Limited Information Strategies and Discrete Selectivity

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Tkachuk's Question

Is the closed discrete selection game on the space of continuous functions $f : X \to \mathbb{R}$ equivalent to the point-open game on X?

Notation

Suppose X is a topological space. X is assumed to be $T_{3.5}$.

- $C_p(X)$ is the space of continuous functions $f : X \to \mathbb{R}$
- It is endowed with the topology of pointwise convergence.
- If $f : X \to \mathbb{R}$ is continuous, $F = \{x_1, \cdots, x_n\} \subseteq X$, and $\varepsilon > 0$, then

$$[f, F, \varepsilon] = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq n\}$$

Closed Discrete Selections

The concept of a closed discrete selection was isolated by Sanchez and Tkachuk in 2017.

Closed Discrete Selection (Tkachuk, 2017)

X satisfies closed discrete selection if:
For every sequence $(U_n : n \in \omega)$ of open subsets of X, there are points $x_n \in U_n$ so that $\{x_n : n \in \omega\}$ is closed and discrete.

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Note that if X is first countable (or even has one point with a countable neighborhood basis), then it cannot have closed discrete selection. The converse is not generally true.

Closed Discrete Selections of Functions

Theorem (Tkachuk, 2017)

Suppose X is Hausdorff and completely regular. Then the following are equivalent:

- $C_p(X)$ fails to satisfy closed discrete selection
- X is countable
- $C_p(X)$ is first-countable

Selection Principles

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections of sets.

$S_1(\mathcal{A}, \mathcal{B})$

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are $C_n \in A_n$ so that $\{C_n : n \in \omega\} \in \mathcal{B}$

$S_{FIN}(\mathcal{A}, \mathcal{B})$

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$

Common Selection Principles

Let $\mathcal{O}$ be the open covers of X. Then $S_1(\mathcal{O}, \mathcal{O})$ and $S_{FIN}(\mathcal{O}, \mathcal{O})$ are both strengthenings of the Lindelof property.

- $S_1(\mathcal{O}, \mathcal{O})$ is called the Rothberger property.
- $S_{FIN}(\mathcal{O}, \mathcal{O})$ is called the Menger property.

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$$S_1(\mathcal{O},\mathcal{O}) \Rightarrow S_{FIN}(\mathcal{O},\mathcal{O}) \Rightarrow \text{Lindelof}$$

and

$$\text{Compact} \Rightarrow \sigma-Compact \Rightarrow S_{FIN}(\mathcal{O},\mathcal{O}) \Rightarrow \text{Lindelof}$$

$S_{\square}(\mathcal{A}, \mathcal{B})$ can be turned into a two-player game.

- The game is played over rounds indexed by the naturals.
- At round n, player I plays a set A_n from $\mathcal{A}$ and player II responds by playing a selection C_n from A_n.
- If those selections are singletons, then the game is $G_1(\mathcal{A}, \mathcal{B})$. If they are finite sets, then the game is $G_{FIN}(\mathcal{A}, \mathcal{B})$.
- Player II wins a given run of the game $(A_0, C_0, A_1, C_1, \cdots)$ if $\bigcup_n C_n \in \mathcal{B}$.
- If player II does not win, then player I wins.

Perfect Information Strategies

Fix a game $G_\square(\mathcal{A}, \mathcal{B})$.

- A perfect information strategy for player I takes in a run of $G_\square(\mathcal{A}, \mathcal{B})$ up to some round n and outputs a set $A_{n+1} \in \mathcal{A}$.
- A perfect information strategy for player II takes in a run of $G_\square(\mathcal{A}, \mathcal{B})$ up to some round n and outputs a selection C_n from I's most recent move.

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- A perfect information strategy for player II takes in a run of $G_\square(\mathcal{A}, \mathcal{B})$ up to some round n and outputs a selection C_n from I's most recent move.
- A strategy σ for player I is winning if the run $(\sigma(\emptyset), C_0, \sigma(C_0), C_1, \cdots)$ wins for I no matter what selections II makes. If I has a winning strategy we write $I \uparrow G_\square(\mathcal{A}, \mathcal{B})$.
- A strategy τ for player II is winning if the run $(A_0, \tau(A_0), A_1, \tau(A_0, A_1), \cdots)$ wins for II no matter what sets player I plays. If II has a winning strategy we write $II \uparrow G_\square(\mathcal{A}, \mathcal{B})$.

