The Independence of Choice

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Our goal is to create a forcing-like construction so that starting with a model of ZFC, we can create a model of ZF+¬C. Standard forcing is a bit too rigid, as we have seen that it preserves choice. We begin by changing perspectives on the forcing construction.

1 Forcing via Boolean Algebras

1.1 Boolean Valued Models

Definition 1. A Boolean algebra is a 6-tuple $\langle B, +, \cdot, -, 0, 1\rangle$ where B is a set, $+, \cdot$ are binary operation and $-$ is a unary operation s.t. $\forall a, b, c \in B$

B1. $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

B2. $a + b = b + a$ and $a \cdot b = b \cdot a$

B3. $a + (b \cdot c) = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

B4. $a + a = a$ and $a \cdot a = a$

B5. $-(a + b) = (-a) \cdot (-b)$ and $-(a \cdot b) = (-a) + (-b)$

B6. $a + (-a) = 1$ and $a \cdot (-a) = 0$

B7. $-(-a) = a$

We say that $a \leq b$ iff $a = b \cdot a$ (iff $a + b = b$). This is a partial order. We also define the binary operation $\to$ by $a \to b = -a + b$.

Definition 2. A Boolean algebra B is complete iff every subset of B has a supremum and infimum with respect to $\leq$. For $A \subseteq B$, we then can define $\sum \{a : a \in A\} = \sup(A)$ and $\prod \{a : a \in A\} = \inf(A)$.

For the following, let M be a model of ZFC and $B \in M$ be a fixed complete (wrt M) Boolean algebra. We will work inside of M for the rest of this section.

Definition 3. We define a class M^B, a Boolean valued model, recursively:

- $M^B_0 = \emptyset$
- $M^B_{\alpha+1} = \{x : x \text{ is a function } \wedge \text{dom}(x) \subseteq M^B_\alpha \wedge \text{ran}(x) \subseteq B\}$
- $M^B_\delta = \bigcup_{\alpha<\delta} M^B_\alpha$ if $\lim(\delta)$.

Finally we set $M^B = \bigcup_{\alpha \in \text{On}} M^B_\alpha$.

Lemma 1. *There is a natural embedding from M to M^B.*

Proof. We define $\hat{}: M \to M^B$ by recursion on $\in$.

- $\hat{\emptyset} = \emptyset$
- $\hat{x} = \{(\hat{y}, 1) : y \in x\}$

This map is clearly 1-1 and if $y \in x$, then $\hat{x}(\hat{y}) = 1$. □

Definition 4. For $x \in M^B$, let $\rho(x) = \min\left\{\alpha : x \in M^B_{\alpha+1}\right\}$.

Definition 5. For every formula $\varphi(x_1, \cdots, x_n)$ with variables in M^B we will define its Boolean value, $[\![\varphi]\!] \in B$. We proceed recursively both by formula and by ρ:

- $[\![x \in y]\!] = \sum_{x \in \mathrm{dom}(y)} (y(z) \cdot [\![z = x]\!])$
- $[\![x = y]\!] = \prod_{z \in \mathrm{dom}(x)} (x(z) \to [\![z \in y]\!]) \cdot \prod_{z \in \mathrm{dom}(y)} (y(z) \to [\![z \in x]\!])$
- $[\![\neg\varphi]\!] = -[\![\varphi]\!]$
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cdot [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!]$
- $[\![\forall x \varphi]\!] = \prod\left\{[\![\varphi(x)]\!] : x \in M^B\right\}$
- $[\![\exists x \varphi]\!] = \sum\left\{[\![\varphi(x)]\!] : x \in M^B\right\}$

Lemma 2. *The following are true:*

- $[\![x = x]\!] = 1$
- $x(y) \le [\![y \in x]\!]$
- $[\![x = y]\!] = [\![y = x]\!]$
- $[\![x = y]\!] \cdot [\![y = x]\!] \le [\![x = z]\!]$
- $[\![x = x_1]\!] \cdot [\![x \in y]\!] \le [\![x_1 \in y]\!]$
- $[\![x = x_1]\!] \cdot [\![y \in x]\!] \le [\![y \in x_1]\!]$

Now let φ be a formula. Then

- $[\![x = y]\!] \cdot [\![\varphi(x)]\!] \le [\![\varphi(y)]\!]$
- $[\![\exists y \in x \varphi(y)]\!] = \sum\{(x(y) \cdot [\![\varphi(x)]\!]) : y \in dom(x)\}$
- $[\![\forall y \in x \varphi(y)]\!] = \prod\{(x(y) \to [\![\varphi(x)]\!]) : y \in dom(x)\}$

Definition 6. Let φ be a formula with variables in M^B. Then φ is valid in M^B iff $[\![\varphi]\!] = 1$.

