

Limited Information Strategies and Discrete Selectivity

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1 Definitions

Definition 1. The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n < \omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n < \omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

Definition 3. Let \mathcal{O}_X be the collection of open covers for a topological space X . Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for X (M for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

Definition 4. An ω -cover \mathcal{U} for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

Definition 5. Let Ω_X be the collection of ω -covers for a topological space X . Then $S_{fin}(\Omega_X, \Omega_X)$ is the Ω -Menger property for X (ΩM for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the Ω -Menger game.

Definition 6. Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \text{cl } A$. (Call A a *blade* of x .) Then $S_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness property* for X at x (CFT_x for short), and $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness game* for X at x .

Definition 7. A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.

Definition 8. Let \mathcal{D}_X be the collection of dense subsets of a topological space X . Then $S_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness game* for X at x .

Definition 9. A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Definition 10. $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for X (SS for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for X .

Definition 11. A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$ for $\langle A_0, \dots, A_n \rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \dots, A_n \rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 12. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write $\text{II} \uparrow_{\text{mark}} G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 13. In some instances, player I will be able to win a game regardless of what II is playing. In this case, it is possible to have a strategy for I which depends only on the round of the game. We say I has a *pre-determined strategy* and write $\text{I} \uparrow_{\text{pre}} G$.

Notation 14. If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property P , then we say $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^+ (“strategically P ”), and $\text{II} \uparrow_{\text{mark}} G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^{+\text{mark}}$ (“Markov P ”). Of course, $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$.

Notation 15. Let $S_1(\mathcal{A}, \mathcal{B}), G_1(\mathcal{A}, \mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A}, \mathcal{B}), G_{fin}(\mathcal{A}, \mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the “strong Menger” property is commonly known as “Rothberger”. We will thus call “strong Ω -Menger” “ Ω -Rothberger” and shorten it with ΩR , and otherwise attach the prefix “s” when abbreviating to all other strong variants.

In addition to pure selection games, we also will be playing various point-picking games.

Definition 16. Set $T(X)$ to be the non-empty open subsets of X . The *point-open game* for X , denoted $PO(X)$, is played as follows. Each round, player I plays a point $x_n \in X$ and player II plays an open sets U_n with the property that $x_n \in U_n$. I wins the play of the game if $X = \bigcup_n U_n$.

The *finite-open game* for X , denoted $FO(X)$, is played similarly, except that I now plays finite subsets of X , and II’s open sets must cover I’s finite set. Note that $PO(X)$ is just $sFO(X)$. $\Omega FO(X)$ and $\Omega PO(X)$ are defined according to convention: I now wins if $\{U_n : n \in \omega\}$ forms an ω -cover of X .

Definition 17. Let $x \in X$. *Gruenhage’s W -game* for x , denoted $Gru_{\vec{O}, P}(X, x)$, is played as follows. Each round, player I plays an open set U_n with the property that $x \in U_n$ and player II plays a point $x_n \in U_n$. I wins if $x_n \rightarrow x$.

The *closure game* for x , denoted $CL(X, x)$, is played the same as Gruenhage’s W -game, but now I wins if $x \in \overline{\{x_n : n \in \omega\}}$. Note that this is $G_1(T(X), \Omega_{X, x})$.

The *discrete selectivity game*, denoted $CD(X)$, is also played the same as Gruenhage’s W -game, but now II wins if $\{x_n : n \in \omega\}$ is closed and discrete. Note that this is $G_1(T(X), CD)$ if we let CD denoted the closed discrete subsets of X .

Definition 18. A space X is *discretely selective* iff

2 2-marks in $CD(X)$

Let $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$.

Game 19. Let G be the following game. During round n , player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

For $f \in \omega^\alpha$, let $f^\leftarrow[n] = \{\beta < \alpha : f(\beta) < n\}$.

Proposition 20. $\Pi \uparrow_{2\text{-mark}} G$.

