Class-in exam: 02:30-4:25 pm 25th.

Name: Number:

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1. [16] Prove or disprove the convexity or concavity of the following problem explicitly
   (a) [4] Consider two convex sets S_1 and S_2. Show whether a new set $S = S_1 - S_2$ is convex or not
   (b) [4] Show whether the function $f(x_1, x_2) = 10 - 3(x_2 - x_1^2)^2$ over $S = \{(x_1, x_2)| - 1 \leq x_2 \leq 1, -1 \leq x_2 \leq 1\}$ is convex or concave.
   (c) [4] $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^p$ for $p \geq 1$
   (d) [4] Let x be a real-valued random variable which takes values in $a_1, \dots, a_n$, whereas $a_1 < \cdots < a_n$ and $\Pr(x = a_i) = p_i$. Show whether the variance of $\boldsymbol{x}$ is convex or not.

(a) Suppose $x_1, x_2 \in S_1$ and $y_1, y_2 \in S_2$. Then, S can have $x_1 - y_1$ and $x_2 - y_2$. For S to be convex, $\alpha(x_1 - y_1) + (1 - \alpha)(x_2 - y_2)$ must belong to S. To see this,

$$
\begin{aligned}
&\alpha(x_1 - y_1) + (1 - \alpha)(x_2 - y_2) \\
=&\underbrace{\alpha x_1 + (1 - \alpha)x_2}_{\in S_1} - \underbrace{[\alpha y_1 + (1 - \alpha)y_2]}_{\in S_2}
\end{aligned}
$$

Thus, S is convex.

(b) Hessian of f is

$$
6\begin{bmatrix} -6x_1^2 + 2x_2 & 2x_1 \\ 2x_1 & -1 \end{bmatrix}
$$

whose determinant is $6(6x_1^2 - 2x_2 - 4x_1^2) = 12(x_1^2 - x_2)$. Thus, it is neither convex, nor concave

(c) $f(\boldsymbol{x})$ is viewed as a composition of two functions, $f(\boldsymbol{x}) = g(h(\boldsymbol{x}))$ with $g(t) = t^p$ and $h(\boldsymbol{x}) = \|\boldsymbol{x}\|$. Note that $g(t) = t^p$ is convex for $p \geq 1$ and $t \geq 0$, whereas $h(\boldsymbol{x}) = \|\boldsymbol{x}\|$ is convex. Thus, $f(\boldsymbol{x})$ is convex.

(d)

$$
\text{var}\,\boldsymbol{x} = \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2
$$

which is a concave quadratic function of p_i

2. [10] Using the steepest descent method for the problem

$$\text{minimize} \quad f(x_1, x_2) = 4x_1^2 + x_2^2$$

(a) [6] If $x_1^{(0)} = 1, x_2^{(0)} = 4$, find the expression for $x_1^{(k)}, x_2^{(k)}$ explicitly.

(b) [4] What is the rate of convergence of the sequence $f(\boldsymbol{x}^{(k)}) - f(\boldsymbol{x}^*)$?

(a) Gradient and Hessian of f is

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 8x_1 \\ 2x_2 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

Thus, $\boldsymbol{x}^{(k)}$ is updated as

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \frac{64x_1^2 + 4x_2^2}{8(64x_1^2 + x_2^2)} \begin{bmatrix} 8x_1 \\ 2x_2 \end{bmatrix} = \boldsymbol{x}^{(k)} - \frac{16x_1^2 + x_2^2}{64x_1^2 + x_2^2} \begin{bmatrix} 4x_1 \\ x_2 \end{bmatrix}$$

By induction, we have

$$\boldsymbol{x}^{(k)} = (0.6)^k \begin{bmatrix} (-1)^k \\ 4 \end{bmatrix}$$

(b) Note that the optimal value is 0. To get the rate of convergence, we write

$$\frac{4(-1)^{2k+2}(0.6)^{2k+2} + (0.6)^{2k+2}16}{4(-1)^{2k}(0.6)^{2k} + (0.6)^{2k}16} = 0.36$$

3. [12] Using KKT conditions, solve

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & x_1^2 - x_1 x_2 + 2x_2^2 - 4x_1 - 5x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Confirm that your solution is a global minimizer with the second-order KKT condition.

