Generalizing Mycielski

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1 Introduction

In 1964, J. Mycielski proved that there are uncountable sets of reals which are algebraically independent. In doing so, he invented a general construction principle which has been used in a variety of applications. In this talk, I will be generalizing this construction to create a very specific kind of uncountable set. I have used this to prove coloring theorems for $\mathbb{R}$ and certain quotients of $\mathbb{R}$.

Throughout this talk, I will be working in 2^ω (infinite strings of 0s and 1s).

2 Perfect Sets

Recall the following definition.

Definition 1. Let $(X, \mathcal{T})$ be a topological space. A set $P \subseteq X$ is **perfect** if P is closed and P has no isolated points.

In 2^ω, perfect sets have cardinality the continuum, and every perfect set is actually homeomorphic to 2^ω. In fact, perfect sets correspond to specific construction methods. To describe these, I need to discuss trees.

Definition 2. Let $2^{<\omega}$ be the set of finite strings of 0s and 1s. For $s, t \in 2^{<\omega}$, say $s \sqsubseteq t$ if s is an initial segment of t. $T \subseteq 2^{<\omega}$ is a **tree** iff whenever $t \in T$ and $s \sqsubseteq t$, $s \in T$ as well.

The basic clopen sets for 2^ω correspond to these strings. For $s \in 2^{<\omega}$, set

$$N(s) = \{x \in 2^\omega : s \sqsubseteq x\}.$$

Trees can be used to generate arbitrary closed subsets of 2^ω.

Definition 3. If T is a tree, set $[T] = \{x \in 2^\omega : (\forall n \in \omega)[x|_n \in T]\}$. $[T]$ is the infinite branches of T. T is **pruned** if whenever $s \in T$, there is a $t \in T$ so that $s \sqsubseteq T$.

Proposition 1. *$F \subseteq 2^\omega$ is closed iff there is a unique pruned tree T so that $F = [T]$.*

The property of being perfect is reflected in the tree which generates the closed set.

Definition 4. A tree T is **perfect** iff whenever $s \in T$, there is an extension $s \sqsubseteq t \in T$ so that $t^\frown 0, t^\frown 1 \in T$.

Proposition 2. *$P \subseteq 2^\omega$ is perfect iff there is a unique pruned perfect tree T so that $P = [T]$.*

I want to view these trees as continuously generated, as it makes the picture clearer now and is important later. A tree T can be considered as generated by a function $s \mapsto \sigma_s$ on $2^{<\omega}$. T is pruned if whenever s is properly extended by t, σ_s is properly extended by σ_t. In this case, the construction leads to a continuous relation $\phi \subseteq 2^\omega \times 2^\omega$ given by

$$\phi(a) = \lim_{n \to \infty} \sigma_{a|_n}.$$

T is perfect if

$$\sigma_{s^\frown 0}(|s|) \neq \sigma_{s^\frown 1}(|s|)$$

for all $s \in 2^{<\omega}$. If T is perfect, the map ϕ is well-defined and is a 1-1 function. Note that $[T] = \phi[2^\omega]$.

Note that perfect sets do not enjoy very many nice closure properties. While finite unions of perfect sets are perfect, finite intersections need not be. While non-empty, finite, descending intersections are perfect again, countable descending intersections need not be perfect. Perfect sets are, however, closed under an operation which mimics the construction of perfect sets.

Definition 5. Suppose that P_s for $s \in 2^{<\omega}$ are perfect sets so that

- each $P_s \neq \emptyset$,
- the diameter of each P_s is bounded by $2^{-|s|}$,
- if $s \sqsubseteq t$, then $P_t \subseteq P_s$, and
- $P_{s^\frown 0} \cap P_{s^\frown 1} = \emptyset$ for all s.

The **fusion** of the P_s is $P := \bigcup_{a \in 2^\omega} \bigcap_{n \in \omega} P_{a|_n}$.

