A Horror of Topology without AC: The Independence of Stone's Theorem from ZF

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This work is a modified presentation of Good, Tree and Watson in [1].

1 Introduction

Recall the following notions from topology.

Definition 1. Let X be a topological space and $\mathcal{U}$, $\mathcal{V}$ be covers of X.

- $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if for all $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ so that $V \subseteq U$.
- $\mathcal{V}$ is **locally finite** if for all $x \in X$ there is a neighborhood U of X so that U has nonempty intersection with only finitely many elements of $\mathcal{V}$.
- X is **paracompact** if every open cover admits a locally finite open refinement.

Recall the statement of Stone's Theorem.

Theorem 1. *(ZFC) Every metric space is paracompact.*

The contemporary proof of Stone's theorem, discovered by Rudin [2] features a conspicuous use of AC. It is thus natural to ask whether or not AC is required for Stone's theorem to hold. Weaker versions of Stone's theorem, which require the metric space to be either second countable or separable, are theorems of ZF. We will show that it is consistent with ZF for the full theorem to be false. That is we will construct a model N and a metric space (X, d) so that $N \models (X$ is not paracompact).

2 The Model and the Space

Let $P = \mathrm{FN}(\omega \times \mathbb{R} \times \omega_1 \times \omega_1, 2, \omega_1)$. That is partial functions p from $\omega \times \mathbb{R} \times \omega_1 \times \omega_1$ to 2 with $|p|$ at most countable. Let G be generic for P. In $V[G]$ define the following:

- For $(n, r, \alpha) \in \omega \times \mathbb{R} \times \omega_1$, let $x_{nr\alpha} = \{\beta \in \omega_1 : \exists p \in G(p \Vdash G(n, r, \alpha, \beta) = 1)\}$
- For $(n, r) \in \omega \times \mathbb{R}$, let $A_{nr} = \{x_{nr\alpha} : \alpha \in \omega_1\}$
- For $n \in \omega$, let $R_n = \{A_{nr} : r \in \mathbb{R}\}$
- Let $R = \{R_n : n \in \omega\}$

All of these sets have canonical names in the standard way. Set $N = L(\mathrm{trcl}(\{R\}))$ and $X = \bigcup_n R_n$. Note that the elements of X are the A_{nr}, i.e. X is composed of countably many real lines whose points are ω_1 sequences of subsets of ω_1.

3 Basic Properties of N and X

Lemma 1. *For all n, r, α, $x_{nr\alpha} \in N$. For all n, r, $A_{nr} \in N$. For all n, $R_n \in N$. Everything in N is definable from finitely many ordinals and parameters from trcl($\{R\}$).*

Lemma 2. $\mathbb{R}^V = \mathbb{R}^N = \mathbb{R}^{V[G]}$.

Proof. This follows as P is countably closed. □

Lemma 3. *X can be given a metric so that each of the R_n are connected.*

Proof. We define the metric as follows:

$$\left[d(A_{nr}, A_{ns}) = \frac{|r-s|}{1+|r-s|}\right] \wedge [m \neq n \implies d(A_{mr}, A_{ns}) = 1]$$

One can check that this works. □

Theorem 2. *There is no set $S \in N$ with $\emptyset \neq S \cap R_n \subsetneq R_n$ for all n.*

Proof. We will show that there is no function $f \in N$ so that $\mathrm{dom}(f) = R$ and $f(R_n)$ is a proper nonempty subset of R_n for all n. This suffices as $f(R_n) = S \cap R_n$ would be such a function. We proceed by way of contradiction. Let f in N be such a function. Let $p_0 \in G$ be so that

$$p_0 \Vdash (f \text{ is a function}) \wedge \forall n(\emptyset \neq f(R_n) \subsetneq R_n)$$

