CHAPTER

# INTERPLANETARY TRAJECTORIES

# 8

## 8.1 INTRODUCTION

In this chapter, we consider some basic aspects of planning interplanetary missions. We begin by considering Hohmann transfers, which are the easiest to analyze and the most energy efficient. The orbits of the planets involved must lie in the same plane and the planets must be positioned just right for a Hohmann transfer to be used. The time between such opportunities is derived. The method of patched conics is employed to divide the mission up into three parts: the hyperbolic departure trajectory relative to the home planet, the cruise ellipse relative to the sun, and the hyperbolic arrival trajectory relative to the target planet.

The use of patched conics is justified by calculating the radius of a planet's sphere of influence and showing how small it is on the scale of the solar system. Matching the velocity of the spacecraft at the home planet's sphere of influence to that required to initiate the outbound cruise phase and then specifying the periapse radius of the departure hyperbola determines the delta-v requirement at departure. The sensitivity of the target radius to the burnout conditions is discussed. Matching the velocities at the target planet's sphere of influence and specifying the periapse of the arrival hyperbola yields the delta-v at the target for a planetary rendezvous or the direction of the outbound hyperbola for a planetary flyby. Flyby maneuvers are discussed, including the effect of leading- and trailing-side flybys, and some noteworthy examples of the use of gravity assist maneuvers are presented.

The chapter concludes with an analysis of the situation in which the planets' orbits are not coplanar and the transfer ellipse is tangent to neither orbit. This is akin to the chase maneuver in Chapter 6 and requires the solution of Lambert's problem using Algorithm 5.2.

## 8.2 INTERPLANETARY HOHMANN TRANSFERS

As can be seen from Table A.1, the orbits of most of the planets in the solar system lie very close to the earth's orbital plane (the ecliptic plane). The innermost planet, Mercury, and the formerly outermost planet, Pluto (which was reclassified by the International Astronomical Union as a *dwarf planet* in 2006), differ most in inclination (7° and 17°, respectively). The orbital planes of the other planets lie within 3.5° degrees of the ecliptic. It is also evident from Table A.1 that most of the planetary orbits have small eccentricities, the exceptions once again being Mercury and Pluto.

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Besides Pluto, there are currently four other IAU-recognized dwarf planets orbiting the Sun (namely, Ceres (the smallest), Haumea, Makemake, and Eris (the largest)). Ceres lies in the asteroid belt between Mars and Jupiter. Reclassified as a dwarf planet in 2006, it was visited by NASA's *Dawn* spacecraft in 2015. The three other dwarfs lie at the outer reaches of the solar system, beyond Neptune's orbit. Eris is roughly the size of Pluto, but its highly elliptical orbit takes it far beyond that of Pluto. All of the dwarf planets have orbits that are significantly inclined to the ecliptic, from 10.6° for Ceres to 44.2° for Eris.

To simplify the beginning of our study of interplanetary trajectories, we will focus on the eight major planets and assume that all of their orbits are circular and coplanar. In Section 8.10, we will relax this assumption.

The most energy-efficient way for a spacecraft to transfer from one planet's orbit to another is to use a Hohmann transfer ellipse (Section 6.3). Consider Fig. 8.1, which shows a Hohmann transfer from an inner planet 1 to an outer planet 2. The departure point $D$ is at periapsis (perihelion) of the transfer ellipse and the arrival point is at apoapsis (aphelion). The circular orbital speed of planet 1 relative to the sun is given by Eq. (2.63),

$$V_1 = \sqrt{\frac{\mu_{\text{sun}}}{R_1}} \tag{8.1}$$

The specific angular momentum $h$ of the transfer ellipse relative to the sun is found from Eq. (6.2), so that the heliocentric speed $V_D^{(v)}$ of the space vehicle on the transfer ellipse at the departure point $D$ is

$$V_D^{(v)} = \frac{h}{R_1} = \sqrt{2\mu_{\text{sun}}}\sqrt{\frac{R_2}{R_1(R_1+R_2)}} \tag{8.2}$$

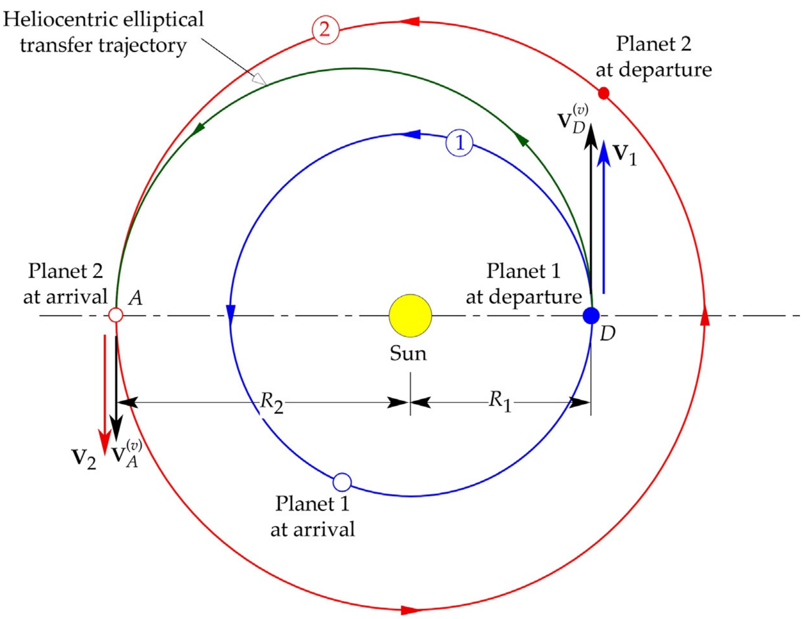


**FIG. 8.1**

Hohmann transfer from inner planet 1 to outer planet 2. $V_D^{(v)} > V_1$ and $V_A^{(v)} < V_2$.

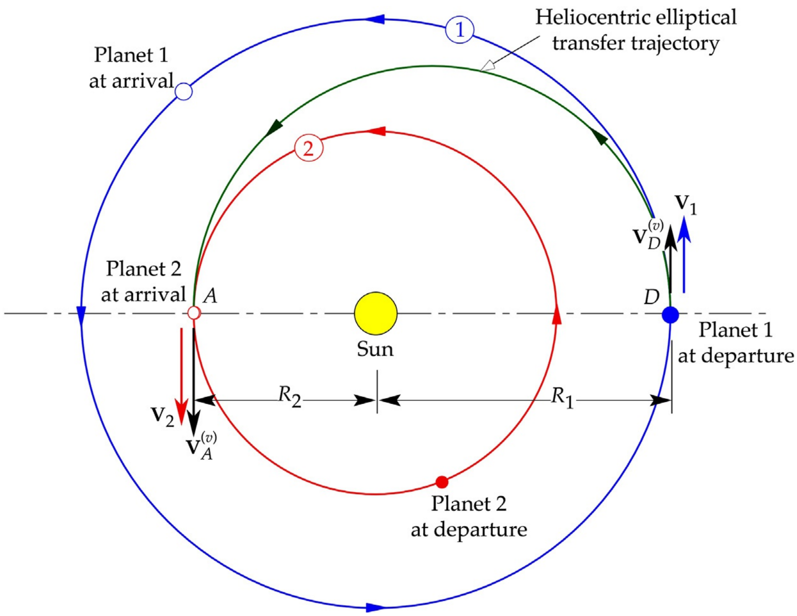


**FIG. 8.2**

Hohmann transfer from outer planet 1 to inner planet 2. $V_D^{(v)} < V_1$ and $V_A^{(v)} > V_2$.

This is greater than the speed $V_1$ of the planet. Therefore, the required spacecraft delta-v at $D$ is

$$\Delta V_D = V_D^{(v)} - V_1 = \sqrt{\frac{\mu_{\text{sun}}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}} - 1\right) \tag{8.3}$$

Likewise, the delta-v at the arrival point $A$ is

$$\Delta V_A = V_2 - V_A^{(v)} = \sqrt{\frac{\mu_{\text{sun}}}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1+R_2}}\right) \tag{8.4}$$

This velocity increment, like that at point $D$, is positive since planet 2 is traveling faster than the spacecraft at point $A$.

For a mission from an outer planet to an inner planet, as illustrated in Fig. 8.2, the delta-v's computed using Eqs. (8.3) and (8.4) will both be negative instead of positive. This is because the departure point and the arrival point are now at aphelion and perihelion, respectively, of the transfer ellipse. The speed of the spacecraft must be reduced for it to drop into the lower energy transfer ellipse at the departure point $D$, and it must be reduced again at point $A$ to arrive in the lower energy circular orbit of planet 2.

## 8.3 RENDEZVOUS OPPORTUNITIES

The purpose of an interplanetary mission is for the spacecraft to not only intercept a planet's orbit but also to rendezvous with the planet when it gets there. For rendezvous to occur at the end of a Hohmann transfer, the location of planet 2 in its orbit at the time of the spacecraft's departure from planet 1 must

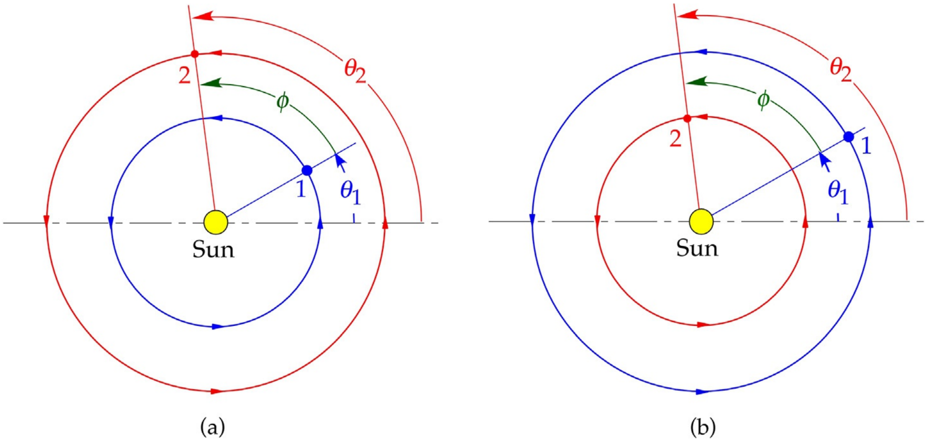


**FIG. 8.3**

Planets in circular orbits around the sun. (a) Planet 2 outside the orbit of planet 1. (b) Planet 2 inside the orbit of planet 1.

be such that planet 2 arrives at the apse line of the transfer ellipse at the same time as the spacecraft does. Phasing maneuvers (Section 6.5) are clearly not practical, especially for manned missions, due to the large periods of the heliocentric orbits.

Consider planet 1 and planet 2 in circular orbits around the sun, as shown in Fig. 8.3. Since the orbits are circular, we can choose a common horizontal apse line from which to measure the true anomaly $\theta$. The true anomalies of planets 1 and 2, respectively, are

$$\theta_1 = \theta_1)_0 + n_1 t \tag{8.5}$$

$$\theta_2 = \theta_2)_0 + n_2 t \tag{8.6}$$

where $n_1$ and $n_2$ are the mean motions (angular velocities) of the planets and $\theta_1)_0$, and $\theta_2)_0$ are their true anomalies at time $t = 0$. The phase angle between the position vectors of the two planets is defined as

$$\phi = \theta_2 - \theta_1 \tag{8.7}$$

$\phi$ is the angular position of planet 2 relative to planet 1. Substituting Eqs. (8.5) and (8.6) into Eq. (8.7) we get

$$\phi = \phi_0 + (n_2 - n_1)t \tag{8.8}$$

where $\phi_0$ is the phase angle at time zero, and $n_2 - n_1$ is the orbital angular velocity of planet 2 relative to planet 1. If the orbit of planet 1 lies inside that of planet 2, as in Fig. 8.3a, then $n_1 > n_2$. Therefore, the relative angular velocity $n_2 - n_1$ is negative, which means planet 2 moves clockwise relative to planet 1. On the other hand, if planet 1 is outside planet 2, then $n_2 - n_1$ is positive, so that the relative motion is counterclockwise.

The phase angle obviously varies linearly with time according to Eq. (8.8). If the phase angle is $\phi_0$ at $t = 0$, how long will it take to become $\phi_0$ again? The answer: when the position vector of planet 2 rotates through $2\pi$ radians relative to planet 1. The time required for the phase angle to return to its initial value is called the synodic period, which is denoted $T_{syn}$. For the case shown in Fig. 8.3a, in which the relative motion is clockwise, $T_{syn}$ is the time required for $\phi$ to change from $\phi_0$ to $\phi_0 - 2\pi$. From Eq. (8.8) we have

$$\phi_0 - 2\pi = \phi_0 + (n_2 - n_1)T_{syn}$$

so that

$$T_{syn} = \frac{2\pi}{n_1 - n_2} \quad (n_1 > n_2)$$

For the situation illustrated in Fig. 8.3b ($n_2 > n_1$), $T_{syn}$ is the time required for $\phi$ to go from $\phi_0$ to $\phi_0 + 2\pi$, in which case Eq. (8.8) yields

$$T_{syn} = \frac{2\pi}{n_2 - n_1} \quad (n_2 > n_1)$$

Both cases are covered by writing

$$T_{syn} = \frac{2\pi}{|n_1 - n_2|} \tag{8.9}$$

Recalling Eq. (3.9), we can write $n_1 = 2\pi/T_1$ and $n_2 = 2\pi/T_2$. Thus, in terms of the orbital periods of the two planets,

$$T_{syn} = \frac{T_1 T_2}{|T_1 - T_2|} \tag{8.10}$$

Observe that $T_{syn}$ is the orbital period of planet 2 relative to planet 1.

## EXAMPLE 8.1

Calculate the synodic period of Mars relative to that of the earth.

### Solution

In Table A.1 we find the orbital periods of earth and Mars:

$$T_{earth} = 365.26\,\text{days}\,(1\,\text{year})$$
$$T_{Mars} = 1\text{ year plus }321.73\text{ days} = 687.99\text{ days}$$

Hence,

$$T_{syn} = \frac{T_{earth}T_{Mars}}{|T_{earth} - T_{Mars}|} = \frac{365.26 \times 687.99}{|365.26 - 687.99|} = \boxed{777.9\,\text{days}}$$

These are earth days (1 day = 24 h). Therefore, it takes 2.13 years for a given configuration of Mars relative to the earth to occur again.

Fig. 8.4 depicts a mission from planet 1 to planet 2. Following a heliocentric Hohmann transfer, the spacecraft intercepts and undergoes rendezvous with planet 2. Later it returns to planet 1 by means of another Hohmann transfer. The major axis of the heliocentric transfer ellipse is the sum of the radii of

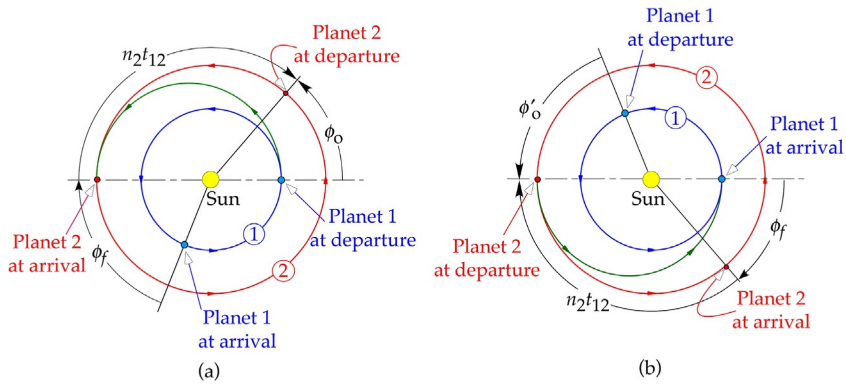


**FIG. 8.4**

Round trip mission, with layover, to planet 2. (a) Departure and rendezvous with planet 2. (b) Return and rendezvous with planet 1.

the two planets' orbits, $R_1 + R_2$. The time $t_{12}$ required for the transfer is one-half the period of the ellipse. Hence, according to the period formula (Eq. 2.83),

$$t_{12} = \frac{\pi}{\sqrt{\mu_{\text{sun}}}}\left(\frac{R_1 + R_2}{2}\right)^{3/2} \tag{8.11}$$

During the time it takes the spacecraft to fly from orbit 1 to orbit 2, through an angle of $\pi$ radians, planet 2 must move around its circular orbit and end up at a point directly opposite planet 1's position when the spacecraft departed. Since planet 2's angular velocity is $n_2$, the angular distance traveled by the planet during the spacecraft's trip is $n_2t_{12}$. Hence, as can be seen from Fig. 8.4a, the initial phase angle $\phi_0$ between the two planets is

$$\phi_0 = \pi - n_2t_{12} \tag{8.12}$$

When the spacecraft arrives at planet 2, the phase angle will be $\phi_f$, which is found using Eqs. (8.8) and (8.12).

$$\begin{aligned}\phi_f &= \phi_0 + (n_2 - n_1)t_{12} = (\pi - n_2t_{12}) + (n_2 - n_1)t_{12} \\ \phi_f &= \pi - n_1t_{12}\end{aligned} \tag{8.13}$$

For the situation illustrated in Fig. 8.4, planet 2 ends up being behind planet 1 by an amount equal to the magnitude of $\phi_f$.

At the start of the return trip, illustrated in Fig. 8.4b, planet 2 must be $\phi_0'$ radians ahead of planet 1. Since the spacecraft flies the same Hohmann transfer trajectory back to planet 1, the time of flight is $t_{12}$, the same as the outbound leg. Therefore, the distance traveled by planet 1 during the return trip is the same as the outbound leg, which means

$$\phi_0' = -\phi_f \tag{8.14}$$

In any case, the phase angle at the beginning of the return trip must be the negative of the phase angle at arrival from planet 1. The time required for the phase angle to reach its proper value is called the wait time, $t_{\text{wait}}$. Setting time equal to zero at the instant we arrive at planet 2, Eq. (8.8) becomes

$$\phi = \phi_f + (n_2 - n_1)t$$

$\phi$ becomes $-\phi_f$ after the time $t_{\text{wait}}$. That is,

$$-\phi_f = \phi_f + (n_2 - n_1)t_{\text{wait}}$$

or

$$t_{\text{wait}} = \frac{-2\phi_f}{n_2 - n_1} \tag{8.15}$$

where $\phi_f$ is given by Eq. (8.13). Eq. (8.15) may yield a negative result, which means the desired phase relation occurred in the past. Therefore, we must add or subtract an integral multiple of $2\pi$ to the numerator to get a positive value for $t_{\text{wait}}$. Specifically, if $N = 0, 1, 2, \ldots$, then

$$t_{\text{wait}} = \frac{-2\phi_f - 2\pi N}{n_2 - n_1} \quad (n_1 > n_2) \tag{8.16}$$

$$t_{\text{wait}} = \frac{-2\phi_f + 2\pi N}{n_2 - n_1} \quad (n_1 < n_2) \tag{8.17}$$

where $N$ is chosen to make $t_{\text{wait}}$ positive. $t_{\text{wait}}$ would probably be the smallest positive number thus obtained.

## EXAMPLE 8.2

Calculate the minimum wait time for initiating a return trip from Mars to earth.

