ω_1 is Countably Reasonable

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We assume AD for these notes.

1 Pointclasses

Definition 1.0.1. We say that $\mathbf{\Gamma} \subseteq \mathcal{P}(\mathbb{R})$ is a **pointclass** if whenever $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $A \in \mathbf{\Gamma}$, $f^{-1}[A] \in \mathbf{\Gamma}$.

If $\mathbf{\Gamma}$ is a pointclass, then $\check{\mathbf{\Gamma}} = \{\mathbb{R} - A : A \in \mathbf{\Gamma}\}$; we also set $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

$\mathbf{\Gamma}$ is **selfdual** iff $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$.

Definition 1.0.2. Suppose $\mathbf{\Gamma}$ is a pointclass. $A \subseteq \mathbb{R}$ is **universal** for $\mathbf{\Gamma}$ iff whenever $B \in \mathbf{\Gamma}$, there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ so that $B = f^{-1}[A]$. If A is universal for $\mathbf{\Gamma}$, then $A \notin \check{\mathbf{\Gamma}}$.

Definition 1.0.3. A **regular norm** on a set $A \subseteq \omega^\omega$ is a map ϕ from A onto an ordinal.

Let $\mathbf{\Gamma}$ be a pointclass. A norm $\phi : A \to \mathrm{On}$ is a **$\mathbf{\Gamma}$-norm** iff there are $\mathbf{\Gamma}$,$\check{\mathbf{\Gamma}}$ binary relations $\leq_\phi^\Gamma$, $\leq_\phi^{\check{\Gamma}}$ so that for all $y \in A$,

$$\forall x \left[(x \in A \wedge \phi(x) \leq \phi(y)) \iff x \leq_\phi^\Gamma y \iff x \leq_\phi^{\check{\Gamma}} y\right].$$

A pointclass $\mathbf{\Gamma}$ has the **prewellordering property** (pwo($\mathbf{\Gamma}$)) if every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-norm.

Definition 1.0.4. A prewellorder is a relation $\preceq$ satisfying all of the properties of a well-order except that we allow $a \preceq b$ and $b \preceq a$ when $a \neq b$.

The existence of a regular norm on a set is equivalent to the existence of a prewellorder.

$\preceq$ is a $\mathbf{\Gamma}$-prewellorder iff the derived norm is a $\mathbf{\Gamma}$-norm.

If $\preceq$ is a prewellorder on A, then $|\preceq|$ is the rank of $\preceq$.

Remark 1.0.1. $\mathbf{\Pi}_1^1$ has the prewellordering property.

Definition 1.0.5. Suppose that $\mathbf{\Gamma}$ has the prewellordering property. Then

$$\delta(\mathbf{\Gamma}) = \sup\{|\preceq| : \preceq \text{ is a } \mathbf{\Delta} \text{ norm}\}.$$

Theorem 1.0.1. $\delta(\mathbf{\Pi}_1^1) = \omega_1$.

Proposition 1.0.1. *Suppose $\mathbf{\Gamma}$ is a non-selfdual pointclass closed under $\forall^1$, $\vee$ and that pwo($\mathbf{\Gamma}$). Suppose $P \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$, and $\phi : P \to \alpha$ is a regular $\mathbf{\Gamma}$-norm. Let $\gamma < \alpha$. Then*

$$\{x \in P : \phi(x) \leq \gamma\} \in \mathbf{\Delta}$$

Proof. Let $A = \{x \in P : \phi(x) \leq \gamma\}$. Let $y \in P$ be so that $\phi(y) = \gamma$. Then

$$x \in A \iff x \in P \wedge \phi(x) \leq \phi(y).$$

But then $x \in A \iff x \leq_\phi^\Gamma y$ and $x \in A$ iff $x \leq_\phi^{\check{\Gamma}} y$. This shows that $A \in \mathbf{\Delta}$. $\square$

Theorem 1.0.2. *Let $\mathbf{\Gamma}$ be non-selfdual and closed under $\forall^1$, $\wedge$, $\vee$, and assume pwo($\mathbf{\Gamma}$). Suppose P is a universal $\mathbf{\Gamma}$ set and $\phi : P \to \delta(\mathbf{\Gamma})$ is a $\mathbf{\Gamma}$-norm. Suppose $A \subseteq P$ is in $\check{\mathbf{\Gamma}}$. Then $\phi[A]$ is bounded below $\delta(\mathbf{\Gamma})$.*

