Characterizing First Order Logic

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September 16, 2014

We are following the presentation of Chang and Keisler.

1 A Brief Review of First Order Logic

Definition 1. A **language** $\mathcal{L}$ is a collection of symbols, which are broken into three groups: relation symbols, function symbols and constant symbols.

Definition 2. Let $\mathcal{L}$ be a language. A **model** for $\mathcal{L}$ is a pair $\mathfrak{A} = (A, \mathcal{I})$, where we call $A \neq \emptyset$ the **universe** of $\mathfrak{A}$ and $\mathcal{I}$ an interpretation function. $\mathcal{I}$ must be as follows:

- Each n-ary function symbol F gets taken to an n-ary function $G : A^n \to A$,
- Each n-ary relation symbols R gets taken to an n-ary relation $P \subseteq A^n$, and
- Each constant symbol c gets taken to a constant $x \in A$.

Proposition 1. *Suppose $\mathfrak{A}$ is a model for the language $\mathcal{L}$. Let X be a set. Then we can expand $\mathfrak{A}$ to a model of $\mathcal{L}' = \mathcal{A} \cup X$.*

Proof. WLOG X is disjoint from $\mathcal{L}$. Let $\mathcal{I}'$ be an interpretation on X. Then $\mathfrak{A}' = (A, \mathcal{I} \cup \mathcal{I}')$ suffices. $\square$

Definition 3. For $\mathfrak{A}$, $\mathfrak{A}'$, $\mathcal{L}$, $\mathcal{L}'$, and X as above, we call $\mathfrak{A}'$ the **expansion** of $\mathfrak{A}$ to $\mathcal{L}'$ and $\mathfrak{A}$ the **reduct** of $\mathfrak{A}'$ to $\mathcal{L}$.

Definition 4. Two models $\mathfrak{A} = (A, \mathcal{I})$ and $\mathfrak{B} = (B, \mathcal{J})$ of $\mathcal{L}$ are **isomorphic** if there is a function $f : A \to B$ so that

- f is a bijection,
- For each relation symbol R, if R is n-ary, then for all $x_1, \cdots, x_n \in A$
$$\mathcal{I}(R)(x_1, \cdots, x_n) \iff \mathcal{J}(R)(f(x_1), \cdots, f(x_n)),$$
- For each function symbol F, if F is n-ary, then for all $x_1, \cdots, x_n \in A$
$$\mathcal{I}(F)(x_1, \cdots, x_n) = \mathcal{J}(F)(f(x_1), \cdots, f(x_n)),$$
- For each constant symbol c, $f(\mathcal{I}(c)) = \mathcal{J}(c)$.

We write $f : \mathfrak{A} \cong \mathfrak{B}$.

Definition 5. Suppose $\mathfrak{A} = (A, \mathcal{I})$ and $\mathfrak{B} = (B, \mathcal{J})$ are models of $\mathcal{L}$. Then $\mathfrak{B}$ is a submodel of $\mathfrak{A}$ if $B \subseteq A$ and the following hold:

- for R an n-ary relation symbol, $\mathcal{J}(R) = \mathcal{I}(R) \cap B^n$,
- for F an n-ary function symbol, $\mathcal{J}(F) = \mathcal{I}(f)|_{B^n}$, and
- for c a constant symbol, $\mathcal{J}(c) = \mathcal{I}(c)$.

We now define first order logic, recursively. First the allowable symbols: We have parentheses (,); variables $v_1, \cdots, v_n, \cdots$; connectives $\wedge$ and $\neg$; and the quantifier $\forall$. Note that these are used in addition to the symbols of $\mathcal{L}$ and the symbol $=$.

Definition 6. The collection of **terms** is defined recursively as follows:

1. A variable is a term,
2. a constant symbol is a term, and
3. if F is an n-ary function symbol, and $t_1, \cdots, t_n$ are terms, then $F(t_1, \cdots, t_n)$ is a term.

Definition 7. The collection of **atomic formulas** is defined recursively as follows:

1. if t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic formula,
2. if R is an n-ary relation symbol and $t_1, \cdots, t_n$ are terms, then $R(t_1, \cdots, t_n)$ is an atomic formula.

Definition 8. The collection of **formulas** is defined recursively as follows:

1. atomic formulas are formulas,
2. if ϕ and ψ are formulas, then $\phi \wedge \psi$ and $\neg\phi$ are formulas,
3. if v is a variable, and ϕ is a formula, then $\forall v \phi$ is a formula.

