

A Brief History of the Axiom of Choice

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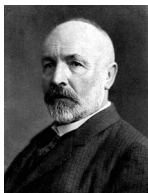
My Beginnings

...
iom of Choice.)

We shall follow the common practice of assuming the axiom of choice (and hence the validity of the well-ordering principle).⁵ However, we should point out that while the axiom of choice seems self-evident the well-ordering principle leads quickly to some baffling conclusions: one only needs to spend a little time trying to imagine what a well-ordering of the reals might look like!

⁵It can be proved that in an appropriate formulation of the axioms of set theory, the axiom of choice is independent of the other axioms; thus we are free to accept its validity.

Georg Cantor(1845 - 1918)



Cantor is considered to be the father of set theory. His work was initially on trigonometric series.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(x) + b_n \cos(x).$$

In, 1870, he proved the following:

Theorem

If a trigonometric series converges to 0 for all reals, then all of the coefficients are 0.

Extending this result, Cantor began looking at the limit points of a set $P \subseteq \mathbb{R}$, denoted P' . Cantor iterated this operation, letting

- ▶ $P^{(0)} = P$, and
- ▶ $P^{(n+1)} = (P^{(n)})'$.

If a trigonometric series converges to 0 off of a set P which has the property that $P^{(n)} = \emptyset$ for some n , then the coefficients are all 0 (“Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen”, 1872).

Extending this notion further, Cantor considered the following:

- ▶ $P^{(\infty)} = \bigcap_n P^{(n)},$
- ▶ $P^{(\infty+1)} = (P^{(\infty)})',$
- ▶ $P^{(\infty+2)} = (P^{(\infty+1)})',$
- ▶ \vdots
- ▶ $P^{(\infty \cdot 2)} = \bigcap_n P^{(n)} \cap \bigcap_n P^{(\infty+n)}$
- ▶ \vdots
- ▶ $P^{(\infty^\infty)} = \bigcap_n P^{(\infty^n)}$
- ▶ \vdots
- ▶ $P^{(\infty^{\infty^\infty})}$
- ▶ \vdots

Transfinite Numbers

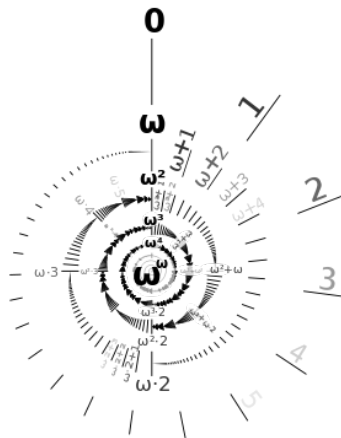
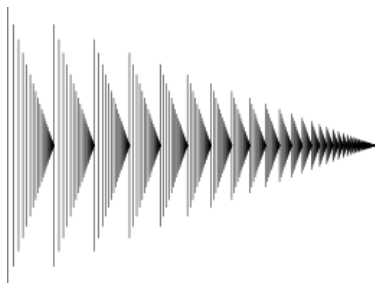
In 1883, in the “Grundlagen einer allgemeinen Mannigfaltigkeitslehre”, Cantor introduced the notion of well-ordering.

Definition

A **well-ordering** of set X is a total order \leq on X , which has the property that every subset of X has a least element with respect to \leq .

This notion generalizes the ordering of the natural numbers. The indexes for his sequences of limit sets are all well-ordered, and he now wrote them as

$$0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega \cdot 2, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots$$



- ▶ The natural numbers were number class (I)
- ▶ The numbers whose collection of predecessors was countable formed number class (II) (Cantor showed that this collection is uncountable)
- ▶ If the collection of predecessors for a number was in bijection with class (II), Cantor said the number was of class (III),
- ▶ \vdots

Then Cantor asserts

“It is always possible to bring any well-defined set into the form of a well-ordered set.”

This became known as the well-ordering principle.

