24th November 2016

Total Maximum Marks: **40**

PYL113/EPL103, Mathematical Physics, Major, Physics Department (IIT, Delhi)

Q1 to Q3 are compulsory. Solve any one out of **Q4 and Q5**. Maximum marks for each question are given in **bold** numericals. Symbols carry usual meaning unless otherwise stated.

(Q1) (a) **(5 marks)** We know that the integral $\int_0^\infty \frac{\sin(x)}{x}\, dx = \pi/2$. Now calculate $\int_{-\infty}^{\infty} \frac{\sin(bx)}{x}\, dx$, with $b > 0$ a constant. What happens when $b < 0$?

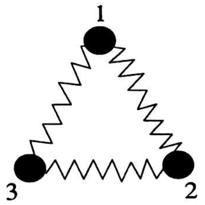
(b) **(5 marks)** Calculate $\int_0^{2\pi} \frac{\cos(\theta)}{2+\sin(\theta)}\, d\theta$, using contour integration.

(Q2) Prove the following vector identities using Levi-Civita symbols. Write each step clearly.

(a) **(5 marks)** $\vec{\nabla} \times (\phi\vec{A}) = \phi\vec{\nabla} \times \vec{A} + \vec{\nabla}\phi \times \vec{A}$

(b) **(5 marks)** $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

(Q3) **(10 marks)** Consider a one dimensional coupled harmonic oscillator system shown in the figure below. Let $\{x_1(t), x_2(t), x_3(t)\}$ represent the displacements of three particles respectively as indicated. Let m be the mass of each particle and k be the spring constant of each spring. Write down the equations of motion for these displacements. Convert this problem into an eigenvalue problem and find all the normal modes (eigenvalues) and corresponding eigenvectors. *(Hint:- You may assume solution of the form $e^{i\omega t}$).*



Solve any one out of (Q4) and (Q5), if you attempt both only the first will be checked.

(Q4) **(10 marks)** Consider the following differential equation.

$$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = 2e^{-x},$$

with initial conditions $u(0) = 2$ and $\frac{du}{dx}|_{x=0} = 1$. Find the corresponding Green's function $G(x, x')$ for regions $x' < x$ and $x' > x$. Using $G(x, x')$ solve the corresponding integral equation to get the solution $u(x)$ for the inhomogeneous differential equation above. Also show that the obtained solution satisfies the given inhomogeneous differential equation. *(Hint:- Use the limits of integration in the integral equation as* $(0, x), x' < x$ *and* $(x, \infty), x' > x$.*)*

(Q5) **(10 marks)** The motion of the body falling in a viscous medium is written as

$$m\frac{d^2x(t)}{dt^2} = mg - c\,\frac{dx(t)}{dt},$$

where m is the mass, g is acceleration due to gravity and c is a constant proportional to the viscosity of the medium. Using **Laplace transforms**, find $x(t)$ and $dx(t)/dt$ with the initial conditions $x(0) = x_0$ and $dx/dt = v_0$ at $t = 0$. Hence obtain the terminal velocity.

Some useful formulae

- Laplace transform (L.T.) of a function $f(t)$ is:

$$\mathcal{L}(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

- Inverse Laplace transform:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\mathcal{L}(s)\, ds,$$

  such that all the singularities of $\mathcal{L}(s)$ lie to the left of the line $x = \gamma$.

- Laplace Transform of $f^{(n)}(t) = d^n f(t)/dt^n$ is:

$$= s^n\mathcal{L}(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \ldots\ldots\ldots - f^{(n-1)}(0),$$

  where $f^{(n)}(\tau)$ represents n^{th} derivative of $f(t)$ calculated at $t = \tau$.

- For a general functions $y_1(x)$ and $y_2(x)$ we have:

$$G(x, x') = a_1 y_1(x) + b_1 y_2(x) - \left(\frac{y_1(x)y_2(x') - y_2(x)y_1(x')}{p(x')W(x')}\right) \quad \text{for } x' < x,$$

  and

$$G(x, x') = a_1 y_1(x) + b_1 y_2(x) \quad \text{for } x < x'$$