{\mathbf{J}} - 0.413\,321\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.23 From the MATLAB output listed in the previous problem solution, the iteratively improved state vector is

$$\underline{\mathbf{r} = -2351.59\hat{\mathbf{I}} + 3440.39\hat{\mathbf{J}} + 5301.1\hat{\mathbf{K}}\,(\text{km})} \quad \underline{\mathbf{v} = -6.624\,03\hat{\mathbf{I}} - 3.8885\hat{\mathbf{J}} - 0.414\,013\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.24 The following MATLAB script uses $\mathbf{r}$ and $\mathbf{v}$ from Problem 5.23 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
%
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_24
% ~~~~~~~~~~~~
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.23.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% ------------------------------------------------------------------
-

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.24:
r = [-2351.59, 3440.39,    5301.1];
v = [-6.62403, -3.8885, -0.414013];
```

```
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.24: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                       = [%g  %g  %g]', ...
                                          r(1), r(2), r(3))
fprintf('\n v (km/s)                     = [%g  %g  %g]', ...
                                          v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                  = %g', T)
    fprintf('\n   Minutes                  = %g', T/60)
    fprintf('\n   Hours                    = %g', T/3600)
    fprintf('\n   Days                     = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.24: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                    = [-2351.59  3440.39  5301.1]
 v (km/s)                  = [-6.62403  -3.8885  -0.414013]

 Angular momentum (km^2/s) = 51868.3
 Eccentricity              = 0.000957299
 Right ascension (deg)     = 28.0006
 Inclination (deg)         = 51.9999
 Argument of perigee (deg) = 88.9231
 True anomaly (deg)        = 4.99817
 Semimajor axis (km):      = 6749.43
 Period:
   Seconds                 = 5518.38
   Minutes                 = 91.973
   Hours                   = 1.53288
   Days                    = 0.0638701
-----------------------------------------------------
```

Problems 5.25 The following MATLAB script uses Equations 5.55 and 5.56 to convert the data given in Problem 5.25 into three tracking site position vectors ($\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$) and three space object direction cosine vectors $\left(\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2, \hat{\boldsymbol{\rho}}_3\right)$. These vectors together with the three observation times are then handed off to the M-function gauss (Appendix D.15). gauss implements both the Gauss *Algorithm 5.5* to compute an approximation of the state vector $(\mathbf{r}, \mathbf{v})$ and *Algorithm 5.6.* which iteratively improves it. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_25
% ~~~~~~~~~~~~
%
% This program uses Algorithms 5.5 and 5.6 (Gauss's method) to compute
% the state vector from the angles only data provided in Problem 5.25.
%
% deg          - factor for converting between degrees and radians
% pi           - 3.1415926...
% mu           - gravitational parameter (km^3/s^2)
% Re           - earth's equatorial radius (km)
% f            - earth's flattening factor
% H            - elevation of observation site (km)
% phi          - latitude of site (deg)
% t            - vector of observation times t1, t2, t3 (s)
% ra           - vector of topocentric equatorial right ascensions
%                at t1, t2, t3 (deg)
% dec          - vector of topocentric equatorial right declinations
%                at t1, t2, t3 (deg)
% theta        - vector of local sidereal times for t1, t2, t3 (deg)
% R            - matrix of site position vectors at t1, t2, t3 (km)
% rho          - matrix of direction cosine vectors at t1, t2, t3
% fac1, fac2   - common factors
% r_old, v_old - the state vector without iterative improvement (km,
km/s)
% r, v         - the state vector with iterative improvement (km,
km/s)
%
% User M-function required: gauss
% ----------------------------------------------------------------------

clear

global mu

deg = pi/180;
mu  = 398600;
Re  = 6378;
f   = 1/298.26;

%...Data declaration for Problem 5.25:
H     = 0.5;
phi   = 60*deg;
t     = [      0      300       600];
ra    = [157.783  159.221   160.526]*deg;
dec   = [24.2403  27.2993   29.8982]*deg;
theta = [     150  151.253   152.507]*deg;
%...

%...Equations 5.56, 5.57:
fac1 = Re/sqrt(1-(2*f - f*f)*sin(phi)^2);
fac2 = (Re*(1-f)^2/sqrt(1-(2*f - f*f)*sin(phi)^2) + H)*sin(phi);
for i = 1:3
    R(i,1) = (fac1 + H)*cos(phi)*cos(theta(i));
    R(i,2) = (fac1 + H)*cos(phi)*sin(theta(i));
    R(i,3) = fac2;
    rho(i,1) = cos(dec(i))*cos(ra(i));
```

```
    rho(i,2) = cos(dec(i))*sin(ra(i));
    rho(i,3) = sin(dec(i));
end

%...Algorithms 5.5 and 5.6:
[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                              R(1,:),    R(2,:),  R(3,:), ...
                              t(1),      t(2),    t(3));

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problems 5.25 and 5.26: Orbit determination')
fprintf('\n                         by the Gauss method\n')
fprintf('\n Radius of earth (km)                = %g', Re)
fprintf('\n Flattening factor                   = %g', f)
fprintf('\n Gravitational parameter (km^3/s^2) = %g', mu)
fprintf('\n\n Input data:\n');
fprintf('\n Latitude (deg) of tracking site = %g', phi/deg);
fprintf('\n Altitude (km) above sea level   = %g', H);
fprintf('\n\n Observations:')
fprintf('\n                Right')
fprintf('                                  Local')
fprintf('\n   Time (s)   Ascension (deg)   Declination (deg)')
fprintf('   Sidereal time (deg)')
for i = 1:3
    fprintf('\n %9.4g %11.4f %19.4f %20.4f', ...
                t(i), ra(i)/deg, dec(i)/deg, theta(i)/deg)
end

fprintf('\n\n Solution:\n')

fprintf('\n Without iterative improvement (Problem 5.25)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n');
fprintf('\n\n With iterative improvement (Problem 5.26)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
fprintf('\n-----------------------------------------------------\n'
```

```
 -----------------------------------------------------
  Problems 5.25 and 5.26: Orbit determination
                          by the Gauss method

  Radius of earth (km)                = 6378
  Flattening factor                   = 0.00335278
  Gravitational parameter (km^3/s^2) = 398600

  Input data:

  Latitude (deg) of tracking site = 60
  Altitude (km) above sea level   = 0.5

  Observations:
               Right                                  Local
    Time (s)   Ascension (deg)   Declination (deg)   Sidereal time
 (deg)
         0    157.7830            24.2403             150.0000
       300    159.2210            27.2993             151.2530
       600    160.5260            29.8982             152.5070
```

```
Solution:

Without iterative improvement (Problem 5.25)...

 r (km)   = [-19050.2, 7702.56, 14469.6]
 v (km/s) = [-3.27477, -0.482844, 5.07464]


With iterative improvement (Problem 5.26)...

 r (km)   = [-19081, 7714.25, 14486.6]
 v (km/s) = [-3.27846, -0.484358, 5.08206]
-----------------------------------------------------
```

Approximate state vector:

$$\underline{\mathbf{r} = -19050.2\hat{\mathbf{I}} + 7702.56\hat{\mathbf{J}} + 14469.6\hat{\mathbf{K}}\,(\text{km})} \quad \underline{\mathbf{v} = -3.27477\hat{\mathbf{I}} - 0.482844\hat{\mathbf{J}} + 5.07464\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.26 From the MATLAB output listed in the previous problem solution, the iteratively improved state vector is

$$\underline{\mathbf{r} = -19081\hat{\mathbf{I}} + 7714.25\hat{\mathbf{J}} + 14486.6\hat{\mathbf{K}}\,(\text{km})} \quad \underline{\mathbf{v} = -3.27846\hat{\mathbf{I}} - 0.484\,358\hat{\mathbf{J}} + 5.082\,06\hat{\mathbf{K}}\,(\text{km/s})}$$

Problem 5.27 The following MATLAB script uses **r** and **v** from Problem 5.26 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_27
% ~~~~~~~~~~~~
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.26.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% --------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.27:
r = [  -19081,   7714.25, 14486.6];
v = [-3.27846, -0.484358, 5.08206];
%...
```

```
%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.27: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                     = [%g  %g  %g]', ...
                                         r(1), r(2), r(3))
fprintf('\n v (km/s)                   = [%g  %g  %g]', ...
                                         v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity              = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                  = %g', T)
    fprintf('\n   Minutes                  = %g', T/60)
    fprintf('\n   Hours                    = %g', T/3600)
    fprintf('\n   Days                     = %g', T/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 5.27: Orbital elements from state vector

 Gravitational parameter (km^3/s^2) = 398600

 State vector:

 r (km)                    = [-19081  7714.25  14486.6]
 v (km/s)                  = [-3.27846  -0.484358  5.08206]

 Angular momentum (km^2/s) = 76005.8
 Eccentricity              = 1.08937
 Right ascension (deg)     = 136.949
 Inclination (deg)         = 62.9772
 Argument of perigee (deg) = 287.335
 True anomaly (deg)        = 112.915
 Semimajor axis (km):      = -77612.3
```

Problem 5.28 The following MATLAB script uses the given three tracking site position vectors $(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3)$ and three space object direction cosine vectors $(\hat{\boldsymbol{\rho}}_1, \hat{\boldsymbol{\rho}}_2, \hat{\boldsymbol{\rho}}_3)$ together with the three observation times to find the state vector $(\mathbf{r}, \mathbf{v})$ by means of *Algorithm 5.5* and then iteratively improve it using *Algorithm 5.6*. Both algorithms are implemented in the MATLAB M-function `gauss` in Appendix D.15. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_28
% ~~~~~~~~~~~~
%
```

```
% This program uses Algorithms 5.5 and 5.6 (Gauss's method) to compute
% the state vector from the angles only data provided in Problem 5.28.
%
% deg          - factor for converting between degrees and radians
% pi           - 3.1415926...
% mu           - gravitational parameter (km^3/s^2)
% t            - vector of observation times t1, t2, t3 (s)
% theta        - vector of local sidereal times for t1, t2, t3 (deg)
% R            - matrix of site position vectors at t1, t2, t3 (km)
% rho          - matrix of direction cosine vectors at t1, t2, t3
% r_old, v_old - the state vector without iterative improvement (km,
km/s)
% r, v         - the state vector with iterative improvement (km,
km/s)
%
% User M-function required: gauss
% ----------------------------------------------------------------------

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.28:
t     = [       0        300        600];

R     = [ 5582.84          0    3073.90
          5581.50    122.122    3073.90
          5577.50    244.186    3073.90];

rho   = [ 0.846428,         0,  0.532504
          0.749290,  0.463023,  0.473470
          0.529447,  0.777163,  0.340152];
%...

%...Algorithms 5.5 and 5.6:
[r, v, r_old, v_old] = gauss(rho(1,:), rho(2,:), rho(3,:), ...
                               R(1,:),   R(2,:),   R(3,:), ...
                                 t(1),     t(2),     t(3));

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problems 5.28 and 5.29: Orbit determination')
fprintf('\n                          by the Gauss method\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g', mu)
fprintf('\n\n Input data:\n');
fprintf('\n Site position vector (R) and space object')
fprintf('\n direction cosine vector (rho) at three times:\n')
for i = 1:3
    fprintf('\n  t = %g s:\n',t(i))
   fprintf('\n   R   = [%g  %g  %g]', R(i,1), R(i,2), R(i,3))
fprintf('\n   rho = [%g  %g  %g]', rho(i,1), rho(i,2), rho(i,3))
                           disp(' ')
end
fprintf('\n\n Solution:\n')
fprintf('\n Without iterative improvement (Problem 5.28)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r_old(1), r_old(2), r_old(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v_old(1), v_old(2), v_old(3))
fprintf('\n');
fprintf('\n\n With iterative improvement (Problem 5.29)...\n')
fprintf('\n  r (km)   = [%g, %g, %g]', r(1), r(2), r(3))
fprintf('\n  v (km/s) = [%g, %g, %g]', v(1), v(2), v(3))
```

```
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problems 5.28 and 5.29: Orbit determination
                         by the Gauss method

 Gravitational parameter (km^3/s^2) = 398600

 Input data:

 Site position vector (R) and space object
 direction cosine vector (rho) at three times:

  t = 0 s:

   R   = [5582.84  0  3073.9]
   rho = [0.846428  0  0.532504]

  t = 300 s:

   R   = [5581.5  122.122  3073.9]
   rho = [0.74929  0.463023  0.47347]

  t = 600 s:

   R   = [5577.5  244.186  3073.9]
   rho = [0.529447  0.777163  0.340152]


 Solution:

 Without iterative improvement (Problem 5.28)...

  r (km)   = [8282.6, 1791.26, 4780.7]
  v (km/s) = [-1.07108, 5.89508, -0.618321]


 With iterative improvement (Problem 5.29)...

  r (km)   = [8306.27, 1805.89, 4795.66]
  v (km/s) = [-1.07872, 5.94219, -0.622807]
-----------------------------------------------------
```

Approximate state vector

$$\underline{\mathbf{r} = 8282.6\hat{\mathbf{I}} + 1791.26\hat{\mathbf{J}} + 4780.7\hat{\mathbf{K}}\ (\text{km})} \qquad \underline{\mathbf{v} = -1.07108\hat{\mathbf{I}} + 5.89508\hat{\mathbf{J}} - 0.618321\hat{\mathbf{K}}\ (\text{km/s})}$$

Problem 5.29 From the MATLAB output listed in the previous problem solution, the iteratively improved state vector is

$$\underline{\mathbf{r} = 8306.27\hat{\mathbf{I}} + 1805.89\hat{\mathbf{J}} + 4795.66\hat{\mathbf{K}}\ (\text{km})} \qquad \underline{\mathbf{v} = -1.07872\hat{\mathbf{I}} + 5.94219\hat{\mathbf{J}} - 0.622807\hat{\mathbf{K}}\ (\text{km/s})}$$

Problem 5.30 The following MATLAB script uses **r** and **v** from Problem 5.29 to compute the orbital elements by means of *Algorithm 4.1,* which is implemented as the M-function `coe_from_sv` in Appendix D.8. The output to the MATLAB Command Window is listed afterwards.

```
%
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_5_30
% ~~~~~~~~~~~~
```

```
%
% This program uses Algorithm 4.1 to obtain the orbital
% elements from the state vector obtained in Problem 5.29.
%
% pi   - 3.1415926...
% deg  - factor for converting between degrees and radians
% mu   - gravitational parameter (km^3/s^2)
% r    - position vector (km) in the geocentric equatorial frame
% v    - velocity vector (km/s) in the geocentric equatorial frame
% coe  - orbital elements [h e RA incl w TA a]
%        where h    = angular momentum (km^2/s)
%              e    = eccentricity
%              RA   = right ascension of the ascending node (rad)
%              incl = orbit inclination (rad)
%              w    = argument of perigee (rad)
%              TA   = true anomaly (rad)
%              a    = semimajor axis (km)
% T    - Period of an elliptic orbit (s)
%
% User M-function required: coe_from_sv
% --------------------------------------------------------------------
-

clear
global mu
deg = pi/180;
mu  = 398600;

%...Data declaration for Problem 5.30:
r = [ 8306.27, 1805.89,   4795.66];
v = [-1.07872, 5.94219, -0.622807];
%...

%...Algorithm 4.1:
coe = coe_from_sv(r,v);

%...Echo the input data and output results to the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 5.30: Orbital elements from state vector\n')
fprintf('\n Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n State vector:\n')
fprintf('\n r (km)                         = [%g  %g  %g]', ...
                                          r(1), r(2), r(3))
fprintf('\n v (km/s)                       = [%g  %g  %g]', ...
                                          v(1), v(2), v(3))
disp(' ')
fprintf('\n Angular momentum (km^2/s) = %g', coe(1))
fprintf('\n Eccentricity                   = %g', coe(2))
fprintf('\n Right ascension (deg)     = %g', coe(3)/deg)
fprintf('\n Inclination (deg)         = %g', coe(4)/deg)
fprintf('\n Argument of perigee (deg) = %g', coe(5)/deg)
fprintf('\n True anomaly (deg)        = %g', coe(6)/deg)
fprintf('\n Semimajor axis (km):      = %g', coe(7))

%...if the orbit is an ellipse, output its period (Equation 2.73):
if coe(2)<1
    T = 2*pi/sqrt(mu)*coe(7)^1.5;
    fprintf('\n Period:')
    fprintf('\n   Seconds                   = %g', T)
    fprintf('\n   Minutes                   = %g', T/60)
    fprintf('\n   Hours                     = %g', T/3600)
    fprintf('\n   Days                      = %g', T/24/3600)
end
```

```
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
Problem 5.30: Orbital elements from state vector

Gravitational parameter (km^3/s^2) = 398600

State vector:

r (km)                    = [8306.27  1805.89  4795.66]
v (km/s)                  = [-1.07872  5.94219  -0.622807]

Angular momentum (km^2/s) = 59242.6
Eccentricity              = 0.0995646
Right ascension (deg)     = 270
Inclination (deg)         = 30.0002
Argument of perigee (deg) = 269.945
True anomaly (deg)        = 190.718
Semimajor axis (km):      = 8893.18
Period:
  Seconds                 = 8346.36
  Minutes                 = 139.106
  Hours                   = 2.31843
  Days                    = 0.0966014
-----------------------------------------------------
```

Problem 6.1

Orbit 1 (circle):

$$v_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6378 + 300}} = 7.726 \text{ km / s}$$

Orbit 2 (ellipse):

$$e = \frac{r_{\text{apogee}} - r_{\text{perigee}}}{r_{\text{apogee}} + r_{\text{perigee}}} = \frac{(6378 + 300) - (6378 + 200)}{(6378 + 300) - (6378 + 200)} = 0.003758$$

$$r_{\text{apogee}} = \frac{h^2}{\mu}\frac{1}{1 - e}$$

$$6378 + 300 = \frac{h^2}{398\,600}\frac{1}{1 - 0.003758} \quad \Rightarrow \quad h = 51\,500 \ \text{km}^2/\text{s}$$

$$v_{\text{apogee}} = \frac{h}{r_{\text{apogee}}} = \frac{51\,500}{6678} = 7.711 \ \text{km/s}$$

$$\Delta v = 7.726 - 7.711 = 0.01453 \ \ \text{km/s} \ \ = 14.53 \ \ \text{m/s}$$

(a)

$$Thrust \cdot \Delta t = m\Delta v$$

$$53\,400 \cdot \Delta t = 125\,000 \cdot 14.53 \quad \Rightarrow \quad \underline{\Delta t = 34.01 \text{ s}}$$

$$v_{avg} = \frac{v_{\text{circle}} + (v_{\text{circle}} + \Delta v)}{2} = v_{\text{circle}} + \frac{\Delta v}{2} = 7.726 + \frac{14.53}{2} = 7.733 \ \ \text{m/s}$$

(b)

$$\Delta s = v_{avg}\Delta t = 7.733 \cdot 34.01 = \underline{263 \text{ km}}$$

(c)

$$\frac{\Delta s}{\text{orbit circumference}} = \frac{263}{2\pi \cdot 6600} = \underline{0.006\,268 \ \text{ or } \ 0.63\%}$$

Problem 6.2

$$v_{\text{perigee}_1} = 8.2 \ \ \text{km/s}$$

$$r_{\text{perigee}_1} = 6378 + 480 = 6858 \text{ km}$$

$$r_{\text{apogee}_2} = r_{\text{perigee}_1} = 6858 \text{ km}$$

$$r_{\text{perigee}_2} = 6378 + 160 = 6538 \text{ km}$$

$$e_2 = \frac{r_{\text{apogee}_2} - r_{\text{perigee}_2}}{r_{\text{apogee}_2} + r_{\text{perigee}_2}} = \frac{6858 - 6538}{6858 + 6538} = 0.023\,89$$

$$r_{\text{apogee}_2} = \frac{{h_2}^2}{\mu}\frac{1}{1 - e_2}$$

$$6858 = \frac{{h_2}^2}{398\,600}\frac{1}{1 - 0.023\,89} \quad \Rightarrow h_2 = 51\,660 \ \ \text{km}^2/\text{s}$$

$$v_{\text{apogee}_2} = \frac{h_2}{r_{\text{apogee}_2}} = \frac{51\,660}{6858} = 7.532 \text{ km/s}$$

$$\Delta v = v_{\text{apogee}_2} - v_{\text{perigee}_1} = 7.532 - 8.2 = \underline{-0.6678 \ \ \text{km/s}}$$

Problem 6.3

(a)

Orbit 1 (circle):

$$v_1 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6378 + 300}} = 7.726 \text{ km/s}$$

Orbit 2 (transfer ellipse):

$$e_2 = \frac{r_{\text{apogee}_2} - r_{\text{perigee}_2}}{r_{\text{apogee}_2} + r_{\text{perigee}_2}} = \frac{(6378 + 3000) - (6378 + 300)}{(6378 + 3000) + (6378 + 300)} = 0.1682$$

$$r_{\text{perigee}_2} = \frac{{h_2}^2}{\mu}\frac{1}{1 + e_2}$$

$$6678 = \frac{{h_2}^2}{398\,600}\frac{1}{1 + 0.1682} \Rightarrow h_2 = 55760 \text{ km}^2/\text{s}$$

$$v_{\text{perigee}_2} = \frac{h_2}{r_{\text{perigee}_2}} = \frac{55760}{6678} = 8.35 \text{ km/s}$$

$$v_{\text{apogee}_2} = \frac{h_2}{r_{\text{apogee}_2}} = \frac{55760}{9378} = 5.946 \text{ km/s}$$

Orbit 3 (circle):

$$v_3 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6378 + 3000}} = 6.519 \text{ km/s}$$

$$\Delta v_1 = v_{\text{perigee}_2} - v_1 = 8.350 - 7.726 = 0.6244 \text{ km/s}$$

$$\Delta v_2 = v_3 - v_{\text{apogee}_2} = 6.519 - 5.946 = 0.5734 \text{ km/s}$$

$$\Delta v_{\text{total}} = \Delta v_1 + \Delta v_2 = \underline{1.198 \text{ km/s}}$$

(b)

$$a_2 = \frac{1}{2}\left(r_{\text{perigee}_2} + r_{\text{apogee}_2}\right) = \frac{1}{2}(6678 + 9378) = 8028 \text{ km/s}$$

$$T_2 = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398\,600}}8028^{3/2} = 7159 \text{ s} \ \left(\text{period of tranfer ellipse}\right)$$

$$t_{\text{perigee to apogee}} = \frac{T_2}{2} = 3579 \text{ s} = \underline{59.65 \text{ m}}$$

Problem 6.4 To determine where the projectile B impacts the earth we need the orbital elements.

$$r_{\text{apogee}_B} = 7000 \text{ km}$$

$$v_{\text{apogee}_B} = 7.1 \text{ km/s}$$

$$h_B = r_{\text{apogee}_B} v_{\text{apogee}_B} = 7000 \cdot 7.1 = 49700 \text{ km}^2/\text{s}$$

$$r_{\text{apogee}_B} = \frac{{h_B}^2}{\mu}\frac{1}{1 + e_B}$$

$$7000 = \frac{49700^2}{398\,600}\frac{1}{1 + e_B} \Rightarrow e_B = 0.1147$$

$$T_B = \frac{2\pi}{\mu^2}\left(\frac{h_B}{\sqrt{1 - {e_B}^2}}\right)^3 = \frac{2\pi}{398\,600^2}\left(\frac{49700}{\sqrt{1 - 0.1147^2}}\right)^3 = 4952 \text{ s} \ \left(\text{period of } B\text{'s orbit}\right)$$

At impact, $r_B = R_{\text{earth}}$.

$$R_{\text{earth}} = \frac{h_B^2}{\mu}\frac{1}{1 + e_B\cos\theta_{impact}} \quad \left(\text{At impact, } r_B = R_{\text{earth}}\right)$$

$$6378 = \frac{49\,700^2}{398\,600}\frac{1}{1 + 0.1147\cos\theta_{impact}} \Rightarrow \theta_{impact} = 104.3° \ \left(\text{from perigee of } B\text{'s elliptical orbit}\right)$$

Determine the time of flight (tof) to impact by first finding t_{impact}, the time from perigee to B's impact point.

$$\tan\frac{E_{impact}}{2} = \sqrt{\frac{1-e_B}{1+e_B}}\tan\frac{\theta_{impact}}{2} = \sqrt{\frac{1-0.1147}{1+0.1147}}\tan\frac{104.3°}{2} \ \Rightarrow \ E_{impact} = 1.708 \text{ rad}$$
$$M_{impact} = E_{impact} - e_B\sin E_{impact} = 1.708 - 0.1147\sin 1.708 = 1.594 \text{ rad}$$
$$t_{impact} = T_B\frac{M_{impact}}{2\pi} = 4952\frac{1.594}{2\pi} = 1257 \text{ s (from impact point to perigee)}$$

Then

$$tof = \frac{T_B}{2} - t_{impact} = \frac{4952}{2} - 1257 = 1220 \text{ s}$$

Find the orbital elements of spacecraft S trajectory.

$$r_{\text{perigee}_S} = 7000 \text{ km}$$
$$v_{\text{perigee}_S} = 1.3v_{esc} = 1.3\sqrt{\frac{2\mu}{r_{\text{perigee}_S}}} = 1.3\sqrt{\frac{2\cdot 398\,600}{7000}} = 13.97 \text{ km/s}$$
$$h_S = r_{\text{perigee}_S}v_{\text{perigee}_S} = 7000\cdot 13.97 = 97\,110 \text{ km}^2/\text{s}$$
$$r_{\text{perigee}_S} = \frac{h_S^2}{\mu}\frac{1}{1+e_S}$$
$$7000 = \frac{97\,110^2}{398\,600}\frac{1}{1+e_S} \ \Rightarrow \ e_S = 2.38$$

Location of S on its hyperbolic trajectory when B impacts the earth:

$$M_h = \frac{\mu^2}{h_S^3}\left(e_S^2 - 1\right)^{3/2} tof = \frac{398\,600^2}{97\,110^3}\cdot\left(2.38^2 - 1\right)^{3/2}\cdot 1220 = 2.131 \text{ rad}$$
$$e_S\sinh F - F = M_h$$
$$2.38\sinh F - F = 2.131 \ \Rightarrow \ F = 1.118 \ (\text{Algorihm 3.2})$$
$$\tan\frac{\theta_S}{2} = \sqrt{\frac{e_S+1}{e_S-1}}\tanh\frac{F}{2} = \sqrt{\frac{2.38+1}{2.38-1}}\tanh\frac{1.118}{2} \ \Rightarrow \ \theta_S = 76.87°$$
$$r_S = \frac{h_S^2}{\mu}\frac{1}{1+e_S\cos\theta_s} = \frac{97\,110^2}{398\,600}\frac{1}{1+0.28\cos 76.87°} = 15\,360 \text{ km}$$

$$distance = r_S - 6378 = \underline{8978 \text{ km}}$$

Problem 6.5

(a) For the transfer ellipse

$$a = \frac{1}{2}\left(r_{\text{Mars}} + r_{\text{earth}}\right) = \frac{1}{2}(227.9 + 149.6)\times 10^6 = 188.8\times 10^6 \quad \text{km}$$

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{132.7\times 10^9}}\left(188.8\times 10^6\right)^{3/2} = 44.73\times 10^6 \text{ s} = 517.7 \text{ days}$$

$$tof = \frac{T}{2} = \underline{258.8 \text{ days}} \ \left(\text{time of flight from earth to Mars}\right)$$

(b) Period of Mars in its orbit,

$$T_{\text{Mars}} = \frac{2\pi}{\sqrt{\mu}}r_{\text{Mars}}{}^{3/2} = \frac{2\pi}{\sqrt{132.7\times 10^9}}\left(227.9\times 10^6\right)^{3/2} = 59.34\times 10^6 \text{ s} = 686.8 \text{ days}$$

$$\therefore \frac{180° - \alpha}{180°} = \frac{tof}{\frac{T_{\text{Mars}}}{2}} = \frac{258.8}{343.4} = 0.7537$$

$$\underline{\alpha = 44.33°}$$

Problem 6.6

$$r_A = 7000 \text{ km} \quad r_C = 32\,000 \text{ km}$$

$$e_1 = 0.3$$

$$e_1 = \frac{r_B - r_A}{r_B + r_A}$$

$$0.3 = \frac{r_B - 7000}{r_B + 7000} \Rightarrow r_B = 13\,000 \text{ km}$$

$$e_2 = 0.5$$

$$e_2 = \frac{r_D - r_C}{r_D + r_C}$$

$$0.5 = \frac{r_D - 32\,000}{r_D + 32\,000} \Rightarrow r_D = 96\,000 \text{ km}$$

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2\cdot 398\,600}\sqrt{\frac{7000\cdot 13\,000}{7000 + 13\,000}} = 60\,230 \text{ km}^2/\text{s}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_C r_D}{r_C + r_D}} = \sqrt{2\cdot 398\,600}\sqrt{\frac{32\,000\cdot 96\,000}{32\,000 + 96\,000}} = 138\,300 \text{ km}^2/\text{s}$$

$$h_3 = \sqrt{2\mu}\sqrt{\frac{r_A r_D}{r_A + r_D}} = \sqrt{2\cdot 398\,600}\sqrt{\frac{7000\cdot 96\,000}{7000 + 96\,000}} = 72\,120 \text{ km}^2/\text{s}$$

$$h_4 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2\cdot 398\,600}\sqrt{\frac{13\,000\cdot 32\,000}{13\,000 + 32\,000}} = 85\,850 \text{ km}^2/\text{s}$$

$$v_{A1} = \frac{h_1}{r_A} = \frac{60\,230}{7000} = 8.604 \text{ km/s} \qquad v_{A3} = \frac{h_3}{r_A} = \frac{72\,120}{7000} = 10.3 \text{ km/s}$$

$$v_{B1} = \frac{h_1}{r_B} = \frac{60\,230}{13\,000} = 4.633 \text{ km/s} \qquad v_{B4} = \frac{h_4}{r_B} = \frac{85\,850}{13\,000} = 6.604 \text{ km/s}$$

$$v_{C2} = \frac{h_2}{r_C} = \frac{138\,300}{32\,000} = 4.323 \text{ km/s} \qquad v_{C4} = \frac{h_4}{r_C} = \frac{85\,850}{32\,000} = 2.683 \text{ km/s}$$

$$v_{D2} = \frac{h_2}{r_D} = \frac{138\,300}{96\,000} = 1.441 \text{ km/s} \qquad v_{D3} = \frac{h_3}{r_D} = \frac{72\,120}{96\,000} = 0.7512 \text{ km/s}$$

$$T_3 = \frac{2\pi}{\sqrt{\mu}}\left(\frac{r_A + r_D}{2}\right)^{3/2} = \frac{2\pi}{\sqrt{398\,600}}\left(\frac{7000 + 96\,000}{2}\right)^{3/2} = 116\,300 \text{ s} = 32.31 \text{ h}$$

$$T_4 = \frac{2\pi}{\sqrt{\mu}}\left(\frac{r_B + r_C}{2}\right)^{3/2} = \frac{2\pi}{\sqrt{398\,600}}\left(\frac{13\,000 + 32\,000}{2}\right)^{3/2} = 33\,590 \text{ s} = 9.33 \text{ h}$$

(a) Transfer orbit 3:

$$\begin{aligned}
\Delta v_A &= v_{A3} - v_{A1} = 10.3 - 8.604 = 1.699 \text{ km/s}\\
\Delta v_D &= v_{D2} - v_{D3} = 1.441 - 0.7512 = 0.6896 \text{ km/s}\\
\Delta v_{\text{total}} &= \Delta v_A + \Delta v_D = \underline{2.389 \text{ km/s}}\\
tof_3 &= \frac{T_3}{2} = \frac{32.32}{2} = \underline{16.15 \text{ h}}
\end{aligned}$$

(b) Transfer orbit 4:

$$\begin{aligned}
\Delta v_B &= v_{B4} - v_{B1} = 6.604 - 4.633 = 1.971 \text{ km/s}\\
\Delta v_C &= v_{C2} - v_{C4} = 4.323 - 2.683 = 1.64 \text{ km/s}\\
\Delta v_{\text{total}} &= \Delta v_B + \Delta v_C = \underline{3.611 \text{ km/s}}\\
tof_4 &= \frac{T_4}{2} = \frac{9.33}{2} = \underline{4.665 \text{ h}}
\end{aligned}$$

Problem 6.7 Orbit 1 is the original circular orbit and orbit 2 is the impact trajectory.

$$v_{A1} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6378 + 500}} = 7.613 \text{ km/s}$$

$$r_{\text{apogee}_2} = \frac{h_2^2}{\mu}\frac{1}{1 - e_2}$$

$$6378 + 500 = \frac{h_2^2}{\mu}\frac{1}{1 - e_2} \Rightarrow \frac{h_2^2}{\mu} = 6878(1 - e_2)$$

$$r_{B2} = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos\theta_B}$$

$$6378 = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos 60°} \Rightarrow \frac{h_2^2}{\mu} = 6378(1 + 0.5e_2)$$

$$\frac{h_2^2}{\mu} = \frac{h_2^2}{\mu} \Rightarrow 6378(1 + 0.5e_2) = 6878(1 - e_2) \Rightarrow e_2 = 0.04967$$

$$\therefore h_2 = \sqrt{6878\mu(1 - e_2)} = \sqrt{6878 \cdot 398\,600 \cdot (1 - 0.04967)} = 51\,040 \text{ km}^2/\text{s}$$

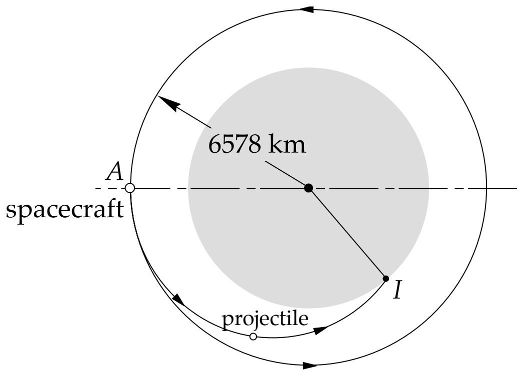
(a)

$$\begin{aligned}
v_{A2} &= \frac{h_2}{r_A} = \frac{51\,040}{6878} = 7.421 \text{ km/s}\\
\Delta v &= v_{A2} - v_{A1} = 7.421 - 7.613 = \underline{-0.1915 \text{ km/s}}
\end{aligned}$$

(b) To fall through the point directly below, we must remove completely the transverse component of velocity:

$$\Delta v = 0 - v_{A1} = \underline{-7.613 \text{ km/s}}$$

Problem 6.8 A is apogee of impact trajectory, I is the impact point, a is the semimajpr axis.



$$r_A = a(1+e)$$

$$\therefore e = \frac{r_A}{a} - 1 \qquad (1)$$

From Equation 3.22,

$$\begin{aligned} r_I &= a(1 - e\cos E) \\ &= a\left[1 - \left(\frac{r_A}{a} - 1\right)\cos E\right] \\ &= (1 + \cos E)a - r_A \cos E \end{aligned}$$

$$\therefore a = \frac{r_I + r_A \cos E}{1 + \cos E} \qquad (2)$$

Substitute (2) into (1) to get

$$e = \frac{r_A}{\dfrac{r_I + r_A\cos E}{1+\cos E}} - 1 = \frac{r_A - r_I}{r_A \cos E + r_I} \qquad (3)$$

Mean anomaly of the impact point (measured ccw from perigee) is

$$M = 2\pi \frac{\left(\dfrac{T}{2} + t\right)}{T} = \pi + 2\pi\frac{t}{T} = \pi + 2\pi \frac{t}{\dfrac{2\pi}{\sqrt{\mu}}a^{3/2}} = \pi + \frac{\sqrt{\mu}t}{a^{3/2}}$$

Let $f(E) = M - E + e\sin E$. Then Kepler's equation is $f(E) = 0$.

$$f(E) = M - E + e\sin E$$

$$f(E) = \pi + \frac{\sqrt{\mu}t}{a^{3/2}} - E + e\sin E = \pi + \frac{\sqrt{\mu}t}{\left(\dfrac{r_i + r_a\cos E}{1+\cos E}\right)^{3/2}} - E + \frac{r_a - r_i}{r_a \cos E + r_i}\sin E$$

Setting $r_A = 6578$ km, $r_I = 6378$ km, $t = 30 \cdot 60 = 1800\,\text{s}$,

$$f(E) = \pi + \frac{1136\,400}{\left(\dfrac{6378 + 6578\cos E}{1 + \cos E}\right)^{3/2}} - E + \frac{200}{6578\cos E + 6378}\sin E$$