### Solution

From Tables A.1 and A.2 we have

$$R_{\text{earth}} = 149.6 \times 10^6 \text{ km}$$
$$R_{\text{Mars}} = 227.9 \times 10^6 \text{ km}$$
$$\mu_{\text{sun}} = 132.71 \times 10^9 \text{ km}^3/\text{s}^2$$

According to Eq. (8.11), the time of flight from earth to Mars is

$$t_{12} = \frac{\pi}{\sqrt{\mu_{\text{sun}}}}\left(\frac{R_{\text{earth}} + R_{\text{Mars}}}{2}\right)^{3/2}$$
$$= \frac{\pi}{\sqrt{132.71 \times 10^9}}\left(\frac{149.6 \times 10^6 + 227.9 \times 10^6}{2}\right)^{3/2} = 2.2362 \times 10^7 \text{ s}$$

or

$$t_{12} = 258.82 \text{ days}$$

From Eq. (3.9) and the orbital periods of earth and Mars (see Example 8.1 above) we obtain the mean motions of the earth and Mars

$$n_{\text{earth}} = \frac{2\pi}{365.26} = 0.017202 \text{ rad/day}$$
$$n_{\text{Mars}} = \frac{2\pi}{687.99} = 0.0091327 \text{ rad/day}$$

The phase angle between earth and Mars when the spacecraft reaches Mars is given by Eq. (8.13).

$$\phi_f = \pi - n_{\text{earth}} t_{12} = \pi - 0.017202 \cdot 258.82 = -1.3107\,\text{rad}$$

Since $n_{\text{earth}} > n_{\text{Mars}}$, we choose Eq. (8.16) to find the wait time

$$t_{\text{wait}} = \frac{-2\phi_f - 2\pi N}{n_{\text{Mars}} - n_{\text{earth}}} = \frac{-2(-1.3107) - 2\pi N}{0.0091327 - 0.017202} = 778.65N - 324.85\,\text{days}$$

$N = 0$ yields a negative value, which we cannot accept. Setting $N = 1$, we find

$$\boxed{t_{\text{wait}} = 453.8\,\text{days}}$$

This is the minimum wait time (1.24 years). Obviously, we could set $N = 2, 3, \ldots$ to obtain longer wait times.

For a spacecraft to depart on a mission to Mars by means of a Hohmann (minimum energy) transfer, the phase angle between earth and Mars must be that given by Eq. (8.12). Using the results of Example 8.2, we find it to be

$$\phi_0 = \pi - n_{\text{Mars}} t_{12} = \pi - 0.0091327 \cdot 258.82 = 0.7778\,\text{rad} = 44.57°$$

This opportunity occurs once every synodic period, which we found to be 2.13 years in Example 8.1. In Example 8.2, we found that the time to fly to Mars is 258.8 days, followed by a wait time of 453.8 days, followed by a return trip time of 258.8 days. Hence, the minimum total time for a manned Mars mission, using Hohmann transfers is

$$t_{\text{total}} = 258.8 + 453.8 + 258.8 = 971.4\,\text{days} = 2.66\,\text{years}$$

## 8.4 SPHERE OF INFLUENCE

The sun, of course, is the dominant celestial body in the solar system. It is over 1000 times more massive than the largest planet, Jupiter, and has a mass of over 300,000 earths. The sun's gravitational pull holds all the planets in its grasp according to Newton's law of gravity (Eq. 1.31). However, near a given planet, the influence of its own gravity exceeds that of the sun. For example, at its surface the earth's gravitational force is over 1600 times greater than the sun's. The inverse-square nature of the law of gravity means that the force of gravity $F_g$ drops off rapidly with distance $r$ from the center of attraction. If $F_{g0}$ is the gravitational force at the surface of a planet with radius $r_0$, then Fig. 8.5 shows how rapidly the force diminishes with distance. At 10 body radii, the force is 1% of its value at the surface. Eventually, the force of the sun's gravitational field overwhelms that of the planet.

To estimate the radius of a planet's gravitational sphere of influence, consider the three-body system comprising a planet $p$ of mass $m_p$, the sun $s$ of mass $m_s$, and a space vehicle $v$ of mass $m_v$ illustrated in Fig. 8.6. The position vectors of the planet and spacecraft relative to an inertial frame centered at the sun are $\mathbf{R}$ and $\mathbf{R}_v$, respectively. The position vector of the space vehicle relative to the planet is $\mathbf{r}$. (Throughout this chapter we will use uppercase letters to represent position, velocity, and acceleration measured relative to the sun and lowercase letters when they are measured relative to a planet.) The gravitational force exerted on the vehicle by the planet is denoted $\mathbf{F}_p^{(v)}$, and that exerted by the

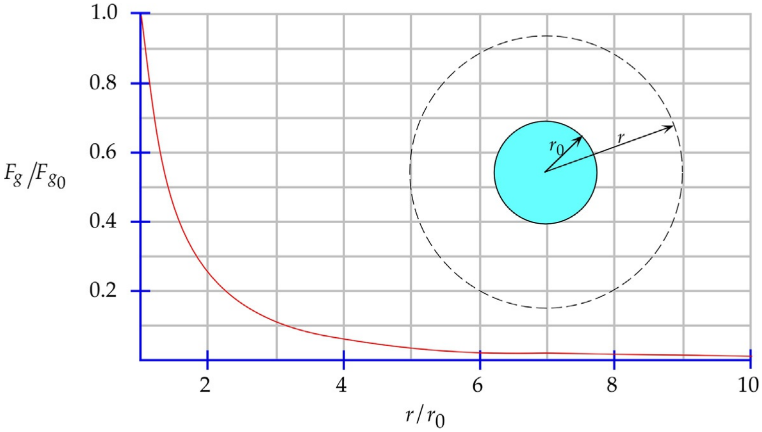


**FIG. 8.5**

Decrease of gravitational force with distance from a planet's surface.

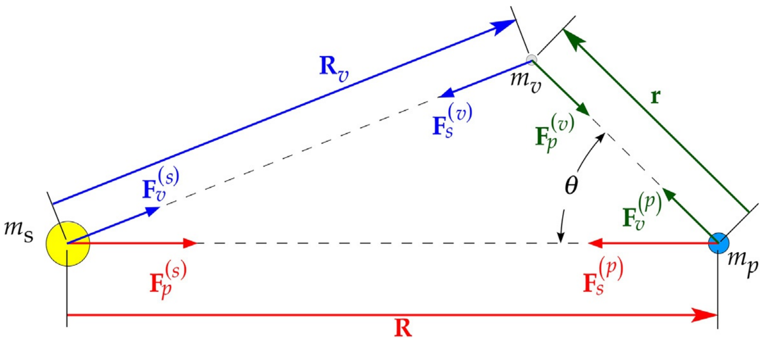


**FIG. 8.6**

Relative position and gravitational force vectors among the three bodies.

sun is $\mathbf{F}_s^{(v)}$. Likewise, the forces on the planet are $\mathbf{F}_s^{(p)}$ and $\mathbf{F}_v^{(p)}$, whereas on the sun we have $\mathbf{F}_v^{(s)}$ and $\mathbf{F}_p^{(s)}$. According to Newton's law of gravitation (Eq. 2.10), these forces are

$$\mathbf{F}_p^{(v)} = -\frac{Gm_v m_p}{r^3}\mathbf{r} \tag{8.18a}$$

$$\mathbf{F}_s^{(v)} = -\frac{Gm_v m_s}{R_v{}^3}\mathbf{R}_v \tag{8.18b}$$

$$\mathbf{F}_s^{(p)} = -\frac{Gm_p m_s}{R^3}\mathbf{R} \tag{8.18c}$$

Observe that

$$\mathbf{R}_v = \mathbf{R} + \mathbf{r} \tag{8.19}$$

From Fig. 8.6 and the law of cosines we see that the magnitude of $\mathbf{R}_v$ is

$$R_v = \left(R^2 + r^2 - 2Rr\cos\theta\right)^{1/2} = R\left[1 - 2(r/R)\cos\theta + (r/R)^2\right]^{1/2} \tag{8.20}$$

We expect that within the planet's sphere of influence, $r/R \ll 1$. In that case, the terms involving $r/R$ in Eq. (8.20) can be neglected, so that, approximately,

$$R_v = R \tag{8.21}$$

The equation of motion of the spacecraft relative to the sun-centered inertial frame is

$$m_v \ddot{\mathbf{R}}_v = \mathbf{F}_s^{(v)} + \mathbf{F}_p^{(v)}$$

Solving for $\ddot{\mathbf{R}}_v$ and substituting the gravitational forces given by Eqs. (8.18a) and (8.18b), we get

$$\ddot{\mathbf{R}}_v = \frac{1}{m_v}\left(-\frac{Gm_v m_s}{R_v{}^3}\mathbf{R}_v\right) + \frac{1}{m_v}\left(-\frac{Gm_v m_p}{r^3}\mathbf{r}\right) = -\frac{Gm_s}{R_v{}^3}\mathbf{R}_v - \frac{Gm_p}{r^3}\mathbf{r} \tag{8.22}$$

Let us write this as

$$\ddot{\mathbf{R}}_v = \mathbf{A}_s + \mathbf{P}_p \tag{8.23}$$

where

$$\mathbf{A}_s = -\frac{Gm_s}{R_v{}^3}\mathbf{R}_v \qquad \mathbf{P}_p = -\frac{Gm_p}{r^3}\mathbf{r} \tag{8.24}$$

$\mathbf{A}_s$ is the primary gravitational acceleration of the vehicle due to the sun, and $\mathbf{P}_p$ is the secondary or perturbing acceleration due to the planet. The magnitudes of $\mathbf{A}_s$ and $\mathbf{P}_p$ are

$$A_s = \frac{Gm_s}{R^2} \qquad P_p = \frac{Gm_p}{r^2} \tag{8.25}$$

where we made use of the approximation given by Eq. (8.21). The ratio of the perturbing acceleration to the primary acceleration is, therefore,

$$\frac{P_p}{A_s} = \frac{\dfrac{Gm_p}{r^2}}{\dfrac{Gm_s}{R^2}} = \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2 \tag{8.26}$$

The equation of motion of the planet relative to the inertial frame is

$$m_p \ddot{\mathbf{R}} = \mathbf{F}_v^{(p)} + \mathbf{F}_s^{(p)}$$

Solving for $\ddot{\mathbf{R}}$, noting that $\mathbf{F}_v^{(p)} = -\mathbf{F}_p^{(v)}$, and using Eqs. (8.18b) and (8.18c) yields

$$\ddot{\mathbf{R}} = \frac{1}{m_p}\left(\frac{Gm_v m_p}{r^3}\mathbf{r}\right) + \frac{1}{m_p}\left(-\frac{Gm_p m_s}{R^3}\mathbf{R}\right) = \frac{Gm_v}{r^3}\mathbf{r} - \frac{Gm_s}{R^3}\mathbf{R} \tag{8.27}$$

Subtracting Eq. (8.27) from Eq. (8.22) and collecting terms, we find

$$\ddot{\mathbf{R}}_v - \ddot{\mathbf{R}} = -\frac{Gm_p}{r^3}\mathbf{r}\left(1 + \frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v{}^3}\left[\mathbf{R}_v - \left(\frac{R_v}{R}\right)^3\mathbf{R}\right]$$

Recalling Eq. (8.19), we can write this as

$$\ddot{\mathbf{r}} = -\frac{Gm_p}{r^3}\mathbf{r}\left(1+\frac{m_v}{m_p}\right) - \frac{Gm_s}{R_v{}^3}\left\{\mathbf{r} + \left[1-\left(\frac{R_v}{R}\right)^3\right]\mathbf{R}\right\} \tag{8.28}$$

This is the equation of motion of the vehicle relative to the planet. By using Eq. (8.21) and the fact that $m_v \ll m_p$, we can write this in approximate form as

$$\ddot{\mathbf{r}} = \mathbf{a}_p + \mathbf{p}_s \tag{8.29}$$

where

$$\mathbf{a}_p = -\frac{Gm_p}{r^3}\mathbf{r} \qquad \mathbf{p}_s = -\frac{Gm_s}{R^3}\mathbf{r} \tag{8.30}$$

In this case $\mathbf{a}_p$ is the primary gravitational acceleration of the vehicle due to the planet and $\mathbf{p}_s$ is the perturbation caused by the sun. The magnitudes of these vectors are

$$a_p = \frac{Gm_p}{r^2} \qquad p_s = \frac{Gm_s}{R^3}r \tag{8.31}$$

The ratio of the perturbing acceleration to the primary acceleration is

$$\frac{p_s}{a_p} = \frac{Gm_s\dfrac{r}{R^3}}{\dfrac{Gm_p}{r^2}} = \frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 \tag{8.32}$$

For motion relative to the planet, the ratio $p_s/a_p$ is a measure of the deviation of the vehicle's orbit from the Keplerian orbit arising from the planet acting by itself ($p_s/a_p = 0$). Likewise, $P_p/A_s$ is a measure of the planet's influence on the orbit of the vehicle relative to the sun. If

$$\frac{p_s}{a_p} < \frac{P_p}{A_s} \tag{8.33}$$

then the perturbing effect of the sun on the vehicle's orbit around the planet is less than the perturbing effect of the planet on the vehicle's orbit around the sun. We say that the vehicle is therefore within the planet's sphere of influence. Substituting Eqs. (8.26) and (8.32) into Eq. (8.33) yields

$$\frac{m_s}{m_p}\left(\frac{r}{R}\right)^3 < \frac{m_p}{m_s}\left(\frac{R}{r}\right)^2$$

which means

$$\left(\frac{r}{R}\right)^5 < \left(\frac{m_p}{m_s}\right)^2$$

or

$$\frac{r}{R} < \left(\frac{m_p}{m_s}\right)^{2/5}$$

Let $r_{\mathrm{SOI}}$ be the radius of the sphere of influence. Within the planet's sphere of influence, defined by

$$\frac{r_{\mathrm{SOI}}}{R} = \left(\frac{m_p}{m_s}\right)^{2/5} \tag{8.34}$$

the motion of the spacecraft is determined by its equations of motion relative to the planet (Eq. 8.28). Outside the sphere of influence, the path of the spacecraft is computed relative to the sun (Eq. 8.22).

The sphere of influence radius presented in Eq. (8.34) is not an exact quantity. It is simply a reasonable estimate of the distance beyond which the sun's gravitational attraction dominates that of a planet. The spheres of influence of all the planets and the earth's moon are listed in Table A.2.

### EXAMPLE 8.3

Calculate the radius of the earth's sphere of influence.

#### Solution

In Table A.1 we find

$$m_{\text{earth}} = 5.974\left(10^{24}\right)\text{kg}$$
$$m_{\text{sun}} = 1.989\left(10^{30}\right)\text{kg}$$
$$R_{\text{earth}} = 149.6\left(10^{6}\right)\text{km}$$

Substituting these data into Eq. (8.34) yields

$$r_{\text{SOI}} = 149.6\times10^{6}\left(\frac{5.974\times10^{24}}{1.989\times10^{24}}\right)^{2/5} = \boxed{925\times10^{6}\,\text{km}}$$

Since the radius of the earth is 6378 km,

$$r_{\text{SOI}} = 145\,\text{earth radii}$$

Relative to the earth, its sphere of influence is very large. However, relative to the sun it is tiny, as illustrated in Fig. 8.7.

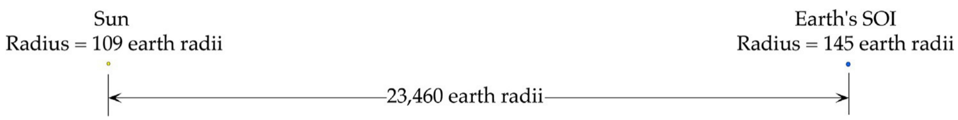


**FIG. 8.7**

The earth's sphere of influence and the sun, drawn to scale.

## 8.5 METHOD OF PATCHED CONICS

"Conics" refers to the fact that two-body, or Keplerian, orbits are conic sections with the focus at the attracting body. To study an interplanetary trajectory, we assume that when the spacecraft is outside the sphere of influence of a planet it follows an unperturbed Keplerian orbit around the sun. Because interplanetary distances are so vast, for heliocentric orbits we may neglect the size of the spheres of influence and consider them, like the planets they surround, to be just points in space coinciding with the planetary centers. Within each planetary sphere of influence, the spacecraft travels an unperturbed Keplerian path about the planet. While the sphere of influence appears as a mere speck on the scale of the solar system, from the point of view of the planet it is very large indeed and may be considered to lie at infinity.

To analyze a mission from planet 1 to planet 2 using the method of patched conics, we first determine the heliocentric trajectory, such as the Hohmann transfer ellipse discussed in Section 8.2, that will intersect the desired positions of the two planets in their orbits. This trajectory takes the spacecraft from the sphere of influence of planet 1 to that of planet 2. At the spheres of influence, the heliocentric velocities of the transfer orbit are computed relative to the planet to establish the velocities "at infinity," which are then used to determine planetocentric departure trajectory at planet 1 and arrival trajectory at planet 2. In this way, we "patch" together the three conics, one centered at the sun and the other two centered at the planets in question.

Whereas the method of patched conics is remarkably accurate for interplanetary trajectories, such is not the case for lunar rendezvous and return trajectories. The orbit of the moon is determined primarily by the earth, whose sphere of influence extends well beyond the moon's 384,400-km orbital radius. To apply patched conics to lunar trajectories we ignore the sun and consider the motion of a spacecraft as influenced by just the earth and moon, as in the restricted three-body problem discussed in Section 2.12. The size of the moon's sphere of influence is found using Eq. (8.34), with the earth playing the role of the sun:

$$r_{\mathrm{SOI}} = R\left(\frac{m_{\mathrm{moon}}}{m_{\mathrm{earth}}}\right)^{2/5}$$

where $R$ is the radius of the moon's orbit. Thus, using Table A.1,

$$r_{\mathrm{SOI}} = 384{,}400\left[\frac{73.48\left(10^{21}\right)}{5974\left(10^{21}\right)}\right]^{2/5} = 66{,}200\,\mathrm{km}$$

as recorded in Table A.2. The moon's sphere of influence extends out to over one-sixth of the distance to the earth. We can hardly consider it to be a mere speck relative to the earth. Another complication is the fact that the earth and the moon are somewhat comparable in mass, so that their center of mass lies almost three-quarters of an earth radius from the center of the earth. The motion of the moon cannot be accurately described as rotating around the center of the earth.

Complications such as these place the analysis of cislunar trajectories beyond the scope of this chapter. (In Example 2.18, we did a lunar trajectory calculation not by using patched conics but by integrating the equations of motion of a spacecraft within the context of the restricted three-body problem.) We extend the patched conic technique to lunar trajectories in Chapter 9.

## 8.6 PLANETARY DEPARTURE

To escape the gravitational pull of a planet, the spacecraft must travel a hyperbolic trajectory relative to the planet, arriving at its sphere of influence with a relative speed $v_\infty$ (hyperbolic excess speed) greater than zero. On a parabolic trajectory, according to Eq. (2.91), the spacecraft will arrive at the sphere of influence ($r = \infty$) with a relative speed of zero. In that case, the spacecraft remains in the same orbit as the planet and does not embark upon a heliocentric elliptical path.

