Set Theoretic Generalizations of Algebra

Jared Holshouser

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1 Jónsson Algebras

From a set theory standpoint all of the usual algebraic objects have a common form: a set and operations that set is closed under. Anything which fits into this pattern is referred to as an algebra. Formally,

Definition 1. A **set theoretic algebra (ST algebra)** is a tuple $(A, f_1, f_2, \cdots)$, where each $f_n : A^{k_n} \to A$ for some k_n, and there are up to countably many f_n.

This definition is very general, and covers many objects studied in algebra.

Example 1.
- groups, monoids, semigroups, magmas, etc... are all ST algebras with a single function.
- rings, IDs, UFDs, PIDs, and fields are all ST algebras with two functions.
- $(\mathbb{N}, +, \cdot, \hat{})$ is an ST algebra with three functions.

Tetration, and further iterations are a natural way to keep producing more functions. Including all of these would produce an ST algebra with infinitely many operations $(\mathbb{N}, +, \cdot, \uparrow, \uparrow\uparrow, \uparrow\uparrow\uparrow, \cdots)$. The possibility of having infinitely many operations comes into play in set theory through the application of Skolem functions.

It may seem that the definition has not covered modules, vector spaces, or algebras. In fact they are also ST algebras. Consider the case of modules. This is a group $(G, *)$ acting on a ring $(R, +, \cdot)$. The structure is partially captured as an algebra by $(G \times R, f_1, f_2, f_3, f_4)$ where

- $f_1 : (G \times R)^2 \to G \times R$ is given by $f_1((g, x), (h, y)) = (g * h, r_0)$, where r_0 is some fixed element of R,
- $f_2 : (G \times R)^2 \to G \times R$ is given by $f_2((g, x), (h, y)) = (g_0, x + y)$, where g_0 is some fixed element of G,
- $f_3 : (G \times R)^2 \to G \times R$ is given by $f_3((g, x), (h, y)) = (g_0, x \cdot y)$,
- $f_4 : (G \times R)^2 \to G \times R$ is given by $f_4((g, x), (h, y)) = (g_0, gy)$.

To get properties of the identity and inverses and so on, simply add more maps. The same thing works for vector spaces, and something similar works for algebras.

An ST algebra can have substructures in at least the most general way. Ideas such as normal or ideal seem to be more specific.

Definition 2. Let $(A, f_1, f_2, \cdots)$ be an ST algebra. Then B is a **subalgebra** of A iff $B \subseteq A$ and $f_n[B^{k_n}] \subseteq B$ for all n.

The study of subalgebras and generating sets is of course helpful for understanding a given ST algebra. The existence of large subalgebras is of particular interest to set theorists; large in the sense of the same size as the original algebra. Note that these cannot always be found. Finite groups are one obvious example. Bjarni Jónsson studied these in the 60's and 70's in the course of studying universal algebra.

Definition 3. An ST algebra $(A, f_1, f_2, \cdots)$ is **Jónsson** iff there are no proper subalgebras B so that $|B| = |A|$.

The shift in perspective from algebra to set theory is the following: the set theorist is interested in the collection of possible algebras on a set rather than in particular algebras. Towards this end, B. Jónsson wondered if there were sets which admitted no Jónsson algebras.

Definition 4. A set A is **Jónsson** iff there are no Jónsson algebras on A. Equivalently, A is Jónsson iff whenever $(A, f_1, f_2, \cdots)$ is an ST algebra, there is a subalgebra $B \subsetneq A$ so that $|B| = |A|$.

Question: Are any sets Jónsson?

Example 2. $\mathbb{N}$ is NOT Jónsson. Define $f_n : \mathbb{N} \to \mathbb{N}$ by $f_n(i) = n$. Any set B closed under all of the f_n must be $\mathbb{N}$ again.

These results are all well and good, but they hardly correspond to a systematic attempt to find a Jónsson set. Note that if $|A| = |B|$, then A is Jónsson iff B is Jónsson (any algebra on A turns into an algebra on B through the bijection and vice versa). So the Jónsson property has more to do with the size of the set than the set itself. So to systematically check for Jónsson sets it suffices to check one set of each possible size. This leads us naturally to cardinal numbers.