Graphing $f(E)$ reveals that $f(E) = 0$ at $E = 5.319\,\text{rad}$. Substituting this into (1) and (2) yields

$$e = 0.01975 \qquad a = 6451\ \text{km}$$

True anomaly of the impact point:

$$\tan\frac{\theta_I}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_I}{2} = \sqrt{\frac{1+0.01975}{1-0.01975}}\tan\frac{5.319}{2} \Rightarrow \theta = 303.8° \quad (123.8° \text{ cw from apogee})$$

$$h = \sqrt{\mu a(1-e^2)} = \sqrt{398\,600 \cdot 6451 \cdot (1 - 0.01975^2)} = 50700\ \text{km}^2/\text{s}$$

$$v_A = \frac{h}{r_A} = \frac{50\,700}{6578} = 7.707 \text{ km/s} \quad \text{velocity of projectile at apogee.}$$

$$v_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6578}} = 7.784 \text{ km/s} \quad \text{velocity of spacecraft in circular orbit.}$$

$$\Delta v = v_a - v_c = 7.707 - 7.784 = \underline{-0.07725 \text{ km/s}}$$

Problem 6.9

$$r_{\text{apogee}_1} = 6378 + 302 = 6680 \text{ km} \qquad r_{\text{perigee}_1} = 6378 + 296 = 6674 \text{ km}$$

$$r_{\text{apogee}_2} = 6378 + 291 = 6669 \text{ km} \qquad r_{\text{perigee}_2} = 6378 + 259 = 6637 \text{ km}$$

$$r_3 = 6378 + 259 = 6637 \text{ km}$$

$$r_{\text{apogee}_4} = 6378 + 255 = 6633 \text{ km} \qquad r_{\text{perigee}_4} = 6378 + 194 = 6572 \text{ km}$$

$$h_1 = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee}_1} r_{\text{perigee}_1}}{r_{\text{apogee}_1} + r_{\text{perigee}_1}}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6680 \cdot 6674}{6680 + 6674}} = 51\,590 \text{ km}^2/\text{s}$$

$$v_{\text{apogee}_1} = \frac{h_1}{r_{\text{apogee}_1}} = \frac{51\,590}{6680} = 7.723 \text{ km/s}$$

$$v_{\text{perigee}_1} = \frac{h_1}{r_{\text{perigee}_1}} = \frac{51\,590}{6674} = 7.730 \text{ km/s}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee}_2} r_{\text{perigee}_2}}{r_{\text{apogee}_2} + r_{\text{perigee}_2}}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6669 \cdot 6637}{6669 + 6637}} = 51\,500 \text{ km}^2/\text{s}$$

$$v_{\text{apogee}_2} = \frac{h_2}{r_{\text{apogee}_2}} = \frac{51\,500}{6669} = 7.722 \text{ km/s}$$

$$v_{\text{perigee}_2} = \frac{h_1}{r_{\text{perigee}_2}} = \frac{51\,500}{6637} = 7.759 \text{ km/s}$$

$$v_3 = \sqrt{\frac{\mu}{r_3}} = \sqrt{\frac{398\,600}{6637}} = 7.750 \text{ km/s}$$

$$h_4 = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee}_4} r_{\text{perigee}_4}}{r_{\text{apogee}_4} + r_{\text{perigee}_4}}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6633 \cdot 6572}{6633 + 6572}} = 51\,300 \text{ km}^2/\text{s}$$

$$v_{\text{apogee}_4} = \frac{h_4}{r_{\text{apogee}_4}} = \frac{51\,300}{6633} = 7.734 \text{ km/s}$$

$$v_{\text{perigee}_4} = \frac{h_4}{r_{\text{perigee}_4}} = \frac{51\,300}{6572} = 7.806 \text{ km/s}$$

Apogee of orbit 1 to perigee of orbit 2:

$$h_{12} = \sqrt{2\mu}\sqrt{\frac{r_{\text{apogee}_1} r_{\text{perigee}_2}}{r_{\text{apogee}_1} + r_{\text{perigee}_2}}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6680 \cdot 6637}{6680 + 6637}} = 51\,520 \text{ km}^2/\text{s}$$

$$\Delta v_{12} = \left|\frac{h_{12}}{r_{\text{apogee}_1}} - v_{\text{apogee}_1}\right| + \left|\frac{h_{12}}{r_{\text{perigee}_2}} - v_{\text{perigee}_2}\right| = |7.712 - 7.723| + |7.762 - 7.759| = 0.013\,93 \text{ km/s}$$

Perigee of orbit 2 to orbit 3 (tangent):

$$\Delta v_{23} = v_{\text{perigee}_2} - v_3 = 7.759 - 7.750 = 0.009\,313 \text{ km/s}$$

Orbit 3 to perigee of orbit 4:

$$h_{34} = \sqrt{2\mu}\sqrt{\frac{r_3 r_{\text{perigee}_4}}{r_3 + r_{\text{perigee}_4}}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6572 \cdot 6637}{6572 + 6637}} = 51\,310 \text{ km}^2/\text{s}$$

$$\Delta v_{34} = \left|\frac{h_{34}}{r_3} - v_3\right| + \left|\frac{h_{34}}{r_{\text{perigee}_4}} - v_{\text{perigee}_4}\right| = |7.731 - 7.75| + |7.807 - 7.806| = 0.020\,26 \text{ km/s}$$

$$\Delta v_{total} = \Delta v_{12} + \Delta v_{23} + \Delta v_{34} = 0.013\,93 + 0.009\,313 + 0.020\,26 = \underline{0.04351 \text{ km/s}}$$

Problem 6.10 $r_A = 6878$ km $r_B = 7378$ km

$$v_{A_1} = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{398\,600}{6878}} = 7.613 \text{ km/s}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6878 \cdot 7378}{6878 + 7378}} = 53\,270 \text{ km}^2/\text{s}$$

$$v_{A_2} = \frac{h_2}{r_A} = \frac{53\,270}{6878} = 7.745 \text{ km/s}$$

$$v_{B_2} = \frac{h_2}{r_B} = \frac{53\,270}{7378} = 7.220 \text{ km/s}$$

Alternatively, the energy equation, $v^2/2 - \mu/r = -\mu/(2a)$, implies

$$v = \sqrt{\left(\frac{2}{r} - \frac{1}{a}\right)\mu}$$

so that, since $a = (r_A + r_B)/2 = 7128$ km,

$$v_{A_2} = \sqrt{\left(\frac{2}{r_A} - \frac{1}{a}\right)\mu} = \sqrt{\left(\frac{2}{6878} - \frac{1}{7128}\right) \cdot 398\,600} = 7.745 \text{ km/s}$$

$$v_{B_2} = \sqrt{\left(\frac{2}{r_B} - \frac{1}{a}\right)\mu} = \sqrt{\left(\frac{2}{7378} - \frac{1}{7128}\right) \cdot 398\,600} = 7.220 \text{ km/s}$$

$$v_{B_3} = \sqrt{\frac{\mu}{r_B}} = \sqrt{\frac{398\,600}{7378}} = 7.35 \text{ km/s}$$

$$\Delta v = |v_{A_2} - v_{A_1}| + |v_{B_3} - v_{B_2}| = 0.1323 + 0.1300 = \underline{0.2624 \text{ km/s}}$$

Problem 6.11

$$v_{A_1} = \sqrt{\frac{\mu}{r}}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r(12r)}{r + 12r}} = 1.359\sqrt{\mu r}$$

$$v_{A_2} = \frac{h_2}{r} = 1.359\sqrt{\frac{\mu}{r}}$$

$$v_{B_2} = \frac{h_2}{12r} = 0.1132\sqrt{\frac{\mu}{r}}$$

Alternatively, using the energy equation,

$$a_2 = \frac{r + 12r}{2} = 6.5r$$

$$v_{A_2} = \sqrt{\left(\frac{2}{r} - \frac{1}{a_2}\right)\mu} = \sqrt{1.846\frac{\mu}{r}} = 1.359\sqrt{\frac{\mu}{r}}$$

$$v_{B_2} = \sqrt{\left(\frac{2}{12r} - \frac{1}{a_2}\right)\mu} = \sqrt{0.01282\frac{\mu}{r}} = 0.1132\sqrt{\frac{\mu}{r}}$$

$$v_{B_3} = \sqrt{\frac{\mu}{12r}} = 0.2887\sqrt{\frac{\mu}{r}}$$

$$\Delta v = \left|v_{A_2} - v_{A_1}\right| + \left|v_{B_3} - v_{B_2}\right| = 0.3587\sqrt{\frac{\mu}{r}} + 0.1754\sqrt{\frac{\mu}{r}} = \underline{0.5342\sqrt{\frac{\mu}{r}}}$$

Problem 6.12

$$v_{A_1} = \sqrt{\frac{\mu}{r}} \qquad v_{A_2} = \sqrt{\frac{2\mu}{r}} = 1.414\sqrt{\frac{\mu}{r}}$$

$$v_{B_3} = \sqrt{\frac{2\mu}{12r}} = 0.4082\sqrt{\frac{\mu}{r}} \qquad v_{B_4} = \sqrt{\frac{\mu}{12r}} = 0.2887\sqrt{\frac{\mu}{r}}$$

$$\Delta v = \left|v_{A_2} - v_{A_1}\right| + \left|v_{B_4} - v_{B_3}\right| = 0.4142\sqrt{\frac{\mu}{r}} + 0.1196\sqrt{\frac{\mu}{r}} = \underline{0.5338\sqrt{\frac{\mu}{r}}}$$

Problem 6.13 $r_A = r \qquad r_B = 3r$

$$v_{A_1} = v_1 = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{\mu}{r}}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2\mu}\sqrt{\frac{r(3r)}{r + 3r}} = 1.225\sqrt{\mu r}$$

$$v_{A_2} = \frac{h_2}{r_A} = \frac{1.225\sqrt{\mu r}}{r} = 1.225\sqrt{\frac{\mu}{r}}$$ (Alternatively, use the energy equation.)

$$v_{B_2} = \frac{h_2}{r_B} = \frac{1.225\sqrt{\mu r}}{3r} = 0.4082\sqrt{\frac{\mu}{r}}$$

$$v_{B_3} = v_3 = \sqrt{\frac{\mu}{r_B}} = \sqrt{\frac{\mu}{3r}} = 0.5774\sqrt{\frac{\mu}{r}}$$

$$\Delta v = \left|v_{A_2} - v_{A_1}\right| + \left|v_{B_3} - v_{B_2}\right| = 0.2247\sqrt{\frac{\mu}{r}} + 0.1691\sqrt{\frac{\mu}{r}} = 0.3938\sqrt{\frac{\mu}{r}} = \underline{0.3938 v_1}$$

Problem 6.14 $r_A = 6678$ km $\qquad r_C = 9378$ km

(a)

Orbit 1:

$$v_{A_1} = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{398\,600}{6678}} = 7.726 \text{ km/s}$$

Orbit 2:

$$\frac{r_B - r_A}{r_B + r_A} = e_2$$

$$\frac{r_B - 6678}{r_B + 6678} = 0.3 \quad \Rightarrow \quad r_B = 12\,402 \text{ km}$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_A r_B}{r_A + r_B}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{6678 \cdot 12\,402}{6678 + 12\,402}} = 58\,830 \text{ km}^2/\text{s}$$

$$v_{A2} = \frac{h_2}{r_A} = \frac{58\,830}{6678} = 8.809 \text{ km/s}$$

$$v_{B2} = \frac{h_2}{r_B} = \frac{58\,830}{12\,402} = 4.743 \text{ km/s}$$

Orbit 3:

$$h_3 = \sqrt{2\mu}\sqrt{\frac{r_B r_C}{r_B + r_C}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{12\,402 \cdot 9378}{12\,402 + 9378}} = 65\,250 \text{ km}^2/\text{s}$$

$$v_{B3} = \frac{h_3}{r_B} = \frac{65\,250}{12\,402} = 5.261 \text{ km/s}$$

$$v_{C3} = \frac{h_3}{r_C} = \frac{65\,250}{9378} = 6.957 \text{ km/s}$$

Orbit 4:

$$v_{C4} = \sqrt{\frac{\mu}{r_C}} = \sqrt{\frac{398\,600}{9\,378}} = 6.519 \text{ km/s}$$

$$\Delta v_{total} = |v_{A2} - v_{A1}| + |v_{B3} - v_{B2}| + |v_{C4} - v_{C3}| = 1.083 + 0.5177 + 0.4379 = \underline{2.039 \text{ km/s}}$$

(b)

$$T_2 = \frac{2\pi}{\sqrt{\mu}}\left(\frac{r_A + r_B}{2}\right)^{3/2} = \frac{2\pi}{\sqrt{398\,600}}\left(\frac{6678 + 12\,402}{2}\right)^{3/2} = 9273 \text{ s}$$

$$T_3 = \frac{2\pi}{\sqrt{\mu}}\left(\frac{r_B + r_C}{2}\right)^{3/2} = \frac{2\pi}{\sqrt{398\,600}}\left(\frac{12\,402 + 9378}{2}\right)^{3/2} = 11\,310 \text{ s}$$

$$t_{total} = \frac{1}{2}(T_1 + T_2) = 10\,290 \text{ s} = \underline{2.859 \text{ hr}}$$

Problem 6.15 $r_A = r_C = r_E = 15\,000$ km $\quad r_B = 22\,000$ km $\quad r_D = 6878$ km

(a)

Orbit 1:

$$v_A)_1 = \sqrt{\frac{\mu}{r_A}} = \sqrt{\frac{398\,600}{15\,000}} = 5.155 \text{ km/s}$$

$$\gamma_{A1} =$$

Orbit 2:

$$e_2 = \frac{r_B - r_D}{r_B + r_D} = \frac{22\,000 - 6878}{22\,000 + 6878} = 0.5237$$

$$h_2 = \sqrt{2\mu}\sqrt{\frac{r_B r_D}{r_B + r_D}} = \sqrt{2 \cdot 398\,600}\sqrt{\frac{22\,000 \cdot 6878}{22\,000 + 6878}} = 64\,630 \text{ km}^2/\text{s}$$

At the maneuvering point A:

$$r_A = \frac{h_2^{\,2}}{\mu}\frac{1}{1+e_2\cos\theta_A}$$

$$15\,000 = \frac{64\,630^2}{398\,600}\frac{1}{1+0.5237\cos\theta_A} \quad\Rightarrow\quad \theta_A = 125.1°$$

$$v_{A2}\big)_\perp = \frac{h_2}{r_A} = \frac{64\,630}{15\,000} = 4.309\ \text{km/s}$$

$$v_{A2}\big)_r = \frac{\mu}{h_2}e_2\sin\theta_A = \frac{398\,600}{64\,630}0.5237\sin 125.1° = 2.641\ \text{km/s}$$

$$v_{A2} = \sqrt{v_{A2}\big)_\perp^{\,2} + v_{A2}\big)_r^{\,2}} = \sqrt{4.309^2 + 2.641^2} = 5.054\ \text{km/s}$$

$$\gamma_{A2} = \tan^{-1}\frac{v_{A2}\big)_r}{v_{A2}\big)_\perp} = \tan^{-1}\frac{2.641}{4.309} = 0.5499 \quad\Rightarrow\quad \gamma_{A2} = 31.51°$$

$$\Delta\gamma_A = \gamma_{A2} - \gamma_{A1} = 31.51° - 0 = 31.51°$$

$$\Delta v_A = \sqrt{v_{A1}^{\,2} + v_{A2}^{\,2} - 2v_{A1}v_{A2}\cos\Delta\gamma_A} = \sqrt{5.155^2 + 5.054^2 - 25.1555.054\cos\Delta\gamma_A} = \underline{2.773\ \text{km/s}}\text{T}$$

(b)
Try Hohmann transfer (orbit 3) from point E on orbit 1 to point B on orbit 2.

$$h_3 = \sqrt{2\mu}\sqrt{\frac{r_E r_B}{r_E + r_B}} = \sqrt{2\cdot 398\,600}\sqrt{\frac{15\,000\cdot 22\,000}{15\,000 + 22\,000}} = 84\,320\ \ \text{km}^2/\text{s}$$

$$v_{E_1} = v_{A1} = 5.155\ \text{km/s}$$

$$v_{E3} = \frac{h_3}{r_E} = \frac{84\,320}{15\,000} = 5.621\ \text{km/s}$$

$$v_{B3} = \frac{h_3}{r_B} = \frac{84\,320}{22\,000} = 3.833\ \text{km/s}$$

$$v_{B2} = \frac{h_2}{r_B} = \frac{64\,630}{22\,000} = 2.938\ \text{km/s}$$

$$\Delta v_{total} = \left|v_{E3} - v_{E_1}\right| + \left|v_{B2} - v_{B3}\right| = 0.4665 + 0.985 = 1.362\ \text{km/s}$$

Try Hohmann transfer (orbit 4) from point C on orbit 1 to point D on orbit 2.

$$h_4 = \sqrt{2\mu}\sqrt{\frac{r_C r_D}{r_C + r_D}} = \sqrt{2\cdot 398\,600}\sqrt{\frac{15\,000\cdot 6878}{15\,000 + 6878}} = 61\,310\ \ \text{km}^2/\text{s}$$

$$v_{C1} = v_{A1} = 5.155\ \text{km/s}$$

$$v_{C4} = \frac{h_4}{r_C} = \frac{61\,310}{15\,000} = 4.088\ \text{km/s}$$

$$v_{D4} = \frac{h_4}{r_D} = \frac{61\,310}{6878} = 8.914\ \text{km/s}$$

$$v_{D4} = \frac{h_2}{r_D} = \frac{64\,630}{6878} = 9.397\ \text{km/s}$$

$$\Delta v_{total} = \left|v_{C4} - v_{C_1}\right| + \left|v_{D2} - v_{D4}\right| = 1.067 + 0.4824 = 1.55\ \text{km/s}$$

This is larger than the total computed above; thus for minimum Hohmann transfer

$$\underline{\Delta v = 1.362\ \text{km/s}}$$

Problem 6.16

Orbit 1:

$$r_{\text{perigee}_1} = 6378 + 1270 = 7648 \text{ km}$$

$$v_{\text{perigee}_1} = 9 \text{ km/s}$$

$$h_1 = r_{\text{perigee}_1} v_{\text{perigee}_1} = 7648 \cdot 9 = 68\,832 \text{ km}^2/\text{s}$$

$$r_{\text{perigee}_1} = \frac{h_1^2}{\mu}\frac{1}{1+e_1}$$

$$7648 = \frac{68\,832^2}{398\,600}\frac{1}{1+e_1} \Rightarrow e_1 = 0.5542$$

At the maneuver point, $\theta = 100°$.

$$r = \frac{h_1^2}{\mu}\frac{1}{1+e_1\cos\theta} = \frac{68\,832^2}{398\,600}\frac{1}{1+0.5542\cos 100°} = 13\,150 \text{ km}$$

$$v_{1_\perp} = \frac{h_1}{r} = \frac{68\,832}{13\,150} = 5.234 \text{ km/s}$$

$$v_{1_r} = \frac{\mu}{h_1}e_1\sin\theta = \frac{398\,600}{68\,832}0.5542\sin 100° = 3.16 \text{ km/s}$$

$$v_1 = \sqrt{{v_{1_\perp}}^2 + {v_{1_r}}^2} = \sqrt{5.234^2 + 3.16^2} = 6.114 \text{ km/s}$$

$$\gamma_1 = \tan^{-1}\frac{v_{1_r}}{v_{1_\perp}} = \tan^{-1}\frac{3.16}{5.234} = 31.13$$

Orbit 2:

$$e_2 = 0.4$$

$$r = \frac{h_2^2}{\mu}\frac{1}{1+e_2\cos\theta}$$

$$13\,150 = \frac{h_2^2}{398\,600}\frac{1}{1+0.4\cos 100°} \Rightarrow h_2 = 69\,840 \text{ km}^2/\text{s}$$

$$v_{2_\perp} = \frac{h_2}{r} = \frac{69\,840}{13\,150} = 5.311 \text{ km/s}$$

$$v_{2_r} = \frac{\mu}{h_2}e_2\sin\theta = \frac{398\,600}{69\,840}0.4\sin 100° = 2.248 \text{ km/s}$$

$$v_2 = \sqrt{{v_{2_\perp}}^2 + {v_{2_r}}^2} = \sqrt{5.311^2 + 2.248^2} = 5.767 \text{ km/s}$$

$$\gamma_2 = \tan^{-1}\frac{v_{2_r}}{v_{2_\perp}} = \tan^{-1}\frac{2.248}{5.767} = 22.94°$$

$$\Delta\gamma = \gamma_2 - \gamma_1 = 22.94° - 31.13° = \underline{-8.181°}$$

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\Delta\gamma} = \sqrt{6.114^2 + 5.767^2 - 2\cdot 6.114\cdot 5.767\cos(-8.181)} = \underline{0.9155 \text{ km/s}}$$

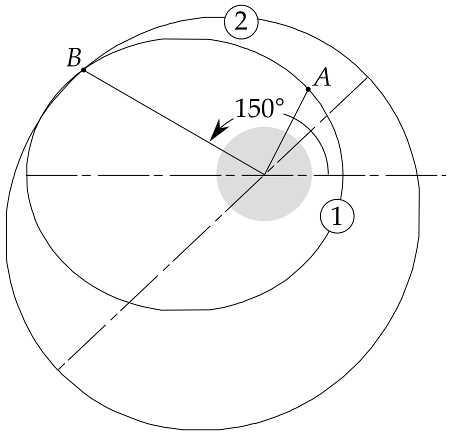
Problem 6.17

$$r_A = 12\,756 \text{ km} \qquad v_A = 6.5992 \text{ km} \qquad \gamma_A = 20°$$

Orbit 1:

$$v_{A_\perp} = v_A\cos\gamma_A = 6.5992\cos 20° = 6.20122 \text{ km/s}$$

$$\therefore h_1 = r_A v_{A_\perp} = 12\,756 \cdot 6.20122 = 79\,102.8 \text{ km}^2/\text{s}$$



$$v_{A_r} = v_A \sin\gamma_A = 6.5992 \cdot \sin 20° = 2.25706 \text{ km/s}$$

$$v_{A_r} = \frac{\mu}{h_1} e_1 \sin\theta_A$$

$$2.25706 = \frac{398\,600}{79\,102.8} e_1 \sin\theta_A \;\Rightarrow\; e_1 \sin\theta_A = 0.447\,917$$

$$r_A = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos\theta_A}$$

$$12756 = \frac{79\,102.8^2}{398\,600}\frac{1}{1 + e_1\cos\theta_A} \;\Rightarrow\; e_1\cos\theta_A = 0.230\,641$$

$$\therefore e_1^2\left(\sin^2\theta_A + \cos^2\theta_A\right) = 0.447\,917^2 + 0.230\,641^2$$

$$e_1^2 = 0.253\,825 \;\Rightarrow\; e_1 = 0.503\,81$$

$$\therefore \sin\theta_A = \frac{0.447\,917}{0.503\,81} = 0.889\,058 \;\Rightarrow\; \theta_A = 62.7552° \text{ or } \theta_A = 117.235°$$

Since $\cos\theta_A > 0$, $\theta_A = 62.7552°$.

$$r_B = \frac{h_1^2}{\mu}\frac{1}{1 + e_1\cos\theta_{B1}} = \frac{79\,102.8^2}{398\,600}\frac{1}{1 + 0.503\,81\cos 150°} = 27\,848.9 \text{ km}$$

$$\left.v_{B\perp}\right)_1 = \frac{h_1}{r_B} = \frac{79\,102.8}{27\,848.9} = 2.840\,43 \text{ km/s}$$

$$\left.v_{B_r}\right)_1 = \frac{\mu}{h_1} e_1 \sin\theta_{B1} = \frac{398\,600}{79\,102.8}\cdot 0.503\,81 \cdot \sin 150° = 1.269\,45 \text{ km/s}$$

Orbit 2:

$$\Delta v_{B\perp} = 0.758\,20 \text{ km/s}$$

$$\therefore \left.v_{B\perp}\right)_2 = \left.v_{B\perp}\right)_1 + \Delta v_{B\perp} = 2.840\,43 + 0.758\,20 = 3.598\,63 \text{ km/s}$$

$$h_2 = r_B \left.v_{B\perp}\right)_2 = 27\,848.9 \cdot 3.598\,63 = 100\,218 \text{ km}^2/\text{s}$$

$$\Delta v_{B_r} = 0$$

$$\therefore \left.v_{B_r}\right)_2 = \left.v_{B_r}\right)_1 + \Delta v_{B\perp} = 1.269\,45 + 0 = 1.269\,45 \text{ km/s}$$

$$\left.v_{B_r}\right)_2 = \frac{\mu}{h_2} e_2 \sin\theta_{B2}$$

$$1.269\,45 = \frac{398\,600}{100\,218} e_2 \sin\theta_{B2} \;\Rightarrow\; e_2 \sin\theta_{B2} = 0.319\,146$$

$$r_B = \frac{h_2^2}{\mu}\frac{1}{1 + e_2\cos\theta_{B2}}$$

$$27\,848.9 = \frac{100\,218^2}{398\,600}\frac{1}{1 + e_2\cos\theta_{B2}} \;\Rightarrow\; e_2\cos\theta_{B2} = -0.095\,216\,6$$

$$e_2^{\,2}\left(\sin^2\theta_{B2}+\cos^2\theta_{B2}\right)=0.319146^2+\left(-0.0952166\right)^2$$
$$e_2^{\,2}=0.110921$$
$$e_2=0.333048$$

$$\therefore \sin\theta_{B2}=\frac{0.319146}{0.333048}=0.958261 \;\Rightarrow\; \theta_{B2}=73.3877° \text{ or } 106.612°$$

Since $\cos\theta_{B2}<0$, $\theta_{B2}=106.612°$.

$$\Delta\theta=150-106.612°=\underline{43.3877°}$$

That is, the apse line is rotated 43.387 7° ccw from that of orbit 1.

Problem 6.18

$$r_1=r_2$$
$$\frac{h^2}{\mu}\frac{1}{1+e\cos\theta}=\frac{h^2}{\mu}\frac{1}{1+e\cos(\eta-\theta)}$$
$$\cos\theta=\cos(\eta-\theta)$$

$$\therefore \theta=\eta-\theta \;\Rightarrow\; 2\theta=\eta \;\Rightarrow\; \underline{\theta=\frac{\eta}{2}}$$

Problem 6.19

Orbit 1:

$$r_{P_1}=\frac{h_1^{\,2}}{\mu}\frac{1}{1+e}$$

Orbit 2:

$$r_{P_1}=\frac{h_2^{\,2}}{\mu}\frac{1}{1+e\cos 90°}=\frac{h_2^{\,2}}{\mu}$$

$$\therefore \frac{h_2^{\,2}}{\mu}=\frac{h_1^{\,2}}{\mu}\frac{1}{1+e} \;\Rightarrow\; \underline{h_2=\frac{h_1}{\sqrt{1+e}}}$$

Problem 6.20

At A:

$$r=\frac{h^2}{\mu}$$
$$v_{r1}=\frac{\mu}{h}e\sin 90°=\frac{\mu}{h}e$$
$$v_{r2}=\frac{\mu}{h}e\sin(-90°)=-\frac{\mu}{h}e$$
$$v_{\perp 1}=v_{\perp 2}=\frac{h}{r}$$
$$\therefore \Delta v_{\perp}=0$$
$$\Delta v_r=v_{r2}-v_{r1}=-2\frac{\mu}{h}e$$

$$\therefore \Delta v = |\Delta v_r| = \underline{\frac{2\mu e}{h}}$$

Problem 6.21 For the circular orbit of the space station,

$$r = 6728 \text{ km} \quad v_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6728}} = 7.697 \text{ km/s} \quad T_c = \frac{2\pi}{\sqrt{\mu}} r^{3/2} = \frac{2\pi}{\sqrt{398\,600}} 6728^{3/2} = 5492 \text{ s} = 91.54 \text{ m}$$

(a) The time required for spacecraft A to reach the space station is the time it takes for the space station to fly around to the original position of spacecraft A.

$$t_{SA} = T_c \frac{2\pi r - 600}{2\pi r} = 5492 \frac{2\pi \cdot 6728 - 600}{2\pi \cdot 6728} = 5414 \text{ s} = \underline{90.2 \text{ min}}$$

The time required for spacecraft B to reach the space station is the time it takes for the space station to fly around to the original position of spacecraft B.

$$t_{BS} = T_c \frac{2\pi r + 600}{2\pi r} = 5492 \frac{2\pi \cdot 6728 + 600}{2\pi \cdot 6728} = 5570 \text{ s} = \underline{92.8 \text{ min}}$$

(b)

The period of spacecraft A's phasing orbit, is t_{SA}, which determines the semimajor axis of that orbit:

$$5414 = \frac{2\pi}{\sqrt{398\,600}} a_A^{3/2} \Rightarrow a_A = 6664 \text{ km}$$

Spacecraft A is at the *apogee* of its phasing orbit. From the energy equation

$$v_A = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_A}\right)} = \sqrt{398\,600\left(\frac{2}{6728} - \frac{1}{6664}\right)} = 7.660 \text{ km/s}$$

The delta-v required to drop into the phasing orbit is

$$\Delta v_A = v_A - v_c = 7.660 - 7.697 = -0.036\,94 \text{ km/s}$$

Spacecraft A must therefore slow down in order to speed up (i.e., catch the space station). After one circuit of its phasing orbit, this delta-v must be *added* in order to rejoin the circular orbit. Thus

$$\Delta v_{A_{total}} = 2|\Delta v_A| = \underline{0.073\,88 \text{ km/s}}$$

The period of spacecraft B's phasing orbit, is t_{BS}, which determines the semimajor axis of that orbit:

$$5570 = \frac{2\pi}{\sqrt{398\,600}} a_B^{3/2} \Rightarrow a_B = 6791 \text{ km}$$

Spacecraft B is at the *perigee* of its phasing orbit. From the energy equation

$$v_B = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a_B}\right)} = \sqrt{398\,600\left(\frac{2}{6728} - \frac{1}{6791}\right)} = 7.733 \text{ km/s}$$

The prograde delta-v required to enter the phasing orbit is

$$\Delta v_B = v_B - v_c = 7.733 - 7.697 = +0.03576\ \text{km/s}$$

Spacecraft B must therefore speed up in order to slow down (i.e, allow the space station to catch up). After one circuit of its phasing orbit, this delta-v must be *subtracted* in order to rejoin the circular orbit. Thus

$$\Delta v_{B_{total}} = 2|\Delta v_B| = \underline{0.071\,53\ \text{km/s}}$$

Problem 6.22

$$T_{\text{phasing}} = \frac{T}{2}$$

$$\frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{1}{2}\frac{2\pi}{\sqrt{\mu}}r^{3/2}$$

$$a^{3/2} = \frac{1}{2}r^{3/2}$$

$$a = \left(\frac{1}{2}r^{3/2}\right)^{2/3} \quad \Rightarrow \quad \underline{a = 0.63r}$$

Problem 6.23 $r_A = 13\,000\ \text{km} \qquad r_P = 8000\ \text{km}$

Orbit 1:

$$a_1 = \frac{r_A + r_P}{2} = 10\,500\ \text{km}$$

$$e_1 = \frac{r_A - r_P}{r_A + r_P} = 0.2381$$

$$h_1 = \sqrt{\mu(1+e)r_P} = \sqrt{398\,600(1+0.2381)\cdot 8000} = 62\,830\ \ \text{km}^2/\text{s}$$

$$T_1 = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\sqrt{398\,600}}10\,500^{3/2} = 10710\ \text{s}$$

Time of flight from P to C:

$$E_C = \tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_C}{2}\right) = \tan^{-1}\left(\sqrt{\frac{1-0.2381}{1+0.2381}}\tan\frac{30°}{2}\right) = 0.4144\ \text{rad}$$

$$M_C = E_C - e_1\sin E_C = 0.4144 - 0.2381\sin 0.4144 = 0.3185\ \text{rad}$$

$$t_C = \frac{M_C}{2\pi}T = \frac{0.3185}{2\pi}\cdot 10710 = 542.8\ \text{s}$$

Time of flight from P to D:

$$E_D = \tan^{-1}\left(\sqrt{\frac{1-e_1}{1+e_1}}\tan\frac{\theta_D}{2}\right) = \tan^{-1}\left(\sqrt{\frac{1-0.2381}{1+0.2381}}\tan\frac{90°}{2}\right) = 1.330\ \text{rad}$$

$$M_D = E_D - e_1\sin E_d = 1.330 - 0.2381\sin 1.330 = 1.099\ \text{rad}$$

$$t_D = \frac{M_D}{2\pi}T = \frac{1.099}{2\pi}\cdot 10710 = 1873\ \text{s}$$

Time of flight from C to D:

$$t_{CD} = t_D - t_C = 1873 - 542.8 = 1330\ \text{s}$$

To determine the trajectory from P to D is Lambert's problem. Note that

$$r_D = \frac{h_1^2}{\mu}\frac{1}{1+e_1\cos\theta_D} = \frac{62\,830^2}{398\,600}\frac{1}{1+0.2381\cos 90°} = 9905\ \text{km}$$

so that in perifocal coordinates

$$\mathbf{r}_P = 8000\hat{\mathbf{p}}\ \text{km} \quad \mathbf{r}_D = 9905\hat{\mathbf{q}}\ \text{km}$$

Note as well, that on orbit 1,

$$\mathbf{v}_{P1} = \frac{\mu}{h_1}\left[-\sin\theta_P\hat{\mathbf{p}} + \left(e+\cos\theta_P\right)\hat{\mathbf{q}}\right] = \frac{398\,600}{62\,830}\left[-\sin 0\hat{\mathbf{p}} + \left(0.2381+\cos 0\right)\hat{\mathbf{q}}\right]$$
$$= 7.854\hat{\mathbf{q}}\left(\text{km/s}\right)$$
$$\mathbf{v}_{D1} = \frac{\mu}{h_1}\left[-\sin\theta_D\hat{\mathbf{p}} + \left(e+\cos\theta_D\right)\hat{\mathbf{q}}\right] = \frac{398\,600}{62\,830}\left[-\sin 90°\hat{\mathbf{p}} + \left(0.2381+\cos 90°\right)\hat{\mathbf{q}}\right]$$
$$= -6.344\hat{\mathbf{p}} + 1.510\hat{\mathbf{q}}\left(\text{km/s}\right)$$

The following MATLAB script calls upon Algorithm 5.2, implemented as the M-function lambert in Appendix D.11, to solve Lamberts' problem for the velocities on orbit 2 at P and D. The output to the MATLAB Command Window is listed afterwards.

```
%
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem 6_23
% ~~~~~~~~~~~~
%
% This program uses Algorithm 5.2 to solve Lambert's problem for the
% data of in Problem 6.23.

% deg    - factor for converting between degrees and radians
% pi     - 3.1415926...
% mu     - gravitational parameter (km^3/s^2)
% r1, r2 - initial and final radii (km)
% dt     - time between r1 and r2 (s)
% dtheta - change in true anomaly during dt (degrees)
% R1, R2 - initial and final position vectors (km)
% string - = 'pro' if the orbit is prograde
%          = 'retro if the orbit is retrograde
% V1, V2 - initial and final velocity vectors (km/s)

% User M-function required: lambert
% -----------------------------------------------------------
clear
global  mu
mu     = 398600;       %km^3/s^2
deg    = pi/180;

r1     = 8000;   %km
r2     = 9905;   %km
dt     = 1330;        %sec
dtheta = 90;           %degrees

R1 = [r1  0  0];
R2 = [r2*cos(dtheta*deg)  r2*sin(dtheta*deg)   0];

%...Algorithm 5.2:
string = 'pro';
[V1 V2] = lambert(R1, R2, dt, string);

%...Echo the input data and output results to the command window:
fprintf('\n-----------------------------------------------------')
fprintf('\n Problem 6.23: Lambert''s Problem\n')

fprintf('\n Input data:\n');
```

```
fprintf('\n   Gravitational parameter (km^3/s^2) = %g\n', mu)
fprintf('\n   Radius 1 (km)                      = %g', r1)
fprintf('\n   Position vector R1 (km)            = [%g  %g  %g]\n',...
                                               R1(1), R1(2), R1(3))
fprintf('\n   Radius 2 (km)                      = %g', r2)
fprintf('\n   Position vector R2 (km)            = [%g  %g  %g]\n',...
                                               R2(1), R2(2), R2(3))
fprintf('\n   Elapsed time (s)                   = %g', dt)
fprintf('\n   Change in true anomaly (deg) = %g', dtheta)

fprintf('\n\n Solution:\n')
fprintf('\n  Velocity vector V1 (km/s) = [%g  %g  %g]',...
                                       V1(1), V1(2), V1(3))
fprintf('\n  Velocity vector V2 (km/s) = [%g  %g  %g]',...
                                       V2(1), V2(2), V2(3))
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 6.23: Lambert's Problem

 Input data:

   Gravitational parameter (km^3/s^2) = 398600

   Radius 1 (km)                = 8000
   Position vector R1 (km)      = [8000  0  0]

   Radius 2 (km)                = 9905
   Position vector R2 (km)      = [-0  9905  0]

   Elapsed time (s)             = 1330
   Change in true anomaly (deg) = 90

 Solution:

  Velocity vector V1 (km/s) = [-2.53168  9.57638  0]
  Velocity vector V2 (km/s) = [-7.73458  4.37347  0]
-----------------------------------------------------
```

$$\mathbf{v}_{P2} = -2.532\hat{\mathbf{p}} + 9.576\hat{\mathbf{q}}\,(\text{km/s})$$
$$\mathbf{v}_{D2} = -7.734\hat{\mathbf{p}} + 4.373\hat{\mathbf{q}}\,(\text{km/s})$$

$$\Delta\mathbf{v}_P = \mathbf{v}_{P2} - \mathbf{v}_{P1} = (-2.532\hat{\mathbf{p}} + 9.576\hat{\mathbf{q}}) - 7.854\hat{\mathbf{q}} = -2.532\hat{\mathbf{p}} + 1.722\hat{\mathbf{q}}\,(\text{km/s})$$
$$\|\Delta\mathbf{v}_P\| = \sqrt{(-2.532)^2 + 1.722^2} = 3.062\,(\text{km/s})$$

$$\Delta\mathbf{v}_D = \mathbf{v}_{D2} - \mathbf{v}_{D1} = (-7.734\hat{\mathbf{p}} + 4.373\hat{\mathbf{q}}) - (-6.344\hat{\mathbf{p}} + 1.510\hat{\mathbf{q}}) = -1.391\hat{\mathbf{p}} + 2.863\hat{\mathbf{q}}\,(\text{km/s})$$
$$\|\Delta\mathbf{v}_D\| = \sqrt{(-1.391)^2 + 2.863^2} = 3.183\,(\text{km/s})$$

$$\Delta v_{total} = \|\Delta\mathbf{v}_P\| + \|\Delta\mathbf{v}_D\| = 3.062 + 3.183 = \underline{6.245\text{ km/s}}$$

Problem 6.24

$$h = \sqrt{\mu a(1-e^2)} = \sqrt{398\,600 \cdot 15\,000 \cdot (1-0.5^2)} = 66\,960 \quad \text{km}^2/\text{s}$$
$$r_{\text{ascending node}} = \frac{h^2}{\mu}\frac{1}{1+e\cos(-\omega)} = \frac{66\,960^2}{398\,600}\frac{1}{1+0.5\cos(-30°)} = 7851\text{ km}$$

$$r_{\text{descending node}} = \frac{h^2}{\mu}\frac{1}{1+e\cos(-\omega+\pi)} = \frac{66\,960^2}{398\,600}\frac{1}{1+0.5\cos(-30°+180°)} = 19\,840\text{ km}$$

Rotate the orbital plane 10 degrees around the node line. That means hold v_r fixed and rotate $v_\perp$ 10 degrees. For minimum delta-v, do this maneuver at the furthest distance from the focus (at the descending node, rather than the ascending node).

$$v_\perp = \frac{h}{r_{\text{descending node}}} = \frac{66\,960}{19\,840} = 3.375\text{ km/s}$$

$$\Delta v = 2v_\perp \sin\frac{\Delta i}{2} = 2\cdot 3.375\cdot\sin\frac{10°}{2} = \underline{0.5883\text{ km/s}}$$

(Note: if the maneuver is done at the ascending node,

$$v_\perp = \frac{h}{r_{\text{ascending node}}} = \frac{66\,960}{7851} = 8.53\text{ km/s}$$

$$\Delta v = 2v_\perp \sin\frac{\Delta i}{2} = 2\cdot 8.53\cdot\sin\frac{10°}{2} = 1.487\text{ km/s}$$

Over twice the delta-v requirement.)

