Using Non-standard Analysis to Realize Lebesgue Measure as a Counting Measure

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1 The Hyperreals

Using the compactness theorem for first order logic, it is possible to form a model of arithmetic with infinite numbers. By a model of arithmetic, we mean a space $(N, S, +, \cdot, 0, 1, \leq)$ satisfying the Peano axioms.

We can also form such a model of arithmetic using an ultrapower construction, the details of which we will not discuss here. The ultrapower construction can be extended to expand the reals to an ordered field we call the hyperreals. The hyperreals have infinite and infinitesimal numbers.

Definition 1. Let ${}^*\mathbb{N}$ be a model of arithmetic with infinite numbers, and ${}^*\mathbb{R}$ be the hyperreals.

Definition 2. If $x \in {}^*\mathbb{R}$ is so that $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$, then we say that x is an **infinitesimal**. If $x \in {}^*\mathbb{R}$ is so that $|x| > N$ for all $n \in \mathbb{N}$, then x is said to be **infinite**. Let ${}^*\mathbb{N}_\infty$ be the infinite numbers in ${}^*\mathbb{N}$.

Definition 3. Say $x, y \in {}^*\mathbb{R}$ are **near**, written $x \simeq y$ if $|x - y|$ is infinitesimal. For those $x \in {}^*\mathbb{R}$ that are finite, there is a unique $y \in \mathbb{R}$ so that $x \simeq y$. We say that $st(x) = y$.

Definition 4. Let $x \in \mathbb{R}$, then $\mu(x) = \{y \in {}^*\mathbb{R} : x \simeq y\}$. This is called the **monad** of x.

Definition 5. For $A \subseteq \mathbb{R}$, ${}^*A$ is the restriction of the extension of $\mathbb{R}$ to A. We can also lift functions $f : \mathbb{R} \to \mathbb{R}$ to functions ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$.

In general, this is a bit hard to describe, but for well-described sets its simple:

- ${}^*(0, 1) = \{x \in {}^*\mathbb{R} : 0 < x < 1\}$,
- ${}^*\{\frac{1}{n}\}_{n\in\mathbb{N}} = \{\frac{1}{n} : n \in {}^*\mathbb{N}\}$.

Proposition 1. *${}^*(A \cup B) = {}^*A \cup {}^*B$ and ${}^*(A^c) = ({}^*A)^c$.*

Definition 6. Let ϕ be a mathematical sentence about $\mathbb{R}$. ${}^*\phi$ is the transformation of that sentence which replaces all references to sets and functions by their $*$.

Lets see some examples:

- If ϕ is $(\forall x, y \in \mathbb{R})[x + y = y + x]$, then ${}^*\phi$ is $(\forall x, y \in {}^*\mathbb{R})[x + y = y + x]$,
- If ϕ is $(\exists x \in \mathbb{R})[\sin(x) = 1]$, then ${}^*\phi$ is $(\exists x \in {}^*\mathbb{R})[{}^*\sin(x) = 1]$,
- If ϕ is $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})[|x| < n]$, then ${}^*\phi$ is $(\forall x \in {}^*\mathbb{R})(\exists n \in {}^*\mathbb{N})[|x| < n]$.

Theorem 1 (Transfer)**.** *A statement ϕ is true for $\mathbb{R}$ if and only ${}^*\phi$ is true for ${}^*\mathbb{R}$.*

Proposition 2. $x_n \to L$ *iff* $(\forall N \in {}^*\mathbb{N}_\infty)[x_n \simeq L]$.

Proof. Suppose that $x_n \to L$. Then

$$(\forall \varepsilon > 0(\text{in } \mathbb{R}))(\exists N \in \mathbb{N})(\forall m \geq n(\text{in } \mathbb{N}))[|x_m - L| < \varepsilon]$$

Then if $N \in {}^*\mathbb{N}_\infty$ we have that $|x_N - L| < \varepsilon$ for all $\varepsilon > 0$. So $x_N \simeq L$ for all $N \in {}^*\mathbb{N}_\infty$.

