Major: Part B | **PYL112: Quantum Mechanics** | **Time: 1.5 hours**
09-05-2016 | **II Semester 2015-16** | **Max. Marks: 30**

Answer any FIVE questions

NOTE: All symbols used have their usual meaning. Each question carries 6 marks. Some useful mathematical expressions are given at the end of the questions.

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1) Consider a hydrogen atom in its *ground* state.

a) Find the *Clebsch-Gordan coefficients* associated with the coupling of spins of the electron and the proton.

b) Find the transformation matrix which is formed by the C-G coefficients. Check whether this matrix is unitary or not.

[4+2 = 6 Marks]

2) Consider a system of total angular momentum $j = 1$. The operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ (in the usual eigen basis of $\hat{\vec{J}}^2$ and $\hat{J}_z$) are given by

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{J}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) What are the possible values when measuring $\hat{J}_x$?

b) Calculate $\langle \hat{J}_z \rangle$ if the system is in the state $J_x = -\hbar$.

c) If the system was initially in the state $|\psi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{3} \\ 2\sqrt{2} \\ \sqrt{3} \end{pmatrix}$, what values will one obtain when measuring $\hat{J}_x$ and with what probabilities?

[2+1+3 = 6 Marks]

3) The wave function of an electron in a hydrogen atom is given by

$$|\psi_{21m_lm_s}(r,\theta,\varphi)\rangle = R_{21}(r)\left[\frac{1}{\sqrt{3}}Y_{10}(\theta,\varphi)\left|\frac{1}{2},+\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}Y_{11}(\theta,\varphi)\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right].$$

Where $\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle$ are the spin state vectors.

a) Calculate $\hat{J}_z|\psi_{21m_lm_s}(r,\theta,\varphi)\rangle$, i.e., the *z-component* of the electron's *total* angular momentum.

b) If you measure $\hat{\vec{J}}^2$, what values will you obtain? What are the corresponding probabilities?

c) If you measure the *z-component* of the electron's *orbital* angular momentum, what values will you obtain? What are the corresponding probabilities?

d) Calculate $\langle\psi_{21m_lm_s}|\hat{L}_-|\psi_{21m_lm_s}\rangle$ and $\langle\psi_{21m_lm_s}|\hat{S}_z|\psi_{21m_lm_s}\rangle$.

[1+2+1+2 = 6 Marks]

$+iJ_y$ $-iJ_y$ $= 2J_x$

$J_+ + J_- = 2J_x$

$J_x + iJ_y$

4) Consider the electron's spin operator: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, where the components of $\vec{\sigma}$ are the Pauli spin matrices.

a) Find the eigenvalues and eigenstates of $\vec{S}$ in the direction of a unit vector $\vec{n}$; assume $\vec{n}$ lies in the xz plane.

b) Find the probability of measuring $\hat{S}_z = +\frac{\hbar}{2}$.

[4+2 = 6 Marks]

5) Consider a hydrogen atom whose wave function is given at time $t = 0$ by

$$\Psi(\vec{r},0) = \frac{A}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} + \frac{1}{\sqrt{2\pi}}\left(\frac{z-\sqrt{2}x}{r}\right)R_{21}(r),$$

Where A is a real constant, a_0 is the Bohr radius, and $R_{21}(r)$ is the radial wave function: $R_{21}(r) = \frac{1}{\sqrt{6}}\left(\frac{1}{a_0}\right)^{3/2}\left(\frac{r}{2a_0}\right)e^{-r/2a_0}$.

a) Write down $\Psi(\vec{r},0)$ in terms of $\sum_{n,l,m}\psi_{nlm}(\vec{r})$ where $\psi_{nlm}(\vec{r}) = R_{nl}(r)\,Y_{lm}(\theta,\varphi)$ is the hydrogen atom wave function

b) Find A so that $\Psi(\vec{r},0)$ is normalized, i.e., $\int \psi^*_{n'l'm'}(\vec{r})\,\psi_{nlm}(\vec{r})\,d^3r = \delta_{n'n}\,\delta_{l'l}\,\delta_{m'm}$.

c) Write down the wave function $\Psi(\vec{r},t)$ at some later time t.

d) Is $\Psi(\vec{r},0)$ an eigenfunction of $\hat{L}^2$ and $\hat{L}_z$? If yes, what are the eigenvalues?

e) If a measurement of the energy is made, what value(s) could be found and with what probability?

f) What is the probability that a measurement of $\hat{L}_z$ yields $1\hbar$?

[1+1+1+1+1+1 = 6 Marks]

6) Consider a system of *two spinless particles* of reduced mass μ that is subject to a finite, central potential well

$$V(r) = \begin{cases} -V_0 & for\ \ 0 \le r \le a \\ 0 & for\ \ r > a \end{cases}$$

Where V_0 is positive. The purpose of this problem is to show how to find the minimum value of V_0 so that the potential well has one $l = 0$ bound state.

a) Find the solution of the radial Schrödinger equation in both regions, $0 \le r \le a$ and $r > a$, in the case where the particle has zero angular momentum and its energy is located in the range $-V_0 < E < 0$.

b) Show that the continuity condition of the radial function at $r = a$ can be reduced to a transcendental equation in E.

c) Find the minimum value of V_0 so that the system has (i) one bound state, and (ii) two bound states. [*Hint:* Use the continuity condition in (b) OR the method of graphical solution of the transcendental equation derived in (b)].

