Final-term **Network Performance Analysis and Modeling (ELL 894)** 2nd. May, 2017

Class-in exam: 03:30-5:30 pm, LH612.

Name: Number:

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1. [15] Show whether the following set or function is convex or not

   (a) [3] $f(x) = \ln(e^{x_1} + \cdots + e^{x_n})$

   (b) [3] The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x| \|x-a\|_2 \leq \theta\|x-b\|_2\}$.

   (c) [3] If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then the product fg is convex.

   (d) [3] $f(x) = e^{\beta \boldsymbol{x}^T A \boldsymbol{x}}$, where A is a positive semidefinite symmetric $n \times n$ matrix and β is a positive scalar.

   (e) [3] $f(\boldsymbol{x}) = -\log(-\log(\sum_{i=1}^m e^{a_i^T \boldsymbol{x}+b_i}))$ on **dom** $f = \{\boldsymbol{x}| \sum_{i=1}^m e^{a_i^T \boldsymbol{x}+b_i} < 1\}$.

(a)

$$z' \nabla^2 f(x) z = \frac{1}{\beta(x)^2} \sum_{i=1}^n \sum_{j=1}^n e^{(x_i + x_j)} (z_i - z_j)^2 \geq 0$$

Thus, f is convex.

(b) The set is convex, in fact a ball

$$\begin{aligned} &\{x| \|x-a\|_2 \leq \theta\|x-b\|_2\} \\ &= \{x| \|x-a\|_2^2 \leq \theta^2\|x-b\|_2^2\} \\ &= \{x|(1-\theta^2)x^T x - 2(a-\theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\} \end{aligned}$$

(c)

$$\begin{aligned} f(\theta x + (1-\theta)y) g(\theta x + (1-\theta)y) &\leq (\theta f(x) + (1-\theta) f(y))(\theta g(x) + (1-\theta) g(y)) \\ &= \theta f(x) g(x) + (1-\theta) f(y) g(y) + \theta(1-\theta)(f(y) - f(x))(g(x) - g(y)) \end{aligned}$$

The third term is less than or equal to zero if f and g are both increasing or both decreasing. Therefore,

$$\begin{aligned} f(\theta x + (1-\theta)y) g(\theta x + (1-\theta)y) &\leq (\theta f(x) + (1-\theta) f(y))(\theta g(x) + (1-\theta) g(y)) \\ &= \theta f(x) g(x) + (1-\theta) f(y) g(y) \end{aligned}$$

(d) This is a composition, $g(f(\boldsymbol{x}))$, where $g(t) = e^{\beta t}$ and $f(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$

(e) $g(\boldsymbol{x}) = \log(\sum_{i=1}^m e^{a_i^T \boldsymbol{x}+b_i})$ is convex from (a) together with affine mapping. So, $-g$ is concave. The function $h(y) = -\log y$ is convex and decreasing. Thus, $f = h(-g(x))$ is convex.

2. [12] Consider the equality constrained least-squares:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 \\ \text{subject to} \quad & G\boldsymbol{x} = \boldsymbol{h} \end{aligned}$$

(a) [6] Find the minimizer $\boldsymbol{x}^*$ while all the Lagrangian multipliers associated with it are explicitly determined

(b) [6] Write a pseudocode (Matlab-wise) of a numerical algorithm to solve the dual problem

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(a) We can write

$$\begin{aligned} L &= (A\boldsymbol{x} - \boldsymbol{b})^T(A\boldsymbol{x} - \boldsymbol{b}) + \boldsymbol{\lambda}^T(G\boldsymbol{x} - \boldsymbol{h}) \\ &= \boldsymbol{x}^T A^T A\boldsymbol{x} + (\boldsymbol{\lambda}^T G - 2\boldsymbol{b}^T A)\boldsymbol{x} - \boldsymbol{\lambda}^T\boldsymbol{h} + \boldsymbol{b}^T\boldsymbol{\lambda} \end{aligned}$$

The dual feasibility yields

$$\frac{dL}{d\boldsymbol{x}} = 2A^T A\boldsymbol{x} + (G^T\boldsymbol{\lambda} - 2\boldsymbol{b}^T A) = 0$$

so that we have

$$\boldsymbol{x}^\star = \frac{1}{2}(A^T A)^{-1}\left(2\boldsymbol{b}^T A - G^T\boldsymbol{\lambda}^\star\right)$$

