

HW 4

written questions

Problem 1a)

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} \frac{11}{9} & -\frac{1}{3} & -\frac{2}{9} \\ -2 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

$$C = A A^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{11} = A_{11} A_{11}^{-1} + A_{12} A_{21}^{-1} + A_{13} A_{31}^{-1}$$

$$C_{12} = A_{11} A_{12}^{-1} + A_{12} A_{22}^{-1} + A_{13} A_{32}^{-1}$$

$$C_{13} = A_{11} A_{13}^{-1} + A_{12} A_{23}^{-1} + A_{13} A_{33}^{-1}$$

$$C_{21} = A_{21} A_{11}^{-1} + A_{22} A_{21}^{-1} + A_{23} A_{31}^{-1}$$

$$C_{22} = A_{21} A_{12}^{-1} + A_{22} A_{22}^{-1} + A_{23} A_{32}^{-1}$$

$$C_{23} = A_{21} A_{13}^{-1} + A_{22} A_{23}^{-1} + A_{23} A_{33}^{-1}$$

$$C_{31} = A_{31} A_{11}^{-1} + A_{32} A_{21}^{-1} + A_{33} A_{31}^{-1}$$

$$C_{32} = A_{31} A_{12}^{-1} + A_{32} A_{22}^{-1} + A_{33} A_{32}^{-1}$$

$$C_{33} = A_{31} A_{13}^{-1} + A_{32} A_{23}^{-1} + A_{33} A_{33}^{-1}$$

$$C_{11} = 3\left(\frac{11}{9}\right) + 1(-2) + 2\left(-\frac{1}{3}\right) = 1$$

$$C_{12} = 3\left(-\frac{1}{3}\right) + 1(1) + 2(0) = 0$$

$$C_{13} = 3\left(\frac{2}{9}\right) + 1(0) + 2\left(\frac{1}{3}\right) = 0$$

$$C_{21} = 6\left(\frac{11}{9}\right) + 3(-2) + 4\left(\frac{1}{3}\right) = 0$$

$$C_{22} = 6\left(-\frac{1}{3}\right) + 3(1) + 4(0) = 1$$

$$C_{23} = 6\left(-\frac{2}{9}\right) + 3(0) + 4\left(\frac{1}{3}\right) = 0$$

$$C_{31} = 3\left(\frac{11}{9}\right) + 1(-2) + 5\left(-\frac{1}{3}\right) = 0$$

$$C_{32} = 3\left(-\frac{1}{3}\right) + 1(1) + 5(0) = 0$$

$$C_{33} = 3\left(-\frac{2}{9}\right) + 1(0) + 5\left(\frac{1}{3}\right) = 0$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \|A\|_\infty = 13, \quad \|A^{-1}\|_\infty = 3, \quad \text{cond}(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 39$$

$$6+3+4 = 13$$

$$2+1 = 3$$

$$c) \quad \underline{x}_c = (-.99, 1.01, .99) \quad \underline{x} = (-1, 1, 1) \quad \underline{A} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$\underline{r} = \underline{b} - \underline{A} \underline{x}_c$$

$$\text{Error magnification factor: } \frac{\frac{\|\underline{x} - \underline{x}_c\|_\infty}{\|\underline{x}\|_\infty}}{\frac{\|\underline{b}\|_\infty}{\|\underline{r}\|_\infty}} = \frac{\frac{.01}{1}}{\frac{3}{.05}} = \frac{.01}{60}$$

$$\underline{A} \underline{x}_c = \begin{bmatrix} -.99(3) + -.99(1) + -.99(2) \\ 1.01(6) + 1.01(3) + 1.01(4) \\ .99(3) + .99(1) + .99(5) \end{bmatrix} = \begin{pmatrix} -.02 \\ 1.05 \\ 2.99 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -.02 \\ 1.05 \\ 2.99 \end{pmatrix} = \begin{pmatrix} .02 \\ -.05 \\ -.01 \end{pmatrix}$$

$$\|\underline{r}\|_\infty = .05$$

$$\text{EMF} = \frac{.05}{3} = .016$$

This seems like a very good EMF, considering an EMF of 39 is possible for this system.

Problem 2

a) Points: $(0, 0)$ $(1, 1)$ $(2, 2)$ $(3, 7)$

$$x = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 7 \end{pmatrix}$$

$$L_k = \prod_{\substack{i=1 \\ i \neq k}}^n \frac{x-x_i}{x_k-x_i}$$

$$\begin{aligned}
 k=1 \quad L_1 &= \frac{x-1}{0-1} \frac{x-2}{0-2} \frac{x-3}{0-3} = P_3(x) = \sum_{k=1}^n y_k L_k(x) \\
 k=2 \quad L_2 &= \frac{x-0}{1-0} \frac{x-2}{1-2} \frac{x-3}{1-3} = P_3(x) = 0 + 1 \left(\frac{x}{1} \frac{x-2}{-1} \frac{x-3}{-2} \right) \\
 k=3 \quad L_3 &= \frac{x-0}{2-0} \frac{x-1}{2-1} \frac{x-3}{2-3} = + 2 \left(\frac{x}{2} \frac{x-1}{-1} \frac{x-3}{-1} \right) \\
 k=4 \quad L_4 &= \frac{x-0}{3-0} \frac{x-1}{3-1} \frac{x-2}{3-2} = + 7 \left(\frac{x}{3} \frac{x-1}{2} \frac{x-2}{1} \right)
 \end{aligned}$$