Definition 9. A **sentence** is a formula with no free variables.

We now define model satisfaction:

Definition 10. Fix a model $\mathfrak{A}$ of a language $\mathcal{L}$. We will define $\mathfrak{A} \models \phi[\vec{x}]$ for formulas ϕ, and $t[\vec{x}]$ for terms t. We begin with terms.

- If t is v_i, then $t[\vec{x}] := x_i$.
- If t is c, then $t[\vec{x}] := \mathcal{I}(c)$.
- If t is $F(t_1, \cdots, t_n)$, then $t[\vec{x}] = \mathcal{I}(F)(t_1[\vec{x}], \cdots, t_n[\vec{x}])$.

Now we do atomic formulas.

- Suppose ϕ is $t_1 = t_2$. Then $\mathfrak{A} \models \phi(\vec{x}) \iff t_1[\vec{x}] = t_2[\vec{x}]$.
- Suppose ϕ is $R(t_1, \cdots, t_n)$. Then $\mathfrak{A} \models \phi(\vec{x}) \iff \mathcal{I}(R)(t_1[\vec{x}], \cdots, t_n[\vec{x}])$.

Finally we do formulas.

- $\mathfrak{A} \models (\phi \wedge \psi)$ iff $\mathfrak{A} \models \phi$ and $\mathfrak{A} \models \psi$.
- $\mathfrak{A} \models \neg\phi$ iff it is false that $\mathfrak{A} \models \phi$.
- $\mathfrak{A} \models \forall v_i \phi[x_1, \cdots, x_m]$ (for $i \leq m$) iff for all $a \in A$, $\mathfrak{A} \models \phi[x_0, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_m]$

Remark 1. For sentences ϕ, we can write $\mathfrak{A} \models \phi$ instead of $\mathfrak{A} \models \phi[\vec{x}]$, as it does not depend on $\vec{x}$.

Definition 11. Two models $\mathfrak{A}$ and $\mathfrak{B}$ of a language $\mathcal{L}$ are **elementarily equivalent** iff whenever ϕ is a sentence we have that $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$. We write $\mathfrak{A} \equiv \mathfrak{B}$.

Proposition 2. *If* $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{A} \equiv \mathfrak{B}$*.*

Definition 12. Let Σ be a set of sentences of $\mathcal{L}$. Then Σ **has a model** if there is a model $\mathfrak{A}$ of $\mathcal{L}$ so that $\mathfrak{A} \models \phi$ for all $\phi \in \Sigma$.

Theorem 1 (Compactness Theorem)**.** *A set of sentences* Σ *has a model iff every finite subset of* Σ *has a model.*

Theorem 2 (Downward Lowehnheim-Skolem Theorem)**.** *Every every consistent set of sentences in the language* $\mathcal{L}$ *has a model of size* $\leq \max\{\omega, |\mathcal{L}|\}$*. In particular, every valid sentence has a countable model.*

2 Abstract Logics

From here on, in order to simplify the argument, we restrict the discussion to lanaguages without function symbols.

Definition 13. An **abstract logic** is a pair of classes $(l, \models_l)$ with the following properties: l is the class of sentences and $\models_l$ is the **satisfaction relation** of the logic $(l, \models_l)$.

1. The Occurrence Property: For each $\phi \in l$ there is associated a finite language $\mathcal{L}_\phi$ called the set of **symbols occuring in** ϕ. The relation $\mathfrak{A} \models_l \phi$ is a relation between sentences ϕ of l and models $\mathfrak{A}$ for languages $\mathcal{L}$ which contain $\mathcal{L}_\phi$.

2. The Expansion Property: The relation $\mathfrak{A} \models_l \phi$ depends only on the reduct of $\mathfrak{A}$ to $\mathcal{L}_\phi$. I.e. if $\mathfrak{A} \models_l \phi$ and $\mathfrak{B}$ is an expansion of $\mathfrak{A}$ to a larger language, then $\mathfrak{B} \models_l \phi$.

3. The Isomorphism Property: The relation $\mathfrak{A} \models_l \phi$ is preserved under isomorphism.

4. The Renaming Property: The relation $\mathfrak{A} \models_l \phi$ is preserved under renaming. That is, if we relabel the symbols of the logic, and modify the model accordingly, $\models_l$ is preserved.

