GYRE Equations & Variables

Preliminaries

GYRE radial eigenfunctions are expressed in terms of a set of dimensionless variables $y_i(x)$ $(i = 1, 2, \ldots)$, where $x \equiv r/R$ is the dimensionless radial coordinate. The equations governing these eigenfunctions depend on the underlying stellar structure, and on the dimensionless oscillation frequency in the co-rotating frame,

$$\omega_{\rm c} = \omega - m\Omega(x).$$

Here, $\Omega(x)$ is the rotation angular frequency, m is the azimuthal order, and ω is the corresponding dimensionless frequency in an inertial frame.

The equations also depend on the effective harmonic degree $\ell_{\rm e}$. In the non-rotating limit, $\ell_{\rm e}$ reduces to the ordinary spherical harmonic degree ℓ. Within the traditional approximation of rotation (TAR), $\ell_{\rm e}$ is obtained by solving

$$\ell_{\rm e}(\ell_{\rm e} + 1) = \lambda(\ell, m; \nu),$$

where $\lambda(\ell, m; \nu)$ is the eigenvalue of Laplace's tidal equation (see, e.g., Townsend, 2003) for the indicated ℓ and m, and for spin parameter $\nu \equiv 2\Omega/\omega_{\rm c}$. Due to its dependence on Ω (both directly, and through $\omega_{\rm c}$), $\ell_{\rm e}$ varies with position in a differentially rotating star. The value of $\ell_{\rm e}$ at the inner boundary is denoted $\ell_{\rm i}$, and the dependent variables y_i are scaled using $\ell_{\rm i}$ so that they approach constant values at this boundary.

Structure Coefficients

The properties of the underlying stellar structure are described by a set of dimensionless structure coefficients, which largely follow the definitions in Unno et al. (1989).

Mechanical

$$V = -\frac{{\rm d}\ln P}{{\rm d}\ln r} \qquad A^* = \frac{1}{\Gamma_1}\frac{{\rm d}\ln P}{{\rm d}\ln r} - \frac{{\rm d}\ln \rho}{{\rm d}\ln r} \qquad U = \frac{{\rm d}\ln M_r}{{\rm d}\ln r} \qquad c_1 = \frac{r^3}{R_*^3}\frac{M_*}{M_r} \qquad \Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S$$

Thermal

$$\nabla = \frac{{\rm d}\ln T}{{\rm d}\ln P} \qquad \nabla_{\rm ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S \qquad \delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P$$

$$c_{\rm rad} = x^{-3}\frac{L_{\rm rad}}{L_*} \qquad \partial c_{\rm rad} = \frac{{\rm d}\ln c_{\rm rad}}{{\rm d}\ln r}$$

$$c_{\epsilon,{\rm ad}} = x^{-3}\frac{4\pi r^3 \epsilon_{\rm ad}\rho}{L_*} \qquad c_{\epsilon,S} = x^{-3}\frac{4\pi r^3 \epsilon_S \rho}{L_*}$$

$$c_{\rm dif} = (\kappa_{\rm ad} - 4\nabla_{\rm ad})\, V\nabla + \nabla_{\rm ad}\left(V + \frac{{\rm d}\ln \nabla_{\rm ad}}{{\rm d}\ln r}\right)$$

$$c_{\rm thn} = \frac{c_p}{ac\kappa T^3}\sqrt{\frac{GM_*}{R_*^3}} \qquad \partial c_{\rm thn} = \frac{{\rm d}\ln c_{\rm thn}}{{\rm d}\ln r} \qquad c_{\rm thk} = x^{-3}\frac{4\pi r^3 c_p T \rho}{L_*}\sqrt{\frac{GM_*}{R_*^3}}$$

$$\kappa_{\rm ad} = \left(\frac{\partial \ln \kappa}{\partial \ln P}\right)_S \qquad \kappa_S = c_p\left(\frac{\partial \ln \kappa}{\partial S}\right)_P$$

$$\epsilon_{\rm ad} = \left(\frac{\partial \epsilon}{\partial \ln P}\right)_S \qquad \epsilon_S = c_p\left(\frac{\partial \epsilon}{\partial S}\right)_P$$