2 Partition Properties

Definition 2.0.6. If $f : \alpha \to \mathrm{On}$, then f has **uniform cofinality** ω iff there is a function $f' : \omega \times \alpha \to \mathrm{On}$ so that

$$(\forall \beta < \alpha)\, [f(\beta) = \sup\{f'(\beta', n) : n < \omega\}]$$

and

$$(\forall \beta < \alpha)(\forall n, m \in \omega)\, [n < m \implies f'(\beta, n) < f'(\beta, m)]\, .$$

Definition 2.0.7. We say $f : \alpha \to \mathrm{On}$ is **of the correct type** if f is increasing, everywhere discontinuous, and of uniform cofinality ω.

Definition 2.0.8. Let κ be a cardinal and $\lambda \leq \kappa$. Then $[\kappa]^\lambda$ is the set of increasing function from λ to κ.

We write $\kappa \to (\kappa)^\lambda$ to mean the following: for every $P : [\kappa]^\lambda \to 2$, there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and P is constant on $[H]^\lambda$.

We write $\kappa \overset{cub}{\to} (\kappa)^\lambda$ to mean the following: for every $P : [\kappa]^\lambda \to 2$, there is a club $C \subseteq \kappa$ and an $i \in 2$ so that for all $f : \lambda \to C$ of the correct type, $P(f) = i$.

Proposition 2.0.2. *For all cardinals κ and ordinals $\lambda \leq \kappa$:*

1. *if $\kappa \overset{cub}{\to} (\kappa)^\lambda$, then $\kappa \to (\kappa)^\lambda$, and*

2. *if $\kappa \to (\kappa)^{\omega\lambda}$, then $\kappa \overset{cub}{\to} (\kappa)^\lambda$.*

Proof. We prove part 1. So suppose that $\kappa \overset{cub}{\to} (\kappa)^\lambda$. Let $P : [\kappa]^\lambda \to 2$. Then there is a club $C \subseteq \kappa$ an $i \in 2$ so that for all $f : \lambda \to C$ of the correct type, $P(f) = i$. Let $f_C : \kappa \to C$ be the enumeration of C. Define $g : \kappa \to \kappa$ by

$$g(\alpha) = \sup\{f_C(\beta) : \beta < \alpha + \omega\}$$

Let $H = g[\kappa]$. Note that $H \subseteq C$ and $|H| = \kappa$. Let $f : \lambda \to H$ be increasing. Define $f' : \lambda \times \omega \to \kappa$ by

$$f'(\beta, n) = f_C(g^{-1}(f(\beta)) + n).$$

Then f' witnesses that f has uniform cofinality ω. Also, f is everywhere discontinuous. So $P(f) = i$. $\square$

Definition 2.0.9. κ has the **strong partition property** if $\kappa \to (\kappa)^\kappa$. κ has the **weak partition property** if $\kappa \to (\kappa)^\lambda$ for all $\lambda < \kappa$.

Remark 2.0.2. Under the axiom of choice, $\kappa \to (\kappa)^\omega$ is false for all cardinals κ.

3 Reasonability

Definition 3.0.10. Let κ be a regular cardinal, $\lambda \in \mathrm{On}$, $\lambda \leq \kappa$. We say κ is **λ-reasonable** iff there is a non-selfdual pointclass $\boldsymbol{\Gamma}$ closed under $\exists^1$, and a map $\phi : \omega^\omega \to \mathcal{P}(\lambda \times \kappa)$ so that

1. $(\forall F : \lambda \to \kappa)(\exists x \in \omega^\omega)[\phi(x) = F]$,
2. $(\forall \beta < \lambda)(\forall \gamma < \kappa)[R_{\beta,\gamma} \in \boldsymbol{\Delta}]$, where
$$x \in R_{\beta,\gamma} \iff [(\beta,\gamma) \in \phi(x) \wedge (\forall \gamma' < \kappa)[(\beta,\gamma') \in \phi(x) \implies \gamma' = \gamma]],$$
3. Suppose $\beta < \lambda$, $A \in \exists^1\boldsymbol{\Delta}$, and $A \subseteq R_\beta$, where $R_\beta = \bigcup_{\gamma<\kappa} R_{\beta,\gamma}$. Then
$$(\exists \gamma_0 < \kappa)(\forall x \in A)(\exists \gamma < \gamma_0)[x \in R_{\beta,\gamma}].$$

We κ is **reasonable** iff if is κ-reasonable.

