Experiment S4: Measuring Harmonic Oscillators

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Contents

1 Introduction

This experiment concerned itself with damped harmonic oscillations and their modelling, specifically the parallels that can be drawn between different types of damped harmonic systems. The experiment also considered resonance in the case of forced oscillations for the spring-cart system. Practically, it involved filming oscillations with a camera placed perpendicular to the motion in line with the origin. The footage was then saved and sent to the Tracker video analysis software, which returned values for the displacement from the origin with respect to time.

1.1 Initial Calibration

We initially tested the experimental setup with a simple example. The cart was fully loaded with weights and the springs with the lowest spring constant were used so that the oscillation might be as slow as possible and therefore as easy as possible for the software to analyse. Being attached by strings at both ends, the cart rests at equilibrium. A stable marker was set up behind the cart to mark this point. The cart was then pulled to the side by a small distance and released with the camera rolling. The film was stopped when the cart came to a complete stop, and the footage was sent to Tracker. Below is an image of the initial conditions of the Tracker video:

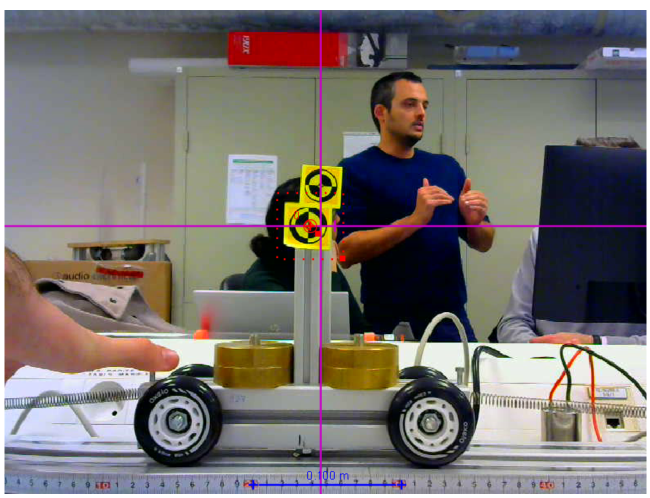

Figure 1: Initial Conditions of the Tracker Software

The blue line with a labelled distance represents the reference distance, which the software uses to calibrate itself. The purple axes represent the x and y directions, as well as the 0 in each respective direction. Finally, the red point and outline represents the object being tracked.

Below is an example of the data output from the Tracker system:

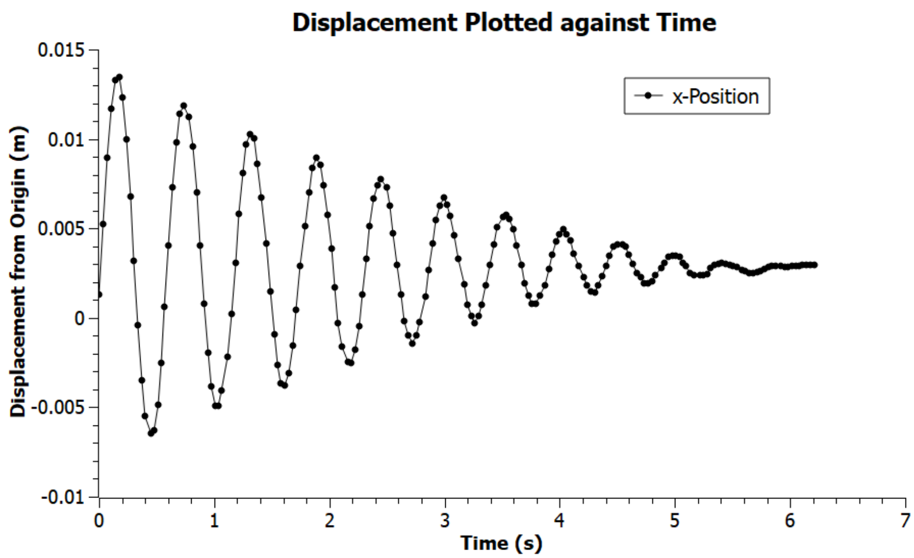

Figure 2: The Plot of an Exemplar Oscillation

1.2 Preliminary Remarks

Let us define the units of the important values in the experiment:

Quantities in the Experiment and their Units	
Quantity	Unit
f	s^{-1}
ω	s^{-1}
k	kgs^{-1}
M	kg

Figure 3: Table of Units: Experiment 1

We will also clarify the masses of the objects involved:
The spring constant of the springs used was calculated and found to be $132N/m$.

2 Experiment 1: Free Mass-Spring Oscillator

This experiment was concerned specifically with the analysis of the motion of a wheeled cart attached with springs at each end, and set in motion by releasing it from rest at a certain distance from its equilibrium. It then goes through a sinusoidal motion that is damped by frictional forces.

Experimental Objects and their Mases	
Object	Mass (kg)
Cart	0.9370
Mass 1	0.3261
Mass 2	0.3261
Mass 3	0.3261
Mass 4	0.3271

Figure 4: Table of Masses

2.1 Experimental Setup

The Experimental setup was as described in the initial calibration.

2.2 Uncertainty of the Measurement

This experiment sought to determine the degree of uncertainty of the measurements that were being made within the experiment.

2.2.1 Experimental Protocol

As described above, we filmed the motion of a cart that had been displaced by a certain amount, in this case exactly $3cm$ to the left. This experiment was repeated and filmed 5 times. We then followed the same procedure as outlined in Section 1.1.

2.2.2 Presentation of Results

For each trial, two amplitude peaks were chosen where the data point was as close as possible to the esti mated peak of the wave in order to reduce the uncertainty. The time difference was then divided by the number of oscillations between the two points to get the time period of one wave. The reciprocal of this value was taken to calculate the frequency. The five trials and their respective estimated frequencies are tabulated below.

Frequency Estimation	
Trial Number (#)	Frequency (Hz)
1	1.724
2	1.721
3	1.724
4	1.721
5	1.724

Figure 5: Experiment 1a Results Table: Estimated Frequencies of Oscillation.