Proof. Let $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark σ for Π by

$$\sigma(\langle \alpha \rangle, 0) = \emptyset$$

and

$$\sigma(\langle \alpha, \beta \rangle, n+1) = f_\beta^\leftarrow[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let ν be an attack by I against σ , and let $\gamma < \nu(n)$ for $N \leq n < \omega$. If $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N \leq n < \omega$, then $\gamma \in \sigma(\langle \nu(n), \nu(n+1) \rangle, n+1)$ for infinitely-many $N \leq n < \omega$. Otherwise $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$ for cofinitely-many $N \leq n < \omega$, so $\gamma \in \sigma(\langle \nu(n), \nu(n+1) \rangle, n+1)$ for cofinitely-many $N \leq n < \omega$. Therefore σ is a winning 2-mark. \square

Theorem 21. $I \uparrow_{2\text{-mark}} CD(C_p(\omega_1 + 1))$

Proof. Let σ be a winning 2-mark for Π in G .

Given a point $f \in C_p(\omega_1 + 1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f \rangle, 1) = [\mathbf{0}; \sigma(\langle \alpha_f \rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g \rangle, n+2) = [\mathbf{0}; \sigma(\langle \alpha_f, \alpha_g \rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let ν be a legal attack by Π against σ . For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle \alpha_{\nu(n)}, \alpha_{\nu(n+1)} \rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \text{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \text{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \text{cl}\{\nu(n) : n < \omega\}$. \square

3 Combining game results

Theorem 22. *The following are equivalent for $T_{3,5}$ spaces X .*

- a) X is R^+ , that is, $\Pi \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$.
- b) $I \uparrow PO(X)$.
- c) $I \uparrow FO(X)$.
- d) $I \uparrow \Omega FO(X)$.
- e) $I \uparrow Gru_{\vec{\mathcal{O}}, P}(C_p(X), \mathbf{0})$.
- f) $I \uparrow Gru_{\vec{\mathcal{O}}, P}(C_p(X), \mathbf{0})$.
- g) $I \uparrow CL(C_p(X), \mathbf{0})$.
- h) $I \uparrow CD(C_p(X))$.
- i) X is ΩR^+ , that is, $\Pi \uparrow G_1(\Omega_X, \Omega_X)$.
- j) $C_p(X)$ is $sCFT^+$, that is, $\Pi \uparrow G_1(\Omega_{C_p(X), \mathbf{0}}, \Omega_{C_p(X), \mathbf{0}})$.
- k) $C_p(X)$ is $sCDFT^+$, that is, $\Pi \uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X), \mathbf{0}})$.

Proof. (a) \Leftrightarrow (b) is a well-known result of Galvin.

(b) \Leftrightarrow (c) is 4.3 of [Telgarksy 1975].

(d) \Leftrightarrow (c) is clear, but we need to show that (a) \Leftrightarrow (d). So assume X is R^+ , which is equivalent to ΩR^+ . Let σ be a winning strategy for II in $G_1(\Omega_X, \Omega_X)$. Let $T(X)$ be the non-empty open sets of X , and let $s \in T(X)^{<\omega}$. Assume $\tau(t) \in [X]^{<\omega}$ is defined for all $t < s$, and $\mathcal{U}_t \in \Omega_X$ is defined for all $\emptyset < t \leq s$.

Suppose that for all $F \in [X]^{<\omega}$, there existed $U_F \in T(X)$ containing F such that for all $\mathcal{U} \in \Omega_X$, $U_F \neq \sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, U_s, \mathcal{U} \rangle)$. Let $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$. Then $\sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, U_s, \mathcal{U} \rangle)$ must equal some U_F , demonstrating a contradiction.

So there exists $\tau(s) \in [X]^{<\omega}$ such that for all $U \in T(X)$ containing $\tau(s)$, there exists $\mathcal{U}_{s \smallfrown \langle U \rangle} \in \Omega_X$ such that $U = \sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s \smallfrown \langle U \rangle} \rangle)$. (To complete the induction, $\mathcal{U}_{s \smallfrown \langle U \rangle}$ may be chosen arbitrarily for all other $U \in T(X)$.)

