Choice Grab Bag

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September 2, 2013

1 The Ultrafilter Theorem

Definition 1. The Ultrafilter Theorem (UT) says that every filter can be extended to an ultrafilter.

Remark 1. UT is strictly weaker than AC, but implies the ordering principle, and AC for well-founded sets. It is not implied by and does not imply DC, and it is false under AD.

Theorem 1. *The following are equivalent:*

1. *UT*
2. *A product of compact Hausdorff spaces is compact.*
3. *If Σ is a finitely satisfiable set of sentences in propositional logic, then Σ is satisfiable.*

Proof. We first see that 1) implies 2). If we consult the proof of Tychonoff's theorem, we see that full choice is only used to select a point which the ultrafilters are converging to. If the spaces are T2, then convergence is unique, and so AC is not necessary.

Now we see that 2) implies 3). Let $\mathcal{L}$ be the language for Σ, and Φ be the set of sentences in $\mathcal{L}$. Let $X = 2^{\Phi}$. Then by hypothesis X is compact. For $f \in X$, say f is consistent if for all $\varphi_1, \cdots, \varphi_n, \psi \in \Phi$

$$(f(\varphi_1) = \cdots = f(\varphi_n) = 1 \wedge \{\varphi_1, \cdots, \varphi_n\} \vdash \psi) \implies f(\psi) = 1$$

For $\Gamma \subseteq \Sigma$ finite, let $F_\Gamma = \{f \in X : f \text{ is consistent } \wedge f\restriction_\Gamma = 1\}$. Then F_Γ is closed. Now $F_\Gamma \neq \emptyset$ as Σ is finitely satisfiable. Finally, $\{F_\Gamma : \Gamma \in \Sigma^{<\omega}\}$ is closed under finite intersection. Thus by compactness, $\bigcap\{F_\Gamma : \Gamma \in \Sigma^{<\omega}\} \neq \emptyset$. Let f witness this. Then f is consistent and $f\restriction_\Sigma = 1$. So Σ is satisfiable.

Finally we see that 3) implies 1). Let F be a filter on X. Let $\mathcal{L}$ be the language containing constants for all the members of $\mathcal{P}(X)$, and a unary relation U. Let Σ be the set of sentences consisting of

- $\neg U(\emptyset)$,
- $(U(A_1) \wedge \cdots \wedge U(A_n)) \implies U(A_1 \cap \cdots \cap A_n)$ for all $A_1, \cdots, A_n \in \mathcal{P}(X)$,
- $U(A) \wedge U(X \smallsetminus A)$ for all $A \in \mathcal{P}(X)$, and
- $U(A)$ for all $A \in F$

Then Σ is finitely satisfiable. Thus Σ is satisfiable. Let U witness this. Then U is an ultrafilter on X extending F. $\square$

2 Dependent Choice

Definition 2. The axiom of dependent choice (DC) states the following: If R is a binary relation on a nonempty set X such that $\forall x \in X \exists y \in X(xRy)$, then $\exists f \in X^\omega \forall n \in \omega(f(n)Rf(n+1))$.

Definition 3. A topological space X is Baire iff the countable intersection of open dense sets is non-empty. BCT2 states the following: Every compact Hausdorff space is Baire.

Definition 4. $\mathcal{G} \subseteq \mathcal{P}(X)$ is a grill iff

- $X \in \mathcal{G}$ and $\emptyset \notin \mathcal{G}$
- $A \cup B \in \mathcal{G}$ iff $A \in \mathcal{G}$ or $B \in \mathcal{G}$

$\mathcal{G} \to x$ iff $\mathcal{N}_x \subseteq \mathcal{G}$.