Problem 6.25 For the circular orbit

$$v_1 = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6778}} = 7.668\text{ km/s}$$

Assume the maneuver is done at apogee of the ellipse (orbit 2).

$$r = \frac{h_2^{\,2}}{\mu}\frac{1}{1-e_2}$$

$$6778 = \frac{h_2^{\,2}}{398\,600}\frac{1}{1-0.5} \Rightarrow h_2 = 36\,750\text{ km}^2/\text{s}$$

Then

$$r_{\text{perigee}} = \frac{h_2^{\,2}}{\mu}\frac{1}{1+e_2} = \frac{36\,750^2}{398\,600}\frac{1}{1+0.5} = 2259\text{ km}$$

which is inside the earth. So the maneuver cannot occur at apogee. Assume it occurs at perigee.

$$r = \frac{h_2^{\,2}}{\mu}\frac{1}{1+e_2}$$

$$6778 = \frac{h_2^{\,2}}{398\,600}\frac{1}{1+0.5} \Rightarrow h_2 = 63\,660\text{ km}^2/\text{s}$$

$$v_2 = \frac{h_2}{r} = \frac{63\,660}{6778} = 9.392\text{ km/s}$$

$$\Delta v = \sqrt{v_1^{\,2}+v_2^{\,2}-2v_1v_2\cos\delta} = \sqrt{7.668^2+9.392^2-2\cdot 7.668\cdot 9.392\cos\delta} = \underline{3.414\text{ km/s}}$$

Problem 6.26

$$r_{\text{apogee}} = \frac{h^2}{\mu}\frac{1}{1-e} = \frac{60\,000^2}{398\,600}\frac{1}{1-0.3} = 12\,900 \text{ km}$$

$$v_{\text{apogee}} = \frac{h}{r_{\text{apogee}}} = \frac{60\,000}{12\,900} = 4.65 \text{ km/s}$$

$$\Delta v = 2v_{\text{apogee}} \sin\frac{\delta}{2} = 2 \cdot 4.65 \sin\frac{90°}{2} = \underline{6.577 \text{ km/s}}$$

Problem 6.27

$$v_{B_1} = \sqrt{\frac{\mu}{r_o}}$$

$$r_o = \frac{{h_2}^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{{h_2}^2}{\mu}\frac{1}{1+0.25\cos(-90°)} = \frac{{h_2}^2}{\mu} \Rightarrow h_2 = \sqrt{\mu r_o}$$

$$v_{B_{\perp 2}} = \frac{h_2}{r_o} = \frac{\sqrt{\mu r_o}}{r_o} = \sqrt{\frac{\mu}{r_o}}$$

$$v_{B_{r2}} = \frac{\mu}{h_2} e_2 \sin\theta = \frac{\mu}{\sqrt{\mu r_o}} \cdot 0.25 \cdot \sin(-90°) = -0.25\sqrt{\frac{\mu}{r_o}}$$

$$\Delta v = \sqrt{\left(v_{B_{r2}} - v_{B_{r1}}\right)^2 + {v_{B_{\perp 1}}}^2 + {v_{B_{\perp 2}}}^2 - 2v_{B_{\perp 1}}v_{B_{\perp 2}}\cos\delta}$$

$$\Delta v = \sqrt{\left(-0.25\sqrt{\frac{\mu}{r_o}} - 0\right)^2 + \sqrt{\frac{\mu}{r_o}}^2 + \sqrt{\frac{\mu}{r_o}}^2 - 2\sqrt{\frac{\mu}{r_o}}\sqrt{\frac{\mu}{r_o}}\cos(-90°)}$$

$$\Delta v = \sqrt{0.0625\frac{\mu}{r_o} + \frac{\mu}{r_o} + \frac{\mu}{r_o} - 0}$$

$$\Delta v = \sqrt{2.0625\frac{\mu}{r_o}}$$

$$\Delta v = \underline{1.436\sqrt{\frac{\mu}{r_o}}}$$

Problem 6.28 The initial and target orbits are "1" and "2", respectively, and "3" is the transfer orbit.

$$r_1 = 6678 \text{ km}$$

$$v_1 = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{398\,600}{6678}} = 7.726 \text{ km/s}$$

$$r_2 = 6978 \text{ km}$$

$$v_2 = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{398\,600}{6978}} = 7.558 \text{ km/s}$$

$$a_3 = \frac{r_1 + r_2}{2} = \frac{6678 + 6978}{2} = 6828 \text{ km}$$

$$v_{\text{perigee}_3} = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_3}\right)} = \sqrt{398\,600\left(\frac{2}{6678} - \frac{1}{6828}\right)} = 7.810 \text{ km/s}$$

$$v_{\text{apogee}_3} = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_3}\right)} = \sqrt{398\,600\left(\frac{2}{6978} - \frac{1}{6828}\right)} = 7.474 \text{ km/s}$$

(a)

$$\Delta v = \left(v_{\text{perigee}_3} - v_1\right) + \left(v_2 - v_{\text{apogee}_3}\right) + 2 \cdot v_2 \sin\frac{\Delta i}{2}$$
$$= (7.810 - 7.726) + (7.558 - 7.474) + 2 \cdot 7.558 \sin\frac{20°}{2}$$
$$= 0.0844 + 0.08348 + 2.625 = \underline{2.793 \text{ km/s}}$$

(b)

$$\Delta v = \left(v_{\text{perigee}_3} - v_1\right) + \sqrt{{v_{\text{apogee}_3}}^2 + {v_2}^2 - 2v_{\text{apogee}_3} v_2 \cos\Delta i}$$
$$= (7.810 - 7.726) + \sqrt{7.474^2 + 7.588^2 - 2 \cdot 7.474 \cdot 7.558 \cos 20°}$$
$$= 0.0844 + 2.612 = \underline{2.696 \text{ km/s}}$$

(c)

$$\Delta v = \sqrt{{v_{\text{perigee}_3}}^2 + {v_1}^2 - 2v_{\text{perigee}_3} v_1 \cos\Delta i} + \left(v_2 - v_{\text{apogee}_3}\right)$$
$$= \sqrt{7.81^2 + 7.726^2 - 2 \cdot 7.81 \cdot 7.726 \cos 20°} + (7.558 - 7.474)$$
$$= 2.699 + 0.08348 = \underline{2.783 \text{ km/s}}$$

Problem 6.29 Design problem.

Problem 6.30

$$A = \sin^{-1}\left(\frac{\cos i}{\cos\phi}\right)$$

(a) $$A = \sin^{-1}\left(\frac{\cos 116.57°}{\cos 28.5°}\right) = \sin^{-1}(-0.5088) = \underline{329.4°}$$

(b) $$A = \sin^{-1}\left(\frac{\cos 116.57°}{\cos 34.5°}\right) = \sin^{-1}(-0.5427) = \underline{327.1°}$$

(c) $$A = \sin^{-1}\left(\frac{\cos 116.57°}{\cos 34.5°}\right) = \sin^{-1}(-0.4493) = \underline{333.3°}$$

Problem 7.1

$$\hat{\mathbf{i}} = 0\hat{\mathbf{I}} + \hat{\mathbf{J}} + 0\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = 0\hat{\mathbf{I}} + 0\hat{\mathbf{J}} + \hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = \hat{\mathbf{I}} + 0\hat{\mathbf{J}} + 0\hat{\mathbf{K}}$$
$$\therefore [\mathbf{Q}]_{Xx} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{r}_B = (6378 + 250)\hat{\mathbf{K}} = 6628\hat{\mathbf{K}}(\text{ km})$$
$$\mathbf{v}_B = -\sqrt{\frac{\mu}{r_B}}\hat{\mathbf{J}} = -\sqrt{\frac{398\,600}{6628}}\hat{\mathbf{J}} = -7.754\ 92\hat{\mathbf{J}}(\text{km/s})$$
$$\mathbf{a}_B = -\frac{{v_B}^2}{r_B}\hat{\mathbf{K}} = -\frac{7.755^2}{6628}\hat{\mathbf{K}} = -0.009\ 073\ 45\hat{\mathbf{K}}\left(\text{km}^2/\text{s}\right)$$

$$\mathbf{r}_A = (6378 + 300)\hat{\mathbf{K}} = 6678\hat{\mathbf{J}}(\text{ km})$$
$$\mathbf{v}_A = \sqrt{\frac{\mu}{r_A}}\hat{\mathbf{K}} = -\sqrt{\frac{398\,600}{6678}}\hat{\mathbf{K}} = 7.725\,84\hat{\mathbf{K}}(\text{km/s})$$
$$\mathbf{a}_A = -\frac{{v_A}^2}{r_A}\hat{\mathbf{J}} = -\frac{7.755^2}{6628}\hat{\mathbf{J}} = -0.008\ 938\ 08\hat{\mathbf{J}}\left(\text{km}^2/\text{s}\right)$$

$$\mathbf{\Omega}_{xyz} = \frac{v_A}{r_A}\hat{\mathbf{I}} = \frac{7.726}{6678}\hat{\mathbf{I}} = 0.001\ 156\ 91\hat{\mathbf{I}}(\text{rad/s})$$
$$\dot{\mathbf{\Omega}}_{xyz} = 0$$

$$\mathbf{r}_{rel} = \mathbf{r}_B - \mathbf{r}_A = -6678\hat{\mathbf{J}} + 6628\hat{\mathbf{K}}(\text{ km})$$
$$\{\mathbf{r}_{rel}\}_{xyz} = [\mathbf{Q}]_{Xx}\{\mathbf{r}_{rel}\}_{XYZ} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -6678 \\ 6628 \end{Bmatrix} = \begin{Bmatrix} -6678 \\ 6628 \\ 0 \end{Bmatrix}$$
$$\underline{\mathbf{r}_{rel} = -6678\hat{\mathbf{i}} + 6628\hat{\mathbf{j}}(\text{ km})}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega}_{xyz} \times \mathbf{r}_{rel} + \mathbf{v}_{rel}$$
$$-7.754\,92\hat{\mathbf{J}} = 7.725\,84\hat{\mathbf{K}} + (0.001\ 156\ 91\hat{\mathbf{I}}) \times (-6678\hat{\mathbf{J}} + 6628\hat{\mathbf{K}}) + \mathbf{v}_{rel}$$
$$-7.754\,92\hat{\mathbf{J}} = 7.725\,84\hat{\mathbf{K}} + (-7.667\ 99\hat{\mathbf{J}} - 7.725\ 84\hat{\mathbf{K}}) + \mathbf{v}_{rel}$$
$$\mathbf{v}_{rel} = -0.086\ 931\ 6\hat{\mathbf{J}}(\text{km/s})$$
$$\{\mathbf{v}_{rel}\}_{xyz} = [\mathbf{Q}]_{Xx}\{\mathbf{v}_{rel}\}_{XYZ} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.086\ 931\ 6 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.086\ 931\ 6 \\ 0 \\ 0 \end{Bmatrix}$$
$$\mathbf{v}_{rel} = \underline{-0.086\,931\,6\hat{\mathbf{i}}\ (\text{km/s})}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}}_{xyz} \times \mathbf{r}_{rel} + \mathbf{\Omega}_{xyz} \times (\mathbf{\Omega}_{xyz} \times \mathbf{r}_{rel}) + 2\mathbf{\Omega}_{xyz} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}$$
$$-0.009\ 073\ 45\hat{\mathbf{K}} = -0.008\ 983\ 08\hat{\mathbf{J}} + 0 + (0.001\ 156\ 91\hat{\mathbf{I}}) \times \left[(0.001\ 156\ 91\hat{\mathbf{I}}) \times (-6678\hat{\mathbf{J}} + 6628\hat{\mathbf{K}})\right]$$
$$+ 2\left[(0.001\ 156\ 91\hat{\mathbf{I}}) \times (-0.086\ 931\ 6\hat{\mathbf{J}})\right] + \mathbf{a}_{rel}$$

$$-0.009\,073\,45\hat{\mathbf{K}} = -8.938\,08\left(10^{-3}\right)\hat{\mathbf{J}} + 0 + \left[1.156\,91\left(10^{-3}\right)\hat{\mathbf{I}}\right] \times \left(-7.667\,99\hat{\mathbf{J}} - 7.725\,84\hat{\mathbf{K}}\right)$$
$$+ 2\left[-0.000\,100\,572\hat{\mathbf{K}}\right] + \mathbf{a}_{rel}$$

$$-0.009\,073\,45\hat{\mathbf{K}} = -8.938\,08\left(10^{-3}\right)\hat{\mathbf{J}} + 0 + \left(0.008\,938\,08\hat{\mathbf{J}} - 0.008\,871\,16\hat{\mathbf{K}}\right)$$
$$+ 2\left(-0.000\,100\,572\hat{\mathbf{K}}\right) + \mathbf{a}_{rel}$$

$$-0.009\,073\,45\hat{\mathbf{K}} = -0.009\,072\,31\hat{\mathbf{K}} + \mathbf{a}_{rel}$$

$$\mathbf{a}_{rel} = -1.140\,18 \times 10^{-6}\hat{\mathbf{K}}\left(\text{km/s}^2\right)$$

$$\left\{\mathbf{a}_{rel}\right\}_{xyz} = \left[\mathbf{Q}\right]_{Xx}\left\{\mathbf{a}_{rel}\right\}_{XYZ} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -1.140\,18 \times 10^{-6} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1.140\,18 \times 10^{-6} \\ 0 \end{Bmatrix}$$

$$\mathbf{a}_{rel} = \underline{-1.140\,18 \times 10^{-6}\hat{\mathbf{j}}\left(\text{km/s}^2\right)}$$

Problem 7.2

$$\mathbf{v}_A = \sqrt{\frac{\mu}{r_A}}\hat{\mathbf{j}} = -\sqrt{\frac{398\,600}{8000}}\hat{\mathbf{j}} = 7.058\,68\hat{\mathbf{j}}\left(\text{km/s}\right)$$
$$\mathbf{v}_B = \sqrt{\frac{\mu}{r_B}}\hat{\mathbf{j}} = -\sqrt{\frac{398\,600}{7000}}\hat{\mathbf{j}} = 7.546\,05\hat{\mathbf{j}}\left(\text{km/s}\right)$$

$$\mathbf{\Omega}_{xyz} = \frac{v_A}{r_A}\hat{\mathbf{k}} = \frac{7.058\,68}{8000}\hat{\mathbf{k}} = 0.000\,882\,335\hat{\mathbf{k}}\left(\text{rad/s}\right)$$
$$\dot{\mathbf{\Omega}}_{xyz} = 0$$

$$\mathbf{r}_{rel} = \mathbf{r}_B - \mathbf{r}_A = -1000\hat{\mathbf{i}}\left(\text{km}\right)$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega}_{xyz} \times \mathbf{r}_{rel} + \mathbf{v}_{rel}$$
$$7.546\,05\hat{\mathbf{j}} = 7.058\,68\hat{\mathbf{j}} + \left(0.000\,882\,335\hat{\mathbf{k}}\right) \times \left(-1000\hat{\mathbf{i}}\right) + \mathbf{v}_{rel}$$
$$7.546\,05\hat{\mathbf{j}} = 7.058\,68\hat{\mathbf{j}} - 0.882\,335\hat{\mathbf{j}} + \mathbf{v}_{rel}$$
$$\mathbf{v}_{rel} = 1.369\,70\hat{\mathbf{j}}\ \left(\text{km/s}\right)$$

$$\mathbf{a}_A = -\frac{{v_A}^2}{r_A}\hat{\mathbf{i}} = -\frac{7.058\,68^2}{8000}\hat{\mathbf{i}} = -0.006\,228\,12\hat{\mathbf{i}}\left(\text{km/s}^2\right)$$
$$\mathbf{a}_B = -\frac{{v_B}^2}{r_B}\hat{\mathbf{i}} = -\frac{7.546\,05^2}{7000}\hat{\mathbf{i}} = -0.008\,134\,69\hat{\mathbf{i}}\left(\text{km/s}^2\right)$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}}_{xyz} \times \mathbf{r}_{rel} + \mathbf{\Omega}_{xyz} \times \left(\mathbf{\Omega}_{xyz} \times \mathbf{r}_{rel}\right) + 2\mathbf{\Omega}_{xyz} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}$$

$$-0.008\,134\,69\hat{\mathbf{i}} = -0.006\,228\,12\hat{\mathbf{i}} + 0 + \left(0.000\,882\,335\hat{\mathbf{k}}\right) \times \left[\left(0.000\,882\,335\hat{\mathbf{k}}\right) \times \left(-1000\hat{\mathbf{i}}\right)\right]$$
$$+ 2\left(0.000\,882\,335\hat{\mathbf{k}}\right) \times \left(1.369\,70\hat{\mathbf{j}}\right) + \mathbf{a}_{rel}$$

$$-0.008\,134\,69\hat{\mathbf{i}} = -0.006\,228\,12\hat{\mathbf{i}} + \left(0.000\,882\,335\hat{\mathbf{k}}\right)\times\left(-0.882\,335\hat{\mathbf{j}}\right)$$
$$+\,2\left(-0.001\,208\,54\hat{\mathbf{i}}\right) + \mathbf{a}_{rel}$$

$$0.008\,134\,69\hat{\mathbf{i}} = -0.006\,228\,12\hat{\mathbf{i}} + 0.000\,778\,516\hat{\mathbf{k}}$$
$$-\,0.002\,417\,07\hat{\mathbf{i}} + \mathbf{a}_{rel}$$

$$-0.008\,134\,69\hat{\mathbf{i}} = -0.007\,866\,68\hat{\mathbf{i}} + \mathbf{a}_{rel}$$
$$\mathbf{a}_{rel} = \underline{-0.000\,268\,012\hat{\mathbf{i}}\left(\text{km/s}^2\right)}$$

Problem 7.3 $\mathbf{r} = \mathbf{r}_o + \delta\mathbf{r}$

(a)

$$r = \left[\left(\mathbf{r}_o + \delta\mathbf{r}\right)\cdot\left(\mathbf{r}_o + \delta\mathbf{r}\right)\right]^{1/2}$$
$$= \left(\mathbf{r}_o\cdot\mathbf{r}_o + 2\mathbf{r}_o\cdot\delta\mathbf{r} + \delta\mathbf{r}\cdot\delta\mathbf{r}\right)^{1/2}$$
$$= \left(r_o^{\,2} + 2\mathbf{r}_o\cdot\delta\mathbf{r} + \delta r^2\right)^{1/2}$$
$$= r_o\left[1 + \frac{2\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,2}} + \left(\frac{\delta r}{r_o}\right)^2\right]^{1/2}$$

Keep only terms of the first order in δr:

$$r = r_o\left(1 + \frac{2\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,2}}\right)^{1/2}$$

$$\therefore \sqrt{r} = \sqrt{r_o}\left(1 + \frac{2\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,2}}\right)^{1/4}$$

By means of the binomial theorem,

$$\sqrt{r} = \sqrt{r_o}\left(1 + \frac{1}{4}\frac{2\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,2}}\right) = \sqrt{r_o} + \frac{1}{2}\frac{\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,3/2}}$$
$$\therefore \underline{O(\delta\mathbf{r}) = \frac{1}{2}\frac{\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,3/2}}}$$

(b)

$$\mathbf{r}_o = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}} \qquad \delta\mathbf{r} = 0.01\hat{\mathbf{i}} - 0.01\hat{\mathbf{j}} + 0.03\hat{\mathbf{k}} \quad \left(\delta r = 0.033\,166\,2\right)$$
$$r_o = \left(3^2 + 4^2 + 5^2\right)^{1/2} = 7.071\,07$$
$$\sqrt{r_o} = 2.659\,15$$

$$O(\delta\mathbf{r}) = \frac{1}{2}\frac{\mathbf{r}_o\cdot\delta\mathbf{r}}{r_o^{\,3/2}} = \frac{1}{2}\frac{\left(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}\right)\cdot\left(0.01\hat{\mathbf{i}} - 0.01\hat{\mathbf{j}} + 0.03\hat{\mathbf{k}}\right)}{7.071\,07^{3/2}} = \frac{1}{2}\frac{0.140\,000}{18.803\,00} = 0.003\,722\,81$$

$$\mathbf{r} = 3.01\hat{\mathbf{i}} + 3.99\hat{\mathbf{j}} + 5.03\hat{\mathbf{k}}$$
$$r = \left(3.01^2 + 3.99^2 + 5.03^2\right)^{1/2} = 7.090\,92$$
$$\sqrt{r} = 2.662\,88$$
$$\sqrt{r} - \sqrt{r_o} = 0.003\,729\,58$$

$$\left|\frac{\left(\sqrt{r}-\sqrt{r_o}\right)-O(\delta\mathbf{r})}{\sqrt{r}-\sqrt{r_o}}\right|\cdot 100=\left|\frac{0.003\,729\,58-0.003\,722\,81}{0.003\,729\,58}\right|\cdot 100=\underline{0.181\,565\%}$$

Thus, $O(\delta\mathbf{r})$ is within 0.2% of $\sqrt{r}-\sqrt{r_o}$,

(c) $\delta\mathbf{r}=\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$ $\left(\delta r=3.316\,62,\ \ \delta r/r_o=4.69\times 10^{-1}\right)$

$$O(\delta\mathbf{r})=\frac{1}{2}\frac{\mathbf{r}_o\cdot\delta\mathbf{r}}{{r_o}^{3/2}}=\frac{1}{2}\frac{\left(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}}\right)\cdot\left(\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}\right)}{7.071\,07^{3/2}}=\frac{1}{2}\frac{14}{18.803\,00}=0.372\,281$$

$$\mathbf{r}=\mathbf{r}_o+\delta\mathbf{r}=4\hat{\mathbf{i}}+3\hat{\mathbf{j}}+8\hat{\mathbf{k}}$$
$$r=\left(4^2+3^2+8^2\right)^{1/2}=9.433\,98$$
$$\sqrt{r}=3.071\,48$$
$$\sqrt{r}-\sqrt{r_o}=0.412\,331$$

$$\left|\frac{\left(\sqrt{r}-\sqrt{r_o}\right)-O(\delta\mathbf{r})}{\sqrt{r}-\sqrt{r_o}}\right|\cdot 100=\left|\frac{0.412\,331-0.372\,281}{0.412\,331}\right|\cdot 100=\underline{9.713\,08\%}$$

In this case $O(\delta\mathbf{r})$ is a poor approximation, exceeding $\sqrt{r}-\sqrt{r_o}$ by almost 10 percent.

Problem 7.4 For $e<<1$:

$$\begin{aligned}r&=\frac{a\left(1-e^2\right)}{1+e\cos\theta}\\&\approx a\left(1-e^2\right)(1-e\cos\theta)\\&=a(1-e\cos\theta)-ae^2+ae^3\cos\theta\\&\approx\underline{a(1-e\cos\theta)}\end{aligned}$$

Problem 7.5 $\ddot{x}+9x=10$

$$x=A\sin 3t+B\cos 3t+\frac{10}{9}$$
$$\dot{x}=3A\cos 3t-3B\sin 3t$$

At $t=0$, x=5:

$$5=A\sin(0)+B\cos(0)+\frac{10}{9}$$
$$5=B+\frac{10}{9}$$
$$\therefore B=\frac{35}{9}$$

At $t=0,\ \dot{x}=-3$:

$$-3=3A\cos(0)-3\cdot\frac{35}{9}\sin(0)$$
$$-3=3A$$
$$A=-1$$

Thus

$$x = -\sin 3t + \frac{35}{9}\cos 3t + \frac{10}{9}$$
$$\dot{x} = -3\cos 3t - \frac{35}{3}\sin 3t$$

At $t = 1.2$:

$$x = -\sin(3 \cdot 1.2) + \frac{35}{9}\cos(3 \cdot 1.2) + \frac{10}{9} = \underline{-1.934}$$
$$\dot{x} = -3\cos(3 \cdot 1.2) - \frac{35}{3}\sin(3 \cdot 1.2) = \underline{7.853}$$

Problem 7.6

$$\ddot{x} + 10x + 2\dot{y} = 0 \quad (1)$$
$$\ddot{y} + 3\dot{x} = 0 \quad (2)$$

Initial conditions (at $t = 0$):

$$x_0 = 1 \quad (3)$$
$$\dot{x}_0 = -3 \quad (4)$$
$$y_0 = 2 \quad (5)$$
$$\dot{y}_0 = 4 \quad (6)$$

From (2)

$$\frac{d}{dt}(\dot{y} + 3x) = 0$$
$$\dot{y} + 3x = \text{const}$$
$$\dot{y} + 3x = \dot{y}_0 + 3x_0 = 4 + 3 \cdot 1 = 7$$

$$\dot{y} = 7 - 3x \quad (7)$$

Substitute (7) into (1):

$$\ddot{x} + 10x + 2(7 - 3x) = 0$$
$$\ddot{x} + 4x = -14$$

General solution:

$$x = A\sin 2t + B\cos 2t - \frac{7}{2} \quad (8)$$
$$\dot{x} = 2A\cos 2t - 2B\sin 2t \quad (9)$$

Evaluating (8) at $t = 0$ and using (3),

$$1 = A\sin(0) + B\cos(0) - \frac{7}{2}$$
$$1 = B - \frac{7}{2}$$
$$B = \frac{9}{2} \quad (10)$$

Evaluating (9) at $t = 0$ and using (4),

$$-3 = 2A\cos(0) - 2B\sin(0)$$
$$A = -\frac{3}{2} \tag{11}$$

With (10) and (11), (8) becomes

$$x = -\frac{3}{2}\sin 2t + \frac{9}{2}\cos 2t - \frac{7}{2} \tag{12}$$

Substituting (12) into (7) yields

$$\dot{y} = 7 - 3\left(-\frac{3}{2}\sin 2t + \frac{9}{2}\cos 2t - \frac{7}{2}\right)$$
$$\dot{y} = \frac{9}{2}\sin 2t - \frac{27}{2}\cos 2t + \frac{35}{2} \tag{13}$$

Then

$$y = -\frac{9}{4}\cos 2t - \frac{27}{4}\sin 2t + \frac{35}{2}t + C \tag{14}$$

Evaluating (14) at $t = 0$ and using (5), we get

$$2 = -\frac{9}{4}\cos(0) - \frac{27}{4}\sin(0) + \frac{35}{2}(0) + C$$

$$2 = -\frac{9}{4} + C$$

$$C = \frac{17}{4} \tag{15}$$

Substituting this into (14) yields

$$y = -\frac{9}{4}\cos 2t - \frac{27}{4}\sin 2t + \frac{35}{2}t + \frac{17}{4} \tag{16}$$

At $t = 5$, (12) and (14) yield, respectively,

$$x = -\frac{3}{2}\sin(2\cdot 5) + \frac{9}{2}\cos(2\cdot 5) - \frac{7}{2} = \underline{-6.460}$$
$$y = -\frac{9}{4}\cos(2\cdot 5) - \frac{27}{4}\sin(2\cdot 5) + \frac{35}{2}\cdot 5 + \frac{17}{4} = \underline{97.31}$$

Problem 7.7

$$n = \frac{2\pi}{90\cdot 60} = 0.0011636 \text{ s}^{-1}$$
$$t = 15\cdot 60 = 900 \text{ s}$$

$$\left[\mathbf{\Phi}_{rr}(t)\right] = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 2.5 & 0 & 0 \\ -1.087 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{rv}}(t)\right] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 744.29 & 859.44 & 0 \\ -859.44 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\{\delta\mathbf{r}_0\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \text{ (km)} \qquad \{\delta\mathbf{v}_0\} = \begin{Bmatrix} 0 \\ 0.01 \\ 0 \end{Bmatrix} \text{(km/s)}$$

$$\{\delta\mathbf{r}\} = \left[\mathbf{\Phi}_{\text{rr}}(t)\right]\{\delta\mathbf{r}_0\} + \left[\mathbf{\Phi}_{\text{rv}}(t)\right]\{\delta\mathbf{v}_0\}$$

$$\begin{aligned}\{\delta\mathbf{r}\} &= \left[\mathbf{\Phi}_{\text{rr}}(t)\right]\{\delta\mathbf{r}_0\} + \left[\mathbf{\Phi}_{\text{rv}}(t)\right]\{\delta\mathbf{v}_0\} \\ &= \begin{bmatrix} 2.5 & 0 & 0 \\ -1.087 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 744.29 & 859.44 & 0 \\ -859.44 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.01 \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} 2.5 \\ -1.087 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 8.5944 \\ 2.7718 \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} 11.094 \\ 1.6847 \\ 0 \end{Bmatrix}\end{aligned}$$

$$\|\delta\mathbf{r}\| = \underline{11.222 \text{ km}}$$

Problem 7.8

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{6600}} = 7.7714 \text{ km/s}$$

$$n = \frac{v}{r} = \frac{7.7714}{6600} = 0.001177\,5 \text{ s}^{-1}$$

$$T = \frac{2\pi}{n} = 5446.1 \text{ s}$$

$$t = \frac{T}{3} = 1778.7 \text{ s}$$

$$\left[\mathbf{\Phi}_{\text{rr}}(t)\right] = \begin{bmatrix} 4-3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 5.5 & 0 & 0 \\ -7.3702 & 1 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{rv}}(t)\right] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 735.49 & 2547.8 & 0 \\ -2547.8 & -2394.2 & 0 \\ 0 & 0 & 735.49 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{vr}}(t)\right] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0.003\,059\,2 & 0 & 0 \\ -0.010\,597 & 0 & 0 \\ 0 & 0 & -0.001\,0197 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{vv}}(t)\right] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} -0.5 & 1.7321 & 0 \\ -1.7321 & -5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}$$

$$\left\{\delta\mathbf{r}_f\right\} = \left[\mathbf{\Phi}_{rr}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{rv}\right]\left\{\delta\mathbf{v}_0^{\,+}\right\}$$

$$\begin{Bmatrix}0\\0\\0\end{Bmatrix} = \begin{bmatrix}5.5 & 0 & 0\\ -7.3702 & 1 & 0\\ 0 & 0 & -0.5\end{bmatrix}\begin{Bmatrix}1\\1\\1\end{Bmatrix} + \begin{bmatrix}735.49 & 2547.8 & 0\\ -2547.8 & -2394.2 & 0\\ 0 & 0 & 735.49\end{bmatrix}\left\{\delta\mathbf{v}_0^{\,+}\right\}$$

$$\begin{bmatrix}735.49 & 2547.8 & 0\\ -2547.8 & -2394.2 & 0\\ 0 & 0 & 735.49\end{bmatrix}\left\{\delta\mathbf{v}_0^{\,+}\right\} = -\begin{bmatrix}5.5 & 0 & 0\\ -7.3702 & 1 & 0\\ 0 & 0 & -0.5\end{bmatrix}\begin{Bmatrix}1\\1\\1\end{Bmatrix} = \begin{Bmatrix}-5.5\\6.3702\\0.5\end{Bmatrix}$$

$$\left\{\delta\mathbf{v}_0^{\,+}\right\} = \begin{bmatrix}735.49 & 2547.8 & 0\\ -2547.8 & -2394.2 & 0\\ 0 & 0 & 735.49\end{bmatrix}^{-1}\begin{Bmatrix}-5.5\\6.3702\\0.5\end{Bmatrix} = \begin{Bmatrix}-0.000\,647\,34\\-0.001\,971\,8\\0.000\,679\,82\end{Bmatrix}(\text{km/s})$$

$$\left\{\Delta\mathbf{v}_1\right\} = \left\{\delta\mathbf{v}_0^{\,+}\right\} - \left\{\delta\mathbf{v}_0^{\,-}\right\} = \begin{Bmatrix}-0.000\,647\,34\\-0.001\,971\,8\\0.000\,679\,82\end{Bmatrix} - \begin{Bmatrix}0\\0\\0.005\end{Bmatrix} = \begin{Bmatrix}-0.000\,647\,34\\-0.001\,971\,8\\-0.004\,320\,2\end{Bmatrix}(\text{km/s})$$

$$\Delta v_1 = \left\|\Delta\mathbf{v}_1\right\| = 0.004\,792\,8 \ \text{ km/s}$$

$$\left\{\delta\mathbf{v}_f^{\,-}\right\} = \left[\mathbf{\Phi}_{vr}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{vv}\right]\left\{\delta\mathbf{v}_0^{\,+}\right\}$$

$$\left\{\delta\mathbf{v}_f^{\,-}\right\} = \begin{bmatrix}0.003\,059\,2 & 0 & 0\\ -0.010\,597 & 0 & 0\\ 0 & 0 & -0.001\,019\,7\end{bmatrix}\begin{Bmatrix}1\\1\\1\end{Bmatrix} + \begin{bmatrix}-0.5 & 1.7321 & 0\\ -1.7321 & -5 & 0\\ 0 & 0 & -0.5\end{bmatrix}\begin{Bmatrix}-0.000\,647\,34\\-0.001\,971\,8\\0.000\,679\,82\end{Bmatrix}$$

$$\left\{\delta\mathbf{v}_f^{\,-}\right\} = \begin{Bmatrix}0.003\,059\,2\\-0.010\,597\\-0.001\,019\,7\end{Bmatrix} + \begin{Bmatrix}-0.003\,091\,7\\0.010\,98\\-0.000\,339\,91\end{Bmatrix}$$

$$\left\{\delta\mathbf{v}_f^{\,-}\right\} = \begin{Bmatrix}-0.000\,032\,48\\0.000\,383\,12\\-0.001\,359\,6\end{Bmatrix}(\text{km/s})$$

$$\left\{\delta\mathbf{v}_f^{\,+}\right\} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$\left\{\Delta\mathbf{v}_2\right\} = \left\{\delta\mathbf{v}_f^{\,+}\right\} - \left\{\delta\mathbf{v}_f^{\,-}\right\} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} - \begin{Bmatrix}-0.000\,032\,48\\0.000\,383\,12\\-0.001\,359\,6\end{Bmatrix} = \begin{Bmatrix}0.000\,032\,48\\-0.000\,383\,12\\0.001\,359\,6\end{Bmatrix}(\text{km/s})$$

$$\Delta v_2 = \left\|\Delta\mathbf{v}_2\right\| = 0.001\,413 \ \text{ km/s}$$

$$v_{total} = \Delta v_1 + \Delta v_2 = 0.004\,792\,8 + 0.001\,413 = \underline{0.006\,205\,8 \ \text{ km/s}} = \underline{6.206 \ \text{ m/s}}$$