So τ is a strategy for I in $\Omega FO(X)$. Let ν legally attack τ , so $\tau(\nu \upharpoonright n) \subseteq \nu(n)$ for all $n < \omega$. It follows that $\nu(n) = \sigma(\langle \mathcal{U}_{\nu \upharpoonright 1}, \dots, \mathcal{U}_{\nu \upharpoonright n}, \mathcal{U}_{\nu \upharpoonright n+1} \rangle)$. Since $\langle \mathcal{U}_{\nu \upharpoonright 1}, \mathcal{U}_{\nu \upharpoonright 2}, \dots \rangle$ is a legal attack against σ , it follows that $\{\sigma(\langle \mathcal{U}_{\nu \upharpoonright 1}, \dots, \mathcal{U}_{\nu \upharpoonright n+1} \rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$ is an ω -cover. Therefore τ is a winning strategy, verifying $I \uparrow \Omega FO(X)$.

The equivalence of (b), (e), (g), and (h) are given as 3.8 of [Tkachuk 2017].

The equivalence of (e) is (f) are given in [[https://doi.org/10.1016/0016-660X\(76\)90024-6](https://doi.org/10.1016/0016-660X(76)90024-6)].

The equivalence of (i), (j), and (k) are due to Clontz's Menger preprint.

(k) \Leftrightarrow (g) follows from 3.18a of [Tkachuk 2017]. \square

Tkachuk showed the following in [CLOSEDDISCRETESELECTIONS].

Theorem 23. *The following are equivalent for $T_{3.5}$ spaces X .*

a) X is uncountable.

b) $C_p(X)$ has discrete selectivity, that is, $I \not\uparrow_{pre} CD(C_p(X))$.

Clontz came across these in grad school (didn't make it into the dissertation):

Theorem 24. $I \uparrow_{pre} PO(X)$ if and only if $II \uparrow_{mark} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Let σ be a winning Markov strategy for II in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover \mathcal{U} , $\sigma(\mathcal{U}, n) \neq U_x$. Then $\sigma(\{U_x : x \in X\}, n) \not\subseteq \{U_x : x \in X\}$, a contradiction.

So for each $n < \omega$, there exists $\tau(n) \in X$ such that for any open neighborhood U of $\tau(n)$, there exists an open cover \mathcal{U}_n such that $\sigma(\mathcal{U}_n, n) = U$. Then τ is a predetermined strategy for I in $PO(X)$.

It is also winning: for every attack f against τ , note that $f(n)$ is an open neighborhood of $\tau(n)$, so choose \mathcal{U}_n such that $\sigma(\mathcal{U}_n, n) = f(n)$. Then since $\langle \mathcal{U}_0, \mathcal{U}_1, \dots \rangle$ is a legal attack against σ , it follows that $\{f(n) : n < \omega\}$ is an open cover of X . Therefore τ is a winning predetermined strategy.

Now let σ be a winning predetermined strategy for I in $PO(X)$. For an open cover \mathcal{U} of X and $n < \omega$, let $\tau(\mathcal{U}, n)$ be any open set in \mathcal{U} containing $\sigma(n)$. It follows that τ is a winning Markov strategy for II in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. \square

Theorem 25. $II \uparrow_{mark} PO(X)$ if and only if $I \uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Let σ be a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. For $x \in X$ and $n < \omega$, let $\tau(x, n)$ be any open set in $\sigma(n)$ containing x . It follows that τ is a winning Markov strategy for II in $PO(X)$.

Now let σ be a winning Markov strategy for II in $PO(X)$. We may defined the open cover $\tau(n) = \{\sigma(x, n) : x \in X\}$ of X . It follows that τ is a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. \square

Combining with several other results in the literature, we may observe the following.

Theorem 26. *The following are equivalent for $T_{3,5}$ spaces X .*

- a) X is countable.
- b) X is R^{+mark} .
- c) $I \uparrow_{pre} PO(X)$.
- d) $I \uparrow_{pre} FO(X)$.
- e) $I \uparrow_{pre} \Omega FO(X)$.
- f) $C_p(X)$ is first-countable.
- g) $I \uparrow_{pre} Gru_{\vec{O}, P}(C_p(X), \mathbf{0})$.
- h) $I \uparrow_{pre} Gru_{\vec{\tilde{O}}, P}(C_p(X), \mathbf{0})$.
- i) $I \uparrow_{pre} CL(C_p(X), \mathbf{0})$.
- j) $I \uparrow_{pre} CD(C_p(X))$.
- k) X is ΩR^{+mark} .
- l) $C_p(X)$ is $sCFT^{+mark}$.
- m) $C_p(X)$ is $sCDFT^{+mark}$.