Cantor's Cardinals and Ordinals

In “Beiträge zur Begründung der transfiniten Mengenlehre”, written in 1895, Cantor solidified the transfinite numbers into two notions, cardinals and ordinals.

- ▶ The cardinals were the sizes of sets, as measured by bijections,
- ▶ The ordinals were the different order-types of well-orders.

\aleph_0 is the cardinal number corresponding to \mathbb{N} , \aleph_1 is the cardinal number corresponding to number class (II), and so on.

2^{\aleph_0} is the cardinal corresponding to \mathbb{R} .

The Well-Ordering Principle is False?



In 1904, Julius König(1849 - 1913) presented a proof that \mathbb{R} cannot be well-ordered at the third International Congress of Mathematicians at Heidelberg

Ernst Zermelo(1871 - 1953), quickly found a flaw in König's proof. It depended on some cardinal arithmetic that just wasn't true.



The Axiom of Choice

Zermelo then set out to prove that the well-ordering principle was reasonable. To do this, he related it to a principle called the axiom of choice.

Definition

The axiom of choice (AC) asserts the following: for every non-empty collection of non-empty sets $\{X_a : a \in A\}$, there is a function f with domain A that picks out elements from each set, i.e. $f(a) \in X_a$ for all $a \in A$.

Zermelo referred to the axiom of choice as a “logical principle which “is applied without hesitation everywhere in mathematical deduction.”

The Well-Ordering Principle is Reasonable?

In 1904, in his “Beweis, dass jede Menge wohlgeordnet werden kann (Aus einem an Herrn Hilbert gerichteten Briefe)” Zermelo proved that AC was equivalent to the well-ordering principle.

His proof was not received without criticism. His construction was viewed as ambiguous, and his use of arbitrary sets functions was controversial.

In response to his critics, Zermelo released a second proof in 1908. In the “Neuer Beweis für die Möglichkeit einer Wohlordnung,” He first establishes a list of axioms for sets. It is over these axioms that his proof of the equivalence of AC and the well-ordering principle is clear. These axioms, called Z, are the basis of the modern axioms of set theory.

Z

- ▶ AXIOM I. Axiom of extensionality (Axiom der Bestimmtheit) "If every element of a set M is also an element of N and vice versa ... then $M \equiv N$. Briefly, every set is determined by its elements."
- ▶ AXIOM II. Axiom of elementary sets (Axiom der Elementarmengen) "There exists a set, the null set, \emptyset , that contains no element at all. If a is any object of the domain, there exists a set $\{a\}$ containing a and only a as an element. If a and b are any two objects of the domain, there always exists a set $\{a, b\}$ containing as elements a and b but no object x distinct from them both."
- ▶ AXIOM III. Axiom of separation (Axiom der Aussonderung) "Whenever the propositional function $-(x)$ is definite for all elements of a set M , M possesses a subset M' containing as elements precisely those elements x of M for which $-(x)$ is true."
- ▶ AXIOM IV. Axiom of the power set (Axiom der Potenzmenge) "To every set T there corresponds a set T' , the power set of T , that contains as elements precisely all subsets of T ."
- ▶ AXIOM V. Axiom of the union (Axiom der Vereinigung) "To every set T there corresponds a set $\cup T$, the union of T , that contains as elements precisely all elements of the elements of T ."
- ▶ AXIOM VI. Axiom of choice (Axiom der Auswahl) "If T is a set whose elements all are sets that are different from \emptyset and mutually disjoint, its union $\cup T$ includes at least one subset S_1 having one and only one element in common with each element of T ."
- ▶ AXIOM VII. Axiom of infinity (Axiom des Unendlichen) "There exists in the domain at least one set Z that contains the null set as an element and is so constituted that to each of its elements a there corresponds a further element of the form $\{a\}$, in other words, that with each of its elements a it also contains the corresponding set $\{a\}$ as element."