2 Ordinals and Cardinals

The way to create a hierarchy of sizes is to understand the world of mathematics through a single binary relation, $\in$. This is a partial order on mathematical objects. Sets A and B are equal precisely when

$$x \in A \iff x \in B$$

for all x. It is an axiom of set theory that $\in$ is a well-founded relation, that is every $\in$-chain has a $\in$-minimal element. So although there are situations where $a_1 \in a_2 \in a_3 \in \cdots$, there are no situations where $\cdots \in a_3 \in a_2 \in a_1$. The goal is to use $\in$ to simulate the order $\leq$ on $\mathbb{N}$, and then to extend it. $\leq$ has two properties that $\in$ does not have in general, it is transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$) and it is total (either $a \leq b$ or $b \leq a$).

Definition 5. A set A is **transitive** iff whenever $x \in A$ and $y \in x$, $y \in A$ as well. α is an **ordinal** iff α is transitive and $\in$ is a total order on α.

The collection of ordinals is transitive and totally ordered with respect to $\in$. The natural numbers are recreated as an initial segment of the ordinals.

Example 3. Consider the sequence of sets

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \cdots$$

Each of these is $\in$-transitive, and $\in$ is total on them. Since nothing is an element of $\in$, we identify $\emptyset$ with 0. Then $\{\emptyset\}$ is 1, $\{\emptyset, \{\emptyset\}\}$ is 2 and so on. Note that $0 \in 1 \in 2 \in \cdots$, and in particular, $n = \{0, \cdots, n-1\}$ for all n. This sequence is the set theoretic simulation of $\mathbb{N}$. It is referred to as ω or $\aleph_0$.

The construction principle on display here is quite general. If α is an ordinal, let $\alpha + 1 = \alpha \cup \{\alpha\}$. Note that $\alpha \in \alpha + 1$ and if $\beta \in \alpha$, then $\beta \in \alpha + 1$. Moreover, if $\beta \in \alpha + 1$, then either $\beta = \alpha$ or $\beta \in \alpha$. So this $+1$ is a true successor operation.

These finite ordinals are special in that none of them are in bijection with each other. Infinite ordinals can easily be in bijection with each other. Consider ω and $\omega + 1$. The map $f : \omega + 1 \to \omega$ by $f(\omega) = 0$ and $f(n) = n + 1$ is a bijection. In fact, there are uncountably many countable ordinals.

Definition 6. An ordinal κ is a **cardinal number** (or just cardinal) if κ is not in bijection with and $\alpha \in \kappa$.

The cardinals can be indexed by the ordinals. The first uncountable cardinal, or first cardinal above ω is $\aleph_1$. The next cardinal above that is $\aleph_2$ and so on. The least cardinal larger than all of the $\aleph_n$ is $\aleph_\omega$. The next cardinal after that would be $\aleph_{\omega+1}$. There is a cardinal corresponding to every ordinal and the axiom of choice implies that every set is in bijection with some cardinal.

Note that the construction of these cardinals has nothing to do with the construction of $\mathbb{R}$. The continuum hypothesis is the assertion that $\mathbb{R}$ is in bijection with $\aleph_1$. Deep theorems in set theory established in the 1960's showed that the standard axiom of set theory do not establish a fixed cardinal which $\mathbb{R}$ is in bijection with. In fact, it can be almost anything!

Question Are there any Jónsson cardinals?

It turns out that the best way to try and answer this question is by turning it into a more explicitly combinatorial problem

3 Finite Partition Properties

A finite partition property is a kind of coloring property. Recall Ramsey's theorem on the naturals: for any fixed dimension k, you want to color, and fixed finite number of colors p, and any finite size m you want to stay larger than, there is some n so that whenever c colors the increasing k-tuples from n with p colors, there is a subset of size m whose tuples all get the same color. Saying this in words is clearly clumsy, so its good to introduce some notation.