5. The Closure Property: l contains all atomic sentences, l is closed under the usual first order connectives, $\models_l$ satisfies the usual rules for satisfaction of atomic formulas and first order connectives, and the set of symbols $\mathcal{L}_\phi$ behaves as expected for atomic sentences and first order connectives.

6. The Quantifier Property: l is closed under universal and existential quantifiers, and behaves as expected for them.

7. The Relativization Property: For each sentence $\phi \in l$ and relation $R(x, b_1, \cdots, b_n)$ with $R, b_1, \cdots, b_n$ not in $\mathcal{L}_\phi$, there is a new sentence $\phi|R(x, b_1, \cdots, b_n)$, read ϕ relativized to $R(x, b_1, \cdots, b_n)$ which has the set of symbols $\mathcal{L}_\phi \cup \{R, b_1, \cdots, b_n\}$ and is such that whenever $\mathfrak{B}$ is the submodel of a model $\mathfrak{A}$ for $\mathcal{L}_\phi$ with universe

   $$B = \{a \in A : R(a, b_1, \cdots, b_n)\},$$

   we have

   $$(\mathfrak{A}, R, b_1, \cdots, b_n) \models_l \phi|R(x, b_1, \cdots, b_n) \text{ iff } \mathfrak{B} \models_l \phi$$

Remark 2. This notion of abstract logic does not have free variables, so l should be considered as a collection of sentences.

Proposition 3. *First order logic is an abstract logic.*

Proof. Properties 1 - 6 are either standard theorems or obviously true. We will prove that property 7 holds. Let ϕ be a sentence of first order logic, and $R(x, b_1, \cdots, b_n)$ be a relation. Then the relativization of ϕ is formed as follows. Replace the quantifier $\forall x\psi$ by

$$\forall x[R(x, b_1, \cdots, b_n) \implies \psi]$$

and replace the quantifier $\exists x\psi$ by

$$\exists x[R(x, b_1, \cdots, b_n) \wedge \psi]$$

□

Remark 3. Every abstract logic l contains first order logic by the closure and quantifier properties.

Definition 14. Two abstract logics l, l' are **equivalent** if for all $\phi \in l$, there is a $\psi \in l'$ so that $\mathcal{L}_\phi = \mathcal{L}_\psi$ and $\mathfrak{A} \models_l \phi$ iff $\mathfrak{A} \models_{l'} \psi$ for all $\mathfrak{A}$; and similarly for all $\psi \in l'$.

Definition 15. By a **model** of a set T of sentences of an abstract logic l we mean a model $\mathfrak{A}$ so that $\mathfrak{A} \models_l \phi$ for all $\phi \in T$.

Two models $\mathfrak{A}$ and $\mathfrak{B}$ for the same language l are **l-elementarily equivalent** if for each sentence $\phi \in l$, $\mathfrak{A} \models_l \phi$ iff $\mathfrak{B} \models_l \phi$.

Definition 16. An abstract logic l is **countably compact** iff for every countable set $T \subseteq l$, if every finite subset of T has a model, then T has a model.

The **Lowenheim number** of l is the least cardinal α such that every sentence of l which has a model, has a model of power at most α.

Remark 4. If the Lowenheim number of a logic l exists, it must be at least ω, as l contains every sentence of first order logic.

Proposition 4. *Let l be an abstract logic such that for each finite language $\mathcal{L}$, the class $\{\phi \in l : \mathcal{L}_\phi \subseteq \mathcal{L}\}$ is a set. Then the Lowenheim number of l exists.*

Proof. By renaming, we can WLOG assume that if $\phi \in l$, then $\mathcal{L}_\phi \in V_\omega$. This makes l a set. Define a map $\alpha : l \to ON$ by

$$\alpha(\phi) = \begin{cases} \omega & \text{if } \phi \text{ has no models} \\ \kappa & \text{if } \kappa \text{ is the size of the smallest model of } \phi \end{cases}$$

Then $\sup\{\alpha(\phi) : \phi \in l\}$ exists, and is the Lowenheim number of l. $\square$