The objective function is

$$\frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - [4\ 5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Its minimum without constraints is obtained as

$$\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Thus, it is highly likely that the objective function hits the constraint $x_1 \leq 2$ at its minimum.
Now, consider dual feasibility is

$$\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \lambda_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If the second is active; $\lambda_1 = \lambda_3 = \lambda_4 = 0$. Then, we have

$$\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 - \lambda_2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 - \lambda_2 \\ 5 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 21 - 4\lambda_2 \\ 14 - \lambda_2 \end{bmatrix}$$

Since we assume that $x_1 = 2$, we obtain $\lambda_2 = 7/4$, with which we also have $x_2 = 7/4$
The basis of the null space for the gradient of the second constraint is $[0\ 1]^T$, while Hessian of Lagrangian is $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$. We can check

$$[0\ 1] \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 > 0$$

Thus, it is the global minimizer

4. [12] Solve the following problem with the feasible direction method

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & 2x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1 - 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & x_1 + 5x_2 \le 5 \\ & -x_1 \le 0, -x_2 \le 0 \end{aligned}$$

Use the initial point, $x_1^{(0)} = 0$ and $x_2^{(0)} = 0$. At each step, specify d_1, d_2, optimal step size α, $x_1^{(k)}$ and $x_2^{(k)}$.

- At iteration 1, since $\nabla f(\boldsymbol{x}) = \begin{bmatrix} 4x_1 - 2x_2 - 4 \\ 4x_2 - 2x_1 - 6 \end{bmatrix}$, we solve

$$\begin{aligned} \underset{\boldsymbol{d}}{\text{minimize}} \quad & -4d_1 - 6d_2 \\ \text{subject to} \quad & -d_1 \le 0 \\ & -d_2 \le 0 \\ & -1 \le d_1 \le 1, -1 \le d_2 \le 1 \end{aligned}$$

We have $d_1 = 1$ and $d_2 = 1$, while $\alpha_{\max} = 5/6$
The step size is determined by finding the minimum of $f(\alpha) = 2\alpha^2 - 10\alpha$ for $0 \le \alpha \le 5/6$. The step size is $\alpha = 2/5$ so that we have $x_1^{(1)} = 5/6$ and $x_2^{(1)} = 5/6$

- At iteration 2, we need to solve

$$\begin{aligned} \underset{\boldsymbol{d}}{\text{minimize}} \quad & -\frac{7}{3}d_1 - \frac{13}{3}d_2 \\ \text{subject to} \quad & d_1 + 5d_2 \le 0 \\ & -1 \le d_1 \le 1, -1 \le d_2 \le 1 \end{aligned}$$

We have $d_1 = 1$ and $d_2 = -1/5$. Furthermore, we find α of minimizing $f(\alpha) = \frac{62}{25}\alpha^2 - \frac{22}{15}\alpha - \frac{125}{8}$ for $0 \le \alpha \le 5/12$; $\alpha = 55/186$. Thus, $x_1^{(2)} = \frac{5}{6} - \frac{55}{186}1 = \frac{35}{31}$ and $x_2^{(2)} = \frac{5}{6} + \frac{55}{186}\frac{1}{5} = \frac{24}{31}$

- At iteration 3, we solve

$$\begin{aligned} \underset{\boldsymbol{d}}{\text{minimize}} \quad & -\frac{32}{31}d_1 - \frac{160}{31}d_2 \\ \text{subject to} \quad & d_1 + 5d_2 \le 0 \\ & -1 \le d_1 \le 1, -1 \le d_2 \le 1 \end{aligned}$$

which yields $d_1 = 1$ and $d_2 = -1/5$. So this is a KKT point.

5. [10] Consider the bound-contrainted problem ($l_i < u_i$)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} \quad & l_i \leq x_i \leq u_i \ \text{ for } \ i = 1, 2, \ldots, n \end{aligned}$$

Using the first-order KKT conditions, find the minimizer x_i^* and the minimum.

Lagrangian is

$$\sum_{i=1}^{n} (c_i + \lambda_i(-x_i + l_i) + \mu_i(x_i - u_i)) = 0$$

whereas dual feasibility condition is

$$c_i - \lambda_i + \mu_i = 0$$

- If $x_i^* = u_i$, then $\lambda_i = 0$, such that we have $\mu_i = -c_i > 0$. Thus, c_i must be negative.
- If $x_i^* = l_i$, then $\mu_i = 0$, such that we have $\lambda_i = c_i > 0$. Thus, c_i must be positive.
- If $c_i = 0$, then we can see $l_i < x_i < \mu_i$. Thus, the optimium value is

$$\sum_{i=1}^{n} (l_i \max(c_i, 0) + u_i \max(-c_i, 0))$$