An Introduction to Non Standard Topology

Jared Holshouser

September 12, 2014

1 Superstructures and Monomorphisms

Let $\mathbb{N} \subseteq X$.

Definition 1. Let $V_0(X) = X$ and $V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X))$. The superstructure over X is $V(X) = \bigcup_{n\in\mathbb{N}} V_n(X)$.

Definition 2. The language $\mathcal{L}_X$ for V_X is $\mathcal{L}_X = \{\varepsilon\}$. Note that $=$ is interpreted as set equality on $V(X) \setminus X$.

A formula of $\mathcal{L}_X$ is built up from

- Atomic formulas: $x \in y$, $x = y$, $\langle x_1, \cdots, x_n\rangle \in y$, $\langle x_1, \cdots, x_n\rangle = y$, $\langle\langle x_1, \cdots, x_n\rangle, x\rangle \in y$ and $\langle\langle x_1, \cdots, x_n\rangle, x\rangle = y$,
- Logical Connectives, and
- $\forall x \in y\varphi$ and $\exists x \in y\varphi$.

Example 1. An example of a formula in $\mathcal{L}_X$ is

$$\forall x \in b(x \in y \wedge \langle y, a\rangle = b)$$

Let $\mathbb{N} \subseteq Y$ and let $* : V(X) \to V(Y)$ be an injection.

Definition 3. If φ is a formula in $\mathcal{L}_X$, the $*$-transform ${}^*\varphi$ of φ is the formula in $\mathcal{L}_Y$ obtained by replacing each constant c occuring in φ with ${}^*c$.

Definition 4. $* : V(X) \to V(Y)$ is a monomorphism iff

1. ${}^*\emptyset = \emptyset$
2. If $a \in X$, then ${}^*a \in Y$ and if $n \in \mathbb{N}$, then ${}^*n = n$.
3. If $a \in V_{n+1}(X) \setminus V_n(X)$, then ${}^*a \in V_{n+1}(Y) \setminus V_n(Y)$.
4. If $a \in^* V_n(X)$ for $n \geq 1$ and $b \in a$, then $b \in^* V_{n-1}(X)$. This is called Strictness.
5. For any φ in $\mathcal{L}_X$, φ holds in $V(X)$ iff ${}^*\varphi$ holds in $V(Y)$. This is called the transfer principle.

Theorem 1. *Suppose that $*$ is a monomorphism.*

1. *Let $a, b, a_1, \cdots, a_n \in V(X)$. Then*
   (a) $^*\{a_1, \cdots, a_n\} = \{^*a_1, \cdots, {}^*a_n\}$
   (b) $^*\langle a_1, \cdots, a_n\rangle = \langle ^*a_1, \cdots, {}^*a_n\rangle$
   (c) $a \in b$ *iff* $^*a \in {}^*b$
   (d) $a = b$ *iff* $^*a = {}^*b$
   (e) $a \subseteq b$ *iff* $^*a \subseteq {}^*b$
   (f) $^*\left(\bigcup_{i=1}^n a_i\right) = \bigcup_{i=1}^n {}^*a_i$ *and* $^*\left(\bigcap_{i=1}^n a_i\right) = \bigcap_{i=1}^n {}^*a_i$
   (g) $^*(a_1 \times \cdots \times a_n) = {}^*a_1 \times \cdots \times {}^*a_n$.
2. *If P is a relation on $a_1 \times \cdots \times a_n$, then $^*P$ is a relation on $^*a_1 \times \cdots \times {}^*a_n$. For $n = 2$, $^*(dom(P)) = dom({}^*P)$ and $^*(ran(P)) = ran({}^*P)$.*
3. *If $f : a \to b$, then $^*f : {}^*a \to {}^*b$ and $^*[f(c)] = {}^*f({}^*c)$ for each $c \in a$. Also, f is 1-1 iff $^*f$ is 1-1.*

Proof. All of these result from transferring the appropriate sentence. For instance consider 1a). Here we would transform

$$\forall x \in b\,(x = a_1 \vee \cdots \vee x = a_n) \wedge a_1 \in b \wedge \cdots \wedge a_n \in b$$

For 1f) (with $n = 2$) we would transfer $\forall x \in c(x \in a \vee x \in b)$ and $\forall x \in a(x \in c)$ and $\forall x \in b(x \in c)$. $\square$

Let $\mathcal{U}$ be an ultrafilter on an index set I.

