The Independence of the Axiom of Choice from the Ultrafilter Theorem

Jared Holshouser

February 27, 2014

It is known that the axiom of choice (AC) and the prime ideal theorem (PIT) are equivalent to many important theorems. Halpern and Levy [2] showed in 1971 that PIT does not imply AC. We will prove the same result here, but modify the techniques and language so as to be readily understood by a reader with a basic knowledge of forcing. The form of PIT we will use here is as follows: every filter can be extended to an ultrafilter. The paper has essentially three parts. We will 1. create an inner model of a generic extension which fails choice, 2. use properties of the inner model to create a potential ultrafilter, and 3. use the Halpern-Lauchli theorem to verify that it truly is an ultrafilter.

1 Forcing Preliminaries

Let $P = FN(\omega \times \omega, 2)$ and let G be L-generic for P.

Definition 1. G is a total function from $\omega \times \omega \to 2$. For each $n \in \omega$, set $x_n = G(n, \cdot)$. Set $A = \{x_1, x_n, \cdots\}$.

Proposition 1. *A and the x_n have canonical names.*

Proof. For each n, m, set $\dot{x_n}(m) = \left\{\left(\widehat{(m, n)}, p\right) : p(m, n) = 1\right\}$. Set $\dot{A} = \{(\dot{x_n}, 1_P) : n \in \omega\}$. $\square$

Proposition 2. *The x_n are distinct.*

Proof. By way of contradiction, suppose that $p \Vdash \dot{x_i} = \dot{x_j}$. Let m be large enough so that $(j, m), (i, m) \notin \mathrm{dom}(p)$. Let $p \subseteq q$ be so that $q(i, m) = 1$ and $q(j, m) = 0$. Then $q \Vdash \dot{x_i}\,(\hat{m}) = 1$ and $q \Vdash \dot{x_j}\,(\hat{m}) = 1$. So $q \Vdash \dot{x_i} \neq \dot{x_j}$. This is a contradiction as $q \leq p$. $\square$

2 Some Information About our Inner Model

We will work in $L(A \cup \{A\})$. Recall the following:

Definition 2. We define $L(A \cup \{A\})$ as follows:

- $L_0(A \cup \{A\}) = tr(A \cup \{A\})$, the transitive closure of $A \cup \{A\}$.
- $L_{\alpha+1}(A \cup \{A\}) = \mathcal{D}\left(L_\alpha(A \cup \{A\})\right)$.
- If β is a limit, then $L_\beta(A \cup \{A\}) = \bigcup_{\alpha<\beta} L_\alpha(A \cup \{A\})$.

$L(A \cup \{A\}) = \bigcup_{\alpha \in \mathrm{On}} L_\alpha(A \cup \{A\})$.

Theorem 1. *$L(A \cup \{A\})$ is a transitive model of ZF and $L \subseteq L(A \cup \{A\}) \subseteq L[G]$.*

Proposition 3. *Let $X \in L(A \cup \{A\})$. Then X is definable from finitely many ordinals $\alpha_1, \cdots, \alpha_m$, finitely many $x_1, \cdots, x_n$ and A. We will usually write $x \in X$ iff $\varphi(x_1, \cdots, x_n, x)$ and suppress the ordinals and A.*

Theorem 2. *A is not well-orderable in $L(A \cup \{A\})$. Fix $x_1, \cdots, x_n \in A$. Let X be the class of elements in $L(A \cup \{A\})$ which are definable from $x_1, \cdots, x_n$. Then X is well-ordered.*

Proof. By way of contradiction suppose that there is a bijection $f : \omega \to A$ in $L(A \cup \{A\})$. Say that $f(k) = y$ iff $\varphi(x_1, \cdots, x_n, y, k)$. Suppose that $p \Vdash$ "f is a bijection". Let $i > n$ be so that $(i, m) \notin \text{dom}(p)$ for all m. Then there is a $q \leq p$ and a k so that $q \Vdash \varphi(\dot{x}_1, \cdots, \dot{x}_n, \dot{x}_i, k)$. Let $j > i$ be so that $(j, m) \notin \text{dom}(q)$ for all m. Let π be a permutation of ω which fixes $1, \cdots, n$ and takes i to j. Then π extends to an automorphism of P which fixes $x_1, \cdots, x_n$ and A and take x_i to x_j. Then $\pi(q)$ and q are compatible, but $\pi(q) \Vdash \varphi(\dot{x}_1, \cdots, \dot{x}_n, \dot{x}_j, k)$. This is a contradiction.