[2+2+2 = 6 Marks]

Table 1: Spherical Harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta,\varphi)$	$Y_{lm}(x,y,z)$
$Y_{00}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}$	$Y_{00}(x,y,z)=\frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_{10}(x,y,z)=\sqrt{\frac{3}{4\pi}}\frac{z}{r}$
$Y_{1,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$	$Y_{1,\pm1}(x,y,z)=\mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$
$Y_{20}(\theta,\varphi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$Y_{20}(x,y,z)=\sqrt{\frac{5}{16\pi}}\frac{3z^2-r^2}{r^2}$
$Y_{2,\pm1}(\theta,\varphi)=\mp\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\theta\cos\theta$	$Y_{2,\pm1}(x,y,z)=\mp\sqrt{\frac{15}{8\pi}}\frac{(x\pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta,\varphi)=\sqrt{\frac{15}{32\pi}}e^{\pm2i\varphi}\sin^2\theta$	$Y_{2,\pm2}(x,y,z)=\mp\sqrt{\frac{15}{32\pi}}\frac{x^2-y^2\pm2ixy}{r^2}$

Table 2: First few radial wave functions $R_{nl}(r)$ of the hydrogen atom

$R_{10}(r)=2a_0^{-3/2}e^{-r/a_0}$	$R_{21}(r)=\frac{1}{\sqrt{6a_0^3}}\frac{r}{2a_0}e^{-r/2a_0}$
$R_{20}(r)=\frac{1}{\sqrt{2a_0^3}}\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}$	$R_{31}(r)=\frac{8}{9\sqrt{6a_0^3}}\left(1-\frac{r}{6a_0}\right)\left(\frac{r}{3a_0}\right)e^{-r/3a_0}$
$R_{30}(r)=\frac{2}{3\sqrt{3a_0^3}}\left(1-\frac{2r}{3a_0}+\frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$	$R_{32}(r)=\frac{4}{9\sqrt{30a_0^3}}\left(\frac{r}{3a_0}\right)^2e^{-r/3a_0}$

Table 3: First few Legendre polynomials and associated Legendre functions.

Legendre Polynomials	Associated Legendre Functions
$P_0(\cos\theta)=1$	$P_1^1(\cos\theta)=\sin\theta$
$P_1(\cos\theta)=\cos\theta$	$P_2^1(\cos\theta)=3\cos\theta\sin\theta$
$P_2(\cos\theta)=(3\cos^2\theta-1)/2$	$P_2^2(\cos\theta)=3\sin^2\theta$
$P_3(\cos\theta)=(5\cos^3\theta-3\cos\theta)/2$	$P_3^1(\cos\theta)=3\sin\theta(5\cos^3\theta-1)/2$
$P_4(\cos\theta)=(35\cos^4\theta-30\cos^2\theta+3)/8$	$P_3^2(\cos\theta)=15\sin^2\theta\cos\theta$
$P_5(\cos\theta)=(63\cos^5\theta-70\cos^3\theta+15\cos\theta)/8$	$P_3^3(\cos\theta)=15\sin^3\theta$

$-\psi_{111}=\sqrt{\frac{3}{8\pi}}\,x$

$\psi_{11-1}=\sqrt{\frac{3}{8\pi}}$

Table 4: First few Laguerre polynomials and associated Laguerre polynomials.

Laguerre polynomials $L_k(r)$	Associated Laguerre polynomials $L_k^N(r)$
$L_0 = 1$	$L_1^1 = -1$
$L_1 = 1 - r$	$L_2^1 = -4 + 2r, \quad L_2^2 = 2$
$L_2 = 2 - 4r + r^2$	$L_3^1 = -18 + 18r - 3r^2, \; L_3^2 = 18 - 6r, \; L_3^3 = -6$
$L_3 = 6 - 18r + 9r^2 - r^3$	$L_4^1 = -96 + 144r - 48r^2 + 4r^3,$
$L_4 = 24 - 96r + 72r^2 - 16r^3 + r^4$	$L_4^2 = 144 - 96r + 12r^2, \; L_4^3 = 24r - 96, \; L_4^4 = 24$
$L_5 = 120 - 600r + 600r^2 - 200r^3 + 25r^4 - r^5$	$L_5^1 = -600 + 1200r - 600r^2 + 100r^3 - 5r^4,$
	$L_5^2 = 1200 - 1200r + 300r^2 - 20r^3,$
	$L_5^3 = -1200 + 600r - 60r^2, \; L_5^4 = 600 - 120r,$
	$L_5^5 = -120$