Definition 5. The bounded ultrapower of $V(X)$ is

$$\prod_{\mathcal{U}}^{0} V(X) := \bigcup_{n\in\mathbb{N}} \prod_{\mathcal{U}} [V_n(X) \setminus V_{n-1}(X)]$$

Define $e : V(X) \to \prod_{\mathcal{U}}^0 V(X)$ by $e(a) = [c_a]$ where $c_a(i) = a$ for all $i \in I$.

Definition 6. We define the Mostowski collapse $M : \prod_{\mathcal{U}}^0 V(X) \to V({}^*X)$ as follows: First set

$$^*X = \prod_{\mathcal{U}} X$$

Then for $b \in \prod_{\mathcal{U}} X$, set $M([b]) = [b]$ and for $b \in \prod_{\mathcal{U}} [V_n(X) \setminus V_{n-1}(X)]$, set

$$M([b]) = \left\{ M([a]) : [a] \in \bigcup_{k=1}^{n-1} \prod_{\mathcal{U}} [V_k(X) \setminus V_{k-1}(X)] \wedge [a] \in_{\mathcal{U}} [b] \right\}$$

Lemma 1. *e and M have the following important properties:*

1. *e and M are 1-1.*
2. *e maps X into ${}^*X$ and M maps ${}^*X$ onto ${}^*X$.*
3. *e maps $V_{n+1}(X) \smallsetminus V_n(X)$ into $\prod_{\mathcal{U}} [V_{n+1}(X) \smallsetminus V_n(X)]$ and M maps $\prod_{\mathcal{U}} [V_{n+1}(X) \smallsetminus V_n(X)]$ into $V_{n+1}({}^*X) \smallsetminus V_n({}^*X)$.*
4. *$a \in b$ iff $e(a) \in_{\mathcal{U}} e(b)$ and $[a] \in_{\mathcal{U}} [b]$ iff $M([a]) \in M([b])$.*
5. *$e(X) = [c_X]$ and $M([c_X]) = {}^*X$.*
6. *Let $[a], [b] \in \prod^0_{\mathcal{U}} V(X)$ and put $c_i = \{a_i, b_i\}$ for all $i \in I$. Then $[c] \in \prod^0_{\mathcal{U}} V(X)$ and $M([c]) = \{M([a]), M([b])\}$, where $c(i) = c_i$. We can do the same thing with more terms. We can also replace $\{\}$ with $\langle\rangle$ and $=$ with $\in$. In other words, the atomic formulas of $\mathcal{L}_X$ push through in the obvious way.*
7. *If $[b] \in_{\mathcal{U}} e(a)$ and $a \in V_n(X) \smallsetminus V_{n-1}(X)$, then $[b] \in_{\mathcal{U}} e(V_{n-1}(X))$.*

Proof. We show select non-obvious parts. Since M is the identity on ${}^*X$ it is clearly 1-1 there. Now suppose $[a] \neq [b]$ are not in ${}^*X$. Then set

$$U_a = \{i \in I : \exists v_i \in a_i (v_i \notin b_i)\} \text{ and } U_b = \{i \in I : \exists v_i \in b_i (v_i \notin a_i)\}$$

As $[a] \neq [b]$, either $U_a \in \mathcal{U}$ or $U_b \in \mathcal{U}$. WLOG $U_a \in \mathcal{U}$. Choose $v_i \in a_i \smallsetminus b_i$ for all $i \in U_a$. let $v(i) = v_i$ if $i \in U_a$ and $v(i) = c$ otherwise, for some constant c. Then $M([v]) \in M([a]) \smallsetminus M([b])$.