Problem 7.9

$$t = \frac{T_0}{2}$$

$$nt = \frac{2\pi}{T_0} = \pi$$

$$\left[\mathbf{\Phi}_{rr}(t)\right] = \begin{bmatrix}4 - 3\cos nt & 0 & 0\\ 6(\sin nt - nt) & 1 & 0\\ 0 & 0 & \cos nt\end{bmatrix} = \begin{bmatrix}7 & 0 & 0\\ -6\pi & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$[\mathbf{\Phi}_{\mathrm{rv}}(t)] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{n} & 0 \\ -\frac{4}{n} & -\frac{3\pi}{n} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{\Phi}_{\mathrm{vr}}(t)] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -12n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{\Phi}_{\mathrm{vv}}(t)] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a)

$$\{\delta\mathbf{r}_f\} = [\mathbf{\Phi}_{\mathrm{rr}}]\{\delta\mathbf{r}_0\} + [\mathbf{\Phi}_{\mathrm{rv}}]\{\delta\mathbf{v}_0{}^+\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ -6\pi & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{Bmatrix} \delta\rho \\ \delta y_0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & \frac{4}{n} & 0 \\ -\frac{4}{n} & -\frac{3\pi}{n} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ \delta v_0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7\delta\rho \\ -6\pi\delta\rho + \delta y_0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{4\delta v_0}{n} \\ -\frac{3\pi\delta v_0}{n} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 7\delta\rho + \frac{4\delta v_0}{n} \\ -6\pi\delta\rho - \frac{3\pi\delta v_0}{n} + \delta y_0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$7\delta\rho + \frac{4\delta v_0}{n} = 0 \;\Rightarrow\; \delta v_0 = -\frac{7}{4}n\delta\rho$$

$$\delta y_0 = 6\pi\delta\rho + \frac{3\pi}{n}\left(-\frac{7}{4}n\delta\rho\right) = \frac{3}{4}\pi\delta\rho$$

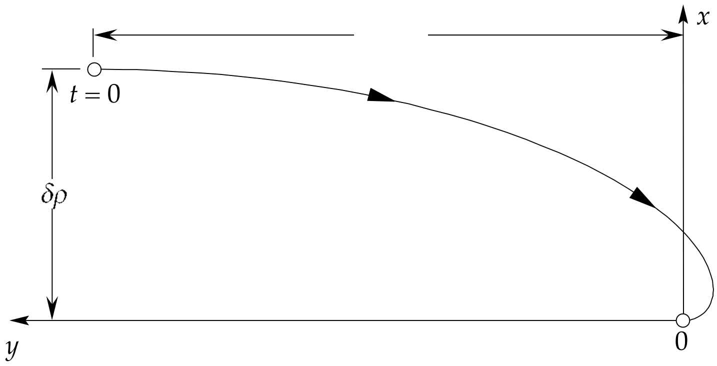
$$\therefore \underline{\{\delta\mathbf{r}_0\} = \begin{Bmatrix} \delta\rho \\ \frac{3}{4}\pi\delta\rho \\ 0 \end{Bmatrix}} \qquad \{\delta\mathbf{v}_0{}^+\} = \begin{Bmatrix} 0 \\ -\frac{7}{4}v\delta\rho \\ 0 \end{Bmatrix}$$

(b)

$$\{\delta\mathbf{v}_f{}^-\} = [\mathbf{\Phi}_{\mathrm{vr}}]\{\delta\mathbf{r}_0\} + [\mathbf{\Phi}_{\mathrm{vv}}]\{\delta\mathbf{v}_0{}^+\}$$

$$\{\delta\mathbf{v}_f{}^-\} = \begin{bmatrix} 0 & 0 & 0 \\ -12n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \delta\rho \\ \frac{3}{4}\pi\delta\rho \\ 0 \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{Bmatrix} 0 \\ -\frac{7}{4}v\delta\rho \\ 0 \end{Bmatrix}$$

$$\{\delta\mathbf{v}_f{}^-\} = \begin{Bmatrix} 0 \\ -12n\delta\rho \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{49}{4}n\delta\rho \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{1}{4}n\delta\rho \\ 0 \end{Bmatrix} = \underline{\begin{Bmatrix} 0 \\ \frac{1}{2}\frac{\pi\delta\rho}{T_0} \\ 0 \end{Bmatrix}}$$



Problem 7.10

$$nt = 2\pi \qquad n = \frac{2\pi}{T}$$

$$\left[\mathbf{\Phi}_{\text{rr}}(t)\right] = \begin{bmatrix} 4-3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -12\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{rv}}(t)\right] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{6\pi}{n} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{vr}}(t)\right] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{vv}}(t)\right] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left\{\delta\mathbf{r}_f\right\} = \left[\mathbf{\Phi}_{\text{rr}}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{\text{rv}}\right]\left\{\delta\mathbf{v}_0{}^+\right\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -12\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \delta y_0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{6\pi}{n} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \delta u_0{}^+ \\ \delta v_0{}^+ \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \delta y_0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{6\pi\delta v_0{}^+}{n} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \delta y_0 - \frac{6\pi\delta v_0{}^+}{n} \\ 0 \end{Bmatrix} \Rightarrow \delta v_0{}^+ = \frac{n\delta y_0}{6\pi}$$

Thus, assuming a Hohmann transfer, $\delta u_0{}^+ = 0$,

$$\left\{\delta\mathbf{v}_0{}^+\right\} = \begin{Bmatrix} 0 \\ \frac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix}$$

$$\{\Delta\mathbf{v}_1\} = \{\delta\mathbf{v}_0{}^+\} - \{\delta\mathbf{v}_0{}^-\} = \begin{Bmatrix} 0 \\ \dfrac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix}$$

$$\Delta v_1 = \|\Delta\mathbf{v}_1\| = \frac{n\delta y_0}{6\pi}$$

$$\{\delta\mathbf{v}_f{}^-\} = [\mathbf{\Phi}_{\text{vr}}]\{\delta\mathbf{r}_0\} + [\mathbf{\Phi}_{\text{vv}}]\{\delta\mathbf{v}_0{}^+\}$$

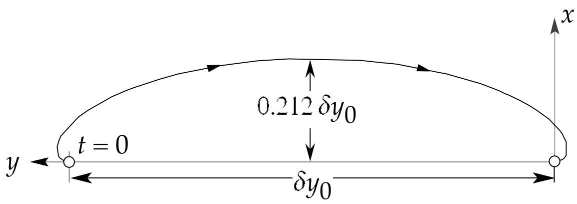
$$\{\delta\mathbf{v}_f{}^-\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ \delta y_0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \dfrac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix}$$

$$\{\delta\mathbf{v}_f{}^-\} = \begin{Bmatrix} 0 \\ \dfrac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix}$$

$$\{\Delta\mathbf{v}_2\} = \{\delta\mathbf{v}_f{}^+\} - \{\delta\mathbf{v}_f{}^-\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ \dfrac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dfrac{n\delta y_0}{6\pi} \\ 0 \end{Bmatrix}$$

$$\Delta v_2 = \|\Delta\mathbf{v}_2\| = \frac{n\delta y_0}{6\pi}$$

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = \frac{n\delta y_0}{6\pi} + \frac{n\delta y_0}{6\pi} = \frac{n\delta y_0}{3\pi} = \underline{\frac{2\delta y_0}{3T}}$$



Problem 7.11

$$n = \frac{2\pi}{2\cdot 3600} = 0.000\,872\,66\ \text{s}^{-1}$$

$$t = 30\cdot 60 = 1800\ \text{s}$$

$$nt = \frac{\pi}{2}$$

$$[\mathbf{\Phi}_{\text{rr}}(t)] = \begin{bmatrix} 4-3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -3.425 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{\Phi}_{\text{rv}}(t)] = \begin{bmatrix} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1-\cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 1146 & 2292 & 0 \\ -2292 & -816.3 & 0 \\ 0 & 0 & 1146 \end{bmatrix}$$

$$[\mathbf{\Phi}_{\text{vr}}(t)] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0.002\,618 & 0 & 0 \\ -0.005\,236 & 0 & 0 \\ 0 & 0 & -0.000\,8727 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{vv}(t)\right] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{\delta\mathbf{r}_f\right\} = \left[\mathbf{\Phi}_{rr}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{rv}\right]\left\{\delta\mathbf{v}_0{}^+\right\}$$

$$\left\{\delta\mathbf{r}_f\right\} = \begin{bmatrix} 4 & 0 & 0 \\ -3.425 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 6 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1146 & 2292 & 0 \\ -2292 & -816.3 & 0 \\ 0 & 0 & 1146 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.003 \\ 0 \end{Bmatrix}$$

$$\left\{\delta\mathbf{r}_f\right\} = \begin{Bmatrix} 0 \\ 6 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -6.875 \\ 2.449 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -6.875 \\ 8.449 \\ 0 \end{Bmatrix} (\text{km})$$

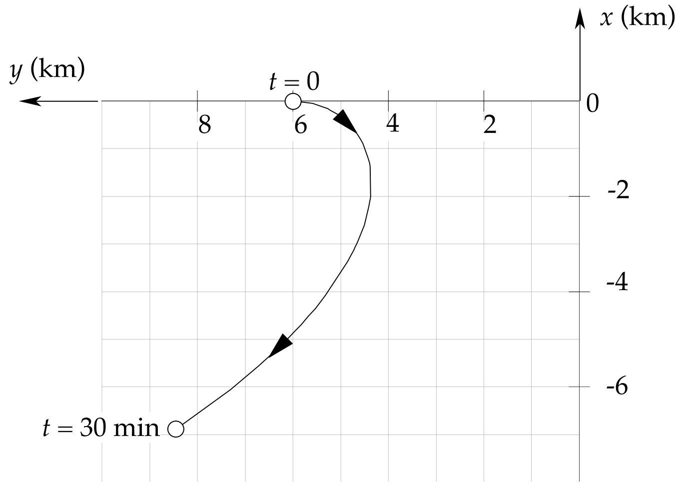
$$\underline{\left\|\delta\mathbf{r}_f\right\| = 10.89 \text{ km}}$$

$$\left\{\delta\mathbf{v}_f\right\} = \left[\mathbf{\Phi}_{vr}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{vv}\right]\left\{\delta\mathbf{v}_0{}^+\right\}$$

$$\left\{\delta\mathbf{v}_f\right\} = \begin{bmatrix} 0.002618 & 0 & 0 \\ -0.005236 & 0 & 0 \\ 0 & 0 & -0.0008727 \end{bmatrix}\begin{Bmatrix} 0 \\ 6 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.003 \\ 0 \end{Bmatrix}$$

$$\left\{\delta\mathbf{v}_f\right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -0.006 \\ 0.009 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -0.006 \\ 0.009 \\ 0 \end{Bmatrix} (\text{km/s})$$

$$\underline{\left\|\delta\mathbf{v}_f\right\| = 0.01082 \text{ km/s}}$$



Problem 7.12 Use C-W frame attached to original location of satellite in GEO.

$$n = \frac{2\pi + \dfrac{2\pi}{365.26}}{24 \cdot 3600} = 7.292 \times 10^{-5} \text{ s}^{-1}$$

First determine the relative position and velocity after two hours.

$$t = 2 \cdot 3600 = 7200 \text{ s}$$
$$nt = 0.1671\pi$$

$$\left[\boldsymbol{\Phi}_{\mathrm{rr}}(t)\right] = \begin{bmatrix} 4-3\cos nt & 0 \\ 6(\sin nt - nt) & 1 \end{bmatrix} = \begin{bmatrix} 1.404 & 0 \\ -0.1427 & 1 \end{bmatrix}$$

$$\left[\boldsymbol{\Phi}_{\mathrm{rv}}(t)\right] = \begin{bmatrix} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1-\cos nt) \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) \end{bmatrix} = \begin{bmatrix} 6874 & 3694 \\ -3694 & 5895 \end{bmatrix}$$

$$\left[\boldsymbol{\Phi}_{\mathrm{vr}}(t)\right] = \begin{bmatrix} 3n\sin nt & 0 \\ 6n(\cos nt - 1) & 0 \end{bmatrix} = \begin{bmatrix} 10.97\times10^{-5} & 0 \\ -5.893\times10^{-5} & 0 \end{bmatrix}$$

$$\left[\boldsymbol{\Phi}_{\mathrm{vv}}(t)\right] = \begin{bmatrix} \cos nt & 2\sin nt \\ -2\sin nt & 4\cos nt - 3 \end{bmatrix} = \begin{bmatrix} 0.8653 & 1.002 \\ -1.002 & 0.4612 \end{bmatrix}$$

$$\{\delta\mathbf{r}_2\} = [\boldsymbol{\Phi}_{\mathrm{rr}}]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{\mathrm{rv}}]\{\delta\mathbf{v}_0\}$$

$$\begin{Bmatrix} -10 \\ 10 \end{Bmatrix} = \begin{bmatrix} 1.404 & 0 \\ -0.1427 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 6874 & 3694 \\ -3694 & 5895 \end{bmatrix}\{\delta\mathbf{v}_0\}$$

$$\begin{bmatrix} 6874 & 3694 \\ -3694 & 5895 \end{bmatrix}\{\delta\mathbf{v}_0\} = \begin{Bmatrix} -10 \\ 10 \end{Bmatrix}$$

$$\{\delta\mathbf{v}_0\} = \begin{bmatrix} 6874 & 3694 \\ -3694 & 5895 \end{bmatrix}^{-1}\left(\begin{Bmatrix} -10 \\ 10 \end{Bmatrix}\right) = \begin{Bmatrix} -0.00177 \\ 0.000\,587 \end{Bmatrix}(\mathrm{km/s})$$

$$\{\delta\mathbf{v}_2{}^-\} = [\boldsymbol{\Phi}_{\mathrm{vr}}]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{\mathrm{vv}}]\{\delta\mathbf{v}_0\}$$

$$\{\delta\mathbf{v}_2{}^-\} = \begin{bmatrix} 10.97\times10^{-5} & 0 \\ -5.893\times10^{-5} & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0.8653 & 1.002 \\ -1.002 & 0.4612 \end{bmatrix}\begin{Bmatrix} -0.00177 \\ 0.000\,587 \end{Bmatrix}$$

$$\{\delta\mathbf{v}_2{}^-\} = \begin{Bmatrix} -0.000\,943\,4 \\ 0.002\,045 \end{Bmatrix}\ (\mathrm{km/s})$$

Initiate rendezvous with the origin.

$$t = 6\cdot3600 = 21\,600\ \mathrm{s}$$
$$nt = 0.5014\pi$$

$$\{\delta\mathbf{r}_6\} = [\boldsymbol{\Phi}_{\mathrm{rr}}]\{\delta\mathbf{r}_2\} + [\boldsymbol{\Phi}_{\mathrm{rv}}]\{\delta\mathbf{v}_2{}^+\}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 4.013 & 0 \\ -3.451 & 1 \end{bmatrix}\begin{Bmatrix} -10 \\ 10 \end{Bmatrix} + \begin{bmatrix} 13\,710 & 27\,540 \\ -27\,540 & -9947 \end{bmatrix}\{\delta\mathbf{v}_2{}^+\}$$

$$\begin{bmatrix} 13\,710 & 27\,540 \\ -27\,540 & -9947 \end{bmatrix}\{\delta\mathbf{v}_2{}^+\} = -\begin{bmatrix} 4.013 & 0 \\ -3.451 & 1 \end{bmatrix}\begin{Bmatrix} -10 \\ 10 \end{Bmatrix}$$

$$\begin{bmatrix} 13\,710 & 27\,540 \\ -27\,540 & -9947 \end{bmatrix}\{\delta\mathbf{v}_2{}^+\} = \begin{Bmatrix} 40.13 \\ -44.51 \end{Bmatrix}$$

$$\{\delta\mathbf{v}_2{}^+\} = \begin{bmatrix} 13\,710 & 27\,540 \\ -27\,540 & -9947 \end{bmatrix}^{-1}\begin{Bmatrix} 40.13 \\ -44.51 \end{Bmatrix} = \begin{Bmatrix} 0.001\,329 \\ 0.000\,795\,4 \end{Bmatrix}\ (\mathrm{km/s})$$

$$\{\delta\mathbf{v}_6{}^-\} = [\boldsymbol{\Phi}_{\mathrm{vr}}]\{\delta\mathbf{r}_0\} + [\boldsymbol{\Phi}_{\mathrm{vv}}]\{\delta\mathbf{v}_2{}^+\}$$

$$\{\delta\mathbf{v}_6{}^-\} = \begin{bmatrix} 0.000\,218\,8 & 0 \\ -0.000\,439\,4 & 0 \end{bmatrix}\begin{Bmatrix} -10 \\ 10 \end{Bmatrix} + \begin{bmatrix} -0.004\,3 & 2 \\ -2 & -3.017 \end{bmatrix}\begin{Bmatrix} 0.001\,329 \\ 0.000\,795\,4 \end{Bmatrix}$$

$$\{\delta\mathbf{v}_6{}^-\} = \begin{Bmatrix} -0.002\,188 \\ 0.004\,394 \end{Bmatrix} + \begin{Bmatrix} 0.001\,585 \\ -0.005\,057 \end{Bmatrix} = \begin{Bmatrix} -0.000\,602\,5 \\ -0.000\,663 \end{Bmatrix}\ (\mathrm{km/s})$$

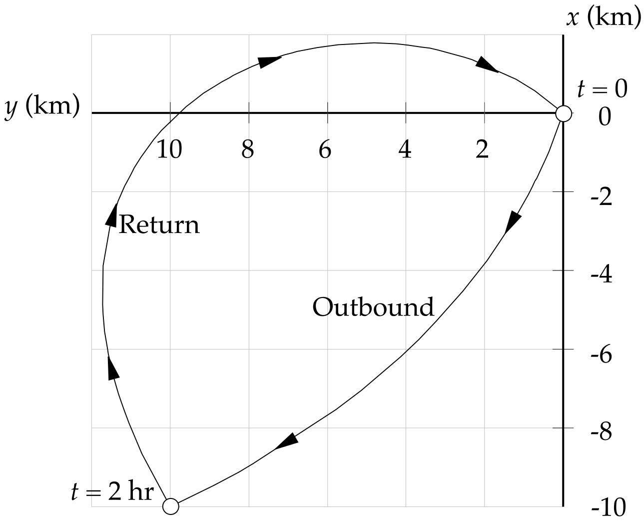
$$\{\Delta\mathbf{v}_2\} = \{\delta\mathbf{v}_2^{+}\} - \{\delta\mathbf{v}_2^{-}\} = \begin{Bmatrix} 0.001329 \\ 0.0007954 \end{Bmatrix} - \begin{Bmatrix} -0.000\,943\,4 \\ 0.002\,045 \end{Bmatrix} = \begin{Bmatrix} 0.002\,272 \\ -0.001\ \ 25 \end{Bmatrix} \text{(km/s)}$$

$$\Delta v_2 = \|\Delta\mathbf{v}_2\| = 0.002\,593 \text{ km/s}$$

$$\{\Delta\mathbf{v}_6\} = \{\delta\mathbf{v}_6^{+}\} - \{\delta\mathbf{v}_6^{-}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -0.000\,602\,5 \\ -0.000\ \ 663 \end{Bmatrix} = \begin{Bmatrix} 0.000\,602\,5 \\ 0.000\ \ 663 \end{Bmatrix} \text{(km/s)}$$

$$\Delta v_6 = \|\Delta\mathbf{v}_6\| = 0.000\,8958 \text{ km/s}$$

$$\Delta v_{total} = \Delta v_2 + \Delta v_6 = 0.002\,593 + 0.000\,8958 = 0.003\,489 \text{ km/s} = \underline{3.489 \text{ m/s}}$$



Problem 7.13 Design problem.

Problem 7.14

$$n = \frac{\sqrt{\dfrac{398\,600}{6600}}}{6600} = 0.001\,177\,5 \text{ s}^{-1}$$

$$\delta v = -\frac{3}{2}n\delta x = -\frac{3}{2}\cdot 0.001\,177\,5\cdot 5 = 0.008\,8311 \text{ km/s} = \underline{8.8311 \text{ m/s}}$$

Problem 7.15

$$nt = \frac{\pi}{2}$$

$$[\mathbf{\Phi}_{rr}(t)] = \begin{bmatrix} 4-3\cos nt & 0 \\ 6(\sin nt - nt) & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3.425 & 1 \end{bmatrix}$$

$$[\mathbf{\Phi}_{rv}(t)] = \begin{bmatrix} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1-\cos nt) \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{n} & \dfrac{2}{n} \\ -\dfrac{2}{n} & -\dfrac{0.7124}{n} \end{bmatrix}$$

$$\{\delta\mathbf{r}_f\} = [\mathbf{\Phi}_{rr}]\{\delta\mathbf{r}_0\} + [\mathbf{\Phi}_{rv}]\{\delta\mathbf{v}_0\}$$

$$\{\delta\mathbf{r}_f\} = \begin{bmatrix} 4 & 0 \\ -3.425 & 1 \end{bmatrix}\begin{Bmatrix} \delta r \\ \pi\delta r \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{n} & \dfrac{2}{n} \\ -\dfrac{2}{n} & -\dfrac{0.7124}{n} \end{bmatrix}\begin{Bmatrix} \dfrac{n\pi\delta r}{16} \\ -\dfrac{7n\delta r}{4} \end{Bmatrix}$$

$$\{\delta\mathbf{r}_f\} = \begin{Bmatrix} 4\delta r \\ -0.2832\delta r \end{Bmatrix} + \begin{Bmatrix} -3.304\delta r \\ 0.854\delta r \end{Bmatrix} = \begin{Bmatrix} 0.6963\delta r \\ 0.85708\delta r \end{Bmatrix}$$

$$d = \left\|\delta\mathbf{r}_f\right\| = \underline{0.9004\delta r}$$

Problem 7.16

$nt = \pi \ nt = \pi$

$$\left[\mathbf{\Phi}_{\text{rr}}(t)\right] = \begin{bmatrix} 4-3\cos nt & 0 \\ 6(\sin nt - nt) & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ -18.85 & 1 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{rv}}(t)\right] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{n} \\ -\frac{4}{n} & -\frac{9.425}{n} \end{bmatrix}$$

$$\left\{\delta\mathbf{r}_f\right\} = \left[\mathbf{\Phi}_{\text{rr}}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{\text{rv}}\right]\left\{\delta\mathbf{v}_0\right\}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 7 & 0 \\ -18.85 & 1 \end{bmatrix}\begin{Bmatrix} \delta x_0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & \frac{4}{n} \\ -\frac{4}{n} & -\frac{9.425}{n} \end{bmatrix}\left\{\delta\mathbf{v}_0\right\}$$

$$\begin{bmatrix} 0 & \frac{4}{n} \\ -\frac{4}{n} & -\frac{9.425}{n} \end{bmatrix}\left\{\delta\mathbf{v}_0\right\} = \begin{Bmatrix} -7\delta x_0 \\ 18.85\delta x_0 \end{Bmatrix}$$

$$\left\{\delta\mathbf{v}_0\right\} = \begin{bmatrix} 0 & \frac{4}{n} \\ -\frac{4}{n} & -\frac{9.425}{n} \end{bmatrix}^{-1}\begin{Bmatrix} -7\delta x_0 \\ 18.85\delta x_0 \end{Bmatrix}$$

$$\left\{\delta\mathbf{v}_0\right\} = \begin{Bmatrix} -0.589n\delta x_0 \\ -1.75n\delta x_0 \end{Bmatrix}$$

or

$$\underline{\delta\mathbf{v}_0 = -0.589n\delta x_0\hat{\mathbf{i}} - 1.75n\delta x_0\hat{\mathbf{j}}\ (\text{km/s})}$$

Problem 7.17

$$nt = \frac{\pi}{4}$$

$$\left[\mathbf{\Phi}_{\text{rr}}(t)\right] = \begin{bmatrix} 4-3\cos nt & 0 \\ 6(\sin nt - nt) & 1 \end{bmatrix} = \begin{bmatrix} 1.879 & 0 \\ -0.4697 & 1 \end{bmatrix}$$

$$\left[\mathbf{\Phi}_{\text{rv}}(t)\right] = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) \end{bmatrix} = \begin{bmatrix} \frac{0.7071}{n} & \frac{0.5858}{n} \\ -\frac{0.5858}{n} & \frac{0.4722}{n} \end{bmatrix}$$

$$\left\{\delta\mathbf{r}_f\right\} = \left[\mathbf{\Phi}_{\text{rr}}\right]\left\{\delta\mathbf{r}_0\right\} + \left[\mathbf{\Phi}_{\text{rv}}\right]\left\{\delta\mathbf{v}_0\right\}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1.879 & 0 \\ -0.4697 & 1 \end{bmatrix}\begin{Bmatrix} -d \\ \delta y_0 \end{Bmatrix} + \begin{bmatrix} \frac{0.7071}{n} & \frac{0.5858}{n} \\ -\frac{0.5858}{n} & \frac{0.4722}{n} \end{bmatrix}\begin{Bmatrix} 0 \\ \delta v_0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.879d \\ 0.4697d + \delta y_0 \end{Bmatrix} + \begin{Bmatrix} \frac{0.5858\delta v_0}{n} \\ \frac{0.4722\delta v_0}{n} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{0.5858\delta v_0}{n} - 1.879d \\ \frac{0.4722\delta v_0}{n} + 0.4697d + \delta y_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & \dfrac{0.5858}{n} \\ 1 & \dfrac{0.4722}{n} \end{bmatrix} \begin{Bmatrix} \delta y_0 \\ \delta v_0 \end{Bmatrix} = \begin{Bmatrix} 1.879d \\ -0.4697d \end{Bmatrix}$$

$$\begin{Bmatrix} \delta y_0 \\ \delta v_0 \end{Bmatrix} = \begin{bmatrix} 0 & \dfrac{0.5858}{n} \\ 1 & \dfrac{0.4722}{n} \end{bmatrix}^{-1} \begin{Bmatrix} 1.879d \\ -0.4697d \end{Bmatrix}$$

$$\begin{Bmatrix} \delta y_0 \\ \delta v_0 \end{Bmatrix} = \begin{bmatrix} -0.8062 & 1 \\ 1.707n & 0 \end{bmatrix} \begin{Bmatrix} 1.879d \\ -0.4697d \end{Bmatrix} = \begin{Bmatrix} -1.984d \\ 3.207nd \end{Bmatrix}$$

$$\underline{\delta y_0 = -1.984d}$$

Problem 8.1

$$\mu_{\text{sun}} = 132.7 \times 10^9 \quad \text{km}^3/\text{s}^2$$
$$\mu_{\text{earth}} = 398\,600 \quad \text{km}^3/\text{s}^2$$
$$R_{\text{earth}} = 147.4 \times 10^6 \text{ km}$$
$$r_{\text{earth}} = 6378 \text{ km}$$

$$a = \frac{1}{2}\left(R_{\text{earth}} + R_2\right) = \frac{1}{2}\left(147.4 \times 10^6 + 120 \times 10^6\right) = 138.7 \times 10^6 \text{ km}$$

Heliocentric spacecraft velocity at earth's sphere of influence:

$$V^{(v)} = \sqrt{\mu_{\text{sun}}\left(\frac{2}{R_{\text{earth}}} - \frac{1}{a}\right)} = \sqrt{132.7 \times 10^9\left(\frac{2}{147.4 \times 10^6} - \frac{1}{138.7 \times 10^6}\right)} = 28.43 \text{ km/s}$$

Heliocentric velocity of earth:

$$V_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{earth}}}} = \sqrt{\frac{132.7 \times 10^9}{149.6 \times 10^6}} = 30.06 \text{ km/s}$$

$$\therefore v_\infty = V_{\text{earth}} - V^{(v)} = 30.06 - 28.43 = 1.579 \text{ km/s}$$

Geocentric spacecraft velocity of spacecraft at perigee of departure hyperbola:

$$v_p = \sqrt{{v_\infty}^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{1.579^2 + \frac{2 \cdot 398\,600}{6378 + 200}} = 11.12 \text{ km/s}$$

Geocentric spacecraft velocity in its circular parking orbit:

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r}} = \sqrt{\frac{398\,600}{6378 + 200}} = 7.784 \text{ km/s}$$

$$\therefore \Delta v = v_p - v_c = 11.12 - 7.784 = \underline{3.337 \text{ km/s}}$$

Problem 8.2

$$\mu_{\text{sun}} = 132.7 \times 10^9 \quad \text{km}^3/\text{s}^2$$
$$\mu_{\text{earth}} = 398\,600 \quad \text{km}^3/\text{s}^2$$
$$\mu_{\text{Mercury}} = 22\,930 \quad \text{km}^3/\text{s}^2$$

$$R_{\text{earth}} = 149.6 \times 10^6 \text{ km}$$
$$R_{\text{Mercury}} = 57.91 \times 10^6 \text{ km}$$

$$r_{\text{earth}} = 6378 \text{ km}$$
$$r_{\text{Mercury}} = 2440 \text{ km}$$

$$V_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{earth}}}} = \sqrt{\frac{132.7 \times 10^9}{149.6 \times 10^6}} = 29.78 \text{ km/s}$$

$$V_{\text{Mercury}} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Mercury}}}} = \sqrt{\frac{132.7 \times 10^9}{57.91 \times 10^6}} = 47.87 \text{ km/s}$$

Semimajor axis of Hohmann transfer ellipse:

$$a = \frac{1}{2}\left(R_{\text{earth}} + R_{\text{Mercury}}\right) = \frac{1}{2}\left(149.6\times10^6 + 57.91\times10^6\right) = 103.8\times10^6 \text{ km}$$

Departure from earth:

Spacecraft heliocentric velocity:

$$V^{(v)} = \sqrt{\mu_{\text{sun}}\left(\frac{2}{R_{\text{earth}}} - \frac{1}{a}\right)} = \sqrt{132.7\times10^9\left(\frac{2}{149.6\times10^6} - \frac{1}{103.8\times10^6}\right)} = 22.25 \text{ km/s}$$

$$\therefore v_\infty = V_{\text{earth}} - V^{(v)} = 29.78 - 22.25 = 7.532 \text{ km/s}$$

Spacecraft geocentric velocity in circular parking orbit:

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_p}} = \sqrt{\frac{398\,600}{6378+150}} = 7.814 \text{ km/s}$$

Spacecraft geocentric velocity at perigee of departure hyperbola:

$$v_p = \sqrt{v_\infty{}^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{7.532^2 + \frac{2\cdot 398\,600}{6378+150}} = 13.37 \text{ km/s}$$

$$\therefore \Delta v_1 = v_p - v_c = 13.37 - 7.814 = 9.611 \text{ km/s}$$

Arrival at Mercury:

Spacecraft heliocentric velocity:

$$V^{(v)} = \sqrt{\mu_{\text{sun}}\left(\frac{2}{R_{\text{Mercury}}} - \frac{1}{a}\right)} = \sqrt{132.7\times10^9\left(\frac{2}{57.91\times10^6} - \frac{1}{103.8\times10^6}\right)} = 57.48 \text{ km/s}$$

$$\therefore v_\infty = V^{(v)} - V_{\text{Mercury}} = 57.48 - 47.87 = 9.611 \text{ km/s}$$

Spacecraft velocity relative to Mercury at periapse of approach hyperbola:

$$v_p = \sqrt{v_\infty{}^2 + \frac{2\mu_{\text{Mercury}}}{r_p}} = \sqrt{9.611^2 + \frac{2\cdot 22\,930}{2440+150}} = 10.49 \text{ km/s}$$

Spacecraft parking orbit speed relative to Mercury:

$$v_c = \sqrt{\frac{\mu_{\text{Mercury}}}{r_p}} = \sqrt{\frac{22\,930}{2440+150}} = 2.975 \text{ km/s}$$

$$\therefore \Delta v_2 = v_p - v_c = 10.49 - 2.975 = 7.516 \text{ km/s}$$

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 9.611 + 7.516 = \underline{15.03 \text{ km/s}}$$

Problem 8.3

$$r_{SOI} = R\left(\frac{m_p}{m_{\text{sun}}}\right)$$

$$m_{\text{sun}} = 1.989\times10^{30} \text{ kg}$$

Mercury:

$$R = 57.91 \times 10^6 \text{ km}$$
$$m_p = 3.302 \times 10^{23} \text{ kg}$$
$$r_{SOI} = 57.91 \times 10^6 \left(\frac{3.302 \times 10^{23}}{1.989 \times 10^{30}} \right) = \underline{1.124 \times 10^5 \text{ km}}$$

Venus:

$$R = 108.2 \times 10^6 \text{ km}$$
$$m_p = 4.869 \times 10^{24} \text{ kg}$$
$$r_{SOI} = 108.2 \times 10^6 \left(\frac{4.869 \times 10^{24}}{1.989 \times 10^{30}} \right) = \underline{6.162 \times 10^5 \text{ km}}$$

Mars:

$$R = 227.9 \times 10^6 \text{ km}$$
$$m_p = 6.419 \times 10^{23} \text{ kg}$$
$$r_{SOI} = 227.9 \times 10^6 \left(\frac{6.419 \times 10^{23}}{1.989 \times 10^{30}} \right) = \underline{5.771 \times 10^5 \text{ km}}$$

Jupiter:

$$R = 778.6 \times 10^6 \text{ km}$$
$$m_p = 1.899 \times 10^{27} \text{ kg}$$
$$r_{SOI} = 778.6 \times 10^6 \left(\frac{1.899 \times 10^{27}}{1.989 \times 10^{30}} \right) = \underline{4.882 \times 10^7 \text{ km}}$$

Problem 8.4

$$r_{SOI} = R \left(\frac{m_p}{m_{\text{sun}}} \right)$$
$$m_{\text{sun}} = 1.989 \times 10^{30} \text{ kg}$$

Saturn:

$$R = 1433 \times 10^6 \text{ km}$$
$$m_p = 5.685 \times 10^{26} \text{ kg}$$
$$r_{SOI} = 1433 \times 10^6 \left(\frac{5.685 \times 10^{26}}{1.989 \times 10^{30}} \right) = \underline{5.479 \times 10^7 \text{ km}}$$

Uranus:

$$R = 2872 \times 10^6 \text{ km}$$
$$m_p = 8.683 \times 10^{25} \text{ kg}$$
$$r_{SOI} = 2872 \times 10^6 \left(\frac{8.683 \times 10^{25}}{1.989 \times 10^{30}} \right) = \underline{5.178 \times 10^7 \text{ km}}$$

Neptune:

$$R = 4495 \times 10^6 \text{ km}$$
$$m_p = 1.024 \times 10^{26} \text{ kg}$$
$$r_{SOI} = 1.024 \times 10^{26} \left(\frac{1.024 \times 10^{26}}{1.989 \times 10^{30}} \right) = \underline{8.658 \times 10^7 \text{ km}}$$

Pluto:

$$R = 5870 \times 10^6 \text{ km}$$

$$m_p = 1.25 \times 10^{22} \text{ kg}$$

$$r_{SOI} = 5870 \times 10^6 \left(\frac{1.25 \times 10^{22}}{1.989 \times 10^{30}} \right) = \underline{3.076 \times 10^6 \text{ km}}$$

Problem 8.5

$$\mu_{\text{sun}} = 132.7 \times 10^9 \text{ km}^3/\text{s}^2$$

$$\mu_{\text{Jupiter}} = 126.7 \times 10^6 \text{ km}^3/\text{s}^2$$

$$R_{\text{earth}} = 149.6 \times 10^6 \text{ km}$$

$$R_{\text{Jupiter}} = 778.6 \times 10^6 \text{ km}$$

$$r_{\text{Jupiter}} = 71\,490 \text{ km}$$

Semimajor axis of Hohmann transfer ellipse:

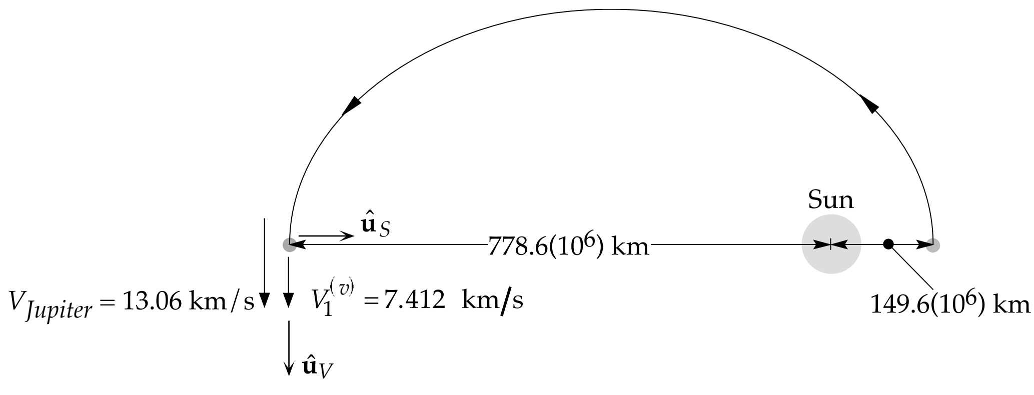
$$a_1 = \frac{1}{2}\left(R_{\text{earth}} + R_{\text{Jupiter}}\right) = \frac{1}{2}\left(149.6 \times 10^6 + 778.6 \times 10^6\right) = 464.1 \times 10^6 \text{ km}$$

$$V_{\text{Jupiter}} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Jupiter}}}} = \sqrt{\frac{132.7 \times 10^9}{778.6 \times 10^6}} = 13.06 \text{ km/s}$$

Use the energy equation to obtain the spacecraft's velocity upon arrival at Jupiter's sphere of influence:

$$V_1^{(v)} = \sqrt{\mu_{\text{sun}}\left(\frac{2}{R_{\text{Jupiter}}} - \frac{1}{a_1}\right)} = \sqrt{132.7 \times 10^9 \left(\frac{2}{778.6 \times 10^6} - \frac{1}{464.1 \times 10^6}\right)} = 7.412 \text{ km/s}$$

$$v_\infty = V_{\text{Jupiter}} - V_1^{(v)} = 13.06 - 7.412 = 5.643 \text{ km/s}$$



Eccentricity of hyperbolic swing by trajectory:

$$e = 1 + \frac{r_p {v_\infty}^2}{\mu_{\text{Jupiter}}} = 1 + \frac{271\,490 \cdot 5.643^2}{126.7 \times 10^6} = 1.068$$

Turn angle:

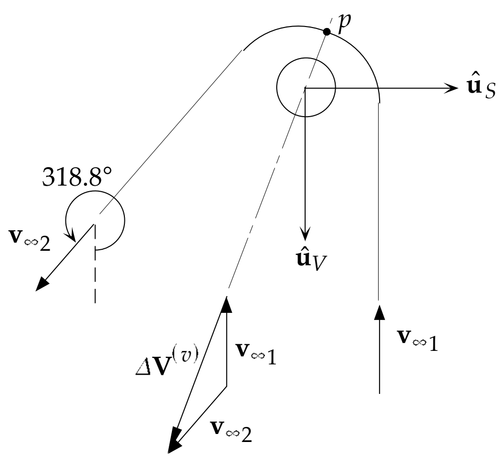
$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.068}\right) = 138.8°$$

Angle between $\mathbf{V}_{\text{Jupiter}}$ and $\mathbf{v}_\infty$ at inbound crossing: $\phi_1 = 180°$. At the outbound crossing,

$$\phi_2 = 180° + \delta = 318.8°$$

At the outbound crossing

$$\begin{aligned}
\mathbf{V}_2^{(v)} &= \mathbf{V}_{\text{Jupiter}} + \mathbf{v}_{\infty 2} \\
&= V_{\text{Jupiter}}\hat{\mathbf{u}}_V + \left(v_\infty \cos\phi_2 \hat{\mathbf{u}}_V + v_\infty \sin\phi_2\right)\hat{\mathbf{u}}_S \\
&= 13.06\hat{\mathbf{u}}_V + \left(5.643 \cdot \cos 318.8°\hat{\mathbf{u}}_V + 5.643 \sin 318.8°\right)\hat{\mathbf{u}}_S \\
&= 17.30\hat{\mathbf{u}}_V - 3.716\hat{\mathbf{u}}_S \ \left(\text{km/s}\right)
\end{aligned}$$



$$\begin{aligned}
\Delta\mathbf{V}^{(v)} &= \mathbf{V}_2^{(v)} - \mathbf{V}_1^{(v)} \\
&= \left(17.30\hat{\mathbf{u}}_V - 3.716\hat{\mathbf{u}}_S\right) - 7.412\hat{\mathbf{u}}_V \\
&= 9.890\hat{\mathbf{u}}_V - 3.716\hat{\mathbf{u}}_S \ \left(\text{km/s}\right)
\end{aligned}$$

$$\therefore \Delta V^{(v)} = \left\|\Delta\mathbf{V}^{(v)}\right\| = \underline{10.57 \ \text{km/s}}$$

Alternatively, $\Delta V^{(v)} = 2v_\infty \sin\dfrac{\delta}{2} = 2 \cdot 5.643 \cdot \sin\dfrac{138.8°}{2} = \underline{10.57 \ \text{km/s}}$.