Limited Information Strategies

- A Markov tactic for II is a strategy $\tau(A, n)$ which takes in only the round number and the most recent move of I. If II has a winning Markov tactic we write $II \uparrow_{mark} G_{\square}(\mathcal{A}, \mathcal{B})$.
- A pre-determined strategy for I is a strategy $\sigma(n)$ which takes in only the round number. If I has a winning pre-determined strategy we write $I \uparrow_{pre} G_{\square}(\mathcal{A}, \mathcal{B})$.

Easy Implications

$$I \uparrow_{pre} G(\mathcal{A}, \mathcal{B}) \Rightarrow I \uparrow G(\mathcal{A}, \mathcal{B}) \Rightarrow II \not\uparrow G(\mathcal{A}, \mathcal{B})$$

$$II \uparrow_{mark} G(\mathcal{A}, \mathcal{B}) \Rightarrow II \uparrow G(\mathcal{A}, \mathcal{B}) \Rightarrow I \not\uparrow G(\mathcal{A}, \mathcal{B})$$

$$II \uparrow G(\mathcal{A}, \mathcal{B}) \Rightarrow S(\mathcal{A}, \mathcal{B})$$

$$I \not\uparrow_{pre} G(\mathcal{A}, \mathcal{B}) \iff S(\mathcal{A}, \mathcal{B})$$

Useful Selection Games

- $G_1(\mathcal{O}, \mathcal{O})$ is the Rothberger game
- $G_1(\mathcal{P}, \neg\mathcal{O})$ is the point-open game, where $\mathcal{P}$ is the collection of point bases.
- $G_1(\mathcal{T}, CD)$ is the closed discrete selection game, where $\mathcal{T}$ is the collection of open sets and CD is the collection of closed discrete subsets of X.
- $G_1(\mathcal{N}(x), \neg\Gamma_{X,x})$ is Gruenhage's point picking game, where $\mathcal{N}(x)$ is the neighborhoods of x and $\Gamma_{X,x}$ is the sequences which converge to x.
- $G_1(\Omega_{X,x}, \Omega_{X,x})$ is the strong countable fan tightness game, where $\Omega_{X,x} = \{A \subseteq X : x \in \overline{A}\}$.

Useful Selection Games

- $G_1(\mathcal{P}, \neg\mathcal{O})$ and $G_1(\mathcal{O}, \mathcal{O})$ are dual (Galvin).
- $I \uparrow_{pre} G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if X is countable.
- $I \uparrow G_1(\mathcal{T}, CD) \iff I \uparrow G_1(\mathcal{N}(x), \neg\Gamma_{X,x}) \iff II \uparrow G_1(\Omega_{X,x}, \Omega_{X,x})$ (Clontz and Tkachuk)

Closed Discrete Selections of Functions II

Tkachuk, 2017

The following are equivalent:

- $I \uparrow_{pre} G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$
- $I \uparrow_{pre} G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$
- $C_p(X)$ is first-countable

Tkachuk, 2017

The following are equivalent:

- $I \uparrow G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$
- $II \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$
- $I \uparrow G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$
- $I \uparrow G_1(\mathcal{N}(\mathbf{0}), \neg\Gamma_{C_p(X),\mathbf{0}})$
- $II \uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$

Tkachuk's Question

Tkachuk, 2017

If $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$, then $II \uparrow G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$.

Is it true that if $II \uparrow G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$, then $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$?

ω-Covers and Games

Consider a topological space $(X, \mathcal{T})$.

- $\mathcal{U}$ is an ω-cover of X if whenever $F \subseteq X$ is finite, there is a $U \in \mathcal{U}$ so that $F \subseteq U$.
- Let Ω_X be the collection of ω-covers of X.
- For $F \subseteq X$ finite, let $\mathcal{N}(F) = \{U \in \mathcal{T} : F \subseteq U\}$. We can then consider $\mathcal{F}_X = \{\mathcal{N}[F] : F \subseteq X \text{ is finite}\}$.

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- $G_1(\mathcal{F}, \neg\mathcal{O})$ is the finite-open game. We will also need its variant $G_1(\mathcal{F}, \neg\Omega)$
- $G_1(\Omega, \Omega)$ is the ω-Rothberger game.