Replacement



In 1922, following correspondence with Zermelo, Abraham Fraenkel (1891 - 1965) proposed an additional axiom to fix gaps in Z. The axiom of replacement asserts the following:

“If M is a set and each element of M is replaced by [a set or an urelement] then M turns into a set again.”

This addition by Fraenkel is where “Zermelo-Fraenkel set theory” gets its name.

Replacement



That same year, Thoralf Skolem(1887 - 1963) proposed his own axiom of replacement, using the newly developed first order logic. His axiom of replacement asserts the following:

“Let U be a definite proposition that holds for certain pairs (a, b) in the domain B ; assume further, that for every a there exists at most one b such that U is true. Then, as a ranges over the elements of a set M_a , b ranges over all elements of a set M_b .”

Although it is called Zermelo-Fraenkel set theory, Skolem's version is the accepted version of the axiom used today. Zermelo never actually accepted Skolem's axiom, calling it “set theory for the impoverished.”

The Modern Transfinite



At this point, John Von Neumann(1903 - 1957) enters the picture. In 1923, in the paper “Zur Einführung der transfiniten Zahlen,” he greatly simplified Cantor’s ordinals, realizing them as individual sets instead of classes of equivalent sets.

- ▶ $0 = \emptyset,$
- ▶ $1 = \{0\},$
- ▶ $2 = \{0, 1\},$
- ▶ \vdots
- ▶ $\omega = \{0, 1, 2, \dots\},$

It is worthwhile to note that this work was preceded by Mirimanoff in 1917 and seems to have been known by Zermelo prior to Von Neumann’s discovery.

Foundation

In 1929, Von Neumann introduced the axiom of foundation. While it is little used outside of set theory, this axiom gives concrete shape to the kind of object that the axioms of set theory are describing.

Definition

The **Axiom of Foundation** asserts that \in is a well-founded relation. In other words, there is no way to build a sequence of sets x_n with

$$\cdots \in x_n \in \cdots x_2 \in x_1 \in x_0$$

In 1930, Zermelo stated the official axioms of Zermelo-Fraenkel set theory (ZFC). Building off of Z, it includes replacement and foundation.

Using the Axiom of Choice

After the well-ordering theorem, AC was first used by set theorists for cardinal arithmetic. Eventually, it saw use in Algebra and Topology. AC was used to prove the following:

- ▶ Every vector space has a basis(1932). (Felix Hausdorff(1868 - 1942) in “Zur Theorie der linearen metrischen Räume”)
- ▶ Products of compact spaces are compact, i.e. Tychonoff's theorem(1935/1937). (Andrey Tychonoff(1906 - 1993) in “Über die topologische Erweiterung von Räumen”/Eduard Čech in “On bicomact spaces”)
- ▶ Hahn-Banach theorem(1927/1929). (Hans Hahn(1879 - 1934) in “Über linearer Gleichungssysteme in linearer Räumen”/Stefan Banach(1892 - 1945) in “Sur les fonctionelles linéaires”)
- ▶ Rings have maximal ideals, i.e. Krull's Theorem(1929). (Wolfgang Krull(1899 - 1971) in “Idealtheorie in Ringen ohne Endlichkeitsbedingungen”)

Zorn's Lemma



Modern proofs of the above theorems will probably not use the axiom of choice. Instead, they will use Zorn's lemma.

Definition

Zorn's Lemma is the following assertion: "If (P, \leq) is partial order and every totally ordered subset has a maximum element, then P has a maximal element."

This formulation was given by Max Zorn(1906 - 1993) in 1935, although it has predecessors in work of Kuratowski and Hausdorff. However, Kanamori gives credit for this principle to Wacław Sierpinski(1882 - 1969) (Hypothèse du Continu (1934)).

Measure Theory



In 1902, Henri Lebesgue(1875 - 1941) extended the notion of length to more subsets of \mathbb{R} and the notion of integral to more real functions (“Intégrale, longueur, aire”). Sets which are given measure in this scheme are called Lebesgue measurable.