Now suppose that $(\forall N \in {}^*\mathbb{N}_\infty)[x_n \simeq L]$. Let $\varepsilon > 0$ be in $\mathbb{R}$. Then

$$(\exists N \in {}^*\mathbb{N})(\forall m \geq n(\text{in } {}^*\mathbb{N}))[|x_m - L| < \varepsilon]$$

So by transfer,

$$(\exists N \in \mathbb{N})(\forall m \geq n(\text{in } \mathbb{N}))[|x_m - L| < \varepsilon]$$

Since $\varepsilon > 0$ was arbitrary, $x_n \to L$. $\square$

2 Superstructures

To do measure theory, probability theory, functional analysis, etc... we will need to be able to look at collections of subsets of $\mathbb{R}$ and collections of those and so on. We incorporate these into whats a called a superstructure.

Definition 7. The superstructure of a set X is defined recursively:

- $V_0(X) = X$,
- $V_{n+1}(X) = \mathcal{P}(V_n(X))$.

We let $V(X) = \bigcup_{n\in\mathbb{N}} V_n(X)$, and call $V(X)$ the **superstructure** of X.

Let V be the superstructure or $\mathbb{R}$ and W be the superstructure of ${}^*\mathbb{R}$. We want to extend the embedding of $\mathbb{R}$ into ${}^*\mathbb{R}$ to an embedding of V into W.

Proposition 3. *There is a map* $* : V \to W$ *so that*

1. $*(\mathbb{R}) = {}^*\mathbb{R}$,
2. $*$ *is 1-1, preserves* $\in$ *and preserves rank,*
3. $*$ *preserves finite set operations, and*
4. *(Transfer) A sentence* φ *holds in* $V(X)$ *iff* ${}^*\varphi$ *holds in* $V({}^*X)$.

Definition 8. $Y \in W$ is said to be **standard** if there is an $X \in V$ so that $Y = {}^*X$. $Y \in W$ is said to be **internal** if for some n, $Y \in {}^*V_n$.

Proposition 4. *The union of internal sets is internal, and the complement of an internal set is internal. All standard sets are internal.*

Let's see some examples:

- ${}^*\mathbb{R}$ is internal and ${}^*\mathbb{N}$ is internal.
- Fix $N \in {}^*\mathbb{N}$. Then $\{k \in {}^*\mathbb{N} : k \leq N\}$ is internal.
- $\mathbb{N}$ is external and ${}^*\mathbb{N}_\infty$ is external.
- Fix $x \in \mathbb{R}$. Then $\mu(x)$ is external.

Definition 9. W is **countably saturated** if whenever $R \in W$ is a binary relation and $A \subseteq \mathrm{dom}(R)$ is countable and is so that for all $a_1, \cdots, a_n \in A$ there is b so that $R(a_i, b)$ for $1 \leq i \leq n$, then there is a b so that $R(a, b)$ for all $a \in A$.

We assume that W is countably saturated from here on out.

Proposition 5. *Let* C *be internal. Then*

1. *If* $A \subseteq C$ *is internal, and* $A \subseteq \bigcup_{k\in\mathbb{N}} B_k$*, where each* B_k *is internal, then there is a* $k \in \mathbb{N}$ *so that* $A \subseteq B_1 \cup \cdots \cup B_k$.
2. *If* $(A_k)_{k\in\mathbb{N}}$ *is a strictly increasing or decreasing sequence of internal subsets of* C*, then* $\bigcup_{k\in\mathbb{N}} A_k$ *and* $\bigcap_{k\in\mathbb{N}}$ *are external.*

Proof. We prove part 1. We proceed by contradiction. Let $R = \{(B_k, a) : a \in A \wedge a \notin B_k\}$. Then for any $B_1, \cdots, B_k$ there is an $a \in A$ so that $R(B_i, a)$ for $1 \leq i \leq k$. So there is an $a \in A$ so that $a \notin B_k$ for all $k \in \mathbb{N}$. So $A \not\subseteq \bigcup_{k\in\mathbb{N}} B_k$. □

Proposition 6 (Luxemburg)**.** *Let* $A \subseteq {}^*\mathbb{R}$ *be internal and bounded. Then* $st[A]$ *is closed.*

Proof. Let $x \in \overline{st[A]}$. Then for each n, there is a $y \in B(x, \frac{1}{k}) \cap st[A]$ for $1 \leq k \leq n$. So there is a $y \in^* B(x, \frac{1}{k}) \cap A$ for $1 \leq k \leq n$. So by saturation, there is a $y \in \mu(x) \cap A$. Then $x = st(y) \in st[A]$. □

3 Loeb Measure Spaces

Let Λ be an internal nonempty set in W and let $\mathcal{C}$ be an internal algebra on Λ. Assume μ is an internal finitely additive measure defined on $\mathcal{C}$ with values in $^*[0, S]$ for some $S \in \mathbb{R}$.