Definition 7. For a cardinal κ and $n \in \mathbb{N}$, let $[\kappa]^n = \{(\alpha_1, \cdots, \alpha_n) : \alpha_1 < \cdots < \alpha_n \in \kappa\}$. I.e. $[\kappa]^n$ is the increasing n-tuples from κ.

Definition 8. Fix cardinals $\delta, \kappa, \epsilon, \lambda$ and $n, m \in \mathbb{N}$. Then $[\delta]^n_\kappa \to [\epsilon]^n_\lambda$ if whenever $c : [\delta]^n \to \kappa$, there is an $A \subseteq \delta$ so that $|A| = \epsilon$ and $|c[[A]^m]| = \lambda$. Ramsey's theorem can be interpreted as saying that for any $k, p, m \in \mathbb{N}$, there is an $n \in \mathbb{N}$ so that $[n]^k_p \to [m]^k_1$. The famous pigeonhole principle can interpreted as whenever $n < m$, $[m]^1_n \to [2]^1_1$.

There is an infinite version of Ramsey's theorem. This states that for fixed $k, p \in \mathbb{N}$, whenever the k-tuples of $\mathbb{N}$ are colored with p colors, there is an infinite $A \subseteq \mathbb{N}$ so that $[A]^k$ is monochromatic. It is tempting to ask for this to hold simultaneously for all tuple lengths k, not in the sense that tuples of all lengths get the same color, but there is one color per dimension. This fails for $\mathbb{N}$ but is not completely out of the realm of possibility.

Definition 9. For a cardinal κ, $[\kappa]^{<\omega} = \bigcup_{n \in \mathbb{N}} [\kappa]^n$.

Definition 10. Fix cardinals $\delta, \kappa, \epsilon, \lambda$. Then $[\delta]^{<\omega}_\kappa \to [\epsilon]^{<\omega}_\lambda$ if whenever $c : [\delta]^{<\omega} \to \kappa$, there is an $A \subseteq \delta$ so that $|A| = \epsilon$ and $|c[[A]^n]| = \lambda$ for all n. The strengthening we had in mind for $\mathbb{N}$ that is false could be written as $[\aleph_0]^{<\omega}_p \to [\aleph_0]^{<\omega}_1$.

Definition 11. A cardinal κ is **Ramsey** if $[\kappa]^{<\omega}_2 \to [\kappa]^{<\omega}_1$.

The existence of Ramsey cardinal cannot be proved from classical mathematics. In fact, a Ramsey cardinal will have sufficient topological closure that it can simulate all of classical mathematics. In more technical phraseology

$$\kappa \text{ is Ramsey} \implies \text{Con(ZFC)}.$$

This is not to say that Ramsey cardinals are unable to be studied, just that explicit examples cannot be constructed using the techniques of classical mathematics.

What we want is a combinatorial characterization of the Jónsson property.

Theorem 1. *The following are equivalent:*

1. *κ is Jónsson, and*
2. *whenever $f : [\kappa]^{<\omega} \to \kappa$, then there is an $A \subseteq \kappa$ so that $|A| = \kappa$ and $f[[A]^{<\omega}] \neq \kappa$.*

Proof. We will show that 1 implies 2. For $n \in \mathbb{N}$, let f_n be the restriction of f to $[\kappa]^n$. Then $(\kappa, f_1, f_2, \cdots)$ is an ST algebra. Since κ is Jónsson, there is a proper subalegbra of cardinality κ. Let B be this subalgebra. Then $f[[B]^{<\omega}] \subseteq B \neq A$.