We prove 6) now. By definition, $M([c]) = \{M([y]) : y_i \in \{a_i, b_i\} \text{ a.e}\}$. If $y_i \in \{a_i, b_i\}$ a.e. let $A = \{i : y_i = a_i\}$ and $B = \{i : y_i = b_i\}$. Then $A \cup B \in \mathcal{U}$ so $A \in \mathcal{U}$ or $B \in \mathcal{U}$. Therefore

$$M([c]) = \{M([y]) : y_i = a_i \text{ a.e}\} \cup \{M([y]) : y_i = b_i \text{ a.e}\} = \{M([a]), M([b])\}$$

□

Theorem 2 (Los's Theorem)**.** *If $\varphi(x_1, \cdots, x_n)$ is a formula in $\mathcal{L}_X$ with $x_1, \cdots, x_n$ free and $[a_1], \cdots, [a_n] \in \prod^0_{\mathcal{U}} V(X)$, then ${}^*\varphi(M([a_1]), \cdots, M([a_n]))$ is true in ${}^*V(X)$ iff*

$$\{i \in I : \varphi(a_1(i), \cdots, a_n(i))\} \in \mathcal{U}$$

Proof. This follows by induction on the formula φ. □

Theorem 3. *Let $* : V(X) \to V({}^*X)$ by $* = M \circ e$. Then $*$ is a monomorphism.*

Proof. The first four properties are true based off of lemma 1. We will show that the transfer principle holds. Let φ be a sentence of $\mathcal{L}_X$. Then by Los's theorem, ${}^*\varphi$ is true in $V({}^*X)$ iff $\{i \in I : \varphi\} \in \mathcal{U}$. Since φ is a sentence, it has no free variables, its either true or false for $V(X)$. If its true, then $\{i \in I : \varphi\} = I \in \mathcal{U}$. If its false, then $\{i \in I : \varphi\} = \emptyset \notin \mathcal{U}$. So ${}^*\varphi$ is true iff φ is true. □

2 Enlargements

Definition 7. If $A \in V_n(X) \setminus V_0(X)$ for some $n \geq 1$, we set $\mathcal{P}_F(A) = \{B \subseteq A : B \text{ is finite}\}$. We call ${}^*\mathcal{P}_F(A)$ the hyperfinite subsets of A.

Example 2. Let $j \in^* \mathbb{N}$. Then $J = \{n \in {}^*\mathbb{N} : n \leq j\}$ is a hyperfinite subset of $\mathbb{R}$.

The transfer principle tells us that elementary results that hold for finite sets also hold for hyperfinite sets.

Definition 8. Elements of $V(X)$ and elements of the form ${}^*b$ for some $b \in V(X)$ are called standard. Everything else is non-standard.

Example 3. If $X = \mathbb{R}$, real numbers are standard. If $a < b \in \mathbb{R}$, then $I = \{x \in {}^*\mathbb{R} : a \leq x \leq b\}$ is standard. If $\alpha < \beta \in {}^*\mathbb{R}$ are infinitesimal and $J = \{x \in {}^*\mathbb{R} : \alpha \leq x \leq \beta\}$, then J is non-standard.

Definition 9. $V({}^*X)$ is an enlargement of $V(X)$ iff for all $A \in V(X)$ there is a $B \in {}^*\mathcal{P}_F(A)$ such that ${}^*a \in B$ for all $a \in A$.

Example 4. Let $j \in^* \mathbb{N}$ be infinite. Then $\{n \in {}^*\mathbb{N} : 0 \leq n \leq j\}$ contains every natural number.

We aim to construct an index set and a filter so that $V({}^*X)$ is an enlargement. Let

$$J = \{a \in V(X) : a \neq \emptyset \wedge \text{ a is finite}\}$$

For $a \in J$, let $J_a = \{a \in J : a \subseteq b\}$. Let $\mathcal{F}$ be the filter generated by the J_a. Let $\mathcal{V}$ be an ultrafilter extending $\mathcal{F}$. Note that $\mathcal{F}$ is non-principal.

