28^{th} August 2016

Total Maximum Marks: **20**

PYL113/EPL103, Mathematical Physics, Minor 1, Physics Department (IIT, Delhi)

All questions are compulsory, maximum marks for each question are given in **bold** numericals. Symbols carry usual meaning unless otherwise stated.

(Q1) (a) **(3 marks)** Consider an analytic function $f(z) = u(x, y) + iv(x, y)$. Show that $f^*(z^*)$ is also analytic.

(b) **(2 marks)** If $f(z)$ is analytic. Show explicitly that the derivative of $f(z)$ with respect to z^* exists only when $f(z)$ is constant.

*Hint:- * operation implies complex conjugate.*

(Q2) (a) **(2 marks)** Expand $f(z) = \frac{1}{1-z}$ in Taylor series about $z = i$ and find the radius of convergence.

(b) **(3 marks)** Find the Laurent series expansion of $f(z) = e^z/(z+1)$ such that $0 < |z+1| < \infty$. What is the residue (*meaning coefficient* b_1)? Now use Cauchy's integral formula to evaluate:

$$I = \frac{1}{2\pi i} \oint_C \frac{e^z}{(z+1)} \, dz$$

where C is a **positively oriented** contour encircling $z = -1$. Show that the answer matches with the one obtained using residue b_1 from Laurent series expansion above.

(c) **(2 marks)** Evaluate $I = \frac{1}{2\pi i} \oint_C \pi \cot(\pi z) \, dz$ where C is a **negatively** oriented contour enclosing singularity at $z = 0$ only.

| > <

(Q4) (a) **(3 marks)** Consider a set of orthonormal basis vectors $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, ..., |e_n\rangle\}$. Show that the projection operator defined as $\hat{P}_m = \sum_{j=1}^{m} |e_j\rangle\langle e_j|$, over a subset of $m \leq n$ vectors is idempotent that is $\hat{P}_m^2 = \hat{P}_m$. Hence show that the **eigenvalues** of $\hat{P}_m$ are 0 or 1.

$e_j \cdot e_i$

(b) **(2 marks)** Consider a **real** vector space of dimension 3 formed by **column** vectors. For this vector space write down a set of **orthonormal** basis vectors. Prove the completeness that is $\hat{P}_3 = \sum_{j=1}^{3} |e_j\rangle\langle e_j| = I$, where I is an identity matrix.

(c) **(3 marks)** Show that any real 3×3 matrix B can be written as a linear combination of entities of the form $|e_i\rangle\langle e_j|$, with $i, j = 1, 2, 3$ and $|e_i\rangle$'s are the basis vectors found in (b) above.

(P. T. O.)

Some useful formulae

- Cauchy-Riemann (CR) conditions $u_x = v_y$ and $u_y = -v_x$.
- Taylor expansion of $e^z = \sum_{n=0}^{\infty} z^n/n!$.
- $1/(1-z) = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$.
- Cauchy's integral formula

$$I = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)}\, dz = \begin{cases} f(z_0), & \text{if } C \text{ includes } z_0. \\ 0, & \text{otherwise} \end{cases}$$

- Laurent series expansion in the neighbourhood of point z_0, of a function $f(z)$, analytic in an annular region is:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n\,+1} + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^n} \quad \text{and} \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^{-n+1}}$$