Large Large Cardinals and Inconsistency

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1 Measurable Cardinals

Definition 1. A cardinal κ is measurable iff there is a non-principal κ-complete ultrafilter on κ.

We usually characterize measurable cardinals differently.

Definition 2. Let M and N be inner models of ZFC. Then $j : M \to N$ is an elementary embedding iff j is 1-1 and

$$M \models \varphi(x_1, \cdots, x_n) \iff N \models \varphi(j(x_1), \cdots, j(x_n))$$

for all sentences φ. If j is not the identity, then there is a least cardinal λ so that $j(\lambda) \neq \lambda$. We call λ the critical point of j and write $\mathrm{crit}(j) = \lambda$. Note that $j(\lambda) > \lambda$.

Theorem 1. *κ is measurable iff there is an elementary embedding $j : V \to M$ so that $\mathrm{crit}(j) = \kappa$.*

Proof. We sketch the proof. If κ is measurable with measure U, then we can take the ultrapower of V by U, call it $\mathrm{Ult}(V, U)$. This is a well-founded model as U is κ and thus countably complete. Let M be the transitive collapse of $\mathrm{Ult}(V, U)$ and $j : V \to M$ be the composition of the ultrapower embedding and the collapse map. This is the desired embedding.

Conversely suppose that $j : V \to M$ is so that $\mathrm{crit}(j) = \kappa$. Define U by $X \in U$ iff $\kappa \in j(X)$. This is a non-principal, κ-complete, and normal ultrafilter on κ. $\square$

Remark 1. If U on κ is normal, and we construct $j : V \to M$ as in the proof above, then $j([\mathrm{id}_\kappa]) = \kappa$. This allows us to prove various things about κ. For instance we can see that $\{\lambda < \kappa : \lambda \text{ is weakly compact}\} \in U$.

Remark 2. Note that in the definition of measurable above, $U, j \in V$. This and the minimality of L show us that L cannot have any measurable cardinals. In fact measurable cardinals transcend L in a much stronger way. If there is a measurable cardinal, then $\left|\mathcal{P}(\alpha)^L\right| = |\alpha|$ for all $\alpha \geq \omega$.

Proposition 1. *Let κ be measurable and let $j : V \to M$ be constructed from a κ-complete ultrafilter U over κ. Then $M^\kappa \subseteq M$, $V_{\kappa+1} = M_{\kappa+1}$ and $V_{\kappa+2} \nsubseteq M$.*

Proof. Again we sketch the proof. That $M^\kappa \subseteq M$ follows from the κ-completeness of U. Let $X \subseteq V_\kappa$. Then

$$j(X) \cap V_\kappa = X$$

So $V_{\kappa+1} = M_{\kappa+1}$. Finally $U \in V_{\kappa+2}$, but $U \notin M$. $\square$

2 Large Large Cardinals

The last proposition showed that an embedding generated by a measurable cardinals is limited in how much of V it can capture. The way to define large large cardinals is to assert the existence of embeddings which capture more and more of V. We state some of these axioms in increasing consistency strength.

Definition 3.

- Let α be an ordinal. κ is α-strong iff there is an embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ so that $\alpha < j(\kappa)$ and $V_{\kappa+\alpha} \subseteq M$.
- κ is strong iff κ is α-strong for all α.
- κ is Woodin iff κ for all $f : \kappa \to \kappa$ there is an $\alpha < \kappa$ so that $f''\alpha \subseteq \alpha$ and an embedding $j : V \to M$ with $\mathrm{crit}(j) = \alpha$ and $V_{j(f)(\alpha)} \subseteq M$.
- κ is superstrong iff there is an embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ so that $V_{j(\kappa)} \subseteq M$.

Definition 4.

- Let α be an ordinal. κ is α-supercompact iff there is an embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ so that $\alpha < j(\kappa)$ and $M^\alpha \subseteq M$.
- κ is supercompact iff κ is α-supercompact for all $\alpha \geq \kappa$.

Remark 3. κ is measurable if κ is 1-strong and κ is measurable if κ is κ-supercompact. A Woodin cardinal has a stationary set of measurable cardinals below it.

Remark 4. It is overkill, but the existence of a supercompact cardinal is enough to ensure that AD holds in $L(\mathbb{R})$, and thus that every constuctible set of reals is Lebesgue measurable, Baire measurable, and has the perfect set property.

3 Inconsistency

A natural question at this point is whether or not one can find a definable non-trivial elementary embedding $j : V \to M$ so that $V \subseteq M$. We will show that this is impossible.

Theorem 2 (Kunen)**.** *Suppose that $j : V \to M$ is a non-trivial elementary embedding. Then $V \neq M$.*

Proof. Let $\kappa = \mathrm{crit}(j)$ and $\lambda = \sup\{j^n(\kappa) : n \in \omega\}$. Let $A = \{\xi < \lambda^+ : \mathrm{cf}(\xi) = \omega\}$. This set is stationary. By a standard theorem, we can partition A into κ many stationary sets S_α. Let $f : \kappa \to \mathcal{P}(\lambda^+)$ by $f(\alpha) = S_\alpha$. Note that $j(\lambda) = \lambda$ from how we have defined λ. Thus

$$\lambda^+ \leq j\left(\lambda^+\right) = \left(\lambda^+\right)^M \leq \lambda^+$$

and so $\lambda^+ = j(\lambda^+)$. Consider $j(f)$. This is creates partition of $j(A)$ into $j(\kappa)$ many stationary (in M) sets by elementarity. By the above argument, $j(A) = A$. Thus $j(f)(\kappa) \subseteq A$ is stationary in M.

By way of contradiction suppose that $V = M$. Then $j(f)(\kappa)$ is stationary in V. Therefore we have some $\alpha_0 < \kappa$ so that $j(f)(\kappa) \cap j(\alpha_0)$ is stationary as $j(f)(\kappa) = \bigcup_{\alpha<\kappa}[j(f)(\kappa) \cap f(\alpha)]$, $\lambda > \kappa$ and the club filter on λ^+ is λ^+-complete. Let

$$C = \left\{\xi < \lambda^+ : \mathrm{cf}(\xi) = \omega \wedge j(\xi) = \xi\right\}$$

Then C is unbounded in $\lambda+$, and C is closed under countable sequences. Borrowing again from infinite combinatorics, this suffices to guarantee that $C \cap j(f)(\kappa) \cap f(\alpha_0) \neq \emptyset$ as all elements of $j(f)(\kappa)$ have cofinality ω. Let $\xi_0 \in C \cap j(f)(\kappa) \cap f(\alpha_0)$. Then

$$\xi_0 = j(\xi_0) \in j(f(\alpha_0)) = j(f)(j(\alpha_0))$$

So $\xi_0 \in j(f)(\kappa) \cap j(f)(j(\alpha_0))$. This contradicts the fact that $j(f)$ created a partition of A. $\square$