2.2.3 Conclusion

From the data above, we can calculate an average frequency of $1.723Hz$. This also allows us to calculate the standard deviation of the measurements, for which the formula is:

$$\sqrt{\frac{\sum_{i=1}^{n}(X^i - \bar{X})^2}{n-1}}$$

This yields a value for the uncertainty of: $1.658 \times 10^{-3}Hz$. The final result is therefore:

$$f = 1.723Hz \pm 0.002Hz$$

One of the main sources of error in the system may be the Tracker software, which introduces a certain amount of noise to the measurement, as the software does not perfectly track the point but instead follows it as best it can.

2.3 Influence of Initial Conditions

We now want to determine in what way the initial conditions of the system, particularly the initial displacement of the cart, affect the behaviour of the system.

2.3.1 Experimental Protocol

In order to do this, for each of 5 different initial displacements from the origin, (1,2,3,4 and 5 cm), the cart was released and filmed three times. Each of these videos was then analysed in Tracker to determine the average time-period of one oscillation, and from this, the frequency. The three frequencies were then averaged to get a result for each initial condition.

2.3.2 Data Analysis

The data for the initial displacement $x_0 = 3$ was not re-measured, with instead the result obtained in Section 2.2.3 being used. Below is the table of results:

Frequency against Initial Displacement	
Initial Displacement (cm)	Frequency (Hz)
1	1.733
2	1.719
3	1.724
4	1.727
5	1.722

Figure 6: Experiment 1b Results Table: Frequency against Initial Displacement.

2.3.3 Presentation of Results

Below is the graph of the data above, with a line inserted at $f = 1.725Hz$, the average of the 5 frequencies.

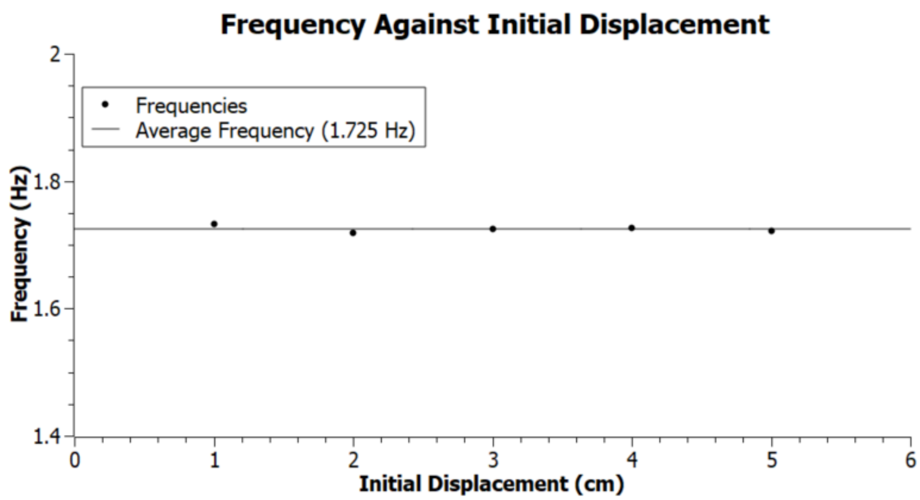

Figure 7: The Plot of Frequency against Initial Displacement

2.3.4 Conclusion

Given the fact that the variations in the frequency with respect to the initial displacement are of the order of 1% and that there is no discernible trend to what little variation there is, we might suggest that the initial condition has no bearing on the frequency, a conclusion that is in line with the theory governing the system, which would dictate that the frequency changes only with respect to the mass of the cart and the stiffness of the springs.
In theory, we know that the angular velocity of the system is defined as:

$$\omega = \sqrt{\frac{k}{m}}$$

Where k is the stiffness of the spring and m is the mass. The time period is defined as:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

So the frequency is defined as:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Thus, if we have the frequency and the mass, we can find k, which is defined as:

$$k = m(2\pi f)^2$$
$$k = 2.2424 \times (2 \times \pi \times 1.725)^2$$
$$k = 263.42 N/m$$

By taking the standard deviation of the data above, we can find that the uncertaity of the frequency is: $0.005 Hz$ (equivalent to 0.3%). The uncertainty of the mass, assuming it was weighed on a normal scientific scale, would be $1g$, which is $\approx 0.05\%$. This yields an uncertainty on the stiffness of 0.35%, which in real terms is $0.92 N/m$. Therefore:

$$k = 263.42 N/m \pm 0.35\%$$
$$k = 263 N/m \pm 1 N/m$$

This is in line with the result that we initially had for the stiffness of the spring when it was calculated on its own by adding masses and measuring the distance stretched. That gave a value of $132 N/m$ for one spring, and here, the two spring system is giving a value of $263 N/m$, which is $\approx 2 \times 132$.

2.4 Influence of the Mass

We now want to determine to what extent variations in the mass of the cart for a fixed spring constant affect the frequency of oscillation of the system.

2.4.1 Experimental Protocol

In order to determine the variations, the cart was filmed three times in each of five different configurations, the first configuration was with the cart totally unweighted, and so carrying only its own mass. Each successive configuration added one plate, whose weights, along with the weight of the cart, are enumerated in Figure 4. These fifteen films were then analysed in Tracker to determine the average time period of an oscillation at each mass, and the reciprocal was taken to find the frequency.

2.4.2 Data Analysis

Below is a table comparing the total mass of the system and the mean frequency of its oscillations.

2.4.3 Presentation of Results

Below is a graphical representation of the table above.

Frequency against System Mass	
Total Mass (g)	Frequency (Hz)
937.0	2.674
1263.1	2.280
1589.2	2.042
1915.3	1.863
2242.4	1.720

Figure 8: Experiment 1c Results Table: Frequency against System Mass.