Theorem 3.0.3 (Martin)**.** *If κ is $\omega\lambda$-reasonable, then $\kappa \to (\kappa)^\lambda$.*

Proof. Let $\boldsymbol{\Gamma}, \boldsymbol{\Delta}$, and ϕ be witnesses to the fact that κ is $\omega\lambda$-reasonable. We will show that $\kappa \overset{cub}{\to} (\kappa)^\lambda$.
Claim 1. $\boldsymbol{\Delta}$ is closed under $< \kappa$ unions and intersections. We do not prove this.

Let $P : [\kappa]^\lambda \to 2$. Consider the following game:

$$\begin{array}{c|cccccc}
\mathrm{I} & x_0 & & x_1 & & \cdots & \to x \\
\hline
\mathrm{II} & & y_0 & & y_1 & \cdots & \to y
\end{array}$$

where II wins under the following criteria:

- If there is a least $\beta < \omega\lambda$ so that $x \notin R_\beta$ or $y \notin R_\beta$, then II wins if $x \notin R_\beta$.
- Otherwise, let $f_x, f_y : \omega\lambda \to \kappa$ be the functions x and y determine. Define $f_{x,y} : \lambda \to \kappa$ by
$$f_{x,y}(\beta) = \sup\{\max(f_x(\beta'), f_y(\beta')) : \beta' < \omega(\beta+1)\}.$$
II wins iff $P(f_{x,y}) = 1$.

First assume that II has a winning strategy τ. For $\beta < \omega\lambda$ and $\gamma < \kappa$, define $S_{\beta,\gamma} \subseteq \omega^\omega$ by
$$x \in S_{\beta,\gamma} \iff (\forall \beta' \leq \beta)(\exists \gamma' \leq \gamma)[x \in R_{\beta',\gamma'}].$$
Note that $S_{\beta,\gamma} \in \boldsymbol{\Delta}$. Hence, for all $\beta < \omega\lambda$ and $\gamma < \kappa$, $\tau[S_{\beta,\gamma}] \in \exists^1\boldsymbol{\Delta}$. Note also that $\tau[S_{\beta,\gamma}] \subseteq R_\beta$. Thus there is a $\delta_0 < \kappa$ so that for all $y \in \tau[S_{\beta,\gamma}]$, there is a $\delta < \delta_0$ so that $y \in R_{\beta,\delta}$. Define $\psi : \omega\lambda \times \kappa \to \kappa$ by
$$\psi(\beta,\gamma) = \sup\{\phi(y)(\beta) : y \in \tau[S_{\beta,\gamma}]\}.$$
Note that if $\gamma \leq \gamma'$, then $\psi(\beta,\gamma) \leq \psi(\beta,\gamma')$. For $\beta < \omega\lambda$, let $C_\beta \subseteq \kappa$ be the set of points closed under $\psi(\beta,\cdot)$. Let $C = \bigcap_{\beta<\omega\lambda} C_\beta$. Note that C is a club and if $\beta < \omega\lambda$ and $\gamma \in C$, then $\psi(\beta,\gamma) \in C$. Let C' be the limit points of C.

Suppose $F : \lambda \to C'$ is of the correct type. We will show that $P(F) = 1$. Let x be so that $\phi(x)$ determines a function $f_x : \omega\lambda \to C$ with
$$F(\beta) = \sup\{f_x(\beta') : \beta' < \omega(\beta+1)\}$$
and so that $f_x(\beta) \geq \beta$ for all β. Let $y = \tau(x)$. As II is winning and $x \in R_\beta$ for all β, $y \in R_\beta$ for all β as well. So y determines a function $f_y : \omega\lambda \to \kappa$. Fix $\beta < \omega\lambda$. Then $x \in S_{\beta,f_x(\beta)}$ and $x \in S_{\beta+1,f_x(\beta+1)}$, and $f_x(\beta)$, $f_x(\beta+1)$ are least with this property. Now $y \in \tau[S_{\beta,f_x(\beta)}]$, and $\psi(\beta, f_x(\beta)) = \min\{\alpha \in C : \alpha \geq f_x(\beta)\} \leq f_x(\beta+1)$. Thus $f_y(\beta) \leq f_x(\beta+1)$. So $F = f_{x,y}$, and therefore $P(F) = 1$. $\square$