Theorem 4. *If $V({}^*X)$ is constructed from $V(X)$ using $\mathcal{V}$ and J, then it is an enlargement.*

Proof. Let $A \in V(X)$. Define $\Gamma : J \to \mathcal{P}_F(A)$ by $\Gamma(a) = a \cap A$. Let $B = M([\Gamma])$. Then $B \in {}^*\mathcal{P}_F(A)$. If $x \in A$, then

$$J_{(x)} = \{a \in J : x \in a\} = \{a \in J : x \in a \cap A\} \in \mathcal{V}$$

So $[c_x] \in_{\mathcal{V}} [\Gamma]$, so ${}^*x = M \circ e(x) \in B$. $\square$

Definition 10. A binary relation P is concurrent on $A \subseteq \text{dom}(P)$ iff for each $a \in A^{<\omega}$, there is a $y \in \text{ran}(P)$ so that $\langle a_i, y \rangle \in P$ for all $i < \ell(a)$. P is concurrent iff P is concurrent on $\text{dom}(P)$.

Theorem 5. *The following are equivalent*

1. *$V({}^*X)$ is an enlargement of $V(X)$*
2. *For each concurrent relation $P \in V(X)$ there is a $b \in ran({}^*P)$ so that $\langle {}^*x, b \rangle \in {}^*P$ for all $x \in dom(P)$.*

Proof. We will show that 1) implies 2). Let $B \in {}^*\mathcal{P}_F(\text{dom}(P))$ be so that for each $x \in \text{dom}(P)$, ${}^*x \in B$. Then in $V(X)$

$$\forall w \in \mathcal{P}_F(\text{dom}(P)) \exists y \in \text{ran}(P) \forall x \in w(\langle x, y \rangle \in P)$$

So by transfer, the $*$-transform of this statement is true in $V({}^*X)$. So there is a $b \in \text{ran}({}^*P)$ so that $\langle z, b \rangle \in {}^*P$ for each $z \in B$. $\square$

3 Topology

Let $(X, \mathcal{T})$ be a topological space and assume that $\mathbb{N} \subseteq X$.

We will use the following as a black box for what follows:

Theorem 6. *Set $V_0(X) = X$ and for $n \in \omega$, set $V_{n+1}(X) = \mathcal{P}(V_n(X)) \cup V_n(X)$. The supserstructure of X, $V(X) = \bigcup_{n\in\omega} V_n(X)$. Let ${}^*X = X^J/\mathcal{U}$, where J is an index set and $\mathcal{U}$ is an ultrafilter on J. We have a map $* : V(X) \to V({}^*X)$ which has the following properties:*

1. *$*$ is 1-1, preserves $\in$ and preserves rank.*
2. *$*$ preserves finite set operations.*
3. *(Transfer) A sentence φ holds in $V(X)$ iff ${}^*\varphi$ holds in $V({}^*X)$.*
4. *(Enlargement) If $P \in V(X)$ is a relation so that whenever $x_1, \cdots, x_n \in dom(P)$ there is a $y \in ran(P)$ with $\langle x_i, y\rangle \in P$ for $1 \le i \le n$, then there is a $y \in ran({}^*P)$ so that $\langle {}^*x, y\rangle \in {}^*P$ for all $x \in dom(P)$.*

3.1 Basic Notions

Definition 11. The sets in ${}^*\mathcal{T}$ are called $*$-open sets. The monad of a point $x \in X$ is

$$m(x) = \bigcap \{ {}^*U : U \in \mathcal{N}_x \}$$

$y \in {}^*X$ is said to be near $x \in X$ if $y \in m(x)$. We write $y \simeq x$ or x is a standard part of y. We say $y \in {}^*X$ is near standard if there is an $x \in X$ so that $y \simeq x$.

Example 5. In $\mathbb{R}$, $m(0) = \bigcap_{r\in\mathbb{R}^+} {}^*(-r, r)$ is the set of infinitesimals.