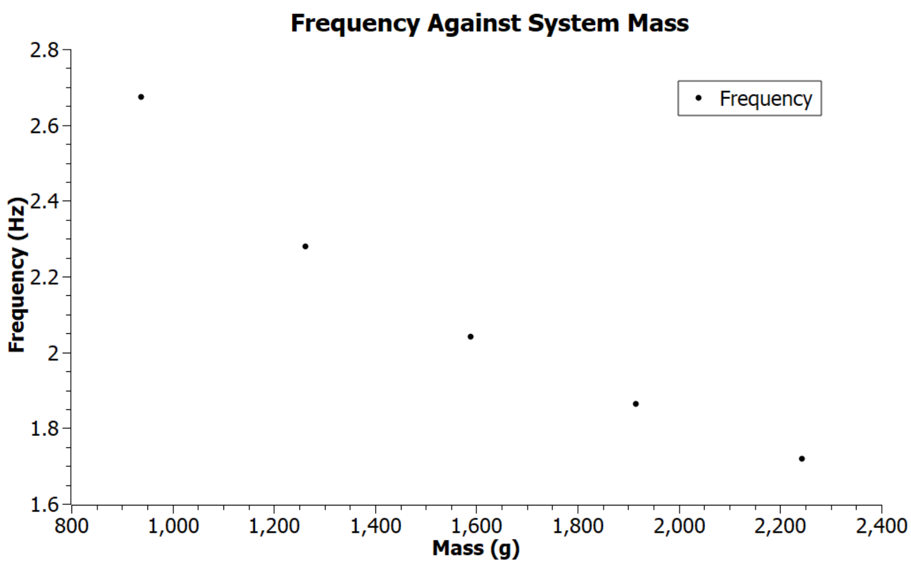

Figure 9: The Plot of Frequency against System Mass

Given that we know that in principle, the frequency obeys an inverse square root relation ship with the mass, that is to say:

$$f \propto \frac{1}{\sqrt{M}}$$

We can redraw this graph by plotting frequency against the inverse of the square root of the mass, shown below:

2.4.4 Conclusion

Given that in fact, the angular velocity is defined as:

$$\omega = \sqrt{\frac{k}{m}}$$

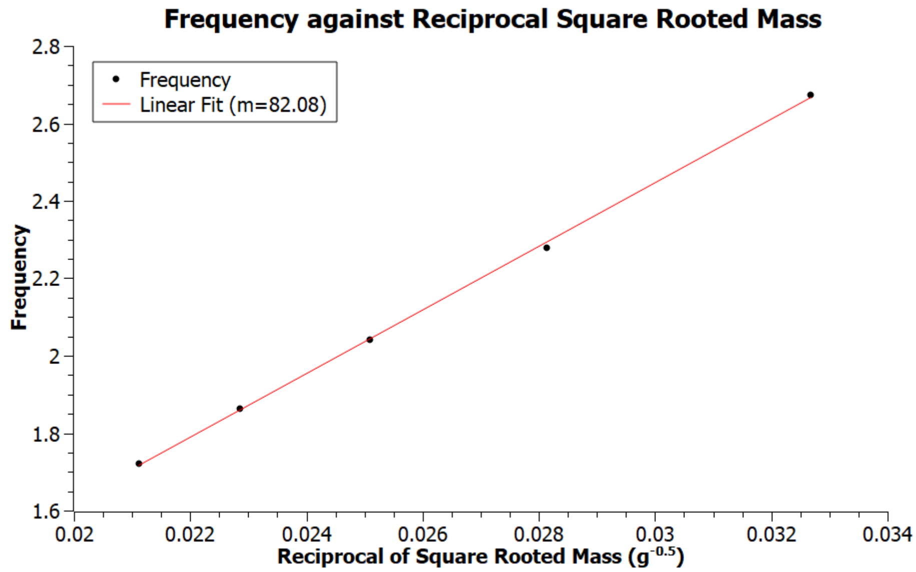

Figure 10: The Plot of Frequency against Reciprocal Rooted System Mass

And therefore the frequency is defined as:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This would mean that the constant of linearity in the graph above is:

$$\frac{1}{2\pi}\sqrt{k}$$

By readjusting the curve to use kg values instead of g values, we get a constant of linearity of 2.596. Therefore:

$$\begin{aligned} 2.596 &= \frac{1}{2\pi}\sqrt{k} \\ \sqrt{k} &= 2\pi \times 2.596 \\ k &= (2\pi \times 2.596)^2 \\ k &= 266.05 N/m \end{aligned}$$

As for the calculation of the stiffness in a previous question, this is very close to the value obtained when we double the spring constant for one spring alone, as $266 \approx 2 \times 132$.

By this same reasoning, given a system sprung with two different constants, k_1 and k_2, we can predict that the spring constant of the system would be $k_1 + k_2$.

A small car such as a fiat panda will weigh approximately $1,000kg$ with one occupant, compared to approximately $2000kg$ for a large car such as a Mercedes S-Class in the same configuration. Adding three passengers (With assumed average weight $80kg$) will represent a 24% and a 12% increase respectively. By taking the reciprocal root of this increase, we find that this increase in weight represents a decrease in oscillating frequency of 10.2% for a small car, and 5.5% in the case of a large car.

3 Experiment 2: Forced Oscillations

This experiment sought to investigate the motion of a forced harmonic system, specifically a cart sprung at both ends on a laterally oscillating support.

3.1 Experimental Setup

This experiment involved setting the cart on a support that oscillated in a sinusoidal manner at a set frequency that we could regulate. The camera was placed as before, but the significant difference was that the motion of both the cart and the support were now being tracked and then compared.

3.1.1 Experimental Protocol

The support was powered by an electric motor whose voltage could be varied. it gave no motion output below $5V$. The voltage was varied by $0.5V$ and a video was taken once the system was in equilibrium. The voltages were then increased in this manner until $8.5V$ where, thinking we were near the resonant frequency, we increased in increments on $0.1V$ until $8.7V$ we then made additional measurements at $9V$ and at $10V$. Unfortunately it was difficult to make measurements above the resonant frequency as the experimental set up seemed to introduce a second signal over the initial one. This made measurements of amplitude nearly impossible. A graph of the x-position of the cart in this situation is shown below:

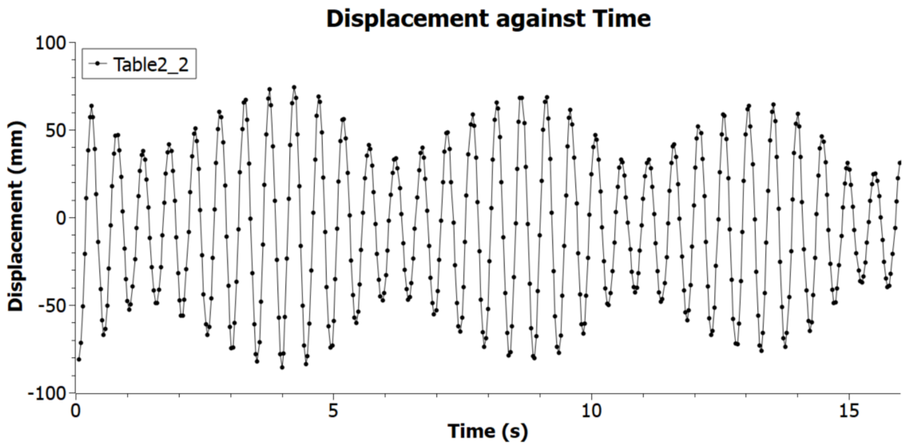

Figure 11: The Plot of Displacement against Time for Superposed Waves

Aside from this issue, the rest of the data was taken without problems, and the videos analysed in Tracker.

