An Undecidable Algebra Problem

Jared Holshouser

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W-groups

Note that all groups in this presentation are Abelian groups.

Definition

A surjective homomorphism $\varphi : H \to G$ is said to split iff there is a homomorphism $\psi : G \to H$ s.t. $\varphi \circ \psi = \mathrm{id}_G$.

Definition

G is a W-group iff whenever $\varphi : H \to G$ is a surjective homomorphism and $\ker(\varphi) \cong \mathbb{Z}$, φ splits.

Theorem

Countable W-groups are free.

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Theorem

- *ZFC+V=L implies that every W-group of cardinality ω_1 is free.*
- *ZFC+MA+$2^\omega > \omega_1$ implies that there is a W-group of cardinality ω_1 that is not free.*

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- *A subgroup of a free group is free.*
- *A finitely generated torsion free group is free.*

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Suppose H is a subgroup of G such that H and G/H are both free. Then G is free. Moreover, any basis of H extends to a basis of G.

Chains

Definition

A smooth chain of groups is an ascending chain

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots$$

for $\alpha < \lambda$ such that G_α is a subgroup of $G_{\alpha+1}$ and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for limit α.

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Theorem

Let G be the union of a smooth chain of groups $\{G_\alpha : \alpha < \lambda\}$ such that G_0 is free and for all $\alpha < \lambda$, $G_{\alpha+1}/G_\alpha$ is free. Then G is free. Moreover, G/G_α is free for all $\alpha < \lambda$.

Theorem

A subgroup of W-group is a W-group.

Theorem

Every W-group is torsion free.

Theorem

If H is a subgroup of G such that G is a W-group but G/H is not a W-group, then there is a homomorphism $\varphi : H \to \mathbb{Z}$ which does not extend to a homomorphism from G to $\mathbb{Z}$.

Chase's Condition

Definition

A group G is ω_1-free iff every countable subgroup is free. If G is a ω_1-free, H a subgroup of G is ω_1-pure iff G/H is ω_1-free.

Corollary

Every W-group is ω_1-free.

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Corollary

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Definition

G satisfies Chase's condition iff G is an ω_1-free group such that every countable subgroup of G is contained in a countable ω_1-pure subgroup of G.

Lemma

If G is a group of cardinality ω_1, G satisfies Chase's condition iff G is the union of a smooth chain of countable free groups

$$G_0 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots$$

for $\alpha < \omega_1$ so that $G_0 = \{0\}$ and for all $\alpha < \omega_1$, $G_{\alpha+1}$ is ω_1-pure in G.

We endow ω_1 with the order topology.

Definition

$C \subseteq \omega_1$ is club iff C is closed and unbounded. $E \subseteq \omega_1$ is stationary iff E meets every club set C.

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$f : \omega_1 \to \omega_1$ is normal iff f is continuous and increasing.

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Theorem

$C \subseteq \omega_1$ is club iff there is a normal function $f : \omega_1 \to \omega_1$ so that $C = f[\omega_1]$.

Suppose that G is the union of the smooth chain of countable free groups $\{G_\alpha : \alpha < \omega_1\}$. Set

$$E = \{\alpha < \omega_1 : \alpha \text{ is a limit and } G_\alpha \text{ is not } \omega_1 - \text{pure}\}$$

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Theorem

G is free iff E is not stationary.

Theorem (V=L)

Let C be a set which is the union of a strictly increasing smooth chain of countable sets $\{C_\alpha : \alpha < \omega_1\}$, and E be a stationary subset of ω_1. Then there is a sequence $\{S_\alpha : \alpha < \omega_1\}$ such that $S_\alpha \subseteq C_\alpha$ for all $\alpha \in E$ and such that for any $X \subseteq C$, $\{\alpha \in E : X \cap C_\alpha = S_\alpha\}$ is stationary in ω_1.

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Theorem (V=L)

Let G be the union of a strictly increasing smooth chain $\{G_\alpha : \alpha < \omega_1\}$ of countable free groups such that $E' = \{\alpha < \omega_1 : G_{\alpha+1}/G_\alpha \text{ is not free}\}$ is stationary in ω_1. Then G is not a W-group.

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Theorem

ZFC+V=L implies that every W-group of cardinality ω_1 is free.

Theorem (MA+$2^\omega > \omega_1$)

Let A and B be sets of cardinality $< 2^\omega$, and let P be a family of functions with the following properties:

- *Every $f \in P$ is a function from a subset of A into B.*
- *For all $a \in A$ and $f \in P$ there is a $g \in P$ so that $f \subseteq g$ and $a \in dom(g)$.*
- *For every uncountable $P' \subseteq P$ there are $f_1, f_2 \in P'$ and $f_3 \in P$ so that $f_1 \neq f_2$ and f_3 extends both f_1 and f_2.*

Then there is a function $g : A \to B$ so that for all finite $F \subseteq A$ there is an $f \in P$ with $F \subseteq dom(f)$ and $g \restriction_F = f \restriction_F$.

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Theorem (MA+$2^\omega > \omega_1$)

Let G be a group of cardinality ω_1 which satisfies Chase's condition. Then G is a W-group.

Theorem

There is a group G of cardinality ω_1 which satisfies Chase's condition but is not free.

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There is a group G of cardinality ω_1 which satisfies Chase's condition but is not free.

Theorem

ZFC+MA+$2^\omega > \omega_1$ implies that there is a W-group of cardinality ω_1 that is not free.

References

Eklof, Paul C. "Whitehead's Problem is Undecidable." The American Mathematical Monthly 83.10: 775-778. Print.