Dimensionless Variables

For non-radial non-adiabatic calculations, GYRE uses a set of six dimensionless variables:

$$
\begin{aligned}
x &= \frac{r}{R_*}, \\
y_1 &= x^{2-\ell_{\mathrm{i}}} \, \frac{\xi_r}{r}, \\
y_2 &= x^{2-\ell_{\mathrm{i}}} \, \frac{P'}{\rho g r}, \\
y_3 &= x^{2-\ell_{\mathrm{i}}} \, \frac{\Phi'}{g r}, \\
y_4 &= x^{2-\ell_{\mathrm{i}}} \, \frac{1}{g} \frac{\mathrm{d}\Phi'}{\mathrm{d}r}, \\
y_5 &= x^{2-\ell_{\mathrm{i}}} \, \frac{\delta S}{c_p}, \\
y_6 &= x^{-1-\ell_{\mathrm{i}}} \, \frac{\delta L_{\mathrm{rad}}}{L_*}.
\end{aligned}
$$

Here, ξ_r is the radial displacement perturbation, primes indicate Eulerian perturbations, and δ denotes the Legrangian perturbation. As discussed previously, the $x^{\cdots}$ scaling of the variables ensures that they approach constant values at the inner boundary.

For non-radial adiabatic calculations, only the first four variables are used; and for radial adiabatic calculations with `reduce_order=.TRUE.` , only the first two.

Differential Equations

For non-radial non-adiabatic calculations, GYRE solves a system of six coupled, first-order differential equations:

$$
\begin{aligned}
x \frac{\mathrm{d}y_1}{\mathrm{d}x} &= \left(\frac{V}{\Gamma_1} - 1 - \ell_{\mathrm{i}} \right) y_1 + \left(\frac{\lambda}{c_1 \omega^2} - \frac{V}{\Gamma_1} \right) y_2 + \alpha_{\mathrm{gr}} \frac{\lambda}{c_1 \omega^2} y_3 + \delta y_5, \\
x \frac{\mathrm{d}y_2}{\mathrm{d}x} &= (c_1 \omega^2 - A^*) y_1 + (3 - U + A^* - \ell_{\mathrm{i}}) y_2 - \alpha_{\mathrm{gr}} y_4 + \delta y_5, \\
x \frac{\mathrm{d}y_3}{\mathrm{d}x} &= \alpha_{\mathrm{gr}} (3 - U - \ell_{\mathrm{i}}) y_3 + \alpha_{\mathrm{gr}} y_4, \\
x \frac{\mathrm{d}y_4}{\mathrm{d}x} &= \alpha_{\mathrm{gr}} A^* U y_1 + \alpha_{\mathrm{gr}} \frac{V}{\Gamma_1} U y_2 + \alpha_{\mathrm{gr}} \lambda y_3 - \alpha_{\mathrm{gr}} (U + \ell_{\mathrm{i}} - 2) y_4 - \alpha_{\mathrm{gr}} \delta U y_5, \\
x \frac{\mathrm{d}y_5}{\mathrm{d}x} &= \frac{V}{f_{\mathrm{rh}}} \left[\nabla_{\mathrm{ad}} (U - c_1 \omega^2) - 4 (\nabla_{\mathrm{ad}} - \nabla) + c_{\mathrm{dif}} \right] y_1 + \\
&\quad \frac{V}{f_{\mathrm{rh}}} \left[\frac{\lambda}{c_1 \omega^2} (\nabla_{\mathrm{ad}} - \nabla) - c_{\mathrm{dif}} \right] y_2 + \alpha_{\mathrm{gr}} \frac{V}{f_{\mathrm{rh}}} \left[\frac{\lambda}{c_1 \omega^2} (\nabla_{\mathrm{ad}} - \nabla) \right] y_3 + \alpha_{\mathrm{gr}} \frac{V \nabla_{\mathrm{ad}}}{f_{\mathrm{rh}}} y_4 + \\
&\quad \left[\frac{V \nabla}{f_{\mathrm{rh}}} (4 f_{\mathrm{rh}} - \kappa_S) + \partial f_{\mathrm{rh}} + 2 - \ell_{\mathrm{i}} \right] y_5 - \frac{V \nabla}{f_{\mathrm{rh}} c_{\mathrm{rad}}} y_6, \\
x \frac{\mathrm{d}y_6}{\mathrm{d}x} &= \left[\alpha_{\mathrm{hf}} \lambda \left(\frac{\nabla_{\mathrm{ad}}}{\nabla} - 1 \right) c_{\mathrm{rad}} - V c_{\epsilon,\mathrm{ad}} \right] y_1 + \left[V c_{\epsilon,\mathrm{ad}} - \lambda c_{\mathrm{rad}} \left(\alpha_{\mathrm{hf}} \frac{\nabla_{\mathrm{ad}}}{\nabla} - \frac{3 + \partial c_{\mathrm{rad}}}{c_1 \omega^2} \right) \right] y_2 + \\
&\quad \alpha_{\mathrm{gr}} \left[\lambda c_{\mathrm{rad}} \frac{3 + \partial c_{\mathrm{rad}}}{c_1 \omega^2} \right] y_3 + \left[c_{\epsilon,S} - \alpha_{\mathrm{hf}} \frac{\lambda c_{\mathrm{rad}}}{\nabla V} + \mathrm{i} \omega c_{\mathrm{thk}} \right] y_5 - \left[1 + \ell_{\mathrm{i}} \right] y_6.
\end{aligned}
$$

