Hausdorff Dimension of Metric Spaces and Lipschitz Maps Onto Cubes

Jared Holshouser

September 6, 2016

1 Introduction

We follow the work of Kelete, Mathe, and Zindulka [3].

When can a metric space be mapped onto the $[0,1]^k$? If our notion of map is too loose, the question has some bizarre answers. For instance, in 1890, Peano built a continuous function from $[0,1]$ onto $[0,1]^2$. Later work in this direction revealed that for any k, there are continuous functions from $[0,1]$ onto $[0,1]^k$, and they can even be taken to be $\frac{1}{k}$-Holder continuous. To restrict the problem to a reasonable category, we will require that our maps be Lipschitz. The following proposition gives us one constraint.

Proposition 1. *Let (X,d) and (Y,ρ) be metric spaces, $d \geq 0$ be a real number, and $f : X \to Y$ be Lipschitz. Then there is a constant $C > 0$ so that*

$$\mathcal{H}^d(f[S]) \leq C\mathcal{H}^d(S)$$

for all Borel $S \subseteq X$.

So it must be that $\mathcal{H}^k(X) > 0$. Does this suffice? In 1933, Kolmogorov [4] conjectured that if $X \subseteq \mathbb{R}^n$ is so that $\mathcal{H}^1(X) > 0$, then there is a Lipschitz map from X onto $[0,1]$. Sadly, this is false. In 1963, Vitushkin, Ivanov, and Melnikov [9] constructed a compact subset of the plane with positive 1-dimensional Hausdorff measure that cannot be mapped onto a segment by a Lipschitz map. We will show that it does suffice to assume that $\dim_H(X) > k$.

2 Preliminaries

Frostman's Lemma is useful tool for computing Hausdorff dimension.

Theorem 1 (Frostman's Lemma)**.** *Let (X,ρ) be a compact metric space, $S \subseteq X$ be Borel, and $d > 0$. Then the following are equivalent:*

- *$\mathcal{H}^d(S) > 0$, and*
- *there is a finite Borel measure μ with $\mu(S) > 0$ so that $\mu(E) \leq |E|^s$ for any Borel $E \subseteq S$.*

Zindulka introduced the following notion in [10].

Definition 1. A metric space (X,d) is **monotone** iff there is a linear order $\prec$ and a constant C so that

$$(\forall a,b \in X)[\operatorname{diam}([a,b]_\prec) \leq Cd(a,b)].$$

If this holds for a given C, we say the space is C-monotone.

Nekvinda and Zindulka [8] proved that that every ultrametic space in monotone. We can say even more for compact ultrametric spaces.

Lemma 1. *Any compact ultrametric space* (X, d) *is* 1*-monotone.*

Proof. Let $D = |X|$. If $D = 0$, then it is a singleton, so this is trivial. Otherwise, since d is an ultrametric, the relation $d(x, y) < D$ is an equivalence relation. The equivalence classes are open, and X is compact, so there are only finitely many equivalence classes: $X_1, \cdots, X_k$. These then are closed and compact as well. Since X is compact and $D \neq 0$, there are two points with distance D, so $k \geq 2$. Note also that if a and b are from distinct equivalence classes, then $d(a, b) = D$.

Since each equivalence class is a compact metric space, we do the same for each of them. So we get a tree of clopen sets $X_{i_1,\cdots,i_m}$ with the property that, for a fixed m, these sets form a partition of X, and if $a \in X_{i_1,\cdots,i_m,j}$ and $b \in X_{i_1,\cdots,i_m,j'}$ and $j \neq j'$, then $d(a, b) = \text{diam}\,(X_{i_1,\cdots,i_m})$. So for any $x \in X$, there is a unique $\vec{s} \in \omega^\omega$ so that $x \in X|_{\vec{s}|_m}$ for all m. Similarly, each such $\vec{s}$ defines a unique point (This requires some argument). So define $F : X \to \omega^\omega$ by $F(x)$ is the unique $\vec{s}$ with $x \in X|_{\vec{s}|_m}$ for all m.