To determine $\boldsymbol{\lambda}^\star$, we use the constraint as

$$G\boldsymbol{x} = \frac{1}{2}G(A^T A)^{-1}\left(2\boldsymbol{b}^T A - G^T\boldsymbol{\lambda}\right) = \boldsymbol{h}$$

We can further write

$$\boldsymbol{\lambda}^\star = \left(G(A^T A)^{-1}G^T\right)^{-1}\left(G(A^T A)^{-1}\boldsymbol{b}^T A - \boldsymbol{h}\right)$$

(b) Plugging $\boldsymbol{x} = \frac{1}{2}(A^T A)^{-1}\left(2\boldsymbol{b}^T A - G^T\boldsymbol{\lambda}\right)$ into L, we have the dual function as

$$\begin{aligned} D(\boldsymbol{\lambda}) = & \left[\frac{1}{2}(A^T A)^{-1}\left(2\boldsymbol{b}^T A - G^T\boldsymbol{\lambda}\right)\right]^T A^T A\left[\frac{1}{2}(A^T A)^{-1}\left(2\boldsymbol{b}^T A - G^T\boldsymbol{\lambda}\right)\right] \\ & + (\boldsymbol{\lambda}^T G - 2\boldsymbol{b}^T A)\left[\frac{1}{2}(A^T A)^{-1}\left(2\boldsymbol{b}^T A - G^T\boldsymbol{\lambda}\right)\right] - \boldsymbol{\lambda}^T\boldsymbol{h} + \boldsymbol{b}^T\boldsymbol{\lambda} \end{aligned}$$

which is a quadratic programming of $\boldsymbol{\lambda}$, where no sign restriction on $\boldsymbol{\lambda}$ is required. Newton method can solve this at one iteration.

3. [12] Consider the minimization problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = 0 \\ & \|\boldsymbol{x}\|_2 \leq 1 \end{aligned}$$

(a) [9] Find the minimizer $\boldsymbol{x}^*$ while all the Lagrangian multipliers associated with it are explicitly determined

(b) [3] Construct the dual problem and discuss the duality gap

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(a)

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \mu) = \boldsymbol{c}^T \boldsymbol{x} - \boldsymbol{\lambda}^T A\boldsymbol{x} + \mu(\boldsymbol{x}^T \boldsymbol{x} - 1)$$

$$\frac{dL(\boldsymbol{x}, \boldsymbol{\lambda}, \mu)}{d\boldsymbol{x}} = \boldsymbol{c} - A^T \boldsymbol{\lambda} + 2\mu \boldsymbol{x} = 0$$

Thus, we have $\boldsymbol{x} = \frac{1}{2\mu}\left(A^T \boldsymbol{\lambda} - \boldsymbol{c}\right)$. Plugging this into the first equality condition, we have

$$A\boldsymbol{x} = \frac{1}{2\mu}\left(AA^T \boldsymbol{\lambda} - A\boldsymbol{c}\right) = 0$$

which yields $\boldsymbol{\lambda}^* = \left(AA^T\right)^{-1} A\boldsymbol{c}$. To determine μ, Plugging $\boldsymbol{x}$ into the second inequality, we have

$$\sqrt{\frac{1}{4\mu^2}\left(A^T\boldsymbol{\lambda} - \boldsymbol{c}\right)^T \left(A^T \boldsymbol{\lambda} - \boldsymbol{c}\right)} = 1 \Rightarrow \mu = \frac{1}{2}\|A^T\boldsymbol{\lambda} - \boldsymbol{c}\|_2$$

Thus, the minimizer $\boldsymbol{x}^*$ is

$$\boldsymbol{x} = \frac{A^T \boldsymbol{\lambda}^* - \boldsymbol{c}}{\|A^T \boldsymbol{\lambda}^* - \boldsymbol{c}\|_2}$$