The α_{gr} coefficient is set to zero in the Cowling (1941) approximation (`cowling_approx=.TRUE.`), and to one otherwise. Likewise, the α_{hf} coefficient is set to zero in the NARF approximation (`narf_approx=.TRUE.`;

see Townsend, 2005), and to one otherwise. Finally,

$$f_{\rm rh} \equiv 1 - \alpha_{\rm rh} \frac{{\rm i}\omega c_{\rm thn}}{4}, \qquad \partial f_{\rm rh} \equiv \frac{\partial \ln f_{\rm rh}}{\partial \ln x} = -\alpha_{\rm rh} \frac{{\rm i}\omega c_{\rm thn} \partial c_{\rm thn}}{4 f_{\rm rh}}, \tag{1}$$

with the $\alpha_{\rm rh}$ set to one in the Eddington approximation (`eddingon_approx=.TRUE.`) and zero otherwise.

For non-radial adiabatic calculations, the last two equations are set aside and the y_5 terms dropped from the first four equations. For radial adiabatic calculations with `reduce_order=.TRUE.`, the last four equations are set aside and the first two replaced by

$$\begin{aligned} x \frac{{\rm d}y_1}{{\rm d}x} &= \left(\frac{V}{\Gamma_1} - 1 \right) y_1 - \frac{V}{\Gamma_1} y_2, \\ x \frac{{\rm d}y_2}{{\rm d}x} &= (c_1 \omega^2 + U - A^*) y_1 + (3 - U + A^*) y_2. \end{aligned}$$

Boundary Conditions

Inner Boundary

When `inner_bound='REGULAR'`, GYRE applies regularity-enforcing conditions at the inner boundary:

$$\begin{aligned} c_1 \omega^2 y_1 - \ell y_2 - \alpha_{\rm gr} y_3 &= 0, \\ \alpha_{\rm gr} \ell y_3 - (2\alpha_{\rm gr} - 1) y_4 &= 0, \\ y_5 &= 0. \end{aligned}$$

When `inner_bound='ZERO_R'`, the first and second conditions are replaced with zero radial displacement conditions,

$$\begin{aligned} y_1 &= 0, \\ y_4 &= 0. \end{aligned}$$

Likewise, when `inner_bound='ZERO_H'`, the first and second conditions are replaced with zero horizontal displacement conditions,

$$\begin{aligned} y_2 - y_3 &= 0, \\ y_4 &= 0. \end{aligned}$$

Outer Boundary

When `outer_bound='VACUUM'`, GYRE applies vacuum surface pressure conditions at the outer boundary:

$$\begin{aligned} y_1 - y_2 &= 0 \\ \alpha_{\rm gr} U y_1 + (\alpha_{\rm gr} \ell_{\rm e} + 1) y_3 + \alpha_{\rm gr} y_4 &= 0 \\ (2 - 4\nabla_{\rm ad} V) y_1 + 4 \nabla_{\rm ad} V y_2 + 4 f_{\rm rh} y_5 - y_6 &= 0 \end{aligned}$$