Fig. 8.8 shows a spacecraft departing on a Hohmann trajectory from planet 1 toward a target planet 2, which is farther away from the sun (as in Fig. 8.1). On crossing the sphere of influence, the heliocentric velocity $\mathbf{V}_D^{(v)}$ of the spacecraft is parallel to the asymptote of the departure hyperbola as well as to the planet's heliocentric velocity vector $\mathbf{V}_1$. $\mathbf{V}_D^{(v)}$ and $\mathbf{V}_1$ must be parallel and in the same direction for

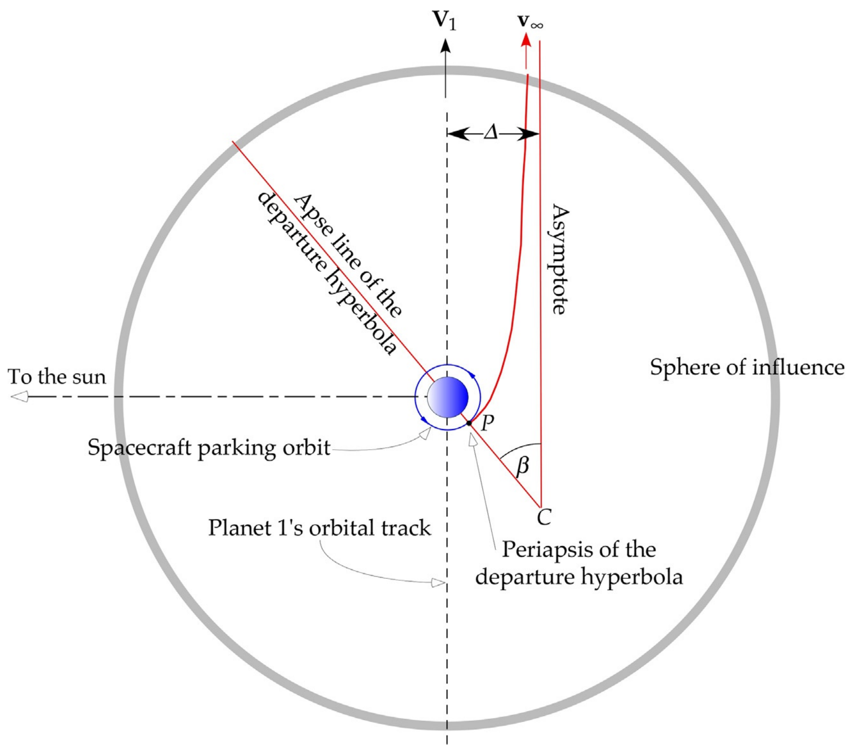


**FIG. 8.8**

Departure of a spacecraft on a mission from an inner planet to an outer planet.

a Hohmann transfer such that $\Delta V_D$ in Eq. (8.3) is positive. Clearly, $\Delta V_D$ is the hyperbolic excess speed of the departure hyperbola,

$$v_\infty = \sqrt{\frac{\mu_{\text{sun}}}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}}-1\right) \tag{8.35}$$

It would be well at this point for the reader to review Section 2.9 on hyperbolic trajectories and compare Fig. 8.8 and Fig. 2.25. Recall that point $C$ is the center of the hyperbola.

A space vehicle is ordinarily launched into an interplanetary trajectory from a circular parking orbit. The radius of this parking orbit equals the periapse radius $r_p$ of the departure hyperbola. According to Eq. (2.50), the periapsis radius is given by

$$r_p = \frac{h^2}{\mu_1}\frac{1}{1+e} \tag{8.36}$$

where $h$ is the angular momentum of the departure hyperbola (relative to the planet), $e$ is the eccentricity of the hyperbola, and $\mu_1$ is the planet's gravitational parameter. The hyperbolic excess speed is found in Eq. (2.115), from which we obtain

$$h = \frac{\mu_1\sqrt{e^2-1}}{v_\infty} \tag{8.37}$$

Substituting this expression for the angular momentum into Eq. (8.36) and solving for eccentricity yields

$$e = 1 + \frac{r_p {v_\infty}^2}{\mu_1} \tag{8.38}$$

We place this result back into Eq. (8.37) to obtain the following expression for the angular momentum:

$$h = r_p\sqrt{{v_\infty}^2 + \frac{2\mu_1}{r_p}} \tag{8.39}$$

Since the hyperbolic excess speed is specified by the mission requirements (Eq. 8.35), choosing a departure periapsis $r_p$ yields the parameters $e$ and $h$ of the departure hyperbola. From the angular momentum, we get the periapsis speed,

$$v_p = \frac{h}{r_p} = \sqrt{{v_\infty}^2 + \frac{2\mu_1}{r_p}} \tag{8.40}$$

which can also be found from an energy approach using Eq. (2.113). With Eq. (8.40) and the speed of the circular parking orbit (Eq. 2.63),

$$v_c = \sqrt{\frac{\mu_1}{r_p}} \tag{8.41}$$

we can calculate the delta-v required to put the vehicle onto the hyperbolic departure trajectory,

$$\Delta v = v_p - v_c = v_c\left(\sqrt{2 + \left(\frac{v_\infty}{v_c}\right)^2} - 1\right) \tag{8.42}$$

The location of periapsis, where the delta-v maneuver must occur, is found using Eq. (2.99) and Eq. (8.38),

$$\beta = \cos^{-1}\left(\frac{1}{e}\right) = \cos^{-1}\left(\frac{1}{1 + \dfrac{r_p {v_\infty}^2}{\mu_1}}\right) \tag{8.43}$$

$\beta$ gives the orientation of the apse line of the hyperbola to the planet's heliocentric velocity vector.

It should be pointed out that the only requirement on the orientation of the plane of the departure hyperbola is that it must contain the center of mass of the planet as well as the relative velocity vector $\mathbf{v}_\infty$. Therefore, as shown in Fig. 8.9, the hyperbola can be rotated about a line *A–A*, which passes through the planet's center of mass and is parallel to $\mathbf{v}_\infty$ (or $\mathbf{V}_1$, which of course is parallel to $\mathbf{v}_\infty$ for Hohmann transfers). Rotating the hyperbola in this way sweeps out a surface of revolution on which all possible departure hyperbolas lie. The periapsis of the hyperbola traces out a circle which, for the specified periapsis radius $r_p$, is the locus of all possible points of injection into a departure trajectory toward the target planet. This circle is the base of a cone with its vertex at the center of the planet. From Fig. 2.25 we can determine that its radius is $r_p \sin\beta$, where $\beta$ is given just above in Eq. (8.43).

The plane of the parking orbit, or direct ascent trajectory, must contain the line *A–A* and the launch site at the time of launch. The possible inclinations of a prograde orbit range from a minimum of $i_{\min}$, where $i_{\min}$ is the latitude of the launch site, to $i_{\max}$, which cannot exceed 90°. Launch site safety

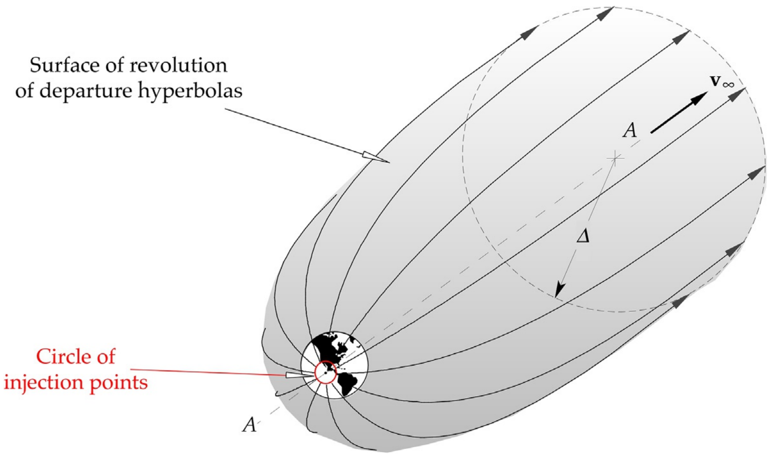


**FIG. 8.9**

Locus of possible departure trajectories for a given $v_\infty$ and $r_p$.

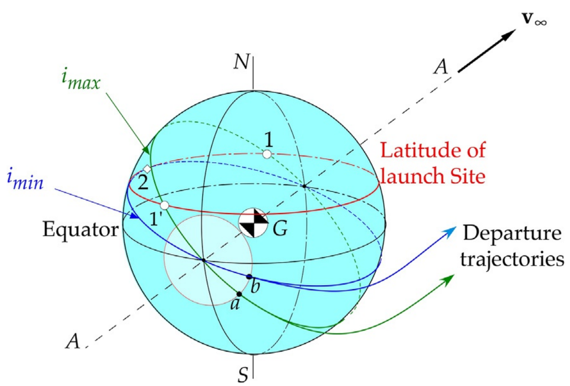


**FIG. 8.10**

Parking orbits and departure trajectories for a launch site at a given latitude.

considerations may place additional limits on this range. For example, orbits originating from the Kennedy Space Center in Florida (latitude 28.5°), are limited to inclinations between 28.5° and 52.5°. For the scenario illustrated in Fig. 8.10, the location of the launch site limits access to just the departure trajectories having periapses lying between *a* and *b*. The figure shows that there are two times per day—when the planet rotates the launch site through positions 1 and 1′—that a spacecraft can be launched into a parking orbit. These times are closer together (the launch window is smaller), the lower the inclination of the parking orbit.

Once a spacecraft is established in its parking orbit, then an opportunity for launch into the departure trajectory occurs at each orbital circuit.

If the mission is to send a spacecraft from an outer planet to an inner planet, as in Fig. 8.2, then the spacecraft's heliocentric speed $V_D^{(v)}$ at departure must be less than that of the planet. That means the

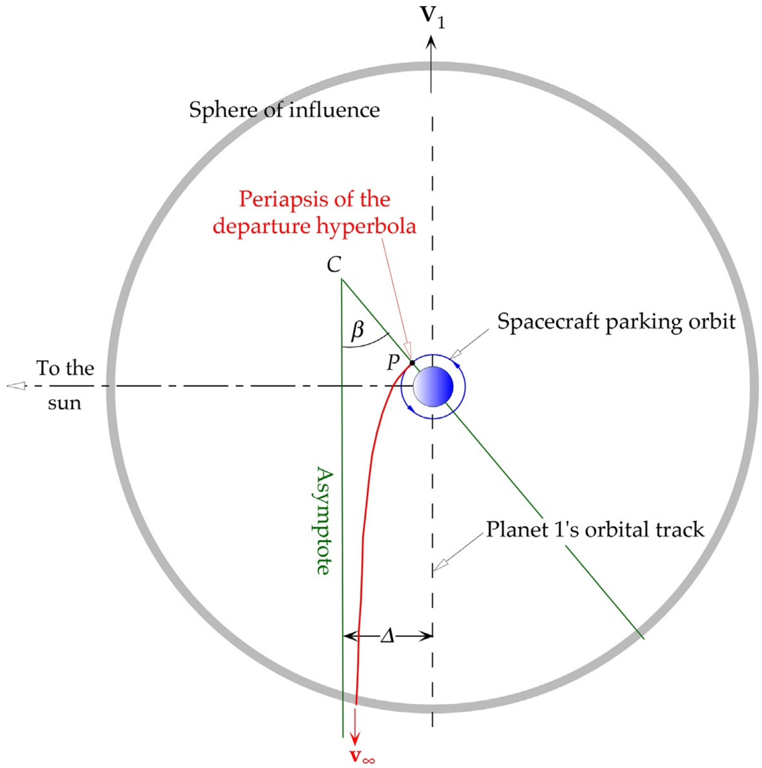


**FIG. 8.11**

Departure of a spacecraft on a trajectory from an outer planet to an inner planet.

spacecraft must emerge from the backside of the sphere of influence with its relative velocity vector $\mathbf{v}_\infty$ directed opposite to $\mathbf{V}_1$, as shown in Fig. 8.11. Figs. 8.9 and 8.10 apply to this situation as well.

## EXAMPLE 8.4

A spacecraft is launched on a mission to Mars starting from a 300-km circular parking orbit. Calculate (a) the delta-v required, (b) the location of perigee of the departure hyperbola, and (c) the amount of propellant required as a percentage of the spacecraft mass before the delta-v burn, assuming a specific impulse of 300 s.

### Solution

From Tables A.1 and A.2, we obtain the gravitational parameters for the sun and the earth,

$$\mu_{\text{sun}} = 1.327\left(10^{11}\right)\text{km}^3/\text{s}^2$$
$$\mu_{\text{earth}} = 398{,}600\,\text{km}^3/\text{s}^2$$

and the orbital radii of the earth and Mars,

$$R_{\text{earth}} = 149.6\left(10^6\right)\text{km}$$
$$R_{\text{Mars}} = 227.9\left(10^6\right)\text{km}$$

(a) According to Eq. (8.35), the hyperbolic excess speed is

$$v_\infty = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{earth}}}}\left(\sqrt{\frac{2R_{\text{Mars}}}{R_{\text{earth}}+R_{\text{Mars}}}}-1\right) = \sqrt{\frac{1.327(10^{11})}{149.6(10^6)}}\left(\sqrt{\frac{2\cdot 227.9(10^6)}{149.6(10^6)+227.9(10^6)}}-1\right)$$

from which

$$v_\infty = 2.943\,\text{km/s}$$

The speed of the spacecraft in its 300-km circular parking orbit is given by Eq. (8.41),

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{R_{\text{earth}}+300}} = \sqrt{\frac{398{,}600}{6678}} = 7.726\,\text{km/s}$$

Finally, we use Eq. (8.42) to calculate the delta-v required to step up to the departure hyperbola

$$\Delta v = v_c\left(\sqrt{2+\left(\frac{v_\infty}{v_c}\right)^2}-1\right) = 7.726\left(\sqrt{2+\left(\frac{2.943}{7.726}\right)^2}-1\right)$$

$$\boxed{\Delta v = 3.590\,\text{km/s}}$$

(b) Perigee of the departure hyperbola, relative to the earth's orbital velocity vector, is found using Eq. (8.43),

$$\beta = \cos^{-1}\left(\frac{1}{1+\dfrac{r_p {v_\infty}^2}{\mu_{\text{earth}}}}\right) = \cos^{-1}\left(\frac{1}{1+\dfrac{6678\cdot 2.943^2}{368{,}600}}\right)$$

$$\boxed{\beta = 29.16^\circ}$$

Fig. 8.12 shows that the perigee can be located on either the sunlit or the dark side of the earth. It is likely that the parking orbit would be a prograde orbit (west to east), which would place the burnout point on the dark side.

(c) From Eq. (6.1), we have

$$\frac{\Delta m}{m} = 1 - \exp\left(-\frac{\Delta v}{I_{\text{sp}} g_0}\right)$$

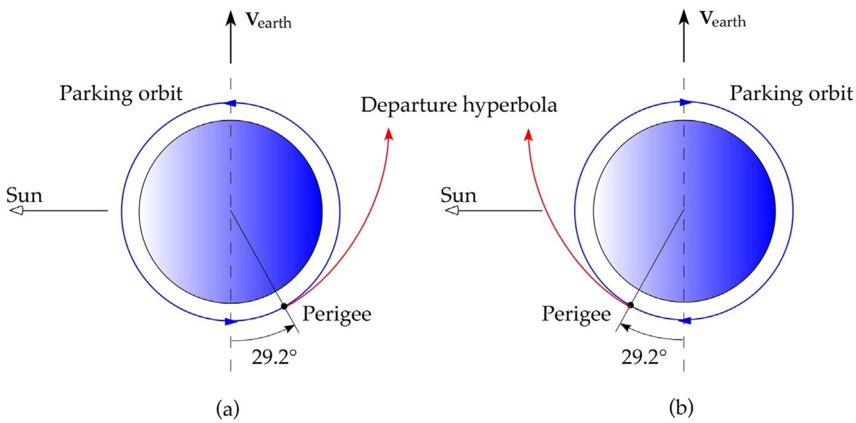


**FIG. 8.12**

Departure trajectory to Mars initiated from (a) the dark side and (b) the sunlit side of the earth (Example 8.4).

Substituting $\Delta v = 3.590\,\text{km/s}$, $I_{sp} = 300\,\text{s}$, and $g_0 = 9.81(10^{-3})\,\text{km/s}^2$, this yields

$$\boxed{\frac{\Delta m}{m} = 0.705}$$

That is, prior to the delta-v maneuver over, 70% of the spacecraft mass must be propellant.

## 8.7 SENSITIVITY ANALYSIS

The initial maneuvers required to place a spacecraft on an interplanetary trajectory occur well within the sphere of influence of the departure planet. Since the sphere of influence is just a point on the scale of the solar system, we may ask what effects small errors in position and velocity at the maneuver point have on the trajectory. Assuming the mission is from an inner to an outer planet, let us consider the effect that small changes in the burnout velocity $v_p$ and radius $r_p$ have on the target radius $R_2$ of the heliocentric Hohmann transfer ellipse (see Figs. 8.1 and 8.8).

$R_2$ is the radius of aphelion, so we use Eq. (2.70) to obtain

$$R_2 = \frac{h^2}{\mu_{\text{sun}}}\frac{1}{1-e}$$

Substituting $h = R_1 V_D^{(v)}$ and $e = (R_2 - R_1)/(R_2 + R_1)$, and solving for $R_2$, yields

$$R_2 = \frac{R_1^2\left(V_D^{(v)}\right)^2}{2\mu_{\text{sun}} - R_1\left(V_D^{(v)}\right)^2} \tag{8.44}$$

(This expression holds as well for a mission from an outer to an inner planet.) The change $\delta R_2$ in $R_2$ due to a small variation $\delta V_D^{(v)}$ of $V_D^{(v)}$ is

$$\delta R_2 = \frac{dR_2}{dV_D^{(v)}}\delta V_D^{(v)} = \frac{4R_1^2\mu_{\text{sun}}}{\left[2\mu_{\text{sun}} - R_1\left(V_D^{(v)}\right)^2\right]^2} V_D^{(v)}\delta V_D^{(v)}$$

Dividing this equation by Eq. (8.44) leads to

$$\frac{\delta R_2}{R_2} = \frac{2}{1 - \dfrac{R_1\left(V_D^{(v)}\right)^2}{2\mu_{\text{sun}}}}\frac{\delta V_D^{(v)}}{V_D^{(v)}} \tag{8.45}$$

The departure speed $V_D^{(v)}$ of the space vehicle is the sum of the planet's speed $V_1$ and excess speed $v_\infty$

$$V_D^{(v)} = V_1 + v_\infty$$

We can solve Eq. (8.40) for $v_\infty$,

$$v_\infty = \sqrt{{v_p}^2 - \frac{2\mu_1}{r_p}}$$

Hence

$$V_D^{(v)} = V_1 + \sqrt{{v_p}^2 - \frac{2\mu_1}{r_p}} \tag{8.46}$$

The change in $V_D^{(v)}$ due to variations $\delta r_p$ and $\delta v_p$ of the burnout position (periapse) $r_p$ and speed $v_p$ is given by

$$\delta V_D^{(v)} = \frac{\partial V_D^{(v)}}{\partial r_p}\delta r_p + \frac{\partial V_D^{(v)}}{\partial v_p}\delta v_p \tag{8.47}$$

From Eq. (8.46), we obtain

$$\frac{\partial V_D^{(v)}}{\partial r_p} = \frac{\mu_1}{v_\infty r_p^2} \qquad \frac{\partial V_D^{(v)}}{\partial v_p} = \frac{v_p}{v_\infty}$$

Therefore,

$$\delta V_D^{(v)} = \frac{\mu_1}{v_\infty r_p^2}\delta r_p + \frac{v_p}{v_\infty}\delta v$$

Once again making use of Eq. (8.40), this can be written as follows:

$$\frac{\delta V_D^{(v)}}{V_D^{(v)}} = \frac{\mu_1}{V_D^{(v)} v_\infty r_p}\frac{\delta r_p}{r_p} + \frac{v_\infty + \dfrac{2\mu_1}{r_p v_\infty}}{V_D^{(v)}}\frac{\delta v_p}{v_p} \tag{8.48}$$

Substituting this into Eq. (8.45) finally yields the desired result: an expression for the variation of $R_2$ due to variations in $r_p$ and $v_p$

$$\frac{\delta R_2}{R_2} = \frac{2}{1 - \dfrac{R_1\left(V_D^{(v)}\right)^2}{2\mu_{\text{sun}}}}\left(\frac{\mu_1}{V_D^{(v)} v_\infty r_p}\frac{\delta r_p}{r_p} + \frac{v_\infty + \dfrac{2\mu_1}{r_p v_\infty}}{V_D^{(v)}}\frac{\delta v_p}{v_p}\right) \tag{8.49}$$

Consider a mission from earth to Mars, starting from a 300-km parking orbit. We have

$$\begin{aligned}
\mu_{\text{sun}} &= 1.327\left(10^{11}\right)\text{km}^3/\text{s}^2 \\
\mu_1 &= \mu_{\text{earth}} = 398{,}600\,\text{km}^3/\text{s}^2 \\
R_1 &= 149.6\left(10^6\right)\text{km} \\
R_2 &= 227.9\left(10^6\right)\text{km} \\
r_p &= 6678\,\text{km}
\end{aligned}$$

In addition, from Eqs. (8.1) and (8.2),

$$V_1 = V_{\text{earth}} = \sqrt{\frac{\mu_{\text{sun}}}{R_1}} = \sqrt{\frac{1.327\left(10^{11}\right)}{149.6\left(10^6\right)}} = 29.78\,\text{km/s}$$

$$\begin{aligned}
V_D^{(v)} &= \sqrt{2\mu_{\text{sun}}}\sqrt{\frac{R_2}{R_1(R_1+R_2)}} = \sqrt{2\cdot 1.327\left(10^{11}\right)}\sqrt{\frac{227.9\left(10^6\right)}{149.6\left(10^6\right)\left[149.6\left(10^6\right)+227.9\left(10^6\right)\right]}} \\
&= 32.73\,\text{km/s}
\end{aligned}$$

Therefore,

$$v_\infty = V_D^{(v)} - V_{\text{earth}} = 2.943\,\text{km/s}$$

and, from Eq. (8.40),

$$v_p = \sqrt{v_\infty{}^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{2.943^2 + \frac{2 \cdot 398{,}600}{6678}} = 11.32\,\text{km/s}$$

Substituting these values into Eq. (8.49) yields

$$\frac{\delta R_2}{R_2} = 3.127\frac{\delta r_p}{r_p} + 6.708\frac{\delta v_p}{v_p}$$

This expression shows that a 0.01% variation (1.1 m/s) in the burnout speed $v_p$ changes the target radius $R_2$ by 0.067% or 153,000 km! Likewise, an error of 0.01% (0.67 km) in burnout radius $r_p$ produces an error of over 70,000 km. Thus, small errors that are likely to occur in the launch phase of the mission must be corrected by midcourse maneuvers during the coasting flight along the elliptical transfer trajectory.