$$\begin{aligned}
 P_3(x) &= x(-x+2)(-\frac{1}{2}x+\frac{3}{2}) + x(x-1)(-x+3) + \frac{7}{3}x(\frac{x}{2}-\frac{1}{2})(x-2) \\
 &= (-x^2+2x)(-\frac{1}{2}x+\frac{3}{2}) + (x^2-x)(-x+3) + (\frac{7}{6}x^2-\frac{7}{6}x)(x-2) \\
 &= \frac{1}{2}x^3 - \frac{3}{2}x^2 - x^2 + 3x + -x^3 + 3x^2 + x^2 - 3x + \frac{7}{6}x^3 - \frac{7}{6}x^2 - \frac{7}{3}x^2 + \frac{7}{3}x \\
 &= .66\bar{6}x^3 - 2x^2 + 2.33\bar{3}x + 0
 \end{aligned}$$

b) no, there is no parabola that will go through these points.

x	$f[x]$
0	0
1	1
2	2
3	7

$1 = f[x_1, x_2]$
 $0 = f[x_1, x_2, x_3]$
 $1 = f[x_2, x_3]$
 $2 = f[x_2, x_3, x_4]$
 $5 = f[x_3, x_4]$

$$.66\bar{6} = f[x_1, x_2, x_3, x_4] = c$$

$$\begin{aligned}
 P(x) &= 0 + 1(x-0) + 0(x-0)(x-1) + .66\bar{6}(x-0)(x-1)(x-2) \\
 &= x + .66\bar{6}(x^2-x)(x-2) = x + .66\bar{6}(x^3 - 2x^2 - x^2 + 2x) \\
 &= x + .66\bar{6}x^3 - 1.33\bar{3}x^2 - .66\bar{6}x^2 + 1.33\bar{3}x = .66\bar{6}x^3 - 2x^2 + 2.33\bar{3}x
 \end{aligned}$$

Problem 2

c) cont

yes, I got the same result as from lagrange method.

This makes sense because there exists 1 unique interpolating polynomial of degree $n-1$ for these n data points. If the points all layed on a line or parabola this wouldn't be true.

d) add an extra point to the list of points to get a polynomial degree 4 that interpolates the original ~~4~~ and the new 5th points, where $P_3(x) = -66x^3 + -2x^2 + 2.33x$, $P_4(x) =$

$$\begin{array}{|c|c|} \hline x & f(x) \\ \hline 0 & 0 \\ \hline \end{array}$$

$$P_3(x) + f[x_1 x_2 x_3 x_4 x_5](x)(x-1)(x-2)(x-3)$$

will interpolate the new list

	1	1	0	
		1		.66
2	2	2	2	
				$f[x_1 x_2 x_3 x_4 x_5] = \frac{y_5 - \frac{y_1}{2} - \frac{y_2}{3}}{4}$
3	3	5		$f[x_2 x_3 x_4 x_5] = \frac{y_5 - y_2}{2}$
				$f[x_3 x_4 x_5] = \frac{y_5 - y_3}{2}$
4	4			$f[x_4 x_5] = y_5 - 7$

$$P_4(x) = P_3(x) + \left(\frac{y_5}{24} - \frac{10}{12}\right) x(x-1)(x-2)(x-3)$$

$$\begin{aligned} x(x-1)(x-2)(x-3) &= (x^2 - x)(x-2)(x-3) = (x^3 - 2x^2 - x^2 + 2x)(x-3) \\ &= (x^3 - 3x^2 + 2x)(x-3) = x^4 - 3x^3 + 2x^2 - 3x^3 + 9x^2 - 6x = \\ &= x^4 - 6x^3 + 11x^2 - 6x \end{aligned}$$

choose y_5 to be 0, $P_4(x) = .66x^3 - 2x^2 + 2.33x - \frac{5}{6}x^4 + 5x^3 - 9.166x^2 + 5$

$$\therefore - .833x^4 + 5.66x^3 - 11.166x^2 + 7.333x$$

choose y_5 to be 8 $\therefore P_4(x) = .66x^3 - 2x^2 + 2.33x - .5x^4 + 3x^3 - 5.5x^2 + 3x$

$$\therefore .5x^4 + 3.66x^3 - 7.5x^2 + 5.333x$$

e) no, since $P_3(x) = 0.66x^3 - 2x^2 + 2.33x$ is the unique polynomial of degree 3 or less which interpolates then only if $(4, 2)$ lies on $P_3(x)$, i.e. if $P_3(4) = 2$, would there be a polynomial of degree 3 or less that interpolates $\underline{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 7 \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\underline{y}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 7 \\ 2 \end{pmatrix}$. $P_3(4)$ doesn't equal 2, $P_3(4) = 20$, so no polynomial degree 3 or less interpolates $(\underline{x}_2, \underline{y}_2)$