We now define the order on X. $x \prec y$ iff $F(x) <_{lex} F(y)$. We need to show that for any $x \prec y \in X$, $\text{diam}([x, y]_\prec) = d(x, y)$. Let $\vec{s} \in \omega^{<\omega}$ so that $x, y \in X_s$. Then $d(x, y) = \text{diam}(X_s)$. But also, $[x, y]_\prec \subseteq X_s$, so

$$d(x, y) \leq \text{diam}([x, y]_\prec) \leq \text{diam}(X_s) = d(x, y).$$

This completes the proof. □

3 Nice Large Metric Spaces Can be Mapped Onto Cubes

Theorem 2. *If (X, d) is a compact monotone metric space with positive s-dimensional Hasudorff measure, where $s > 0$, then X can be mapped onto a non-degenerate interval by an s-Holder function.*

Proof. By Frostman's lemma, we can choose a non-zero finite Borel measure μ on X so that $\mu(E) \leq |E|^s$ for any Borel $E \subseteq X$. Since X is a monotone metric space there is a linear order $\prec$ and a constant C so that

$$\forall a, b \in X(\operatorname{diam}([a, b]_\prec) \leq Cd(a, b)).$$

Claim 1. Any open interval $(a, b)_\prec$ is open, and thus Borel.

Reason. We proceed by contradiction. So there is an $x \in (a, b)_\prec$ so that for all n, there is an $x_n \in B(x, \frac{1}{n})$ so that $x_n \preceq a$ or $b \preceq x_n$. Let N be so that $\frac{1}{N} < \frac{1}{C} \min\{\operatorname{diam}([a, x]_\prec), \operatorname{diam}([x, b]_\prec)\}$. First suppose that $x_N \preceq a$. By assumption,

$$\operatorname{diam}([x_N, x]_\prec) \leq Cd(x_N, x) \leq C\frac{1}{N} < \operatorname{diam}([a, x]_\prec),$$

but this cannot be, as $[a, x]_\prec \subseteq [x_N, x]_\prec$. Now suppose that $b \preceq x_N$. Again,

$$\operatorname{diam}([x, x_N]_\prec) \leq Cd(x_N, x) \leq C\frac{1}{N} < \operatorname{diam}([x, b]_\prec),$$

which is a contradiction as $[x, b]_\prec \subseteq [x, x_N]_\prec$. Thus the interval is open. □

For $x \in X$, let $(-\infty, x)_\prec = \{y \in X : y \prec x\}$, and let $g(x) = \mu((-\infty, x)_\prec)$. Then g is s-Holder, since for any $a \prec b \in X$,

$$0 \leq g(b) - g(a) = \mu([a, b)_\prec) \leq \operatorname{diam}([a, b)_\prec)^s \leq (Cd(a, b))^s.$$

Thus $g[X]$ is compact. Since μ is not the zero measure and X is separable, $g[X]$ is not a singleton.

We will now show that $g[X]$ is connected. Since $g[X]$ is closed, all we need to prove is that there are no $u, v \in g[X]$ with $u < v$ and $(u, v) \cap g[X] = \emptyset$. Suppose otherwise, and let u and v witness it. Let $D \subseteq X$ be countable and dense. Let $D_1 = \{x \in D : g(x) \leq u\}$ and $D_2 = \{x \in D : g(x) \geq v\}$. Since $(u, v) \cap g[X] = \emptyset$, $D = D_1 \cup D_2$. We also get that $\mu\left(\bigcup_{x \in D_1}(-\infty, x)\right) \leq u$ and $\mu\left(\bigcap_{x \in D_2}(-\infty, x)\right) \geq v$. Let

$$A = \left(\bigcap_{x \in D_2}(-\infty, x)\right) - \left(\bigcup_{x \in D_1}(-\infty, x)\right).$$

Then $\mu(A) \geq v - u > 0$. On the other hand, A cannot have more than two points. For if $x \prec y \prec z \in A$, then $(x, z)_\prec \neq \emptyset$, but $(x, z)_\prec \cap D = \emptyset$, which contradicts the denseness of D. Since $\mu(E) \leq |E|^s$ for any Borel $E \subseteq X$, the singletons are measure zero. Hence $\mu(A) = 0$. This is a contradiction. □

Corollary 1. *Let X be a compact monotone metric space and let $k > 0$ be an integer. Then X can be mapped onto the k-dimension cube $[0, 1]^k$ by a Lipschitz map iff X has positive k-dimensional Hausdorff measure.*

Proof. Clearly it is necessary that $\mathcal{H}^k(X) > 0$. So suppose that $\mathcal{H}^k(X) > 0$. Then by the theorem, there is a k-Holder continuous map $g : X \to \mathbb{R}$ so that $g[X] = [0, 1]$. It is known that there is a $\frac{1}{k}$-Holder Peano curve $h : [0, 1] \to [0, 1]^k$, the classical construction works. Then the composition $h \circ g$ is a Lipschitz map from X onto $[0, 1]^k$. □

In 2013, Mendel and Naor [6] proved some results about approximating sets in the context of dimension. The following is a weak version of one thing they showed.