3.1.2 Data Analysis

Below is a table of the results from the experiment:

Analysis of Forced Oscillation					
Voltage (V)	f_{plate} (Hz)	f_{cart} (Hz)	A_{plate} (m)	A_{cart} (m)	$\Delta\phi$ (rad)
5	0.769	0.768	0.00464	0.00728	0
5.5	0.693	0.695	0.00457	0.00867	0
6	0.635	0.632	0.00454	0.01066	0
6.5	0.579	0.58	0.00449	0.01578	0
7	0.536	0.537	0.00452	0.02758	0
7.5	0.502	0.506	0.00489	0.07784	0
8	0.495	0.496	0.00488	0.16113	0
8.5	0.488	0.488	0.00472	0.19348	0
8.6	0.482	0.478	0.00482	0.19564	π
8.7	0.479	0.476	0.00461	0.19414	π
9	0.407	0.409	0.00471	0.01148	π
10	0.368	0.373	0.00461	0.06773	π

Figure 12: Experiment 2 Results Table: Forced Oscillations.

3.1.3 Presentation of Results

Below is the graph of the ratio of amplitude amplification against f_{plate}.

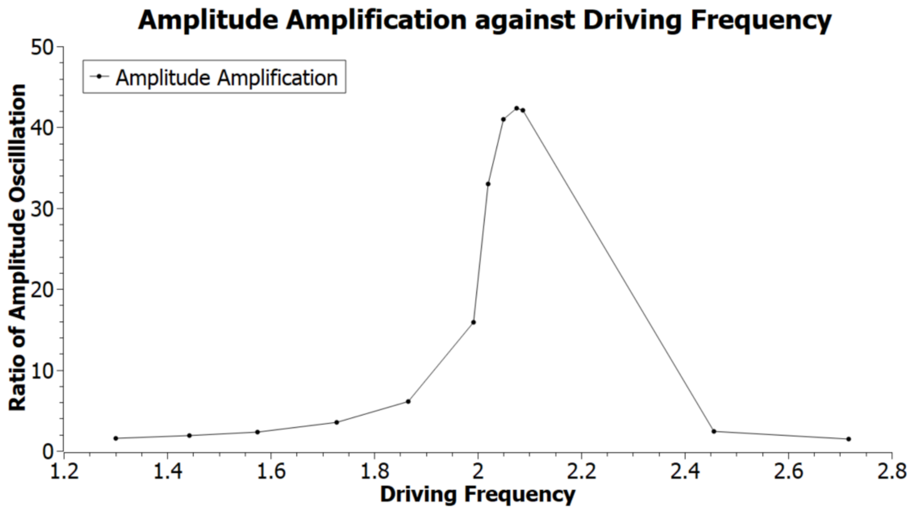

Figure 13: The Plot of Amplitude Amplification against Driving Frequency

Below are the three graphs of both the plate and the cart at a very low frequency ($1.28Hz$), near the resonant frequency ($2.07Hz$), and at a high frequency ($2.72Hz$).