(b) Substituting $\boldsymbol{x} = \frac{1}{2\mu}\left(A^T \boldsymbol{\lambda} - \boldsymbol{c}\right)$ into the Lagrangian function, we get

$$\begin{aligned} \underset{\boldsymbol{\lambda}, \mu}{\text{maximize}} \quad & -\left(\frac{1}{\mu} - \frac{1}{\mu^2}\right) \boldsymbol{\lambda}^T (AA^T) \boldsymbol{\lambda} + 2\left(\frac{1}{\mu} - \frac{1}{\mu^2}\right)(A\boldsymbol{c})^T \boldsymbol{\lambda} \\ \text{subject to} \quad & \mu \geq 0 \end{aligned}$$

When we determine $\boldsymbol{\lambda}$ and μ, we can determine $\boldsymbol{x}$

4. [15] Consider the following LP

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & z = 5x_1 + 2x_2 + 8x_3 \\ \text{subject to} \quad & 2x_1 - 3x_2 + 2x_3 \geq 3 \\ & -x_1 + x_2 + x_3 \geq 5 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

(a) [6] Solve the above LP with a proper tableau method: determine x_1, x_2, x_3 and z

(b) [3] What will the objective value z and the solution x_1, x_2, and x_3 be if we replace the first constraint with $2x_1 - 3x_2 + 2x_3 \geq 8$?

(c) [3] What will the objective value z and the solution x_1, x_2, and x_3 be if we replace the objective function with $5x_1 + 2x_2 - 7x_3$?

(d) [3] What will the objective value z and the solution x_1, x_2, and x_3 be if we add a new constraint $3x_1 + 5x_2 - x_3 \geq 2$?

(a) At iteration 0,

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	5	2	8	0	0	0
x_4	2	-3	2	-1	0	3
x_5	-1	1	1	0	-1	5

Multiplying -1 and see that x_2 is the leaving variable and x_5 is the entering variable: We can perform the row operation (pivoting)

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	5	2	8	0	0	0
x_4	-2	3	-2	1	0	-3
x_5	1	-1	-1	0	1	-5

at iteration 1, we have

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	7	0	6	0	2	-10
x_4	1	0	-5	1	3	-18
x_2	-1	1	1	0	-1	5

Now, x_4 is the entering variable and x_3 is the leaving variable so that we get

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	$\frac{41}{5}$	0	0	$\frac{6}{5}$	$\frac{28}{5}$	-31.6
x_3	$-\frac{1}{5}$	0	1	$-\frac{1}{5}$	$-\frac{3}{5}$	3.6
x_2	-0.8	1	0	0.2	-0.8	1.4

Thus, $x_1 = 0$, $x_2 = 1.4$, $x_3 = 3.6$ and $z = 31.6$

(b) The objective value will be increased by $\Delta b y_1$ (y_1 is Lagrangian multiplier). Thus, we have $31.6 + 5\frac{6}{5} = 37.6$

$$\boldsymbol{x} = \boldsymbol{x}^* + B^{-1}\begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}0.4\\4.6\end{bmatrix}$$

(c) Using the basis that we have in the tableau, we can see that the problem is no longer feasible

$$\boldsymbol{c}_N - \boldsymbol{c}_N B^{-1} N \succeq 0$$

(d) The additional constraint is inactive at the solution. Thus, the corresponding Lagrangian multiplier is zero. Then, the objective value does not change.

5. [6] Show that a point $\boldsymbol{x} = [0 \;\; 3.12 \;\; 0.92]^T$ of the following LP

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & -3x_1 + 13x_2 + 9x_3 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 5 \\ & x_1 + 3x_2 + 2x_3 \leq 11.2 \\ & 2x_2 + 3x_3 \leq 9 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

is optimal.

The dual feasibility of this LP is

$$\begin{bmatrix} 3 \\ -13 \\ -9 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \mu_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \mu_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \mu_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since x_2 and x_3 are not zero, we can set $\mu_2 = \mu_3 = 0$. In addition, only the first constraint is inactive, i.e., $\lambda_1 = 0$, we can have

$$\begin{bmatrix} 3 \\ -13 \\ -9 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \mu_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} -3 \\ 13 \\ 9 \end{bmatrix}$$

We can get $\lambda_2 = 4.2$, $\lambda_3 = 0.2$, and $\mu_1 = 7.2$. It satisfies KKT conditions, thus it is optimal.