Solovay's Inaccessible: The Cost of Measurability

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1 Introduction

Solovay was the first the create a model of ZF where every set of reals is measurable. His model is easy to describe, but it takes some work to show that all sets of reals are measurable in it. His technique is as follows: Let κ be inaccessible and G be $\text{coll}(\omega, \kappa)$ generic for V. Solovay's model is $L(\mathbb{R})^{V[G]}$. Solovay showed the following with this model:

Theorem 1. *$Con(ZFC + \exists\kappa(\kappa \text{ inac.})) \implies Con(ZFC + \text{All sets of reals are meas.})$*

It is natural to ask if the inaccessible cardinal is necessary. In 1984, Shelah [3] showed that the inaccessible is in fact necessary. J. Raisonnier [1] then presented a more straightforward proof of this result. The theorem that they both show is as follows:

Theorem 2. *Assume $ZF + DC+$ every Σ^1_3 set is measurable. Then ω_1^V is inaccessible in L.*

The main goal is a slightly easier theorem. We will be loosely following the presentation of B. Semmes [2].

Theorem 3 (ZF + DC)**.** *If there is an uncountable well-ordered set of reals, then there is a non-measurable set.*

From this we get a weaker version of Shelah's result.

Theorem 4. *Assume $ZF + DC+$ every set is measurable. Then $V \neq L$ and ω_1^V is inaccessible in L.*

Proof. $V \neq L$ as L has a Δ^3_1 non-measurable set. For the second part, it suffices to show that $\omega_1^{L[a]} < \omega_1^V$ for all $a \in 2^\omega$. We proceed by way of contradiction. Suppose $a \in 2^\omega$ is so that $\omega_1^{L[a]} = \omega_1^V$. Then $2^\omega \cap L[a]$ has cardinality ω_1, and is thus a well-ordered uncountable set of reals. Thus by theorem 2, there is a non-measurable set of reals. This is a contradiction. $\square$

2 Rapid Filters and Non-Measurable Sets

Recall the following classical result.

Theorem 5. *Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then if we view $\mathcal{U}$ as a subset of 2^ω, $\mathcal{U}$ is non-measurable.*

Ideally we would cite this result and be done. Unfortunately in ZF+DC we are not guaranteed the existence of a non-principal ultrafilter on ω. We replace the notion of an ultrafilter with that of a rapid filter. This notion was first introduced by M. Talagrand [4].

Definition 1. A filter $F \subseteq 2^\omega$ is **rapid** if for every increasing $f : \omega \to \omega$ there is an $a \in F$ so that $|a \cap f(k)| \leq k$ for all $k \in \omega$. Here we see a as a subset of ω.

We show that rapid filters suffice to produce non-measurable sets.

Proposition 1. *Rapid filters are non-measurable.*

Proof Sketch. We first show that a measurable filter F cannot have measure 1. Let $O : 2^\omega \to 2^\omega$ be the bit-flip map. Note that $O[F]$ is the corresponding ideal defined by F. Then O is measure preserving and $F \cap O[F] = \emptyset$. Thus it cannot be that $m(F) = 1$. We proceed by showing that every measurable rapid filter is measure 1. For this it suffices to show that F meets every closed subset with positive measure. This is shown in [2]. $\square$

We now focus on the task of creating a rapid filter. Let $A \subseteq 2^\omega$ have order type ω_1.

Definition 2. Let $a \in 2^\omega$ and $W \subseteq 2^{<\omega}$. W **captures** a iff $\exists n \forall m \geq n (a \restriction_m \in W)$. W **captures** A iff W captures every a in A.

Definition 3. Let $W \subseteq 2^{<\omega}$ and $n \in \omega$. W **splits** on n if there is an $s \in W$ so that $lh(s) = n$ and $s^\frown 0, s^\frown 1 \in W$.

Define F as follows:

$$a \in F \iff \exists W \subseteq 2^{<\omega} (W \text{ captures } A \text{ and } W \text{ splits on } n \text{ iff } a(n) = 1)$$

Lemma 1. *F is a filter on ω.*

Proof. We first show that F is not empty. Let $a = \omega$ and $W = 2^{<\omega}$. Then W witnesses that $a \in F$. If $a \in F$ and $a \subseteq b$, then it is clear that $b \in F$. We now show that F is closed under intersections. Let $a, b \in F$. Let W_a and W_b witness this. Then $W_a \cap W_b$ captures a, and $W_a \cap W_b$ splits on n iff $a(n) = b(n) = 1$. So $a \cap b \in F$. Finally we show that $\emptyset \notin F$. To see this it suffices to show that any W which never splits cannot capture A. Suppose W never splits. Then any $s \in W$ can only help W capture at most one sequence $a \in 2^\omega$. But W is countable; so W can capture only countably many $a \in 2^\omega$. Since A is uncountable, this means W cannot capture A. Thus no W is a witness that $\emptyset \in F$ and therefore $\emptyset \notin F$. So F is a filter. $\square$

We have some work to do before showing that F is rapid. The following lemma is left unproven.