## 8.8 PLANETARY RENDEZVOUS

A spacecraft arrives at the sphere of influence of the target planet with a hyperbolic excess velocity $\mathbf{v}_\infty$ relative to the planet. In the case illustrated in Fig. 8.1, a mission from an inner planet 1 to an outer planet 2 (e.g., earth to Mars), the spacecraft's heliocentric approach velocity $\mathbf{V}_A^{(v)}$ is smaller in magnitude than that of the planet, $\mathbf{V}_2$. Therefore, it crosses the forward portion of the sphere of influence, as shown in Fig. 8.13. For a Hohmann transfer, $\mathbf{V}_A^{(v)}$ and $\mathbf{V}_2$ are parallel, so the magnitude of the hyperbolic excess velocity is, simply,

$$v_\infty = V_2 - V_A^{(v)} \tag{8.50}$$

If the mission is as illustrated in Fig. 8.2, from an outer planet to an inner one (e.g., earth to Venus), then $V_A^{(v)}$ is greater than $V_2$, and the spacecraft must cross the rear portion of the sphere of influence, as shown in Fig. 8.14. In that case

$$v_\infty = V_A^{(v)} - V_2 \tag{8.51}$$

What happens after crossing the sphere of influence depends on the nature of the mission. If the goal is to impact the planet (or its atmosphere), the aiming radius $\Delta$ of the approach hyperbola must be such that the hyperbola's periapsis radius $r_p$ equals essentially the radius of the planet. If the intent is to go into orbit around the planet, then $\Delta$ must be chosen so that the delta-v burn at periapsis will occur at the correct altitude above the planet. If there is no impact with the planet and no drop into a capture orbit around the planet, then the spacecraft will simply continue past periapsis on a flyby trajectory, exiting the sphere of influence with the same relative speed $v_\infty$ as it entered, but with the velocity vector rotated through the turn angle $\delta$, given by Eq. (2.100),

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) \tag{8.52}$$

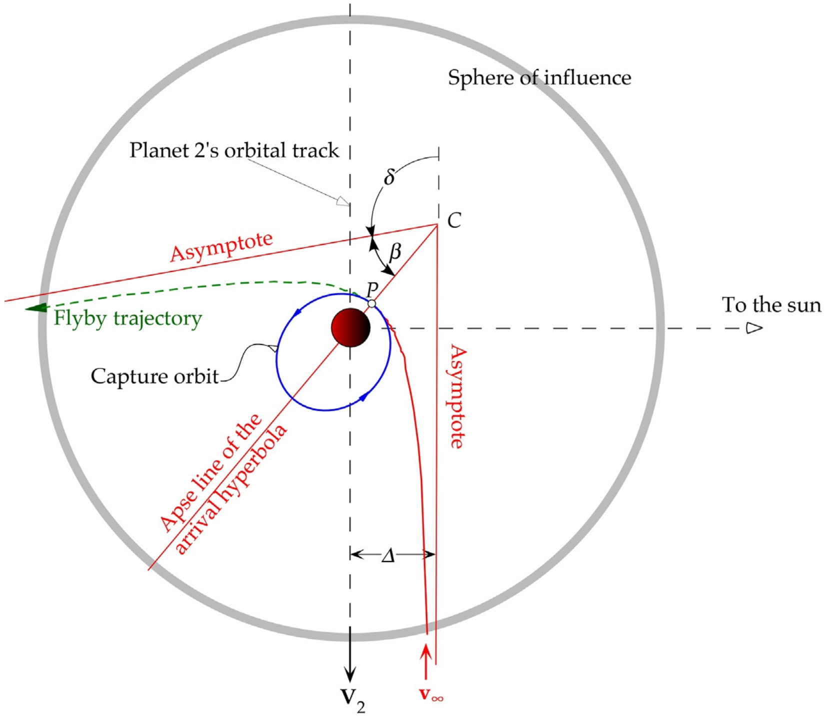


**FIG. 8.13**

Spacecraft approach trajectory for a Hohmann transfer to an outer planet from an inner one. *P* is the periapse of the approach hyperbola.

With the hyperbolic excess speed $v_\infty$ and the periapse radius $r_p$ specified, the eccentricity of the approach hyperbola is found from Eq. (8.38),

$$e = 1 + \frac{r_p {v_\infty}^2}{\mu_2} \tag{8.53}$$

where $\mu_2$ is the gravitational parameter of planet 2. Hence, the turn angle is

$$\delta = 2\sin^{-1}\left(\frac{1}{1 + \dfrac{r_p {v_\infty}^2}{\mu_2}}\right) \tag{8.54}$$

We can combine Eqs. (2.103) and (2.107) to obtain the following expression for the aiming radius:

$$\Delta = \frac{h^2}{\mu_2}\frac{1}{\sqrt{e^2 - 1}} \tag{8.55}$$

The angular momentum of the approach hyperbola relative to the planet is found using Eq. (8.39),

$$h = r_p\sqrt{{v_\infty}^2 + \frac{2\mu_2}{r_p}} \tag{8.56}$$

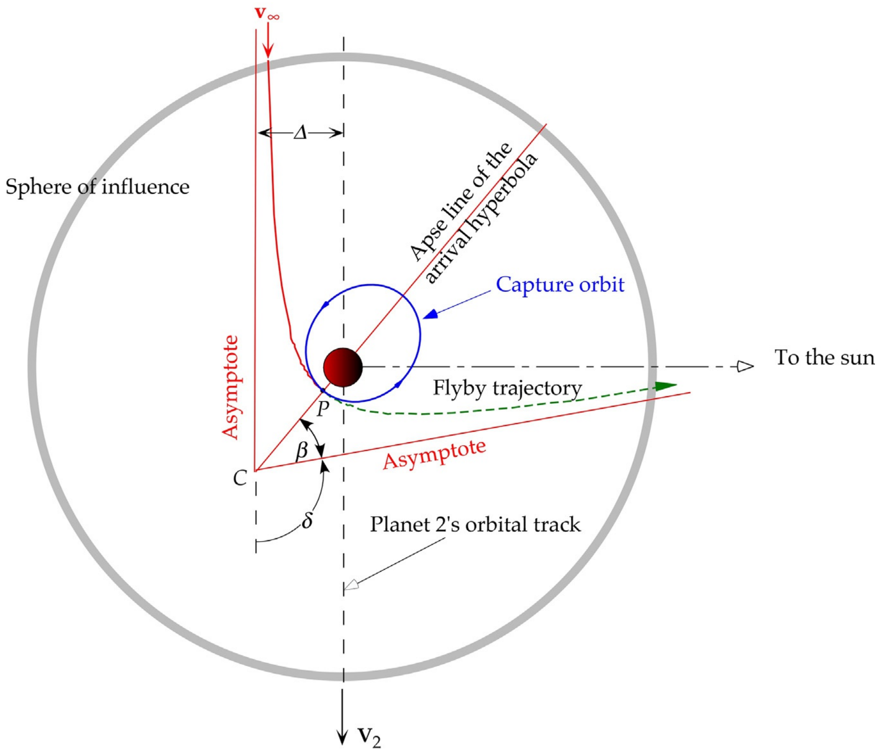


**FIG. 8.14**

Spacecraft approach trajectory for a Hohmann transfer to an inner planet from an outer one. *P* is the periapse of the approach hyperbola.

Substituting Eqs. (8.53) and (8.56) into Eq. (8.55) yields the aiming radius in terms of the periapse radius and the hyperbolic excess speed,

$$\Delta = r_p\sqrt{1+\frac{2\mu_2}{r_p v_\infty{}^2}} \tag{8.57}$$

Just as we observed when discussing departure trajectories, the approach hyperbola does not lie in a unique plane. We can rotate the hyperbolas illustrated in Figs. 8.11 and 8.12 about a line *A–A* parallel to $\mathbf{v}_\infty$ and passing through the target planet's center of mass, as shown in Fig. 8.15. The approach hyperbolas in that figure terminate at the circle of periapses. Fig. 8.16 is a plane through the solid of revolution revealing the shape of hyperbolas having a common $v_\infty$ but varying Δ.

Let us suppose that the purpose of the mission is to enter an elliptical orbit of eccentricity $e$ around the planet. This will require a delta-v maneuver at periapsis *P* (Figs. 8.13 and 8.14), which is also periapsis of the ellipse. The speed in the hyperbolic trajectory at periapsis is given by Eq. (8.40),

$$v_p)_{\text{hyp}} = \sqrt{v_\infty{}^2 + \frac{2\mu_2}{r_p}} \tag{8.58}$$

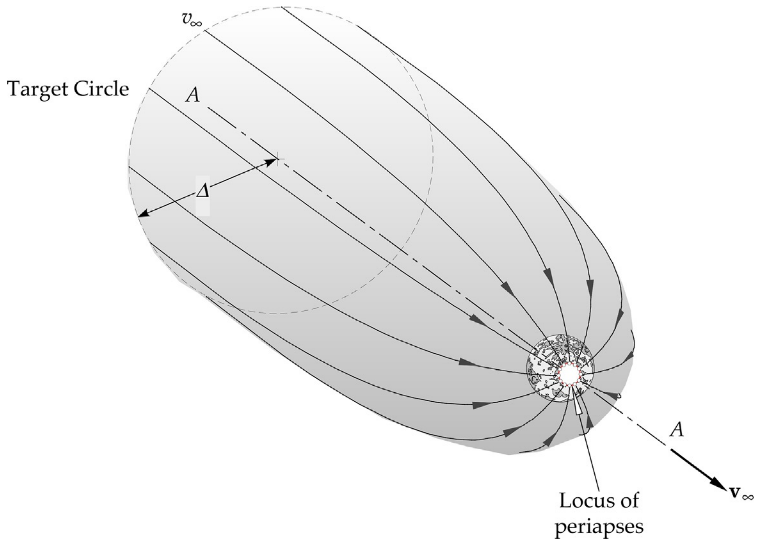


**FIG. 8.15**

Locus of approach hyperbolas to the target planet.

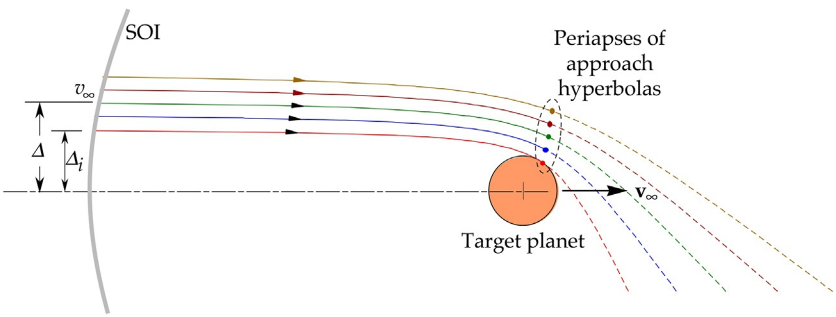


**FIG. 8.16**

Family of approach hyperbolas having the same $v_\infty$ but different Δ.

The velocity at periapsis of the capture orbit is found by setting $h = r_p v_p$ in Eq. (2.50) and solving for $v_p$

$$v_p)_{\text{capture}} = \sqrt{\frac{\mu_2(1+e)}{r_p}} \tag{8.59}$$

Hence, the required delta-v is

$$\Delta v = v_p)_{\text{hyp}} - v_p)_{\text{capture}} = \sqrt{v_\infty{}^2 + \frac{2\mu_2}{r_p}} - \sqrt{\frac{\mu_2(1+e)}{r_p}} \tag{8.60}$$

For a given $v_\infty$, $\Delta v$ clearly depends on the choice of periapse radius $r_p$ and capture orbit eccentricity $e$. Requiring the maneuver point to remain the periapsis of the capture orbit means that $\Delta v$ is maximum for a circular capture orbit and decreases with increasing eccentricity until $\Delta v = 0$, which, of course, means no capture (flyby).

To determine optimal capture radius, let us write Eq. (8.60) in nondimensional form as

$$\frac{\Delta v}{v_\infty} = \sqrt{1 + \frac{2}{\xi}} - \sqrt{\frac{1+e}{\xi}} \tag{8.61}$$

where

$$\xi = \frac{r_p {v_\infty}^2}{\mu_2} \tag{8.62}$$

The first and second derivatives of $\Delta v / v_\infty$ with respect to $\xi$ are

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\Delta v}{v_\infty} = \left( -\frac{1}{\sqrt{\xi+2}} + \frac{\sqrt{1+e}}{2} \right) \frac{1}{\xi^{3/2}} \tag{8.63}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\frac{\Delta v}{v_\infty} = \left( \frac{2\xi+3}{(\xi+2)^{3/2}} - \frac{3}{4}\sqrt{1+e} \right) \frac{1}{\xi^{5/2}} \tag{8.64}$$

Setting the first derivative equal to zero and solving for $\xi$ yields

$$\xi = 2\frac{1-e}{1+e} \tag{8.65}$$

Substituting this value of $\xi$ into Eq. (8.64), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\frac{\Delta v}{v_\infty} = \frac{\sqrt{2}}{64}\frac{(1+e)^3}{(1-e)^{3/2}} \tag{8.66}$$

This expression is positive for elliptical orbits ($0 \le e < 1$), which means that when $\xi$ is given by Eq. (8.65), $\Delta v$ is a minimum. Therefore, from Eq. (8.62), the optimal periapse radius as far as fuel expenditure is concerned is

$$r_p = \frac{2\mu_2}{{v_\infty}^2}\frac{1-e}{1+e} \tag{8.67}$$

We can combine Eqs. (2.50) and (2.70) to get

$$\frac{1-e}{1+e} = \frac{r_p}{r_a} \tag{8.68}$$

where $r_a$ is the apoapsis radius. Thus, Eq. (8.67) implies

$$r_a = \frac{2\mu_2}{{v_\infty}^2} \tag{8.69}$$

That is, the apoapsis of this capture ellipse is independent of the eccentricity and equals the radius of the optimal circular orbit.

Substituting Eq. (8.65) back into Eq. (8.61) yields the minimum $\Delta v$,

$$\Delta v = v_\infty \sqrt{\frac{1-e}{2}} \tag{8.70}$$

Finally, placing the optimal $r_p$ into Eq. (8.57) leads to an expression for the aiming radius required for minimum $\Delta v$,

$$\Delta = 2\sqrt{2}\frac{\sqrt{1-e}}{1+e}\frac{\mu_2}{{v_\infty}^2} = \sqrt{\frac{2}{1-e}}r_p \tag{8.71}$$

Clearly, the optimal $\Delta v$ (and periapsis height) are reduced for highly eccentric elliptical capture orbits ($e \to 1$). However, it should be pointed out that the use of optimal $\Delta v$ may have to be sacrificed in favor of a variety of other mission requirements.

## EXAMPLE 8.5

After a Hohmann transfer from earth to Mars, calculate
(a) the minimum delta-v required to place a spacecraft in orbit with a period of 7 h
(b) the periapsis radius
(c) the aiming radius
(d) the angle between periapsis and Mars' velocity vector.

### Solution

The following data are required from Tables A.1 and A.2:

$$\begin{aligned}
\mu_{\text{sun}} &= 1.327\left(10^{11}\right)\text{km}^3/\text{s}^2\\
\mu_{\text{Mars}} &= 42{,}830\,\text{km}^3/\text{s}^2\\
R_{\text{earth}} &= 149.6\left(10^6\right)\text{km}\\
R_{\text{Mars}} &= 227.9\left(10^6\right)\text{km}\\
r_{\text{Mars}} &= 3396\,\text{km}
\end{aligned}$$

(a) The hyperbolic excess speed is found using Eq. (8.4),

$$v_\infty = \Delta V_A = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Mars}}}}\left(1-\sqrt{\frac{2R_{\text{earth}}}{R_{\text{earth}}+R_{\text{Mars}}}}\right) = \sqrt{\frac{1.327\left(10^{11}\right)}{227.9\left(10^6\right)}}\left(1-\sqrt{\frac{2\cdot 149.6\left(10^6\right)}{149.6\left(10^6\right)+227.9\left(10^6\right)}}\right)$$

$$\therefore v_\infty = 2.648\,\text{km/s}$$

We can use Eq. (2.83) to express the semimajor axis $a$ of the capture orbit in terms of its period $T$,

$$a = \left(\frac{T\sqrt{\mu_{\text{Mars}}}}{2\pi}\right)^{2/3}$$

Substituting $T = 7\cdot 3600\,\text{s}$ yields

$$a = \left(\frac{25{,}200\sqrt{42{,}830}}{2\pi}\right)^{2/3} = 8832\,\text{km}$$

From Eq. (2.73) we obtain

$$a = \frac{r_p}{1-e}$$

On substituting the optimal periapsis radius (Eq. 8.67) this becomes

$$a = \frac{2\mu_{\text{Mars}}}{{v_\infty}^2}\frac{1}{1+e}$$

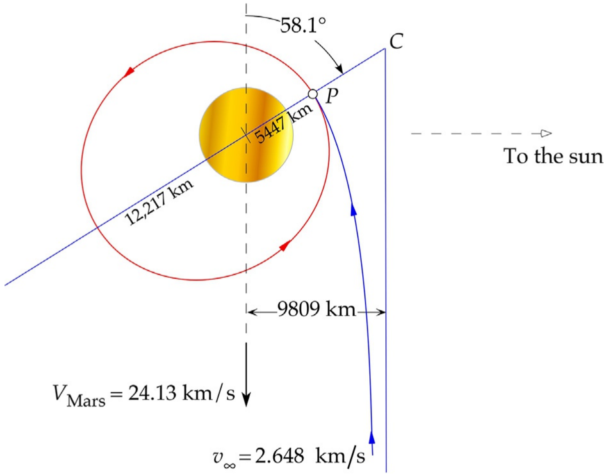


**FIG. 8.17**

An optimal approach to a Mars capture orbit with a 7-h period ($r_{\text{Mars}} = 3396$ km).

from which

$$e = \frac{2\mu_{\text{Mars}}}{a{v_\infty}^2} - 1 = \frac{2 \cdot 42{,}830}{8832 \cdot 2.648^2} - 1 = 0.3833$$

Thus, using Eq. (8.70), we find

$$\Delta v = v_\infty \sqrt{\frac{1-e}{2}} = 2.648\sqrt{\frac{1-03833}{2}} = \boxed{1.470\text{km/s}}$$

(b) From Eq. (8.66), we obtain the periapse radius

$$r_p = \frac{2\mu_{\text{Mars}}}{{v_\infty}^2}\frac{1-e}{1+e} = \frac{2 \cdot 42{,}830}{2.648^2}\frac{1-0.3833}{1+0.3833} = \boxed{5447\text{km}}$$

(c) The aiming radius is given by Eq. (8.71),

$$\Delta = r_p\sqrt{\frac{2}{1-e}} = 5447\sqrt{\frac{2}{1-0.3833}} = \boxed{9809\text{km}}$$

(d) Using Eq. (8.43), we get the angle to periapsis

$$\beta = \cos^{-1}\left(\frac{1}{1+\dfrac{r_p {v_\infty}^2}{\mu_{\text{Mars}}}}\right) = \cos^{-1}\left(\frac{1}{1+\dfrac{5447 \cdot 2.648^2}{42{,}830}}\right) = \boxed{58.09°}$$

Mars, the approach hyperbola, and the capture orbit are shown to scale in Fig. 8.17. The approach could also be made from the dark side of the planet instead of the sunlit side. The approach hyperbola and capture ellipse would be the mirror image of that shown, as is the case in Fig. 8.12.

