

HW 6 Problem 1

$$1) \quad J = \begin{bmatrix} m \frac{R^2}{4} + m \frac{h^2}{3} & 0 & 0 \\ 0 & m \frac{R^2}{4} + m \frac{h^2}{3} & 0 \\ 0 & 0 & m \frac{R^2}{2} \end{bmatrix}$$

$$r = \begin{pmatrix} 8 \\ h/2 \end{pmatrix}$$

$$I = J + m r^x r^x = J + \begin{bmatrix} -\frac{h^2}{4} m & 0 & 0 \\ 0 & -\frac{h^2}{4} m & 0 \\ 0 & 0 & m \frac{R^2}{2} \end{bmatrix}$$

$$I = \begin{bmatrix} \frac{1}{12} m (3R^2 + h^2) & 0 & 0 \\ 0 & \frac{1}{12} m (3R^2 + h^2) & 0 \\ 0 & 0 & m \frac{R^2}{2} \end{bmatrix} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$I_y = I_x$$

$$2) \quad \underline{I}_c = \begin{pmatrix} T_x \\ T_y \\ 0 \end{pmatrix}, \quad \underline{\omega}_0 = \begin{pmatrix} \omega_{x0} \\ \omega_{y0} \\ \omega_{z0} \end{pmatrix} \neq 0$$

$$\dot{\omega} = -\underline{I}^{-1} (\underline{\omega}^x \underline{I} \underline{\omega} - \underline{T}_c)$$

$$I_x \dot{\omega}_x + (I_z - I_x) \omega_z \omega_y = T_x$$

$$I_y \dot{\omega}_y + (I_x - I_z) \omega_x \omega_z = T_y$$

$$I_z \dot{\omega}_z + (I_x - I_x) \omega_y \omega_x = T_z = 0$$

$$\dot{\omega}_z = 0$$

Re define problem to be simpler

$$I_z - I_x = \beta, \quad I_x - I_z = -\beta, \quad I = I_x = I_y$$

$$I \dot{\omega}_x + \beta \omega_z \omega_y = T_x, \quad \dot{\omega}_x = \frac{T_x}{I} - \omega_y \frac{\beta \omega_z}{I}$$

$$I \dot{\omega}_y - \beta \omega_x \omega_z = T_y, \quad \dot{\omega}_y = \frac{T_y}{I} - \omega_x \frac{-\beta \omega_z}{I}$$

$$d = \frac{\beta \omega_z}{I}$$

$$\dot{W}_x = \frac{I_x}{I} - \alpha W_y, \quad \ddot{W}_x = -\alpha \dot{W}_y = -\alpha \frac{I_y}{I} - \alpha^2 W_x$$

$$\dot{W}_y = \frac{I_y}{I} + \alpha W_x, \quad \ddot{W}_y = \alpha \dot{W}_x = \alpha \frac{I_x}{I} + \alpha^2 W_y$$

$$W_x = A_1 \cos(\alpha t) + B_1 \sin(\alpha t) + \alpha \frac{I_x}{I} \quad @t=0 \quad A_1 = W_{x0} - \alpha \frac{I_y}{I}$$

$$W_y = A_2 \cos(\alpha t) + B_2 \sin(\alpha t) - \alpha \frac{I_y}{I} \quad \longrightarrow \quad A_2 = W_{y0} + \alpha \frac{I_x}{I}$$

$$W_x = \left(W_{x0} - \alpha \frac{I_y}{I}\right) \cos(\alpha t) + B_1 \sin(\alpha t) + \alpha \frac{I_x}{I}$$

$$W_y = \left(W_{y0} + \alpha \frac{I_x}{I}\right) \cos(\alpha t) + B_2 \sin(\alpha t) - \alpha \frac{I_y}{I}$$

↓ derivative

$$\dot{W}_x = \left(W_{x0} - \alpha \frac{I_y}{I}\right) \alpha \sin(\alpha t) - B_1 \alpha \sin(\alpha t)$$

$$\dot{W}_y = \left(W_{y0} + \alpha \frac{I_x}{I}\right) \alpha \sin(\alpha t) - B_2 \alpha \sin(\alpha t)$$