Lemma 2. *Suppose that the union of any sequence of ω_1 null sets is null. Then for any increasing $f : \omega \to \omega$, there is a $W \subseteq 2^{<\omega}$ so that captures W captures A and for all k*

$$|\{s \in W : lh(s) = f(k)\}| \leq k$$

Lemma 3. *Suppose that for any increasing $f : \omega \to \omega$ there is a $W \subseteq 2^{<\omega}$ so that W captures A and for all k $|\{s \in W : lh(s) = f(k)\}| \leq k$. Then F is rapid.*

Proof. Let $f : \omega \to \omega$ be increasing. Let W be as given by the hypothesis. Set

$$a(k) = 1 \iff W \text{ splits on } k$$

Then $a \in F$ as witnessed by W. WLOG suppose that for any $s \in W$ with $lh(s) = k$, there is a $t \in W$ so that $s \subseteq t$ and $lh(t) = k + 1$. Then W can split on k iff

$$|\{s \in W : lh(s) = k\}| < |\{s \in W : lh(s) = k + 1\}|$$

Thus

$$\begin{aligned}
|a \cap f(k)| &= |\{n < f(k) : W \text{ splits on } n\}| \\
&= |\{n < f(k) : \exists s \in W(lh(s) = n \wedge s \frown 0, s \frown 1 \in W)\}| \\
&\leq |\{s \in W : lh(s) < f(k) \wedge s \frown 0, s \frown 1 \in W)\}| \\
&\leq |\{s \in W : lh(s) = f(k)\}| \\
&\leq k
\end{aligned}$$

$\square$

Proposition 2. *If there is an $A \subseteq 2^\omega$ with ordertype ω_1 and the union of ω_1 null sets is null, then a rapid filter exists.*

3 Proof of the Main Theorem

Definition 4. Let $T \subseteq 2^\omega$. T is a **tail set** if

$$(a \in T \wedge \exists n \forall m \geq n(a(m) = b(m))) \implies b \in A$$

Theorem 6 (0-1 Law)**.** *If T is a measurable tail set, then $m(T) = 0$ or $m(T) = 1$.*

Proposition 3. *Assume that every set of reals is measurable. Then the union of ω_1 null sets is null.*

Proof. Let $T_\alpha \subseteq 2^\omega$ be null for $\alpha < \omega_1$ and $T = \bigcup_\alpha A_\alpha$. WLOG each T_α is tail set and the T_α are pairwise disjoint. Then T is a tail set. Since T is measurable, it suffices by the 0-1 law to show that T does not have measure 1. We proceed by contradiction. Let

$$U = \{\langle a, b\rangle : \exists \alpha, \beta \in \omega_1(\alpha < \beta \wedge a \in T_\alpha \wedge b \in T_\beta)\}$$

Then U is also a tail set. We will show that U cannot have measure 0 or 1. Let $a \in T$, say $a \in T_\alpha$. Since $m\left(\bigcup_{\xi \leq \alpha} T_\xi\right) = 0$, it must be that

$$m\left(\bigcup_{\xi > \alpha} T_\xi\right) = m(\{b : \langle a, b\rangle \in U\}) = 1$$

So for all $a \in T$, $m(\{b : \langle a, b\rangle \in U\}) = 1$. By Fubini's theorem it follows that U cannot have measure 0. Similarly, we can show that for all $b \in T$, $m(\{a : \langle a, b\rangle \in U\}) = 0$. But then for all $b \in U$ we have that $m(\{a : \langle a, b\rangle \notin U\}) = 1$. So again by Fubini's theorem we have that $2^\omega - U$ cannot have measure 0. This is the desired contradiction. □

Theorem 7 (ZF + DC)**.** *If there is an uncountable well-ordered set of reals, then there is a non-measurable set.*

Proof. We proceed by contrapositive. Suppose that every set of reals is measurable. Then by proposition 3, any uncountable union of null sets is null. But then by prop 2, there is a rapid filter on ω if there is an uncountable set of reals. By proposition 1, rapid filters are non-measurable. So it must be that there is no uncountable set of reals. □

References

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