Theorem 3. *(Mendel and Naor) For every compact metric space (X, d) and $\varepsilon > 0$ there is a closed subset $Y \subseteq X$ so that $dim_H(Y) \geq (1 - \varepsilon) dim_H(X)$ and (Y, d) is bi-Lipschitz equivalent to an ultrametric space.*

Theorem 4. *Let A be an analytic subset of a separable complete metric space (X, d), and let k be an integer. If $dim_H(A) > k$, then A can be mapped onto the k dimensional cube $[0, 1]^k$ by a Lipschitz map.*

Proof. Let $s \in (k, \dim_H(A)) \subseteq \mathbb{R}$. By the theorem of Howroyd, A has a compact subset C with finite and positive s-dimensional Hausdorff measure. Then by the Mendel-Naor theorem, C has a subset E with $\dim_H(E) > k$ that is bi-Lipschitz equivalent to an ultrametric space. Say $f : E \to (Z, \rho)$ is bi-Lipschitz. Applying the Howroyd theorem again (this time to $k \in (k - 1, \dim_H(E))$), we get a compact subset B of E with positive and finite k-dimensional Hausdorff measure. B is bi-Lipschitz equivalent to a compact ultrametric space Y, via $f|_B$. Now Y can be mapped onto $[0, 1]^k$ by a Lipschitz map, say $g : Y \to [0, 1]^k$. Then $h = g \circ f|_B$ is Lipschitz and $h : B \to [0, 1]^k$ is onto. Write $h(x) = (h_1(x), \cdots, h_k(x))$. Then each of the h_i are Lipschitz and real-valued. So we can extend each h_i to a Lipschitz $H_i : A \to [0, 1]$. Then $H(x) = (H_1(x), \cdots, H_k(x))$ is a Lipschitz function from A onto $[0, 1]^k$. □

It was conjectured by Laczkovich [5] in 1991 that if $A \subseteq \mathbb{R}^n$ has positive Lebesgue measure, then A can be mapped onto $[0, 1]^n$ with a Lipschitz map. This has been shown for $n \leq 2$ [1], but is still open for higher n.

4 Large Metric Spaces that Cannot Be Mapped Onto A Segment

Theorem 5. *Assume that, in $\mathbb{R}^n$, less than continuum many closed sets of measure zero and a set of measure zero cannot cover $\mathbb{R}^n$. Then there is a non-Lebesgue-null $A \subseteq \mathbb{R}^n$ so that for any continuous function $f : \mathbb{R}^n \to \mathbb{R}$, $f[A]$ does not contain any interval.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ and $\{N_\alpha : \alpha < \mathfrak{c}\}$ enumerate the collection of $\mathbb{R}^n \to \mathbb{R}$ continuous functions and the collection of Lebesgue null Borel subsets of $\mathbb{R}^n$, respectively.

By transfinite induction, for every $\alpha < \mathfrak{c}$, we construct points $x_\alpha \in \mathbb{R}^n$ and $y_\alpha \in (0, 1)$ so that

1. $x_\alpha \notin N_\alpha$,
2. $x_\alpha \notin \bigcup_{\beta<\alpha} f_\beta^{-1}(y_\beta)$,
3. $y_\alpha \notin f_\alpha[\{x_\beta : \beta \leq \alpha\}]$, and
4. $f_\alpha^{-1}(y_\alpha)$ is Lebesgue null.

x_0 and y_0 can be chosen essentially arbitrarily (up to conditions 1 and 3). Suppose we have completed all steps β for $\beta < \alpha$. By our main assumption, and since $f_\beta^{-1}(y_\beta)$ is a closed Lebesgue null set for each $\beta < \alpha$, we can choose an x_α so that conditions 1 and 2 hold. Now it cannot be that more than countably many $f_\alpha^{-1}(t)$ for $t \in (0, 1)$ have positive measure, and $f_\alpha[\{x_\beta : \beta \leq \alpha\}]$ is null by our main assumption. So the set

$$\{y \in (0, 1) : f_\alpha^{-1}(y_\alpha) \text{ is Lebesgue null} \wedge y \notin f_\alpha[\{x_\beta : \beta \leq \alpha\}]\}$$

has full measure in $(0, 1)$. Thus we can find a y_α satisfying properties 3 and 4. This completes the recursive step.