In 1905, Giuseppe Vitali(1875 - 1932) showed that AC implies the existence of a non-measurable set (“Sul problema della misura dei gruppi di punti di una retta”).



Equivalences

The following are all equivalent to the axiom of choice.

- ▶ Cardinal comparison
- ▶ Onto functions have inverses
- ▶ Zorn's Lemma
- ▶ All partially ordered sets have antichains
- ▶ Every vector space has a basis
- ▶ Krull's theorem
- ▶ All sets can be turned into groups
- ▶ Cartesian products of non-empty sets are non-empty
- ▶ Closed games are determined

Different Forms of Choice

The common forms of choice split into two hierarchies. The first depends on the structure of the sets:

- ▶ Choice for pairs
- ▶ Choice for triples
- ▶ \vdots
- ▶ Choice for finite sets
- ▶ Choice for well-ordered sets
- ▶ Every partial order can be extended to a total order
- ▶ Every filter can be extended to an ultrafilter

The second depends on the structure of the index set:

- ▶ Every set either contains a copy of \mathbb{N} or is contained in a copy of \mathbb{N}
- ▶ Countable choice: if $\{X_n : n \in \mathbb{N}\}$ are non-empty sets, then there is a choice function for them
- ▶ Dependent choice: if R is a binary relation on the set X so that for all $n \in \mathbb{N}$ and $x_0, \dots, x_n \in X$ there is an $x \in X$ so that $x_k R x$ for all $1 \leq k \leq n$, then there is a sequence $\langle x_n : n \in \mathbb{N} \rangle$ so that $x_n R x_{n+1}$ for all n .
(Paul Bernays(1888-1977), 1942, "A system of axiomatic set theory. III. Infinity and enumerability. Analysis")
- ▶ AC_κ : if $\{X_\alpha : \alpha < \kappa\}$ are non-empty sets, then there is a choice function for them
- ▶ AC_{WO} : if $\{X_a : a \in A\}$ is a well-ordered collection of non-empty sets, then there is a choice function for them

More Equivalences

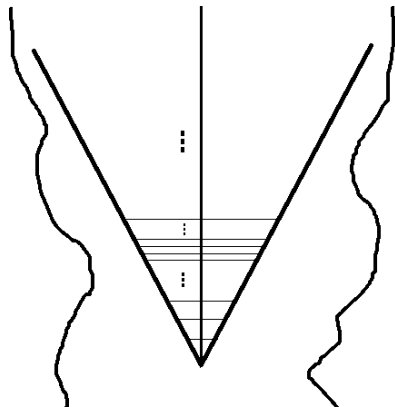
The following all exist somewhere within the hierarchy of choice.

- ▶ The countable union of countable sets is countable
- ▶ The **eight** definitions of finite are equivalent
- ▶ There exists a non-measurable set
- ▶ Every field has an algebraic closure
- ▶ Every subgroup of a free group is free
- ▶ $(\mathbb{R}, +) \cong (\mathbb{C}, +)$
- ▶ Every Hilbert space has an orthonormal basis
- ▶ Baire Category Theorem
- ▶ Every $T_{3\frac{1}{2}}$ -space has a Stone-Čech compactification

Universes

A Set theory universe is a class of sets which satisfy the ZFC axioms (or some fragment of them). Von Neumann built the fundamental universe to justify his axiom of foundation. Starting with a class of sets which may not satisfy foundation, you can build his universe V :

- ▶ $V_0 = \emptyset$,
- ▶ $V_1 = \mathcal{P}(V_0) (= \{\emptyset\})$,
- ▶ $V_2 = \mathcal{P}(V_1) (= \{\emptyset, \{\emptyset\}\})$,
- ▶ \vdots
- ▶ $V_\omega = \bigcup_n V_n$ (\mathbb{N} and all of its finite subsets),
- ▶ $V_{\omega+1} = \mathcal{P}(V_\omega)$ (contains \mathbb{R}),
- ▶ \vdots