Theorem 1 (Fundamental Theorem of Boolean-Valued Models)**.** *The following are true:*

1. *Every axiom of predicate logic is valid in M^B. The rules of inference of the predicate logic, if applied to formulas valid in M^B result in formulas valid in M^B.*
2. *Every axiom of ZFC is valid in M^B. Consequently, every provable statement in ZFC is valid in M^B.*

1.2 Generic Extensions

Definition 7. A subset $A \subseteq B$ is called a partition of B if $\sum\{a : a \in A\} = 1$ and $a \cdot a' = 0$ for all $a \neq a'$ in A.

Definition 8. A set $U \subseteq B$ is an ultrafilter on B if

U1. $0 \notin U$ and $U \neq \emptyset$

U1. If $a, b \in U$, then $a \cdot b \in U$

U3. If $a \in U$ and $b \geq a$, then $b \in U$

U4. For all $a \in B$, either $a \in U$ or $-a \in U$

Definition 9. A set $G \subseteq B$, G not necessarily in M, is an M-generic ultrafilter on B if

G1. G is an ultrafilter on B, and

G2. If $A \subseteq G$ and $A \in M$, then $\prod\{a : a \in A\} \in G$, or

G2'. If $A \in M$ is a partition of B, then there is a unique $a \in A$ so that $a \in G$.

Fix an M-generic ultrafilter G on B.

Definition 10. We define the interpretation map, $i_G : M^B \to V$, of M^B by G recursively on $\rho(x)$:

- $i_G(\emptyset) = \emptyset$
- $i_G(x) = \{i_G(y) : x(y) \in G\}$.

The generic extension of M by G is the range of i_G. $M[G] = \left\{i_G(x) : x \in M^B\right\}$.

We will simply write i_G as i since there is no chance for confusion.

Lemma 3. *For each $x \in M$, $i(\hat{x}) = x$. So $M \subseteq M[G]$. Also $G \in M[G]$.*

Proof. We proceed by induction of $\in$. $i(\emptyset) = \emptyset$. Also, if $i(\hat{y}) = y$ for all $y \in x$, then as $\mathrm{dom}(\hat{x}) = \{\hat{y} : y \in x\}$,

$$i(\hat{x}) = \{i(\hat{y}) : \hat{x}(\hat{y}) \in G\} = \{y : \hat{x}(\hat{y}) \in G\} = \{y : y \in x\} = x$$

We now define the canonical generic ultrafilter $\underline{G} \in M^B$ as follows. $\mathrm{dom}\left(\underline{G}\right) = \{\hat{u} : u \in B\}$ and $\underline{G}(\hat{a}) = a$ for all $a \in B$. Then

$$i\left(\underline{G}\right) = \{i(x) : \underline{G}(x) \in G\} = \{i(\hat{x}) : \underline{G}(\hat{x}) \in G\} = \{x : \underline{G}(\hat{x}) \in G\} = \{x : x \in G\} = G$$

This completes the proof. $\square$

Definition 11. If $x \in M[G]$, we say that $\underline{x} \in M^B$ is a name for x if $i\left(\underline{x}\right) = x$.

Lemma 4. *If $\underline{x}, \underline{y}$ are names for x, y, then $x \in y$ iff $[\![\underline{x} \in \underline{y}]\!] \in G$ and $x = y$ iff $[\![\underline{x} = \underline{y}]\!] \in G$.*

Proof. We proceed by induction on $(\rho(x), \rho(y))$, proving both claims simultaneously.