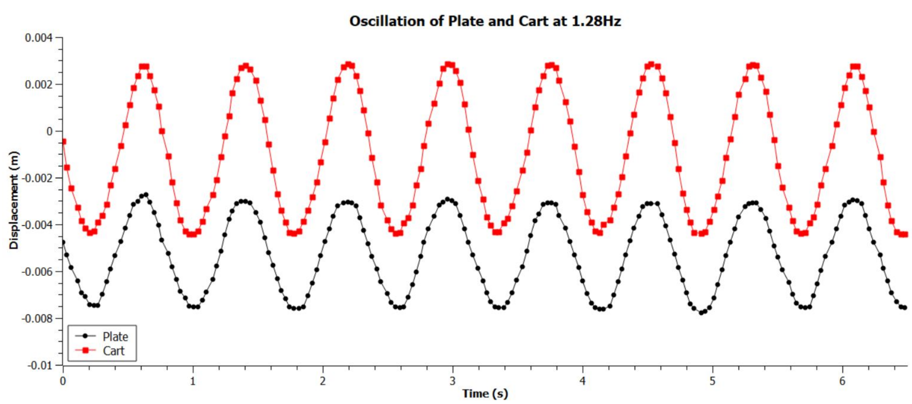

Figure 14: Cart and Plate Oscillations at 1.28 Hz

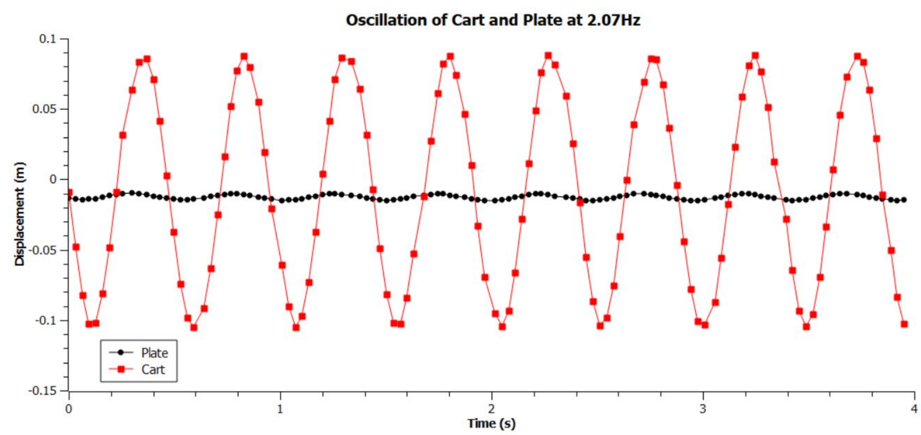

Figure 15: Cart and Plate Oscillations at 2.07 Hz

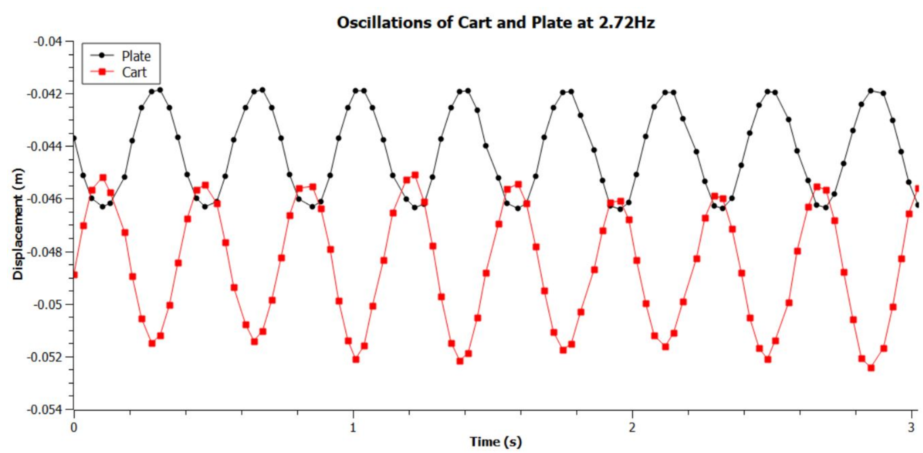

Figure 16: Cart and Plate Oscillations at 2.72 Hz

3.1.4 Conclusion

At nearly all of the forcing frequencies that were measured, the frequency of the cart was very close to the same as the frequency of the plate. The amplitude however varied signifiantly; at low frequencies the cart amplitude was only slightly higher than the plate amplitude. Near the resonant frequency, the amplitude of the cart was siginificantly larger than that of the cart, and at higher frequencies, the two amplitudes were similar. The resonant frequency is very close to the frequency that was found for an undamped system of the same mass.

A common example of resonance phenomena might be a children's swing, which will swing very well at a certain frequency, but will resist being pushed at higher or lower frequencies. Another might be a tuning fork, which when hit generates a note based on its resonant frequency.

By varying the mass and from the theory, we can observe or determine that the mass does in fact affect the resonant frequency, with a larger mass causing a decrease in the resonant frequency, specifically, we can say this about the relation between the two:

$$f_{resonance} \propto \frac{1}{\sqrt{M}}$$

The phase difference is 0 at all points up to the resonance frequency, after which the waves de-phase, and develop a phase difference of π radians.

4 Fluid Oscillations in a U-Tube

4.1 Preliminary Remarks

Given that the restoring force is linearly related to the mass, specificallly it is:

$$F = -\frac{2mg}{l}y$$

where y is the vertical displacement from the origin, the acceleration is not at all dependent on the mass, and thus the density of the liquid.

4.2 Experimental Setup

A u-tube was set up and filled to a certain height with water, the tube was then pushed to the side, and one of the ends blocked, such that when the tube is righted, there is a difference in height between the water in both sides of the u-tube. The blocked end is then released, and the water is left to oscillate. The oscillations are filmed, and the motion analysed in Tracker to get the frequency.

4.3 Measuring the Frequency

4.3.1 Experimental Protocol

The experiment was carried out as described above, using a constant amount of water, and varying the initial displacement, starting at $2cm$ and increasing in increments of $2cm$ up to $12cm$.

4.3.2 Data Analysis

Below is the table of results for the first part of this experiment:

Analysis of U-Tube Oscillation		
Initial Displacement (m)	Frequency (Hz)	First Oscillation Amplitude (m)
0.02	0.8429	0.0228
0.04	0.8391	0.0367
0.06	0.8470	0.0552
0.08	0.8460	0.0742
0.10	0.8426	0.0909
0.12	0.8387	0.1063

Figure 17: Experiment 3 Results Table: Frequency and Amplitude against Initial Displacement.

4.3.3 Presentation of Results

Below are the graphs for frequency and first amplitude oscillation as a function of the initial displacement. In each case, y-uncertainty was calculated as the standard deviation of the data, and x-uncertainty was taken to be 5mm.