Angular momentum of the new orbit after flyby:

$$h_2 = R_{\text{Jupiter}} V_\perp^{(v)} = 778.6 \times 10^6 \cdot 17.30 = \underline{13.47 \times 10^9 \ \ \text{km}^2/\text{s}}$$

Obtain the semimajor axis a_2 from the energy equation:

$$\frac{V_2^{(v)2}}{2} - \frac{\mu_{\text{sun}}}{R_{\text{Jupiter}}} = -\frac{\mu_{\text{sun}}}{2a_2}$$

$$\frac{17.70^2}{2} - \frac{132.7 \times 10^9}{778.6 \times 10^6} = -\frac{132.7 \times 10^9}{2a_2} \quad \Rightarrow \quad \underline{a_2 = 4.791 \times 10^9 \ \text{km}}$$

Observe that $a_2/a_1 = 10.32$. The new orbit dwarfs the original one in size and, therefore, energy.

Use Equation 3.61 to find the eccentricity:

$$a_2 = \frac{{h_2}^2}{\mu_{\text{sun}}}\frac{1}{1 - {e_2}^2}$$

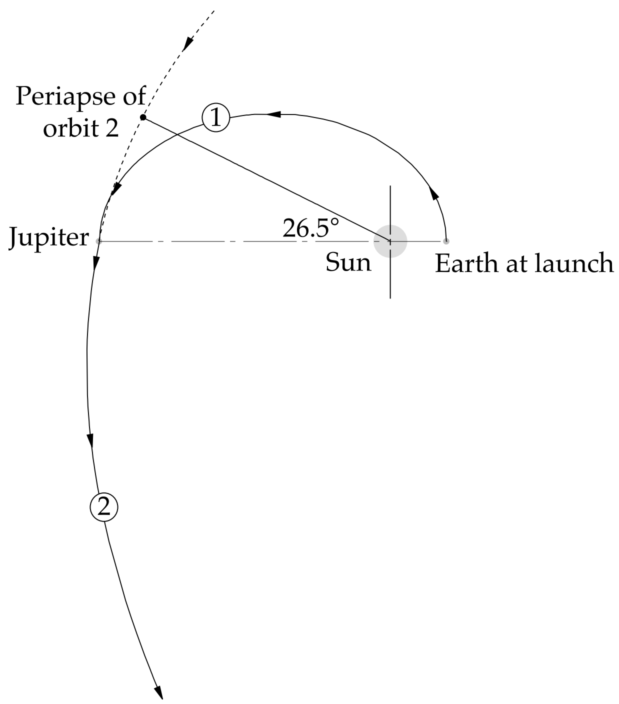
$$4.791 \times 10^9 = \frac{13.47 \times 10^{9^2}}{132.7 \times 10^9}\frac{1}{1 - {e_2}^2} \quad \Rightarrow \quad \underline{e_2 = 0.8453}$$

Use the orbit equation to find the true anomaly in the new orbit:

$$R_{\text{Jupiter}} = \frac{{h_2}^2}{\mu_{\text{sun}}}\frac{1}{1 + e_2 \cos\theta_2}$$

$$778.6 \times 10^6 = \frac{13.47 \times 10^{9^2}}{132.7 \times 10^9}\frac{1}{1 + 0.8453\cos\theta_2} \quad \Rightarrow \quad \theta_2 = 26.5°$$

The new and original heliocentric orbits are illustrated below.



Problem 8.6

Algorithm 8.1, which makes use of the data in Table 8.1, is implemented in MATAB as the M-function `planet_elements_and_sv` in Appendix D.17. The MATLAB script `Example_8_07`, which also appears in Appendix D.17, calls upon `planet_elements_and_sv` to calculate the orbital elements of earth on the date specified in Example 8.7. The output to the Command Window is also listed in Appendix D.17 for comparison with the results presented in Example 8.7. By changing `planet_id` to 4, the following Command Window output is obtained for Mars.

```
-----------------------------------------------------
 Problem 8.6

 Input data:

   Planet: Mars
   Year  : 2003
   Month : August
   Day   : 27
   Hour  : 12
   Minute: 0
   Second: 0

   Julian day: 2452879.000

 Orbital elements:

  Angular momentum (km^2/s)                     = 5.47595e+09
  Eccentricity                                  = 0.0934167
  Right ascension of the ascending node (deg) = 49.5682
  Inclination to the ecliptic (deg)             = 1.85035
  Argument of perihelion (deg)                  = 286.488
  True anomaly (deg)                            = 358.131
  Semimajor axis (km)                           = 2.27936e+08

  Longitude of perihelion (deg)                 = 336.057
  Mean longitude (deg)                          = 334.513
  Mean anomaly (deg)                            = 358.457
  Eccentric anomaly (deg)                       = 358.298
```

```
State vector:

 Position vector (km) = [1.85954e+08  -8.99155e+07  -6.45661e+06]
 Magnitude            = 2.06653e+08

 Velocity (km/s)      = [11.4744  23.8842  0.218255]
 Magnitude            = 26.4984
-----------------------------------------------------
```

Problem 8.7 The following MATLAB script calls upon Algorithm 8.1, implemented as the MATLAB M-function `planet_elements_and_sv` in Appendix D.17, to compute the distance of the earth from the sun on the first day of each month of the year 2005, at 12:00:00 UT. The output to the MATLAB command window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_8_07a
% ~~~~~~~~~~~~~
%
% This program uses Algorithm 8.1 to compute the orbital elements
% and state vector of Mars at the date and time specified
% in Example 8.7.
%
% mu        - gravitational parameter of the sun (km^3/s^2)
%
% coe       - vector of heliocentric orbital elements
%             [h  e  RA  incl  w  TA  a  w_hat  L  M  E],
%             where
%              h     = angular momentum                   (km^2/s)
%              e     = eccentricity
%              RA    = right ascension                    (deg)
%              incl  = inclination                        (deg)
%              w     = argument of perihelion             (deg)
%              TA    = true anomaly                       (deg)
%              a     = semimajor axis                     (km)
%              w_hat = longitude of perihelion ( = RA + w) (deg)
%              L     = mean longitude ( = w_hat + M)      (deg)
%              M     = mean anomaly                       (deg)
%              E     = eccentric anomaly                  (deg)
%
% r         - heliocentric position vector (km)
% v         - heliocentric velocity vector (km/s)
%
% planet_id - planet identifier:
%              1 = Mercury
%              2 = Venus
%              3 = Earth
%              4 = Mars
%              5 = Jupiter
%              6 = Saturn
%              7 = Uranus
%              8 = Neptune
%              9 = Pluto
%
% year      - range: 1901 - 2099
% month     - range: 1 - 12
% day       - range: 1 - 31
% hour      - range: 0 - 23
% minute    - range: 0 - 60
% second    - range: 0 - 60
%
% User M-functions required: planet_elements_and_sv,
%                            month_planet_names
```

```
% ----------------------------------------------------------------------

global mu
mu  = 1.327124e11;
deg = pi/180;

%...Data declaration for Problem 8.6 :
planet_id = 3;
year      = 2005;
day       = 1;
hour      = 12;
minute    = 0;
second    = 0;
%...

fprintf('\n-----------------------------------------------------\n')
fprintf(' Problem 8.7a: Determine the month\n')
fprintf('\n Year = %g  Time = 12:00:00 UT\n', year)

for month = 1:12
    %...Algorithm 8.1:
    [coe, r, v, jd] = planet_elements_and_sv ...
              (planet_id, year, month, day, hour, minute, second);
    %...Convert the month numbers into names for output:
    [month_name, planet_name] = month_planet_names(month, planet_id);
    fprintf('\n %10s 1st:   Distance = %11.5e (km)', month_name,
norm(r))
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 8.7a: Determine the month

 Year = 2005  Time = 12:00:00 UT

  January   1st:   Distance = 1.47100e+08 (km)
  February  1st:   Distance = 1.47417e+08 (km)
  March     1st:   Distance = 1.48243e+08 (km)
  April     1st:   Distance = 1.49505e+08 (km)
  May       1st:   Distance = 1.50745e+08 (km)
  June      1st:   Distance = 1.51707e+08 (km)
  July      1st:   Distance = 1.52093e+08 (km)
  August    1st:   Distance = 1.51830e+08 (km)
  September 1st:   Distance = 1.50963e+08 (km)
  October   1st:   Distance = 1.49758e+08 (km)
  November  1st:   Distance = 1.48464e+08 (km)
  December  1st:   Distance = 1.47505e+08 (km)
-----------------------------------------------------
```

This list reveals that the greatest distance occurs in the month of July. We can modify the above MATLAB script to loop through the days of July:

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_8_07b
% ~~~~~~~~~~~~~
%
% This program uses Algorithm 8.1 to determine the distance of the
% earth from the sun, according to Problem 8.7
%
% User M-functions required: planet_elements_and_sv,
%                            month_planet_names
% ----------------------------------------------------------------------
```

```
global mu
mu  = 1.327124e11;

%...Data declaration for Problem 8.7 :
planet_id = 3;
year      = 2005;
month     = 7;
hour      = 12;
minute    = 0;
second    = 0;
%...

fprintf('\n--------------------------------------------------------\n')
fprintf(' Problem 8.7b: Determine the day\n')
fprintf('\n Year = %g  Time = 12:00:00 UT\n', year)
%...Convert the planet_id and month numbers into names for output:
  [month_name, planet_name] = month_planet_names(month, planet_id);

for day = 1:31
    %...Algorithm 8.1:
    [coe, r, v, jd] = planet_elements_and_sv ...
              (planet_id, year, month, day, hour, minute, second);
    %...Convert the planet_id and month numbers into names for output:
    [month_name, planet_name] = month_planet_names(month, planet_id);
    fprintf('\n %5s %4g:   Distance = %14.7e (km)',...
                                          month_name, day, norm(r))
end
fprintf('\n--------------------------------------------------------\n')
```

The output to the MATLAB command window is:

```
-----------------------------------------------------
 Problem 8.7b: Determine the day

 Year = 2005  Time = 12:00:00 UT

 July         1:   Distance =  1.5209314e+08 (km)
 July         2:   Distance =  1.5209524e+08 (km)
 July         3:   Distance =  1.5209664e+08 (km)
 July         4:   Distance =  1.5209732e+08 (km)
 July         5:   Distance =  1.5209728e+08 (km)
 July         6:   Distance =  1.5209653e+08 (km)
 July         7:   Distance =  1.5209506e+08 (km)
 July         8:   Distance =  1.5209287e+08 (km)
 July         9:   Distance =  1.5208998e+08 (km)
 July        10:   Distance =  1.5208637e+08 (km)
 July        11:   Distance =  1.5208204e+08 (km)
 July        12:   Distance =  1.5207701e+08 (km)
 July        13:   Distance =  1.5207126e+08 (km)
 July        14:   Distance =  1.5206481e+08 (km)
 July        15:   Distance =  1.5205765e+08 (km)
 July        16:   Distance =  1.5204978e+08 (km)
 July        17:   Distance =  1.5204122e+08 (km)
 July        18:   Distance =  1.5203195e+08 (km)
 July        19:   Distance =  1.5202198e+08 (km)
 July        20:   Distance =  1.5201132e+08 (km)
 July        21:   Distance =  1.5199996e+08 (km)
 July        22:   Distance =  1.5198792e+08 (km)
 July        23:   Distance =  1.5197519e+08 (km)
 July        24:   Distance =  1.5196178e+08 (km)
 July        25:   Distance =  1.5194769e+08 (km)
 July        26:   Distance =  1.5193292e+08 (km)
```

```
 July        27:   Distance =  1.5191748e+08 (km)
 July        28:   Distance =  1.5190138e+08 (km)
 July        29:   Distance =  1.5188461e+08 (km)
 July        30:   Distance =  1.5186718e+08 (km)
 July        31:   Distance =  1.5184910e+08 (km)
-----------------------------------------------------
```

The furthest distance occurs on July 4.

Problem 8.8 For the data given in this problem, the following MATLAB script invokes Algorithm 8.2, which is implemented as the MATLAB M-function `interplanetary` in Appendix D.18. `interplanetary` uses Algorithms 8.1 and 5.2 to compute $\mathbf{v}_\infty$ at the home and target planets. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_8_08
% ~~~~~~~~~~~~
%
% This program uses Algorithm 8.2 to obtain v-infinities
% in Problem 8.8.
%
% mu            - gravitational parameter of the sun (km^3/s^2)
% deg           - conversion factor between degrees and radians
% pi            - 3.1415926...
%
% planet_id     - planet identifier:
%                   1 = Mercury
%                   2 = Venus
%                   3 = Earth
%                   4 = Mars
%                   5 = Jupiter
%                   6 = Saturn
%                   7 = Uranus
%                   8 = Neptune
%                   9 = Pluto
%
% year          - range: 1901 - 2099
% month         - range: 1 - 12
% day           - range: 1 - 31
% hour          - range: 0 - 23
% minute        - range: 0 - 60
% second        - range: 0 - 60
%
% depart        - [planet_id, year, month, day, hour, minute, second]
%                 at departure
% arrive        - [planet_id, year, month, day, hour, minute, second]
%                 at arrival
%
% planet1       - [Rp1, Vp1, jd1]
% planet2       - [Rp2, Vp2, jd2]
% trajectory    - [V1, V2]
%
% coe           - orbital elements [h e RA incl w TA]
%                 where
%                   h    = angular momentum (km^2/s)
%                   e    = eccentricity
%                   RA   = right ascension of the ascending
%                          node (rad)
%                   incl = inclination of the orbit (rad)
%                   w    = argument of perigee (rad)
%                   TA   = true anomaly (rad)
```

```
%                   a    = semimajor axis (km)
%
% jd1, jd2      - Julian day numbers at departure and arrival
% tof           - time of flight from planet 1 to planet 2 (days)
%
% Rp1, Vp1      - state vector of planet 1 at departure (km, km/s)
% Rp2, Vp2      - state vector of planet 2 at arrival (km, km/s)
% R1, V1        - heliocentric state vector of spacecraft at
%                 departure (km, km/s)
% R2, V2        - heliocentric state vector of spacecraft at
%                 arrival (km, km/s)
%
% vinf1, vinf2 - hyperbolic excess velocities at departure
%                 and arrival (km/s)
%
% User M-functions required: interplanetary, coe_from_sv,
%                            month_planet_names
% ---------------------------------------------------------------------

clear
global mu
mu  = 1.327124e11;
deg = pi/180;

%...Data declaration for Problem 8.8:

%...Departure
planet_id = 3;
year      = 2005;
month     = 12;
day       = 1;
hour      = 0;
minute    = 0;
second    = 0;
depart = [planet_id  year  month  day  hour  minute  second];

%...Arrival
planet_id = 2;
year      = 2006;
month     = 4;
day       = 1;
hour      = 0;
minute    = 0;
second    = 0;
arrive = [planet_id  year  month  day  hour  minute  second];

%...

%...Algorithm 8.2:
[planet1, planet2, trajectory] = interplanetary(depart, arrive);

R1  = planet1(1,1:3);
Vp1 = planet1(1,4:6);
jd1 = planet1(1,7);

R2  = planet2(1,1:3);
Vp2 = planet2(1,4:6);
jd2 = planet2(1,7);

V1  = trajectory(1,1:3);
V2  = trajectory(1,4:6);

tof = jd2 - jd1;
```

```
%...Use Algorithm 5.1 to find the orbital elements of the
%   spacecraft trajectory based on [Rp1, V1]...
coe  = coe_from_sv(R1, V1);
%   ... and [R2, V2]
coe2 = coe_from_sv(R2, V2);

%...Equations 8.102 and 8.103:
vinf1 = V1 - Vp1;
vinf2 = V2 - Vp2;

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 8.8: Earth to Venus')
fprintf('\n\n Departure:\n');
fprintf('\n   Planet: %s', planet_name(depart(1)))
fprintf('\n   Year  : %g', depart(2))
fprintf('\n   Month : %s', month_name(depart(3)))
fprintf('\n   Day   : %g', depart(4))
fprintf('\n   Hour  : %g', depart(5))
fprintf('\n   Minute: %g', depart(6))
fprintf('\n   Second: %g', depart(7))
fprintf('\n\n   Julian day: %11.3f\n', jd1)
fprintf('\n   Planet position vector (km)    = [%g  %g  %g]', ...
                                                R1(1),R1(2), R1(3))

fprintf('\n   Magnitude                      = %g\n', norm(R1))

fprintf('\n   Planet velocity (km/s)         = [%g  %g  %g]', ...
                                 Vp1(1), Vp1(2), Vp1(3))

fprintf('\n   Magnitude                      = %g\n', norm(Vp1))

fprintf('\n   Spacecraft velocity (km/s)     = [%g  %g  %g]', ...
                                                V1(1), V1(2), V1(3))

fprintf('\n   Magnitude                      = %g\n', norm(V1))

fprintf('\n   v-infinity at departure (km/s) = [%g  %g  %g]', ...
                                        vinf1(1), vinf1(2), vinf1(3))

fprintf('\n   Magnitude                      = %g\n', norm(vinf1))

fprintf('\n\n Time of flight = %g days\n', tof)

fprintf('\n\n Arrival:\n');
fprintf('\n   Planet: %s', planet_name(arrive(1)))
fprintf('\n   Year  : %g', arrive(2))
fprintf('\n   Month : %s', month_name(arrive(3)))
fprintf('\n   Day   : %g', arrive(4))
fprintf('\n   Hour  : %g', arrive(5))
fprintf('\n   Minute: %g', arrive(6))
fprintf('\n   Second: %g', arrive(7))
fprintf('\n\n   Julian day: %11.3f\n', jd2)
fprintf('\n   Planet position vector (km)   = [%g  %g  %g]', ...
                                               R2(1), R2(2), R2(3))

fprintf('\n   Magnitude                     = %g\n', norm(R1))

fprintf('\n   Planet velocity (km/s)        = [%g  %g  %g]', ...
                                 Vp2(1), Vp2(2), Vp2(3))
```

```
fprintf('\n   Magnitude                     = %g\n', norm(Vp2))

fprintf('\n   Spacecraft Velocity (km/s)    = [%g  %g  %g]', ...
                                          V2(1), V2(2), V2(3))

fprintf('\n   Magnitude                     = %g\n', norm(V2))

fprintf('\n   v-infinity at arrival (km/s)  = [%g  %g  %g]', ...
                                 vinf2(1), vinf2(2), vinf2(3))

fprintf('\n   Magnitude                     = %g', norm(vinf2))

fprintf('\n\n\n Orbital elements of flight trajectory:\n')

fprintf('\n  Angular momentum (km^2/s)                    = %g',...
                                                        coe(1))
fprintf('\n  Eccentricity                                 = %g',...
                                                        coe(2))
fprintf('\n  Right ascension of the ascending node (deg) = %g',...
                                                    coe(3)/deg)
fprintf('\n  Inclination to the ecliptic (deg)           = %g',...
                                                    coe(4)/deg)
fprintf('\n  Argument of perihelion (deg)                = %g',...
                                                    coe(5)/deg)
fprintf('\n  True anomaly at departure (deg)             = %g',...
                                                    coe(6)/deg)
fprintf('\n  True anomaly at arrival (deg)               = %g\n',...
                                                   coe2(6)/deg)
fprintf('\n  Semimajor axis (km)                         = %g',...
                                                        coe(7))
% If the orbit is an ellipse, output the period:
if coe(2) < 1
    fprintf('\n  Period (days)                                = %g',...
                                 2*pi/sqrt(mu)*coe(7)^1.5/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 8.8: Earth to Venus

 Departure:

   Planet: Earth
   Year  : 2005
   Month : December
   Day   : 1
   Hour  : 0
   Minute: 0
   Second: 0

   Julian day: 2453705.500

   Planet position vector (km)    = [5.33243e+07  1.37541e+08  -1830.84]
   Magnitude                      = 1.47517e+08

   Planet velocity (km/s)         = [-28.2595  10.6564  -6.01367e-05]
   Magnitude                      = 30.202

   Spacecraft velocity (km/s)     = [-27.0436  6.58196  2.7931]
   Magnitude                      = 27.9729

   v-infinity at departure (km/s) = [1.2159  -4.07444  2.79316]
```

```
  Magnitude                      = 5.08735


Time of flight = 121 days


Arrival:

  Planet: Venus
  Year  : 2006
  Month : April
  Day   : 1
  Hour  : 0
  Minute: 0
  Second: 0

  Julian day: 2453826.500

  Planet position vector (km)   = [-5.74135e+07  -9.1938e+07  2.05581e+06]
  Magnitude                     = 1.47517e+08

  Planet velocity (km/s)        = [29.4613  -18.7099  -1.95644]
  Magnitude                     = 34.955

  Spacecraft Velocity (km/s)    = [30.3918  -22.2324  -3.68154]
  Magnitude                     = 37.8351

  v-infinity at arrival (km/s)  = [0.930541  -3.52248  -1.7251]
  Magnitude                     = 4.0311


Orbital elements of flight trajectory:

 Angular momentum (km^2/s)                    = 4.0914e+09
 Eccentricity                                 = 0.183291
 Right ascension of the ascending node (deg)  = 68.8159
 Inclination to the ecliptic (deg)            = 5.77973
 Argument of perihelion (deg)                 = 142.255
 True anomaly at departure (deg)              = 217.738
 True anomaly at arrival (deg)                = 26.8913

 Semimajor axis (km)                          = 1.30519e+08
 Period (days)                                = 297.66
-----------------------------------------------------
```

The output shows that at departure from earth, $v_\infty = 5.087$ km/s. Hence, the spacecraft velocity at perigee of the departure hyperbola is

$$v_p = \sqrt{{v_\infty}^2 + \frac{2\mu_{\text{earth}}}{{r_p}^2}} = \sqrt{5.087^2 + \frac{2 \cdot 398\,600}{(6378+180)^2}} = 12.14 \text{ km/s}$$

The spacecraft velocity in its circular 180 km parking orbit is

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_p}} = \sqrt{\frac{398\,600}{6378+180}} = 7.796 \text{ km/s}$$

Hence, the delta-v requirement at earth is

$$\Delta v_1 = v_p - v_c = 12.14 - 7.796 = 4.346 \text{ km/s}$$

At Venus ($\mu_{\text{Venus}} = 324\,900\ \text{km}^3/\text{s}^2$, $r_{\text{Venus}} = 6052\ \text{km}$) the above output shows that $v_\infty = 4.031\ \text{km/s}$. The speed at the 300 km altitude periapse on the arrival hyperbola is therefore

$$v_{p_{\text{hyperbola}}} = \sqrt{v_\infty{}^2 + \frac{2\mu_{\text{Venus}}}{r_p{}^2}} = \sqrt{4.031^2 + \frac{2\cdot 324\,900}{(6052+300)^2}} = 10.89\ \text{km/s}$$

The semimajor axis of the elliptical capture orbit is

$$a = \frac{1}{2}\left[\left(r_{\text{Venus}} + 300\right) + \left(r_{\text{Venus}} + 9000\right)\right] = 10\,702$$

Therefore the velocity at periapse on the ellipse is, using the energy equation,

$$v_{p_{\text{ellipse}}} = \sqrt{\mu_{\text{Venus}}\left(\frac{2}{r_p} - \frac{1}{a}\right)} = \sqrt{324\,900\left(\frac{2}{6052+300} - \frac{1}{10\,702}\right)} = 8.482\ \text{km/s}$$

It follows that the delta-v requirement at Venus is

$$\Delta v_2 = v_{p_{\text{hyperbola}}} - v_{p_{\text{ellipse}}} = 10.89 - 8.482 = 2.406\ \text{km/s}$$

The total delta-v requirement is

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 4.346 + 2.406 = \underline{6.753\ \ \text{km/s}}$$

Problem 8.9 For the data given in this problem, the following MATLAB script invokes Algorithm 8.2, which is implemented as the MATLAB M-function `interplanetary` in Appendix D.18. `interplanetary` uses Algorithms 8.1 and 5.2 to compute $\mathbf{v}_\infty$ at the home and target planets. The output to the MATLAB Command Window is listed afterwards.

```
% ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
% Problem_8_09
% ~~~~~~~~~~~~
%
% This program uses Algorithm 8.2 to obtain v-infinities
% in Problem 8.9.
%
% mu           - gravitational parameter of the sun (km^3/s^2)
% deg          - conversion factor between degrees and radians
% pi           - 3.1415926...
%
% planet_id    - planet identifier:
%                  1 = Mercury
%                  2 = Venus
%                  3 = Earth
%                  4 = Mars
%                  5 = Jupiter
%                  6 = Saturn
%                  7 = Uranus
%                  8 = Neptune
%                  9 = Pluto
%
% year         - range: 1901 - 2099
% month        - range: 1 - 12
% day          - range: 1 - 31
% hour         - range: 0 - 23
% minute       - range: 0 - 60
% second       - range: 0 - 60
```

```
%
% depart       - [planet_id, year, month, day, hour, minute, second]
%                at departure
% arrive       - [planet_id, year, month, day, hour, minute, second]
%                at arrival
%
% planet1      - [Rp1, Vp1, jd1]
% planet2      - [Rp2, Vp2, jd2]
% trajectory   - [V1, V2]
%
% coe          - orbital elements [h e RA incl w TA]
%                where
%                  h    = angular momentum (km^2/s)
%                  e    = eccentricity
%                  RA   = right ascension of the ascending
%                         node (rad)
%                  incl = inclination of the orbit (rad)
%                  w    = argument of perigee (rad)
%                  TA   = true anomaly (rad)
%                  a    = semimajor axis (km)
%
% jd1, jd2     - Julian day numbers at departure and arrival
% tof          - time of flight from planet 1 to planet 2 (days)
%
% Rp1, Vp1     - state vector of planet 1 at departure (km, km/s)
% Rp2, Vp2     - state vector of planet 2 at arrival (km, km/s)
% R1, V1       - heliocentric state vector of spacecraft at
%                departure (km, km/s)
% R2, V2       - heliocentric state vector of spacecraft at
%                arrival (km, km/s)
%
% vinf1, vinf2 - hyperbolic excess velocities at departure
%                and arrival (km/s)
%
% User M-functions required: interplanetary, coe_from_sv,
%                            month_planet_names
% ---------------------------------------------------------------------

clear
global mu
mu  = 1.327124e11;
deg = pi/180;

%...Data declaration for Problem 8.9:

%...Departure
planet_id = 3;
year      = 2005;
month     = 8;
day       = 15;
hour      = 0;
minute    = 0;
second    = 0;
depart = [planet_id  year  month  day  hour  minute  second];

%...Arrival
planet_id = 4;
year      = 2006;
month     = 3;
day       = 15;
hour      = 0;
minute    = 0;
second    = 0;
```

```
arrive = [planet_id  year  month  day  hour  minute  second];

%...

%...Algorithm 8.2:
[planet1, planet2, trajectory] = interplanetary(depart, arrive);

R1  = planet1(1,1:3);
Vp1 = planet1(1,4:6);
jd1 = planet1(1,7);

R2  = planet2(1,1:3);
Vp2 = planet2(1,4:6);
jd2 = planet2(1,7);

V1  = trajectory(1,1:3);
V2  = trajectory(1,4:6);

tof = jd2 - jd1;

%...Use Algorithm 5.1 to find the orbital elements of the
%   spacecraft trajectory based on [Rp1, V1]...
coe  = coe_from_sv(R1, V1);
%   ... and [R2, V2]
coe2 = coe_from_sv(R2, V2);

%...Equations 8.102 and 8.103:
vinf1 = V1 - Vp1;
vinf2 = V2 - Vp2;

%...Echo the input data and output the solution to
%   the command window:
fprintf('-----------------------------------------------------')
fprintf('\n Problem 8.9: Earth to Mars')
fprintf('\n\n Departure:\n');
fprintf('\n   Planet: %s', planet_name(depart(1)))
fprintf('\n   Year  : %g', depart(2))
fprintf('\n   Month : %s', month_name(depart(3)))
fprintf('\n   Day   : %g', depart(4))
fprintf('\n   Hour  : %g', depart(5))
fprintf('\n   Minute: %g', depart(6))
fprintf('\n   Second: %g', depart(7))
fprintf('\n\n   Julian day: %11.3f\n', jd1)
fprintf('\n   Planet position vector (km)    = [%g  %g  %g]', ...
                                               R1(1),R1(2), R1(3))

fprintf('\n   Magnitude                      = %g\n', norm(R1))

fprintf('\n   Planet velocity (km/s)         = [%g  %g  %g]', ...
                              Vp1(1), Vp1(2), Vp1(3))

fprintf('\n   Magnitude                      = %g\n', norm(Vp1))

fprintf('\n   Spacecraft velocity (km/s)     = [%g  %g  %g]', ...
                                               V1(1), V1(2), V1(3))

fprintf('\n   Magnitude                      = %g\n', norm(V1))

fprintf('\n   v-infinity at departure (km/s) = [%g  %g  %g]', ...
                                    vinf1(1), vinf1(2), vinf1(3))

fprintf('\n   Magnitude                      = %g\n', norm(vinf1))
```

```
fprintf('\n\n Time of flight = %g days\n', tof)

fprintf('\n\n Arrival:\n');
fprintf('\n   Planet: %s', planet_name(arrive(1)))
fprintf('\n   Year  : %g', arrive(2))
fprintf('\n   Month : %s', month_name(arrive(3)))
fprintf('\n   Day   : %g', arrive(4))
fprintf('\n   Hour  : %g', arrive(5))
fprintf('\n   Minute: %g', arrive(6))
fprintf('\n   Second: %g', arrive(7))
fprintf('\n\n   Julian day: %11.3f\n', jd2)
fprintf('\n   Planet position vector (km)   = [%g  %g  %g]', ...
                                         R2(1), R2(2), R2(3))

fprintf('\n   Magnitude                     = %g\n', norm(R1))

fprintf('\n   Planet velocity (km/s)        = [%g  %g  %g]', ...
                                  Vp2(1), Vp2(2), Vp2(3))

fprintf('\n   Magnitude                     = %g\n', norm(Vp2))

fprintf('\n   Spacecraft Velocity (km/s)    = [%g  %g  %g]', ...
                                         V2(1), V2(2), V2(3))

fprintf('\n   Magnitude                     = %g\n', norm(V2))

fprintf('\n   v-infinity at arrival (km/s)  = [%g  %g  %g]', ...
                               vinf2(1), vinf2(2), vinf2(3))

fprintf('\n   Magnitude                     = %g', norm(vinf2))

fprintf('\n\n\n Orbital elements of flight trajectory:\n')

fprintf('\n  Angular momentum (km^2/s)                    = %g',...
                                                      coe(1))
fprintf('\n  Eccentricity                                 = %g',...
                                                      coe(2))
fprintf('\n  Right ascension of the ascending node (deg) = %g',...
                                                  coe(3)/deg)
fprintf('\n  Inclination to the ecliptic (deg)           = %g',...
                                                  coe(4)/deg)
fprintf('\n  Argument of perihelion (deg)                = %g',...
                                                  coe(5)/deg)
fprintf('\n  True anomaly at departure (deg)             = %g',...
                                                  coe(6)/deg)
fprintf('\n  True anomaly at arrival (deg)               = %g\n',...
                                                 coe2(6)/deg)
fprintf('\n  Semimajor axis (km)                          = %g',...
                                                      coe(7))
% If the orbit is an ellipse, output the period:
if coe(2) < 1
    fprintf('\n  Period (days)                              = %g',...
                             2*pi/sqrt(mu)*coe(7)^1.5/24/3600)
end
fprintf('\n-----------------------------------------------------\n')
```

```
-----------------------------------------------------
 Problem 8.9: Earth to Mars

 Departure:

   Planet: Earth
```

```
  Year  : 2005
  Month : August
  Day   : 15
  Hour  : 0
  Minute: 0
  Second: 0

  Julian day: 2453597.500

  Planet position vector (km)    = [1.19728e+08  -9.28572e+07  798.533]
  Magnitude                      = 1.51517e+08

  Planet velocity (km/s)         = [17.7711  23.4274  -0.000315884]
  Magnitude                      = 29.405

  Spacecraft velocity (km/s)     = [20.6107  25.7677  1.75181]
  Magnitude                      = 33.0431

  v-infinity at departure (km/s) = [2.83967  2.34021  1.75212]
  Magnitude                      = 4.07557


Time of flight = 212 days


Arrival:

  Planet: Mars
  Year  : 2006
  Month : March
  Day   : 15
  Hour  : 0
  Minute: 0
  Second: 0