Definition 17. Let $\mathcal{A}$ and $\mathcal{B}$ be models for a language $\mathcal{L}$. A **partial isomorphism** $I : \mathcal{A} \cong_p \mathcal{B}$ is a relation I on the set of pairs of finite sequences $(a_1, \cdots, a_n)$, $(b_1, \cdots, b_n)$ of elements of A and B of the same length such that

1. $\emptyset I \emptyset$
2. If $(a_1, \cdots, a_n) I (b_1, \cdots, b_n)$, then $(\mathfrak{A}, a_1, \cdots, a_n)$ and $(\mathfrak{B}, b_1, \cdots, b_n)$ satisfy the same atomic sentences of $\mathcal{L}(c_1, \cdots, c_n)$
3. If $(a_1, \cdots, a_n) I (b_1, \cdots, b_n)$, then
$$\forall c \in A \exists d \in B[(a_1, \cdots, a_n, c) I (b_1, \cdots, b_n, d)]$$
and vice versa.

Proposition 5. *Any two finite or countable partially isomorphic models are isomorpic.*

Proposition 6. *Let l be an abstract logic which has Lowenheim number ω. Then any two models which are partially isomorphic are l-elementarily equivalent.*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are models for a language $\mathcal{L}$ and $I : A \cong_p B$. Suppose by way of contradiction that there is a $\phi \in l$ so that $\mathfrak{A} \models_l \phi$ and $\mathfrak{B} \models_l \neg\phi$. WLOG we may assume that $\mathcal{L} = \mathcal{L}_\phi$ and that I is preserved under subsequences. For the first, use reducts, and for the second extend I in the obvious way. Define $F : A^{<\omega} \times A \to A^{<\omega}$ and $F' : A^{<\omega} \times A^{<\omega} \to A^{<\omega}$ by

$$F((a_1, \cdots, a_n), b) = (a_1, \cdots, a_n, b)$$

and

$$F'((a_1, \cdots, a_n), (b_1, \cdots, b_m)) = (a_1, \cdots, a_n, b_1, \cdots, b_m)$$

Let $\mathfrak{A}' = (A \cup A^{<\omega}, \mathfrak{A}, F, F')$ be a model, where F and F' are treated as relations. Similarly build $\mathfrak{B}' = (B \cup B^{<\omega}, \mathfrak{B}, G, G')$. WLOG, by the isomorphism property, A,$A^{<\omega}$, B, and $B^{<\omega}$ are all disjoint from each other. Now consider $\mathfrak{C} = (\mathfrak{A}', \mathfrak{B}', I)$. Then by the closure, quantifier, and relativization properties of l, there is a sentence $\psi \in l$ so that

$$\mathfrak{C} \models_l \psi \text{ and } \psi \implies [(\mathfrak{A} \models_l \phi) \wedge (\mathfrak{B} \models_l \neg\phi) \wedge (\mathfrak{A} \cong_p \mathfrak{B})]$$

Then ψ has a countable model $\mathfrak{C}_0$, which gives us models $\mathfrak{A}_0$ and $\mathfrak{B}_0$ for $\mathcal{L}$. Then $\mathfrak{A}_0 \cong \mathfrak{B}_0$. This contradicts the isomorphism property, as $\mathfrak{C}_0 \models \psi$. $\square$

3 The Main Theorem

Theorem 3 (Lindstrom's Characterization of First Order Logic)**.** *Suppose l is an abstract logic which is countably compact and has Lowenheim number ω. Then l is equivalent to first order logic.*

Proof. Let l be a countably compact abstract logic with Lowenheim number ω. Let ϕ be a sentence of first order logic, and $\mathfrak{A}$ be a model of $\mathcal{L}_\phi$. Then by the closure and quantifier properties, $\mathfrak{A} \models \phi$ iff $\mathfrak{A} \models_l \phi$. Now we must prove the other direction.

We first define some relations between models of a finite language $\mathcal{L}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $\mathcal{L}$. We define relations I_k for $k \in \omega$ between $\mathfrak{A}$ and $\mathfrak{B}$ recursively. Let $(a_1, \cdots, a_n) \in A^n$ and $(B_1, \cdots, b_n) \in B^n$.

- $\vec{a} I_0 \vec{b}$ iff $\vec{a}$ and $\vec{b}$ satisfy the same atomic formulas.
- $\vec{a} I_{k+1} \vec{b}$ iff
  1. $\forall c \in A \exists d \in B((a_1, \cdots, a_n, c) I_k (b_1, \cdots, b_n, d))$ and
  2. $\forall d \in B \exists c \in A((a_1, \cdots, a_n, c) I_k (b_1, \cdots, b_n, d))$.

