Department of Mathematics

Indian Institute of Technology Delhi

MTL 411/MAL245 Functional Analysis: Major Test

Maximum Marks: 50 Time: 2 hours

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1. State whether the statements [(a)-(j)] given below are true or false. Justify your answer.

(a) An open interval (a, b) in $\mathbb{R}$ is of first category.

(b) If $X = \mathcal{R}[a, b]$, the space of all functions $f : [a, b] \to \mathbb{R}$ such that $|f|$ is Riemann integrable, then the function $\|f\|_1 = \int_a^b |f(t)|\, dt$ is not a norm on X.

(c) The set $S = \{f \in C[0, 1] : f(\frac{1}{2}) = 0\}$ is open in $\left(C[0, 1], \|.\|_\infty\right)$.

(d) Let $C[-1, 1]$ be endowed with the norm $\|f\|_1 := \int_{-1}^1 |f(t)|\, dt$. Consider the sequence $(f_n)_{n=2}^\infty$ defined by

$$f_n(t) = \begin{cases} 0 & \text{if } t \notin [\frac{-1}{n}, \frac{-1}{n}], \\ 1 + nt & \text{if } \frac{-1}{n} \leq t \leq 0, \\ 1 - nt & \text{if } 0 \leq t \leq \frac{1}{n}. \end{cases}$$

Then $f_n \to 0$ as $n \to \infty$.

(e) Let $\mathcal{P}$ denote the space of all polynomials with coefficients in $\mathbb{K}$ endowed with the sup-norm. Define $T : \mathcal{P} \to \mathcal{P}$ by $Tp(x) = np(x)$, where n is the degree of the polynomial p. Then T is a linear unbounded operator.

(f) A continuous linear functional f on a normed linear space X is called an extension of a given continuous linear functional g defined on a subspace M of X, if $f(x) = g(x)$ for all $x \in M$. If f is an extension of g, then $\|f\| \geq \|g\|$.

(g) Let $X = \left(C[0, 1], \|.\|_\infty\right)$ and $T : X \to \mathbb{R}$ be defined by $Tf = \int_0^1 f(t)\, dt$. Then T is a bounded linear map. Further, since T is an isometry it follows that $\|T\| = 1$.

(h) Let T be an injective bounded linear operator from a Banach space X onto a Banach space Y. Then T is a homeomorphism.

(i) If $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors in a normed linear X, then for any set $a_1, a_2, \ldots, a_n$ of real numbers, there exists a continuous linear functional f on X such that $f(v_k) = a_k$ for $k = 1, 2, \ldots, n$.

Definition 1 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be 2π-periodic if $f(. + 2\pi) = f(.)$. We define $C_{2\pi} := \{f : \mathbb{R} \to \mathbb{R} | f \text{ is } 2\pi-\text{periodic and continuous on } \mathbb{R}\}$ and endow the space $C_{2\pi}$ with the norm $\|f\|_{\infty,[-\pi,\pi]} := \sup\{|f(x)| : -\pi \leq x \leq \pi\} = \sup\{|f(x)| : x \in \mathbb{R}\}$. A trigonometric polynomial of degree N is a function of the form $p(t) := \frac{a_0}{2} + \sum_{k=1}^N [a_k \cos(kt) + b_k \sin(kt)]$. The collection of all such trigonometric polynomials of degree N is denoted by T_N.*

(j) For every function f in $C_{2\pi}$ there is a $p^* \in T_N$ such that $\|f - p^*\|_{\infty,[-\pi,\pi]} = \inf\{\|f-p\|_{\infty,[-\pi,\pi]} : p \in T_N\}$.

[$10 \times 3 = 30$ Marks]

2. (a) Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Prove that T is injective and $\mathrm{Rg}(T)$ is closed in Y if and only if there exists a constant $C > 0$ such that $\|x\| \leq C\,\|Tx\| \quad \forall\ x \in X$.

(b) Let $X = C[0,1]$ be endowed with the supnorm. Let $T : X \to X$ be a linear map defined by

$$f(t) \mapsto \int_0^t f(s)\ \mathrm{d}s \quad (t \in [0,1]).$$

Is T bounded ? Is T injective ? Is $\mathrm{Rg}(T)$ closed ? Justify.

[$4 + 3 = 7$ Marks]

3. (a) Let $T : X \to Y$ be a linear operator between the Banach spaces X and Y such that

$$x_n \to 0 \text{ and } Tx_n \to y \quad \text{as } n \to \infty \Longrightarrow y = 0.$$

Is T is bounded ? Give reason to support your answer.

(b) Let $C[a,b]$ be endowed with the norm $\|f\|_2 = [\int_a^b |f(t)|^2 \mathrm{d}t]^{1/2}$. Consider a fixed $g \in C[a,b]$ and define $T : C[a,b] \to \mathbb{K}$ by

$$f \mapsto \int_a^b f(t)\overline{g(t)}\ \mathrm{d}t.$$

Prove that T is a bounded linear functional and $\|T\| = \|g\|_2$.

[$3 + 4 = 7$ Marks]

4. Test whether the following linear transformations are compact or not.

(a) Let $C[a,b]$ be endowed with the supnorm and let $A : C[a,b] \to C[a,b]$ be defined by

$$Ax(s) = \int_a^s x(t)\ \mathrm{d}t \quad \forall\ s \in [a,b].$$

(b) Let $B : l^p \to l^p$ be the left-shift operator defined by

$$B(x_1, x_2, x_3 \ldots) = (x_2, x_3, \ldots).$$

[$3 + 3 = 6$ Marks]