  Julian day: 2453809.500

  Planet position vector (km)   = [-8.33472e+07  2.26736e+08  6.79991e+06]
  Magnitude                     = 1.51517e+08

  Planet velocity (km/s)        = [-21.8221  -6.30169  0.40447]
  Magnitude                     = 22.7173

  Spacecraft Velocity (km/s)    = [-20.4825  -4.25753  -0.845206]
  Magnitude                     = 20.9374

  v-infinity at arrival (km/s)  = [1.33959  2.04416  -1.24968]
  Magnitude                     = 2.74496


Orbital elements of flight trajectory:

 Angular momentum (km^2/s)                    = 5.00602e+09
 Eccentricity                                 = 0.246978
 Right ascension of the ascending node (deg) = 322.198
 Inclination to the ecliptic (deg)           = 3.03935
 Argument of perihelion (deg)                = 355.671
 True anomaly at departure (deg)             = 4.33479
 True anomaly at arrival (deg)               = 152.278

 Semimajor axis (km)                          = 2.01098e+08
 Period (days)                                = 569.273
-----------------------------------------------------
```

The output shows that at departure from earth, $v_\infty = 4.076$ km/s. Hence, the spacecraft velocity at perigee of the departure hyperbola is

$$v_p = \sqrt{v_\infty{}^2 + \frac{2\mu_{\text{earth}}}{r_p{}^2}} = \sqrt{4.076^2 + \frac{2 \cdot 398\,600}{(6378 + 190)^2}} = 11.75\ \text{km/s}$$

The spacecraft velocity in its circular 190 km parking orbit is

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_p}} = \sqrt{\frac{398\,600}{6378 + 190}} = 7.790\ \text{km/s}$$

Hence, the delta-v requirement at earth is

$$\Delta v_1 = v_p - v_c = 11.75 - 7.790 = 3.957\ \text{km/s}$$

At Mars ($\mu_{\text{Mars}} = 42\,830\ \text{km}^3/\text{s}^2$, $r_{\text{Mars}} = 3396$ km) the above output shows that $v_\infty = 2.745$ km/s. The speed at the 300 km altitude periapse on the arrival hyperbola is therefore

$$v_{p_{\text{hyperbola}}} = \sqrt{v_\infty{}^2 + \frac{2\mu_{\text{Mars}}}{r_p{}^2}} = \sqrt{2.745^2 + \frac{2 \cdot 42\,830}{(3396 + 300)^2}} = 5.542\ \text{km/s}$$

The semimajor axis of the capture ellipse is found from the required 35 hour period.

$$T = \frac{2\pi}{\sqrt{\mu_{\text{Mars}}}} a^{3/2}$$

$$35 \cdot 3600 = \frac{2\pi}{\sqrt{42\,830}} a^{3/2} \quad \Rightarrow \quad a = 25\,830\ \text{km}$$

Therefore the velocity at periapse on the ellipse is, using the energy equation,

$$v_{p_{\text{ellipse}}} = \sqrt{\mu_{\text{Mars}}\left(\frac{2}{r_p} - \frac{1}{a}\right)} = \sqrt{42\,830\left(\frac{2}{3396 + 300} - \frac{1}{25\,830}\right)} = 4.639\ \text{km/s}$$

It follows that the delta-v requirement at Mars is

$$\Delta v_2 = v_{p_{\text{hyperbola}}} - v_{p_{\text{ellipse}}} = 5.542 - 4.639 = 0.9030\ \text{km/s}$$

The total delta-v requirement is

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 = 3.957 + 0.9030 = \underline{4.860\ \text{km/s}}$$

Problem 8.10

$$R_{\text{earth}} = 149.6 \times 10^6\ \text{km} \qquad R_{\text{Saturn}} = 1433 \times 10^6\ \text{km}$$

Semimajor axis of Hohmann transfer ellipse:

$$a = \frac{1}{2}\left(R_{\text{earth}} + R_{\text{Saturn}}\right) = \frac{1}{2}\left(149.6 \times 10^6 + 1433 \times 10^6\right) = 791.3 \times 10^6\ \text{km}$$

Period of Hohmann transfer ellipse:

$$T = \frac{2\pi}{\sqrt{\mu_{\text{sun}}}} a^{3/2} = \frac{2\pi}{\sqrt{132.7 \times 10^9}} \left(791.3 \times 10^6\right)^{3/2} = 383.9 \times 10^6 \text{ s} = 12.17 \text{ y}$$

Therefore, the time of flight to Saturn's orbit is $T/2 = \underline{6.083 \text{ y}}$. Cassini departed on 15 October 1997 (Julian day 2 450 736.5) and arrived on 1 July 2004 (Julian day 2 453 187.5). The number of years for Cassinni's flight was

$$\frac{2\,453\,187.5 - 2\,450\,736.5}{365.25} = 6.71 \text{ y}$$

Cassini, with its several flyby maneuvers, required a flight time only about 10 percent longer than the Hohmann transfer.

The velocity of the spacecraft at the outbound crossing of the earth's sphere of influence is

$$V^{(v)} = \sqrt{\mu_{\text{sun}}\left(\frac{2}{R_{\text{earth}}} - \frac{1}{a}\right)} = \sqrt{132.7 \times 10^9 \left(\frac{2}{149.6 \times 10^6} - \frac{1}{791.3 \times 10^6}\right)} = 40.08 \text{ km/s}$$

The velocity of earth in its (assumed) circular orbit is

$$V_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{earth}}}} = \sqrt{\frac{132.7 \times 10^9}{149.6 \times 10^6}} = 29.78 \text{ km/s}$$

Thus

$$v_\infty = V^{(v)} - V_{\text{earth}} = 40.08 - 29.78 = 10.3 \text{ km/s}$$

The spacecraft velocity at the 180 km altitude perigee of the departure hyperbola is

$$v_p = \sqrt{{v_\infty}^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{10.3^2 + \frac{2 \cdot 398\,600}{6378 + 180}} = 15.09 \text{ km/s}$$

The velocity in the circular parking orbit is

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_p}} = \sqrt{\frac{398\,600}{6378 + 180}} = 7.796 \text{ km/s}$$

Hence,

$$\Delta v = v_p - v_c = 15.09 - 7.796 = 7.289 \text{ km/s}$$

From Equation 6.1

$$\Delta m = m_o\left[1 - \exp\left(-\frac{\Delta v}{I_{sp} g_o}\right)\right] = m_o\left[1 - \exp\left(-\frac{7.289}{300 \cdot 9.81 \times 10^{-3}}\right)\right] = 0.916 m_o$$

But Δm equals the mass m_p of propellant expended, and the initial mass m_o equals m_p plus the mass of the spacecraft (2000 kg). Thus,

$$m_p = 0.916\left(2000 + m_p\right) \Rightarrow \underline{m_p = 21\,810 \text{ kg}}$$

Problem 9.1

Method 1: Use $(\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ as the basis.

$$\hat{\mathbf{i}} = \sin\theta\sin\phi\hat{\mathbf{I}} - \sin\theta\cos\phi\hat{\mathbf{J}} + \cos\theta\hat{\mathbf{K}} \tag{1}$$

$$\hat{\mathbf{j}} = \cos\phi\hat{\mathbf{I}} + \sin\phi\hat{\mathbf{J}} \tag{2}$$

$$\hat{\mathbf{k}} = -\cos\theta\sin\phi\hat{\mathbf{I}} + \cos\theta\cos\phi\hat{\mathbf{J}} + \sin\theta\hat{\mathbf{K}} \tag{3}$$

$$\boldsymbol{\omega} = 2\hat{\mathbf{K}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{i}} \tag{4}$$

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = 3\dot{\hat{\mathbf{j}}} + 4\dot{\hat{\mathbf{i}}}$$

$$\dot{\hat{\mathbf{j}}} = \left(2\hat{\mathbf{K}}\right)\times\hat{\mathbf{j}} = \left(2\hat{\mathbf{K}}\right)\times\left(\cos\phi\hat{\mathbf{I}} + \sin\phi\hat{\mathbf{J}}\right) = -2\sin\phi\hat{\mathbf{I}} + 2\cos\phi\hat{\mathbf{J}}$$

$$\begin{aligned}\dot{\hat{\mathbf{i}}} &= \left(2\hat{\mathbf{K}} + 3\hat{\mathbf{j}}\right)\times\hat{\mathbf{i}} = \left(2\hat{\mathbf{K}} + 3\cos\phi\hat{\mathbf{I}} + 3\sin\phi\hat{\mathbf{J}}\right)\times\left(\sin\theta\sin\phi\hat{\mathbf{I}} - \sin\theta\cos\phi\hat{\mathbf{J}} + \cos\theta\hat{\mathbf{K}}\right)\\ &= \left(3\cos\theta\sin\phi + 2\sin\theta\cos\phi\right)\hat{\mathbf{I}} + \left(2\sin\theta\sin\phi - 3\cos\theta\cos\phi\right)\hat{\mathbf{J}} - 3\sin\theta\hat{\mathbf{K}}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\alpha} &= 3\left(-2\sin\phi\hat{\mathbf{I}} + 2\cos\phi\hat{\mathbf{J}}\right) + 4\left[\left(3\cos\theta\sin\phi + 2\sin\theta\cos\phi\right)\hat{\mathbf{I}} + \left(2\sin\theta\sin\phi - 3\cos\theta\cos\phi\right)\hat{\mathbf{J}} - 3\sin\theta\hat{\mathbf{K}}\right]\\ &= \left(-6\sin\phi + 12\cos\theta\sin\phi + 8\sin\theta\cos\phi\right)\hat{\mathbf{I}} + \left(6\cos\phi - 12\cos\theta\cos\phi + 8\sin\theta\sin\phi\right)\hat{\mathbf{J}} - 12\sin\theta\hat{\mathbf{K}}\end{aligned}$$

$$\begin{aligned}\alpha &= \sqrt{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\\ &= \sqrt{\left(-6\sin\phi + 12\cos\theta\sin\phi + 8\sin\theta\cos\phi\right)^2 + \left(6\cos\phi - 12\cos\theta\cos\phi + 8\sin\theta\sin\phi\right)^2 + 144\sin^2\theta}\\ &= \sqrt{\left(\sin^2\phi + \cos^2\phi\right)\left(36 + 144\cos^2\theta - 144\cos\theta + 64\sin^2\theta\right) + 144\sin^2\theta}\\ &= \sqrt{36 + 144\left(\cos^2\theta + \sin^2\theta\right) - 144\cos\theta + 64\sin^2\theta}\\ &= \sqrt{36 + 144 - 144\cos\theta + 64\sin^2\theta}\end{aligned}$$

$$\underline{\alpha = \sqrt{180 - 144\cos\theta + 64\sin^2\theta}}$$

Method 2: Use $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ as the basis. Multiply (1) through by $\cos\theta$ and (2) by $\sin\theta$ to obtain

$$\sin\theta\cos\theta\sin\phi\hat{\mathbf{I}} - \sin\theta\cos\theta\cos\phi\hat{\mathbf{J}} + \cos^2\theta\hat{\mathbf{K}} = \cos\theta\hat{\mathbf{i}}$$

$$-\sin\theta\cos\theta\sin\phi\hat{\mathbf{I}} + \sin\theta\cos\theta\cos\phi\hat{\mathbf{J}} + \sin^2\theta\hat{\mathbf{K}} = \sin\theta\hat{\mathbf{k}}$$

Adding these two equations yields

$$\hat{\mathbf{K}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{k}}$$

Then (4) can be written

$$\boldsymbol{\omega} = 2\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{k}}\right) + 3\hat{\mathbf{j}} + 4\hat{\mathbf{i}} = \left(4 + 2\cos\theta\right)\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\sin\theta\hat{\mathbf{k}} \tag{6}$$

$$\boldsymbol{\alpha} = \left.\frac{d\boldsymbol{\omega}}{dt}\right)_{rel} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$$

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right)_{rel} = -2\dot{\theta}\sin\theta\hat{\mathbf{i}} + 2\dot{\theta}\cos\theta\hat{\mathbf{k}} = -6\sin\theta\hat{\mathbf{i}} + 6\cos\theta\hat{\mathbf{k}}$$

$$\mathbf{\Omega} = 2\hat{\mathbf{K}} + 3\hat{\mathbf{j}} = 2\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{k}}\right) + 3\hat{\mathbf{j}}$$

$$\mathbf{\Omega}\times\boldsymbol{\omega} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos\theta & 3 & 2\sin\theta \\ 4+2\cos\theta & 3 & \sin\theta \end{vmatrix} = 8\sin\theta\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$$

$$\therefore \boldsymbol{\alpha} = \left(-6\sin\theta\hat{\mathbf{i}} + 6\cos\theta\hat{\mathbf{k}}\right) + 8\sin\theta\hat{\mathbf{j}} - 12\hat{\mathbf{k}} = -6\sin\theta\hat{\mathbf{i}} + 8\sin\theta\hat{\mathbf{j}} + \left(6\cos\theta - 12\right)\hat{\mathbf{k}}$$

$$\begin{aligned}\alpha = \sqrt{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} &= \sqrt{36\sin^2\theta + 64\sin^2\theta + \left(36\cos^2\theta - 144\cos\theta + 144\right)} \\ &= \sqrt{36\left(\sin^2\theta + \cos^2\theta\right) + 64\sin^2\theta - 144\cos\theta + 144} \\ &= \sqrt{36 + 64\sin^2\theta - 144\cos\theta + 144}\end{aligned}$$

$$\underline{\alpha = \sqrt{180 + 64\sin^2\theta - 144\cos\theta}}$$

Problem 9.2

(a)

$$\boldsymbol{\omega}_{\text{plate}} = \left\{\left[\left(\dot{\theta}\hat{\mathbf{k}}\right) + \dot{\phi}\hat{\mathbf{j}}\right] + \dot{\nu}\hat{\mathbf{m}}\right\} + \dot{\psi}\hat{\mathbf{n}} \qquad (1)$$

$$\hat{\mathbf{m}} = \sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{k}} \qquad \hat{\mathbf{p}} = \hat{\mathbf{m}}\times\hat{\mathbf{j}} = -\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{k}}$$
$$\hat{\mathbf{n}} = \cos\nu\hat{\mathbf{j}} + \sin\nu\hat{\mathbf{p}} = -\cos\phi\sin\nu\hat{\mathbf{i}} + \cos\nu\hat{\mathbf{j}} + \sin\phi\sin\nu\hat{\mathbf{k}}$$

Substituting $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ into (1) yields the result,

$$\boldsymbol{\omega}_{\text{plate}} = \left(\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu\right)\hat{\mathbf{i}} + \left(\dot{\phi} + \dot{\psi}\cos\nu\right)\hat{\mathbf{j}} + \left(\dot{\theta} + \dot{\nu}\cos\phi + \dot{\psi}\sin\phi\sin\nu\right)\hat{\mathbf{k}} \qquad (2)$$

(b)

$$\boldsymbol{\alpha}_{\text{plate}} = \frac{d\boldsymbol{\omega}_{\text{plate}}}{dt} = \left.\frac{d\boldsymbol{\omega}_{\text{plate}}}{dt}\right)_{rel} + \mathbf{\Omega}\times\boldsymbol{\omega}_{\text{plate}} \qquad (3)$$

where $\mathbf{\Omega}$ is the angular velocity of the *xyz* frame, which is $\dot{\theta}\hat{\mathbf{k}}$ That is,

$$\mathbf{\Omega} = \dot{\theta}\hat{\mathbf{k}} \qquad (4)$$

From (2)

$$\left.\frac{d\boldsymbol{\omega}_{\text{plate}}}{dt}\right)_{rel} = \left[\frac{d}{dt}\left(\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu\right)\right]\hat{\mathbf{i}} + \left[\frac{d}{dt}\left(\dot{\phi} + \dot{\psi}\cos\nu\right)\right]\hat{\mathbf{j}} + \left[\frac{d}{dt}\left(\dot{\theta} + \dot{\nu}\cos\phi + \dot{\psi}\sin\phi\sin\nu\right)\right]\hat{\mathbf{k}}$$

Taking the derivatives, bearing in mind that $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$ are all constant, we get

$$\begin{aligned}\left.\frac{d\boldsymbol{\omega}_{\text{plate}}}{dt}\right)_{rel} &= \left[\dot{\phi}\left(\dot{\nu}\cos\phi + \dot{\psi}\sin\phi\sin\nu\right) - \dot{\psi}\dot{\nu}\cos\phi\cos\nu\right]\hat{\mathbf{i}} \\ &\quad - \dot{\psi}\dot{\nu}\sin\nu\hat{\mathbf{j}} + \left[\dot{\phi}\left(-\dot{\nu}\sin\phi + \dot{\psi}\cos\phi\sin\nu\right) + \dot{\psi}\dot{\nu}\cos\nu\sin\phi\right]\hat{\mathbf{k}}\end{aligned} \qquad (5)$$

From (2) and (4),

$$\boldsymbol{\Omega}\times\boldsymbol{\omega}_{\text{plate}} = \dot{\theta}\hat{\mathbf{k}}\times\Big[\left(\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu\right)\hat{\mathbf{i}} + \left(\dot{\phi}+\dot{\psi}\cos\nu\right)\hat{\mathbf{j}} + \left(\dot{\theta}+\dot{\nu}\cos\phi+\dot{\psi}\sin\phi\sin\nu\right)\hat{\mathbf{k}}\Big]$$
$$= -\dot{\theta}\left(\dot{\psi}\cos\nu+\dot{\phi}\right)\hat{\mathbf{i}} + \dot{\theta}\left(\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu\right)\hat{\mathbf{j}} \tag{6}$$

Substituting (5) and (6) into (3) and collecting terms yields the result,

$$\underline{\boldsymbol{\alpha}_{\text{plate}} = \left[\dot{\nu}\left(\dot{\phi}\cos\phi - \dot{\psi}\cos\phi\cos\nu\right) + \dot{\psi}\dot{\phi}\sin\phi\sin\nu - \dot{\psi}\dot{\theta}\cos\nu - \dot{\phi}\dot{\theta}\right]\hat{\mathbf{i}} + \left[\dot{\nu}\left(\dot{\theta}\sin\phi - \dot{\psi}\sin\nu\right) - \dot{\psi}\dot{\theta}\cos\phi\sin\nu\right]\hat{\mathbf{j}} + \left[\dot{\psi}\dot{\nu}\cos\nu\sin\phi + \dot{\psi}\dot{\phi}\cos\phi\sin\nu - \dot{\phi}\dot{\nu}\sin\phi\right]\hat{\mathbf{k}}} \tag{7}$$

(c)

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC}\times\mathbf{r}_{C/B} + \boldsymbol{\omega}_{BC}\times\left(\boldsymbol{\omega}_{BC}\times\mathbf{r}_{C/B}\right) \tag{8}$$

$$\mathbf{r}_{C/B} = l\hat{\mathbf{m}} = l\sin\phi\hat{\mathbf{i}} + \cos\gamma\,\hat{\mathbf{k}} \tag{9}$$

$$\boldsymbol{\omega}_{BC} = \dot{\theta}\hat{\mathbf{k}} + \dot{\phi}\hat{\mathbf{j}} \tag{10}$$

$$\boldsymbol{\alpha}_{BC} = \frac{d\boldsymbol{\omega}_{BC}}{dt} = \left.\frac{d\boldsymbol{\omega}_{BC}}{dt}\right)_{rel} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_{BC} = \mathbf{0} + \dot{\theta}\hat{\mathbf{k}}\times\left(\dot{\theta}\hat{\mathbf{k}}+\dot{\phi}\hat{\mathbf{j}}\right) = -\dot{\theta}\dot{\phi}\hat{\mathbf{i}} \tag{11}$$

$$\mathbf{a}_B = \mathbf{a}_O + \boldsymbol{\alpha}_{AB}\times\mathbf{r}_{B/O} + \boldsymbol{\omega}_{AB}\times\left(\boldsymbol{\omega}_{AB}\times\mathbf{r}_{B/O}\right)$$
$$= \mathbf{0} + \mathbf{0}\times\left(1.25l\hat{\mathbf{j}}\right) + \dot{\theta}\hat{\mathbf{k}}\times\left(\dot{\theta}\hat{\mathbf{k}}\times1.25l\hat{\mathbf{j}}\right) = -1.25\dot{\theta}^2 l\hat{\mathbf{j}} \tag{12}$$

Substituting (9), (10), (11) and (12) into (8) yields

$$\mathbf{a}_C = \left(-1.25\dot{\theta}^2 l\hat{\mathbf{j}}\right) + \left(-\dot{\theta}\dot{\phi}\hat{\mathbf{i}}\right)\times\left(l\sin\phi\hat{\mathbf{i}}+\cos\gamma\hat{\mathbf{k}}\right) + \left(\dot{\theta}\hat{\mathbf{k}}+\dot{\phi}\hat{\mathbf{j}}\right)\times\left[\left(\dot{\theta}\hat{\mathbf{k}}+\dot{\phi}\hat{\mathbf{j}}\right)\times\left(l\sin\phi\hat{\mathbf{i}}+\cos\gamma\hat{\mathbf{k}}\right)\right]$$

Upon expanding and collecting terms, we get the result

$$\underline{\mathbf{a}_C = -l\left(\dot{\phi}^2+\dot{\theta}^2\right)\sin\phi\hat{\mathbf{i}} + \left(2l\dot{\phi}\dot{\theta}\cos\phi - 1.25l\dot{\theta}^2\right)\hat{\mathbf{j}} - l\dot{\phi}^2\cos\phi\hat{\mathbf{k}}}$$

Problem 9.3

$$\mathbf{a}_G = \left.\frac{d\mathbf{v}}{dt}\right)_{rel} + \boldsymbol{\omega}\times\mathbf{v}$$
$$= \frac{d}{dt}\left(t^3\hat{\mathbf{i}}+4\hat{\mathbf{j}}\right) + \left(2t^2\hat{\mathbf{k}}\right)\times\left(t^3\hat{\mathbf{i}}+4\hat{\mathbf{j}}\right)$$
$$= 3t^2\hat{\mathbf{i}} + \left(-8t^2\hat{\mathbf{i}} + 2t^5\hat{\mathbf{j}}\right)$$
$$= -5t^2\hat{\mathbf{i}} + 2t^5\hat{\mathbf{j}}$$

At $t = 2\,\text{s}$

$$\underline{\mathbf{a}_G = -20\hat{\mathbf{i}} + 64\hat{\mathbf{j}}\left(\text{m}^2/\text{s}\right)}$$

Problem 9.4

$$\boldsymbol{\alpha} = \left.\frac{d\boldsymbol{\omega}}{dt}\right)_{rel} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$$

$$\boldsymbol{\alpha} = \frac{d}{dt}\left(\omega_x\hat{\mathbf{i}}+\omega_y\hat{\mathbf{j}}+\omega_z\hat{\mathbf{k}}\right) + \left(\omega_x\hat{\mathbf{i}}+\omega_y\hat{\mathbf{j}}\right)\times\left(\omega_x\hat{\mathbf{i}}+\omega_y\hat{\mathbf{j}}+\omega_z\hat{\mathbf{k}}\right)$$

$$\boldsymbol{\alpha} = 0 + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & 0 \\ \omega_x & \omega_y & \omega_z \end{vmatrix}$$

$$\boldsymbol{\alpha} = \underline{\omega_y \omega_z \hat{\mathbf{i}} - \omega_x \omega_z \hat{\mathbf{j}}}$$

Problem 9.5

About the origin O:

$$\left[\mathbf{I}_i\right] = m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix}$$

$$\left[\mathbf{I}_1\right] = \begin{bmatrix} 20 & -10 & -10 \\ -10 & 20 & -10 \\ -10 & -10 & 20 \end{bmatrix} \left(\text{kg - m}^2\right) \qquad \left[\mathbf{I}_2\right] = \begin{bmatrix} 20 & -10 & -10 \\ -10 & 20 & -10 \\ -10 & -10 & 20 \end{bmatrix} \left(\text{kg - m}^2\right)$$

$$\left[\mathbf{I}_3\right] = \begin{bmatrix} 256 & 128 & -128 \\ 128 & 256 & 128 \\ -128 & 128 & 256 \end{bmatrix} \left(\text{kg - m}^2\right) \qquad \left[\mathbf{I}_4\right] = \begin{bmatrix} 64 & 32 & -32 \\ 32 & 64 & 32 \\ -32 & 32 & 64 \end{bmatrix} \left(\text{kg - m}^2\right)$$

$$\left[\mathbf{I}_5\right] = \begin{bmatrix} 216 & 108 & 108 \\ 108 & 216 & -108 \\ 108 & -108 & 216 \end{bmatrix} \left(\text{kg - m}^2\right) \qquad \left[\mathbf{I}_6\right] = \begin{bmatrix} 216 & 108 & 108 \\ 108 & 216 & -108 \\ 108 & -108 & 216 \end{bmatrix} \left(\text{kg - m}^2\right)$$

$$\left[\mathbf{I}_O\right] = \sum_{i=1}^{6} \left[\mathbf{I}_i\right] = \begin{bmatrix} 792 & 356 & 36 \\ 356 & 792 & -76 \\ 36 & -76 & 792 \end{bmatrix} \left(\text{kg - m}^2\right)$$

$$m = \sum_{i=1}^{6} m_i = 60 \quad \text{kg}$$

$$x_G = \frac{\sum_{i=1}^{6} m_i x_i}{m} = \frac{16}{60} = 0.2667 \text{ m}$$

$$y_G = \frac{\sum_{i=1}^{6} m_i y_i}{m} = \frac{-16}{60} = -0.2667 \text{ m}$$

$$z_G = \frac{\sum_{i=1}^{6} m_i z_i}{m} = \frac{16}{60} = 0.2667 \text{ m}$$

$$\left[\mathbf{I}_{mP}\right] = m \begin{bmatrix} y_G^2 + z_G^2 & -x_G y_G & -x_G z_G \\ -x_G y_G & x_G^2 + z_G^2 & -y_G z_G \\ -x_G z_G & -y_G z_G & x_G^2 + y_G^2 \end{bmatrix} = \begin{bmatrix} 8.533 & 4.267 & -4.267 \\ 4.267 & 8.533 & 4.267 \\ -4.267 & 4.267 & 8.533 \end{bmatrix} \left(\text{kg - m}^2\right)$$

$$\left[\mathbf{I}_G\right] = \left[\mathbf{I}_P\right] - \left[\mathbf{I}_{mP}\right] = \begin{bmatrix} 792 & 356 & 36 \\ 356 & 792 & -76 \\ 36 & -76 & 792 \end{bmatrix} - \begin{bmatrix} 8.533 & 4.267 & -4.267 \\ 4.267 & 8.533 & 4.267 \\ -4.267 & 4.267 & 8.533 \end{bmatrix}$$

$$\left[\mathbf{I}_G\right]=\begin{bmatrix}783.5 & 351.7 & 40.27\\ 351.7 & 783.5 & -80.27\\ 40.27 & -80.27 & 783.5\end{bmatrix}\left(\text{kg - m}^2\right)$$

Problem 9.6

From the previous problem

$$\left[\mathbf{I}_O\right]=\begin{bmatrix}792 & 356 & 36\\ 356 & 792 & -76\\ 36 & -76 & 792\end{bmatrix}\left(\text{kg}\cdot\text{m}^2\right)$$

$$\hat{\mathbf{i}}'=\frac{\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}}{\sqrt{1^2+2^2+2^2}}=0.3333\hat{\mathbf{i}}+0.6667\hat{\mathbf{j}}+0.6667\hat{\mathbf{k}}$$

$$\lfloor\hat{\mathbf{i}}'\rfloor=\lfloor 0.3333\quad 0.6667\quad 0.6667\rfloor$$

$$I_{x'}=\lfloor\hat{\mathbf{i}}'\rfloor\begin{bmatrix}792 & 356 & 36\\ 356 & 792 & -76\\ 36 & -76 & 792\end{bmatrix}\lfloor\hat{\mathbf{i}}'\rfloor^T$$

$$=\lfloor 0.3333\quad 0.6667\quad 0.6667\rfloor\begin{bmatrix}792 & 356 & 36\\ 356 & 792 & -76\\ 36 & -76 & 792\end{bmatrix}\begin{Bmatrix}0.3333\\ 0.6667\\ 0.6667\end{Bmatrix}$$

$$=\lfloor 0.3333\quad 0.6667\quad 0.6667\rfloor\begin{Bmatrix}535.3\\ 596\\ 489.3\end{Bmatrix}$$

$$\underline{I_{x'}=898.7\ \text{kg}\cdot\text{m}^2}$$

Problem 9.7

$$\left[\mathbf{I}_G\right]=\frac{m}{12}\begin{bmatrix}b^2+c^2 & -ab & -ac\\ -ab & a^2+c^2 & -bc\\ -ac & -bc & a^2+b^2\end{bmatrix}$$

$a=l\cos\theta$

$b=l\sin\theta$

$c=0$

$$\left[\mathbf{I}_G\right]=\frac{m}{12}\begin{bmatrix}(l\sin\theta)^2+(0)^2 & -(l\cos\theta)(l\sin\theta) & -(l\cos\theta)(0)\\ -(l\cos\theta)(l\sin\theta) & (l\cos\theta)^2+(0)^2 & -(l\sin\theta)(0)\\ -(l\cos\theta)(0) & -(l\sin\theta)(0) & (l\cos\theta)^2+(l\sin\theta)^2\end{bmatrix}$$

$$\left[\mathbf{I}_G\right]=\frac{ml^2}{12}\begin{bmatrix}\sin^2\theta & -\dfrac{\sin 2\theta}{2} & 0\\ -\dfrac{\sin 2\theta}{2} & \cos^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Problem 9.8

(a)

$$\left[\mathbf{I}_G\right] = \frac{m}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

$$= \frac{1000}{12}\begin{bmatrix} 2^2+1^2 & 0 & 0 \\ 0 & 3^2+1^2 & 0 \\ 0 & 0 & 3^2+2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 416.7 & 0 & 0 \\ 0 & 833.3 & 0 \\ 0 & 0 & 1083 \end{bmatrix} \left(\mathrm{kg}\cdot\mathrm{m}^2\right)$$

$$\left[\mathbf{I}_{mO}\right] = m\begin{bmatrix} {y_G}^2+{z_G}^2 & -x_G y_G & -x_G z_G \\ -x_G y_G & {x_G}^2+{z_G}^2 & -y_G z_G \\ -x_G z_G & -y_G z_G & {x_G}^2+{y_G}^2 \end{bmatrix}$$

$$x_G = \frac{a}{2} = 1.5 \text{ m} \qquad y_G = \frac{b}{2} = 1 \text{ m} \qquad z_G = \frac{c}{2} = 0.5 \text{ m}$$

$$\therefore \left[\mathbf{I}_{mO}\right] = \begin{bmatrix} 1250 & -1500 & -750 \\ -1500 & 2500 & -500 \\ -750 & -500 & 3250 \end{bmatrix} \left(\mathrm{kg}\cdot\mathrm{m}^2\right)$$

$$\left[\mathbf{I}_O\right] = \left[\mathbf{I}_G\right] + \left[\mathbf{I}_{mO}\right] = \begin{bmatrix} 416.7 & 0 & 0 \\ 0 & 833.3 & 0 \\ 0 & 0 & 1083 \end{bmatrix} + \begin{bmatrix} 1250 & -1500 & -750 \\ -1500 & 2500 & -500 \\ -750 & -500 & 3250 \end{bmatrix}$$

$$\underline{\left[\mathbf{I}_O\right] = \begin{bmatrix} 1667 & -1500 & -750 \\ -1500 & 3333 & -500 \\ -750 & -500 & 4333 \end{bmatrix} \left(\mathrm{kg}\cdot\mathrm{m}^2\right)}$$

(b)

$$\begin{vmatrix} 1667-\lambda & -1500 & -750 \\ -1500 & 3333-\lambda & -500 \\ -750 & -500 & 4333-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 9333\lambda^2 + 24.16\left(10^6\right)\lambda - 10.91\left(10^9\right) = 0$$

$$\lambda_1 = 568.9 \text{ kg}\cdot\text{m}^2 \qquad \lambda_2 = 4209 \text{ kg}\cdot\text{m}^2 \qquad \lambda_3 = 4556 \text{ kg}\cdot\text{m}^2$$

Principal direction 1:

$$\begin{bmatrix} 1667-568.9 & -1500 & -750 \\ -1500 & 3333-568.9 & -500 \\ -750 & -500 & 4333-568.9 \end{bmatrix}\begin{Bmatrix} v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1098 & -1500 & -750 \\ -1500 & 2764 & -500 \\ -750 & -500 & 3764 \end{bmatrix}\begin{Bmatrix} 1 \\ v_y^{(1)} \\ v_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2764 & -500 \\ -500 & 3764 \end{bmatrix}\begin{Bmatrix} v_y^{(1)} \\ v_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1500 \\ 750 \end{Bmatrix}$$

$$\begin{Bmatrix} v_y^{(1)} \\ v_z^{(1)} \end{Bmatrix} = \begin{bmatrix} 2764 & -500 \\ -500 & 3764 \end{bmatrix}^{-1} \begin{Bmatrix} 1500 \\ 750 \end{Bmatrix} = \begin{Bmatrix} 0.5929 \\ 0.278 \end{Bmatrix}$$

$$\hat{\mathbf{v}}^{(1)} = \frac{\hat{\mathbf{i}} + 0.5829\hat{\mathbf{j}} + 0.278\hat{\mathbf{k}}}{\sqrt{1^2 + 0.5829^2 + 0.278^2}} = \underline{0.8366\hat{\mathbf{i}} + 0.496\hat{\mathbf{j}} + 0.2326\hat{\mathbf{k}}}$$

Principal direction 2:

$$\begin{bmatrix} 1667 - 4209 & -1500 & -750 \\ -1500 & 3333 - 4209 & -500 \\ -750 & -500 & 4333 - 4209 \end{bmatrix} \begin{Bmatrix} v_x^{(2)} \\ v_y^{(2)} \\ v_z^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -2542 & -1500 & -750 \\ -1500 & -875.5 & -500 \\ -750 & -500 & 124.5 \end{bmatrix} \begin{Bmatrix} 1 \\ v_y^{(2)} \\ v_z^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -875.5 & -500 \\ -500 & 124.5 \end{bmatrix} \begin{Bmatrix} v_y^{(2)} \\ v_z^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1500 \\ 750 \end{Bmatrix}$$

$$\begin{Bmatrix} v_y^{(2)} \\ v_z^{(2)} \end{Bmatrix} = \begin{bmatrix} -875.5 & -500 \\ -500 & 124.5 \end{bmatrix}^{-1} \begin{Bmatrix} 1500 \\ 750 \end{Bmatrix} = \begin{Bmatrix} -1.565 \\ -0.2601 \end{Bmatrix}$$

$$\hat{\mathbf{v}}^{(2)} = \frac{\hat{\mathbf{i}} - 1.565\hat{\mathbf{j}} - 0.2601\hat{\mathbf{k}}}{\sqrt{1^2 + (-1.565)^2 + (-0.2601)^2}} = \underline{0.5333\hat{\mathbf{i}} - 0.8345\hat{\mathbf{j}} - 0.1387\hat{\mathbf{k}}}$$

Principal direction 3:

$$\begin{bmatrix} 1667 - 4556 & -1500 & -750 \\ -1500 & 3333 - 4556 & -500 \\ -750 & -500 & 4333 - 4556 \end{bmatrix} \begin{Bmatrix} v_x^{(3)} \\ v_y^{(3)} \\ v_z^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -2889 & -1500 & -750 \\ -1500 & -1222 & -500 \\ -750 & -500 & -222.3 \end{bmatrix} \begin{Bmatrix} 1 \\ v_y^{(3)} \\ v_z^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -1222 & -500 \\ -500 & -222.3 \end{bmatrix} \begin{Bmatrix} v_y^{(3)} \\ v_z^{(3)} \end{Bmatrix} = \begin{Bmatrix} 1500 \\ 750 \end{Bmatrix}$$

$$\begin{Bmatrix} v_y^{(3)} \\ v_z^{(3)} \end{Bmatrix} = \begin{bmatrix} -1222 & -500 \\ -500 & -222.3 \end{bmatrix}^{-1} \begin{Bmatrix} 1500 \\ 750 \end{Bmatrix} = \begin{Bmatrix} 1.916 \\ -7.685 \end{Bmatrix}$$

$$\hat{\mathbf{v}}^{(3)} = \frac{\hat{\mathbf{i}} + 1.916\hat{\mathbf{j}} - 7.685\hat{\mathbf{k}}}{\sqrt{1^2 + 1.916^2 + (-7.685)^2}} = \underline{0.1253\hat{\mathbf{i}} + 0.2401\hat{\mathbf{j}} - 0.9626\hat{\mathbf{k}}}$$

The following MATLAB script uses the built-in function `eig` to obtain these results, as shown in the Command Window output which follows.