Suppose that $[\![\underline{x} \in \underline{y}]\!] \in G$. Then

$$\sum\left\{\underline{y}(z) \cdot [\![z = \underline{x}]\!] : z \in \mathrm{dom}\left(\underline{y}\right)\right\} \in G$$

Then as G is generic, there is a $z \in \mathrm{dom}\left(\underline{y}\right)$ such that $\underline{y}(z) \cdot [\![z = x]\!] \in G$. Thus $\underline{y}(z) \in G$ and $[\![z = \underline{x}]\!] \in G$. Hence $i(z) \in i\left(\underline{y}\right) = y$. Now by induction, as z is name for $i(z)$ and $\underline{x}$ is a name for x, $i(z) = x$. So $x \in y$. Conversely suppose that $x \in y$. Then by definition,

$$[\![\underline{x} \in \underline{y}]\!] = \sum\left\{\underline{y}(z) \cdot [\![z = \underline{x}]\!] : z \in \mathrm{dom}\left(\underline{y}\right)\right\}$$

So it suffices to show that for some $z \in \operatorname{dom}\left(\underline{y}\right)$, that $\underline{y}(z) \cdot [\![z = \underline{x}]\!] \in G$. As $x \in y$, $i\left(\underline{x}\right) \in i\left(\underline{y}\right)$. So there is some z with $\underline{y}(z) \in G$ so that $i(\underline{x}) = i(z)$. By induction, $[\![z = \underline{x}]\!] \in G$. We are done as G is generic.

The $x = y$ iff $[\![\underline{x} = \underline{y}]\!] \in G$ part is proved similarly. $\square$

Theorem 2. *Let φ be a formula. If $\underline{x_1}, \cdots, \underline{x_n}$ are names for $x_1, \cdots, x_n \in M[G]$, then*

$$M[G] \models \varphi\left(x_1, \cdots, x_n\right) \iff [\![\varphi\left(\underline{x_1}, \cdots, \underline{x_n}\right)]\!] \in G$$

Proof. We proceed by induction on the complexity of φ. The previous lemma covers the base cases. Say $\varphi = \psi \wedge \rho$. Then

$$M[G] \models \varphi \iff M[G] \models \psi \wedge M[G] \models \rho \iff [\![\psi]\!], [\![\rho]\!] \in G \iff [\![\psi]\!] \cdot [\![\rho]\!] \in G \iff [\![\varphi]\!] \in G$$

Say $\varphi = \neg\psi$. Then

$$M[G] \models \varphi \iff \neg(M[G] \models \psi) \iff \neg([\![\psi]\!] \in G) \iff -[\![\psi]\!] \in G \iff [\![\varphi]\!] \in G$$

Finally suppose that $\varphi = \exists x\psi$. Then $[\![\varphi]\!] \in G$ iff $\sum\left\{[\![\psi(x)]\!] : x \in M^B\right\} \in G$. This is true iff there is an $x \in M^B$ so that $[\![\psi(x)]\!] \in G$. By induction, there is an x so that $[\![\psi(x)]\!] \in G$ iff there is an x so that $M[G] \models \psi(x)$. But there is an x such that $M[G] \models \psi(x)$ iff $M[G] \models \varphi(x)$. $\square$

Corollary 1. *$M[G]$ is a model of ZFC and $M[G]$ is the least model of ZFC extending M which contains G.*

Proof. As we saw earlier that all the axioms of ZFC are valid in M^B, it follows that if φ is an axiom of ZFC, then $[\![\varphi]\!] \in G$ and so $M[G] \models \varphi$.

Now suppose that $M \subseteq N$, N is a model of ZF and $G \in N$. Then for all $\alpha \in M$, $M^B_\alpha \in N$ and $i_G \restriction_{M^B_\alpha} \in N$. Then $M^B \subseteq N$ and $M[G] \subseteq N$. $\square$

1.3 The Relationship to Forcing via Partial Orders

Proposition 1. *Let P be dense in $B \setminus \{0\}$. If G is a generic ulatrafilter on B, then $G' = G \cap P$ is M-generic for P. Conversely, if G' is M-generic for P, then $G = \{u \in B : \exists p \in G'(p \leq u)\}$ is a generic ultrafilter on B.*

Proof. First suppose that G is a generic ulatrafilter on B. Suppose that $p \in G'$ and $q \in P$ with $p \leq q$. Then $q \in G$, so $q \in G'$. Let $p, q \in G'$. Then $p \cdot q \leq p, q$, and as P is dense, there is an $r \in P$ so that $r \leq p \cdot q$. So $p \parallel q$. Finally let $D \subseteq P$ be dense. We need to show that $D \cap G \neq \emptyset$. Let $p \in G'$. We define a sequence p_α recursively. Let $p_0 \leq p$ so that $p_0 \in D$. Suppose p_α has been defined and $p_\alpha \in D$. If $p_\alpha \in G$, we are done. Otherwise, $-p_\alpha \in G$ and we let $p_{\alpha+1} \leq -p_\alpha$ be so that $p_{\alpha+1} \in D$. When δ is a limit we have that

$$\prod_{\alpha<\delta} \{-p_\alpha\} \in G$$

as $-p_\alpha \in G$ for each α and G is a generic ultrafilter. Let $p_\delta \in D$ with $p_\delta \leq \prod_{\alpha<\delta}\{-p_\alpha\}$. Now this process has to terminate before $\min\{|G|^+, |D|^+\}$, as otherwise we will deny either ultraness of G or the denseness of D. Thus if the process halts at α, $p_\alpha \in G \cap D$. So G' is M-generic for P.

