FUNCTIONAL ANALYSIS MAL 602

MAJOR TEST

Maximum Credit : 40 May 6, 2016

The numbers on the right indicate maximum credit for the corresponding problems. JUSTIFY YOUR ANSWERS.

All linear spaces in this paper are over the field $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The field $\mathbb{F}$ has its usual topology.

Q.1. Suppose (x_n) is a sequence of pairwise orthogonal non-zero vectors in a Hilbert space H and suppose $\sum_{n=1}^{\infty} \langle x_n, y\rangle$ converges in $\mathbb{F}$ for each $y \in H$. For each n, define the linear operator $\Lambda_n : H \longrightarrow \mathbb{F}$ by letting

$$\Lambda_n(y) = \sum_{i=1}^{n} \langle y, x_i\rangle = \langle y, x_1 + x_2 + \cdots + x_n\rangle, \; y \in H.$$

(a) Give a direct proof to show that $\|\Lambda_n\| = \|x_1 + \cdots + x_n\|$. [3]

(b) Show that for each $y \in H$, $\exists\, M_y > 0$ such that $|\Lambda_n(y)| \leq M_y \;\forall\, n \in \mathbb{N}$. [3]

(c) Show that $\sup_n \|\Lambda_n\| < \infty$. [3]

(d) Show that the series $\sum_{n=1}^{\infty} \|x_n\|^2$ converges. [4]

Q.2. Suppose (a_n) is a real sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every real sequence (b_n) which satisfies $\sum_{n=1}^{\infty} b_n^2 < \infty$. Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges. [7]

(Hints: Consider the real Hilbert space ℓ_2. For each m, define $f_m : \ell_2 \longrightarrow \mathbb{R}$ by $f_m(b) = \sum_{i=1}^{m} a_i b_i$ where $b = (b_n)$ is in ℓ_2. First show that $f_m \in \ell_2^*$ and $\|f_m\| = \left(\sum_{i=1}^{m} a_i^2\right)^{1/2}$. Then try to apply the PUB.)

$T^{2^n}(x)$ $\quad \langle T^{2^{n+1}}(x), x\rangle$

$T^{2^{n+1}}$ $\quad = \langle T^{2^n}(x), T^{2^n}$

Q.3. Let X be a normed linear space, $z \in X$ and $f \in X^*$. For each $x \in X$, define $T: X \to X$ by $T(x) = f(x)z$. By using the sequential characterization of a compact operator, show that T is a compact linear operator. [6]

Q.4. Let H be a Hilbert space and $T: H \to H$ be a non-zero bounded linear map such that $T = T^*$.

(i) Show that T^m is self-adjoint $\forall\, m \in \mathbb{N}$.

(ii) Show that T^2 is non-zero.

(iii) Use induction and (ii) to show that T^{2^n} is non-zero $\forall\, n \in \mathbb{N}$. [~~1+2+3=~~6]

Q.5. Let H be a Hilbert space and $E: H \to H$ be a non-zero idempotent bounded linear operator. If $\langle Eh, h\rangle \geq 0\ \forall\, h \in H$, then show that $\ker E \subseteq (\operatorname{ran} E)^{\perp}$. [5]

Q.6. Let $K: [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. Given $f \in C[a,b]$, define $T(f): [a,b] \to \mathbb{R}$ by $T(f)(x) = \int_a^b K(x,y) f(y)\,dy$ for each $x \in [a,b]$.

(i) Show that $T(f)$ is continuous. [2]

(ii) If $B = \{f \in C[a,b] : \|f\|_\infty < 1\}$, then show that $T(B)$ is equicontinuous. [4]