Theorem 1 (Fusion Lemma)**.** *If P_s are as in the previous definition, then P is a perfect set.*

Viewed in the generative sense ($s \mapsto \sigma_s$), and applied to basic nhoods of 2^ω, the fusion lemma is stated as follows. If

- $|s| \leq |\sigma_s|$ for each s,
- $s \sqsubseteq t$ implies $\sigma_s \sqsubseteq \sigma_t$, and
- for each s, $\sigma_{s^\frown 0}(|s|) \neq \sigma_{s^\frown 1}(|s|)$,

then $\phi[2^\omega]$ is a perfect set. This follows as the $N(\sigma_s)$ are perfect sets.

3 Mycielski's Original Result

Definition 6. Suppose $A \subseteq 2^\omega$ and $n \in \omega$. Then $[A]^n$ is the increasing n-tuples from A.

Theorem 2. *Suppose that $C_m \subseteq (2^\omega)^m$ are comeager for all $m \in \omega$. Then there is a perfect $P \subseteq 2^\omega$ so that for all $m \in \omega$, $[P]^m \subseteq C_m$.*

Proof. I will first prove the that for a particular $m \in \omega$ and comeager $C \subseteq (2^\omega)^m$, there is a perfect set P so that $[P]^m \subseteq C_m$. As C is comeager, we can find descending open dense sets W_n so that $\bigcap_n W_n \subseteq C$. I am going to build σ_s for $s \in 2^{<\omega}$ so that

- $\sigma_\emptyset = \emptyset$,
- $s \sqsubseteq t$ implies $\sigma_s \sqsubseteq \sigma_t$,
- for each s, $\sigma_{s^\frown 0}(|s|) \neq \sigma_{s^\frown 1}(|s|)$, and
- whenever $s_1, \cdots, s_m \in [2^n]$ are distinct, $N(\sigma_{s_1}) \times \cdots \times N(\sigma_{s_m}) \subseteq W_n$.

We then let P be $\phi[2^\omega]$. P is perfect and it is true that $[P]^m \subseteq C$. To see this, let $\vec{x} \in [P]^m$. Then eventually, $x_1|_n, \cdots, x_m|_n$ are distinct. Say this is true for all $n \geq N$. Since $x_1|_n, \cdots, x_m|_n$ are distinct,

$$N(\sigma_{x_1|_n}) \times \cdots \times N(\sigma_{x_m|_n}) \subseteq W_n$$

for all $n \geq N$. Thus $\vec{x} \in \bigcap_n W_n \subseteq C$.

The σ_s are built inductively by length. For n so that $2^n < m$, set $\sigma_s = s$. For n with $m \leq 2^n$, there is actual work to do. Suppose that the σ_s have been defined for all $s \in 2^{<\omega}$ with $|s| \leq n$. For $s \in 2^n$, first set $\sigma^0_{s^\frown i} = \sigma_s{}^\frown i$. Now we enumerate the distinct m-tuples $(s_1, \cdots, s_m)$ from s^{n+1}. Then with a finite induction, we extend the σ^0_s so that

$$N(\sigma_{s_1}) \times \cdots N(\sigma_{s_m}) \subseteq W_{n+1}$$

for all the distinct m-tuples. This completes the inductive step and the proof for a single dimension.

The proof for all dimensions simultaneously is a layering argument. For each m, we find descending open dense sets $W_{m,n}$ so that $\bigcap_n W_{m,n} \subseteq C_m$. Then in construction, the s are built so that if a distinct m-tuple is taken, the corresponding neighborhood product fits inside $W_{m,n}$. $\square$

4 E_0

Recall that E_0 is the equivalence relation defined on 2^ω by

$$xE_0y \iff (\exists N \in \omega)(\forall n \geq N)[x(n) = y(n)].$$

The Glimm-Effros dichotomy tells us the following about E_0.