Let $e \subseteq \omega \times \mathbb{R} \times \omega_1$ be the finite set from which f is defined. Let n be so that $\{n\} \times \mathbb{R} \times \omega_1 \cap e = \emptyset$. Find $r, s \in \mathbb{R}$ and $p \leq p_0$ so that

$$p \Vdash X_{nr} \in f(R_n) \wedge X_{ns} \notin f(R_n)$$

Pick $\beta \in \omega_1$ so that for all $\alpha \geq \beta$ and all t, γ, $(n, t, \alpha, \gamma) \notin dom(p)$. Let ρ be the reflection of $\mathbb{R}$ about $\frac{r+s}{2}$. The collections $[0, \beta]$, and $(\beta, 2\beta]$ are order isomorphic, say by ϕ. Define a permutation π of $\omega \times \mathbb{R} \times \omega_1$ by

$$\pi(m,t,\alpha) = \begin{cases} (m,t,\alpha) & m \neq n \\ (n, \rho(t), \phi(\alpha)) & m = n, \alpha \in [0,\beta] \\ (n, \rho(t), \phi^{-1}(\alpha)) & m = n, \alpha \in (\beta, 2\beta] \\ (n, \rho(t), \alpha) & m = n, \alpha > 2\beta \end{cases}$$

We can extend π to all of P by $\pi(p)(\pi(a_0, a_1, a_2), a_3) = p(a)$ and to V^P from there the standard way. This extension has the following properties:

- $\pi(\hat{f}) = \hat{f}$
- $\pi(\hat{X}_{nt}) = \hat{X}_{n\rho(t)}$
- $\pi(p)$ and p are compatible elements of P.

The first point follows as π fixes e. The second follows by definition. We now prove the third point. Let $a \in \omega \times \mathbb{R} \times \omega_1 \times \omega_1$. We will show that $a_0 \neq n$. BWOC suppose that $a \in \mathrm{dom}(p) \cap \mathrm{dom}(\pi(p))$ and $a_0 = n$. Let $b \in \mathrm{dom}(p)$ be so that $a = \pi(b)$. Then $a_0 = b_0 = n$, $a_1 = \rho(b_1)$, and $a_2 = \phi(b_2)$ as $b_2 < \beta$. But $\phi(b_2) > \beta$ and as $a \in \mathrm{dom}(p)$, $a_2 < \beta$. This is a contradiction. Suppose that $a \in \mathrm{dom}(p) \cap \mathrm{dom}(\pi(p))$. Since, $a_0 \neq n$ we have that$\pi(p)(a) = p(a)$. Now notice that

$$\pi(p) \Vdash X_{ns} \in f(R_n) \wedge X_{nr} \notin f(R_n)$$

This is a contradiction as p and $\pi(p)$ are compatible. □

4 The Main Result

Proposition 1. *$N \models$ (X is not paracompact).*

Proof. Define an open cover $\mathcal{U}$ on X as follows:

$$\mathcal{U} = \{B(x, \varepsilon) : x \in R_n, \varepsilon \in \mathbb{R}^+, n \in \omega\}$$

BWOC suppose $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. Define S by

$$x \in S \iff \forall V \in \mathcal{V}(x \notin \bar{V} - V)$$

For $n \in \omega$, let $S_n = R_n \cap S$. These S_n are non-empty proper subsets of R_n. Let $x \in R_n$ and W be a neighborhood of x so that W meets only $V_0, \cdots, V_k$. Define a finite set F inductively by

$$i \in F \iff W \cap V_i \cap \bigcap\{V_j : j < i \wedge j \in F\} \neq \emptyset$$

Let $O = W \cap \bigcap_{i \in F} V_i$. For $V \in \mathcal{V}$, $V \cap O \neq \emptyset$ iff $V = V_i$ for some $i \leq k$ and $(\bar{V} - V) \cap O = \emptyset$ for all $V \in \mathcal{V}$. So $\emptyset \neq O \subseteq S_n$. But $S_n \subsetneq R_n$ for all n, as given a $V \in \mathcal{V}$ which meets R_n, we can use the connectedness of R_n to find a $z \in \bar{V} - V$. This is a contradiction. □

Theorem 3. *It is consistent with ZF that there is a metric space which is not paracompact.*

References

[1] C. Good, I.J. Tree, and W.S. Watson. "On Stone's Thoeorem and the Axiom of Choice," Proc. Amer. Math. Soc. **126** (1998), 1211 - 1218.

[2] Rudin M.E. "A new proof that metric spaces are paracompact," Proc. Amer. Math. Soc. **20** (1969), 603. MR **38**:5170