## 8.9 PLANETARY FLYBY

A spacecraft that enters a planet's sphere of influence and does not impact the planet or go into orbit around it will continue in its hyperbolic trajectory through periapsis $P$ and exit the sphere of influence. Fig. 8.18 shows a hyperbolic flyby trajectory along with the asymptotes and apse line of the hyperbola. It is a leading-side flyby because the periapsis is on the side of the planet facing into the direction of the planet's motion. Likewise, Fig. 8.19 illustrates a trailing-side flyby. At the inbound crossing point, the heliocentric velocity $\mathbf{V}_1^{(v)}$ of the space vehicle equals the planet's heliocentric velocity $\mathbf{V}$ plus the hyperbolic excess velocity $\mathbf{v}_{\infty_1}$ of the spacecraft (relative to the planet),

$$\mathbf{V}_1^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_1} \tag{8.72}$$

Similarly, at the outbound crossing, we have

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} \tag{8.73}$$

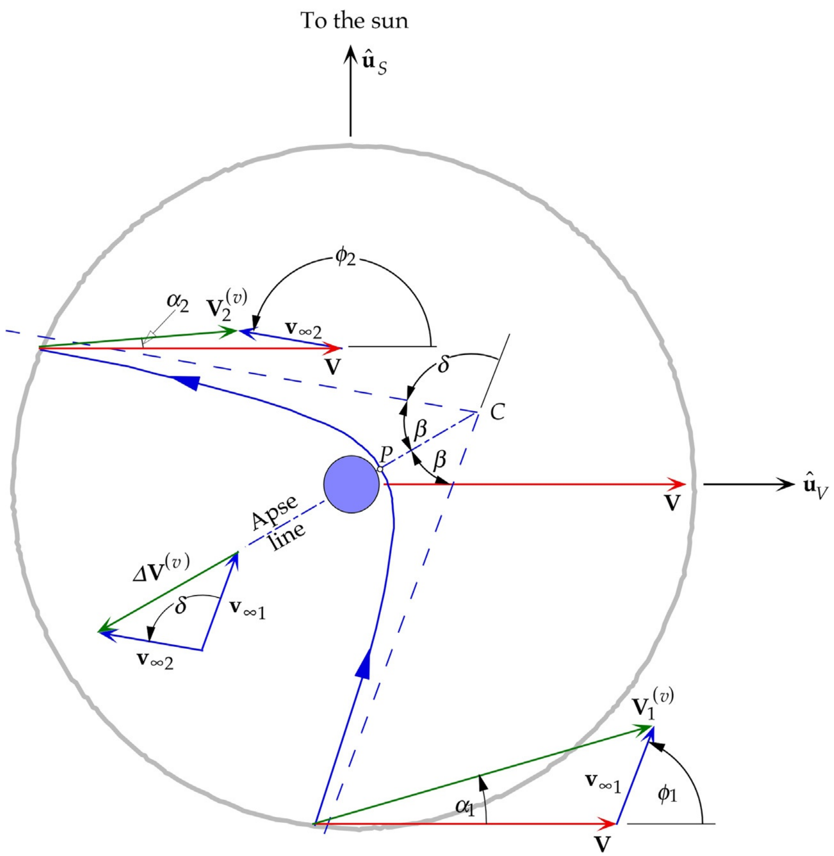


**FIG. 8.18**

Leading-side planetary flyby.

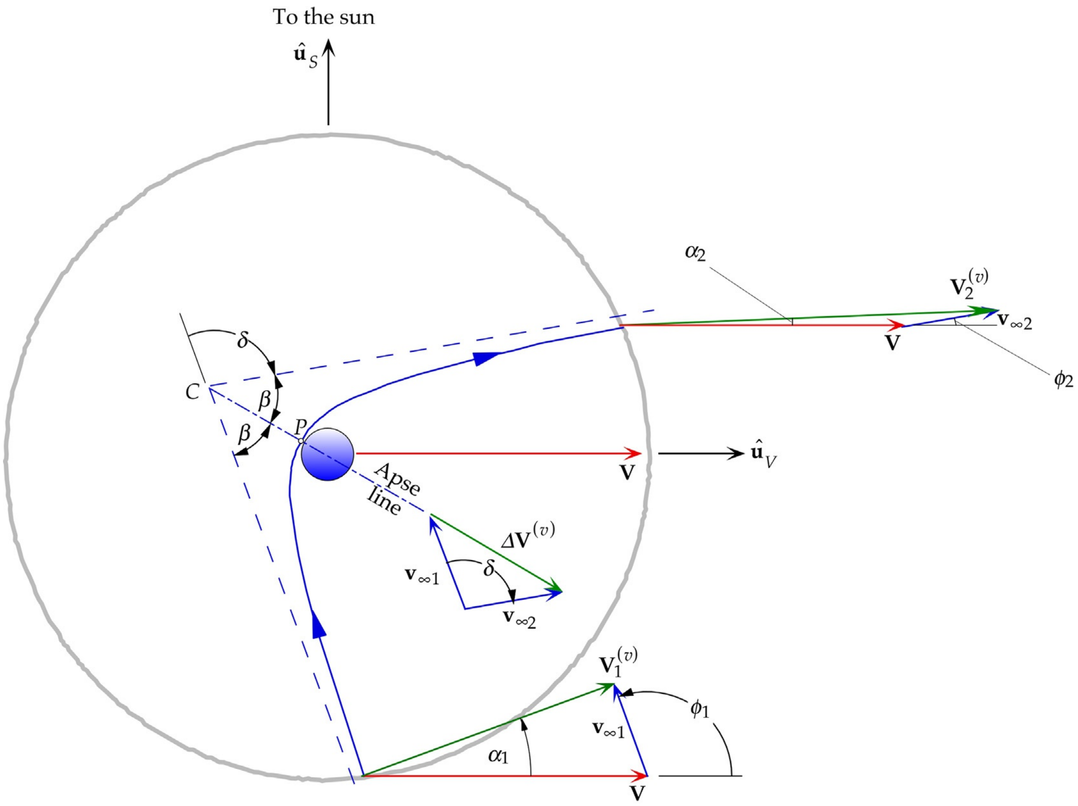


**FIG. 8.19**

Trailing-side planetary flyby.

The change $\Delta\mathbf{V}^{(v)}$ in the spacecraft's heliocentric velocity is

$$\Delta\mathbf{V}^{(v)} = \mathbf{V}_2^{(v)} - \mathbf{V}_1^{(v)} = (\mathbf{V} + \mathbf{v}_{\infty_2}) - (\mathbf{V} + \mathbf{v}_{\infty_1})$$

which means

$$\Delta\mathbf{V}^{(v)} = \mathbf{v}_{\infty_2} - \mathbf{v}_{\infty_1} = \Delta\mathbf{v}_{\infty} \tag{8.74}$$

The hyperbolic excess velocities $\mathbf{v}_{\infty_1}$ and $\mathbf{v}_{\infty_2}$ lie along the asymptotes of the hyperbola and are therefore inclined at the same angle $\beta$ to the apse line (see Fig. 2.25), with $\mathbf{v}_{\infty_1}$ pointing toward and $\mathbf{v}_{\infty_2}$ pointing away from the center $C$. They both have the same magnitude $v_\infty$, with $\mathbf{v}_{\infty_2}$ having simply rotated relative to $\mathbf{v}_{\infty_1}$ by the turn angle $\delta$. Hence, $\Delta\mathbf{v}_\infty$—and therefore $\Delta\mathbf{V}^{(v)}$—is a vector that lies along the apse line and always points away from periapsis, as illustrated in Figs. 8.18 and 8.19. From these figures it can be seen that, in a leading-side flyby, the component of $\Delta\mathbf{V}^{(v)}$ in the direction of the planet's velocity is negative, whereas for the trailing-side flyby, it is positive. This means that a leading-side flyby results in a decrease in the spacecraft's heliocentric speed. On the other hand, a trailing-side flyby increases that speed.

To analyze a flyby problem, we proceed as follows. First, let $\hat{\mathbf{u}}_V$ be the unit vector in the direction of the planet's heliocentric velocity $\mathbf{V}$ and let $\hat{\mathbf{u}}_S$ be the unit vector pointing from the planet to the sun. At the inbound crossing of the sphere of influence, the heliocentric velocity $\mathbf{V}_1^{(v)}$ of the spacecraft is

$$\mathbf{V}_1^{(v)} = V_1^{(v)}\Big)_V \hat{\mathbf{u}}_V + V_1^{(v)}\Big)_S \hat{\mathbf{u}}_S \tag{8.75}$$

where the scalar components of $\mathbf{V}_1^{(v)}$ are

$$V_1^{(v)}\Big)_V = V_1^{(v)} \cos\alpha_1 \qquad V_1^{(v)}\Big)_S = V_1^{(v)} \sin\alpha_1 \tag{8.76}$$

$\alpha_1$ is the angle between $\mathbf{V}_1^{(v)}$ and $\mathbf{V}$. All angles are measured positive counterclockwise. Referring to Fig. 2.12, we see that the magnitude of $\alpha_1$ is the flight path angle $\gamma$ of the spacecraft's heliocentric trajectory when it encounters the planet's sphere of influence (a mere speck) at the planet's distance $R$ from the sun. Furthermore,

$$V_1^{(v)}\Big)_V = V_{\perp_1}^{(v)} \qquad V_1^{(v)}\Big)_S = -V_{r_1}^{(v)} \tag{8.77}$$

${V_{\perp_1}}^{(v)}$ and ${V_{r_1}}^{(v)}$ are furnished by Eqs. (2.48) and (2.49),

$$V_{\perp_1}^{(v)} = \frac{\mu_{\text{sun}}}{h_1}(1 + e_1\cos\theta_1) \qquad V_{r_1}^{(v)} = \frac{\mu_{\text{sun}}}{h_1} e_1 \sin\theta_1 \tag{8.78}$$

in which $e_1$, $h_1$, and $\theta_1$ are the eccentricity, angular momentum, and true anomaly of the heliocentric approach trajectory, respectively.

The velocity of the planet relative to the sun is

$$\mathbf{V} = V\hat{\mathbf{u}}_V \tag{8.79}$$

where $V = \sqrt{\mu_{\text{sun}}/R}$. At the inbound crossing of the planet's sphere of influence, the hyperbolic excess velocity of the spacecraft is obtained from Eq. (8.72),

$$\mathbf{v}_{\infty_1} = \mathbf{V}_1^{(v)} - \mathbf{V}$$

Using this we find

$$\mathbf{v}_{\infty_1} = v_{\infty_1})_V \hat{\mathbf{u}}_V + v_{\infty_1})_S \hat{\mathbf{u}}_S \tag{8.80}$$

where the scalar components of $\mathbf{v}_{\infty_1}$ are

$$v_{\infty_1})_V = V_1^{(v)} \cos\alpha_1 - V \qquad v_{\infty_1})_S = V_1^{(v)} \sin\alpha_1 \tag{8.81}$$

$v_\infty$ is the magnitude of $\mathbf{v}_{\infty_1}$,

$$v_\infty = \sqrt{\mathbf{v}_{\infty_1} \cdot \mathbf{v}_{\infty_1}} = \sqrt{\left(V_1^{(v)}\right)^2 + V^2 - 2V_1^{(v)} V \cos\alpha_1} \tag{8.82}$$

At this point, $v_\infty$ is known, so that upon specifying the periapsis radius $r_p$ we can compute the angular momentum and eccentricity of the flyby hyperbola (relative to the planet), using Eqs. (8.38) and (8.39).

$$h = r_p\sqrt{{v_\infty}^2 + \frac{2\mu}{r_p}} \qquad e = 1 + \frac{r_p {v_\infty}^2}{\mu} \tag{8.83}$$

where $\mu$ is the gravitational parameter of the planet.

The angle between $\mathbf{v}_{\infty_1}$ and the planet's heliocentric velocity $\mathbf{V}$ is $\phi_1$. It is found using the components of $\mathbf{v}_{\infty_1}$ shown in Eq. (8.81),

$$\phi_1 = \tan^{-1}\frac{v_{\infty_1})_S}{v_{\infty_1})_V} = \tan^{-1}\frac{V_1^{(v)}\sin\alpha_1}{V_1^{(v)}\cos\alpha_1 - V} \tag{8.84}$$

At the outbound crossing, the angle between $\mathbf{v}_{\infty_2}$ and $\mathbf{V}$ is $\phi_2$, where

$$\phi_2 = \phi_1 + \delta \tag{8.85}$$

For the leading-side flyby in Fig. 8.18, the turn angle $\delta$ is positive (counterclockwise), whereas in Fig. 8.19 it is negative. Since the magnitude of $\mathbf{v}_{\infty_2}$ is $v_\infty$, we can express $\mathbf{v}_{\infty_2}$ in components as

$$\mathbf{v}_{\infty_2} = v_\infty\cos\phi_2\hat{\mathbf{u}}_V + v_\infty\sin\phi_2\hat{\mathbf{u}}_S \tag{8.86}$$

Therefore, the heliocentric velocity of the spacecraft at the outbound crossing is

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} = V_2^{(v)}\Big)_V\hat{\mathbf{u}}_V + V_2^{(v)}\Big)_S\hat{\mathbf{u}}_S \tag{8.87}$$

where the components of $\mathbf{V}_2^{(v)}$ are

$$V_2^{(v)}\Big)_V = V + v_\infty\cos\phi_2 \qquad V_2^{(v)}\Big)_S = v_\infty\sin\phi_2 \tag{8.88}$$

From this we obtain the spacecraft's radial and transverse heliocentric velocity components,

$$V_{\perp_2}^{(v)} = V_2^{(v)}\Big)_V \qquad V_{r_2}^{(v)} = -V_2^{(v)}\Big)_S \tag{8.89}$$

From these, we finally obtain the three elements $e_2$, $h_2$, and $\theta_2$ of the new heliocentric departure trajectory by means of Eq. (2.21),

$$h_2 = RV_{\perp_2}^{(v)} \tag{8.90}$$

Eq. (2.45),

$$R = \frac{h_2^2}{\mu_{\text{sun}}}\frac{1}{1 + e_2\cos\theta_2} \tag{8.91}$$

and Eq. (2.49),

$$V_{r_2}^{(v)} = \frac{\mu_{\text{sun}}}{h_2}e_2\sin\theta_2 \tag{8.92}$$

Notice that the flyby is considered to be an impulsive maneuver during which the heliocentric radius of the spacecraft, which is confined within the planet's sphere of influence, remains fixed at $R$. The heliocentric velocity analysis is similar to that described in Section 6.7.

## EXAMPLE 8.6

A spacecraft departs earth with a velocity perpendicular to the sun line on a flyby mission to Venus. Encounter occurs at a true anomaly in the approach trajectory of $-30°$. Periapsis altitude is to be 300 km.

(a) For an approach from the dark side of the planet, show that the postflyby orbit is as illustrated in Fig. 8.20.

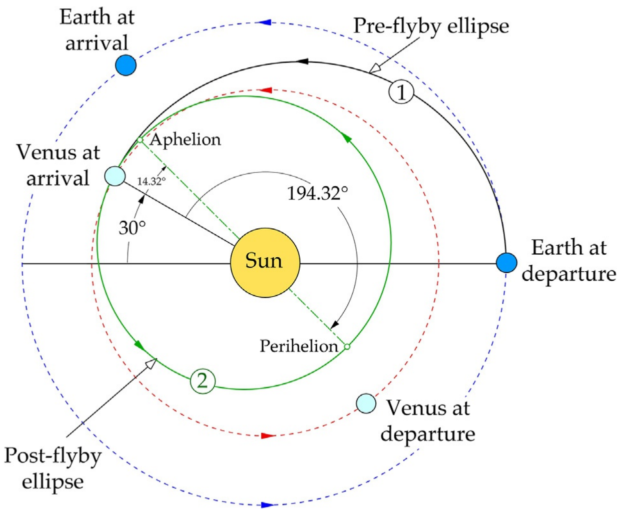


**FIG. 8.20**

Spacecraft orbits before and after a flyby of Venus, approaching from the dark side.

(b) For an approach from the sunlit side of the planet, show that the postflyby orbit is as illustrated in Fig. 8.21.

## Solution

The following data are found in Tables A.1 and A.2:

$$\mu_{sun} = 1.3271\left(10^{11}\right)\text{km}^3/\text{s}^2$$
$$\mu_{Venus} = 324{,}900\text{km}^3/\text{s}^2$$
$$R_{earth} = 149.6\left(10^6\right)\text{km}$$
$$R_{Venus} = 108.2\left(10^6\right)\text{km}$$
$$r_{Venus} = 6052\text{km}$$

*Preflyby ellipse (orbit 1)*

Evaluating the orbit formula (Eq. 2.45) at aphelion of orbit 1 yields

$$R_{earth} = \frac{h_1{}^2}{\mu_{sun}}\frac{1}{1-e_1}$$

Thus,

$$h_1{}^2 = \mu_{sun}R_{earth}(1-e_1) \tag{a}$$

At intercept,

$$R_{Venus} = \frac{h_1{}^2}{\mu_{sun}}\frac{1}{1+e_1\cos(\theta_1)}$$

Substituting Eq. (a) and $\theta_1 = -30°$ into this expression and solving the resulting expression for $e_1$ leads to

$$e_1 = \frac{R_{earth} - R_{Venus}}{R_{earth} + R_{Venus}\cos(\theta_1)} = \frac{149.6\left(10^6\right) - 108.2\left(10^6\right)}{149.6\left(10^6\right) + 108.2\left(10^6\right)\cos(-30°)} = 0.1702$$

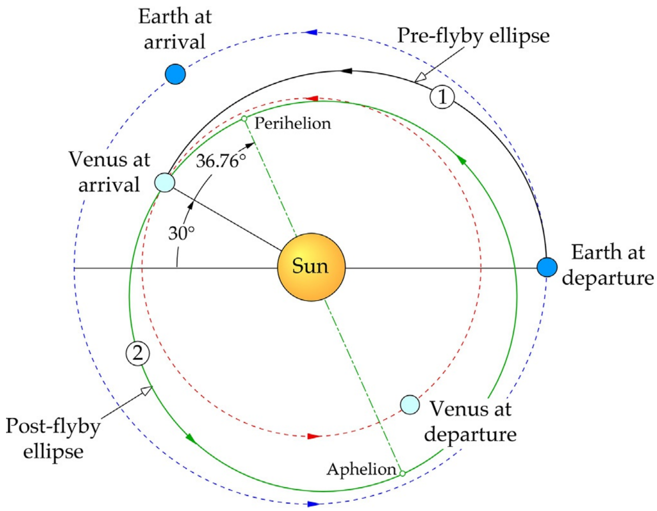


**FIG. 8.21**

Spacecraft orbits before and after a flyby of Venus, approaching from the sunlit side.

