

Major Test :: MAL514 (Complex Analysis)

May 2015

Maximum time: 2 hours

Maximum marks: 40

Marks will be awarded only if appropriate arguments are provided.

1. What is the winding number of a closed rectifiable curve around a point? Produce an example of a closed rectifiable curve γ and a point P such that the winding number $n(\gamma, P)$ of γ around P is -3 . What can you say about continuity of the map $f: \mathbb{C} \setminus \{\sigma\} \rightarrow \mathbb{Z}$ defined by $f(z) = n(\sigma, z)$, for a closed rectifiable curve σ . [5]

2. Let G be an open connected subset of \mathbb{C} and let $f: G \rightarrow \mathbb{C}$ be analytic. Suppose $\{z \in G: f(z) = 0\}$ has a limit point $a \in G$. Show that $f^{(n)}(a) = 0$ for each $n \geq 0$. [4]

3. Suppose g is an analytic function. If g is constant on every circle centered at origin, then show that g is a constant function. [5]

4. State Rouché's theorem. Use it to prove the fundamental theorem of Algebra (that every nonzero polynomial with complex coefficients has as many zeros as its degree). [6 = 2 + 4]

5. Suppose γ_n is the simple closed path defined by the rectangle $[n + 1/2 + ni, -n - 1/2 + ni, -n - 1/2 - ni, n + 1/2 - ni, n + 1/2 + ni]$ (anticlockwise orientation). Let $a \in \mathbb{R} \setminus \mathbb{Z}$. Find $\int_{\gamma_n} (z+a)^{-2} \cot \pi z dz$. Note (do not prove) that $|\cot \pi z| \leq 2$ for $z \in \{\gamma_n\}$ when n is sufficiently large. Show that

$$\lim_{n \rightarrow \infty} \int_{\gamma_n} (z+a)^{-2} \cot \pi z dz = 0.$$

Further deduce that

$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$$

6. Evaluate the integral by using complex integration for $a > 0$.

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}$$

7. Find the zeros and singularities of

$$\frac{1}{\sin(1/z)}$$

Classify the singularities

8. What is an essential singularity? State Casorati-Weierstrass Theorem about essential singularities.