Proposition 1. *For all $x \in X$ there is a $V \in {}^*\mathcal{N}_x$ with $V \subseteq m(x)$.*

Proof. Define P on $\mathcal{N}_x \times \mathcal{N}_x$ by $P(U, V) \iff V \subseteq U$. Then if $U_1, \cdots, U_n \in \mathcal{N}_x$, we can find a $V \in \mathcal{N}_x$ so that $V \subseteq U_1, \cdots, U_n$. Thus by enlargement, there is a $V \in {}^*\mathcal{N}_x$ so that $V \subseteq {}^*U$ for all $U \in \mathcal{N}_x$. So $V \subseteq m(x)$. $\square$

Example 6. Let N be an infinite natural. Then $\left(-\frac{1}{N}, \frac{1}{N}\right) \subseteq m(0)$ and $\left(-\frac{1}{N}, \frac{1}{N}\right) \in {}^*\mathcal{N}_0$

Proposition 2. *Let $A \subseteq X$. Then*

1. *A is open iff $m(x) \subseteq {}^*A$ for each $x \in A$.*
2. *A is closed iff $m(x) \cap {}^*A = \emptyset$ for all $x \in X \setminus A$. (iff whenever $y \in {}^*A$ and $y \simeq x$, then $x \in A$.)*

Proof. It suffices to show 1). First suppose that A is open. Let $x \in A$. Then there is a $U \in \mathcal{N}_x$ so that $U \subseteq A$. Then $m(x) \subseteq {}^*U \subseteq {}^*A$. Now for the converse. Let $x \in A$. Let $V \in {}^*\mathcal{N}_x$ be so that $V \subseteq m(x)$. Then

$$\exists V \in {}^*\mathcal{N}_x \,(V \subseteq {}^*A)$$

So by transfer, $\exists V \in \mathcal{N}_x (V \subseteq A)$. $\square$

Proposition 3. *$x \in \bar{A}$ iff $m(x) \cap {}^*A \neq \emptyset$.*

Proof. First suppose that $x \in \bar{A}$. Then for all $U \in \mathcal{N}_x$, $U \cap A \neq \emptyset$. Define P on $\mathcal{N}_x \times A$ by $P(U, y) \iff y \in U$. Now whenever $U_1, \cdots, U_n \in \mathcal{N}_x$ there is a $y \in A$ so that $y \in U_1 \cap \cdots \cap U_n$. So by enlargement, there is a $y \in {}^*A$ so that $y \in {}^*U$ for all $U \in \mathcal{N}_x$. So $y \in m(x) \cap {}^*A$. Conversely suppose that $y \in m(x) \cap {}^*A$. Let $U \in \mathcal{N}_x$. Then ${}^*U \cap {}^*A \neq \emptyset$. So by transfer, $U \cap A \neq \emptyset$. Thus $x \in \bar{A}$. $\square$

Proposition 4. *X is*

- *T_0 iff for all $x, y \in X$ if $x \simeq y$ and $y \simeq x$, then $x = y$.*
- *T_1 iff for all $x, y \in X$ if $x \simeq y$, then $x = y$.*
- *T_2 iff for all $x, y \in X$ if $x \neq y$, then $m(x) \cap m(y) = \emptyset$.*

Proof. We show the T_2 part. Suppose that X is T_2 and let $x, y \in X$ be so that $x \neq y$. Then there are open sets U and V so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Thus ${}^*U \cap {}^*V = \emptyset$. As $m(x) \subseteq {}^*U$ and $m(y) \subseteq {}^*V$, $m(x) \cap m(y) = \emptyset$. For the converse, let $x, y \in X$ with $x \neq y$. Let U and V be $*$-open so that $U \subseteq m(x)$ and $V \subseteq m(y)$. Then $U \cap V = \emptyset$. So

$$\exists U \in {}^*\mathcal{N}_x \exists V \in {}^*\mathcal{N}_y (U \cap V = \emptyset)$$

Thus by transfer $\exists U \in \mathcal{N}_x \exists V \in \mathcal{N}_y (U \cap V = \emptyset)$. □

Remark 1. Suppose X is T_2. For each $y \in {}^*X$ which is near standard, there is a unique $x \in X$ so that $y \simeq x$. We define $\mathrm{st}(y)$ to be this x.

Proposition 5. *Let X, Y be topological spaces. Then $f : X \to Y$ is continuous iff whenever $x' \simeq x$, ${}^*f(x') \simeq f(x)$.*

Proof. First suppose that f is cont. Then for all $U \in \mathcal{N}_{f(x)}$ there is a $V \in \mathcal{N}_x$ so that $f[V] \subseteq U$. Let $V \in {}^*\mathcal{N}_x$ be so that for all $U \in \mathcal{N}_{f(x)}$, ${}^*f[V] \subseteq {}^*U$. Then ${}^*f[V] \subseteq \bigcap_{U \in \mathcal{N}_{f(x)}} {}^*U = \mu(f(x))$.

