Canonical Formulation

Radial Adiabatic

Variables

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^2 \, \frac{\xi_r}{r} \\
y_2 &= x^2 \, \frac{1}{gr} \left(\frac{p'}{\rho} \right)
\end{aligned}
$$

Differential Equations

$$
\begin{aligned}
x \frac{\mathrm{d}y_1}{\mathrm{d}x} &= \left(\frac{V}{\Gamma_1} - 1 \right) y_1 - \frac{V}{\Gamma_1} y_2 \\
x \frac{\mathrm{d}y_2}{\mathrm{d}x} &= (c_1 \omega^2 + U - A^*) y_1 + (3 - U + A^*) y_2
\end{aligned}
$$

Jump Conditions

At a density discontinuity, y_1 is continuous, while

$$
U^+ y_2^+ - U^- y_2^- = y_1 (U^+ - U^-)
$$

Boundary Conditions

$$
y_1 = 0 \quad \text{as } x \to 0
$$

$$
y_1 - y_2 = 0 \quad \text{as } x \to 1 \qquad \text{(Zero)}
$$

Non-Radial Adiabatic

Variables

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell}\,\frac{\xi_r}{r} \\
y_2 &= x^{2-\ell}\,\frac{P'}{\rho g r} \\
y_3 &= x^{2-\ell}\,\frac{\Phi'}{g r} \\
y_4 &= x^{2-\ell}\,\frac{1}{g}\frac{\mathrm{d}\Phi'}{\mathrm{d}r}
\end{aligned}
$$

Differential Equations

$$
\begin{aligned}
x\frac{\mathrm{d}y_1}{\mathrm{d}x} &= \left(\frac{V}{\Gamma_1} - 1 - \ell\right) y_1 + \left(\frac{\ell(\ell+1)}{c_1\omega^2} - \frac{V}{\Gamma_1}\right) y_2 + \frac{\ell(\ell+1)}{c_1\omega^2} y_3 \\
x\frac{\mathrm{d}y_2}{\mathrm{d}x} &= (c_1\omega^2 - A^*)y_1 + (3 - U + A^* - \ell)y_2 - y_4 \\
x\frac{\mathrm{d}y_3}{\mathrm{d}x} &= (3 - U - \ell)y_3 + y_4 \\
x\frac{\mathrm{d}y_4}{\mathrm{d}x} &= A^* U y_1 + \frac{V}{\Gamma_1} U y_2 + \ell(\ell+1)y_3 - (U + \ell - 2)y_4
\end{aligned}
$$

Jump Conditions

At a density discontinuity, y_1 and y_3 are continuous, while

$$
\begin{aligned}
U^+ y_2^+ - U^- y_2^- &= y_1(U^+ - U^-) \\
y_4^+ - y_4^- &= -y_1(U^+ - U^-)
\end{aligned}
$$

Boundary Conditions

$$
\left.\begin{aligned}
c_1\omega^2 y_1 - \ell y_2 - \ell y_3 &= 0 \\
\ell y_3 - y_4 &= 0
\end{aligned}\right\} \text{as } x \to 0
$$

$$
\left.\begin{aligned}
y_1 - y_2 &= 0 \\
U y_1 + (\ell+1)y_3 + y_4 &= 0
\end{aligned}\right\} \text{as } x \to 1 \qquad \text{(Zero)}
$$

$$
\left.\begin{aligned}
\left\{1 + V^{-1}\left[\frac{\ell(\ell+1)}{c_1\omega^2} - 4 - c_1\omega^2\right]\right\} y_1 - y_2 + \left\{V^{-1}\left[\frac{\ell(\ell+1)}{c_1\omega^2} - \ell - 1\right]\right\} y_3 &= 0 \\
(\ell+1)y_3 + y_4 &= 0
\end{aligned}\right\} \text{as } x \to 1 \qquad \text{(Dziembowski)}
$$

Non-Radial Non-Adiabatic

Variables

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell}\,\frac{\xi_r}{r} \\
y_2 &= x^{2-\ell}\,\frac{P'}{\rho g r} \\
y_3 &= x^{2-\ell}\,\frac{\Phi'}{g r} \\
y_4 &= x^{2-\ell}\,\frac{1}{g}\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \\
y_5 &= x^{2-\ell}\,\frac{\delta S}{c_p} \\
y_6 &= x^{-1-\ell}\,\frac{\delta L_{\mathrm{rad}}}{L_*}
\end{aligned}
$$