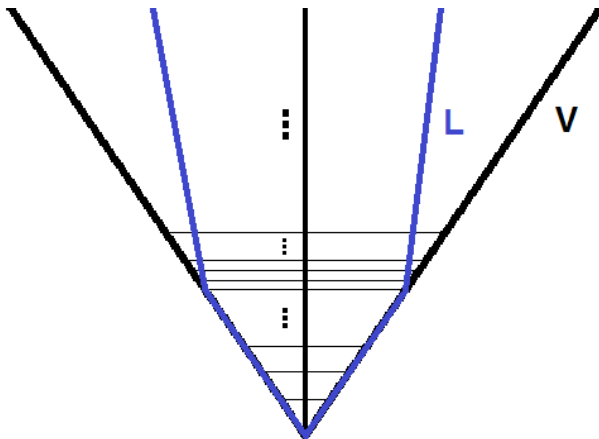


A Universe With Choice

If you start with choice, V will also satisfy choice.

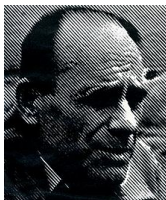


In 1938, Kurt Gödel defined a universe L , which always satisfies the axiom of choice (and the continuum hypothesis). It is formed by restricting the power set operation to “definable” sets. This showed that it was possible to obtain the axiom of choice from an environment that lacked it.

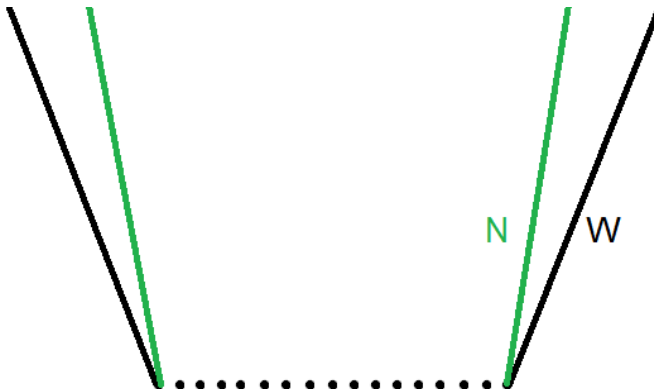


A Universe Without Choice

In 1922, in “Über den Begriff der definiten und die Unabhängigkeit des Auswahlaxioms,” Fraenkel created a universe of set theory with urelements that satisfied all of Z , except for the axiom of choice. He actually is able to create a universe where Z (except choice) is satisfied, but there is a countable family of pairs which has no choice function.



Andrzej Mostowski (1913 - 1975), refined these methods and created a general method of creating permutation models (1939, “Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip”). These are universes of ZF with atoms and were used to explore the relationship of fragments of AC .



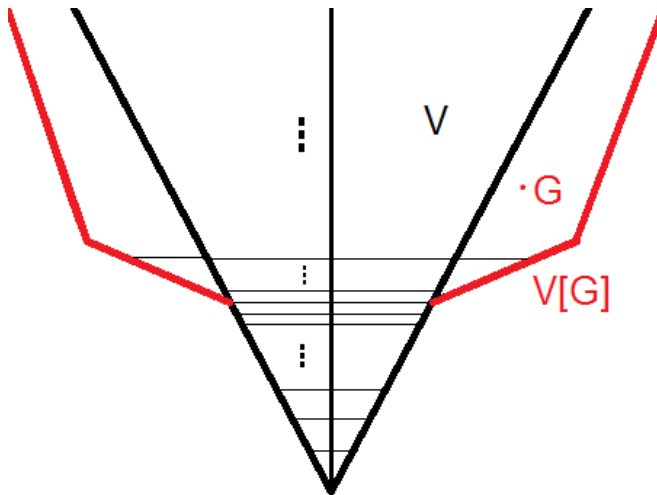
Forcing



In 1963, in order to build a universe of ZFC where the continuum hypothesis is false, Paul Cohen (1934 - 2007) invented the method of forcing ("The independence of the Continuum Hypothesis. I", "The independence of the Continuum Hypothesis. II", and "Independence results in set theory').

