PYL113 MAJOR EXAMINATION

Max. Marks : 50

Attempt all questions.

1. Consider the one dimension heat flow equation or the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = D\frac{\partial u}{\partial t}, \quad x \in [0, l],$$

with $u(x, 0) = f(x)$, $u(0, t) = \alpha$ and $u(l, t) = \beta$. Obtain $u(x, t)$ by the method of separation of variables. (10)

2. The first part carries 8 marks, the second part carries 5 marks.

   (a) Using the Green function technique, obtain the solution of the Helmholtz equation

$$\left(\nabla^2 - p^2\right)\psi(\vec{r}) = f(\vec{r}).$$

   (b) Now consider the Schodinger equation for a simple square well of radius a given by

$$\left(\nabla^2 + \frac{2mE}{\hbar^2}\right)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(r)\psi(\vec{r}),$$

   where $V(r) = -V_o$ for $a \le r \le b$ (with $a, b > 0$), and 0 otherwise. Using the Green function from (i), obtain the solution of this equation at large-r assuming the simplest form of $\psi(\vec{r})$.

3. Each question carries 5 marks.

   (a) Solve the following integral equation

$$\int_0^\infty \cos(xv)\mathrm{y}(v)\mathrm{d}v = \exp\left(-x^2/2\right)$$

   for the function $\mathrm{y}(x)$ assuming that it is even.

   (b) Closely related to the (cylindrical) Bessel functions are the spherical Bessel functions given by:

$$J_\nu(x) = 2^\nu x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n(\nu+n)!}{n!(2n+2\nu+1)!}\, x^{2n}.$$

   Verify directly from this definition that $J_0' = -J_1(x)$. Show with *little* work that

$$J_1(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2}.$$

   (c) By finding the eigenvectors of the Hermitian matrix

$$H = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix},$$

   Construct a unitary matrix U such that $U^\dagger H U = \wedge$, where $\wedge$ is a real diagonal matrix.

4. Solve any 3, each question carries 4 marks. You may solve all if you wish to improve your marks.

   (a) If $z = x + iy$, prove that $|\,x\,| + |\,y\,| \le \sqrt{2}|z|$
   (b) Study the differentiability of $g(z) = \sin(2z)$.
   (c) Expand $\exp(-z^2)\sinh(z+2)$ about $z_o = 0$.
   (d) Obtain the Fourier transform $\mathcal{F}[te^{-\alpha t}H(t)]$, where $H(t)$ is the Heaviside function.
   (e) Use the recurrence relation $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ and the Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = (3x^2-1)/2$ to evaluate $P_3(x)$ and $P_4(x)$.