Now suppose that G_1 is M-generic for P. Clearly, $0 \notin G$ and $G \neq \emptyset$. Let $u, v \in G$. Then there are $p, q \in G'$ so that $p \leq u$ and $q \leq v$. Then $\exists r \in G'$ so that $r \leq p, q$. So $r \leq u, v$ and thus $r \leq u \cdot v$. So $u \cdot v \in G$. Let $u \in G$ and $u \leq v$. Then there is a $p \in G'$ so that $p \leq u$. Thus $p \leq v$, so $v \in G$. Suppose that $u \in B$. Let $D = \{p \in P : p \leq u \wedge p \leq -u\}$. We claim that D is dense. Let $p \in P$. Then $p \cdot u \leq p, u$ and $p \cdot (-u) \leq p, -u$. Since $p \neq 0$, either $p \cdot u \neq 0$ or $p \cdot (-u) \neq 0$. WLOG say $p \cdot u \neq 0$. Then let $q \in P$ be such that $q \leq p \cdot u$. Thus $q \leq u$, so $q \in D$. So as G' is M-generic, there is a $p \in G' \cap D$. So either $u \in G$ or $-u \in G$. Finally let $A \subseteq B$ be a partition. Let

$$A' = \{p \in P : \exists a \in A'(p \leq a)\}$$

We claim that A' is dense. Let $p \in P$. Then as $\sum A = 1$, there is an $a \in A$ so that $p \cdot a \neq 0$. Let $q \in P$ be such that $q \leq p \cdot a$. Then $q \leq a$ and thus $q \in A'$. So A' is dense. Let $p \in G' \cap A'$. Then for some $a \in A$, $p \leq a$. This a is in $G \cap A$. $\square$

Corollary 2. *With G and G' as above, $M[G'] = M[G]$.*

Now let P be a partial order.

Definition 12. We can topologize P by considering the topology generated by the open sets

$$[p] = \{q \in P : q \leq p\}$$

Let $\mathrm{RO}(P)$ be the collection of regularly open sets$\left(\overline{U}^\circ = U\right)$ in this topology on P.

Lemma 5. *$\langle RO(P), +, \cdot, -, \emptyset, P\rangle$ is a complete Boolean algebra, where $U+V = \overline{U \cup V}^\circ$, $U \cdot V = U \cap V$ and $-U = (P \setminus U)^\circ$.*

Definition 13. We define an embedding $e : P \to \mathrm{RO}(P)$ by $e(p) = \overline{[p]}^\circ$.

Lemma 6. *If $p \leq q$, then $e(P) \subseteq e(Q)$. $p \parallel q$ iff $e(P) \cap e(Q) \neq \emptyset$. $e''P$ is dense in $RO(P) \setminus \emptyset$.*

Proof. Suppose that $p \leq q$. Then $[p] \subseteq [q]$, so $e(p) \subseteq e(q)$.

Now suppose that $p \parallel q$. Then $\exists r \in P$ so that $r \leq p, q$, so by what we just showed, $[r] \subseteq [p], [q]$. So certainly $e(P) \cap e(Q) \neq \emptyset$. Now suppose that $p \perp q$ and by way of contradiction suppose that $e(p) \cap e(q) \neq \emptyset$. Let $r \in e(p) \cap e(q)$. Then $[r] \subseteq \overline{[p]}, \overline{[q]}$. Thus there is an $s \leq r$ so that $s \in [p]$ and there is a $t \leq s$ so that $t \in [q]$. Then $t \leq p, q$.