Differential Equations

$$
\begin{aligned}
x\frac{\mathrm{d}y_1}{\mathrm{d}x} &= \left(\frac{V}{\Gamma_1} - 1 - \ell\right) y_1 + \left(\frac{\ell(\ell+1)}{c_1\omega^2} - \frac{V}{\Gamma_1}\right) y_2 + \frac{\ell(\ell+1)}{c_1\omega^2} y_3 + \delta y_5 \\
x\frac{\mathrm{d}y_2}{\mathrm{d}x} &= (c_1\omega^2 - A^*) y_1 + (3 - U + A^* - \ell) y_2 - y_4 + \delta y_5 \\
x\frac{\mathrm{d}y_3}{\mathrm{d}x} &= (3 - U - \ell) y_3 + y_4 \\
x\frac{\mathrm{d}y_4}{\mathrm{d}x} &= A^* U y_1 + \frac{V}{\Gamma_1} U y_2 + \ell(\ell+1) y_3 - (U + \ell - 2) y_4 - \delta U y_5 \\
x\frac{\mathrm{d}y_5}{\mathrm{d}x} &= V\left[\nabla_{\mathrm{ad}}(U - c_1\omega^2) - 4(\nabla_{\mathrm{ad}} - \nabla) + c_{\mathrm{dif}} + \nabla_{\mathrm{ad}}\nabla'_{\mathrm{ad}}\right] y_1 + V\left[\frac{\ell(\ell+1)}{c_1\omega^2}(\nabla_{\mathrm{ad}} - \nabla) - c_{\mathrm{dif}} - \nabla_{\mathrm{ad}}\nabla'_{\mathrm{ad}}\right] y_2 + \\
&\quad V\left[\frac{\ell(\ell+1)}{c_1\omega^2}(\nabla_{\mathrm{ad}} - \nabla)\right] y_3 + V\nabla_{\mathrm{ad}} y_4 + \left[V\nabla(4 - \kappa_S) + 2 - \ell\right] y_5 - \frac{V\nabla}{c_{\mathrm{rad}}} y_6 \\
x\frac{\mathrm{d}y_6}{\mathrm{d}x} &= \left[\ell(\ell+1)\left(\frac{\nabla_{\mathrm{ad}}}{\nabla} - 1\right) c_{\mathrm{rad}} - V c_{\epsilon,\mathrm{ad}}\right] y_1 + \left[V c_{\epsilon,\mathrm{ad}} - \ell(\ell+1) c_{\mathrm{rad}}\left(\frac{\nabla_{\mathrm{ad}}}{\nabla} - \frac{3 + c'_{\mathrm{rad}}}{c_1\omega^2}\right)\right] y_2 + \\
&\quad \left[\ell(\ell+1) c_{\mathrm{rad}} \frac{3 + c'_{\mathrm{rad}}}{c_1\omega^2}\right] y_3 + \left[c_{\epsilon,S} - \frac{\ell(\ell+1) c_{\mathrm{rad}}}{\nabla V} + \mathrm{i}\omega c_{\mathrm{thm}}\right] y_5 - \left[1 + \ell\right] y_6
\end{aligned}
$$

Jump Conditions

At a density discontinuity, y_1, y_3 and y_6 are continuous, while

$$
\begin{aligned}
U^+ y_2^+ - U^- y_2^- &= y_1(U^+ - U^-) \\
y_4^+ - y_4^- &= -y_1(U^+ - U^-) \\
y_5^+ - y_5^- &= -V^+\nabla_{\mathrm{ad}}^+(y_2^+ - y_1) + V^-\nabla_{\mathrm{ad}}^-(y_2^- - y_1)
\end{aligned}
$$

Boundary Conditions

$$\left.\begin{aligned} c_1\omega^2 y_1 - \ell y_2 &= 0 \\ \ell y_3 - y_4 &= 0 \\ y_5 &= 0 \end{aligned}\right\} \text{as } x \to 0$$

Outer

$$\left.\begin{aligned} y_1 - y_2 + y_3 &= 0 \\ U y_1 + (\ell+1) y_3 + y_4 &= 0 \\ (2 - 4\nabla_{\rm ad} V) y_1 + 4\nabla_{\rm ad} V y_2 - 4\nabla_{\rm ad} V y_3 + 4 y_5 - y_6 &= 0 \end{aligned}\right\} \text{as } x \to 1 \qquad \text{(Zero)}$$

$$\left.\begin{aligned} \left\{1 + V^{-1}\left[\frac{\ell(\ell+1)}{\omega^2} - 4 - \omega^2\right]\right\} y_1 - y_2 + \left\{V^{-1}\left[\frac{\ell(\ell+1)}{\omega^2} - \ell - 1\right]\right\} y_3 &= 0 \\ (\ell+1) y_3 + y_4 &= 0 \\ (2 - 4\nabla_{\rm ad} V) y_1 + 4\nabla_{\rm ad} V y_2 + 4 y_5 - y_6 &= 0 \end{aligned}\right\} \text{as } x \to 1 \qquad \text{(Dziembowski)}$$

Alternative Formulations

Radial Adiabatic

JCD Variables Set

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^2 \, \frac{\xi_r}{r} \\
y_2 &= x^2 \, \frac{1}{\sigma^2 r^2} \left(\frac{p'}{\rho} \right)
\end{aligned}
$$