Definition 10. For $A \in \mathcal{C}$ set

$$^\circ\mu(A) = st(\mu(A)).$$

Proposition 7. *$^\circ\mu$ is a countably additive measure on the algebra $\mathcal{C}$. Therefore $^\circ\mu$ can be extended to a measure on the σ-algebra generated by $\mathcal{C}$.*

Proof. Suppose that $A_1, A_2, \cdots \in \mathcal{C}$ and $A = \bigcup_k A_k \in \mathcal{C}$. Since $\{A_n : n \in \mathbb{N}\}$ is countable, and $A \subseteq \bigcup\{A_n : n \in \mathbb{N}\}$, by proposition 5, there are $A_1, \cdots, A_k$ so that $A = A_1 \cup \cdots \cup A_k$. So

$$^\circ\mu(A) = {^\circ\mu}(A_1) + \cdots + {^\circ\mu}(A_k) = {^\circ[}\mu(A_1) + \cdots + \mu(A_k)] = {^\circ\mu}(A_1) + \cdots + {^\circ\mu}(A_k)$$

So $^\circ\mu$ is countably additive. For the rest use Caratheodory's extension theorem. $\square$

Definition 11. For $N \subseteq \Lambda$, set

$$\mu^{outer}(N) = \inf\{ {^\circ\mu}(A) : A \in \mathcal{C} \wedge N \subseteq A\}$$

Say N is **μ_L-null** if $\mu^{outer}(N) = 0$. Set $\mathcal{N}_{\mu_L} = \{N \subseteq \Lambda : N \text{ is } \mu_L - null\}$

Lemma 1. *$\mathcal{N}_{\mu_L}$ is a σ-ideal.*

Proof. Suppose that $N_1, \cdots \in \mathcal{N}_{\mu_L}$. Fix $\varepsilon > 0$ in $\mathbb{R}$. For each $k \in \mathbb{N}$ there is an $A_k \in \mathcal{C}$ so that $N_k \subseteq A_K$ and $^\circ\mu(A_k) < \frac{\varepsilon}{2^k}$. So $\mu(A_k) < \frac{\varepsilon}{2^k}$. Let $k \in \mathbb{N}$. Then $A = A_1 \cup \cdots \cup A_{k+1}$ is in $\mathcal{C}$, $A_i \subseteq A$ for $1 \le i \le k$ and $\mu(A) < \varepsilon$. So by saturation, there is an $A \in \mathcal{C}$ so that $A_k \subseteq A$ for all $k \in \mathbb{N}$ and $\mu(A) < \varepsilon$. But $\bigcup_{k\in\mathbb{N}} N_k \subseteq A$, so since ε was arbitrary, $\bigcup_{k\in\mathbb{N}} N_k$ is μ_L-null. $\square$

Definition 12. Let $B \subseteq \Lambda$ and $A \in \mathcal{C}$. A is a **μ_L-approximation** of B if $A\Delta B$ is μ_L-null. Define

$$L_\mu(\mathcal{C}) = \{B \subseteq \Lambda : B \text{ has a } \mu_L\text{-approximation } A \in \mathcal{C}\}$$

For such B set $\mu_L(B) = {^\circ\mu}(A)$, where A is a μ_L-approximation of B.

