DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MAJOR TEST 2016-2017 FIRST SEMESTER

MTL104 (LINEAR ALGEBRA AND APPLICATIONS)

Time: 2 hours **Max. Marks: 50**

**** Answer to each question should begin on a new page ****

1a. Suppose that $v_1, \ldots\ldots, v_n$ are linearly independent vectors in a vector space $V(F)$. Let $v \in V$ and $v = \sum_{i=1}^{n} \beta_i v_i$. Find the condition under which the vectors $v_1 - v, v_2 - v, \ldots\ldots, v_n - v$ are linearly independent. (3)

1b. Suppose that $W_1, \ldots\ldots, W_k$ are $(n-1)$ dimensional subspaces of an n-dimensional vector space $V(F)$. Show that $\dim\,(W_1 \cap W_2 \ldots\ldots \cap W_k) \geq n - k$. (4)

1c. Suppose $m < n$ and that $f_1, \ldots\ldots, f_m$ are linear functionals on an n-dimensional vector space $V(F)$. Under what conditions on the scalars $\alpha_1, \ldots\ldots, \alpha_m$ it is true that there exists a vector $x \in V$, such that $f_i(x) = \alpha_i, \quad i = 1, \ldots\ldots, m$. (3)

2a. Let $T \in L(V, V)$ where V is a finite dimensional vector space. If $T^2 - 3T + 2I = 0$, examine whether T is invertible or not. (2)

2b. True or false? If a diagonalizable operator has only the characteristic values 0 and 1, it is a projection. Justify your answer. (2)

2c. Prove or disprove that similar matrices have the same minimal polynomial. (3)

3a. Let T be a linear operator on the finite dimensional complex vector space. Suppose that there is a digonalizable operator D on V and a nilpotent operator N on V such that $T = D + N$ with $DN = ND$. Show that the diagonalizable part of the linear operator $T^2 + 4T + 4I$ is $D^2 + 4D + 4I$.

$D^2 + N^2 + 2ND + 4D + 4N + 4I$ (3)

3b. Let V be a finite dimensional vector space over the field F, and let T be a linear operator on V such that $\text{rank}(T) = 1$. Prove that T is either diagonalizable or nilpotent, T can not be both. (3)

4. Let W be a subspace of an inner product space V and let y be a vector in V. Prove that the vector x in W is a best approximation to y by vectors in W if and only if $y - x$ is orthogonal to every vector in W. (4)

5a. Let T be a Normal operator on a finite dimensional inner product space V. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*. (3)

5b. Suppose T is a linear operator on a finite dimensional inner product space V and suppose there exists an orthonormal basis $B = \{x_1, x_2, \ldots\ldots, x_n\}$ for V such that each vector in B is a characteristic vector for T. Then prove or disprove that T is normal. (2)

5c. Let T be a linear operator on the finite dimensional inner product space V, and suppose T is both positive and unitary. Prove or disprove that $T = I$. (2)

6a. Let V be the space of $n \times n$ matrices over the complex numbers, with the inner product $< A, B >=$ trace(AB^*). Let P be a fixed invertible matrix in V, and let T_p be the linear operator on V defined by $T_p(A) = P^{-1}AP$. Find the adjoint of T_p. (3)

6b. Let V be the space of complex $n \times n$ matrices with inner product $< A, B >=$ trace(AB^*). For each M in V, let T_M be the linear operator defined by $T_M(A) = MA$. Prove or disprove that T_M is unitary if and only if M is a unitary matrix. (3)

7a. Let n be a positive integer, and let V be the vector space of all n×n matrices over the field of complex numbers. Let f be the bilinear on V defined by

$$f(A, B) = n\ \text{trace}(AB) - \text{trace}(A)\text{trace}(B).$$

Let V_2 be the subspace of V consisting of all matrices A such that trace$(A) = 0$ and $A^* = -A$ (A^* is the conjugate transpose of A). Denote by f_2 the restriction of f to V_2. Prove or disprove that $f_2(A, A) < 0$ for each nonzero A in V_2. (3)

7b. Prove that real quadratic form

$$Q = \sum_{i,j=1}^{n} a_{ij}x_i x_j = X^T AX$$

can be expressed in the form $Q = X^T BX$ where B is symmetric matrix. (2)

8. Find Singular Value Decomposition of

$$A = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}.$$

(5)