1D Heat Equation Work

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$$\lambda_n = \frac{(2n-1)\pi}{2L} \qquad \text{where} \qquad n = 1, 2, 3, \dots$$

$$b_n = -\frac{2H}{L}\int_0^L x \sin(\lambda_n x)\,dx$$

$$\int_0^L x \sin(\lambda_n x)\,dx$$

$u = x$

$du = dx \qquad dv = \sin(\lambda_n x)\,dx$

$v = \frac{-\cos(\lambda_n x)}{\lambda_n}$

$$\int_0^L x\sin(\lambda_n x)\,dx = \frac{-x\cos(\lambda_n x)}{\lambda_n} - \int_0^L \frac{\cos(\lambda_n x)}{\lambda_n}\,dx$$

$$= \left.\frac{-x\cos(\lambda_n x)}{\lambda_n} - \frac{\sin(\lambda_n x)}{\lambda_n^2}\right|_0^L$$

$$= \frac{-\sin(L\lambda_n)}{\lambda_n^2} - \frac{L\cos(L\lambda_n)}{\lambda_n}$$

$$= \frac{-\cancel{\lambda_n}\sin(L\lambda_n) - \lambda_n^{\cancel{2}} L\cos(L\lambda_n)}{\lambda_n^{\cancel{3}}}$$

$$= \frac{-\sin(L\lambda_n) - \lambda_n L\cos(L\lambda_n)}{\lambda_n^2}$$

* substituting $\lambda_n = \frac{(2n-1)\pi}{2L}$

$$= \frac{-\sin\left[\cancel{L}\left(\frac{(2n-1)\pi}{2\cancel{L}}\right)\right] - \left(\frac{(2n-1)\pi}{2\cancel{L}}\right)\cancel{L}\cos\left[\cancel{L}\left(\frac{(2n-1)\pi}{2\cancel{L}}\right)\right]}{\left[\frac{(2n-1)\pi}{2L}\right]^2}$$

denominator:

$$\frac{[(2n-1)\pi]^2}{4L^2} = \frac{(2n-1)^2\pi^2}{4L^2}$$

$\to 0$ for $n > 0$

numerator:

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$$-\sin\left[\frac{(2n-1)\pi}{2}\right] - \cancel{\left[\frac{(2n-1)\pi}{2}\right]\cos\left[\frac{(2n-1)\pi}{2}\right]} \rightarrow 0 \text{ for } n>0$$

$$-\sin\left[\frac{(2n-1)\pi}{2}\right] = \sin\left(n\pi - \frac{\pi}{2}\right)$$

$$-\sin\left(n\pi - \frac{\pi}{2}\right) = \begin{cases} 1 & \text{for } n \text{ that are odd} \\ -1 & \text{for } n \text{ that are even} \end{cases}$$

$$= -(-1)^n$$

* plugging in:

$$\frac{(-1)^n}{\frac{(2n-1)^2\pi^2}{4L^2}} = \frac{4L^2(-1)^n}{\pi^2(2n-1)^2}$$

$$b_n = \frac{-2H}{\cancel{L}}\left(\frac{4L^{\cancel{2}}(-1)^n}{\pi^2(2n-1)^2}\right)$$

$$\boxed{b_n = \frac{8HL(-1)^n}{\pi^2(2n-1)^2}}$$