Theorem 2 (Loeb)**.** *The following are true abotu μ_L.*

1. *μ_L is well-defined.*
2. *$L_\mu(\mathcal{C})$ is a σ-algebra with $\mathcal{C} \subseteq L_\mu(\mathcal{C})$.*
3. *$\mu_L : L_\mu(\mathcal{C}) \to [0, S]$ is countably-additive.*
4. *$L_\mu(\mathcal{C})$ is complete.*
5. *A subset $B \subseteq \Lambda$ is in $L_\mu(\mathcal{C})$ iff for each $\varepsilon > 0$ in $\mathbb{R}$ there are $A, A' \in \mathcal{C}$ such that $A \subseteq B \subseteq A'$ and $\mu(A' - A) < \varepsilon$.*

Proof. Assume that A and A' are μ_L-approximations of $B \subseteq \Lambda$. Then $A\Delta A'$ is μ_L-null, and so $\mu(A\Delta A') \simeq 0$. Thus

$$\mu(A) \simeq \mu(A) - \mu(A - A') + \mu(A' - A) = \mu(A')$$

Therefore $^\circ\mu(A) = {^\circ\mu}(A')$.

Now each $A \in \mathcal{C}$ is a μ_L-approximation of A, so $\mathcal{C} \subseteq L_\mu(\mathcal{C})$. Note that $\Lambda \in L_\mu(\mathcal{C})$. Suppose that $B, B' \in L_\mu(\mathcal{C})$. Let A, A' be their μ_L-approximations. Then $A - A'$ is a μ_L-approximation of $B - B'$ and $A \cap A'$ is a μ_L-approximation of $B \cap B'$. Now suppose that $B_1, B_2, \cdots \in L_\mu(C)$ are pairwise disjoint. Let A_K be a μ_L-approximation of B_k for each $k \in \mathbb{N}$. Without loss of generality, then A_k are pairwise disjoint. Let

$$s = \sum_{k \in \mathbb{N}} {}^\circ\mu(A_k)$$

Then $s \in [0, S]$ by the countable additivity and monotonicity of ${}^\circ\mu$. Using saturation, we can find an $A \in \mathcal{C}$ so that $\bigcup_{k \in \mathbb{N}} A_k \subseteq A$ and $\mu(A) < s + \frac{1}{k}$ for all $k \in \mathbb{N}$. So $\mu(A) \simeq s$. A is a μ_L-approximation of $\bigcup_{k \in \mathbb{N}} B_k$.

The previous paragraph also shows that μ_L is countably additive.

Suppose that $\mu_L(B) = 0$ and $N \subseteq B$. Let A be a μ_L-approximation of B. Then A is a μ_L-approximation of N as well.

Suppose $B \subseteq \Lambda$ is in $L_\mu(\mathcal{C})$. Let $C \in \mathcal{C}$ be a μ_L-approximation of B and $\varepsilon > 0$ be in $\mathbb{R}$. Then there is a $D \in \mathcal{C}$ so that $C \Delta B \subseteq D$ and $\mu(D) < \varepsilon$. Let $A = C - D$ and $A' = C \cup D$. Now for the converse. Let $B \subseteq \Lambda$. For each $k \in \mathbb{N}$ let $A_k, A'_k \in \mathcal{C}$ be so that $A_k \subseteq B \subseteq A'_k$ and $\mu(A'_k - A_k) < \frac{1}{k}$. For any $k \in \mathbb{N}$, we can find an $A \in \mathcal{C}$ so that $A_i \subseteq A \subseteq A'_i$ for $1 \leq i \leq k$, namely $A = A_1 \cap \cdots \cap A_k$. So by saturation, there is an $A \in \mathcal{C}$ so that $A_k \subseteq A \subseteq A'_k$ for all $k \in \mathbb{N}$. Then A is a μ_L-approximation of B, and so $B \in L_\mu(\mathcal{C})$. $\square$

Definition 13. The measure space $(\Lambda, L_\mu(\mathcal{C}), \mu_L)$ is call the **Loeb space** over $(\Lambda, \mathcal{C}, \mu)$. μ_L called the **Loeb measure** over μ.

4 The Hyperfinite Time Line T

Let G be an infinite natural and let $H = G!$. Set $T = \{1, \cdots, H\}$.

Proposition 8. $[0,1] = \{st(\frac{t}{H}) : t \in T\}$.

Define the measure ν^n for internal $B \subseteq T$ by

$$\nu(B) = \frac{|B|}{H}$$

where $|B|$ is the hyperfinite number of elements of B. The Loeb space over $(T, \{B \subseteq T : B \text{ is internal}\}, \nu)$ is denoted by $(T, L_\nu(T), \nu_L)$.