Let $U \in \mathrm{RO}(P)$ with $U \neq \emptyset$. Then we can find a p so that $[p] \subseteq U$. Then applying closures and interiors we get that $e(p) \subseteq U$ as U is regularly open. Then $e(P) \cap U = e(P)$, so $e(P) \leq U$. $\square$

Theorem 3. *If G is a generic ultrafilter on $RO(P)$, then $G_1 = e^{-1}(G)$ is M-generic for P. If $G_1 \subseteq P$ is M-generic for P, then $G = \{U \in RO(P) : \exists p \in G_1 (e(p) \leq U)\}$ us a generic ultrafilter.*

Corollary 3. *With G and G_1 as above, $M[G] = M[G_1]$.*

Proposition 2. *Let φ be a formula, $p \in P$ and $\underline{x_1}, \cdots, \underline{x_n} \in M^{RO(P)}$. One can define $\Vdash'$ by $p \Vdash' \varphi\left(\underline{x_1}, \cdots, \underline{x_n}\right)$ iff $e(p) \leq \left[\!\left[\varphi\left(\underline{x_1}, \cdots, \underline{x_n}\right)\right]\!\right]$. Then $\Vdash'$ has all the properties of $\Vdash^*$. Conversely suppose that $P \subseteq B$ is dense. Then for $\tau_1, \cdots, \tau_n \in M^P$, we can define a boolean value of a formula φ by*

$$[\![\varphi(\tau_1, \cdots, \tau_n)]\!]' = \sum \{p \in P : p \Vdash \varphi(\tau_1, \cdots, \tau_n)\}$$

It can be shown that $p \Vdash \varphi(\tau_1, \cdots, \tau_n)$ iff $e(p) \leq [\![\varphi(\tau_1, \cdots, \tau_n)]\!]'$ and that $[\![\cdot]\!]'$ has the same properties as $[\![\cdot]\!]$.

2 Symmetric Submodels

Let M be a model of ZFC and let $B \in M$ be a complete Boolean algebra with respect to M.

Definition 14. Let π be an automorphism of B. Then we can extend π to M^B recursively as follows:

- $\pi(\emptyset) = \emptyset$
- $\mathrm{dom}(\pi(x)) = \pi''\mathrm{dom}(x)$ and $\pi(x)(\pi(y)) = \pi(x(y))$

Note that π is 1-1 and $\pi(\hat{x}) = \hat{x}$ for all $x \in M$.

Proposition 3. *$\pi : M^B \to M^B$ is 1-1 and $\pi(\hat{x}) = \hat{x}$ for all $x \in M$.*

Lemma 7. *Let $\varphi(x_1, \cdots, x_n)$ be a formula with variables in M^B. Then*

$$[\![\varphi(\pi(x_1), \cdots, \pi(x_n))]\!] = \pi([\![\varphi(x_1, \cdots, x_n)]\!])$$

Proof. We proceed inductively. By definition and induction on $(\rho(x), \rho(y))$,

$$\begin{aligned}
[\![\pi(x) \in \pi(y)]\!] &= \sum \{\pi(y)(z) \cdot [\![z = \pi(x)]\!] : z \in \mathrm{dom}(\pi(x))\} \\
&= \sum \{\pi(y)(\pi(z)) \cdot [\![\pi(z) = \pi(x)]\!] : z \in \mathrm{dom}(x)\} \\
&= \sum \{\pi(y(z)) \cdot [\![\pi(z) = \pi(x)]\!] : z \in \mathrm{dom}(x)\} \\
&= \sum \{\pi(y(z)) \cdot \pi([\![z = x]\!]) : z \in \mathrm{dom}(x)\} \\
&= \pi\left(\sum \{y(z) \cdot [\![z = x]\!] : z \in \mathrm{dom}(x)\}\right) = \pi([\![x \in y]\!])
\end{aligned}$$

The case for $x = y$ follows similarly. Note that we again are simultaneously handling ε and $=$. The connectives also follow by an easy induction. For example:

$$\begin{aligned}
[\![\varphi(\pi(x_1), \cdots, \pi(x_n)) \vee \psi(\pi(x_1), \cdots, \pi(x_n))]\!] &= [\![\varphi(\pi(x_1), \cdots, \pi(x_n))]\!] + [\![\psi(\pi(x_1), \cdots, \pi(x_n))]\!] \\
&= \pi [\![\varphi(x_1, \cdots, x_n)]\!] + \pi [\![\psi(x_1, \cdots, x_n)]\!] \\
&= \pi([\![\varphi(x_1, \cdots, x_n)]\!] + [\![\psi(x_1, \cdots, x_n)]\!]) \\
&= \pi [\![\varphi(x_1, \cdots, x_n) \vee \psi(x_1, \cdots, x_n)]\!]
\end{aligned}$$

The quantifiers follow easily as well. □

Let $\mathcal{G}$ be a group of automorphisms of B.