```
fprintf('\n-----------------------------------------------------\n')
Matrix = [ 1666.7   -1500    -750
            -1500  3333.3    -500
             -750    -500  4333.3]

[eigvector, eigvalue] = eig(I);

fprintf('\n Eigenvalue                Eigenvector')
for i = 1:3
```

```
fprintf('\n  %g      [%g  %g  %g]', eigvalue(i,i), eigvector(1,i),
eigvector(2,i), eigvector(3,i))
end
fprintf('\n---------------------------------------------------\n')
```

```
---------------------------------------------------

Matrix =

       1666.7         -1500          -750
        -1500        3333.3          -500
         -750          -500        4333.3


 Eigenvalue              Eigenvector
  4208.83      [-0.533304  0.834478  0.138683]
  4555.59      [0.125278  0.240046  -0.962644]
  568.881      [0.836596  0.496008  0.232559]
---------------------------------------------------
```

(c)

$$\hat{\mathbf{i}}' = \frac{3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3^2 + 2^2 + 1^2}} = 0.8018\hat{\mathbf{i}} + 0.5345\hat{\mathbf{j}} + 0.2673\hat{\mathbf{k}}$$

$$\lfloor \hat{\mathbf{i}}' \rfloor = \lfloor 0.8018 \;\; 0.5345 \;\; 0.2673 \rfloor$$

$$I_{x'} = \lfloor \hat{\mathbf{i}}' \rfloor \left[\mathbf{I}_O\right] \lfloor \hat{\mathbf{i}}' \rfloor^T = \lfloor 0.8018 \;\; 0.5345 \;\; 0.2673 \rfloor \begin{bmatrix} 1667 & -1500 & -750 \\ -1500 & 3333 & -500 \\ -750 & -500 & 4333 \end{bmatrix} \begin{Bmatrix} 0.8018 \\ 0.5345 \\ 0.2673 \end{Bmatrix}$$

$$I_{x'} = \lfloor 0.8018 \;\; 0.5345 \;\; 0.2673 \rfloor \begin{Bmatrix} 334.1 \\ 445.4 \\ 239.5 \end{Bmatrix}$$

$$\underline{I_{x'} = 583.3 \text{ kg} \cdot \text{m}^2}$$

Problem 9.9

$$\left\{\mathbf{H}_C\right\} = \left[\mathbf{I}_C\right]\left\{\boldsymbol{\omega}\right\} = \frac{ml^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta & -\dfrac{\sin 2\theta}{2} \\ 0 & -\dfrac{\sin 2\theta}{2} & \cos^2\theta \end{bmatrix}\begin{Bmatrix} \omega \\ 0 \\ \Omega \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{12} ml^2\omega \\ -\dfrac{1}{24} ml^2\Omega\sin 2\theta \\ \dfrac{1}{12} ml^2\Omega\cos^2\theta \end{Bmatrix}$$

$$\mathbf{H}_C = \frac{1}{12} ml^2\omega\hat{\mathbf{i}} - \frac{1}{24} ml^2\Omega\sin 2\theta\hat{\mathbf{j}} + \frac{1}{12} ml^2\Omega\cos^2\theta\hat{\mathbf{k}}$$

$$\mathbf{H}_P = \mathbf{H}_C + \mathbf{r}_{C/P} \times m\mathbf{v}_C$$

$$\mathbf{r}_{C/P} = d\hat{\mathbf{i}} \qquad \mathbf{v}_C = \Omega\hat{\mathbf{k}} \times d\hat{\mathbf{i}} = \Omega d\hat{\mathbf{j}}$$

$$\mathbf{H}_P = \mathbf{H}_C + d\hat{\mathbf{i}} \times m\Omega d\hat{\mathbf{j}} = \mathbf{H}_C + m\Omega d^2\hat{\mathbf{k}}$$

$$\mathbf{H}_P = \left(\frac{1}{12} ml^2\omega\hat{\mathbf{i}} - \frac{1}{24} ml^2\Omega\sin 2\theta\hat{\mathbf{j}} + \frac{1}{12} ml^2\Omega\cos^2\theta\hat{\mathbf{k}}\right) + m\Omega d^2\hat{\mathbf{k}}$$

$$\underline{\mathbf{H}_P = \frac{1}{12} ml^2\omega\hat{\mathbf{i}} - \frac{1}{24} ml^2\Omega\sin 2\theta\hat{\mathbf{j}} + \left(\frac{1}{12} ml^2\cos^2\theta + md^2\right)\Omega\hat{\mathbf{k}}}$$

Problem 9.10

$$\begin{bmatrix} 1000 & 0 & -300 \\ 0 & 1000 & 500 \\ -300 & 500 & 1000 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = 1000 \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$\begin{Bmatrix} 1000\omega_x - 300\omega_z \\ 1000\omega_y + 500\omega_z \\ -300\omega_x + 500\omega_y + 1000\omega_z \end{Bmatrix} = \begin{Bmatrix} 1000\omega_x \\ 1000\omega_y \\ 1000\omega_z \end{Bmatrix}$$

$$\begin{Bmatrix} -300\omega_z \\ 500\omega_z \\ -300\omega_x + 500\omega_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\omega_z = 0$$

$$-300\omega_x + 500\omega_y = 0 \;\Rightarrow\; \omega_y = \frac{3}{5}\omega$$

$$\therefore \boldsymbol{\omega} = \begin{Bmatrix} \omega_x \\ \frac{3}{5}\omega_x \\ 0 \end{Bmatrix}$$

$$\|\boldsymbol{\omega}\| = 1.166\omega_x = 20 \;\Rightarrow\; \omega_x = 17.15$$

$$\therefore \boldsymbol{\omega} = \begin{Bmatrix} 17.15 \\ 10.29 \\ 0 \end{Bmatrix} \;\; or \;\; \underline{\boldsymbol{\omega} = 17.15\hat{\mathbf{i}} + 10.29\hat{\mathbf{j}} \left(\mathrm{s}^{-1}\right)}$$

Problem 9.11

$$\boldsymbol{\omega} = 2t^2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$$

$$\boldsymbol{\alpha} = \left.\frac{d\boldsymbol{\omega}}{dt}\right)_{rel} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \left.\frac{d\boldsymbol{\omega}}{dt}\right)_{rel}$$

$$\boldsymbol{\alpha} = 4t\hat{\mathbf{i}} + 3\hat{\mathbf{k}}$$

$t = 3$:

$$\boldsymbol{\omega} = 18\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$$

$$\boldsymbol{\alpha} = 12\hat{\mathbf{i}} + 3\hat{\mathbf{k}}$$

$$\{\mathbf{M}\} = \left[\mathbf{I}_G\right]\{\boldsymbol{\alpha}\} + \{\boldsymbol{\omega}\} \times \left[\mathbf{I}_G\right]\{\boldsymbol{\omega}\}$$

$$\{\mathbf{M}\} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \begin{Bmatrix} \mathbf{12} \\ 0 \\ 3 \end{Bmatrix} + \begin{Bmatrix} 18 \\ 4 \\ 9 \end{Bmatrix} \times \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \begin{Bmatrix} 18 \\ 4 \\ 9 \end{Bmatrix}$$

$$\{\mathbf{M}\} = \begin{Bmatrix} 120 \\ 0 \\ 90 \end{Bmatrix} + \begin{Bmatrix} 18 \\ 4 \\ 9 \end{Bmatrix} \times \begin{Bmatrix} 180 \\ 80 \\ 270 \end{Bmatrix}$$

$$\mathbf{M} = \left(120\hat{\mathbf{i}} + 90\hat{\mathbf{k}}\right) + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 18 & 4 & 9 \\ 180 & 80 & 270 \end{vmatrix} = \left(120\hat{\mathbf{i}} + 90\hat{\mathbf{k}}\right) + \left(360\hat{\mathbf{i}} - 3240\hat{\mathbf{j}} + 720\hat{\mathbf{k}}\right)$$

$$\mathbf{M} = 480\hat{\mathbf{i}} - 3240\hat{\mathbf{j}} + 810\hat{\mathbf{k}}$$

$$M = \|\mathbf{M}\| = \sqrt{480^2 + (-3240)^2 + 810^2} = \underline{3374 \;\; \mathrm{N \cdot m}}$$

Problem 9.12

$$\{\mathbf{M}_G\} = [\mathbf{I}_G]\{\boldsymbol{\alpha}\} + \{\boldsymbol{\omega}\} \times [\mathbf{I}_G]\{\boldsymbol{\omega}\}$$
$$\{\boldsymbol{\omega}\} = 0$$
$$\mathbf{M}_G = \mathbf{r} \times \mathbf{F} = \left[(0 - 0.075)\hat{\mathbf{I}} + (0 - 0.2536)\hat{\mathbf{J}} + (0 - 0.05714)\hat{\mathbf{K}}\right] \times 100\hat{\mathbf{I}} = 34.29\hat{\mathbf{j}} + 25.36\hat{\mathbf{k}}\ \ (\mathrm{N \cdot m})$$
$$\begin{Bmatrix} 0 \\ 34.29 \\ 25.36 \end{Bmatrix} = \begin{bmatrix} 0.1522 & -0.03975 & 0.0120 \\ -0.03975 & 0.07177 & 0.04057 \\ 0.0120 & 0.04057 & 0.1569 \end{bmatrix} \begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix}$$
$$\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} = \begin{bmatrix} 0.1522 & -0.03975 & 0.0120 \\ -0.03975 & 0.07177 & 0.04057 \\ 0.0120 & 0.04057 & 0.1569 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 34.29 \\ 25.36 \end{Bmatrix} = \underline{\begin{Bmatrix} 143.9 \\ 553.1 \\ 7.61 \end{Bmatrix}\ \left(\mathrm{m/s^2}\right)}$$

Problem 9.13

(a)

$$\sum F_x = m a_{G_x}$$
$$mg\sin\theta = m\left(\frac{v^2}{R}\cos\theta\right)$$
$$\underline{\tan\theta = \frac{v^2}{gR}}$$

(b)

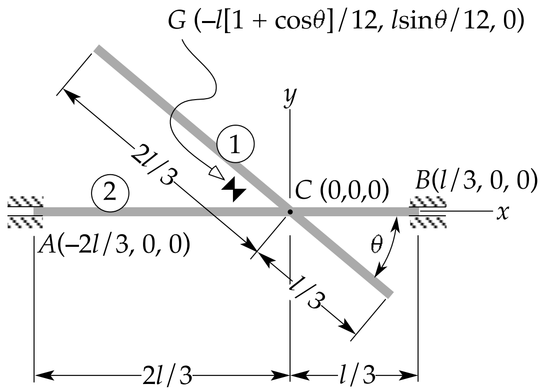
$$\mathbf{M} = \boldsymbol{\omega} \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ A\omega_x & B\omega_y & C\omega_z \end{vmatrix} = (C - B)\omega_y\omega_z\hat{\mathbf{i}} + (A - C)\omega_x\omega_z\hat{\mathbf{j}} + (B - A)\omega_x\omega_y\hat{\mathbf{k}}$$
$$\omega_x = -\frac{v}{R}\sin\theta \qquad \omega_y = 0 \qquad \omega_z = \frac{v}{R}\cos\theta$$
$$\mathbf{M} = (A - C)\left(-\frac{v}{R}\sin\theta\right)\left(\frac{v}{R}\cos\theta\right)\hat{\mathbf{j}}$$
$$M_y = (C - A)\frac{v^2}{R^2}\sin\theta\cos\theta = \underline{(C - A)\frac{v^2}{2R^2}\sin 2\theta}$$

Problem 9.14

$$\mathbf{M} = \boldsymbol{\omega} \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ A\omega_x & B\omega_y & C\omega_z \end{vmatrix} = (C - B)\omega_y\omega_z\hat{\mathbf{i}} + (A - C)\omega_x\omega_z\hat{\mathbf{j}} + (B - A)\omega_x\omega_y\hat{\mathbf{k}}$$
$$\omega_x = 0 \qquad \omega_y = \omega_Z\sin\alpha \qquad \omega_z = \omega_Z\cos\alpha$$
$$\mathbf{M} = (C - B)(\omega_Z\sin\alpha)(\omega_Z\cos\alpha)\hat{\mathbf{i}}$$
$$\underline{M_x = \frac{1}{2}(C - B)\omega_Z{}^2\sin 2\alpha \qquad M_y = 0 \qquad M_z = 0}$$

Problem 9.15

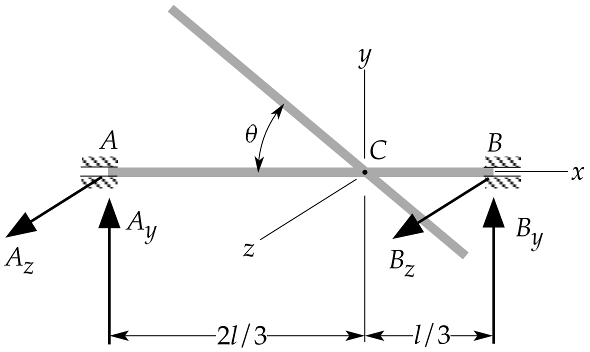
$$x_{G_1} = -\left(\frac{l}{2} - \frac{l}{3}\right)\cos = -\frac{l}{6}\cos\theta$$
$$y_{G_1} = \frac{l}{6}\sin\theta$$
$$z_{G_1} = 0$$



$$x_{G2} = -\frac{l}{6}$$
$$y_{G2} = z_{G2} = 0$$

$$x_G = \frac{mx_{G1} + mx_{G2}}{2m} = \frac{1}{2}\left(-\frac{l}{6}\cos\theta - \frac{l}{6}\right) = -\frac{l}{12}(1+\cos\theta)$$
$$y_G = \frac{my_{G1} + my_{G2}}{2m} = \frac{1}{2}\left(\frac{l}{6}\sin\theta\right) = \frac{l}{12}\sin\theta$$
$$z_G = 0$$

Free-body diagram (no bearing couples and no thrust components of bearing force):



$$\sum F_x = 2ma_{G_x}: \quad 0 = 0$$
$$\sum F_y = 2ma_{G_y}: \quad A_y + B_y = -\frac{m\omega^2 l}{6}\sin\theta \tag{1}$$
$$\sum F_z = 2ma_{G_z}: \quad A_z + B_z = 0 \tag{2}$$

Moments of inertia about G_1 (inferred from results of Exercise 9.7):

$$\left[\mathbf{I}_{G_1}^{(1)}\right] = \frac{ml^2}{12}\begin{bmatrix} \sin^2\theta & \frac{1}{2}\sin 2\theta & 0 \\ \frac{1}{2}\sin 2\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Equation 9.60:

$$\left[\mathbf{I}_{m\,C}^{(1)}\right] = m\begin{bmatrix} {y_{G1}}^2 + {z_{G1}}^2 & -x_{G1}y_{G1} & -x_{G1}z_{G1} \\ -x_{G1}y_{G1} & {x_{G1}}^2 + {z_{G1}}^2 & -y_{G1}z_{G1} \\ -x_{G1}z_{G1} & -y_{G1}z_{G1} & {x_{G1}}^2 + {y_{G1}}^2 \end{bmatrix}$$

$$= m\begin{bmatrix} \left(\frac{l}{6}\sin\theta\right)^2 & -\left(-\frac{l}{6}\cos\theta\right)\frac{l}{6}\sin\theta & 0 \\ -\left(-\frac{l}{6}\cos\theta\right)\frac{l}{6}\sin\theta & \left(-\frac{l}{6}\cos\theta\right)^2 & 0 \\ 0 & 0 & \left(-\frac{l}{6}\cos\theta\right)^2 + \left(\frac{l}{6}\sin\theta\right)^2 \end{bmatrix}$$

$$\left[\mathbf{I}_{m\,C}^{(1)}\right]=\frac{ml^2}{12}\begin{bmatrix}\frac{1}{3}\sin^2\theta & \frac{1}{6}\sin 2\theta & 0\\ \frac{1}{6}\sin 2\theta & \frac{1}{3}\cos^2\theta & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

$$\left[\mathbf{I}_{C}^{(1)}\right]=\left[\mathbf{I}_{G_1}^{(1)}\right]+\left[\mathbf{I}_{m\,C}^{(1)}\right]=\frac{ml^2}{12}\begin{bmatrix}\sin^2\theta & \frac{1}{2}\sin 2\theta & 0\\ \frac{1}{2}\sin 2\theta & \cos^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}+\frac{ml^2}{12}\begin{bmatrix}\frac{1}{3}\sin^2\theta & \frac{1}{6}\sin 2\theta & 0\\ \frac{1}{6}\sin 2\theta & \frac{1}{3}\cos^2\theta & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

$$\left[\mathbf{I}_{C}^{(1)}\right]=\frac{ml^2}{12}\begin{bmatrix}\frac{4}{3}\sin^2\theta & \frac{2}{3}\sin 2\theta & 0\\ \frac{2}{3}\sin 2\theta & \frac{4}{3}\cos^2\theta & 0\\ 0 & 0 & \frac{4}{3}\end{bmatrix}$$

$$\left[\mathbf{I}_{G_2}^{(2)}\right]=\frac{ml^2}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\left[\mathbf{I}_{m\,C}^{(2)}\right]=m\begin{bmatrix}y_{G2}{}^2+z_{G2}{}^2 & -x_{G2}y_{G2} & -x_{G2}z_{G2}\\ -x_{G2}y_{G2} & x_{G2}{}^2+z_{G2}{}^2 & -y_{G2}z_{G2}\\ -x_{G2}z_{G2} & -y_{G2}z_{G2} & x_{G2}{}^2+y_{G2}{}^2\end{bmatrix}=m\begin{bmatrix}0 & 0 & 0\\ 0 & \left(-\frac{l}{6}\right)^2 & 0\\ 0 & 0 & \left(-\frac{l}{6}\right)^2\end{bmatrix}$$

$$\left[\mathbf{I}_{m\,C}^{(2)}\right]=\frac{ml^2}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

$$\left[\mathbf{I}_{C}^{(2)}\right]=\left[\mathbf{I}_{G_2}^{(2)}\right]+\left[\mathbf{I}_{m\,C}^{(2)}\right]=\frac{ml^2}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\frac{ml^2}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

$$\left[\mathbf{I}_{C}^{(2)}\right]=\frac{ml^2}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{4}{3} & 0\\ 0 & 0 & \frac{4}{3}\end{bmatrix}$$

$$\left[\mathbf{I}_{C}\right]=\left[\mathbf{I}_{C}^{(1)}\right]+\left[\mathbf{I}_{C}^{(2)}\right]=\frac{ml^2}{12}\begin{bmatrix}\frac{4}{3}\sin^2\theta & \frac{2}{3}\sin 2\theta & 0\\ \frac{2}{3}\sin 2\theta & \frac{4}{3}\cos^2\theta & 0\\ 0 & 0 & \frac{4}{3}\end{bmatrix}+\frac{ml^2}{12}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{4}{3} & 0\\ 0 & 0 & \frac{4}{3}\end{bmatrix}$$

$$[\mathbf{I}_C] = \begin{bmatrix} \frac{ml^2}{9}\sin^2\theta & \frac{ml^2}{18}\sin 2\theta & 0 \\ \frac{ml^2}{18}\sin 2\theta & \frac{ml^2}{9}\left(1+\cos^2\theta\right) & 0 \\ 0 & 0 & \frac{2ml^2}{9} \end{bmatrix}$$

$$\{\mathbf{H}_C\} = [\mathbf{I}_C]\{\boldsymbol{\omega}\} = \begin{bmatrix} \frac{ml^2}{9}\sin^2\theta & \frac{ml^2}{18}\sin 2\theta & 0 \\ \frac{ml^2}{18}\sin 2\theta & \frac{ml^2}{9}\left(1+\cos^2\theta\right) & 0 \\ 0 & 0 & \frac{2ml^2}{9} \end{bmatrix} \begin{Bmatrix} \omega \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{ml^2\omega}{9}\sin^2\theta \\ \frac{ml^2\omega}{18}\sin 2\theta \\ 0 \end{Bmatrix}$$

$$\mathbf{M}_C = \boldsymbol{\omega}\times\mathbf{H}_C = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega & 0 & 0 \\ \frac{ml^2\omega}{9}\sin^2\theta & \frac{ml^2\omega}{18}\sin 2\theta & 0 \end{vmatrix} = \frac{ml^2\omega^2}{18}\sin 2\theta\hat{\mathbf{k}}$$

$$M_{C_x} = 0 \tag{3}$$
$$M_{C_y} = 0 \tag{4}$$
$$M_{C_z} = \frac{ml^2\omega^2}{18}\sin 2\theta \tag{5}$$

Calculate the moments of the bearing reactions in the above free body diagram:

$$\mathbf{M}_C = \left(-\frac{2}{3}l\hat{\mathbf{i}}\right)\times\left(A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(\frac{l}{3}\hat{\mathbf{i}}\right)\times\left(B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right) = \left(\frac{2}{3}A_z l - \frac{1}{3}B_z l\right)\hat{\mathbf{j}} + \left(-\frac{2}{3}A_y l + \frac{1}{3}B_y l\right)\hat{\mathbf{j}}$$

$$M_{C_x} = 0 \tag{6}$$
$$M_{C_y} = \frac{2}{3}A_z l - \frac{1}{3}B_z l \tag{7}$$
$$M_{C_z} = -\frac{2}{3}A_y l + \frac{1}{3}B_y l \tag{8}$$

From (4) and (7)

$$2A_z - B_z = 0 \tag{9}$$

From (5) and (8)

$$-\frac{2}{3}A_y l + \frac{1}{3}B_y l = \frac{ml^2\omega^2}{18}\sin 2\theta \tag{10}$$

From (1) we have

$$B_y = -\frac{m\omega^2 l}{6}\sin\theta - A_y \tag{11}$$

Substituting this into (10):

$$-\frac{2}{3}A_y l + \frac{1}{3}\left(-\frac{m\omega^2 l}{6}\sin\theta - A_y\right)l = \frac{ml^2\omega^2}{18}\sin 2\theta \;\Rightarrow\; A_y = -\frac{ml^2\omega^2}{18}\sin\theta(1 + 2\cos\theta) \tag{12}$$

Therefore, from (11),

$$B_y = -\frac{m\omega^2 l}{6}\sin\theta - \left[-\frac{ml^2\omega^2}{18}\sin\theta(1+2\cos\theta)\right] = -\frac{m\omega^2 l}{9}\sin\theta(1-\cos\theta) \tag{13}$$

From (2) we have

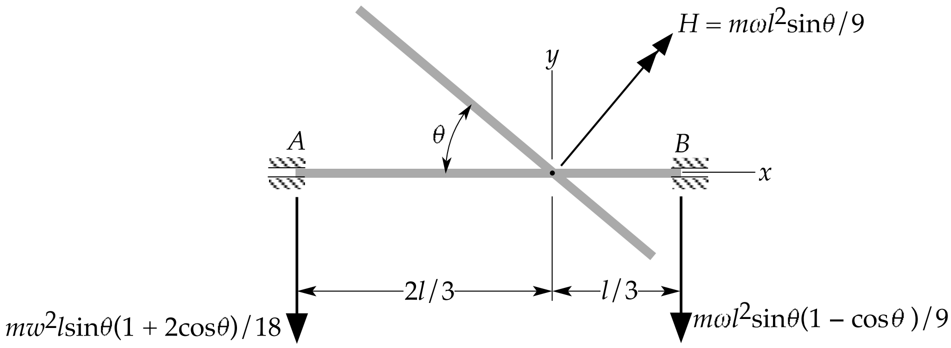
$$B_z = -A_z$$

Substituting this into (9)

$$2A_z - (-A_z) = 0 \;\Rightarrow A_z = 0$$

Therefore, $B_z = 0$.

The only reactions at each bearing are in the plane of the rod and shaft, normal to the shaft, as given by Equations (12) and (13).



Problem 9.16

$$\mathbf{M}_G = \dot{\mathbf{H}}_G\big)_{rel} + \boldsymbol{\omega}\times\mathbf{H}$$

$$\mathbf{M}_G = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ A\omega_x & B\omega_y & C\omega_z \end{vmatrix}$$

$$600\hat{\mathbf{i}} + M_{Gy}\hat{\mathbf{j}} + M_{Gz}\hat{\mathbf{k}} = 5\dot{\omega}_x\hat{\mathbf{i}} + 5(0)\hat{\mathbf{j}} + 10(0)\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0.5 & 100 \\ 5(0) & 5\cdot 0.5 & 10\cdot 100 \end{vmatrix}$$

$$600\hat{\mathbf{i}} + M_{Gx}\hat{\mathbf{j}} + M_{Gy}\hat{\mathbf{k}} = \left(5\dot{\omega}_x + 250\right)\hat{\mathbf{i}}$$

$$5\dot{\omega}_x + 250 = 600 \Rightarrow \underline{\dot{\omega}_x = 70\ \text{rad/s}^2}$$

Problem 9.17

$$\mathbf{M}_O = I_{Ox}\dot{\omega}_x\hat{\mathbf{i}} + I_{Oy}\dot{\omega}_y\hat{\mathbf{j}} + I_{Oz}\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ I_{Ox}\omega_x & I_{Oy}\omega_y & I_{Oz}\omega_z \end{vmatrix}$$

$$I_{Ox} = I_{Oz} = \frac{1}{12}mL^2 + m\left(\frac{L}{6}\right)^2 = \frac{mL^2}{9} \qquad I_{Oy} = 0$$

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

$$\omega_x = 0 \qquad \omega_y = -\omega\cos\theta \qquad \omega_z = \omega\sin\theta$$

$$\mathbf{M}_O = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\omega\cos\theta & \omega\sin\theta \\ 0 & 0 & \dfrac{mL^2}{9}\omega\sin\theta \end{vmatrix} = -\frac{1}{9}m\omega^2L^2\sin\theta\cos\theta\hat{\mathbf{i}}$$

$$\therefore M_{O_x} = -\frac{1}{9}m\omega^2L^2\sin\theta\cos\theta$$

Moment of the weight vector about *O*:

$$M_{O_x} = -mg\frac{L}{6}\sin\theta$$

$$\therefore \quad -\frac{1}{9}m\omega^2L^2\sin\theta\cos\theta = -mg\frac{L}{6}\sin\theta \;\Rightarrow\; \underline{\omega = \sqrt{\frac{3}{2}\frac{g}{L\sin\theta}}}$$

Problem 9.18

$$\mathbf{M}_G = \dot{\mathbf{H}}_G\big)_{rel} + \mathbf{\Omega}\times\mathbf{H}$$

$$\mathbf{M}_G = I_{Gx}\alpha_x\hat{\mathbf{i}} + I_{Gy}\alpha_y\hat{\mathbf{j}} + I_{Gz}\alpha_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Omega_x & \Omega_y & \Omega_z \\ I_{Gx}\omega_x & I_{Gy}\omega_y & I_{Gz}\omega_z \end{vmatrix}$$

$$10\cdot 9.81\cdot 0.25\hat{\mathbf{i}} = 0 + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \omega_p & 630 \\ 0 & 0.01406\omega_p & 0.02812\cdot 630 \end{vmatrix}$$

$$24.52\hat{\mathbf{i}} = 17.72\omega_p\hat{\mathbf{i}} \Rightarrow \underline{\omega_p = 1.384 \text{ rad/s}}$$

Or, using Equation 9.96,

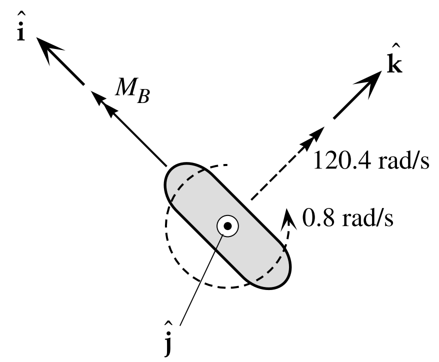
$$\omega_p = \frac{mgd}{I_z\omega_s} = \frac{10\cdot 9.81\cdot 0.25}{0.02812\cdot 630} = \underline{\omega_p = 1.384 \text{ rad/s}}$$

Problem 9.19

$$\omega_{wheel} = \frac{v}{r} = \frac{130\dfrac{1000}{3600}}{0.3} = 120.4 \text{ rad/s}$$

$$I_{wheel} = 25\cdot 0.2^2 = 1\text{ kg}\cdot\text{m}^2$$

$$\mathbf{M} = \boldsymbol{\omega}_p \times \mathbf{H}_s = 0.8\hat{\mathbf{j}}\times\left[1\cdot 120.4\hat{\mathbf{k}}\right] = \underline{96.3\hat{\mathbf{i}}\,(\text{N}\cdot\text{m})}$$



Problem 9.20

$$I_{rotor} = 4 \cdot 0.07^2 = 0.0196 \text{ kg} \cdot \text{m}^2$$

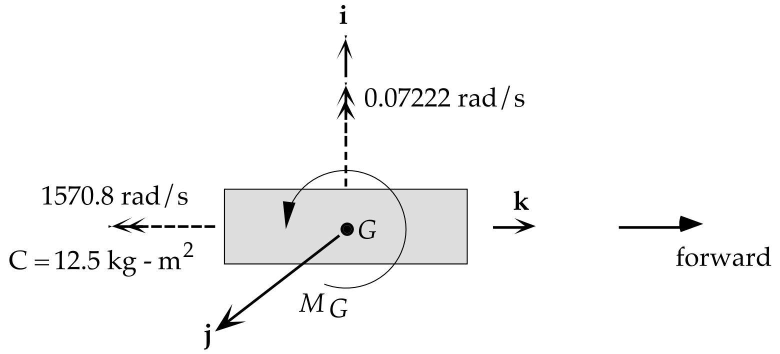
$$\mathbf{M} = \boldsymbol{\omega}_p \times \mathbf{H}_s = 2\hat{\mathbf{i}} \times \left(0.0196 \cdot 10\,000 \cdot \frac{2\pi}{60}\hat{\mathbf{k}}\right) = 41.05\hat{\mathbf{j}}\,(\text{N} \cdot \text{m})$$

$$B_y = F \quad A_y = -F$$

$$0.04F = 41.05$$

$$\underline{F = 1026 \text{ N}}$$

Problem 9.21



$$I_{rotor} = 200 \cdot 0.25^2 = 12.5 \text{ kg} \cdot \text{m}^2$$

$$\omega_s = 15\,000 \cdot \frac{2\pi}{60} = 1571 \text{ rad/s}$$

$$\omega_p = \frac{v}{r} = \frac{\dfrac{650\,000}{3600}}{2500} = 0.072\,22 \text{ rad/s}$$

$$\mathbf{M} = \boldsymbol{\omega}_p \times \mathbf{H}_s = 0.072\,22\hat{\mathbf{i}} \times \left[12.5\left(-1571\hat{\mathbf{k}}\right)\right] = \underline{1418\hat{\mathbf{i}}\,(\text{N} \cdot \text{m})}$$

The moment reaction on the airframe is clockwise, pitching the nose down.

Problem 9.22

$$\mathbf{M}_G = \boldsymbol{\omega}_p \times \mathbf{H}_s$$

$$\boldsymbol{\omega}_p = 20\hat{\mathbf{i}} \ (\text{rad/s})$$

$$\mathbf{H}_s = \frac{1}{2} \cdot 10 \cdot 0.05^2 \cdot 200\hat{\mathbf{k}} = 2.5\hat{\mathbf{k}} \ \left(\text{kg} \cdot \text{m/s}^2\right)$$

$$\mathbf{M}_G = 20\hat{\mathbf{i}} \times 2.5\hat{\mathbf{k}} = -50\hat{\mathbf{j}}\,(\text{N} \cdot \text{m})$$

$$0.6R_B\hat{\mathbf{j}} = -50\hat{\mathbf{j}}$$

$$R_B = \underline{-83.33 \text{ N}} \qquad R_A = -R_B = \underline{83.33 \text{ N}}$$

Problem 9.23

$$\begin{aligned} l_x &= \cos\psi \cos\phi - \sin\phi \sin\psi \cos\theta \\ &= \cos 70^\circ \cos 50^\circ - \sin 50^\circ \sin 70^\circ \cos 25^\circ \\ &= 0.6428 \cdot 0.3420 - 0.7660 \cdot 0.9397 \cdot 0.9063 \\ &= -0.4326 \end{aligned}$$

$$\alpha_{xX} = \cos^{-1}(-0.4326) = \underline{115.6^\circ}$$

Problem 9.24

$$\{\mathbf{H}\} = [\mathbf{I}]\{\boldsymbol{\omega}\} = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} \begin{Bmatrix} 10 \\ 20 \\ 30 \end{Bmatrix} = \underline{\begin{Bmatrix} 0 \\ 500 \\ 1500 \end{Bmatrix} \ (\mathrm{J \cdot s})}$$

$$T = \frac{1}{2}\mathbf{H} \cdot \boldsymbol{\omega} = \frac{1}{2}\lfloor 0 \quad 500 \quad 1500 \rfloor \begin{Bmatrix} 10 \\ 20 \\ 30 \end{Bmatrix} = \underline{23\,000 \ \ \mathrm{J}}$$

Problem 10.1

$$\omega_p = \frac{C}{A-C}\frac{\omega_s}{\cos\theta} = \frac{1200}{2600-1200}\frac{6}{\cos 6°} = 5.171 \text{ rad/s}$$

$$H = A\omega_p = 2600 \cdot 5.171 = \underline{13\,450 \text{ kg}\cdot\text{m}^2/\text{s}}$$

Problem 10.2

$$\omega_p = \frac{C}{A-C}\frac{\omega_s}{\cos\theta} = \frac{500}{300-500}\frac{6}{\cos 10°} = -15.06 \text{ rad/s}$$

$$T = \frac{2\pi}{|\omega_p|} = \frac{2\pi}{15.06} = \underline{0.4173 \text{ s}}$$

Problem 10.3

$$\omega_p = \frac{C}{A-C}\frac{\omega_s}{\cos\theta} = \frac{mr^2}{\frac{1}{2}mr^2 - mr^2}\frac{\omega_s}{\cos\theta} = -2\frac{\omega_s}{\cos\theta}$$

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

$$\frac{1}{\cos\theta} = \left(1 - \frac{\theta^2}{2}\right)^{-1} = 1 + \frac{\theta^2}{2} \qquad (\theta << 0)$$

$$\therefore \underline{\omega_p = -2\omega_s\left(1 + \frac{\theta^2}{2}\right)}$$

Problem 10.4

$$H = A\omega_p = 1000 \cdot 2 = \underline{2000 \text{ kg}\cdot\text{m}^2/s}$$

Problem 10.5

$$\dot{\mathbf{H}}_G\big)_{rel} + \boldsymbol{\omega} \times \mathbf{H}_G = 0$$

$$[\mathbf{I}_G]\{\boldsymbol{\alpha}\} + \boldsymbol{\omega} \times \mathbf{H}_G = 0$$

$$\begin{bmatrix} 385.4 & 0 & 0 \\ 0 & 416.7 & 0 \\ 0 & 0 & 52.08 \end{bmatrix}\{\boldsymbol{\alpha}\} + \begin{Bmatrix} 0.01 \\ -0.03 \\ 0.02 \end{Bmatrix} \times \begin{bmatrix} 385.4 & 0 & 0 \\ 0 & 416.7 & 0 \\ 0 & 0 & 52.08 \end{bmatrix}\begin{Bmatrix} 0.01 \\ -0.03 \\ 0.02 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 385.4 & 0 & 0 \\ 0 & 416.7 & 0 \\ 0 & 0 & 52.08 \end{bmatrix}\{\boldsymbol{\alpha}\} + \begin{Bmatrix} 0.01 \\ -0.03 \\ 0.02 \end{Bmatrix} \times \begin{Bmatrix} 3.854 \\ -12.50 \\ 1.042 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 385.4 & 0 & 0 \\ 0 & 416.7 & 0 \\ 0 & 0 & 52.08 \end{bmatrix}\{\boldsymbol{\alpha}\} + \begin{Bmatrix} 0.2188 \\ 0.0666 \\ -0.00939 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 385.4 & 0 & 0 \\ 0 & 416.7 & 0 \\ 0 & 0 & 52.08 \end{bmatrix}\{\boldsymbol{\alpha}\} + \begin{Bmatrix} 0.2188 \\ 0.0666 \\ -0.00939 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\boldsymbol{\alpha}\} = -\begin{bmatrix} 385.4 & 0 & 0 \\ 0 & 416.7 & 0 \\ 0 & 0 & 52.08 \end{bmatrix}^{-1} \begin{Bmatrix} 0.2188 \\ 0.0666 \\ -0.00939 \end{Bmatrix} = \begin{Bmatrix} -0.0005676 \\ -0.00016 \\ 0.0001803 \end{Bmatrix} \;\left(\text{m/s}^2\right)$$