Proposition 3 (AD)**.** *Suppose $X \subseteq {}^{2^\omega}/_{E_0}$. Then either*

- *X is countable, or*
- *X is in bijection with $\mathbb{R}$, or*
- *X is in bijection with ${}^{2^\omega}/_{E_0}$.*

Since ${}^{2^\omega}/_{E_0}$ has the trivial topology and is not even linearly orderable, to prove partition properties for ${}^{2^\omega}/_{E_0}$ we will need to consider maps lifted to 2^ω. We introduce some notions in order to facilitate the transfer of ideas from 2^ω to ${}^{2^\omega}/_{E_0}$.

Definition 7. $A \subseteq 2^\omega$ has **power $\mathbf{E_0}$** if A is E_0-saturated and $E_0|_A \sim_C E_0$. Notice that this corresponds to saying that ${}^{A}/_{E_0}$ is defined and ${}^{A}/_{E_0}$ is in bijection with ${}^{2^\omega}/_{E_0}$.

Viewed procedurally, a perfect set A has power E_0 iff whenever $\phi(a)E_0\phi(b)$, aE_0b.

5 Mycielski For E_0

Definition 8. For $X \subseteq \mathbb{R}$ and $n \in \omega$, $[X]^n_{E_0} = \{\vec{x} \in [X]^n : |\{[x_1]_{E_0}, \cdots, [x_n]_{E_0}\}| = n\}$. When X is E_0-saturated, this corresponds to $[{}^{X}/_{E_0}]^n$.

Theorem 3. *Suppose $C_m \subseteq (2^\omega)^m$ are comeager for all $m \in \omega$. Then there is an $A \subseteq \mathbb{R}$ with power E_0 so that for all $m \in \omega$, $[A]^m_{E_0} \subseteq C$.*

Proof. Again, I will do the proof with a single dimension. Let $C \subseteq (2^\omega)^m$ be comeager. We can assume without loss of generality that C is E_0-saturated. It will suffice to build an $A \subseteq \mathbb{R}$ so that $E_0|_A \sim_c E_0$ and for $[A]^m_{E_0} \subseteq C$. This will be done through mixture of a fusion argument and a Glimm-Effros argument. In other words we will build a new binary tree using the full binary tree as inputs. There are features of the input strings which we will need to keep track of. For $s, t \in 2^{<\omega}$ with $|s| = |t|$, we set

$$D(s,t) = \max\{n : s(n) \neq t(n)\}.$$

This is the the last entry on which s and t disagree. For $s_1, \cdots, s_m \in 2^{<\omega}$ define

$$\lambda(s_1, \cdots, s_m) = \min\{D(v,w) : v \neq w \in \{s_1, \cdots, s_m\}\}$$

λ characterizes whether or not $(s_1, \cdots, s_m)$ looks more E_0-inequivalent with its most recent entry than it did before.

We can find descending open dense sets W_n so that $\bigcap_n W_n \subseteq C$. We treat these as targets for the tuples of branches of the new binary tree to land inside. Define $S \subseteq (2^{<\omega})^m$ by $\vec{s} \in S$ iff

1. $|s_1| = \cdots = |s_m| \, (= n)$,
2. $\lambda(s_1|_{n-1}, \cdots, s_m|_{n-1}) < \lambda(\vec{s})$, and
3. $m \leq \lambda(\vec{s})$.

S is an enumeration of m-tuples of input strings whose entries seem to be E_0-inequivalent. Well order S lexicographically, say $S = \{(s_{n,1}, \cdots, s_{n,m}) : n \in \omega\}$. Define $S(n,i)$ for $1 \le i \le m$ by

$$v \in S(n,i) \iff s_{n,i} \sqsubseteq v.$$

$S(n,i)$ is the collection of extensions of $s_{n,i}$. Define $S(n) \subseteq (2^{<\omega})^m$ by

$$\vec{v} \in S(n) \iff \left[\exists k \left(\vec{v} \in [2^k]^m\right) \wedge \forall i (v_i \in S(n,i))\right].$$

$S(n)$ is the collection of m-tuples of input strings which extend the nth tuple of S. Define

$$S(n,-1) = 2^{<\omega} - \bigcup_{1 \le i \le m} S(n,i);$$

it is the m-tuples which are not extensions of any part of the nth tuple of S. As a final note, let $N = \min\{n : [2^n]^m \cap S_m(n) \neq \emptyset\}$. N records the level of the full binary tree for which the m-tuples become relevant.

