Integrable Systems - Homework assignment 3

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1 Quantum Heisenberg Model

Consider the Hamiltonian operator of the quantum Heisenberg model (XYZ model)

$$\hat{H} = -\sum_{i=1}^{N}(J_x \hat{S}_i^x \hat{S}_{i+1}^x + J_y \hat{S}_i^y \hat{S}_{i+1}^y + J_z \hat{S}_i^z \hat{S}_{i+1}^z),$$

where

$$\hat{S}_i^\alpha = \mathbb{1} \otimes \cdots \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \cdots \otimes \mathbb{1}$$

is the α component ($\alpha = x, y, z$) of the i-th spin.

(a) Show that $\hat{H}$ is real and symmetric.

We work in the basis formed by the 2N tensor products $|\uparrow\rangle$, $|\downarrow\rangle$.

First note that J_x, J_y, J_z are real constants. Next, we recall that any linear combination of real and symmetric matrices with real coefficients is real and symmetric. Thus, what we need to show is that $\hat{S}_i^\alpha \hat{S}_{i+1}^\alpha$ is real and symmetric for any $i = 1, \ldots, N$, and $\alpha = x, y, z$.

Using properties of tensor products, we see that

$$\begin{aligned}
\hat{S}_i^\alpha \hat{S}_{i+1}^\alpha &= \left(\mathbb{1} \otimes \cdots \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \cdots \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \cdots \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \cdots \otimes \mathbb{1}\right) \\
&= \left((\mathbb{1} \times \mathbb{1}) \otimes \cdots \otimes \left(\frac{\hbar}{2}\hat{\sigma}^\alpha \times \mathbb{1}\right) \otimes \left(\mathbb{1} \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha\right) \otimes \cdots \otimes (\mathbb{1} \times \mathbb{1})\right) \qquad \text{((a).1)} \\
&= \frac{\hbar^2}{4}\left(\mathbb{1} \otimes \cdots \otimes \hat{\sigma}^\alpha \otimes \hat{\sigma}^\alpha \otimes \cdots \otimes \mathbb{1}\right).
\end{aligned}$$

This motivates us to compute $\hat{\sigma}^\alpha \otimes \hat{\sigma}^\alpha$ for $\alpha = x, y, z$.

$$\hat{\sigma}^x \otimes \hat{\sigma}^x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{(a).2}$$

$$\hat{\sigma}^y \otimes \hat{\sigma}^y = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{(a).3}$$

$$\hat{\sigma}^z \otimes \hat{\sigma}^z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{(a).4}$$

We see that these are all real and symmetric. Furthermore, $\mathbb{1}$ is real and symmetric. Since the tensor products of real and symmetric matrices is also real and symmetric, we have that indeed $\hat{S}_i^\alpha \hat{S}_{i+1}^\alpha$ is real and symmetric for all $i = 1, \ldots, N$, and $\alpha = x, y, z$. By our previous reasoning then, $\hat{H}$ is real and symmetric.

(b) Prove that the following commutation relations hold in the isotropic ($J_x = J_y = J_z = J$) Heisenberg model with $N = 2$ spins:

$$[\hat{X} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{X}, \hat{H}] = 0,$$

where $\hat{X} \in \{\hat{\sigma}^z, \hat{\sigma}^+, \hat{\sigma}^-\}$.

First, we want to compute $\hat{H}$. For $N = 2$ in the isotropic model, we have that

$$\hat{H} = -(J\hat{S}_1^x\hat{S}_2^x + J\hat{S}_1^y\hat{S}_2^y + J\hat{S}_1^z\hat{S}_2^z + J\hat{S}_2^x\hat{S}_1^x + J\hat{S}_2^y\hat{S}_1^y + J\hat{S}_2^z\hat{S}_1^z). \tag{(b).1}$$

We can simplify this slightly to obtain

$$\hat{H} = -J(\hat{S}_1^x\hat{S}_2^x + \hat{S}_2^x\hat{S}_1^x + \hat{S}_1^y\hat{S}_2^y + \hat{S}_2^y\hat{S}_1^y + \hat{S}_1^z\hat{S}_2^z + \hat{S}_2^z\hat{S}_1^z) \tag{(b).2}$$

We see a repeating pattern; we see that $\hat{S}_1^\alpha\hat{S}_2^\alpha + \hat{S}_2^\alpha\hat{S}_1^\alpha$ appears for every $\alpha = x, y, z$. For now, we let α be arbitrary, and we see that

$$\hat{S}_1^\alpha\hat{S}_2^\alpha + \hat{S}_2^\alpha\hat{S}_1^\alpha = \left(\frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha\right) + \left(\mathbb{1} \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha\right)\left(\frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \mathbb{1}\right) \tag{(b).3}$$

$$= \left(\left(\frac{\hbar}{2}\hat{\sigma}^\alpha \times 1\right) \otimes \left(1 \times \frac{\hbar}{2}\hat{\sigma}^\alpha\right)\right) + \left(\left(1 \times \frac{\hbar}{2}\hat{\sigma}^\alpha\right) \otimes \left(\frac{\hbar}{2}\hat{\sigma}^\alpha \times 1\right)\right) \tag{(b).4}$$

$$= \left(\frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha\right) + \left(\frac{\hbar}{2}\hat{\sigma}^\alpha \otimes \frac{\hbar}{2}\hat{\sigma}^\alpha\right) \tag{(b).5}$$

$$= \frac{\hbar^2}{2}(\hat{\sigma}^\alpha \otimes \hat{\sigma}^\alpha) \tag{(b).6}$$