With this result, and Eq. (a) yields

$$h_1 = \sqrt{1.327\left(10^{11}\right) \cdot 149.6\left(10^6\right)(1-0.1702)} = 4.059\left(10^9\right)\,\text{km}^2/\text{s}$$

Now we can use Eqs. (2.31) and (2.49) to calculate the radial and transverse components of the spacecraft's heliocentric velocity at the inbound crossing of Venus' sphere of influence

$$V_{\perp_1}^{(v)} = \frac{h_1}{R_{\text{Venus}}} = \frac{4.059\left(10^9\right)}{108.2\left(10^6\right)} = 37.51\,\text{km/s}$$
$$V_{r_1}^{(v)} = \frac{\mu_{\text{sun}}}{h_1} e_1 \sin(\theta_1) = \frac{1.327\left(10^{11}\right)}{4.059\left(10^9\right)} \cdot 0.1702 \cdot \sin(-30°) = -2.782\,\text{km/s}$$

The flight path angle, from Eq. (2.51), is

$$\gamma_1 = \tan^{-1}\frac{V_{r_1}^{(v)}}{V_{\perp_1}^{(v)}} = \tan^{-1}\left(\frac{-2.782}{37.51}\right) = -4.241°$$

The negative sign is consistent with the fact that the spacecraft is flying toward perihelion of the preflyby elliptical trajectory (orbit 1).

The speed of the space vehicle at the inbound crossing is

$$V_1^{(v)} = \sqrt{\left(V_{r_1}^{(v)}\right)^2 + \left(V_{\perp_1}^{(v)}\right)^2} = \sqrt{(-2.782)^2 + 37.51^2} = 37.62\,\text{km/s} \tag{b}$$

*Flyby hyperbola*

From Eqs. (8.75) and (8.77), we obtain

$$\mathbf{V}_1^{(v)} = 37.51\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S\,(\text{km/s})$$

The velocity of Venus in its presumed circular orbit around the sun is

$$\mathbf{V} = \sqrt{\frac{\mu_{\text{sun}}}{R_{\text{Venus}}}}\hat{\mathbf{u}}_V = \sqrt{\frac{1.327(10)^{11}}{108.2(10)^6}}\hat{\mathbf{u}}_V = 35.02\hat{\mathbf{u}}_V\,(\text{km/s}) \tag{c}$$

Hence

$$\mathbf{v}_{\infty_1} = \mathbf{V}_1^{(v)} - \mathbf{V} = (37.51\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S) - 35.02\hat{\mathbf{u}}_V = 2.490\hat{\mathbf{u}}_V + 2.782\hat{\mathbf{u}}_S\,(\mathrm{km/s}) \tag{d}$$

It follows that

$$v_\infty = \sqrt{\mathbf{v}_{\infty_1} \cdot \mathbf{v}_{\infty_1}} = 3.733\,\mathrm{km/s}$$

The periapsis radius is

$$r_p = r_{\mathrm{Venus}} + 300 = 6352\,\mathrm{km}$$

Eqs. (8.38) and (8.39) are used to compute the angular momentum and eccentricity of the planetocentric hyperbola.

$$h = 6352\sqrt{{v_\infty}^2 + \frac{2\mu_{\mathrm{Venus}}}{6352}} = 6352\sqrt{3.733^2 + \frac{2 \cdot 324{,}900}{6352}} = 68{,}480\,\mathrm{km^2/s}$$
$$e = 1 + \frac{r_p {v_\infty}^2}{\mu_{\mathrm{Venus}}} = 1 + \frac{6352 \cdot 3.733^2}{324{,}900} = 1.272$$

The turn angle and true anomaly of the asymptote are

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.272}\right) = 103.6°$$
$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) = \cos^{-1}\left(-\frac{1}{1.272}\right) = 141.8°$$

From Eqs. (2.50), (2.103), and (2.107), the aiming radius is

$$\Delta = r_p\sqrt{\frac{e+1}{e-1}} = 6352\sqrt{\frac{1.272+1}{1.272-1}} = 18{,}340\,\mathrm{km} \tag{e}$$

Finally, from Eqs. (8.84) and (d) we obtain the angle between $\mathbf{v}_{\infty_1}$ and $\mathbf{V}$,

$$\phi_1 = \tan^{-1}\frac{\left.v_{\infty_1}\right)_S}{\left.v_{\infty_1}\right)_V} = \tan^{-1}\frac{2.782}{2.490} = 48.17° \tag{f}$$

There are two flyby approaches, as shown in Fig. 8.22. In the dark-side approach, the turn angle is counterclockwise (+103.6°), whereas for the sunlit side approach, it is clockwise (−103.6°).

(a) *Dark-side approach*
According to Eq. (8.85), the angle between $\mathbf{v}_\infty$ and $\mathbf{V}_{\mathrm{Venus}}$ at the outbound crossing is

$$\phi_2 = \phi_1 + \delta = 48.17° + 103.6° = 151.8°$$

Hence, by Eq. (8.86),

$$\mathbf{v}_{\infty_2} = 3.733(\cos 151.8°\hat{\mathbf{u}}_V + \sin 151.8°\hat{\mathbf{u}}_S) = -3.289\hat{\mathbf{u}}_V + 1.766\hat{\mathbf{u}}_S\,(\mathrm{km/s})$$

Using this and Eq. (c), we compute the spacecraft's heliocentric velocity at the outbound crossing.

$$\mathbf{V}_2^{(v)} = \mathbf{V} + \mathbf{v}_{\infty_2} = 31.73\hat{\mathbf{u}}_V + 1.766\hat{\mathbf{u}}_S\,(\mathrm{km/s})$$

It follows from Eq. (8.89) that

$$V_{\perp_2}^{(v)} = 31.73\,\mathrm{km/s} \qquad V_{r_2}^{(v)} = -1.766\,\mathrm{km/s} \tag{g}$$

The speed of the spacecraft at the outbound crossing is

$$V_2^{(v)} = \sqrt{\left(V_{r_2}^{(v)}\right)^2 + \left(V_{\perp_2}^{(v)}\right)^2} = \sqrt{(-1.766)^2 + 31.73^2} = 31.78\,\mathrm{km/s}$$

This is 5.83 km/s less than the inbound speed.

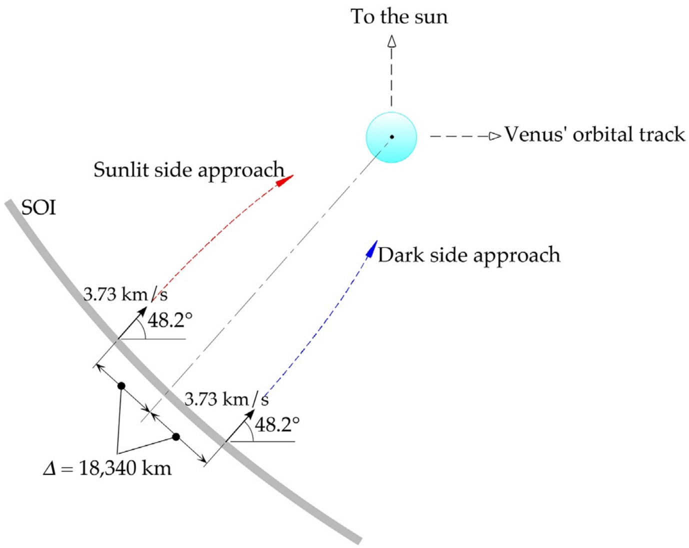


**FIG. 8.22**

Initiation of a sunlit-side approach and dark-side approach at the inbound crossing.

*Postflyby ellipse (orbit 2) for the dark-side approach*

For the heliocentric postflyby trajectory, labeled orbit 2 in Fig. 8.20, the angular momentum is found using Eq. (8.90)

$$h_2 = R_{\text{Venus}} V_{\perp_2}^{(v)} = 108.2(10)^6 \cdot 31.73 = 3.434\left(10^9\right)\left(\text{km}^2/\text{s}\right) \tag{h}$$

From Eq. (8.91),

$$e\cos\theta_2 = \frac{h_2{}^2}{\mu_{\text{sun}} R_{\text{Venus}}} - 1 = \frac{\left[3.434\left(10^9\right)\right]^2}{1.327\left(10^{11}\right)\cdot 108.2\left(10^6\right)} - 1 = -0.1790 \tag{i}$$

and from Eq. (8.92)

$$e\sin\theta_2 = \frac{V_{r_2}^{(v)} h_2}{\mu_{\text{sun}}} = \frac{-1.766 \cdot 3.434(10)^9}{1.327\left(10^{11}\right)} = -0.04569 \tag{j}$$

Thus

$$\tan\theta_2 = \frac{e\sin\theta_2}{e\cos\theta_2} = \frac{-0.04569}{-0.1790} = 0.2553 \tag{k}$$

which means

$$\theta_2 = 14.32° \quad \text{or} \quad 194.32° \tag{l}$$

But $\theta_2$ must lie in the third quadrant since, according to Eqs. (i) and (j), both the sine and cosine are negative. Hence,

$$\theta_2 = 194.32° \tag{m}$$

With this value of $\theta_2$, we can use either Eq. (i) or Eq. (j) to calculate the eccentricity,

$$e_2 = 0.1847 \tag{n}$$

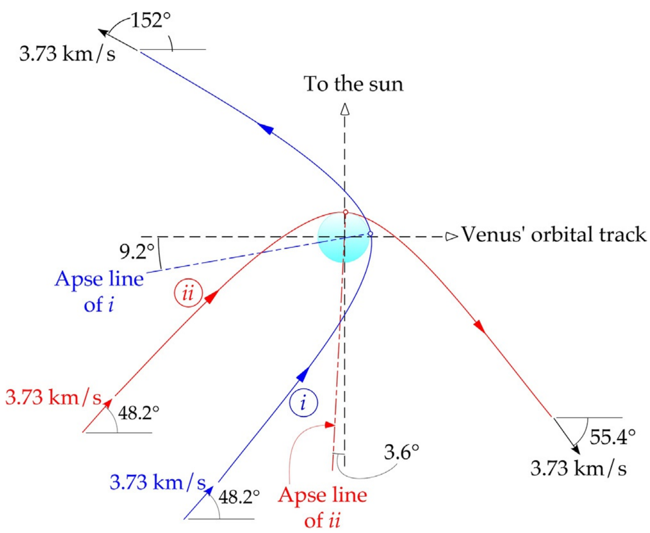


**FIG. 8.23**

Hyperbolic flyby trajectories for (i) the dark-side approach and (ii) the sunlit-side approach.

Perihelion of the departure orbit lies 194.32° clockwise from the encounter point (so that aphelion is 14.32° therefrom), as illustrated in Fig. 8.20. The perihelion radius is given by Eq. (2.50),

$$R_{\text{perihelion}} = \frac{h_2{}^2}{\mu_{\text{sun}}}\frac{1}{1+e_2} = \frac{\left[3.434\left(10^9\right)\right]^2}{1.327\left(10^{11}\right)}\frac{1}{1+0.1847} = 74.98\left(10^6\right)\text{km}$$

which is well within the orbit of Venus.

(b) *Sunlit-side approach*

In this case, the angle between $\mathbf{v}_\infty$ and $\mathbf{V}_{\text{Venus}}$ at the outbound crossing is

$$\phi_2 = \phi_1 - \delta = 48.17° - 103.6° = -55.44°$$

Therefore,

$$\mathbf{v}_{\infty_2} = 3.733[\cos(-55.44°)\hat{\mathbf{u}}_V + \sin(-55.44°)\hat{\mathbf{u}}_S] = 2.118\hat{\mathbf{u}}_V - 3.074\hat{\mathbf{u}}_S\,(\text{km/s})$$

The spacecraft's heliocentric velocity at the outbound crossing is

$$\mathbf{V}_2^{(v)} = \mathbf{V}_{\text{Venus}} + \mathbf{v}_{\infty_2} = 37.14\hat{\mathbf{u}}_V - 3.074\hat{\mathbf{u}}_S\,(\text{km/s})$$

which means

$$V_{\perp_2}^{(v)} = 37.14\text{km/s} \qquad V_{r_2}^{(v)} = 3.074\text{km/s}$$

The speed of the spacecraft at the outbound crossing is

$$V_2^{(v)} = \sqrt{\left(V_{r_2}^{(v)}\right)^2 + \left(V_{\perp_2}^{(v)}\right)^2} = \sqrt{3.050^2 + 37.14^2} = 37.27\text{km/s}$$

This speed is just 0.348 km/s less than the inbound crossing speed. The relatively small speed change is due to the fact that the apse line of this hyperbola is nearly perpendicular to Venus' orbital track, as shown in Fig. 8.23. Nevertheless, the periapses of both hyperbolas are on the leading side of the planet.

*Postflyby ellipse (orbit 2) for the sunlit-side approach*

To determine the heliocentric postflyby trajectory, labeled orbit 2 in Fig. 8.21, we repeat Steps (h) through (n) above.

$$h_2 = R_{\text{Venus}} V_{\perp_2}^{(v)} = 108.2\left(10^6\right) \cdot 37.14 = 4.019\left(10^9\right) \left(\text{km}^2/\text{s}\right)$$

$$e\cos\theta_2 = \frac{h_2^2}{\mu_{\text{sun}} R_{\text{Venus}}} - 1 = \frac{\left[4.019\left(10^9\right)\right]^2}{1.327\left(10^{11}\right) \cdot 108.2\left(10^6\right)} - 1 = 0.1246 \tag{o}$$

$$e\sin\theta_2 = \frac{V_{r_2}^{(v)} h_2}{\mu_{\text{sun}}} = \frac{3.074 \cdot 4.019\left(10^9\right)}{1.327\left(10^{11}\right)} = 0.09309 \tag{p}$$

$$\tan\theta_2 = \frac{e\sin\theta_2}{e\cos\theta_2} = \frac{0.09309}{0.1246} = 0.7469 \Rightarrow \theta_2 = 36.76^\circ \quad \text{or} \quad 216.76^\circ$$

$\theta_2$ must lie in the first quadrant since both the sine and cosine are positive. Hence,

$$\theta_2 = 36.76^\circ \tag{q}$$

With this value of $\theta_2$, we can use either Eq. (o) or (p) to calculate the eccentricity,

$$e_2 = 0.1556$$

Perihelion of the departure orbit lies 36.76° clockwise from the encounter point as illustrated in Fig. 8.21. The perihelion radius is

$$R_{\text{perihelion}} = \frac{h_2{}^2}{\mu_{\text{sun}}} \frac{1}{1+e_2} = \frac{\left[4.019\left(10^9\right)\right]^2}{1.327\left(10^{11}\right)} \frac{1}{1+0.1556} = 105.3\left(10^6\right)\text{km}$$

which is just within the orbit of Venus. Aphelion lies between the orbits of earth and Venus.

---

Gravity assist maneuvers are used to add momentum to a spacecraft over and above that available from a spacecraft's onboard propulsion system. A sequence of flybys of planets can impart the delta-v needed to reach regions of the solar system that would be inaccessible using only existing propulsion technology. The technique can also reduce the flight time. Interplanetary missions using gravity assist flybys must be carefully designed to take advantage of the relative positions of planets.

Pioneer 11, a 260-kg spacecraft launched from Cape Canaveral, Florida, in April 1973, used a December 1974 flyby of Jupiter to gain the momentum required to carry it to the first ever flyby encounter with Saturn on September 1, 1979. Contact with Pioneer 11 was finally lost in September 1995.

Mariner 10 was a 503-kg spacecraft launched from Cape Canaveral on a mission to Mercury on November 3, 1973. It flew by Venus once and Mercury three times. It was deactivated on March 29, 1974, and is probably still in orbit around the sun.

Following its September 1977 launch from Cape Canaveral, the 826-kg Voyager 1, like Pioneer 11 before it, used a flyby of Jupiter (March 1979) to reach Saturn in November 1980. In August 1977, Voyager 2 was launched on its "grand tour" of the outer planets and beyond. This involved gravity assist flybys of Jupiter (July 1979), Saturn (August 1981), Uranus (January 1986), and Neptune (August 1989), after which the spacecraft departed the solar system at an angle of 30° to the ecliptic.

With a mass nine times that of Pioneer 11, the dual-spin Galileo spacecraft was launched from the space shuttle *Atlantis*, departing on October 18, 1989 for an extensive international exploration of Jupiter and its satellites that lasted until the spacecraft was deorbited on September 21, 2003. Galileo used gravity assist flybys of Venus (February 1990), earth (December 1990), and earth again (December 1992) before arriving at Jupiter in December 1995.

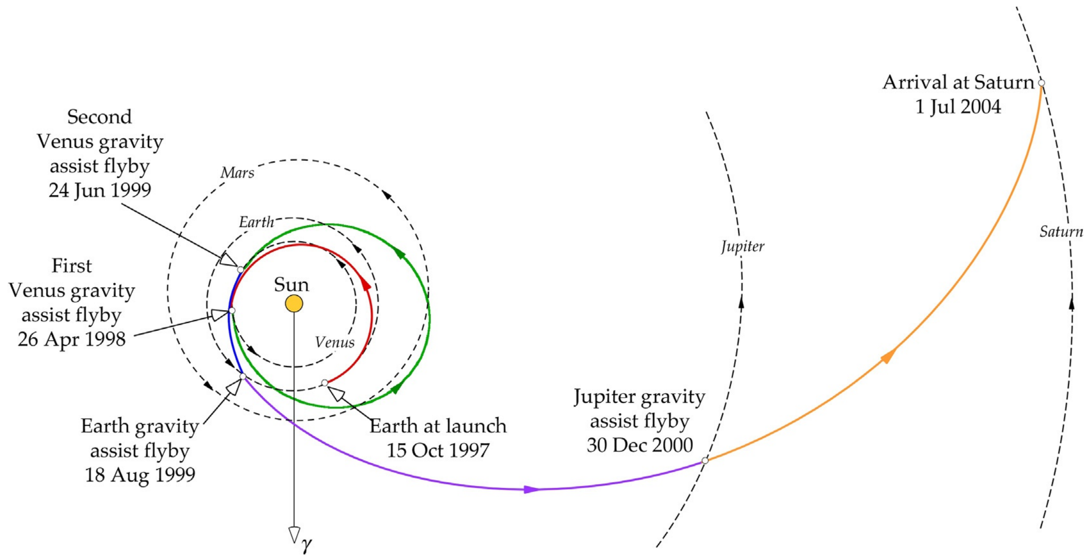


**FIG. 8.24**

Cassini's 7-year mission to Saturn.

Ulysses was a 370-kg international spacecraft launched from the space shuttle *Discovery* in 1990. Its flyby of Jupiter in early February 1992 increased its heliocentric orbital inclination to over 80°. With a period of about six years, this orbit allowed Ulysses to expore the polar regions of the sun until it was decommisioned in June 2009.