To show that 2 implies 1 requires a detour through model theory that we will not go through here. □

4 Situations Where Cardinals are not Jónsson

Proposition 1. *$\aleph_1$ is not Jónsson.*

Proof. For all ordinals α which are between $\aleph_0$ and $\aleph_1$, i.e. all countable ordinals, let $f_\alpha : [\alpha]^{<\omega} \to \alpha$ witness that α is not Jónsson. Define $g : [\aleph_1]^{<\omega} \to \aleph_1$ by

$$g(s) = \begin{cases} f_\alpha(s \setminus \{\max(s)\}) & \text{if } \max(s) = \alpha \geq \aleph_0 \\ 0 & \text{else} \end{cases}$$

We claim that g proves that $\aleph_1$ is not Jónsson. Suppose $B \subseteq \aleph_1$is uncountable. Let $\beta < \aleph_1$. Since B is uncountable, there is some $\alpha \in B$ so that $\beta < \alpha$ and $B \cap \alpha$ is infinite. Since α was not Jónsson, and this was witnessed by f_α, there is a $t \in [B \cap \alpha]^{<\omega}$ so that $f_\alpha(t) = \beta$. Then $g(t^\frown\alpha) = f_\alpha(t) = \beta$. $\square$

Chang, Rowbottom and Erdos-Hajnal all showed that this behavior continues for other cardinals. If a cardinal κ is not Jónsson, then the next cardinal after it will also not be Jónsson. There is still an open question about $\aleph_1$ however. Namely, is there a group which witnesses that $\aleph_1$ is not Jónsson? I.e. is there a group structure on $\aleph_1$ so that every proper subgroup is countable?

Using the result for the next cardinal, this means that $\aleph_n$ is not Jónsson for any $n \in \mathbb{N}$. It is not known if $\aleph_\omega$ can be Jónsson. This is an apparently very difficult question.

Just as there is a smallest group, or a smallest ring, there is a smallest object L so that, restricted to L, $\in$ satisfies all of the axioms of set theory. L is a self-contained universe closed under the operations classical mathematics. L contains all of the ordinals.

Theorem 2. *L has no Jónsson cardinals. More specifically, if κ is so that there is no bijection in L from an ordinal $\alpha < \kappa$ onto κ, then there is an algebra $(\kappa, \vec{f})$ in L so that every subalgebra B in L which is in bijection with κ is actually all of κ.*

This can be interpreted as saying that Jónsson cardinals are not constructible using the techniques of classical mathematics.

5 Situations Where Cardinals Are Jónsson

The best way to approach the result of the previous section is to think about compass-ruler constructions. According to compass-ruler constructions, there is no such thing as the trisection of an angle, and you given a circle, you can't construct a square with the same area. However, more powerful techniques allow us to do these things. The situation is similar here. There is an augmentation of classical mathematics which allows for the construction of Jónsson cardinals.

One such augmentation relates the existence of substructures.

Definition 12. Let V be the collection of all sets. An embedding $j : V \to V$ is **non-trivial** if it is not the identity function. If j is non-trivial, then there is a least cardinal κ so that $j(\kappa) > \kappa$. This is called the **critical point of** j.

A cardinal κ is measurable if is the critical point of a non-trivial embedding.

This makes V act like $\mathbb{N}$ in a sense. $\mathbb{N}$ can non-trivially embed into itself, and such a map pushes some numbers up.

Proposition 2. *Suppose κ is a measurable cardinal. Then κ is a Jónsson cardinal. In fact, there are unboundedly many Jónsson cardinals below κ.*

If κ is measurable, then there is no cardinal λ so that κ is the next cardinal after λ. No technique is currently known which creates a successor cardinal which is Jónsson. Work in cardinal arithmetic by Shelah shows that $\aleph_{\omega+1}$ cannot be Jónsson, and further work in that are has produced a range of conditions which a successor Jónsson cardinal would have to meet.

As a final note, if we abandon the axiom of choice, then the situation around Jónsson cardinals becomes quite different. There is an augmentation of classical mathematics which denies choice called the axiom of determinacy (AD). There is a smallest universe $L(\mathbb{R})$ which satisfies all of the axioms of set theory except choice and which also satisfies AD.

Proposition 3. *In $L(\mathbb{R})$, every uncountable cardinal which can be mapped onto by $\mathbb{R}$ is a Jónsson cardinal.*