Then $\hat{H}$ as in Equation ((b).2) becomes

$$\hat{H} = -\frac{J\hbar^2}{2}\left(\hat{\sigma}^x \otimes \hat{\sigma}^x + \hat{\sigma}^y \otimes \hat{\sigma}^y + \hat{\sigma}^z \otimes \hat{\sigma}^z\right) \tag{(b).7}$$

But we know what these matrices look like from Equations ((a).2), ((a).3), and ((a).4). Subsituting those in, we get that

$$\hat{H} = \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \tag{(b).8}$$

which, as part (a) told us, is real and symmetric.

We next want to compute $\hat{X} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{X}$ for $\hat{X} \in \{\hat{\sigma}^z, \hat{\sigma}^+, \hat{\sigma}^-\}$. Let's do that. For ease of writing, let $\hat{X}^\beta = \hat{\sigma}^\beta \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\sigma}^\beta$ for $\beta = z, +, -$.

We start with $\hat{X} = \hat{\sigma}^z$. Then

$$\hat{X}^z = \hat{\sigma}^z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\sigma}^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{(b).9}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{(b).10}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{(b).11}$$

We continue with $\hat{X} = \hat{\sigma}^+$. Then

$$\hat{X}^+ = \hat{\sigma}^+ \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\sigma}^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{(b).12}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{(b).13}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{(b).14}$$

Finally, we take $\hat{X} = \hat{\sigma}^-$. Then

$$\hat{X}^- = \hat{\sigma}^- \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\sigma}^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{(b).15}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{(b).16}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{(b).17}$$

With all our matrices computed, we can then compute the commutation relations. We start with $\hat{X}^z$, as computed in Equation ((b).11) and $\hat{H}$, as computed in Equation ((b).8).

$$[\hat{X}^z, \hat{H}] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \tag{(b).18}$$

$$- \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -J\hbar^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J\hbar^2 \end{bmatrix} - \begin{bmatrix} -J\hbar^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J\hbar^2 \end{bmatrix} \tag{(b).19}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{(b).20}$$

Hence, the first commutation relation is satisfied.

Next, we look at $\hat{X}^+$, as computed in Equation ((b).14) and $\hat{H}$, as computed in Equation ((b).8).

$$[\hat{X}^+, \hat{H}] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \tag{(b).21}$$

$$- \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{J\hbar^2}{2} & -\frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar}{2} \\ 0 & 0 & 0 & -\frac{J\hbar}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{J\hbar^2}{2} & -\frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{(b).22}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{(b).23}$$

The second commutation relation is hence satisfied as well.

Finally, we look at $\hat{X}^-$, as computed in Equation ((b).17) and $\hat{H}$, as computed in Equation ((b).8).

$$[\hat{X}^+, \hat{H}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \tag{(b).24}$$

$$- \begin{bmatrix} -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & \frac{J\hbar^2}{2} & -J\hbar^2 & 0 \\ 0 & -J\hbar^2 & \frac{J\hbar^2}{2} & 0 \\ 0 & 0 & 0 & -\frac{J\hbar^2}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \tag{(b).25}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & -\frac{J\hbar^2}{2} & -\frac{J\hbar^2}{2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ -\frac{J\hbar^2}{2} & 0 & 0 & 0 \\ 0 & -\frac{J\hbar^2}{2} & -\frac{J\hbar^2}{2} & 0 \end{bmatrix} \tag{(b).26}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{(b).27}$$

The third commutation relation also holds.

(c) Find the ground state of the isotropic, antiferromagnetic ($J < 0$) model with $N = 4$ spins.

From the hint, we know that the ground state is an eigenvector of $\hat{H}$ that corresponds to the smallest eigenvalue. We note that if λ is an eigenvalue of a matrix A with corresponding eigenvector x, and if α is a complex number, then