$$\underline{\|\boldsymbol{\alpha}\| = 0.0006167 \;\; \text{m/s}^2}$$

Problem 10.6

$$\omega_p = \frac{C}{C-A}\frac{\omega_s}{\cos\theta} = \frac{0.72}{0.72-0.36}\frac{30}{\cos 15°} = -62.12 \quad \text{rad/s}$$
$$\omega_n = 0$$

$$\omega_x = \omega_p \sin\theta\sin\psi + \omega_n\cos\psi = -62.12\sin 15°\sin\psi + 0\cdot\cos\psi = -16.08\sin\psi$$
$$\omega_y = \omega_p \sin\theta\cos\psi - \omega_n\cos\psi = -62.12\sin 15°\cos\psi - 0\cdot\cos\psi = -16.08\cos\psi$$
$$\omega_z = \omega_s + \omega_p\cos\theta = 30 + (-62.12)\cos 15° = -30$$

$$\begin{aligned} T_R &= \frac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right) \\ &= \frac{1}{2}\left[0.36(-16.08\sin\psi)^2 + 0.36(-16.08\cos\psi)^2 + 0.72(-30)^2\right] \\ &= \frac{1}{2}\left[93.05\left(\sin^2\psi + \cos^2\psi\right) + 648\right] \\ &= \frac{1}{2}(93.05 + 648) \\ &= \underline{370.5 \;\; \text{J}} \end{aligned}$$

Or,

$$H = A\omega_p = 0.36(-62.12) = -22.36 \;\; \text{kg}\cdot\text{m}^2/s$$

$$\begin{aligned} T_R &= \frac{1}{2}\frac{H^2}{C}\left(1 + \frac{C-A}{A}\sin^2\theta\right) \\ &= \frac{1}{2}\frac{(-22.36)^2}{0.72}\left(1 + \frac{0.72-0.36}{0.36}\sin^2 15°\right) \\ &= \frac{1}{2}\cdot 694.5\cdot(1 + 0.06699) \\ &= \underline{370.5 \;\; \text{J}} \end{aligned}$$

Problem 10.7

$$[\mathbf{I}_G] = \begin{bmatrix} \frac{1}{12}m\left[l^2 + (2l)^2\right] & 0 & 0 \\ 0 & \frac{1}{12}m\left[l^2 + (3l)^2\right] & 0 \\ 0 & 0 & \frac{1}{12}m\left[(2l)^2 + (3l)^2\right] \end{bmatrix} = \begin{bmatrix} \frac{5}{12}ml^2 & 0 & 0 \\ 0 & \frac{5}{6}ml^2 & 0 \\ 0 & 0 & \frac{13}{12}ml^2 \end{bmatrix}$$

$$\{\mathbf{H}_0\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} = \begin{bmatrix} \frac{5}{12}ml^2 & 0 & 0 \\ 0 & \frac{5}{6}ml^2 & 0 \\ 0 & 0 & \frac{13}{12}ml^2 \end{bmatrix} \begin{Bmatrix} 1.5\omega_0 \\ 0.8\omega_0 \\ 0.6\omega_0 \end{Bmatrix} = \begin{Bmatrix} 0.625ml^2\omega_0 \\ 0.6667ml^2\omega_0 \\ 0.65ml^2\omega_0 \end{Bmatrix}$$

$$H_0 = 1.121ml^2\omega_0$$

$$T_0 = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_0 = \frac{1}{2}\lfloor 1.5\omega_0 \quad 0.8\omega_0 \quad 0.6\omega_0 \rfloor \begin{Bmatrix} 0.625ml^2\omega_0 \\ 0.6667ml^2\omega_0 \\ 0.65ml^2\omega_0 \end{Bmatrix} = 0.9305ml^2{\omega_0}^2$$

(a)

$$T = \frac{1}{2}C\omega^2 = \frac{1}{2}\frac{13}{12}ml^2\omega^2 = 0.5417ml^2\omega^2$$

$$T = T_0$$

$$0.5417ml^2\omega^2 = 0.9305ml^2{\omega_0}^2$$

$$\omega = \sqrt{1.718}\omega_0 = \underline{1.311\omega_0}$$

(b)

$$H = C\omega = \frac{13}{12}ml^2\omega$$

$$H = H_0$$

$$\frac{13}{12}ml^2\omega = 1.121ml^2\omega_0$$

$$\underline{\omega = 1.035\omega_0}$$

Problem 10.8

$$H_{initial} = 1000\cdot 6 = 6000 \quad \text{kg}\cdot\text{m}^2/\text{s}$$

$$T_{initial} = \frac{1}{2}\cdot 1000\cdot 6^2 = 18\,000 \quad \text{kg}\cdot\text{m}^2/\text{s}^2$$

$$H_{final} = 5000\omega_{final}$$

$$H_{final} = H_{initial}$$

$$5000\omega_{final} = 6000$$

$$\omega_{final} = 1.2 \quad \text{rad/s}$$

$$T_{final} = \frac{1}{2}\cdot 5000\cdot {\omega_{final}}^2 = \frac{1}{2}\cdot 5000\cdot 1.2^2 = 3600 \quad \text{kg}\cdot\text{m}^2/\text{s}^2$$

$$\Delta T = T_{final} - T_{initial} = 3600 - 18\,000 = \underline{-14\,400 \text{ J}}$$

Problem 10.9

$$\{\mathbf{H}_{G0}\} = \begin{bmatrix} 0.1522 & -0.03975 & 0.012 \\ -0.03975 & 0.07177 & 0.04057 \\ 0.012 & 0.04057 & 0.1569 \end{bmatrix}\begin{Bmatrix} 10 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.522 \\ -0.3975 \\ 0.1200 \end{Bmatrix} \left(\text{kg}\cdot\text{m}^2/\text{s}\right)$$

$$H_{G0} = \|\mathbf{H}_{G0}\| = 1.5776 \quad \text{kg}\cdot\text{m}^2/\text{s}$$

$$T_0 = \frac{1}{2}\mathbf{H}_{G0}\cdot\omega_0 = \frac{1}{2}\lfloor 1.522 \quad -0.3975 \quad 0.1200 \rfloor \begin{Bmatrix} 10 \\ 0 \\ 0 \end{Bmatrix} = 7.610 \text{ J}$$

$$I_{\max} = 0.1747 \text{ kg}\cdot\text{m}^2 \quad (\text{from Example 10.11})$$

$$H_{Gf} = I_{\max}\omega_f = 0.1747\omega_f$$

$$H_{Gf} = H_{G0}$$

$$0.1747\omega_f = 1.578$$

$$\omega_f = 9.03 \quad \text{rad/s}$$

$$T_f = \frac{1}{2}\cdot I_{\max}\omega_f^{\,2} = \frac{1}{2}\cdot 0.1747\cdot 9.03^2 = 7.123 \text{ J}$$
$$\Delta T = T_f - T_0 = 7.123 - 7.610 = \underline{-0.4867 \text{ J}}$$

Problem 10.10

$$C_r\omega_2^{(r)} + C_p\left(\omega_2^{(r)} + \omega_{rel\,2}^{(p)}\right) = C_r\omega_1^{(r)} + C_p\left(\omega_1^{(r)} + \omega_{rel\,1}^{(p)}\right)$$
$$\omega_2^{(r)} = \omega_1^{(r)} + \frac{C_p\left(\omega_{rel\,1}^{(p)} - \omega_{rel\,2}^{(p)}\right)}{C_p + C_r}$$
$$\omega_2^{(r)} = 3 + \frac{500(1-0.5)}{500+1000} = \underline{3.167 \text{ rad/s}}$$

Problem 10.11

$$\omega_p = \frac{C}{A-C}\frac{\omega_s}{\cos\theta} = \frac{1000}{5000-1000}\frac{0.1}{\cos 20^\circ} = 0.0266 \text{ rad/s}$$
$$t = \frac{\pi}{\omega_p} = \frac{\pi}{0.0266} = \underline{118.1 \text{ s}}$$

Problem 10.12

$$\cos\gamma = \frac{A}{\sqrt{A^2 + C^2\tan^2\dfrac{\phi}{2}}}$$
$$A = \frac{m}{12}\left(3r^2 + l^2\right) = \frac{500}{12}\left(3\cdot 0.5^2 + 2^2\right) = 197.9 \text{ kg}\cdot\text{m}^2 \qquad C = \frac{1}{2}mr^2 = \frac{1}{2}500\cdot 0.5^2 = 62.5 \text{ kg}\cdot\text{m}^2$$
$$\phi = 2\tan^{-1}\left[\sqrt{\frac{\left(\dfrac{A}{\cos\gamma}\right)^2 - A^2}{C^2}}\right] = 2\tan^{-1}\left[\sqrt{\frac{\left(\dfrac{197.9}{\cos 5^\circ}\right)^2 - 197.9^2}{62.5^2}}\right] = \underline{30.97^\circ}$$

Problem 10.13

$$Npd = I(\omega_2 - \omega_1)$$
$$\omega_2 = \omega_1 + \frac{Npd}{I}$$
$$\omega_2 = 0.01\cdot 2\pi + \frac{30\cdot 15\cdot 1.5}{2000} = 0.4003 \text{ rad/s} = \underline{0.0637 \text{ rev/s}}$$

Problem 10.14

$$\begin{aligned}\mathbf{H}_{G0} &= A\omega_{0x}\hat{\mathbf{i}} + B\omega_{0y}\hat{\mathbf{j}} + C\omega_{0z}\hat{\mathbf{k}}\\ &= 2000\cdot 0.1\hat{\mathbf{i}} + 4000\cdot 0.3\hat{\mathbf{j}} + 6000\cdot 0.5\hat{\mathbf{k}}\\ &= 200\hat{\mathbf{i}} + 1200\hat{\mathbf{j}} + 3000\hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\mathbf{H}_G &= \mathbf{H}_{G0} + \Delta\mathbf{H}_G\\ &= 200\hat{\mathbf{i}} + 1200\hat{\mathbf{j}} + 3000\hat{\mathbf{k}} + \left(50\hat{\mathbf{i}} - 100\hat{\mathbf{j}} + 300\hat{\mathbf{k}}\right)\\ &= 250\hat{\mathbf{i}} + 1100\hat{\mathbf{j}} + 3300\hat{\mathbf{k}}\end{aligned}$$

$$A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} = 250\hat{\mathbf{i}} + 1100\hat{\mathbf{j}} + 3300\hat{\mathbf{k}}$$

$$\omega_x = \frac{250}{A} = \frac{250}{2000} = 0.125 \text{ rad/s}$$

$$\omega_y = \frac{1100}{B} = \frac{1100}{4000} = 0.275 \text{ rad/s}$$

$$\omega_z = \frac{3300}{C} = \frac{3300}{6000} = 0.55 \text{ rad/s}$$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \underline{0.6275 \text{ rad/s}}$$

Problem 10.15

$$A = \frac{1}{4}mr^2 + \frac{1}{12}ml^2 = \frac{1}{4}300\cdot 1.5^2 + \frac{1}{12}300\cdot 1.5^2 = 225 \text{ kg}\cdot\text{m}^2$$

$$C = \frac{1}{2}mr^2 = \frac{1}{2}300\cdot 1.5^2 = 337.5 \text{ kg}\cdot\text{m}^2$$

$$\mathbf{H}_{G_1} = C\omega_1\hat{\mathbf{k}} = 337.5\cdot\left(1\cdot\frac{2\pi}{60}\hat{\mathbf{k}}\right) = 35.34\hat{\mathbf{k}}\ \left(\text{kg}\cdot\text{m}^2\right)$$

$$\mathbf{H}_{G_2} = \mathbf{H}_{G_1} + \vec{\mathcal{I}}_M$$

$$\mathbf{H}_{G_2} = 35.34\hat{\mathbf{k}} + \left(-\mathcal{I}_M\hat{\mathbf{j}}\right)$$

$$H_{G_2} = \sqrt{35.34^2 + \mathcal{I}_M^2} = \sqrt{1249 + \mathcal{I}_M^2}$$

$$H_{G_2} = A\omega_p$$

$$\sqrt{1249 + \mathcal{I}_M^2} = 225\cdot(0.1\cdot 2\pi) = 141.4$$

$$\mathcal{I}_M^2 = 18740$$

$$\underline{\mathcal{I}_M = 136.9 \text{ N}\cdot\text{m}\cdot\text{s}}$$

Problem 10.16

$$K = 1 + \frac{C}{2mR^2} = 1 + \frac{300}{2\cdot 3\cdot 1.5^2} = 23.22$$

(a)

$$l_f = R\sqrt{K\frac{\omega_0 - \omega_f}{\omega_0 + \omega_f}} = 1.5\sqrt{23.22\cdot\frac{5-1}{5+1}} = \underline{5.902 \text{ m}}$$

$$t = \sqrt{\frac{K}{\omega_0^2}\frac{\omega_0 - \omega_f}{\omega_0 + \omega_f}} = \sqrt{\frac{23.22}{5^2}\frac{5-1}{5+1}} = \underline{0.7869 \text{ s}}$$

(b)

$$l_f = R\sqrt{K\frac{\omega_0 - \omega_f}{\omega_0 + \omega_f}} = 1.5\sqrt{23.22\cdot\frac{5-0}{5+0}} = \underline{7.228 \text{ m}}$$

$$t = \sqrt{\frac{K}{\omega_0^2}\frac{\omega_0 - \omega_f}{\omega_0 + \omega_f}} = \sqrt{\frac{23.22}{5^2}\frac{5-0}{5+0}} = \underline{0.9636 \text{ s}}$$

Problem 10.17

$$T_0 = \frac{1}{2}A\left(\omega_x{}^2 + \omega_y{}^2\right) + \frac{1}{2}C\omega_z{}^2 = \frac{1}{2}A\Omega^2 + \frac{1}{2}C\omega_0{}^2$$

$$\omega_p = \frac{C}{A-C}\frac{\omega_s}{\cos\theta} = \frac{60}{30-60}\frac{2}{\cos 15°} = -4.141 \text{ rad/s}$$
$$\omega_0 = \frac{A}{C}\omega_p\cos\theta = \frac{30}{60}(-4.141)\cos 15° = -2 \text{ rad/s}$$
$$\Omega = \omega_p \sin\theta = -4.141\sin 15° = -1.072 \text{ rad/s}$$

$$\therefore T_0 = \frac{1}{2}\cdot 30(-1.072)^2 + \frac{1}{2}\cdot 60(-2)^2 = 137.2 \text{ J}$$

(a)

$$H_0 = \left|A\omega_p\right| = 30\cdot 4.141 = 124.2 \text{ kg}\cdot\text{m}^2/\text{s}$$
$$H_f = H_0$$
$$60\omega_f = 124.2 \Rightarrow \underline{\omega_f = 2.071 \text{ rad/s}}$$

(b)

$$T_f = \frac{1}{2}C\omega_f{}^2 = \frac{1}{2}\cdot 60\cdot 2.071^2 = 128.6 \text{ J}$$
$$\Delta T = T_f - T_0 = 128.6 - 137.2 = \underline{-8.616 \text{ J}}$$

(c)

$$l_f = R\sqrt{1+\frac{C}{2mR^2}} = 1\cdot\sqrt{1+\frac{60}{2\cdot 7\cdot 1^2}} = \underline{2.299 \text{ m}}$$

Problem 10.18

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}\begin{Bmatrix} 0 \\ n \\ -\omega_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ nA \\ -\omega_s C \end{Bmatrix}$$

$$\mathbf{M}_G = \boldsymbol{\Omega}\times\mathbf{H}_G = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{0} & n & 0 \\ 0 & nA & -\omega_s C \end{vmatrix} = -C\omega_s n\hat{\mathbf{i}}$$

$$\omega_s = 1\cdot\frac{2\pi}{60} = 0.1047 \text{ rad/s}$$

$$n = \frac{2\pi + \dfrac{2\pi}{365.26}}{24\cdot 3600} = 7.292\times 10^{-5} \text{ rad/s}$$

$$C = 550 \text{ kg}\cdot\text{m}^2$$

$$\therefore M_{G_x} = -550\cdot 0.1047\cdot 7.292\times 10^{-5} = \underline{-0.0042 \text{ N}\cdot\text{m}}$$

Problem 10.19 $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_f$ are the initial and final angular velocities of the spacecraft. ω is the angular velocity of the flywheel relative to the vehicle.

$$\left[\mathbf{H}_G^{(v)} + \mathbf{H}_G^{(v)}\right]_0 = \left[\mathbf{H}_G^{(v)} + \mathbf{H}^{(w)}\right]_f$$

$$\left(A\omega_{0x}\hat{\mathbf{i}} + A\omega_{0y}\hat{\mathbf{j}} + C\omega_{0z}\hat{\mathbf{k}}\right) + \left(I_x\omega_{0x}\hat{\mathbf{i}} + I_y\omega_{0y}\hat{\mathbf{j}} + I_z\omega_{0z}\hat{\mathbf{k}}\right)$$
$$= \left(A\omega_x\hat{\mathbf{i}} + A\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}}\right) + \left[I_x\left(\omega_x+\omega\right)\hat{\mathbf{i}} + I_y\omega_y\hat{\mathbf{j}} + I_z\omega_z\hat{\mathbf{k}}\right]$$

$$\left(A+I_z\right)\omega_z = \left(A+I_z\right)\omega_{0_z} \quad \Rightarrow \quad \omega_z = \omega_{0_z} = 0$$
$$\left(A+I_y\right)\omega_y = \left(A+I_y\right)\omega_{0_y} \quad \Rightarrow \quad \omega_y = \omega_{0_y} = 0.05 \text{ rad/s}$$

$$\left(A+I_x\right)\omega_x + I_x\omega = \left(A+I_x\right)\omega_{0_x} \quad \Rightarrow \quad \omega = \left(1+\frac{A}{I_x}\right)\left(\omega_{0_x} - \omega_x\right)$$

$$\therefore \omega = \left(1+\frac{1000}{20}\right)(0.1-0.003) = \underline{4.947 \text{ rad/s}}$$

Problem 10.20

Given: $I_3 > I_2 > I_1$.

Figure 10.29, Stable region I: $I_{roll} > I_{yaw} > I_{pitch}$:

I_1 axis in pitch direction (normal to orbital plane)
I_2 axis in yaw direction (radial)
I_3 axis in roll direction (local horizon)

Figure 10.29, Stable region II (preferred): $I_{pitch} > I_{roll} > I_{yaw}$:

I_1 axis in yaw direction (radial)
I_2 axis in roll direction (local horizon)
I_3 axis in pitch direction (normal to orbital plane)

Problem 11.1

$$m_p = m_{p_{out}} + m_{p_{in}} = m_{p_{out}} + \frac{m_{p_{out}}}{4} = \frac{5}{4} m_{p_{out}}$$

Outbound leg:

$$\Delta v = I_{sp} g_o \ln\left(\frac{m_e + m_p + m_{PL}}{m_e + m_p - m_{p_{out}} + m_{PL}}\right)$$

$$4220 = 430 \cdot 9.81 \cdot \ln\left(\frac{m_e + \frac{5}{4} m_{p_{out}} + 3500}{m_e + \frac{5}{4} m_{p_{out}} - m_{p_{out}} + 3500}\right)$$

$$= 430 \cdot 9.81 \cdot \ln\left(\frac{m_e + \frac{5}{4} m_{p_{out}} + 3500}{m_e + \frac{1}{4} m_{p_{out}} + 3500}\right)$$

$$\frac{m_e + \frac{5}{4} m_{p_{out}} + 3500}{m_e + \frac{1}{4} m_{p_{out}} + 3500} = 2.719$$

$$0.5702 m_{p_{out}} - 1.719 m_e = 6018 \qquad (1)$$

Return from GEO tp LEO:

$$\Delta v = I_{sp} g_o \ln\left(\frac{m_e + \frac{1}{4} m_{p_{out}} + 3500}{m_e}\right)$$

$$4220 = 430 \cdot 9.81 \cdot \ln\left(\frac{m_e + \frac{1}{4} m_{p_{out}}}{m_e}\right)$$

$$\frac{m_e + \frac{1}{4} m_{p_{out}}}{m_e} = 2.719$$

$$m_{p_{out}} = 6.876 m_e \qquad (2)$$

Substitute (2) into(1):

$$0.5702(6.876 m_e) - 1.719 m_e = 6018$$

$$\underline{m_e = 2733 \text{ kg}}$$

Problem 11.2

First stage:

$$c = I_{sp} g_o = 235 \cdot 9.81 = 2943 \text{ m/s}$$

$$v_{bo} = c \ln\left(\frac{m_0}{m_f}\right) - \frac{(m_0 - m_f) g_o}{\dot{m}_e} = 2943 \ln\left(\frac{249.5}{170.1}\right) - \frac{(249.5 - 170.1) \cdot 9.81}{10.61} = 1127 - 73.38 = 1054 \quad \text{m/s}$$

$$h_{bo} = \frac{c}{\dot{m}_e}\left[\ln\left(\frac{m_f}{m_0}\right) m_f + m_0 - m_f\right] - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)^2 g_o$$

$$= \frac{2943}{10.61}\left[\ln\left(\frac{170.1}{249.5}\right) \cdot 170.1 + 249.5 - 170.1\right] - \frac{1}{2}\left(\frac{249.5 - 170.1}{10.61}\right)^2 9.81$$

$$= 3947 - 274.4$$
$$= 3673 \text{ m}$$

After 3 second staging delay:

$$v = v_{bo} - g\Delta t_s = 1054 - 9.81 \cdot 3 = 1024 \text{ m/s}$$
$$h = h_{bo} + v_{bo}\Delta t_s - \frac{1}{2} g\Delta t_s^{\,2} = 3673 + 1054 \cdot 3 - \frac{1}{2} \cdot 9.81 \cdot 3^2 = 3673 + 3117 = 6790 \text{ m}$$

Second stage:

$$v_0 = 1024 \text{ m/s}$$
$$h_0 = 6790 \text{ m}$$

$$c = I_{sp} g_o = 235 \cdot 9.81 = 2305 \text{ m/s}$$

$$v_{bo} = v_0 + c \ln\left(\frac{m_0}{m_f}\right) - \frac{\left(m_0 - m_f\right) g_o}{\dot{m}_e}$$
$$= 1024 + 2305 \ln\left(\frac{113.4}{58.97}\right) - \frac{(113.4 - 58.97) \cdot 9.81}{4.0573}$$
$$= 1024 + 1508 - 131.7$$
$$= 2400 \text{ m/s}$$

$$h_{bo} = h_0 + v_0\left(\frac{m_0 - m_f}{\dot{m}_e}\right) + \frac{c}{\dot{m}_e}\left[\ln\left(\frac{m_f}{m_0}\right) m_f + m_0 - m_f\right] - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)^2 g_o$$

$$= 6790 + 1024\left(\frac{113.4 - 59.97}{4.053}\right) + \frac{2305}{4.053}\left[\ln\left(\frac{59.97}{113.4}\right) \cdot 58.97 + 113.4 - 58.97\right] - \frac{1}{2}\left(\frac{113.4 - 58.97}{4.063}\right)^2 \cdot 9.81$$
$$= 6790 + 13\,760 + 9028 - 884.7$$
$$= 28\,690 \text{ m}$$

Coast to apogee:

$$v_0 = 2400 \text{ m/s}$$
$$h_0 = 28\,690 \text{ m}$$

$$0 = v_0 - g t_{\max} \Rightarrow t_{\max} = \frac{v_0}{g} = \frac{2400}{9.81} = 244.7 \text{ s}$$
$$h_{\max} = h_0 + v_0 t_{\max} - \frac{1}{2} g t_{\max}^{\,2} = 28\,690 + 2400 \cdot 244.7 - \frac{1}{2} \cdot 9.81 \cdot 244.7^2 = \underline{322\,300 \text{ m}}$$

Problem 11.3

$$v_0 = \omega_{\text{earth}} R_{\text{earth}} \cos\phi = 7.292\left(10^{-5}\right) \cdot 6378 \cdot \cos 28° = 0.4107 \text{ km/s}$$
$$\Delta v = \sqrt{\frac{398\,600}{6678}} + 2 - 0.4107 = 9.315 \text{ km/s}$$
$$v_{bo} = \Delta v = 9.315 \text{ km/s}$$
$$v_{bo} = v_{bo1} + v_{bo2}$$
$$v_{bo} = I_{sp_1} g_o \ln\left(\frac{m_{01}}{m_{f_1}}\right) + I_{sp_2} g_o \ln\left(\frac{m_{02}}{m_{f_2}}\right)$$

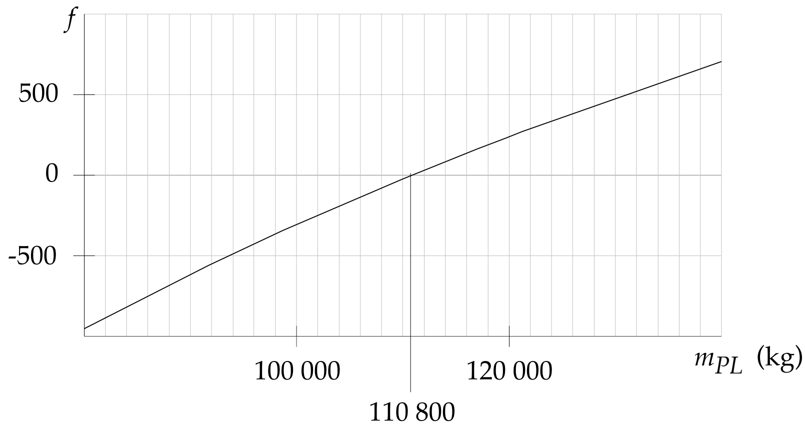
$$9315 = 290\cdot 9.81\cdot \ln\left(\frac{2\cdot 525\,000 + 30\,000 + 600\,000 + m_{PL}}{2\cdot(525000 - 450000) + 3000 + 600000 + m_{PL}}\right)$$

$$+450\cdot 9.81\cdot \ln\left(\frac{30\,000 + 600\,000 + m_{PL}}{30\,000 + m_{PL}}\right)$$

$$9315 = 2845\cdot \ln\left(\frac{1\,680\,000 + m_{PL}}{753\,000 + m_{PL}}\right) + 4414\cdot \ln\left(\frac{630\,000 + m_{PL}}{30\,000 + m_{PL}}\right)$$

To find the value of m_{PL} satisfying this equation, graph the function

$$f = 9315 - 2845\cdot \ln\left(\frac{1\,680\,000 + m_{PL}}{753\,000 + m_{PL}}\right) - 4414\cdot \ln\left(\frac{630\,000 + m_{PL}}{30\,000 + m_{PL}}\right)$$



$f = 0$ when $\underline{m_{PL} = 110\,800 \text{ kg}}$

Problem 11.4

$$\pi_{PL} = \frac{m_{PL}}{m_0} = \frac{10\,000}{150\,000} = 0.06667$$

$$\lambda = \frac{\pi_{PL}^{1/3}}{1-\pi_{PL}^{1/3}} = 0.682$$

$$\varepsilon = \frac{m_E}{m_0 - m_{PL}} = \frac{20\,000}{150\,000 - 10\,000} = 0.1429$$

(a)

$$n = \frac{1+\lambda}{\varepsilon+\lambda} = \frac{1+0.682}{0.1429+0.682} = \underline{2.039}$$

$$\Delta v = I_{sp} g_o \ln n^3 = 310\cdot 0.009\,81\cdot \ln 2.039^3 = \underline{6.5 \text{ km/s}}$$

(b)

$$m_{p_1} = \frac{\left(1-\pi_{PL}^{1/3}\right)(1-\varepsilon)}{\pi_{PL}} m_{PL} = \frac{\left(1-0.06667^{1/3}\right)(1-0.1429)}{0.06667}\cdot 10\,000 = \underline{76\,440 \text{ kg}}$$

$$m_{p_2} = \frac{\left(1-\pi_{PL}^{1/3}\right)(1-\varepsilon)}{\pi_{PL}^{2/3}} m_{PL} = \frac{\left(1-0.06667^{1/3}\right)(1-0.1429)}{0.06667^{2/3}}\cdot 10\,000 = \underline{30\,990 \text{ kg}}$$

$$m_{p_3} = \frac{\left(1-\pi_{PL}^{1/3}\right)(1-\varepsilon)}{\pi_{PL}^{1/3}} m_{PL} = \frac{\left(1-0.06667^{1/3}\right)(1-0.1429)}{0.06667^{1/3}}\cdot 10\,000 = \underline{12\,570 \text{ kg}}$$

(c)

$$m_{E1} = \frac{\left(1-\pi_{PL}{}^{1/3}\right)\varepsilon}{\pi_{PL}} m_{PL} = \frac{\left(1-0.06667^{1/3}\right)\cdot 0.1429}{0.06667} 10\,000 = \underline{12\,740 \text{ kg}}$$

$$m_{E2} = \frac{\left(1-\pi_{PL}{}^{1/3}\right)\varepsilon}{\pi_{PL}{}^{2/3}} m_{PL} = \frac{\left(1-0.06667^{1/3}\right)\cdot 0.1429}{0.06667^{2/3}} 10\,000 = \underline{5166 \text{ kg}}$$

$$m_{E3} = \frac{\left(1-\pi_{PL}{}^{1/3}\right)\varepsilon}{\pi_{PL}{}^{1/3}} m_{PL} = \frac{\left(1-0.06667^{1/3}\right)\cdot 0.1429}{0.06667^{1/3}} 10\,000 = \underline{2095 \text{ kg}}$$

(d)

$$m_{03} = m_{E3} + m_{p_3} + m_{PL} = 2095 + 12\,570 + 10\,000 = \underline{24\,660 \text{ kg}}$$

$$m_{02} = m_{E2} + m_{p_2} + m_{03} = 5166 + 30\,990 + 24\,660 = \underline{60\,820 \text{ kg}}$$

$$m_{01} = m_{E1} + m_{p_1} + m_{02} = 12\,740 + 76\,440 + 60\,820 = \underline{150\,000 \text{ kg}}$$

Problem 11.5

$$c_1 = I_{sp_1} g_o = 300 \cdot 0.009\,81 = 2.943 \text{ km/s}$$

$$c_2 = I_{sp_2} g_o = 235 \cdot 0.009\,81 = 2.305 \text{ km/s}$$

$$\varepsilon_1 = 0.2$$

$$\varepsilon_2 = 0.3$$

$$v_{bo} = 6.2 \text{ km/s}$$

$$v_{bo} = \sum_{i=1}^{2} c_i \ln\left(\frac{c_i\eta - 1}{c_i\varepsilon_i\eta}\right) = c_1 \ln\left(\frac{c_1\eta - 1}{c_1\varepsilon_1\eta}\right) + c_2 \ln\left(\frac{c_2\eta - 1}{c_2\varepsilon_2\eta}\right)$$

$$6.2 = 2.943 \ln\left(\frac{2.943\eta - 1}{2.943 \cdot 0.2\eta}\right) + 2.305 \ln\left(\frac{2.305\eta - 1}{2.305 \cdot 0.3\eta}\right)$$

$$6.2 = 2.943 \ln\left(\frac{2.943\eta - 1}{0.5886\eta}\right) + 2.305 \ln\left(\frac{2.305\eta - 1}{0.6915\eta}\right)$$

To find η, graph the function

$$f = 2.943 \ln\left(\frac{2.943\eta - 1}{0.5886\eta}\right) + 2.305 \ln\left(\frac{2.305\eta - 1}{0.6915\eta}\right) - 6.2$$

As shown below, $f = 0$ when $\eta = 1.726$.

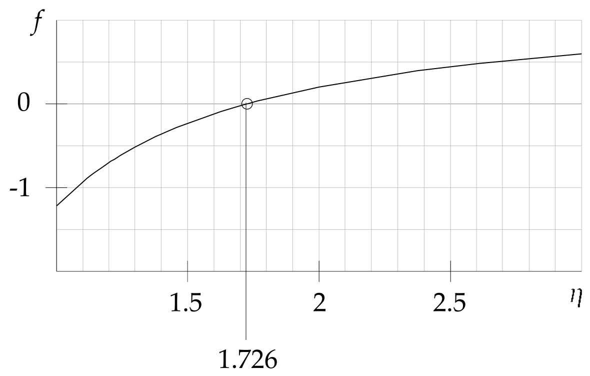
$$n_1 = \frac{c_1\eta - 1}{c_1\varepsilon_1\eta} = \frac{2.943 \cdot 1.726 - 1}{2.943 \cdot 0.2 \cdot 1.726} = 4.016$$

$$n_2 = \frac{c_2\eta - 1}{c_2\varepsilon_2\eta} = \frac{2.305 \cdot 1.726 - 1}{2.305 \cdot 0.3 \cdot 1.726} = 2.496$$

$$m_2 = \frac{n_2 - 1}{1 - \varepsilon_2 n_2} m_{PL} = \frac{2.496 - 1}{1 - 0.3 \cdot 2.496} \cdot 10 = 59.53 \text{ kg}$$

$$m_1 = \frac{n_1 - 1}{1 - \varepsilon_1 n_1}\left(m_2 + m_{PL}\right) = \frac{4.016 - 1}{1 - 0.2 \cdot 4.016}(59.53 + 10) = 1065 \text{ kg}$$

$$M = m_1 + m_2 = 1065 + 59.53 = \underline{1124 \text{ kg}}$$



Problem 11.6

$$z = x^2 + y^2 + 2xy$$
$$g = x^2 - 2x + y^2$$

$$h = z + \lambda g = x^2 + y^2 + 2xy + \lambda\left(x^2 - 2x + y^2\right)$$

$$\frac{\partial h}{\partial x} = 2x + 2y + \lambda(2x - 2) = 0 \ \Rightarrow\ (\lambda + 1)x + y = \lambda$$
$$\frac{\partial h}{\partial y} = 2y + 2x + \lambda(2y) = 0 \ \Rightarrow\ x + (\lambda + 1)y = 0$$

$$\therefore \begin{bmatrix} \lambda + 1 & 1 \\ 1 & \lambda + 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \lambda + 1 & 1 \\ 1 & \lambda + 1 \end{bmatrix}^{-1} \begin{Bmatrix} \lambda \\ 0 \end{Bmatrix} = \frac{1}{\lambda(\lambda + 2)} \begin{bmatrix} \lambda + 1 & -1 \\ -1 & \lambda + 1 \end{bmatrix} \begin{Bmatrix} \lambda \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\lambda + 1}{\lambda + 2} \\ -\dfrac{1}{\lambda + 2} \end{Bmatrix}$$

$$x^2 - 2x + y^2 = 0$$
$$\frac{(\lambda + 1)^2}{(\lambda + 2)^2} - 2\frac{\lambda + 1}{\lambda + 2} + \frac{1}{(\lambda + 2)^2} = 0$$

Multiply through by $(\lambda + 2)^2$ (this is okay since $\lambda + 2 = 0$ clearly does not correspond to a local extremum). Then

$$(\lambda + 1)^2 - 2(\lambda + 1)(\lambda + 2) + 1 = 0$$

or

$$\lambda^2 + 4\lambda + 2 = 0$$

The two roots are -0.5858 and -3.414.

$\lambda = -0.5858$:

$$x = \frac{\lambda + 1}{\lambda + 2} = \frac{-0.5858 + 1}{-0.5858 + 2} = 0.2929$$
$$y = -\frac{1}{\lambda + 2} = -\frac{1}{-0.585 + 2} = -0.7071$$

$$z_1 = x^2 + y^2 + 2xy = 0.2929^2 + (-0.7071)^2 + 2 \cdot 0.2929(-0.7071) = \underline{0.1716}$$

$\lambda = -3.414$:

$$x = \frac{\lambda + 1}{\lambda + 2} = \frac{-3.414 + 1}{-3.414 + 2} = 1.707$$
$$y = -\frac{1}{\lambda + 2} = -\frac{1}{-3.414 + 2} = 0.7071$$
$$z_2 = x^2 + y^2 + 2xy = 1.707^2 + 0.7071^2 + 2 \cdot 1.707 \cdot 0.7071 = \underline{5.828}$$

Note that

$$d^2h = \left(\frac{\partial^2 z}{\partial x^2} + \lambda \frac{\partial^2 g}{\partial x^2}\right)dx^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y} + \lambda \frac{\partial^2 g}{\partial x \partial y}\right)dxdy + \left(\frac{\partial^2 z}{\partial y^2} + \lambda \frac{\partial^2 g}{\partial y^2}\right)dy^2$$
$$d^2h = (2 + \lambda \cdot 2)dx^2 + 2(2 + \lambda \cdot 0)dxdy + (2 + \lambda \cdot 2)dy^2$$
$$d^2h = 2(\lambda + 1)\left(dx^2 + dy^2\right) + 4dxdy$$

For $\lambda = -0.5858$,

$$d^2h = 2(\lambda + 1)\left(dx^2 + dy^2\right) + 4dxdy = 2(-0.5858 + 1)\left(dx^2 + dy^2\right) + 4dxdy$$
$$= 0.8284\left(dx^2 + dy^2\right) + 4dxdy$$

Since $d^2h > 0$, $\underline{z_1 = z_{\text{min}}}$.

For $\lambda = -3.414$,

$$d^2h = 2(\lambda + 1)\left(dx^2 + dy^2\right) + 4dxdy = 2(-3.414 + 1)\left(dx^2 + dy^2\right) + 4dxdy$$
$$= -4.828\left(dx^2 + dy^2\right) + 4dxdy$$

Since $d^2h < 0$, $\underline{z_2 = z_{\text{max}}}$.