Point-Open Versus Finite Open

Proposition

- $G_1(\mathcal{F}, \neg\mathcal{O})$ and $G_1(\mathcal{O}, \mathcal{O})$ are (perfect information) dual (Telgarsky 1975 and Galvin 1978).
- $G_1(\mathcal{F}, \neg\Omega)$ and $G_1(\Omega, \Omega)$ are dual (Clontz, Holshouser).
- $I \uparrow G_1(\mathcal{F}, \neg\mathcal{O})$ if and only if $I \uparrow G_1(\mathcal{F}, \neg\Omega)$ (Clontz, Holshouser).
- $I \uparrow_{pre} G_1(\mathcal{F}, \neg\mathcal{O})$ if and only if $I \uparrow_{pre} G_1(\mathcal{F}, \neg\Omega)$ (Clontz, Holshouser).

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What Tkachuk actually showed was

$$I \uparrow G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)}) \iff I \uparrow G_1(\mathcal{F}_X, \neg\Omega_X)$$

and that

$$II \uparrow G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)}) \implies II \uparrow G_1(\mathcal{F}_X, \neg\Omega_X)$$

Point-Open Versus Finite Open II

Proposition

- If $I \not\uparrow_{pre} G_1(\Omega_X, \Omega_X)$, then $I \not\uparrow_{pre} G_1(\mathcal{O}_{X^n}, \mathcal{O}_{X^n})$ for all n (Scheepers 1997).
- It is consistent with ZFC that there is a $T_{3.5}$ space X where $I \not\uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$ but $I \uparrow_{pre} G_1(\mathcal{O}_{X^2}, \mathcal{O}_{X^2})$ (Babinkostova, Pansera, Scheepers 2013).
- Thus it is consistent with ZFC that there is a space X where $I \not\uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$, but $I \uparrow_{pre} G_1(\Omega_X, \Omega_X)$.
- The existence of pre-determined strategies are equivalent to the existence of perfect information strategies for player I (Pawlikowski 1994 and Scheepers 1997).

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- It is consistent with ZFC that there is a $T_{3.5}$ space X where $I \not\uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$ but $I \uparrow_{pre} G_1(\mathcal{O}_{X^2}, \mathcal{O}_{X^2})$ (Babinkostova, Pansera, Scheepers 2013).
- Thus it is consistent with ZFC that there is a space X where $I \not\uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$, but $I \uparrow_{pre} G_1(\Omega_X, \Omega_X)$.
- The existence of pre-determined strategies are equivalent to the existence of perfect information strategies for player I (Pawlikowski 1994 and Scheepers 1997).

Accounting for this, we should show that if $II \uparrow G_1(\mathcal{F}_X, \neg\Omega_X)$, then $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$.

Answering Tkachuk's Question

Clontz/Holshouser 2018

The following are equivalent.

- $II \uparrow G_1(\mathcal{F}_X, \neg\Omega_X)$
- $I \uparrow G_1(\Omega_X, \Omega_X)$
- $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$
- $II \uparrow G_1(\mathcal{N}(\mathbf{0}), \neg\Gamma_{C_p(X),\mathbf{0}})$
- $I \uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$
- $II \uparrow_{mark} G_1(\mathcal{F}_X, \neg\Omega_X)$
- $I \uparrow_{pre} G_1(\Omega_X, \Omega_X)$, i.e. X is not Rothberger with respect to Ω-covers
- $II \uparrow_{mark} G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$
- $II \uparrow_{mark} G_1(\mathcal{N}(\mathbf{0}), \neg\Gamma_{C_p(X),\mathbf{0}})$
- $I \uparrow_{pre} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$

Proof Sketch

$II \uparrow_{mark} G_1(\mathcal{F}_X, \neg\Omega_X)$ implies $II \uparrow_{mark} G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$:

- $\sigma(F, n)$ a winning Markov strategy for II in $G_1(\mathcal{F}_X, \neg\Omega_X)$.
- $\tau([f, F, \varepsilon], n)$ is a function g so that $g|_F = f|_F$ and $g|_{\sigma(F,n)} = n$.
- Consider a run of $G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$ according to τ: $([f_0, F_0, \varepsilon_0], g_0, \cdots)$.
- $\{\sigma(F_n, n) : n \in \omega\} \notin \Omega_X$. So find $G \subseteq X$ finite so that $G \not\subseteq \sigma(F_n, n)$ for any n.
- Let $f \in C_p(X)$. Find M so that $f|_G \leq M$. The neighborhood

$$\{h \in C_p(X) : h|_G < M\}$$

is an open set around f which misses a tail of the g_n.

Questions

- What happens with $C_p(X, [0,1])$?
- Is there a selection game on $C_p(X)$ which characterizes when X is not Rothberger?
- Is it consistent that the Ω-Rothberger and Rothberger games are equivalent?

Thanks

Thanks for Listening