In Cohen's forcing, one starts with L , and introduces a new set G , to build a universe $L[G]$. The properties of this universe are controlled by L .

The models $L[G]$ satisfy all of ZFC, including choice.

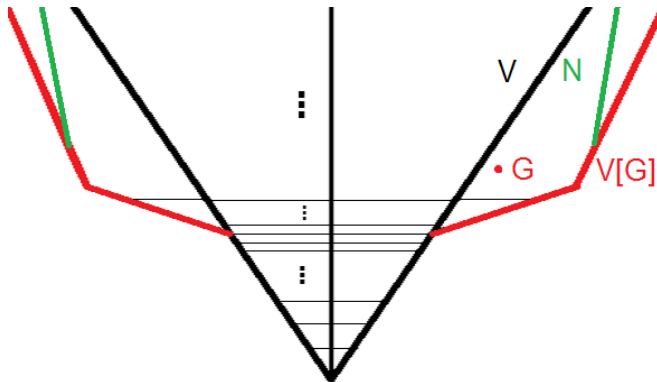


A Universe of ZF Without Choice

Combining his methods with those of Mostowski, Cohen created what are called symmetric submodels. Essentially the forcing adds objects which can behave like atoms over L . With these, Cohen created a model of ZF in which the axiom of choice failed.



It turns out that the symmetric submodel technique largely subsumes the permutation model technique. This was proved in 1966, by Thomas Jech(1944 -) (“On ordering of cardinalities” and “On cardinals and their successors”).



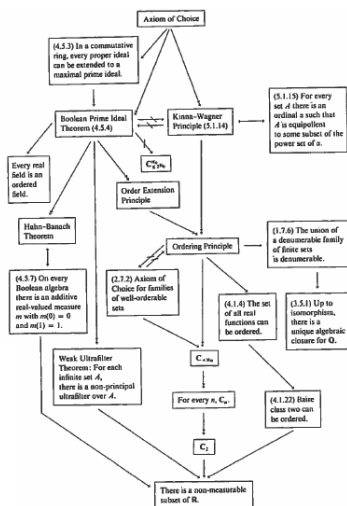
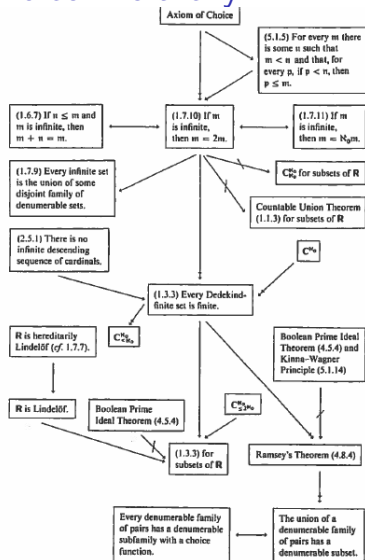
Refinements in Forcing

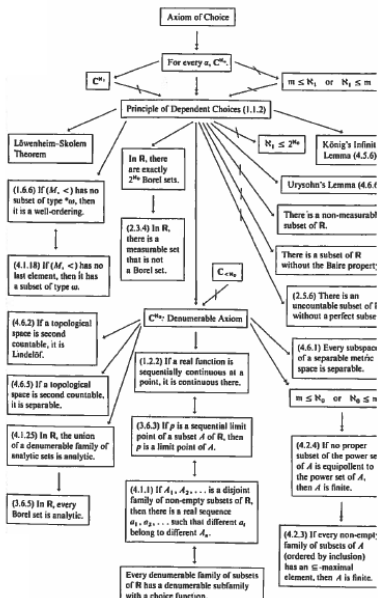
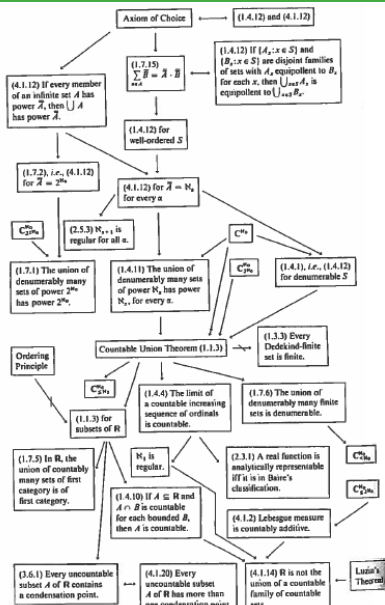