5 Lebesgue Measure as Counting Measure

Definition 14. For all $x \in [0,1]$ let $x_T = \{t \in T : \frac{t}{H} \simeq x\}$. For $A \subseteq [0,1]$, $A_T = \bigcup\{x_T : x \in A\}$.

Theorem 3. *A subset $A \subseteq [0,1]$ is Lebesgue measurable iff $A_T \in L_\nu(T)$, in which case $\lambda(A) = \nu_L(A_T)$.*

Proof. Let $I \subseteq [0,1]$ be an interval with endpoints a, b, where $a \leq b$. We will show that $\nu((a,b)_T) \simeq b - a$. Suppose that $a < b$. By transfer, we can choose $k, m \in {}^*\mathbb{N}$ so that

$$\frac{k-1}{H} < a \leq \frac{k}{H} \text{ and } \frac{k+m-1}{H} < b \leq \frac{k+m}{H}$$

Then $|(a,b)_T| = m$, $a \simeq \frac{k}{H}$, and $b \simeq \frac{k+m}{H}$. Thus

$$\nu((a,b)_T) = \frac{m}{H} = \frac{k+m}{H} - \frac{k}{H} \simeq b - a$$

We now show that for all $c \in [0,1]$, c_T is a ν_L-nullset. Fix $\varepsilon > 0$ in $\mathbb{R}$. Then

$$c_T \subseteq \left(c - \frac{\varepsilon}{3}, c + \frac{\varepsilon}{3}\right)_T$$

and $\nu(\left(c - \frac{\varepsilon}{3}, c + \frac{\varepsilon}{3}\right)_T) < \varepsilon$.

So if $a = b$, then I is a singleton or empty, and thus μ_L-null. In this case $I \in L_\nu(T)$ and $\nu_L(I) = 0 = b - a$. If $a < b$, then $\nu_L(I_T \Delta (a,b)_T) = 0$. Thus $I_T \in L_\nu(T)$ and

$$\nu_L(I_T) =^\circ \nu((a,b)_T) = b - a$$

Therefore if $B \subseteq [0,1]$ is Borel, then $B_T \in L_\nu(T)$ and $\nu_L(B) = \lambda(B)$. Now let B be a Borel subset of $[0,1]$ and $N \subseteq B$ be so that $\lambda(B) = 0$. Then as $L_\nu(T)$ is complete and $N_T \subseteq B_T$, $N_T \in L_\nu(T)$ and $\nu_L(N_T) = 0$. Therefore if $A \subseteq [0,1]$ is Lebesgue measurable, then $A_T \in L_\nu(T)$ and $\nu_L(A_T) = \lambda(A)$.

Now suppose that $A \subseteq [0,1]$ is so that $A_T \in L_\nu(T)$. Then

$$([0,1] - A)_T = T - A_T \in L_\nu(T)$$

Fix $\varepsilon > 0$ in $\mathbb{R}$. Then there are internal sets $C, C' \subseteq T$ so that

$$C \subseteq A_T \text{ and } \nu(A_T - C) < \varepsilon$$

and

$$C' \subseteq ([0,1] - A)_T \text{ and } \nu(([0,1] - A)_T - C') < \varepsilon$$

Now $st[C]$ and $st[C']$ are closed subsets of $[0,1]$. Moreover, $st[C] \subseteq A \subseteq ([0,1] - st[C'])$. Let $F = st[C]$ and $U = [0,1] - st[C']$. We know from before that

$$\begin{aligned}
\lambda(U - F) &= \nu_L((U - F)_T) \\
&= \nu_L(U_T - F_T) \\
&= \nu_L((T - C') - C) \\
&= \nu_L([([0,1] - A)_T - C'] \cup [A_T - C]) \\
&\leq \nu_L(([0,1] - A)_T - C') + \nu_L(A_T - C) \\
&< 2\varepsilon
\end{aligned}$$

So for each $\varepsilon > 0$ there is an open set U and a closed set F so that $F \subseteq A \subseteq U$ and $\lambda(F - U) < \varepsilon$. So A is Lebesgue measurable. $\square$