When `outer_bound='DZIEM'`, the first condition is replaced by the Dziembowski (1971) outer mechanical boundary condition,

$$\left\{ 1 + V^{-1} \left[\frac{\lambda}{c_1 \omega^2} - 4 - c_1 \omega^2 \right] \right\} y_1 - y_2 = 0.$$

When `outer_bound='UNNO'` or `outer_bound='JCD'`, the first condition is replaced by the (possibly-leaky) outer mechanical boundary conditions described by Unno et al. (1989) and Christensen-Dalsgaard (2008), respectively.

Jump Conditions

Across density discontinuities, GYRE enforces conservation of mass, momentum and energy by applying the jump conditions

$$
\begin{aligned}
U^+ y_2^+ - U^- y_2^- &= y_1 (U^+ - U^-) \\
y_4^+ - y_4^- &= -y_1 (U^+ - U^-) \\
y_5^+ - y_5^- &= -V^+ \nabla_{\rm ad}^+ (y_2^+ - y_1) + V^- \nabla_{\rm ad}^- (y_2^- - y_1)
\end{aligned}
$$

Here, + (-) superscripts indicate quantities evaluated on the inner (outer) side of the discontinuity. y_1, y_3 and y_6 remain continuous across discontinuites, and therefore don't need these superscripts.

Alternative Variable Sets

GYRE offers the option to use different sets of dimensionless variables, instead of the canonical set defined above. When `variables_set='DZIEM'`, GYRE uses a set based on the formulation by Dziembowski (1971):

$$
\begin{aligned}
y_1 &= x^{2-\ell_{\rm i}} \, \frac{\xi_r}{r}, \\
y_2 &= x^{2-\ell_{\rm i}} \, \frac{1}{gr} \left(\frac{P'}{\rho} + \Phi' \right), \\
y_3 &= x^{2-\ell_{\rm i}} \, \frac{\Phi'}{gr}, \\
y_4 &= x^{2-\ell_{\rm i}} \, \frac{1}{g} \frac{{\rm d}\Phi'}{{\rm d}r},
\end{aligned}
$$

with y_5 and y_6 defined as before. When `variables_set='JCD'`, GYRE uses a set based on the formulation in the ADIPLS code (Christensen-Dalsgaard, 2008):

$$
\begin{aligned}
y_1 &= x^{2-\ell_{\rm i}} \, \frac{\xi_r}{r}, \\
y_2 &= x^{2-\ell_{\rm i}} \, \frac{\lambda}{r^2 \sigma^2} \left(\frac{P'}{\rho} + \Phi' \right), \\
y_3 &= -x^{2-\ell_{\rm i}} \, \frac{\Phi'}{gr}, \\
y_4 &= -x^{2-\ell_{\rm i}} \, r \frac{\rm d}{{\rm d}r} \left(\frac{\Phi'}{gr} \right),
\end{aligned}
$$

for non-radial calculations, while

$$
\begin{aligned}
y_1 &= x^2 \, \frac{\xi_r}{r}, \\
y_2 &= x^2 \, \frac{1}{r^2 \sigma^2} \left(\frac{P'}{\rho} \right), \\
y_3 &= -x^2 \, \frac{\Phi'}{gr}, \\
y_4 &= -x^2 \, r \frac{\rm d}{{\rm d}r} \left(\frac{\Phi'}{gr} \right),
\end{aligned}
$$

in the radial case when `reduce_order=.FALSE.`. When `variables_set='LAGP'`, GYRE uses a set which replaces the Eulerian pressure perturbation with the Lagrangian one:

$$
\begin{aligned}
y_1 &= x^{2-\ell_{\rm i}} \, \frac{\xi_r}{r}, \\
y_2 &= x^{-\ell_{\rm i}} \, \frac{\delta P}{P}, \\
y_3 &= x^{2-\ell_{\rm i}} \, \frac{\Phi'}{gr}, \\
y_4 &= x^{2-\ell_{\rm i}} \, \frac{1}{g} \frac{{\rm d}\Phi'}{{\rm d}r}.
\end{aligned}
$$

References

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