The international Cassini mission to Saturn made extensive use of gravity assist flyby maneuvers. The 5712-kg Cassini spacecraft was launched on October 15, 1997, from Cape Canaveral, Florida, and arrived at Saturn nearly 7 years later, on July 1, 2004. The mission involved four flybys, as illustrated in Fig. 8.24. A little over 8 months after launch, on April 26, 1998, Cassini flew by Venus at a periapsis altitude of 284 km and received a speed boost of about 7 km/s. This placed the spacecraft in an orbit that sent it just outside the orbit of Mars (but well away from the planet) and returned it to Venus on June 24, 1999, for a second flyby, this time at an altitude of 600 km. The result was a trajectory that vectored Cassini toward the earth for an August 18, 1999, flyby at an altitude of 1171 km. The 5.5-km/s speed boost at earth sent the spacecraft toward Jupiter for its next flyby maneuver. This occurred on December 30, 2000, at a distance of 9.7 million km from Jupiter, boosting Cassini's speed by about 2 km/s and adjusting its trajectory so as to rendezvous with Saturn about three and a half years later. After 13 years in orbit, on September 12, 2015, its fuel exhausted, the Cassini mission ended with the spacecraft plunging into Saturn's upper atmosphere.

Messenger was a 1108-kg spacecraft launched from Cape Canaveral, Florida, on August 3, 2004. It flew by earth, Venus, and Mercury on its way finally to orbit insertion around Mercury on March 2011. Messenger deorbited four years later.

## 8.10 PLANETARY EPHEMERIS

The state vector (**R**, **V**) of a planet is defined relative to the heliocentric ecliptic frame of reference, as illustrated in Fig. 8.25. This is very similar to the geocentric equatorial frame of Fig. 4.7. The sun replaces the earth as the center of attraction, and the plane of the ecliptic replaces the earth's equatorial plane. The vernal equinox continues to define the inertial $X$ axis.

To design realistic interplanetary missions, we must be able to determine the state vector of a planet at any given time. Table 8.1 provides the orbital elements of the planets and their rates of change per century (Cy) with respect to the J2000 epoch (January 1, 2000, 12 h UT). The table, covering the years 1800–2050, is sufficiently accurate for our needs. Alternatively, one can use JPL's online HORIZONS system (JPL Horizons Web-Interface, 2018) or, within MATLAB, the function `planetEphemeris`. From the orbital elements, we can infer the state vector using Algorithm 4.5.

To interpret Table 8.1, observe the following:

- One astronomical unit (1 AU) is $1.49597871(10^8)$ km, the average distance between the earth and the sun.
- One arcsecond ($1''$) is 1/3600 of a degree.
- $a$ is the semimajor axis.
- $e$ is the eccentricity.
- $i$ is the inclination to the ecliptic plane.
- $\Omega$ is the right ascension of the ascending node (relative to the J2000 vernal equinox).
- $\varpi$, the longitude of perihelion, is defined as $\varpi = \omega + \Omega$, where $\omega$ is the argument of perihelion.

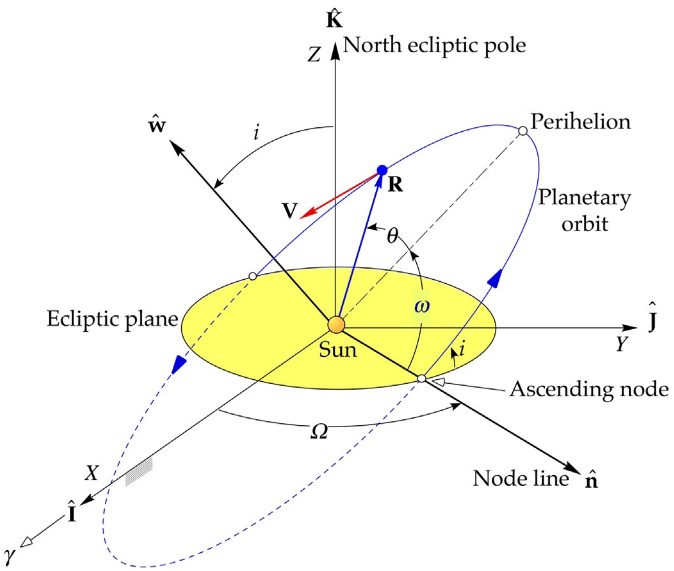


**FIG. 8.25**

Planetary orbit in the heliocentric ecliptic frame.

**Table 8.1 Planetary orbital elements and their centennial rates**

| | $a$ (AU)<br>$\dot{a}$ (AU/Cy) | $e$<br>$\dot{e}$ (1/Cy) | $i$ (°)<br>$\dot{i}$ (°/Cy) | $\Omega$ (°)<br>$\dot{\Omega}$ (°/Cy) | $\varpi$ (°)<br>$\dot{\varpi}$ (°/Cy) | $L$ (°)<br>$\dot{L}$ (°/Cy) |
|---|---|---|---|---|---|---|
| Mercury | 0.38709927 | 0.20563593 | 7.00497902 | 48.33076593 | 77.45779628 | 252.25032350 |
| | 0.00000037 | 0.00001906 | −0.00594749 | −0.12534081 | 0.16047689 | 149,472.67411175 |
| Venus | 0.72333566 | 0.00677672 | 3.39467605 | 76.67984255 | 131.60246718 | 181.97909950 |
| | 0.00000390 | −0.00004107 | −0.00078890 | −0.27769418 | 0.00268329 | 58,517.81538729 |
| Earth | 1.00000261 | 0.01671123 | −0.00001531 | 0.0 | 102.93768193 | 100.46457166 |
| | 0.00000562 | −0.00004392 | −0.01294668 | 0.0 | 0.32327364 | 35,999.37244981 |
| Mars | 1.52371034 | 0.09339410 | 1.84969142 | 49.55953891 | −23.94362959 | −4.55343205 |
| | 0.0001847 | 0.00007882 | −0.00813131 | −0.29257343 | 0.44441088 | 19,140.30268499 |
| Jupiter | 5.20288700 | 0.04838624 | 1.30439695 | 100.47390909 | 14.72847983 | 34.39644501 |
| | −0.00011607 | −0.00013253 | −0.00183714 | 0.20469106 | 0.21252668 | 3034.74612775 |
| Saturn | 9.53667594 | 0.05386179 | 2.48599187 | 113.66242448 | 92.59887831 | 49.95424423 |
| | −0.00125060 | −0.00050991 | 0.00193609 | −0.28867794 | −0.41897216 | 1222.49362201 |
| Uranus | 19.18916464 | 0.04725744 | 0.77263783 | 74.01692503 | 170.95427630 | 313.23810451 |
| | −0.00196176 | −0.00004397 | −0.00242939 | 0.04240589 | 0.40805281 | 428.48202785 |
| Neptune | 30.06992276 | 0.00859048 | 1.77004347 | 131.78422574 | 44.96476227 | −55.12002969 |
| | 0.00026291 | 0.00005105 | 0.00035372 | −0.00508664 | −0.32241464 | 218.45945325 |
| (Pluto) | 39.48211675 | 0.24882730 | 17.14001206 | 110.30393684 | 224.06891629 | 238.92903833 |
| | −0.00031596 | 0.00005170 | 0.00004818 | −0.01183482 | −0.04062942 | 145.20780515 |

*Reproduced with permission from Standish et al. (2013).*

- $L$, the mean longitude, is defined as $L = \varpi + M$, where $M$ is the mean anomaly.
- $\dot{a}$, $\dot{e}$, $\dot{\Omega}$, etc., are the rates of change of the above orbital elements per Julian century. One century (Cy) equals 36,525 days.

**ALGORITHM 8.1**

Determine the state vector of a planet at a given date and time. All angular calculations must be adjusted so that they lie in the range 0° to 360°. Recall that the gravitational parameter of the sun is $\mu = 1.327(10^{11})$ km$^3$/s$^2$. This procedure is implemented in MATLAB as the function *planet_elements_and_sv.m* in Appendix D.35.

1. Use Eqs. (5.47) and (5.48) to calculate the Julian day number $JD$.
2. Calculate $T_0$, the number of Julian centuries between J2000 and the date in question (Eq. 5.49).

$$T_0 = \frac{JD - 2{,}451{,}545.0}{36{,}525}$$

3. If $Q$ is any one of the six planetary orbital elements listed in Table 8.1, then calculate its value at $JD$ by means of the formula

$$Q = Q_0 + \dot{Q}T_0 \tag{8.93b}$$

where $Q_0$ is the value listed for J2000, and $\dot{Q}$ is the tabulated rate. All angular quantities must be adjusted to lie in the range 0°–360°.

4. Use the semimajor axis $a$ and the eccentricity $e$ to calculate the angular momentum $h$ at $JD$ from Eq. (2.71)

$$h = \sqrt{\mu a(1-e^2)}$$

5. Obtain the argument of perihelion $\omega$ and mean anomaly $M$ at $JD$ from the results of Step 3 by means of the definitions

$$\omega = \varpi - \Omega$$
$$M = L - \varpi$$

6. Substitute the eccentricity $e$ and the mean anomaly $M$ at $JD$ into Kepler's equation (Eq. 3.14) and calculate the eccentric anomaly $E$.
7. Calculate the true anomaly $\theta$ using Eq. (3.13).
8. Use $h$, $e$, $\Omega$, $i$, $\omega$, and $\theta$ to obtain the heliocentric position vector $\mathbf{R}$ and velocity $\mathbf{V}$ by means of Algorithm 4.5, with the heliocentric ecliptic frame replacing the geocentric equatorial frame.

## EXAMPLE 8.7

Find the distance between earth and Mars at 12 h UT on August 27, 2003. Use Algorithm 8.1.

Step 1:

According to Eq. (5.48), the Julian day number $J_0$ for midnight (0h UT) of this date is

$$J_0 = 367\cdot 2003 - \text{INT}\left\{\frac{7\left[2003 + \text{INT}\left(\frac{8+9}{12}\right)\right]}{4}\right\} + \text{INT}\left(\frac{275\cdot 8}{9}\right) + 27 + 1{,}721{,}013.5$$
$$= 735{,}101 - 3507 + 244 + 27 + 1{,}721{,}013.5$$
$$= 2{,}452{,}878.5$$

At $UT = 12$, the Julian day number is

$$JD = 2{,}452{,}878.5 + \frac{12}{24} = 2{,}452{,}879.0$$

Step 2:

The number of Julian centuries between J2000 and this date is

$$T_0 = \frac{JD - 2{,}451{,}545}{36{,}525} = \frac{2{,}452{,}879 - 2{,}451{,}545}{36{,}525} = 0.036523\text{Cy}$$

Step 3:

Table 8.1 and Eq. (8.93b) yield the orbital elements of earth and Mars at 12 h UT on August 27, 2003:

| | $a$ (km) | $e$ | $i$ (°) | $\Omega$ (°) | $\varpi$ (°) | $L$ (°) |
|---|---|---|---|---|---|---|
| Earth | $1.4960(10^8)$ | 0.016710 | −0.00048816 | 0.0 | 102.95 | 335.27 |
| Mars | $2.2794(10^8)$ | 0.093397 | 1.8494 | 49.549 | 336.07 | 334.51 |

Step 4:

$$h_{\text{earth}} = 4.4451\left(10^9\right)\text{km}^2/\text{s}$$

$$h_{\text{Mars}} = 5.4760\left(10^9\right)\text{km}^2/\text{s}$$

Step 5:

$$\omega_{\text{earth}} = (\varpi - \Omega)_{\text{earth}} = 102.95 - 0 = 102.95°$$

$$\omega_{\text{Mars}} = (\varpi - \Omega)_{\text{Mars}} = 336.07 - 49.549 = 286.52°$$

$$M_{\text{earth}} = (L - \varpi)_{\text{earth}} = 335.27 - 102.95 = 232.32°$$

$$M_{\text{Mars}} = (L - \varpi)_{\text{Mars}} = 334.51 - 336.07 = -1.56°(358.43°)$$

Step 6:

$$E_{\text{earth}} - 0.016710\sin E_{\text{earth}} = 232.32°(\pi/180) \Rightarrow E_{\text{earth}} = 231.57°$$

$$E_{\text{Mars}} - 0.093397\sin E_{\text{Mars}} = 358.43°(\pi/180) \Rightarrow E_{\text{Mars}} = 358.27°$$

Step 7:

$$\theta_{\text{earth}} = 2\tan^{-1}\left(\sqrt{\frac{1+0.016710}{1-0.016710}}\tan\frac{231.57°}{2}\right) = -129.18 \Rightarrow \theta_{\text{earth}} = 230.8°$$

$$\theta_{\text{Mars}} = 2\tan^{-1}\left(\sqrt{\frac{1+0.093397}{1-0.093397}}\tan\frac{358.27°}{2}\right) = -1.8998° \Rightarrow \theta_{\text{Mars}} = 358.10°$$

Step 8:

From Algorithm 4.5,

$$\mathbf{R}_{\text{earth}} = \left(135.59\hat{\mathbf{I}} - 66.803\hat{\mathbf{J}} - 0.00056916\hat{\mathbf{K}}\right)\left(10^6\right)(\text{km})$$

$$\mathbf{V}_{\text{earth}} = \left(12.680\hat{\mathbf{I}} - 26.610\hat{\mathbf{J}} - 0.00022672\hat{\mathbf{K}}\right)(\text{km/s})$$

$$\mathbf{R}_{\text{Mars}} = \left(185.95\hat{\mathbf{I}} - 89.959\hat{\mathbf{J}} - 6.4534\hat{\mathbf{K}}\right)\left(10^6\right)(\text{km})$$

$$\mathbf{V}_{\text{Mars}} = 11.478\hat{\mathbf{I}} - 23.881\hat{\mathbf{J}} - 0.21828\hat{\mathbf{K}}(\text{km})$$

The distance $d$ between the two planets is therefore

$$\begin{aligned} d &= \|\mathbf{R}_{\text{Mars}} - \mathbf{R}_{\text{earth}}\| \\ &= \sqrt{(185.95-135.59)^2 + [-89.959-(-66.803)]^2 + (-6.4534-0.00056916)^2}\left(10^6\right) \end{aligned}$$

or

$$\boxed{d = 55.80\left(10^6\right)\text{km}}$$

The positions of earth and Mars are illustrated in Fig. 8.26. It is a rare event for Mars to be in opposition (lined up with earth on the same side of the sun) when Mars is at or near perihelion. The two planets had not been this close in recorded history.

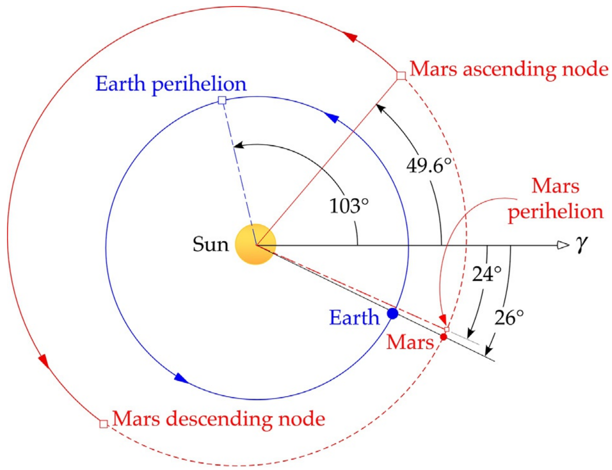


**FIG. 8.26**

Earth and Mars on August 27, 2003. Angles shown are heliocentric latitude, measured in the plane of the ecliptic counterclockwise from the vernal equinox of J2000.

## 8.11 NON-HOHMANN INTERPLANETARY TRAJECTORIES

To implement a systematic patched conic procedure for three-dimensional trajectories, we will use vector notation and the procedures described in Sections 4.4 and 4.6 (Algorithms 4.2 and 4.5), together with the solution of Lambert's problem presented in Section 5.3 (Algorithm 5.2). The mission is to send a spacecraft from planet 1 to planet 2 in a specified time $t_{12}$. As previously discussed in this chapter, we break the mission down into three parts: the departure phase, the cruise phase, and the arrival phase. We start with the cruise phase.

The frame of reference that we use is the heliocentric ecliptic frame, as shown in Fig. 8.27. The first step is to obtain the state vector of planet 1 at departure (time $t$) and the state vector of planet 2 at arrival (time $t + t_{12}$). This is accomplished by means of Algorithm 8.1.

The next step is to determine the spacecraft's transfer trajectory from planet 1 to planet 2. We first observe that, according to the patched conic procedure, the heliocentric position vector of the spacecraft at time $t$ is that of planet 1 ($\mathbf{R}_1$) and at time $t + t_{12}$ its position vector is that of planet 2 ($\mathbf{R}_2$). With $\mathbf{R}_1$, $\mathbf{R}_2$, and the time of flight $t_{12}$ we can use Algorithm 5.2 (Lambert's problem) to obtain the spacecraft's departure and arrival velocities $\mathbf{V}_D^{(v)}$ and $\mathbf{V}_A^{(v)}$ relative to the sun. Either of the state vectors ($\mathbf{R}_1$, $\mathbf{V}_D^{(v)}$) or ($\mathbf{R}_2, \mathbf{V}_A^{(v)}$) can be used to obtain the transfer trajectory's six orbital elements by means of Algorithm 4.2.

The spacecraft's hyperbolic excess velocity on exiting the sphere of influence of planet 1 is

$$\mathbf{v}_\infty)_{\text{Departure}} = \mathbf{V}_D^{(v)} - \mathbf{V}_1 \tag{8.94a}$$

and its excess speed is

$$v_\infty)_{\text{Departure}} = \left\| \mathbf{V}_D^{(v)} - \mathbf{V}_1 \right\| \tag{8.94b}$$

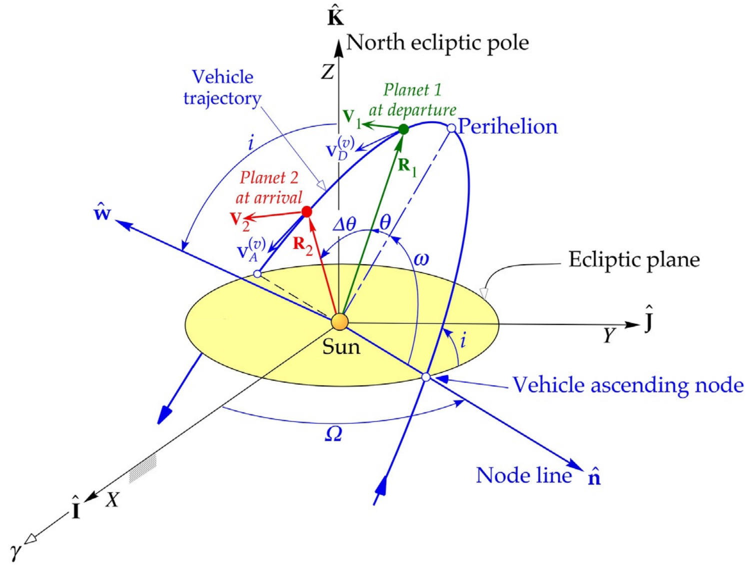


**FIG. 8.27**

Heliocentric orbital elements of a three-dimensional transfer trajectory from planet 1 to planet 2.

Likewise, at the sphere of influence crossing at planet 2,

$$\mathbf{v}_\infty)_{\text{Arrival}} = \mathbf{V}_A^{(v)} - \mathbf{V}_2 \tag{8.95a}$$

$$v_\infty)_{\text{Arrival}} = \left\| \mathbf{V}_A^{(v)} - \mathbf{V}_2 \right\| \tag{8.95b}$$

**ALGORITHM 8.2**

Given the departure and arrival dates (and, therefore, the time of flight), determine the trajectory for a mission from planet 1 to planet 2. This procedure is implemented as the MATLAB function *interplanetary.m* in Appendix D.36.