Now we prove the converse. Let $x \in X$. Then by hypothesis, ${}^*f[\mu(x)] \subseteq \mu(f(x))$. By way of contradiction suppose that there is some $U \in \mathcal{N}_{f(x)}$ so that for all $V \in \mathcal{N}_x$, $f[V] \nsubseteq U$. Then for all $V_1, \cdots, V_n \in \mathcal{N}_x$ there is a $y \in Y$ so that $y \in f[V_i] \smallsetminus U$. So by enlargement we can find a $y \in^* Y$ so that $y \in {}^*f[{}^*V] \smallsetminus {}^*U$ for all $V \in \mathcal{N}_x$. Thus $y \in {}^*f[\mu(x)] \smallsetminus \mu(f(x))$. This is a contradiction. So f is continuous. □

3.2 Products

Recall the following definition for the product topology.

Definition 12. The product topology of $X = \prod_{i \in I} X_i$ is the weak topology generated by projection maps.

Remark 2. ${}^*X = {}^*\left(\prod_{i \in I} X_i\right) = \prod_{i \in {}^*I} Y_i$ for some Y_i so that for $i \in I$, $Y_i = {}^*X_i$. For $i \notin I$, it is not clear what Y_i is.

Lemma 2. *If $x \in X$, then*

$$m(x) = \{y \in {}^*X : y(i) \in m_i(x(i)) \text{ for } i \in I\}$$

Proof. Let $k(x) = \{y \in {}^*X : y(i) \in m_i(x(i)) \text{ for } \in I\}$. We will first show that $m(x) \subseteq k(x)$. The following is true

$$m(x) \subseteq \bigcap_{i \in I} \bigcap_{U \in \mathcal{N}^i_{x(i)}} \left(\left({}^*\pi_i^{-1}\right)'' U \right) = \bigcap_{i \in I} \left[\left({}^*\pi_i^{-1}\right)'' \left(\bigcap_{U \in \mathcal{N}^i_{x(i)}} {}^*U \right) \right] = k(x)$$

Now we will show that $k(x) \subseteq m(x)$. If

$$x \in V = \left(\pi_{i_1}^{-1}\right)'' (U_{i_1}) \cap \cdots \cap \left(\pi_{i_n}^{-1}\right)'' (U_{i_n})$$

then $k(x) \subseteq {}^*V$ as $k(x) \subseteq \left({}^*\pi_i^{-1}\right)'' ({}^*U_i)$ for all $U_i \in \mathcal{N}^i_{x(i)}$ and $*$ moves across finite intersections. □

Theorem 7. *A product of T_2 spaces is T_2.*

Proof. Say $X = \prod_{i \in I} X_i$. If $x, y \in X$ with $m(x) \cap m(y) \neq \emptyset$, let $z \in m(x) \cap m(y)$. Then for all $i \in I$, $z(i) \in m_i({}^*x(i)) \cap m_i({}^*y(i))$. So $x(i) = y(i)$ for all $i \in I$. So $x = y$. □

3.3 Compactness

Theorem 8 (Robinson's Theorem)**.** *$A \subseteq X$ is compact iff for all $y \in {}^*A$ there is an $x \in A$ with $y \simeq x$.*

Proof. We prove the contrapositive. Suppose that for some $y \in {}^*A$, $y \notin m(x)$ for all $x \in A$. Then for all $x \in A$ there is a $U_x \in \mathcal{N}_x$ so that $y \notin {}^*U_x$. The U_x form a cover of A. Let $U_1, \cdots, U_n$ be a finite subcollection. Then ${}^*A$ is not a subset of ${}^*U_1 \cup \cdots \cup {}^*U_n$. So by transfer, A is not a subset of $U_1, \cdots, U_n$. That is the cover $\{U_x : x \in A\}$ has no finite subcover.