Definition 15. For each $x \in M^B$, let $\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}$.

Proposition 4. *For each $x \in M^B$, $sym_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}$. Also, for each $x \in M$, $sym_{\mathcal{G}}(\hat{x}) = \mathcal{G}$.*

Proposition 5. *If $x \in M^B$ and $\pi \in \mathcal{G}$, then $sym_{\mathcal{G}}(\pi(x)) = \pi sym_{\mathcal{G}}(x)\pi^{-1}$.*

Proof. Let $\tau \in \mathrm{sym}_{\mathcal{G}}(x)$. Then

$$\pi \circ \tau \circ \pi^{-1}(\pi(x)) = \pi(\tau(x)) = \pi(x)$$

So $\tau \in \mathrm{sym}_{\mathcal{G}}(\pi(x))$. Conversely, let $\tau \in \mathrm{sym}_{\mathcal{G}}(\pi(x))$. Then $\pi^{-1} \circ \tau \circ \pi \in \mathrm{sym}_{\mathcal{G}}(x)$ and

$$\pi \circ (\pi^{-1} \circ \tau \circ \pi) \circ \pi^{-1} = \tau$$

So $\tau \in \pi \mathrm{sym}_{\mathcal{G}}(x)\pi^{-1}$. □

Definition 16. Let $\mathcal{F}$ be a non-empty collection of subgroups of $\mathcal{G}$. We say that $\mathcal{F}$ is a normal filter iff for all subgroup H, K of $\mathcal{G}$,

- If $K \in \mathcal{F}$ and $K \subseteq H$, then $H \in \mathcal{F}$
- If $H, K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$
- If $\pi \in G$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$

Fix a normal filter $\mathcal{F}$ for $\mathcal{G}$.

Definition 17. We say that $x \in M^B$ is symmetric if $\text{sym}_{\mathcal{G}}(x) \in \mathcal{F}$. We define the class $\text{HS} \subseteq M^B$ of hereditarily symmetric names by recursion:

- $\emptyset \in HS$
- If $\text{dom}(x) \subseteq \text{HS}$ and x is symmetric, then $x \in \text{HS}$.

Note that by the above proposition we have that for each $x \in M$, $\hat{x} \in \text{HS}$.

Proposition 6. *If $x \in HS$ and $\pi \in \mathcal{G}$, then $\pi(x) \in HS$.*

Proof. The fact that $\text{dom}(\pi(x)) \subseteq \text{HS}$ follows by induction on $\rho(x)$. Now if x is symmetric, then $\text{sym}_{\mathcal{G}}(x) \in \mathcal{F}$, so $\pi \text{sym}_{\mathcal{G}}(x) \pi^{-1} \in \mathcal{F}$. $\square$

Let G be an M-generic ultrafilter of B and $i = i_G$.

Definition 18. Define $N = \{i(x) : x \in \text{HS}\}$. Note that $M \subseteq N \subseteq M[G]$.

We introduce an interesting way of showing a transitive class is a model of ZF, which we will apply to N.

Definition 19. A transitive class T is said to be almost universal iff $\forall x \subseteq T \exists y \in T (x \subseteq y)$. We also define the eight Godel operations:

- $F_1(x, y) = \{x, y\}$
- $F_2(x, y) = x \setminus y$
- $F_3(x, y) = x \times y$
- $F_4(x) = \text{dom}(x)$
- $F_5(x) = \in \cap x^2$
- $F_6(x) = \{(a, b, c) : (b, c, a) \in x\}$
- $F_7(x) = \{(a, b, c) : (c, b, a) \in x\}$
- $F_8(x) = \{(a, b, c) : (a, c, b) \in x\}$

Theorem 4. *If a class T is transitive, almost universal and closed under the Godel operations, then T is a model of ZF.*

Theorem 5. *N is a model of ZF.*

Proof. First note that N is transitive, as if $x \in HS$ then $\text{dom}(x) \subseteq HS$. Now let $x, y \in M^B$. Then we can define $z_i \in M^B$ so that $[\![z_i = F_i(x, y)]\!] = 1$, $\text{sym}_{\mathcal{G}}(x) \cap \text{sym}_{\mathcal{G}}(y) \subseteq \text{sym}_{\mathcal{G}}(z_i)$ and $z_i \in \text{HS}$ when $x, y \in \text{HS}$. The definitions are as follows:

$\text{dom}(z_1) = \{x, y\}$ and $\text{ran}(z_1) = 1$	$\text{dom}(z_2) = \{a : a \in x \setminus y\}$ and $\text{ran}(z_2) = 1$
$\text{dom}(z_3) = \{a \times b : a \in x \wedge b \in y\}$ and $\text{ran}(z_3) = 1$	$\text{dom}(z_4) = \{a : \exists b((a, b) \in x)\}$ and $\text{ran}(z_4) = 1$
$\text{dom}(z_5) = \{a \times b : a, b \in x \wedge a \in b\}$ and $\text{ran}(z_5) = 1$	$\text{dom}(z_6) = \{a \times b \times c : (b, c, a) \in x\}$ and $\text{ran}(z_6) = 1$
$\text{dom}(z_7) = \{a \times b \times c : (c, b, a) \in x\}$ and $\text{ran}(z_7) = 1$	$\text{dom}(z_8) = \{a \times b \times c : (a, c, b) \in x\}$ and $\text{ran}(z_8) = 1$

Now let $\underline{x}, \underline{y} \in HS$ be names for x, y. Then $z_k \in HS$ and $i(z_k) = i\left(F_k\left(\underline{x}, \underline{y}\right)\right)$ (or drop the $\underline{y}$ as necessary). Thus we have that N is closed under the Godel operations. Finally we will show that N is almost universal. Note that if X is a subset of N, then $X \subseteq i''\left(\text{HS} \cap M_\alpha^B\right)$ for some α. So it suffices to show that each $Y = i''\left(\text{HS} \cap M_\alpha^B\right)$ is in N. We define $\underline{Y}$ as follows: $\text{dom}(\underline{Y}) = \text{HS} \cap M_\alpha^B$ and $\text{ran}(\underline{Y}) = 1$. Then $\underline{Y}$ is a name for Y, so we just need to show that $\underline{Y} \in \text{HS}$. Now $\text{dom}(\underline{Y}) \subseteq \text{HS}$, so we simply need to check that $\underline{Y}$ is symmetric. If $x \in M_\alpha^B$, then $\pi(x) \in M_\alpha^B$ as π preserves rank. Therefore $\pi''\left(\text{HS} \cap M_\alpha^B\right) = \text{HS} \cap M_\alpha^B$. So $\pi(\underline{Y}) = \underline{Y}$ for all $\pi \in \mathcal{G}$. $\square$

3 The Basic Cohen Model

Let $P = \mathrm{FN}(\omega \times \omega, 2)$ and let $B = \mathrm{RO}(P)$. Let G be an M-generic ultrafilter on B.

Definition 20. For each $n \in \omega$ let $x_n = \{m \in \omega : \exists p(e(p) \in G \wedge p(n,m) = 1)\}$. Let $A = \{x_n : n \in \omega\}$. These objects have canonical names: for all $n, m \in \omega$

$$\underline{x_n}(\hat{m}) = u_{n,m} = \sum \{p \in P : p(n,m) = 1\}$$

whereas $\mathrm{dom}(\underline{A}) = \left\{\underline{x_n} : n \in \omega\right\}$ and $\mathrm{ran}(\underline{A}) = 1$.

Lemma 8. *Let $\pi \in S_\infty$. Then π induces an order-preserving bijection of P. Furthermore, this induces an automorphism of B*

Proof. We define π on P as follows. Let $p \in P$. Then $\mathrm{dom}(\pi(p)) = \{(\pi(n), m) : (n,m) \in \mathrm{dom}(p)\}$ and $\pi(p)(\pi(n), m) = p(n,m)$. This is clearly a bijection as π was and is easily order preserving. Now we define π on B. Let $u \in B$. Then $\pi(u) = \sum\{\pi(p) : p \leq u\}$. $\square$

Let $\mathcal{G}$ be the group of automorphisms of B generated by permutations of ω.

Definition 21. For every $e \in \omega^{<\omega}$, let $\mathrm{fix}(e) = \{\pi \in \mathcal{G} : \pi \restriction_e = \mathrm{id}_e\}$. Let $\mathcal{F}$ be the filter generated by $\{\mathrm{fix}(e) : e \in \omega^{<\omega}\}$.