Lemma 1. *If $\mathcal{G}$ is a grill in $X \times Y$, $\pi_X'' \mathcal{G} \to x$, and Y is compact, then there is a $y \in Y$ such that $\mathcal{G} \to (x, y)$.*

Proof. We proceed by way of contradiction. Let

$$\mathcal{U} = \{U \subseteq Y : U \text{ is open} \wedge \exists V \in \mathcal{N}_x(V \times U \notin \mathcal{G}\}$$

Then $\mathcal{U}$ is an open cover of Y. By compactness, let $\mathcal{U}'$ be a finite subcover. For all $U \in \mathcal{U}'$, let V_U be an nhood of x so that $V_U \times U \notin \mathcal{G}$. Then $V = \bigcap\{V_U : U \in \mathcal{U}'\}$ is an nhood of x with $V \times U \notin G$ for all $U \in \mathcal{U}'$. Since $\mathcal{G}$ is a grill, we then get that

$$\pi_X^{-1}[V] = V \times Y = V \times \bigcup \mathcal{U}' = \bigcup \{V \times U : U \in \mathcal{U}'\} \notin \mathcal{G}$$

So $\pi_X'' \mathcal{G}$ does not converge to x, a contradiction. □

Proposition 1. *DC implies that countable products of compact spaces are compact.*

Remark 2. Let X be a set. Endowing X with the discrete topology, we let αX be the one-point-compactification of X.

Theorem 2. *The following are equivalent:*

1. *Countable products of compact Hausdorff spaces are Baire*
2. *For all X, $(\alpha X)^\omega$ is Baire*
3. *DC*
4. *Countable products of compact Hausdorff spaces are compact and BCT2.*

Proof. Note that 1) implies 2) immediately, as αX is compact and Hausdorff. We now prove that 2) implies 3). Let X be a non-empty set and R a relation on X so that for all $x \in X$ there is a $y \in X$ with xRy. Let $Y = (\alpha X)^\omega$. For all n let

$$B_n = \{f \in X^\omega : \forall m \leq n \exists k \geq m(f(m)Rf(k))\}$$

These sets are dense and open. So by 2), $\bigcap_n B_n \neq \emptyset$. Let $f \in \bigcap_n B_n$. We define a g by induction:

- $g(0) = f(0)$
- $g(n+1) = f(\min\{m : g(n)Rf(m)\})$

This is valid as $f \in \bigcap_n B_n$. Then g is as desired.

We now show that 3) implies 4). The proposition shows that a countable product of compact Hausdorff spaces is compact. If one examines the standard proof of BCT2, they would find that only DC was used. So every compact Hausdorff space is Baire.

Finally we show that 4) implies 1). By 4), a countable product of compact Hausdorff spaces is compact. A product of Hausdorff spaces is always Hausdorff. Thus by 4), a countable product of compact Hausdorff spaces is compact Hausdorff, and thus Baire. □

3 Non-Measurable Sets

Definition 5. C_2 is the statement: For any collection $\{X_\alpha : \alpha \in A\}$ of two-element sets there is a function $f : A \to \prod_{\alpha \in A} X_\alpha$ so that $f(\alpha) \in X_\alpha$ for all α.

Theorem 3. *C_2 implies that there is a non-measurable set.*

Proof. Endow 2^ω with the product measure on 2, as is standard. For $a, b \in 2^\omega$ say that $a \sim b$ iff $|\{n : a(n) \neq b(n)\}| < \omega$. For $a \in 2^\omega$, set $[a] = [a]_\sim$, and define $-a(n) = a(n) + 1(\text{mod } 1)$. Set $-[a] = [-a]$. This is well-defined. Now for all $x \in 2^\omega / \sim$, let $A_x = \{x, -x\}$. Let f be a choice function on $\{A_x : x \in 2^\omega / \sim\}$. Let $A = \bigcup_x f'' A_x$. Then

1. $a \in A$ iff $-a \notin A$;
2. If $a \in A$ and $a \sim b$, then $b \in A$.

We will show that A is not measurable. By 2), if $\mu(A \setminus N_s) = 0$, then if $|t| = |s|$ we have that $\mu(A \setminus N_t) = 0$. So if A is measurable, then $\mu(A) = 0$ or $\mu(A) = 1$. But $\mu(A) = \mu(-A)$, and by 1) $A \cup (-A) = 2^\omega$, where $-A = \{-a : a \in A\}$. Thus A is not measurable. $\square$