The rest of the proof follows analgously to the proof that L is well-ordered. □

Proposition 4. *Suppose that $\psi(x_1, \cdots, x_n, y)$ is true in $L(A \cup \{A\})$ where $x_1, \cdots, x_n, y \in A$. Then there is an initial segment of y, q, so that if y' extends q ($q \preceq y'$), then $\psi(x_1, \cdots, x_n, y')$.*

Proof. Suppose that $p \Vdash \psi^{L(A \cup \{A\})}(\dot{x_1}, \cdots, \dot{x_n}, \dot{y})$. Say that $y = x_k$ for some $k > n$. Let $p' \geq p$ be so

$$(i, j) \in \text{dom}(p') \iff (i, j) \in \text{dom}(p) \wedge i \in \{1, \cdots, n, k\}$$

It suffices to show that $p' \Vdash \psi^{L(A \cup \{A\})}(\dot{x_1}, \cdots, \dot{x_n}, \dot{y})$. By way of contradiction. suppose that there is a $q \leq p'$ so that $q \Vdash \neg\psi(\dot{x_1}, \cdots, \dot{x_n}, \dot{y})$. Let π be a permutation of ω which fixes $1, \cdots, n, k$ and moves the parts of q which make it incompatible with p. Then $\pi(q) \Vdash \neg\psi(\dot{x_1}, \cdots, \dot{x_n}, \dot{y})$ and $\pi(q)$ is compatible with p. This is a contradiction.

□

3 The Basic Setup

Henceforth our calculations take place in $L(A \cup \{A\})$ unless otherwise stated. The main result is as follows:

Theorem 3. *Let X be a set and $\mathcal{F}$ be a filter on X which is definable from $x_1, \cdots, x_n$. Then there is an ultrafilter $\mathcal{U}$ on X which is definable from $x_1, \cdots, x_n$.*

We first create the largest filter possible using only $x_1, \cdots, x_n$.

Definition 3. Recall that the class of elements definable from $x_1, \cdots, x_n$ is well-orderable. Let $\mathcal{U}$ extending $\mathcal{F}$ be the maximal proper filter extending $\mathcal{F}$ which is definable from $x_1, \cdots, x_n$.

We claim that $\mathcal{U}$ is in fact an ultrafilter on X and proceed by way of contradiction. Suppose that there is a $B \subseteq X$ so that B and $X \smallsetminus B$ are not in $\mathcal{U}$. B is definable from $x_1, \cdots, x_n, y_1, \cdots, y_k$ for some k and some $y_1, \cdots, y_k \in A \smallsetminus \{x_1, \cdots, x_n\}$.

4 The Simple Case

We proceed first with $k = 1$. Say that B is defined from $x_1, \cdots, x_n, y'$ by φ. For $y \in A$ set

$$x \in B_y \iff \varphi(x_1, \cdots, x_n, y, x)$$

Lemma 1. *There is a $q \in FN(\omega, 2)$ so that if $y \in A$ with $q \preceq y$, then B_y and $X \smallsetminus B_y$ are not in $\mathcal{U}$.*

Proof. This follows from proposition 4. □

Fix a q as in lemma 1. Let $\mathcal{V}$ be the filter generated by $\mathcal{U}$ and $\{B_y : q \preceq y \wedge y \in A\}$.