We now prove the inverse. Suppose that A is not compact. Let $\mathcal{U}$ witness this. Define P on $\mathcal{U} \times A$ by $P(U, x) \iff x \notin U$. Then whenever $U_1, \cdots, U_n \in \mathrm{dom}(P)$, there is an $x \in A$ so that $x \notin U_1 \cup \cdots \cup U_n$. So by enlargement there is a $y \in {}^*A$ so that $y \notin {}^*U$ for all $U \in \mathcal{U}$. Now for all $x \in A$ there is a $U \in \mathcal{U}$ so that $m(x) \subseteq {}^*U$. Thus $y \notin m(x)$ for any $x \in A$. $\square$

Proposition 6. *If X is compact and $A \subseteq X$ is closed, then A is compact.*

Proof. Let $y \in {}^*A$. Then there is an $x \in X$ with $y \in m(x)$. As A is closed, $x \in A$. $\square$

Proposition 7. *If X is T_2 and $A \subseteq X$ is compact, then A is closed.*

Proof. Let $x \in \bar{A}$. Then $m(x) \cap {}^*A \neq \emptyset$. So let $y \in m(x) \cap {}^*A$. Then there is a $z \in A$ so that $y \in m(z)$. As X is T_2 it follows that $x = z$. So $x \in A$. $\square$

Proposition 8. *Suppose K is compact, Y is a topological space, and $f : K \to Y$ is continuous and onto. Then Y is compact.*

Proof. Let $y' \in {}^*Y$. Say that ${}^*f(x') = y'$. Then for some $x \in K$, $x' \simeq x$. Since f is continuous, ${}^*f(x') \simeq f(x)$. So $y \simeq f(x) \in Y$. $\square$

Theorem 9 (Tychonoff's Theorem)**.** *A product of compact spaces is compact.*

Proof. Say $X = \prod_{i \in I} X_i$. Let $y \in {}^*X$. Then for each standard $i \in I$, $y(i) \in {}^*X_i$. So $y(i)$ is near some point $x_i \in X_i$ for each $i \in I$. Let $x \in X$ be defined by $x(i) = x_i$. Then $y(i) \in m_i(x(i))$ for each $i \in I$. So $y \in m(x)$. $\square$

Theorem 10 (Alexander's Subbase Theorem)**.** *If $\mathcal{S}$ is a subbase for X and every cover of X by members of $\mathcal{S}$ has a finite subcover, then X is compact.*

Proof. We prove the contrapositive. Suppose that X is not compact. Then there is a $y \in {}^*X$ so that for all $x \in X$ there is a $U_x \in \mathcal{N}_x$ so that $y \notin {}^*U_x$. Now each U_x is a finite intersection of V_i from $\mathcal{S}$, so one of the ${}^*V_i$ must omit y. Thus without loss of generality $U_x \in \mathcal{S}$ for all x. Let $\mathcal{U} = \{U_x : x \in X\}$. Then whenever $U_1, \cdots, U_n \in \mathcal{U}$, $y \notin {}^*U_1 \cup \cdots \cup {}^*U_n$, so ${}^*X$ is not covered by any finite collection of ${}^*U_x$. Thus by transfer, X is not covered by any finite collection of U_x. So $\mathcal{U}$ has no finite subcover and therefore X is not compact. $\square$

Theorem 11 (Heine-Borel Theorem)**.** *Let $K \subseteq \mathbb{R}^n$. Then K is compact iff K is closed and bounded.*

Proof. Suppose K is compact. Then since $\mathbb{R}^n$ is T_2, K is closed. By way of contradiction suppose that K is unbounded. Then

$$\forall n \in \mathbb{N} \exists x \in K (\|x\| > n)$$

Let N be an infinite integer. Then by transfer, there is a $y \in {}^*K$ so that $\|y\| > N$. But then y is not near any $x \in K$. So K is not compact. This is a contradiction. So K is bounded.

Conversely, suppose that K is closed and bounded. Let N be this bound. Let $y \in {}^*K$. By transfer, $\|y\| \leq N$. So y is near standard. Let $x \in \mathbb{R}^n$ be so that $y \simeq x$. Since K is closed, $x \in K$. So K is compact. $\square$