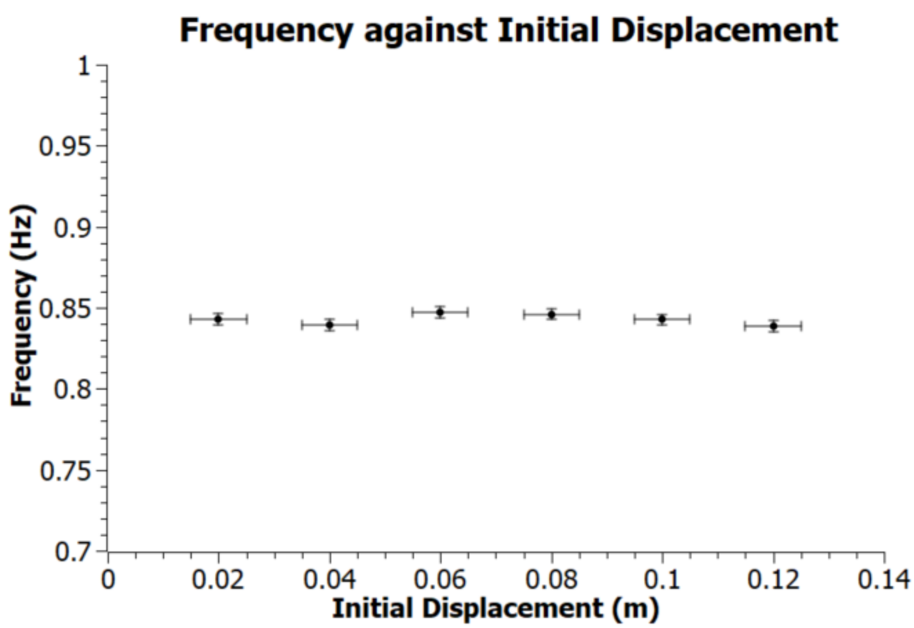

Figure 18: The plot of Frequency against Initial Displacement for a U-Tube

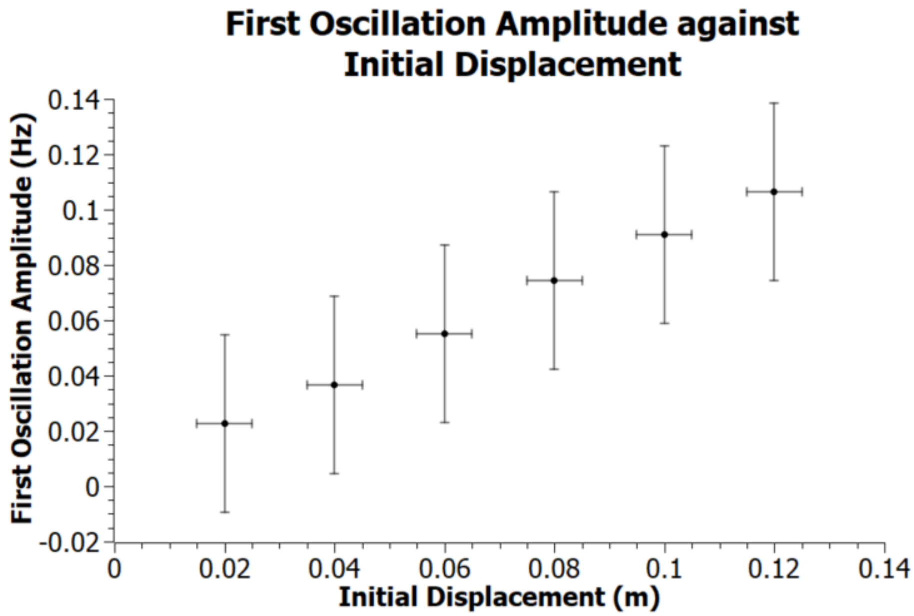

Figure 19: The plot of First Oscillation Amplitude against Initial Displacement for a U-Tube

4.3.4 Conclusion

From the graphs above, we can clearly conlude that the initial displacement has little to no bearing on the oscillating frequency, and that it is linearly related to the amplitude of the first oscillation.

4.4 Influence of the Column Height

4.4.1 Experimental Protocol

As before, the experiment was carried out as described above, but this time instead of varying the initial displacement, this was set at a certain value, and instead the initial height of the column was varied, starting at $13cm$ and increasing in increments of $1cm$ to $20cm$.

4.4.2 Data Analysis

Below is a table of results comparing the initial column height and the frequency of the oscillation.

Frequency against Initial Column Height	
Initial Height (m)	Frequency (Hz)
0.13	0.8844
0.14	0.8681
0.15	0.8601
0.16	0.8466
0.17	0.8308
0.18	0.8242
0.19	0.8134
0.20	0.8009

Figure 20: Experiment 3 Results Table: Frequency against Initial Column Height.

4.4.3 Presentation of Results

Below is the graph of frequency against initial column height, with uncertainties calculated as above:

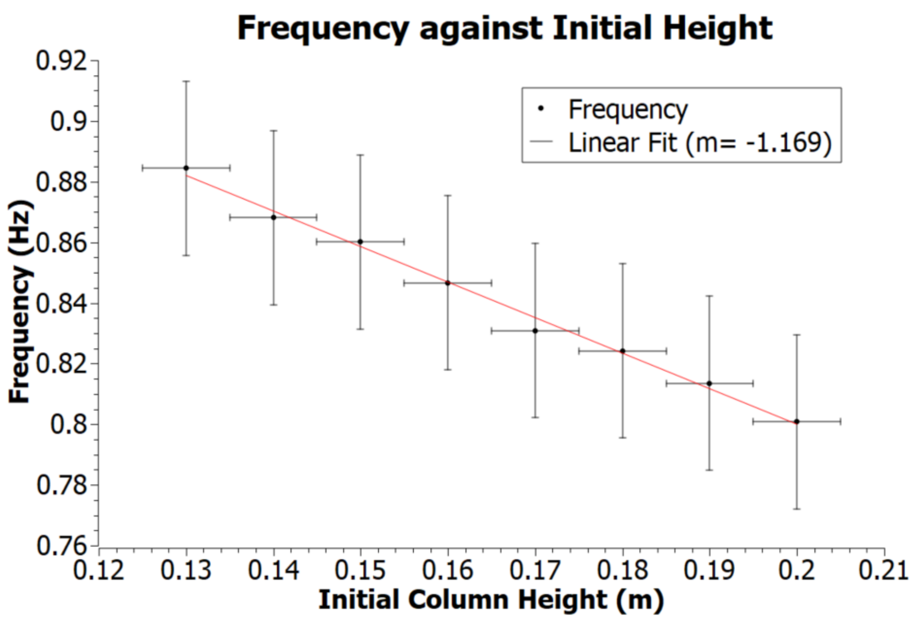

Figure 21: The plot of Frequency against Initial Column height for a U-Tube

4.4.4 Conclusion

In order to get an estimation for the force due to gravity, we can re-plot the graph above but instead of using the height, we can use the reciprocal root of the height, so that the constant of linearity will be $\sqrt{g}$ divided by 4π. Below is the plot of this graph:

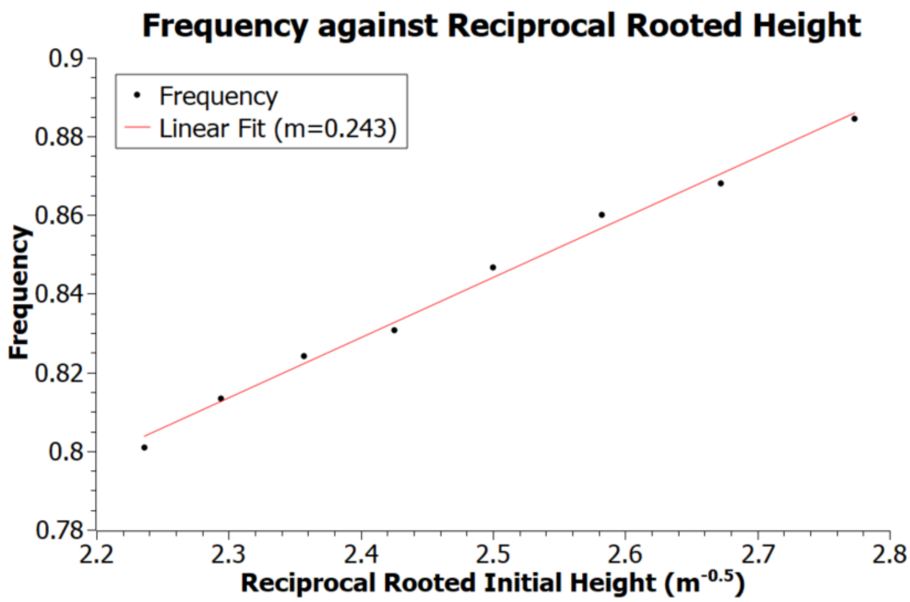

Figure 22: The plot of Frequency against Reciprocal Rooted Initial Height

We can see that:

$$
\begin{aligned}
0.243 &= \frac{1}{4\pi}\sqrt{g} \\
g &= (4\pi \times 0.243)^2 \\
g &= 9.32ms^{-1}
\end{aligned}
$$

4.5 Amplitude Evolution

As in the previous experiment, we can observe that successive oscillations have lower frequencies than those that preceded them. Specifically the ratio between the amplitude of a given wave and that of the wave following seems to be constant for a given mass of water, regardless of initial perturbation or the index of the oscillation. In the U-Tube system, this damping is likely to arise from the viscosity of the liquid, which causes skin friction with the inside of the tube, opposing the motion of the water. The damping could therefore be changed by using a liquid with a different viscosity or a tube made of a different material. For example, a hydrophobic coating on the inside of the tube might significantly reduce damping.

5 General Conclusions

5.1 Parallels between the Experiments

Let us cover the similarities between the two systems we have analysed.

Comparison of Systems		
Property	Mass-Spring System	Fluid System
k	Spring Constant	$\frac{2mg}{l}$
m	$m_c + m_0$	m
f	$\frac{1}{2\pi}\sqrt{\frac{k}{m_c+m_0}}$	$\frac{1}{2\pi}\sqrt{\frac{2g}{l}}$
Restoring Force	$-kx$	$-\frac{2mg}{l}y$
Resisting Force	$F_{friction}$	$F_{skin\_friction}$

Figure 23: General Conclusion Comparison Table.

5.2 Limits of Theoretical Description

As far as the mass-spring system is concerned, one of the main assumptions we make is that the string obeys Hooke's Law throughout its motion, that is to say that its extension is linearly related to the force at all points. For the fluid system, we would have to assume that the surface of the liquid at both ends of the pipe remain flat and surface tension is not broken. These assumptions both hold consistently at low amplitudes, but at higher amplitudes may be broken. However, this is unlikely in the experimental set up we had available as the maximum amplitude possible was severely limited by the size of the plate for the mass-spring system and the length of the tube for the fluid system. Also, given that in both un-forced cases, the initial displacement is also the maximum displacement, the cart, or the water, will never be able to reach the edge of the plate or the end of the tube. Therefore, within the experimental setup available to us in the lab, it will be very difficult to move unforced into a non-harmonic regime.

6 Amplitude Evolution of Mass-Spring System

This section will use the data we have collected in the previous experiments to determine the degree of damping in the systems we analysed.

6.1 Preliminary Remarks

Let us define the units of the relevant parameters

Quantities in the Experiment and their Units	
Quantity	Unit
β	s^{-1}
λ	dimensionless
δ	dimensionless

Figure 24: Table of Units: Amplitude Evolution

6.1.1 Experimental Protocol

No additional experiments took place.

6.1.2 Data Analysis

Below are the five graphs of λ against n for each mass:

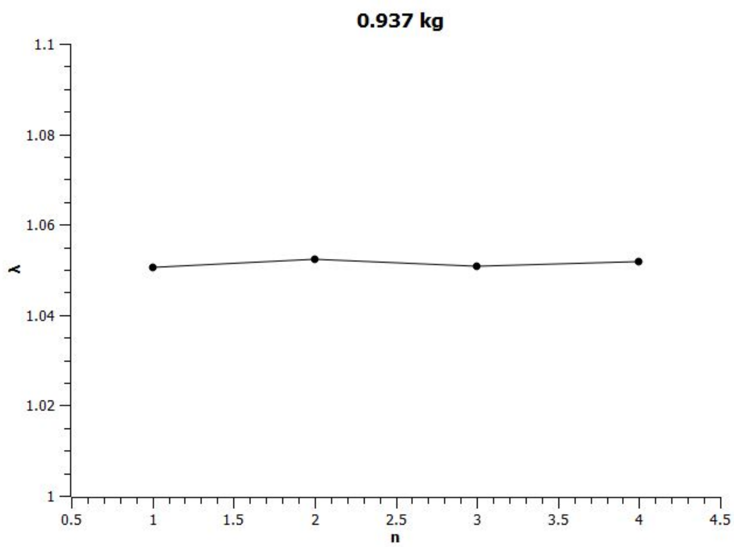

Figure 25: λ against n at 0.937 kg

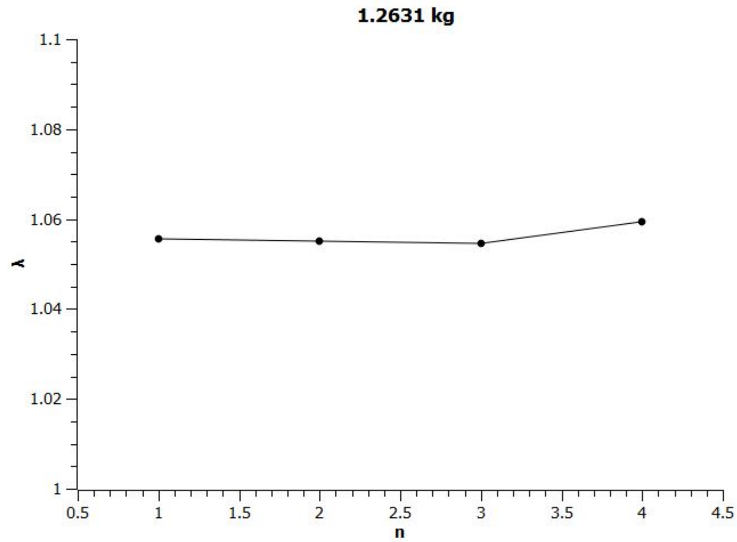

Figure 26: λ against n at 2.2631 kg

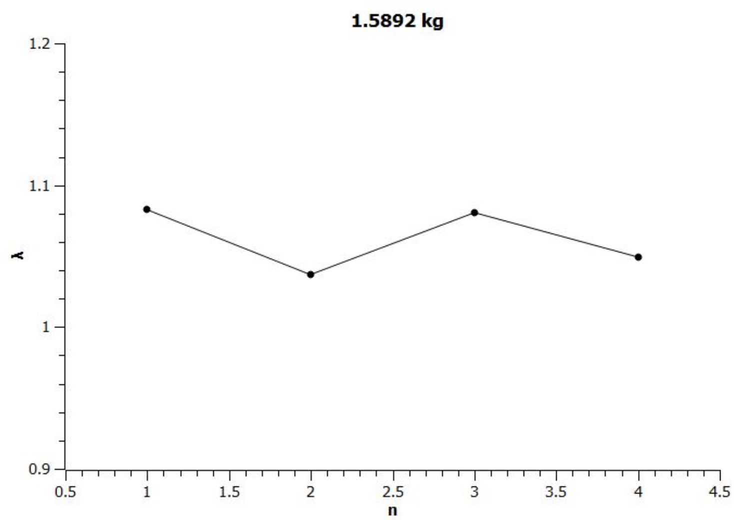

Figure 27: λ against n at 1.5892 kg

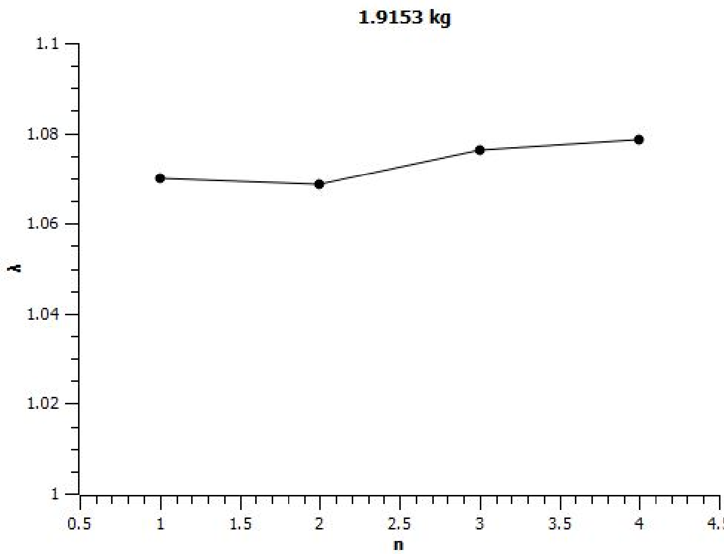

Figure 28: λ against n at 1.9153 kg

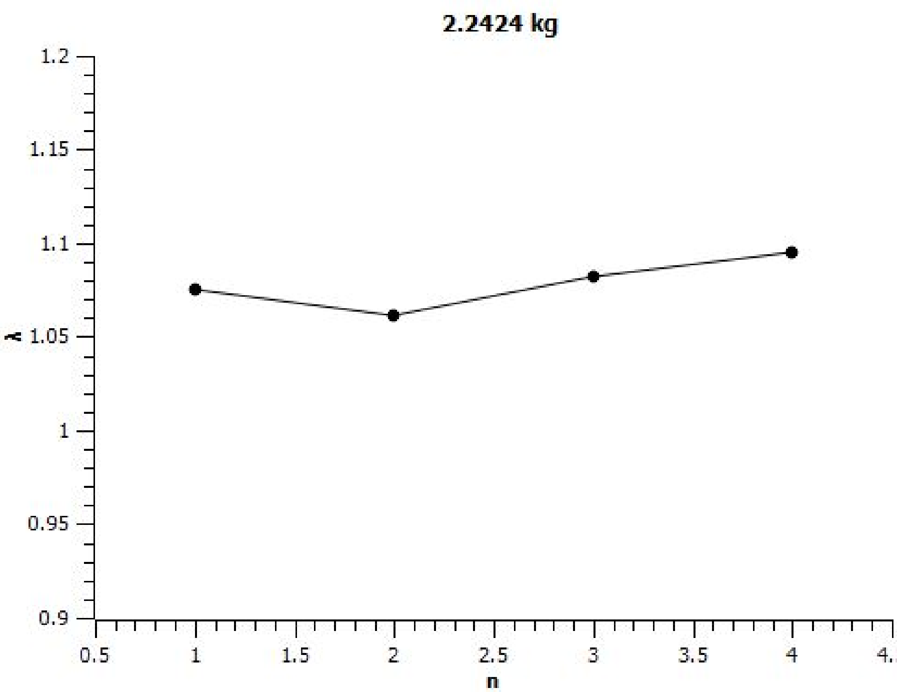

Figure 29: λ against n at 2.2424 kg

6.2 Measuring the Change in Amplitude

Analysis of Forced Oscillation				
M_{total}(kg)	A_0/A_1	A_1/A_2	A_2/A_3	A_3/A_4
0.937	1.0505	1.0523	1.0605	1.0518
1.2631	1.0556	1.0551	1.0546	1.0593
1.5892	1.0827	1.0367	1.0806	1.0492
1.9153	1.0701	1.0686	1.0763	1.0787
2.2424	1.0755	1.0618	1.0817	1.0953

Figure 30: Experiment 2 Results Table: Forced Oscillations.

6.2.1 Conclusion

From the tables above, we can see that the mass does not seem to have an impact on the damping coefficient, this could be because it is a frictional force, which is linearly related to the mass, and as such generates and acceleration that is independent of the mass. This damping could be modified by changing the surface of the plate or the wheels of the cart, or lubrifying the bearings of the wheel axles. These things would change the friction inherent in the system, thereby modifying the damping coefficient.

List of Figures