We will build elements $\sigma_s \in 2^{<\omega}$ for $s \in 2^{<\omega}$ so that

- For $n < N$ and $s \in 2^n$, $\sigma_s = s$, and
- for all $N \le n$, there are $\tau_i \in 2^{<\omega}$ for $1 \le i \le m$, and there are $i(s) \in \{1, \cdots, m\}$ for $s \in 2^n$ so that
  1. if $1 \le i, j \le m$, then $|\tau_i| = |\tau_j|$,
  2. if $|s| = n+1$, then $\sigma_s = \sigma_{s|_n} {}^\frown s(n) {}^\frown \tau_{i(s)}$,
  3. if $|s| = |t| = n$, and there is an $i \ge 1$ so that $s, t \in S(n,i)$, then $i(s) = i(t)$, and
  4. if $|s| = n$, $s \in S(n,-1)$ and $t \in 2^n - S(n,-1)$ is lexicographically least so that $D(s,t)$ is minimized, then $i(s) = i(t)$, and
  5. if $\vec{s} \in [2^n]^m$ and $\vec{s} \in S(n)$, then

$$N(\sigma_{s_1}) \times \cdots \times N(\sigma_{s_m}) \subseteq W_n.$$

and take $A = \bigcup_{a \in 2^\omega} \bigcap_{n \in \omega} N(\sigma_{a|_n})$, i.e. the fusion of the neighborhoods defined by the σ_s.

The construction proceeds as follows. For $n < N$ and $s \in 2^n$, set $\sigma_s = s$. Now for the inductive step. Suppose $N \le n$ and that σ_s have been defined as desired for all $s \in 2^{<\omega}$ with $|s| \le n$. For $s \in 2^n$, set $\sigma^0_{s^\frown i} = \sigma_s {}^\frown i$. We need to define τ_i and $i(s)$. We can find $\tau_1, \cdots, \tau_m \in 2^{<\omega}$ so that $|\tau_i| = |\tau_j|$ for all i, j and

$$N\left(\sigma^0_{s_1} {}^\frown \tau_1\right) \times \cdots \times N\left(\sigma^0_{s_m} {}^\frown \tau_m\right) \subseteq W_{n+1}$$

for all $\vec{s} \in [2^{n+1}]^m \cap S(n+1)$. If $s \in 2^{n+1} \cap S(n+1,i)$ for some $1 \le i \le m$, set $i(s) = i$. We need to define $i(s)$ for $s \in 2^{n+1} \cap S(n+1,-1)$. Fix such an s. Let t_s be the lexicographically least $t \in 2^{n+1} - S(n+1,-1)$ so that $D(s,t)$ is minimized. Then $i(s) = i(t_s)$. Set $\sigma_s = \sigma^0_s {}^\frown \tau_{i(s)}$. This completes the inductive step.

Checking that this construction is sufficient requires work. I first need to show that aE_0b iff $\phi(a)E_0\phi(b)$. Suppose that aE_0b. Let $N_0 = \max\{n : a(n) \neq b(n)\}$. Let

$$Y = \{s \in 2^{<\omega} : (|s| > N_0) \wedge (D(s, a|_{|s|}), D(s, b|_{|s|}) = N_0)\}.$$

Claim 1. The set

$$\{n : (\exists 1 \le i \neq j \le m)(a|_n \in S(n,i) \wedge b|_n \in S(n,j))\}$$

is finite.