Theorem 3.0.4. *Let $\mathbf{\Gamma}$ be a non-selfdual pointclass closed under $\forall^1$, $\wedge$, $\vee$, and assume pwo$(\mathbf{\Gamma})$. Let $\delta = \delta(\mathbf{\Gamma})$. Then $\delta \to (\delta)^\lambda$ for all $\lambda < \omega_1$.*

Proof. Fix λ, and a fix a bijection $\pi : \omega \to \lambda$. Fix a $\mathbf{\Gamma}$-universal set P and a Γ-norm ψ on P. Say ψ is onto δ. Define $\phi : \mathbb{R} \to \mathcal{P}(\lambda \times \delta)$ by

$$(\beta, \gamma) \in \phi(x) \iff \left(x_{\pi^{-1}(\beta)} \in P \wedge \psi\left(x_{\pi^{-1}(\beta)}\right) = \gamma\right).$$

ϕ and $\mathbf{\Gamma}$ will show that δ is λ-reasonable. We show that ϕ satisfies condition 1. Let $F : \lambda \to \delta$. Fix $\beta < \lambda$. Let $n = \pi^{-1}(\beta)$. Let x_n be so that $\psi(x_n) = F(\beta)$. Let $x = \langle x_0, x_1, \cdots \rangle$. Then $\phi(x) = F$. So condition 1 is satisfied. We now check condition 2. Let $\beta < \lambda$ and $\gamma < \delta$. Then

$$\begin{aligned} x \in R_{\beta,\gamma} &\iff [(\beta,\gamma) \in \phi(x) \wedge (\forall \gamma' < \delta)[(\beta,\gamma') \in \phi(x) \implies \gamma' = \gamma]] \\ &\iff \left[\left(x_{\pi^{-1}(\beta)} \in P \wedge \psi\left(x_{\pi^{-1}(\beta)}\right) = \gamma\right) \wedge (\forall \gamma' < \delta)[\psi\left(x_{\pi^{-1}(\beta)}\right) = \gamma' \implies \gamma' = \gamma]\right] \\ &\iff \left[x_{\pi^{-1}(\beta)} \in P \wedge \psi\left(x_{\pi^{-1}(\beta)}\right) = \gamma\right] \end{aligned}$$

which shows that $R_{\beta,\gamma} \in \mathbf{\Delta}$. So condition 3 is satisfied.

We now show that condition 3 is satisfied. Fix $\beta < \lambda$ and $A \in \exists^1 \mathbf{\Delta}$ so that $A \subseteq R_\beta$. Since $\mathbf{\Gamma}$ is closed under $\forall^1$, $\check{\mathbf{\Gamma}}$ is closed under $\exists^1$. Thus $A \in \check{\mathbf{\Gamma}}$. Since $A \subseteq R_\beta$, $A_\beta = \{x_{\pi^{-1}(\beta)} : x \in A\} \subseteq P$. Note also that $A_\beta \in \check{\mathbf{\Gamma}}$. So $\psi[A_\beta]$ is a bounded subset of δ. Therefore there is a $\gamma_0 < \delta$ so that for all $x \in A_\beta$, $\psi(x) < \gamma_0$. But then for all $x \in A$, there is a $\gamma < \gamma_0$ so that $x \in R_{\beta,\gamma}$. So condition 4 is satisfied. $\square$

Corollary 3.0.1. *ω_1 has the weak partition property. I.e., $\omega_1 \to (\omega_1)^\alpha$ for all $\alpha < \omega_1$.*

Proof. Recall that $\mathbf{\Pi}^1_1$ is a non-selfdual pointclass closed under $\wedge$, $\vee$, and $\forall^1$, that pwo$\left(\mathbf{\Pi}^1_1\right)$, and that $\omega_1 = \delta\left(\mathbf{\Pi}^1_1\right)$. $\square$