Lemma 2. *$\mathcal{V}$ is definable from $x_1, \cdots, x_n$ and thus $\mathcal{V} = \mathcal{P}(X)$.*

Proof. This follows by the maximality of $\mathcal{U}$. □

Proposition 5. *We can find $y_1, \cdots, y_\ell$ in A with $q \preceq y_i$ so that*

$$(X \smallsetminus B_{y_1}) \cup \cdots \cup (X \smallsetminus B_{y_\ell}) \in \mathcal{U}$$

Proof. Since $\mathcal{V} = \mathcal{P}(X)$ we can find $y_1, \cdots, y_\ell$ in A with $q \preceq y_i$ and a $U \in \mathcal{U}$ so that

$$(B_{y_1}) \cap \cdots \cap (B_{y_\ell}) \cap U = \emptyset$$

So

$$X \subseteq (X \smallsetminus B_{y_1}) \cup \cdots \cup (X \smallsetminus B_{y_\ell}) \cup (X \smallsetminus U)$$

and thus

$$U \subseteq (X \smallsetminus B_{y_1}) \cup \cdots \cup (X \smallsetminus B_{y_\ell})$$

□

Nothing about what we just did depended on us choosing to use B_y instead of $X \setminus B_y$. So we could run all of the same proofs, and we get the following:

Proposition 6. *We can find $y_1, \cdots, y_\ell$ in A with $q \preceq y_i$ so that*

$$(X \smallsetminus B_{y_1}) \cup \cdots \cup (X \smallsetminus B_{y_\ell}), B_{y_1} \cup \cdots \cup B_{y_\ell} \in \mathcal{U}$$

We now apply proposition 4 again.

Proposition 7. *We can find $q_1, \cdots, q_\ell \leq q$ so that if $\forall i (y_i \in A \wedge q_i \preceq y_i)$, then*

$$(X \smallsetminus B_{y_1}) \cup \cdots \cup (X \smallsetminus B_{y_\ell}), B_{y_1} \cup \cdots \cup B_{y_\ell} \in \mathcal{U}$$

Set $Q_1 = \{q_1, \cdots, q_\ell\}$. For each i we can run the above argument on q_i exactly as we ran the argument for q. So we get $q_i^j \leq q_i$ so that if $y_i^j \in A$ with $q_i^j \preceq y_i^j$ for all $j \leq \ell_i$, then

$$\bigcup_j X \smallsetminus B_{y_i^j}, \bigcup_j B_{y_i^j} \in \mathcal{U}$$

Set $Q_2 = \left\{ q_i^j \right\}_{i,j \leq \ell_i}$

Lemma 3. *Let $y_i^j \in A$ be so that $q_i^j \preceq y_i^j$ for all i, j. Let $h : \left\{ y_i^j \right\}_{i,j} \to 2$. Then*

$$\bigcup \left\{ B_{y_i^j} : h\left(y_i^j\right) = 1 \right\} \in \mathcal{U} \text{ or } \bigcup \left\{ X \smallsetminus B_{y_i^j} : h\left(y_i^j\right) = 0 \right\} \in \mathcal{U}$$

Proof. If there is an i so that h is constant on the y_i^j, then we are done by the comments preceding this lemma. So assume that for all i, h is not constant on y_i^j. Then we can choose j_i for each i so that $h\left(y_i^{j_i}\right) = 1$. Then by the preceding proposition we are done as $q_i \preceq y_i^{j_i}$. □

We claim that this is a contradiction. Fix $y_i^j \in A$ so that $q_i^j \preceq y_i^j$ for all i, j. For $h : \left\{ y_i^j \right\}_{i,j} \to 2$, set

$$g(h, i, j) = \begin{cases} B_{y_i^j} & \text{if } h\left(y_i^j\right) = 1 \\ X \smallsetminus B_{y_i^j} & \text{else} \end{cases}$$

Note that for each $h : \left\{ y_i^j \right\}_{i,j} \to 2$ we have that $\bigcup_{i,j} g(h, i, j) \in \mathcal{U}$. Thus

$$\emptyset = \bigcup_{i,j} \left(B_{y_i^j} \cap (X \smallsetminus B_{y_i^j}) \right) = \bigcap_h \bigcup_{i,j} g(h, i, j) \in \mathcal{U}$$

Therefore $\emptyset \in \mathcal{U}$. This contradicts the properness of $\mathcal{U}$.