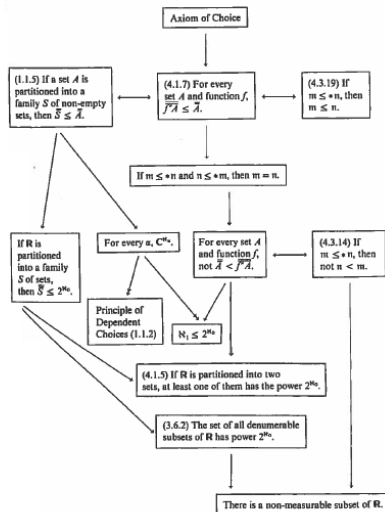
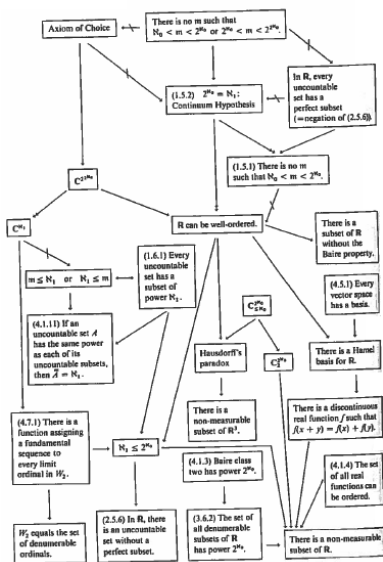
In the mid 60's (1965 - 1967), Dana Scott(1932 -) and Robert Solovay(1938 -), and Petr Vopenka(1935 - 2015) realized the connection between forcing and Boolean algebras ("A proof of the independence of the continuum hypothesis", " 2^{\aleph_0} can be anything it ought to be", "General Theory of ∇ -models"). With this in mind, they were able to simplify the process, and generalize it to work over universes which are not L .

With this modern approach to forcing and the transfer theorems of Jech, the tools were in place to fully explore the choice hierarchy and distinguish forms of choice from each other.

The Choice Hierarchy

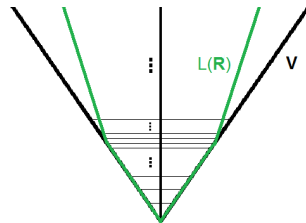






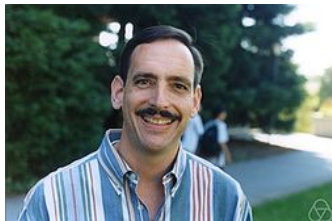
All Definable Sets Can be Measurable

In 1970, Solovay combined forcing techniques with a large cardinal (inaccessible) to produce a universe of ZF where countable choice holds and all sets of reals are measurable (“A model of set theory in which every set of reals is Lebesgue measurable”). The model had the form $L(\mathbb{R})$ inside of a forcing extension.



In 1984, Saharon Shelah(1945 -) showed that Solovay’s result actually required the inaccessible cardinal (“Can you take Solovay’s inaccessible away?”).

All Definable Sets Are Measurable



In the 80's, Hugh Woodin(1955 -) showed that with enough large cardinals, it's not only possible that all sets of reals in $L(\mathbb{R})$ are measurable, but necessary.

Thanks For Listening!