Proof. (a) implies (c) is straight-forward. So let σ be a predetermined strategy for I in $PO(X)$. If $x \notin \{\sigma(n) : n < \omega\}$, let $f(n) = X \setminus \{x\}$ for all $n < \omega$. It follows that f is a legal counter-attack for II defeating σ . Thus not (a) implies not (c).

The equivalence of (b) and (c) was shown above.

Clearly (c) implies (d), so we will see that (d) implies (a). Let $\sigma(n)$ be a pre-determined strategy for I for $FO(X)$. Towards a contradiction, suppose that there is some $x \in X \setminus \bigcup_n \sigma(n)$. II could then play $FO(X)$ as follows. At round n II can play an open set U_n which contains $\sigma(n)$ but excludes x . Then $x \notin \bigcup_n U_n$, and so I has lost. This is a contradiction. So $X = \bigcup_n \sigma(n)$, which means it is countable.

It also clear that (e) implies (d), we will show that (a) implies (e). If X is countable, then so is $[X]^{<\omega}$, enumerate it as $\{s_n : n \in \omega\}$. I's pre-determined strategy for $\Omega FO(X)$ is to play s_n are round n . Clearly whatever II plays will be an ω -cover. Thus (a) - (e) are equivalent.

It is well-known and easy to see that (a) is equivalent to (f).

To see that (f) implies (g), note that we can find a sequence of open sets U_n so that $\mathbf{0} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$ for all n . I simply plays U_n at turn n , and whatever x_n are played by II must converge to x .

Clearly (g) implies (i) which in turn implies (j), which is equivalent to (a) by [CLOSEDDIS-CRETESELECTIONS]

Clontz showed (g) and (h) are equivalent in his dissertation; it's not hard to prove this.

Clontz showed that (k), (l), and (m) are equivalent in his Menger/CFT preprint.

Note that (k) implies (b) implies (a). So the last thing we need to show is that (e) implies (k). Let $\sigma(n)$ be a pre-determined strategy for I for $\Omega FO(X)$. We define a Markov strategy, $\tau(\mathcal{U}, n)$ for II for ΩR as follows. At round n suppose I has played \mathcal{U} . As \mathcal{U} must be an ω -cover, there is a $U \in \mathcal{U}$ so that $\sigma(n) \subseteq U$. II plays such a U_n . Now suppose this game has been played according to τ , and that I has played \mathcal{U}_n for $n < \omega$. Then the sequence of open sets $\tau(\mathcal{U}_n, n)$ forms a legal play against σ for $\Omega FO(X)$. Thus $\{\tau(\mathcal{U}_n, n) : n \in \omega\}$ is an ω -cover of X and so τ is a winning Markov strategy. \square

In that paper, Tkachuk characterizes $\text{II} \uparrow \Omega FO(X)$ as the second player having an “almost winning strategy” (II can prevent I from constructing an ω -cover but perhaps not an arbitrary open cover) in $PO(X)$, which he conflates with $FO(X)$ as they are equivalent for “completely” winning perfect information strategies.

But they cannot be interchanged in general. Note that $\text{II} \uparrow_{\text{tact}} \Omega PO(2)$, where 2 is the two-point discrete space: let $\sigma(\langle x \rangle) = \{x\}$. Since every ω -cover of 2 includes 2, and σ never produces 2, this is a winning tactic. But since 2 is countable, 2 is ΩR^{+mark} . So $\Omega PO(X)$ is a very different game than those described previously.

Now we turn our attention to the opponent.

Theorem 27. *The following are equivalent for all spaces X .*

- a) $\text{II} \uparrow FO(X)$
- b) $\text{II} \uparrow PO(X)$
- c) $\text{I} \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$
- d) X is not R , that is, $\text{I} \uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$
- e) $\text{II} \uparrow_{mark} PO(X)$
- f) $\text{II} \uparrow_{mark} FO(X)$

Proof. (a) \Leftrightarrow (b) is 4.4 of [Telgarksy 1975].