We say $\mathfrak{A} \equiv_k \mathfrak{B}$ if $\emptyset I_k \emptyset$. Since $\mathcal{L}$ is finite and has no function symbols, for each k there is a finite set Γ_k of first order logic in $\mathcal{L}$ such that for all models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{L}$, $\mathfrak{A} \equiv_k \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ agree on Γ_k.

Let $\phi \in l$ be a sentence so that $\mathcal{L}_\phi \subseteq \mathcal{L}$. It suffices to show that there is a k so that for all models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{L}$,

$$(\mathfrak{A} \equiv_k \mathfrak{B} \text{ and } \mathfrak{A} \models_l \phi) \implies \mathfrak{B} \models_l \phi$$

For then we will know that for some boolean combination of sentences in Γ_k, ψ, $\mathcal{L}_\phi = \mathcal{L}_\psi$ and for all models $\mathfrak{A}$ of $\mathcal{L}_\phi$, $\mathfrak{A} \models_l \phi$ iff $\mathfrak{A} \models \psi$.

We proceed by way of contradiction. For each k choose models $\mathfrak{A}_k$ and $\mathfrak{B}_k$ so that

$$\mathfrak{A}_k \equiv_k \mathfrak{B}_k \text{ and } \mathfrak{A} \models_l \phi \text{ and } \mathfrak{B} \models_l \neg\phi \tag{1}$$

Taking a subsequence if necessary, we can assume that the $\mathfrak{A}_k$ satisfy the same atomic sentences. By the isomorphism property, we can assume that the $\mathfrak{A}_k$ interpret the constants of $\mathcal{L}$ the same way. Let $\mathfrak{A} = \bigcup_k \mathfrak{A}_k$. Each $\mathfrak{A}_k$ is a submodel of $\mathfrak{A}$, and we can take the $\mathfrak{A}_k$ so that $A_k \cap \omega = \emptyset$. As in the previous proof, let

$$\mathfrak{A}' = (A \cup A^{<\omega}, \mathfrak{A}, F, F')$$

Similarly define $\mathfrak{B}$ and $\mathfrak{B}'$. Let

$$\mathfrak{C} = (\mathfrak{A}', \mathfrak{B}', R, S, \omega, \leq, I)$$

where R, S and I are so that

$$A_k = \{a \in A : R(a, k)\},\ B_k = \{b \in B : S(b, k)\},$$

$$A_k^{<\omega} = \{\vec{a} \in A^{<\omega} : R(\vec{a}, k)\},\ B_k^{<\omega} = \{\vec{b} \in B^{<\omega} : S(\vec{b}, k)\},$$

and for each k, $I(\vec{a}, \vec{b}, k)$ holds iff $\vec{a} I_k \vec{b}$. By the closure, quantifier, and relativization properties there is a sentence $\theta \in l$ so that $\mathfrak{C} \models_l \theta$ and θ implies that $(\omega, \leq)$ is a total order with immediate successors and predecessors except for the first element, and (1) holds for all $k \in \omega$. By countable compactness, θ has a model

$$\hat{\mathfrak{C}} = (\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, \hat{R}, \hat{S}, \hat{\omega}, \hat{\leq}, \hat{I})$$

so that $\hat{\omega}$ has a nonstandard element H. Then $\hat{\mathfrak{A}}_H \models_l \phi$, $\hat{\mathfrak{B}}_H \models_l \neg\phi$, and $\hat{\mathfrak{A}}_H \equiv_H \hat{\mathfrak{B}}_H$. Define J between $\hat{\mathfrak{A}}_H$ and $\hat{\mathfrak{B}}_H$ by

$$(a_1, \cdots, a_n) J (b_1, \cdots, b_n) \text{ iff } (a_1, \cdots, a_n) \hat{I}_{H-n} (b_1, \cdots, b_n)$$

Then J is a partial isomorphism between $\hat{\mathfrak{A}}_H$ and $\hat{\mathfrak{B}}_H$. But then by the previous proposition and the fact that l has Lowenheim number ω, $\hat{\mathfrak{A}}_H \equiv_l \hat{\mathfrak{B}}_H$. This is a contradiction. $\square$