Proposition 7. *$\mathcal{F}$ is a normal filter.*

Proof. We simply need to check normality. It suffices to show normality for the filter base. Let $e \in \omega^{<\omega}$ and $\pi \in \mathcal{G}$. Let $\tau \in \mathrm{fix}(\pi(e))$. Then $\tau \restriction_{\pi(e)} = \mathrm{id}_{\pi(e)}$. So $\pi^{-1}\tau\pi \restriction_e = \mathrm{id}_e$. Thus $\tau \in \pi\mathrm{fix}(e)\pi^{-1}$. Therefore $\mathrm{fix}(\pi(e)) \subseteq \pi\mathrm{fix}(e)\pi^{-1}$ and thus $\pi\mathrm{fix}(e)\pi^{-1} \in \mathcal{F}$. $\square$

Let N be the symmetric model generated by $B, G, \mathcal{G}, \mathcal{F}$.

Proposition 8. *For all n, $\underline{x_n} \in HS$ and $\underline{A} \in HS$.*

Proof. Suppose that $\pi \in \mathcal{G}$ and $n \in \omega$. Then for all m, $\pi(u_{n,m}) = u_{\pi(n),m}$, so $\pi\left(\underline{x_n}\right) = \underline{x_{\pi(n)}}$. Thus $\mathrm{sym}_{\mathcal{G}}\left(\underline{x_n}\right) = \mathrm{fix}\{n\} \in \mathcal{F}$. This suffices to show that $\underline{x_n} \in \mathrm{HS}$ as $\mathrm{dom}\left(\underline{x_n}\right) = \omega \subseteq \mathrm{HS}$. It now follows that $\underline{A} \in \mathrm{HS}$. $\square$

Theorem 6. *In N, the set of all real numbers cannot be well-ordered.*

Proof. We show that A cannot be well-ordered in N. First notice that the reals x_n are pairwise distinct. We will show that $\left[\!\left[\underline{x_i} = \underline{x_j}\right]\!\right] = 0$ for $i \neq j$. Towards a contradiction, assume that there is a $p \in P$ such that $p \Vdash \underline{x_i} = \underline{x_j}$. Choose m least so that $(i,m),(j,m) \notin \mathrm{dom}(p)$. Let $p \subseteq q$ be so that $q(i,m) = 1$ and $q(j,m) = 0$. Then $q \Vdash \hat{m} \in \underline{x_i}$ and $q \Vdash \hat{m} \notin \underline{x_j}$. So $q \Vdash \underline{x_i} \neq \underline{x_j}$. But $q \leq p$, so this is a contradiction.

We will now show that there is no bijection between ω and A. Towards a contradiction, suppose that f is such a function. Let $\underline{f} \in \mathrm{HS}$ be a name for f. Then for some $p_0 \in G$, $p_0 \Vdash \underline{f} : \hat{\omega} \twoheadrightarrow \underline{A}$. Let $e \in \omega^{<\omega}$ be so that $\mathrm{fix}(e) \subseteq \mathrm{sym}_{\mathcal{G}}\left(\underline{f}\right)$. Then there are $i \in \omega$, $p \leq p_0$ and $n \notin e$ so that $p \Vdash \underline{f}\left(\hat{i}\right) = \underline{x_n}$. Now let n' be least such that $n' \notin e$ and $(n', m) \notin \mathrm{dom}(p)$ for any m. Let $\pi = (n, n') \in S_\infty$. Then $\pi(p)$ and p are compatible, $\pi \in \mathrm{fix}(e)$ and $\pi(n) \neq n$. Thus $\pi\left(\underline{f}\right) = \underline{f}$. Now $\pi(p) \Vdash \left(\pi\left(\underline{f}\right)\right)\left(\pi\left(\hat{i}\right)\right) = \pi\left(\underline{x_n}\right)$. Then $\pi(p) \Vdash \left(\underline{f}\right)\left(\hat{i}\right) = \underline{x_{\pi(n)}}$. Set $q = p \cup \pi(p)$. Then

$$q \Vdash \underline{f}\left(\hat{i}\right) = \underline{x_n} \wedge \left(\underline{f}\right)\left(\hat{i}\right) = \underline{x_{\pi(n)}}$$

and as $\pi(n) \neq n$, $\left[\!\left[\underline{x_n} = \underline{x_{\pi(n)}}\right]\!\right] = 0$. Therefore q forces that f is not a function. But $q \leq p \leq p_0$, which is a contradiction. $\square$

Corollary 4. *AC is independent of ZF.*