Let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. A cannot be Lebesgue null, as property 1 shows that A is not contained in any Borel Lebesgue null set. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $h = g \circ f$, where $g : \mathbb{R} \to \mathbb{R}$ is a linear transformation. Then $h = f_\alpha$ for some $\alpha < \mathfrak{c}$. So $h(x_\beta) \neq y_\alpha$ for any $\beta \leq \alpha$ by property 3, and $h(x_\beta) \neq y_\alpha$ for any $\beta > \alpha$ by property 2. So $y_\alpha \notin h[A]$, and thus $(0, 1) \not\subseteq h[A]$. Therefore $g^{-1}[(0, 1)] \not\subseteq f[A]$. Since g was arbitrary, this means that no interval is contained in $f[A]$. □

The assumption we took for this work would be taken as $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. We will now suppose that $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$.

Theorem 6. *Suppose $cov(\mathcal{M}) < \mathfrak{c}$. For any gauge function φ, there is a separable metric space (X, d) so that $|X| = cov(\mathcal{M})$ with $\mathcal{H}^\varphi((X, d)) > 0$.*

Proof. Fremlin and Miller [7] proved that $\mathrm{cov}(M)$ is the least cardinality of a subspace of (ω^ω, d) that is not a strong measure zero space. So there is an $H \subseteq \omega^\omega$ so that $|H| = \mathrm{cov}(\mathcal{M})$ and (H, d) is not a strong measure zero space. Hence there is a gauge function φ_0 so that $\mathcal{H}^{\varphi_0}((H, d)) > 0$. Let φ be a gauge function and $g : \{1, 2, \cdots\} \to (0, \infty)$ be defined by

$$g(m) = \varphi^{-1}\left(\varphi_0\left(\frac{1}{m}\right)\right).$$

Note that $\varphi\left(\frac{1}{m}\right) = \varphi(g(m))$ for all $m \in \{1, 2, \cdots\}$. Define d_g on ω^ω by

$$d_g(x, y) = g(|x \wedge y| + 1).$$

Then (H, d_g) is a separable metric space and $\mathcal{H}^\varphi((H, d_g)) = \mathcal{H}^{\varphi_0}((H, d))$. Therefore (H, d_g) is as desired. □

Theorem 7 (ZFC)**.** *There is a separable metric space with arbitrarily large Hausdorff dimension than cannot be mapped onto a segment by a uniformly continuous function.*

There is a model of ZFC in which every positive Hausdorff dimensional subset of a Euclidean space can be mapped onto $[0, 1]$ by a uniformly continuous function. [2]

References

[1] Giovanni Alberti, Marianna Csornyei, and David Preiss. Structure of null sets in the plane and applications. *Eur. Math. Soc.*, 2005.

[2] Paul Corazza. The generalized borel conjecture and strongly proper orders. *Trans. Amer. Math. Soc.*, 316, 1989.

[3] Tamas Keleti, Andras Mathe, and Ondrej Zindulka. Hausdorff dimension of metric spaces and lipschitz maps onto cubes. *International Mathematics Research Notices*, 2012.

[4] Andrey Kolmogorov. Beitrage zur matheorie. *Math. Ann.*, 1, 1933.

[5] Miklos Laczkovich. Paradoxical decompositions using lipschitz functions. *Real Anal. Exchange*, 1, 1991.

[6] Mendel Manor and Assaf Naor. Ultrametric subsets with large hausdorff dimension. *Inventiones mathematicae*, 192, 2013.

[7] Arnold W. Miller and David H. Fremlin. On some properties of hurewicz, menger, and rothberger. *Fund. Math.*, 129, 1988.

[8] Ales Nekvinda and Ondrej Zindulka. Monotone metric spaces. *Order*, 2012.

[9] A. G. Vituskina, L. D. Ivanov, and M. S. Melnikov. Incommensurability of the minimal linear measure with the length of a set. *Dokl. Akad. Nauk SSSR*, 151, 1963.

[10] Ondrej Zindulka. Universal measure zero, large hausdorff dimension, and nearly lipschitz maps. *Fund. Math.*, 218, 2012.