Problem 3 a) $y = Be^{Ax}$, $\ln(y) = \ln(B) + \ln e^{Ax} = \ln B + Ax$

$$\ln(y) = Y, \ln(B) = \beta \Rightarrow Y = \beta + \alpha X$$

$$A = \alpha, X = x$$

b)

$$\underline{X} = \begin{pmatrix} -1 & 6.62 \\ 0 & 2.78 \\ 1 & 1.51 \\ 2 & 1.23 \\ 3 & 0.69 \end{pmatrix} \quad \underline{Y} = \begin{pmatrix} -1 & 1.890 \\ 0 & 1.023 \\ 1 & 0.412 \\ 2 & 0.207 \\ 3 & -0.117 \end{pmatrix}$$

c)

$$\underline{A} \underline{s} = \underline{Y} \quad \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 1.890 \\ 1.023 \\ 0.412 \\ 0.207 \\ -0.117 \end{pmatrix}$$

* to avoid confusion between data set and least squares solution, I name \underline{x}^*

Normal eq:

$$\underline{A}^T \underline{A} \underline{s}^* = \underline{A}^T \underline{Y}$$

$$\underline{A}^T \underline{A} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} =$$

$$\underline{A}^T \underline{Y} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \underline{s}^* = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} \underline{s}^* = \begin{pmatrix} 3.4151 \\ -1.4136 \end{pmatrix}$$

Problem 3 c) cont

$$\begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 3.4151 \\ -1.4136 \end{pmatrix}$$

$$\begin{bmatrix} 5 & 5 & 3.4151 \\ 0 & 10 & -4.8287 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0.643 \\ 0 & 1 & -0.4829 \end{bmatrix}$$

$$S_2 = -0.4829 \quad S^* = \begin{pmatrix} 1.1659 \\ -0.4829 \end{pmatrix}$$

$$e) \quad r^* = A S^* - Y$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1.1659 \\ -0.4829 \end{pmatrix} - \begin{pmatrix} 1.8901 \\ 1.0225 \\ 0.4121 \\ 0.2070 \\ -0.1165 \end{pmatrix}$$

$$= \begin{pmatrix} 1.6484 \\ 1.1659 \\ -0.6830 \\ 0.2001 \\ -0.2828 \end{pmatrix} - \begin{pmatrix} 1.8901 \\ 1.0225 \\ -0.4121 \\ 0.2070 \\ -0.1165 \end{pmatrix} = \begin{pmatrix} -0.2413 \\ 0.1434 \\ 0.2709 \\ -0.0669 \\ -0.1663 \end{pmatrix} = \underline{\underline{r^*}}$$

$$\|r^*\|_2 = \sqrt{0.2413^2 + 0.1434^2 + 0.2709^2 + 0.0669^2 + 0.1663^2} \approx 0.4241$$

$$f) 1) S = (0, 0) \quad r = -Y, \quad \|r\|_2 = \|Y\|_2 = \sqrt{1.8901^2 + 1.0225^2 + 0.4121^2 + 0.2070^2 + 0.1165^2} = 2.2009$$

$$f) 2) S = (1, 0) \quad AS = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad r = AS - Y$$

$$r = \begin{pmatrix} -0.8901 \\ -0.0225 \\ 0.5879 \\ 0.7930 \\ 1.1165 \end{pmatrix}, \|r\|_2 = \sqrt{0.8901^2 + 0.0225^2 + 0.5879^2 + 0.7930^2 + 1.1165^2} = 1.7361$$

$$f) 3) s = (1, -0.5)$$

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ -0.5 \end{pmatrix} - \begin{pmatrix} 1.6901 \\ 1.0225 \\ 1.4121 \\ 2.070 \\ -0.1165 \end{pmatrix} = \begin{pmatrix} -0.2413 \\ 0.1434 \\ 0.2709 \\ -0.0669 \\ -0.1662 \end{pmatrix}, \|V\|_2 = \sqrt{0.2413^2 + 0.1434^2 + 0.2709^2 + 0.0669^2 + 0.1662^2} = 0.5914$$

f) as expected the values of s I chose are less good than s^* . This makes sense since s^* by definition reduces r as low as possible

$$g) A = d = s_2 \quad A = s_2 = -0.4829$$

$$\ln(B) = B = s_1 \quad B = e^{s_1} = 3.2088$$