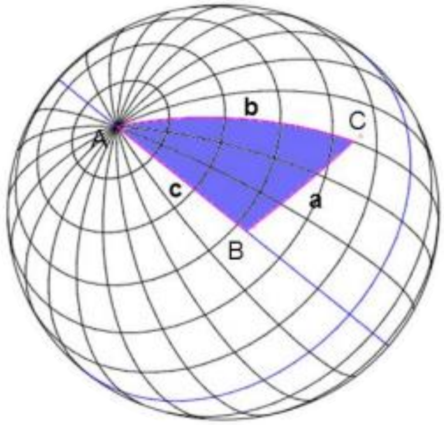

Given two points on a unit sphere, the distance between the two points, points B and C, can be found through spherical geometry. Considering a third point at $(1, 0, 0)$, or at the geographic north pole, point C, to form a spherical triangle, the spherical law of cosines can be utilized to find the distance between the two points. The spherical triangle can be defined on a unit circle to be ΔABC, where the sides of the triangle are defined to be of lengths a, b, c, opposite of points A, B, C, respectively.

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\angle A) \tag{1}$$

Via this process, the distances b and c are the distances from the north pole to each point, respectively, and a is the distance between the two points. $\angle A$ is the interior angle intersection of the arcs at the north pole. Currently, distances a, b, c, are in radians due to being a unit sphere.

Spherical Coordinate System

In a spherical coordinate system (r, θ, ψ), the distances b and c can be expressed as the polar angle θ. Similarly, $\angle A$ in the spherical coordinate system can be expressed as simply the change in azimuth between the two points, $\Delta\psi = \psi_2 - \psi_1$. Substituting in spherical coordinates:

$$\cos(a) = \cos(\theta_B)\cos(\theta_C) + \sin(\theta_B)\sin(\theta_C)\cos(\psi_C - \psi_B) \tag{2}$$

Rearranging and solving for the distance a:

$$a = \cos^{-1}[\cos(\theta_B)\cos(\theta_C) + \sin(\theta_B)\sin(\theta_C)\cos(\psi_C - \psi_B)] \tag{3}$$

Because the distance a is still in radians, simply multiplying by the radius of the sphere in question, being earth, gives the true distance along the surface of the earth:

$$\boxed{a = r_{earth}\cos^{-1}[\cos(\theta_B)\cos(\theta_C) + \sin(\theta_B)\sin(\theta_C)\cos(\psi_C - \psi_B)]} \tag{4}$$

Cylindrical Coordinate System

For a cylindrical coordinate system (ρ, ψ, z), an azimuth angle is still present, but polar angles are not. A simple transformation can be done to substitute a polar angle using the radius and height at a given cylindrical coordinate:

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) \tag{5}$$

Substituting this directly into the spherical coordinate system equation yields the equation for cylindrical coordinates:

$$\boxed{a = r_{earth}\cos^{-1}\left[\cos\left(\tan^{-1}\left(\frac{\rho_B}{z_B}\right)\right)\cos\left(\tan^{-1}\left(\frac{\rho_C}{z_C}\right)\right) + \sin\left(\tan^{-1}\left(\frac{\rho_B}{z_B}\right)\right)\sin\left(\tan^{-1}\left(\frac{\rho_C}{z_C}\right)\right)\cos(\psi_C - \psi_B)\right]} \tag{6}$$

Some trigonometric identities can be substituted here as well if reducing the number of trigonometric functions is desired. It should be noted however that this may result in some cases resulting in division by 0, and if trigonometric functions are used, atan2 is a feasible option.

$$\cos[\tan^{-1}(x)] = \frac{1}{\sqrt{1 + x^2}} \tag{7}$$

$$\sin[\tan^{-1}(x)] = \frac{x}{\sqrt{1 + x^2}} \tag{8}$$

Substituting equations (7) and (8) into equation (6) and simplifying:

$$a = r_{earth}\cos^{-1}\left[\left(\frac{1}{\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}}\right)\left(\frac{1}{\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}}\right) + \left(\frac{\frac{\rho_B}{z_B}}{\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}}\right)\left(\frac{\frac{\rho_C}{z_C}}{\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}}\right)\cos(\psi_C - \psi_B)\right]$$

$$a = r_{earth}\cos^{-1}\left[\frac{1}{\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}} + \left(\frac{\frac{\rho_B}{z_B}}{\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}}\right)\left(\frac{\frac{\rho_C}{z_C}}{\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}}\right)\cos(\psi_C - \psi_B)\right]$$

$$a = r_{earth}\cos^{-1}\left[\frac{1}{\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}} + \frac{\rho_B\rho_C\cos(\psi_C - \psi_B)}{z_B z_C\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}}\right]$$

$$\boxed{a = r_{earth}\cos^{-1}\left[\frac{z_B z_C + \rho_B\rho_C\cos(\psi_C - \psi_B)}{z_B z_C\sqrt{1 + \left(\frac{\rho_B}{z_B}\right)^2}\sqrt{1 + \left(\frac{\rho_C}{z_C}\right)^2}}\right]} \tag{9}$$