↓ Plug in $\dot{W}_x = \frac{I_x}{I} - \alpha W_y, \quad \dot{W}_y = \frac{I_y}{I} + \alpha W_x$, solve for $\frac{I_x}{I}$

$$\left(-W_{x0} \alpha + \frac{I_x}{I} \alpha^2 + B_2 \alpha\right) \sin(\alpha t) + \left(B_1 \alpha + W_{y0} \alpha + \alpha^2 \frac{I_y}{I}\right) \cos(\alpha t) - \alpha^2 \frac{I_y}{I} = \frac{I_x}{I}$$

@t=0 this will simplify $\alpha(B_2 + W_{y0}) = \frac{I_x}{I}, \quad B_2 = \frac{I_x}{\alpha I} - W_{y0}$

$\frac{1}{2} \alpha(B_1 - W_{x0}) = \frac{I_y}{I}, \quad B_1 = \frac{I_y}{\alpha I} + W_{x0}$

$$W_x = \cos(\alpha t) W_{x0} + \alpha \left(-\frac{I_y}{I} \cos(\alpha t) + \frac{I_x}{I}\right) + \sin(\alpha t) \left(\frac{I_x}{\alpha I} + W_{y0}\right)$$

$$W_y = \cos(\alpha t) W_{y0} + \alpha \left(\frac{I_x}{I} \cos(\alpha t) - \frac{I_y}{I}\right) + \sin(\alpha t) \left(\frac{I_y}{\alpha I} + W_{x0}\right)$$

Problem 2

1. $Jx = \lambda x$

$$\bar{x}^T Jx = \bar{x}^T \lambda x$$

$$(J\bar{x})^T = \bar{x}^T J^T = \bar{x}^T J$$

$$(J\bar{x})^T x = \bar{x}^T \lambda x$$

$$(Jx)^T \bar{x} = x^T \bar{\lambda} \bar{x}$$

$$(\lambda x)^T \bar{x} = \bar{\lambda} x^T \bar{x}$$

$$\lambda x^T \bar{x} = \bar{\lambda} x^T \bar{x}$$

$$\boxed{\lambda = \bar{\lambda}} \text{ so } \lambda \text{ is real}$$

2. $x^T Jx > 0$

$$x^T Jx = x^T \lambda x > 0$$

$$x^T \lambda x = \lambda x^T x = \lambda \|x\|^2 > 0 \Rightarrow \boxed{\lambda > 0}$$

3. a $\lambda_1 x_1^T x_2 = \lambda_2 x_1^T x_2 \Leftarrow \text{Prove}$

$$\lambda_1 x_1^T x_2 = (\lambda_1 x_1)^T x_2 = (Jx_1)^T x_2 = x_1^T (Jx_2)$$

3. b $= \boxed{x_1^T \lambda_2 x_2}$, since $\lambda_1 x_1^T x_2 = \lambda_2 x_1^T x_2$, $\lambda_1 \neq \lambda_2$,

$$\boxed{x_1^T x_2 = 0}$$

HW6

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$$J = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{x}^T J \underline{x} = 2x_1^2 - 2x_1x_2 + 3x_2^2 + x_3^2$$

$$|2x_1x_2| < 2x_1^2 \text{ if } x_1 > x_2 \Rightarrow 2x_1^2 - 2x_1x_2 > 0$$

$$|2x_1x_2| < 2x_2^2 \text{ if } x_2 \gg x_1 \Rightarrow 3x_2^2 - 2x_1x_2 \gg 0$$

$$\text{if } x_1 > x_2$$

$$2x_1^2 - 2x_1x_2 + 3x_2^2 + x_3^2 > 0 + 3x_2^2 + x_3^2 > 0$$

$$\text{if } x_2 \gg x_1$$

$$2x_1^2 - 2x_1x_2 + 3x_2^2 + x_3^2 \gg 2x_1^2 + 0 + x_3^2 > 0$$