5 The Halpern-Lauchli Theorem

Definition 4. A tree is **finitistic** if $T \neq \emptyset$, each node of T has finite order, and each level of T is finite.

Definition 5. Let T be a finitistic tree. $M \subseteq T$ is said to be $(m, 1)$**-dense** if there is a $t \in T$ with $|t| = m$ so that whenever $t < s$ and $|s| = m + 1$, there is a $u \in M$ with $s < u$.

If $T_1, \cdots, T_k$ are finitistic trees and $M_i \subseteq T_i$ is $(m, 1)$-dense for each i, we call $M_1 \times \cdots \times M_k$ an $(m, 1)$**-matrix**.

Theorem 4. *Suppose that $T_1, \cdots, T_k$ are finitistic trees with no maximal nodes. Then there is an n so that if $f : T_1 \restriction_n \times \cdots \times T_k \restriction_n \to 2$, then we can find an $m < n$ and an $(m, 1)$-matrix $M_1 \times \cdots \times M_k \subseteq T_1 \restriction_n \times \cdots \times T_k \restriction_n$ which is homogeneous for f.*

Proof. This was originally proved via metamathematical means. A direct proof was given by Argyros, Felouzis and Kanellopoulos in 2002[1]. $\square$

6 The General Case

We now suppose that the set B is defined by $\varphi(x_1, \cdots, x_n, y_1', \cdots, y_k', x)$. We will define k-sequences Q_n for all n. First we find q_i so that if $y_i \in A$ and $q_i \preceq y_i$ for all i and $B_{y_1,\cdots,y_k}$ is defined by $\varphi(x_1, \cdots, x_n, y_1, \cdots y_k, x)$, then $B_{y_1,\cdots,y_k}$ and $x \setminus B_{y_1,\cdots,y_k}$ are not in $\mathcal{U}$. Set $Q_0 = (\{q_1\}, \cdots, \{q_n\})$.

Proposition 8. *We can create k-sequences Q_n for $n \geq 1$ so that*

1. *If $q \in Q_n(i)$, then there is an $r \in Q_{n-1}(i)$ so that $q \leq r$.*
2. *If $r \in Q_{n-1}(i)$, then there are $q, q' \leq r$ so that $q \neq q'$ and $q, q' \in Q_n(i)$.*
3. *If $q, r \in Q_n(i)$, then $q \perp r$ or $q = r$.*
4. *Let $q_i \in Q_{n-1}(i)$ for each i and suppose that $Y \subseteq A^k$ is so that whenever we have $r_i \in Q_n(i)$ with $r_i \leq q_i$ for all i, there is a unique $(y_1, \cdots, y_k) \in Y$ with $q_i \preceq y_i$ for all i. Then*

$$\bigcup \{B_{y_1,\cdots,y_k} : (y_1, \cdots, y_k) \in Y\} \in \mathcal{U}$$

$$\bigcup \{X \setminus B_{y_1,\cdots,y_k} : (y_1, \cdots, y_k) \in Y\} \in \mathcal{U}$$

Proof. The method of construction is recursive. We simply apply the analysis of the simple case to each $(q_1, \cdots, q_k) \in \prod_i Q_{n-1}(i)$ and let $Q_n(i)$ be the compilation of the resulting q_i^j. ($Q_n(i)$ is built from all sequences $(q_1, \cdots, q_k)$) $\square$

Definition 6. For $1 \leq i \leq k$, set $T_i = \bigcup_{n\in\omega} Q_n(i)$. $q \leq_i r$ iff $r \leq q$. Set $T = T_1 \times \cdots \times T_k$.

Note that the nth level of T is $Q_n(1) \times \cdots \times Q_n(k)$ and that T satisfies the conditions of the Halpern-Lauchli theorem.