LAGP Variables Set

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^2 \, \frac{\xi_r}{r} \\
y_2 &= \frac{\delta p}{p}
\end{aligned}
$$

Non-Radial Adiabatic

JCD Set

$$
\left.
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell} \, \frac{\xi_r}{r} \\
y_2 &= x^{2-\ell} \, \frac{\ell(\ell+1)}{r^2 \sigma^2} \left(\frac{p'}{\rho} + \Phi' \right) \\
y_3 &= -x^{2-\ell} \, \frac{\Phi'}{gr} \\
y_4 &= -x^{2-\ell} \, r \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\Phi'}{gr} \right)
\end{aligned}
\right\} \ell \neq 0
$$

$$
\left.
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^2 \, \frac{\xi_r}{r} \\
y_2 &= x^2 \, \frac{1}{r^2 \sigma^2} \left(\frac{p'}{\rho} \right) \\
y_3 &= -x^2 \, \frac{\Phi'}{gr} \\
y_4 &= -x^2 \, r \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\Phi'}{gr} \right)
\end{aligned}
\right\} \ell = 0
$$

LAGP Set

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell}\,\frac{\xi_r}{r} \\
y_2 &= x^{-\ell}\,\frac{\delta p}{p} \\
y_3 &= x^{2-\ell}\,\frac{\Phi'}{gr} \\
y_4 &= x^{2-\ell}\,\frac{1}{g}\frac{\mathrm{d}\Phi'}{\mathrm{d}r}
\end{aligned}
$$

Non-Radial Non-Adiabatic

JCD Set

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell}\,\frac{\xi_r}{r} \\
y_2 &= x^{2-\ell}\,\frac{1}{gr}\left(\frac{p'}{\rho} + \Phi'\right) \\
y_3 &= x^{2-\ell}\,\frac{\Phi'}{gr} \\
y_4 &= x^{2-\ell}\,\frac{1}{g}\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \\
y_5 &= x^{2-\ell}\,\frac{\delta S}{c_p} \\
y_6 &= x^{-1-\ell}\,\frac{\delta L_{\mathrm{rad}}}{L_*}
\end{aligned}
$$

LAGP Set

$$
\begin{aligned}
x &= \frac{r}{R_*} \\
y_1 &= x^{2-\ell}\,\frac{\xi_r}{r} \\
y_2 &= x^{-\ell}\,\frac{\delta p}{p} \\
y_3 &= x^{2-\ell}\,\frac{\Phi'}{gr} \\
y_4 &= x^{2-\ell}\,\frac{1}{g}\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \\
y_5 &= x^{2-\ell}\,\frac{\delta S}{c_p} \\
y_6 &= x^{-1-\ell}\,\frac{\delta L_{\mathrm{rad}}}{L_*}
\end{aligned}
$$

Structure Coefficients

Mechanical

$$V = -\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r} \qquad A^* = \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r} - \frac{\mathrm{d}\ln \rho}{\mathrm{d}\ln r} \qquad U = \frac{\mathrm{d}\ln M_r}{\mathrm{d}\ln r} \qquad D = \frac{\mathrm{d}\ln \rho}{\mathrm{d}\ln r}$$

$$c_1 = \frac{r^3}{R_*^3}\frac{M_*}{M_r} \qquad \Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S$$

Thermal

$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} \qquad \nabla_{\mathrm{ad}} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S \qquad \nabla'_{\mathrm{ad}} = \frac{\mathrm{d}\ln \nabla_{\mathrm{ad}}}{\mathrm{d}\ln r} \qquad \delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p$$

$$c_{\mathrm{rad}} = x^{-3}\frac{L_{\mathrm{rad}}}{L_*} \qquad c'_{\mathrm{rad}} = \frac{\mathrm{d}\ln c_{\mathrm{rad}}}{\mathrm{d}\ln r}$$

$$c_{\epsilon,\mathrm{ad}} = x^{-3}\frac{4\pi r^3 \epsilon_{\mathrm{ad}}\rho}{L_*} \qquad c_{\epsilon,S} = x^{-3}\frac{4\pi r^3 \epsilon_S \rho}{L_*} \qquad c_{\mathrm{thm}} = x^{-3}\frac{4\pi r^3 c_p T\rho}{L_*}\sqrt{\frac{GM_*}{R_*^3}}$$

$$c_{\mathrm{dif}} = (\kappa_{\mathrm{ad}} - 4\nabla_{\mathrm{ad}})\, V\nabla + \nabla_{\mathrm{ad}} V$$

$$\kappa_{\mathrm{ad}} = \left(\frac{\partial \ln \kappa}{\partial \ln P}\right)_S \qquad \kappa_S = c_p T \left(\frac{\partial \ln \kappa}{\partial S}\right)_P$$

$$\epsilon_{\mathrm{ad}} = \left(\frac{\partial \epsilon}{\partial \ln P}\right)_S \qquad \epsilon_S = c_p T \left(\frac{\partial \epsilon}{\partial S}\right)_P$$