1. Use Algorithm 8.1 to determine the state vector $(\mathbf{R}_1, \mathbf{V}_1)$ of planet 1 at departure and the state vector $(\mathbf{R}_2, \mathbf{V}_2)$ of planet 2 at arrival.
2. Use $\mathbf{R}_1$, $\mathbf{R}_2$, and the time of flight in Algorithm 5.2 to find the spacecraft velocity $\mathbf{V}_D^{(v)}$ at departure from planet 1's sphere of influence and its velocity $\mathbf{V}_A^{(v)}$ upon arrival at planet 2's sphere of influence.
3. Calculate the hyperbolic excess velocities at departure and arrival using Eqs. (8.94) and (8.95).

## EXAMPLE 8.8

A spacecraft departs earth's sphere of influence on November 7, 1996 (0 h UT), on a prograde coasting flight to Mars, arriving at Mars' sphere of influence on September 12, 1997 (0 h UT). Use Algorithm 8.2 to determine the trajectory and then compute the hyperbolic excess velocities at departure and arrival.

### Solution

Step 1:

Algorithm 8.1 yields the state vectors for earth and Mars.

$$\mathbf{R}_{\text{earth}} = 1.0499\left(10^8\right)\hat{\mathbf{I}} + 1.0465\left(10^8\right)\hat{\mathbf{J}} + 716.93\hat{\mathbf{K}}\,(\text{km}) \qquad \left(R_{\text{earth}} = 1.4824\left(10^8\right)\text{km}\right)$$

$$\mathbf{V}_{\text{earth}} = -21.515\hat{\mathbf{I}} + 20.958\hat{\mathbf{J}} + 0.00014376\hat{\mathbf{K}}\,(\text{km/s}) \qquad \left(V_{\text{earth}} = 30.055\,\text{km/s}\right)$$

$$\mathbf{R}_{\text{Mars}} = -2.0858\left(10^7\right)\hat{\mathbf{I}} - 2.1842\left(10^8\right)\hat{\mathbf{J}} + 4.06244\left(10^6\right)\hat{\mathbf{K}}\,(\text{km}) \qquad \left(R_{\text{Mars}} = 2.1945\left(10^8\right)\text{km}\right)$$

$$\mathbf{V}_{\text{Mars}} = 25.037\hat{\mathbf{I}} + 0.22311\hat{\mathbf{J}} - 0.62018\hat{\mathbf{K}}\,(\text{km/s}) \qquad \left(V_{\text{Mars}} = 25.046\,\text{km/s}\right)$$

Step 2:

The position vector $\mathbf{R}_1$ of the spacecraft at crossing the earth's sphere of influence is just that of the earth,

$$\mathbf{R}_1 = \mathbf{R}_{\text{earth}} = 1.0499\left(10^8\right)\hat{\mathbf{I}} + 1.0465\left(10^8\right)\hat{\mathbf{J}} + 716.93\hat{\mathbf{K}}\,(\text{km})$$

On arrival at Mars' sphere of influence, the spacecraft's position vector is

$$\mathbf{R}_2 = \mathbf{R}_{\text{mars}} = -2.0858\left(10^7\right)\hat{\mathbf{I}} - 2.1842\left(10^8\right)\hat{\mathbf{J}} - 4.06244\left(10^6\right)\hat{\mathbf{K}}\,(\text{km})$$

According to Eqs. (5.47) and (5.48)

$$JD_{\text{Departure}} = 2{,}450{,}394.5$$
$$JD_{\text{Arrival}} = 2{,}450{,}703.5$$

Hence, the time of flight is

$$t_{12} = 2{,}450{,}703.5 - 2{,}450{,}394.5 = 309\,\text{days}$$

Entering $\mathbf{R}_1$, $\mathbf{R}_2$, and $t_{12}$ into Algorithm 5.2 yields

$$\mathbf{V}_D^{(v)} = -24.429\hat{\mathbf{I}} + 21.782\hat{\mathbf{J}} + 0.94810\hat{\mathbf{K}}\,(\text{km/s}) \quad \left(V_D^{(v)} = 32.743\,\text{km/s}\right)$$
$$\mathbf{V}_A^{(v)} = 22.157\hat{\mathbf{I}} + 0.19959\hat{\mathbf{J}} + 0.45793\hat{\mathbf{K}}\,(\text{km/s}) \quad \left(V_A^{(v)} = 22.162\,\text{km/s}\right)$$

Using the state vector $(\mathbf{R}_1, \mathbf{V}_D^{(v)})$, we employ Algorithm 4.2 to find the orbital elements of the heliocentric transfer trajectory.

$$\boxed{\begin{aligned} h &= 4.8456\left(10^6\right)\text{km}^2/\text{s} \\ e &= 0.20581 \\ \Omega &= 44.898^\circ \\ i &= 1.6622^\circ \\ \omega &= 19.973^\circ \\ \theta_1 &= 340.04^\circ \\ a &= 1.8475\left(10^8\right)\text{km} \end{aligned}}$$

Step 3:

At departure, the hyperbolic excess velocity is

$$\mathbf{v}_\infty)_{\text{Departure}} = \mathbf{V}_D^{(v)} - \mathbf{V}_{\text{earth}} = -2.9138\hat{\mathbf{I}} + 0.79525\hat{\mathbf{J}} + 0.94796\hat{\mathbf{K}}\,(\text{km/s})$$

Therefore, the hyperbolic excess speed is

$$v_\infty)_{\text{Departure}} = \|\mathbf{v}_\infty)_{\text{Departure}}\| = \boxed{3.1656\,\text{km/s}} \tag{a}$$

Likewise, at arrival

$$\mathbf{v}_\infty)_{\text{Arrival}} = \mathbf{V}_A^{(v)} - \mathbf{V}_{\text{Mars}} = -2.58805\hat{\mathbf{I}} + 0.023514\hat{\mathbf{J}} + 0.16254\hat{\mathbf{K}}\,(\text{km/s})$$

so that

$$\mathbf{v}_\infty)_{\text{Arrival}} = \|\mathbf{v}_\infty)_{\text{Arrival}}\| = \boxed{2.8852\,\text{km/s}} \tag{b}$$

For the previous example, Fig. 8.28 shows the orbits of earth, Mars, and the spacecraft from directly above the ecliptic plane. Dotted lines indicate the portions of an orbit that are below the plane. $\lambda$ is the heliocentric longitude measured counterclockwise from the vernal equinox of J2000. Also shown are the position of Mars at departure and the position of the earth at arrival.

The transfer orbit resembles that of the Mars Global Surveyor, which departed earth on November 7, 1996, and arrived at Mars 309 days later, on September 12, 1997.

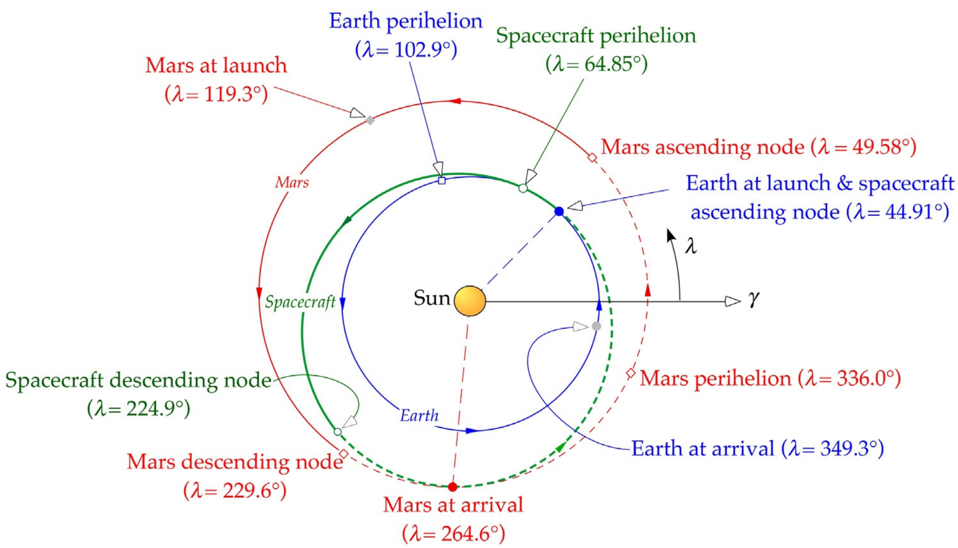


**FIG. 8.28**

The transfer trajectory of Example 8.8, together with the orbits of earth and Mars, as viewed from directly above the plane of the ecliptic.

## EXAMPLE 8.9

In Example 8.8, calculate the delta-v required to launch the spacecraft onto its cruise trajectory from a 180-km circular parking orbit. Sketch the departure trajectory.

### Solution

Recall that

$$r_{\text{earth}} = 6378\,\text{km}$$
$$\mu_{\text{earth}} = 398{,}600\,\text{km}^3/\text{s}^2$$

The radius to periapsis of the departure hyperbola is the radius of the earth plus the altitude of the parking orbit,

$$r_p = 6378 + 180 = 6558\,\text{km}$$

Substituting this and Eq. (a) from Example 8.8 into Eq. (8.40) we get the speed of the spacecraft at periapsis of the departure hyperbola,

$$\left.v_p\right)_{\text{Departure}} = \sqrt{\left[\left.v_\infty\right)_{\text{Departure}}\right]^2 + \frac{2\mu_{\text{earth}}}{r_p}} = \sqrt{3.1651^2 + \frac{2.398{,}600}{6558}} = 11.47\,\text{km/s}$$

The speed of the spacecraft in its circular parking orbit is

$$v_c = \sqrt{\frac{\mu_{\text{earth}}}{r_p}} = \sqrt{\frac{398{,}600}{6558}} = 7.796\,\text{km/s}$$

Hence, the delta-v requirement is

$$\Delta v = \left.v_p\right)_{\text{Departure}} - v_c = \boxed{3.674\,\text{km/s}}$$

The eccentricity of the hyperbola is given by Eq. (8.38),

$$e = 1 + \frac{r_p\left[\left.v_\infty\right)_{\text{Departure}}\right]^2}{\mu_{\text{earth}}} = 1 + \frac{6558\cdot 3.1656^2}{398{,}600} = 1.165$$

If we assume that the spacecraft is launched from a parking orbit of 28° inclination, then the departure appears as shown in the three-dimensional sketch in Fig. 8.29.

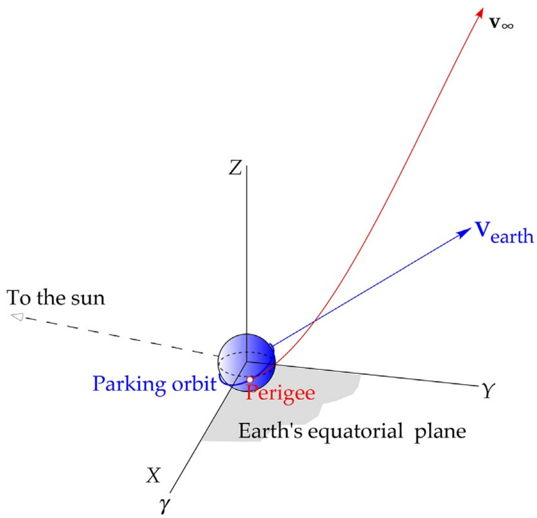


**FIG. 8.29**

The departure hyperbola, assumed to be at 28° inclination to earth's equator.

## EXAMPLE 8.10

In Example 8.8, calculate the delta-v required to place the spacecraft in an elliptical capture orbit around Mars with a periapsis altitude of 300 km and a period of 48 h. Sketch the approach hyperbola.

### Solution

From Tables A.1 and A.2, we know that

$$r_{\text{Mars}} = 3380\,\text{km}$$
$$\mu_{\text{Mars}} = 42{,}830\,\text{km}^3/\text{s}^2$$

The radius to periapsis of the arrival hyperbola is the radius of Mars plus the periapsis of the elliptical capture orbit,

$$r_p = 3380 + 300 = 3680\,\text{km}$$

According to Eq. (8.40) and Eq. (b) of Example 8.8, the speed of the spacecraft at periapsis of the arrival hyperbola is

$$v_p)_{\text{Arrival}} = \sqrt{[v_\infty)_{\text{Arrival}}]^2 + \frac{2\mu_{\text{Mars}}}{r_p}} = \sqrt{2.8852^2 + \frac{2\cdot 42{,}830}{3680}} = 5.621\,\text{km/s}$$

To find the speed $v_p)_{\text{ellipse}}$ at periapsis of the capture ellipse, we use the required period (48 h) to determine the ellipse's semimajor axis, using Eq. (2.83),

$$a_{\text{ellipse}} = \left(\frac{T\sqrt{\mu_{\text{Mars}}}}{2\pi}\right)^{3/2} = \left(\frac{48\cdot 3600\cdot\sqrt{42{,}830}}{2\pi}\right)^{3/2} = 31{,}880\,\text{km}$$

From Eq. (2.73), we obtain

$$e_{\text{ellipse}} = 1 - \frac{r_p}{a_{\text{ellipse}}} = 1 - \frac{3680}{31{,}880} = 0.8846$$

Then Eq. (8.59) yields

$$v_p)_{\text{ellipse}} = \sqrt{\frac{\mu_{\text{Mars}}}{r_p}\left(1+e_{\text{ellipse}}\right)} = \sqrt{\frac{42{,}830}{3680}(1+0.8846)} = 4.683\,\text{km/s}$$

Hence, the delta-v requirement is

$$\Delta v = v_p)_{\text{Arrival}} - v_p)_{\text{ellipse}} = \boxed{0.9382\,\text{km/s}}$$

The eccentricity of the approach hyperbola is given by Eq. (8.38),

$$e = 1 + \frac{r_p\left(v_\infty)_{\text{Arrival}}\right)^2}{\mu_{\text{Mars}}} = 1 + \frac{3680\cdot 2.8852^2}{42{,}830} = 1.715$$

Assuming that the capture ellipse is a polar orbit of Mars, then the approach hyperbola is as illustrated in Fig. 8.30. Note that Mars' equatorial plane is inclined 25° to the plane of its orbit around the sun. Furthermore, the vernal equinox of Mars lies at an angle of 85° from that of the earth.

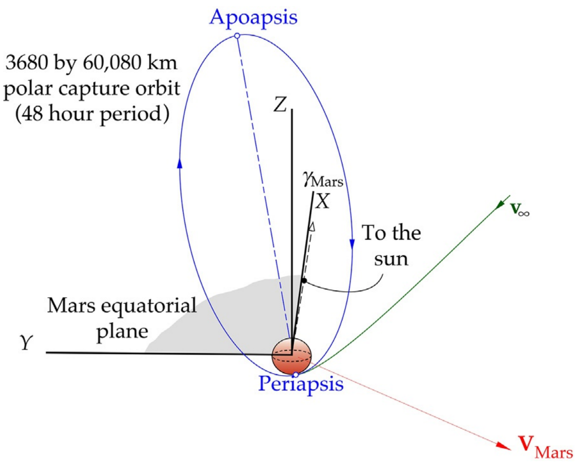


**FIG. 8.30**

The Mars approach hyperbola and capture ellipse.

## PROBLEMS

### Section 8.2

**8.1** Find the total delta-v required for a Hohmann transfer from earth's orbit to Saturn's orbit.
[Ans.: 15.74 km/s]

**8.2** Find the total delta-v required for a Hohmann transfer from Mars' orbit to Jupiter's orbit.
[Ans.: 10.15 km/s]

### Section 8.3

**8.3** Calculate the synodic period of Venus relative to the earth.
{Ans.: 583.9 days}

**8.4** Calculate the synodic period of Jupiter relative to Mars.
{Ans.: 816.6 days}

### Section 8.4

**8.5** Calculate the radius of the spheres of influence of Mercury, Venus, Mars, and Jupiter.
{Ans.: See Table A.2}

**8.6** Calculate the radius of the spheres of influence of Saturn, Uranus, and Neptune.
{Ans.: See Table A.2}

### Section 8.6

**8.7** On a date when the earth was $147.4(10^6)$ km from the sun, a spacecraft parked in a 200-km-altitude circular earth orbit was launched directly into an elliptical orbit around the sun with perihelion of $120(10^6)$ km and aphelion equal to the earth's distance from the sun on the launch date. Calculate the delta-v required and $v_\infty$ of the departure hyperbola.
{Ans.: $\Delta v = 3.34$ km/s, $v_\infty = 1.579$ km/s}

**8.8** Calculate the propellant mass required to launch a 2000-kg spacecraft from a 180-km-altitude circular earth orbit on a Hohmann transfer trajectory to the orbit of Saturn. Calculate the time required for the mission and compare it with that of Cassini. Assume the propulsion system has a specific impulse of 300 s.
{Ans.: 6.03 yr; 21,810 kg}

**Section 8.7**

**8.9** An earth orbit has a perigee radius of 7000 km and a perigee velocity of 9 km/s. Calculate the change in apogee radius due to a change of
**(a)** 1 km in the perigee radius
**(b)** 1 m/s in the perigee speed.
{Ans.: (a) 13.27 km; (b) 10.99 km}

**8.10** An earth orbit has a perigee radius of 7000 km and a perigee velocity of 9 km/s. Calculate the change in apogee speed due to a change of
**(a)** 1 km in the perigee radius
**(b)** 1 m/s in the perigee speed.

{Ans.: (a) −1.81 m/s; (b) −0.406 m/s}

**Section 8.8**

**8.11** Estimate the total delta-v requirement for a Hohmann transfer from earth to Mercury, assuming a 150-km-altitude circular parking orbit at earth and a 150-km circular capture orbit at Mercury. Furthermore, assume that the planets have coplanar circular orbits with radii equal to the semimajor axes listed in Table A.1.
{Ans.: 13.08 km/s}

**Section 8.9**

**8.12** Suppose a spacecraft approaches Jupiter on a Hohmann transfer ellipse from earth. If the spacecraft flies by Jupiter at an altitude of 200,000 km on the sunlit side of the planet, determine the orbital elements of the postflyby trajectory and the delta-v imparted to the spacecraft by Jupiter's gravity. Assume that all the orbits lie in the same (ecliptic) plane.
{Ans.: $\Delta V = 10.6$ km/s, $a = 4.79(10^6)$ km, $e = 0.8453$}

**Section 8.10**

**8.13** Use Table 8.1 to verify the orbital elements for earth and Mars presented in Example 8.7.

**8.14** Use Table 8.1 to determine the day of the year 2005 when the earth was farthest from the sun.
{Ans.: July 4}

**Section 8.11**

**8.15** On December 1, 2005, a spacecraft left a 180-km-altitude circular orbit around the earth on a mission to Venus. It arrived at Venus 121 days later on April 1, 2006, entering a 300-km-by-9000-km capture ellipse around the planet. Calculate the total delta-v requirement for this mission.
{Ans.: 6.75 km/s}

**8.16** On August 15, 2005, a spacecraft in a 190-km, 52°-inclination circular parking orbit around the earth departed on a mission to Mars, arriving at the red planet on March 15, 2006, whereupon retrorockets placed it into a highly elliptic orbit with a periapsis of 300 km and a period of 35 h. Determine the total delta-v required for this mission.
{Ans.: 4.86 km/s}

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JPL Horizons Web-Interface, 2018. https://ssd.jpl.nasa.gov/horizons.cgi. (Accessed 29 June 2018).

Standish, E.M., Williams, J.G., 2013. Orbital ephemerides of the sun, moon and planets. In: Seidelmann, P.K. (Ed.), Explanatory Supplement to the Astronomical Almanac, p. 338. University Science Books.