$$(\alpha A)x = \alpha(Ax) = \alpha(\lambda x) = (\alpha\lambda)x, \tag{(c).1}$$

hence $\alpha\lambda$ is an eigenvalue of the matrix αA, with corresponding eigenvector x. Motivated by this, we define $\tilde{H} = \frac{\hat{H}}{\frac{-J\hbar^2}{4}}$, i.e. H with purely the matrices. We can then find the eigenvectors and eigenvalues of $\tilde{H}$ using Python using the following code:

```
# subquestion c, part 1

import numpy as np
import math

# Define i^2 = -1
i = complex(0, 1)

# Define the Pauli matrices, as well as the plus and minus
sx = np.array([[0,1],[1,0]], dtype=complex)
sy = np.array([[0,-i],[i,0]], dtype=complex)
sz = np.array([[1,0],[0,-1]], dtype=complex)
identity = np.eye(2)

# Define a list of pauli matrices to easily define S
pauli = np.array([sx, sy, sz])

# Define S without the factor of hbar/2
def S_hat(i,a):
    product_matrix = np.array([identity, identity, identity, identity])
    index = (i % 4)
    product_matrix[index] = pauli[a]
    prod = product_matrix[0]
    for i in range(3):
        prod = np.kron(prod, product_matrix[i+1])
    return prod

def S_con(i,a):
    return np.dot(S_hat(i, a) , S_hat(i+1, a))

# Construct H without the factor -Jhbar^2 / 4:
def H_hat():
    sum = np.zeros((16,16), dtype = complex)
    for i in range(4):
        sum += S_con(i, 0) + S_con(i, 1) + S_con(i, 2)
    return sum

eigenvalues, eigenvectors = np.linalg.eig(H_hat())

print(eigenvalues)
```

We note that all the eigenvalues are real, so they are well-ordered and we can find a smallest one. To go from the printed eigenvalues to the eigenvalues of $\hat{H}$, as discussed earlier, we need to multiply the entire array by $\frac{-J\hbar}{4}$. However, note that since $J < 0$, this multiplication does not change which eigenvalue is the smallest. Since the eigenvector does not change upon this multiplication, we can discard it. To find the smallest eigenvalue, we use `np.min`:

```
print(eigenvalues)
print(np.min(eigenvalues))
```

And we find that the smallest eigenvalue is -8, which occurs at the 7th position (since Python starts counting from 0). Then using

```
print(eigenvectors[:,7])
```

gives us that the eigenvector is, after correcting floating point errors and rounding to three significant digits, is given by

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.289 \\ 0 \\ 0.577 \\ -0.289 \\ 0 \\ 0 \\ -0.289 \\ 0.577 \\ 0 \\ -0.289 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{(c).2}$$

(d) Consider the following (spectral parameter dependent) R -matrix:

$$R(u) = u\mathbb{1}_4 + iP = \begin{bmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{bmatrix},$$

where P is a permutation matrix. Verify that R is a solution of the Yang-Baxter equation, that is,

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

Here $R_{12} = R \otimes \mathbb{1}_2$, $R_{23} = \mathbb{1}_2 \otimes R$, and R_{13} act on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Furthermore,

$$\hat{R}_{12} |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle = |\phi_1\rangle \otimes |\psi_2\rangle \otimes |\phi_3\rangle$$

where

$$\hat{R} |\psi_1\rangle \otimes |\psi_3\rangle = |\phi_1\rangle \otimes |\phi_3\rangle .$$

To be able to have the matrix dependent on u and v, we use Sympy instead of Numpy. Constructing R_{12} and R_{23} is fairly easy, since we can just use the kronecker product, which is already embedded into Sympy.

To construct R_{13}, we have to go through more work. Since 1 and 3 are not adjacent, we cannot simply kronecker product it. We first have to make swaps so that sites are next to each other, using a permutation matrix. Then, we have to apply the kronecker product, and finally we permute back. However, we also have the formula $R = u\mathbb{1}_4 + iP$ such that $R_{13} = u\mathbb{1}_4 + iP_{13}$. Hence in our code we first find P_{13} (the matrix permuting the first and third state), by seeing how it acts on each basis vector. The following code finds all the R-matrices.

```
# subquestion d

import sympy as sp

# Define symbols
u, v = sp.symbols('u v', commutative=True)

# Identity matrices
I2 = sp.eye(2)
I4 = sp.eye(4)

# Permutation matrix P (4x4)
P = sp.Matrix([[1,0,0,0],[0,0,1,0],[0,1,0,0],[0,0,0,1]])

# R-matrices
def R(x):
    return x * I4 + sp.I * P

def R12(u):
    return sp.kronecker_product(R(u), I2)

def R23(u):
    return sp.kronecker_product(I2, R(u))
```

```

# Initialize P13 to compute R13
P13 = sp.zeros(8)

# Canonical basis for N=3:
basis = [(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)
    ,(1,1,1)]

# Make it easy to find the right basis element
index = {basis[i]: i for i in range(8)}

# Construct R12
for a,b,c in basis:
    i = index[(a,b,c)]
    j = index[(c,b,a)]
    P13[i,j] = 1

# R13
def R13(u,v):
    return (u+v)*sp.eye(8) + sp.I*P13
```

Now R is a solution of the Yang-Baxter equation if and only if

$$R_{12}(u)R_{13}(u+v)R_{23}(v) - R_{23}(v)R_{13}(u+v)R_{12}(u) = 0. \tag{(d).1}$$

We have the R matrices defined, so we can code this in:

```
print(sp.simplify(R12(u)*R13(u,v)*R23(v) - R23(v)*R13(u,v)*R12(u)))
```

which indeeds gives us the zero matrix. Indeed, R is a solution of the Yang-Baxter equation

End of homework 3