Reason. Note that $\lambda(a|_k, b|_k, u_3, \cdots, u_m) \le N_0$ for all k and all $u_3, \cdots, u_m \in 2^k$. Thus $(a|_k, b|_k, u_3, \cdots, u_m)$ (or some permutation) can be in S only when

$$\lambda(a|_{k-1}, b|_{k-1}, u_3|_{k-1}, \cdots, u_m|_{k-1}) < N_0.$$

This means that for $(a|_k, b|_k, u_3, \cdots, u_m)$ (or some permutation) to be in S, it must be that $k \le N_0$. So there is an $N > N_0$ so that for all $n \ge N$, if there are $i, j \ge 1$ so that $a|_n \in S(n,i)$ and $b|_n \in S(n,j)$, then $i = j$. $\square$

So there is an $N' > N_0$ so that for all $n \geq N'$, if there are $i, j \geq 1$ so that $a|_n \in S(n,i)$ and $b|_n \in S(n,j)$, then $i = j$. We can repeat this argument for all $u \neq v \in Y$. Since Y is finite, this shows that there is an $N_1 > N_0$ so that for $n \geq N_1$, if $v, w \in 2^n \cap Y$ and for some $i, j \geq 1$, $v \in S(n,i)$ and $w \in S(n,j)$, then $i = j$. Let $N_2 = |\sigma_{a_{N_1}}|$.

From this we can show that for $n \geq N_2$, $\phi(a)(n) = \phi(b)(n)$. It suffices to show that for all $n > N_1$, $i(a|_n) = i(b|_n)$. Let $n > N_1$. We proceed in cases.

Case 1: Suppose that there are $i, j \geq 1$ so that $a|_n \in S(n,i)$ and $b|_n \in S(n,j)$. Then we know that $i = j$. Thus $i(a|_n) = i(b|_n)$.

Case 2: Suppose that $a|_n \in S(n,-1)$ and $b|_n \in S(n,i)$ for some $i \geq 1$. Then as $n > N_0$, $t_{a|_n} \in Y$, and thus as $n > N_1$, it must be that $i(t_{a|_n}) = i(b|_n)$. Therefore as $i(a|_n) = i(t_{a|_n})$, $i(a|_n) = i(b|_n)$.

Case 3: Suppose that $a|_n, b|_n \in S(n,-1)$. There are multiple subcases here. It could be that $t_{a|_n} \in Y$ or $t_{b|_n} \in Y$. Then actually $t_{a|_n} \in Y$ and $t_{b|_n} \in Y$. In this case, as $n > N_1$, $i(t_{a|_n}) = i(t_{b|_n})$. Therefore as $i(a|_n) = i(t_{a|_n})$ and $i(b|_n) = i(t_{b|_n})$, it must be that $i(a|_n) = i(b|_n)$. It could also be that $t_{a|_n} \notin Y$. In this case,

$$\min\{D(a|_n, t) : t \in S(n,1) \cup \cdots \cup S(n,m)\} > N_0$$

and so as $D(a|_n, b|_n) = N_0$, it must be that $D(a|_n, t) = D(b|_n, t)$ for all $t \notin S(n,-1)$. Thus $t_{a|_n} = t_{b|_n}$, and so $i(a|_n) = i(b|_n)$.

Inversely, suppose that $(a,b) \notin E_0$. Then for infinitely many n, $a(n) \neq b(n)$. Thus, by construction, for infinitely many n, $\phi(a)(n) \neq \phi(b)(n)$. Therefore $(\phi(a), \phi(b)) \notin E_0$. So ϕ is the desired function.

We finally show that $[A]^m_{E_0} \subseteq C$. Suppose $\vec{x} \in [A]^m_{E_0}$. Say $\phi(a_i) = x_i$. Note that $\vec{a} \in [A]^m_{E_0}$. Thus $\lambda(a_1|_n, \cdots, a_m|_n)$ is both monotonically increasing and unbounded as a function of n. So for infinitely many n, $(a_1|_n, \cdots, a_m|_n) \in S$, and therefore for infinitely many k, $(a_1|_k, \cdots, a_m|_k) \in S(k)$. So $(\phi(a_1), \cdots, \phi(a_m)) \in \bigcap_n W_n$. Thus $\vec{x} \in C_m$. This completes the proof.

The proof for all dimensions simultaneously introduces a few wrinkles into the construction and the proof that it works, but is mostly straightforward. □