$x^4 - p$

$p^4 - y^4$

$x^4 -$

Department of Mathematics
Major Test
Course: MAL 145 (Number Theory)

Duration: 2 hours

Note: All questions are compulsory. **M. Marks: 50**

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1. Prove that any positive integer, not of the form $4^a(8k+7)$, can be written as a sum of three squares. **[8 marks]**

2. Prove that any positive multiple of 8 is a sum of eight odd squares. **[5 marks]**

3. Prove that if x, y, z is a primitive Pythagorean triple in which x and z are consecutive positive integers, then $x = 2t(t+1), y = 2t+1$ and $z = 2t(t+1)+1$ for some integer $t > 0$. **[5 marks]**

4. Prove that the equation $x^4 - y^4 = 2z^2$ has no solution in positive integers x, y, z. **[5 marks]**

5. If p is an odd prime, show that $\sum_{a=1}^{p-2} \left(\frac{a(a+1)}{p}\right) = -1$. From this, deduce that for $p > 5$, there exist integers $1 \leq a, b \leq p-2$ for which $\left(\frac{a}{p}\right) = \left(\frac{a+1}{p}\right) = 1$ and $\left(\frac{b}{p}\right) = \left(\frac{b+1}{p}\right) = -1$. **[4+4 marks]**

6. For a fixed integer $n > 1$, show that all the solvable congruences $x^2 \equiv a \pmod{n}$ with $\gcd(a, n) = 1$ have the same number of solutions. **[5 marks]**

7. Prove the following identity combinatorially:

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-x^n}\right) = 1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)^2(1-x^2)^2 \cdots (1-x^m)^2}.$$

[7 marks]

8. Prove that the density of lattice points in the plane that are visible from the origin is $6/\pi^2$.

$\langle 1, 3, 7, 9 \rangle$ **[7 marks]**

$2 \neq a^2 + b^2$

1, 3, 5