Let n be as guaranteed by Halpern-Lauchli. Let $W = \bigcup_{i<k, m\leq n} Q_m(i)$ and let $H : W \to A$ be so that for all $q \in W$, $q \preceq H(q)$. Finally let $Y_i = H\left[\left(\bigcup_{m\leq n} Q_m(i)\right)\right]$.

Proposition 9. *For all* $h : Y_1 \times \cdots \times Y_k \to 2$,

$$\bigcup \{B_{y_1,\cdots,y_k} : h(y_1, \cdots, y_k) = 1\} \in \mathcal{U}$$

or

$$\bigcup \{X \setminus B_{y_1,\cdots,y_k} : h(y_1, \cdots, y_k) = 0\} \in \mathcal{U}$$

Proof. Define $f : T_1 \restriction_n \times \cdots \times T_k \restriction_n \to 2$ by

$$f(q_1, \cdots, q_n) = h(H(q_1), \cdots, H(q_k))$$

Let $M = M_1 \times \cdots \times M_k$ be an $(m, 1)$-matrix and homogeneous for f. Suppose that $f[M] = 1$. Then

$$h[(H[M_1] \times \cdots \times H[M_k])] = 1$$

We can find $q_i \in Q_m(i)$ so that for all $r \in Q_{m+1}(i)$ if $r \leq q_i$, then there is a $p \in M_i$ so that $p \leq r$. Define $g : Q_{m+1} \to A$ as follows: If there is an i so that $r \leq q_i$, let $g(r) = H(p)$, where $p \leq r$ is in M_i. Otherwise let $g(r)$ be some y so that $r \preceq y$.

Note that $r \preceq g(r)$ for all r. Thus from property 4

$$\bigcup \{B_{y_1,\cdots,y_k} : q_i \preceq y_i \wedge \exists r\,(y_i = g(r))\} \in \mathcal{U}$$

Now, if for all i, $q_i \preceq y_i$ and there is an r so that $y_i = g(r)$, then $r \leq q_i$, and so $y_i = H(p)$ for some $p \in M_i$. So by homogeneity,

$$H(y_1, \cdots, y_k) = 1$$

If $f[M] = 0$, consider $\hat{f} = |1 - f|$. Then $\hat{f}[M] = 1$ and we apply the above proof. □

This is a contradiction just as in the simple case. For $h : Y_1 \times \cdots \times Y_k \to 2$, and $(y_1, \cdots, y_k) \in Y_1 \times \cdots \times Y_k$, set

$$g(h, y_1, \cdots, y_k) = \begin{cases} B_{y_1,\cdots,y_k} & \text{if } h(y_1, \cdots, y_k) = 1 \\ X \setminus B_{y_1,\cdots,y_k} & \text{else} \end{cases}$$

Note that for each $h : Y_1 \times \cdots \times Y_k \to 2$ we have that $\bigcup_{y_1,\cdots,y_k} g(h, y_1, \cdots, y_k) \in \mathcal{U}$. Thus

$$\emptyset = \bigcup_{y_1,\cdots,y_k} (B_{y_1,\cdots,y_k} \cap X \setminus B_{y_1,\cdots,y_k}) = \bigcap_h \bigcup_{y_1,\cdots,y_k} g(h, y_1, \cdots, y_k) \in \mathcal{U}$$

Therefore $\emptyset \in \mathcal{U}$.

7 References

References

[1] Argyros, S.A., V. Felouzis, and V. Kanellopoulos. "A proof of Halpern-Lauchli partition theorem." European Journal of Combinatorics 23.1 (2002): 1-10. Print.

[2] Halpern, J.D., and A. Levy. "THE BOOLEAN PRIME IDEAL THEOREM DOES NOT IMPLY THE AXIOM OF CHOICE." Axiomatic Set Theory 13 (1971): 83-134. Print.

[3] Jech, Thomas J.. The axiom of choice. Dover ed. Mineola, N.Y.: Dover Publications, 2008. Print.

[4] Kunen, Kenneth. Set theory: an introduction to independence proofs. Amsterdam: North-Holland Pub. Co. ;, 1980. Print.