The duality of $PO(X)$ and $G_1(\mathcal{O}_X, \mathcal{O}_X)$ for both players when considering perfect information is a well-known result of Galvin. So (b) is equivalent to (c).

The equivalence of (c) and (d) is just a restatement of Pawlikowski's result that the Rothberger selection principle is equivalent to $\text{I} \not\uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$, since the Rothberger selection principle is equivalent to $\text{I} \not\uparrow_{pre} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

(d) and (e) were shown to be equivalent above.

Finally, (f) implies (e) is obvious. Let $b : \omega^2 \rightarrow \omega$ be a bijection. Given a winning Markov strategy σ for II in $PO(X)$, define $\tau(F_n, n) = \bigcup \{\sigma(x(i, n), b(i, n)) : i < \omega\}$ where $F_n = \{x(i, n) : i < \omega\}$ (this indexing will cause at least one point to be repeated infinitely often, but this won't

be a problem). So given an attack $\langle F_0, F_1, \dots \rangle$ against τ , consider the attack g against σ , where $g(n) = x_{b^{\leftarrow}(n)}$. It follows that

$$X \neq \bigcup \{\sigma(g(n), n) : n < \omega\} = \bigcup \{\sigma(x(i, n), b(i, n)) : i, n < \omega\} = \bigcup \{\tau(F_n, n) : n < \omega\}$$

and therefore τ is a winning Markov strategy for II. Thus (e) implies (f). \square

Theorem 28. *The following are equivalent for all spaces X .*

- a) $\text{II} \uparrow \Omega FO(X)$
- b) $\text{I} \uparrow G_1(\Omega_X, \Omega_X)$
- c) X is not ΩR , that is, $\text{I} \uparrow_{pre} G_1(\Omega_X, \Omega_X)$
- d) $\text{II} \uparrow_{mark} \Omega FO(X)$

Proof. Let σ be a winning strategy for II in $\Omega FO(X)$. For $s \in ([X]^{<\omega})^{<\omega}$, let $\mathcal{U}_s = \{\sigma(s \smallfrown \langle F \rangle : F \in [X]^{<\omega}\}$. Then define the strategy τ for I by $\tau(s) = \sigma(\langle \mathcal{U}_{s \upharpoonright 0}, \dots, \mathcal{U}_s \rangle)$. Then every attack f against τ yields $g \in ([X]^{<\omega})^\omega$ such that $f(n) = \sigma(g \upharpoonright n + 1)$. Thus $\{f(n) : n < \omega\} = \{\sigma(g \upharpoonright n + 1) : n < \omega\}$ is not an ω -cover, so τ is a winning strategy, verifying that (a) implies (b).

The equivalence of (b) and (c) is given by Thm2 of [http://eudml.org/doc/212209].

Let σ be a winning predetermined strategy for I in $G_1(\Omega_x, \Omega_x)$. For $F \in [X]^{<\omega}$ and $n < \omega$, let $\tau(F, n)$ be any open set in $\sigma(n)$ containing F . It follows that τ is a winning Markov strategy for II in $\Omega FO(X)$, verifying that (c) implies (d).

(d) implies (a) is trivial, so the proof is complete. \square

ΩR is equivalent to all finite powers being R : Thm3 of [http://eudml.org/doc/212209]. These notions cannot coincide: see Thm9 of [http://dx.doi.org/10.1016/j.topol.2013.07.022] for a consistent example of a R space whose square is not R , so therefore not ΩR . Note the distinction with strategies for the opponent, as R^+ is equivalent to ΩR^+ and R^{+mark} is equivalent to ΩR^{+mark} .

Corollary 29. *The following are equivalent for all $T_{3.5}$ spaces.*

- a) $\text{II} \uparrow \Omega FO(X)$
- b) $\text{II} \uparrow_{mark} \Omega FO(X)$
- c) $\text{I} \uparrow G_1(\Omega_X, \Omega_X)$
- d) X is not ΩR , that is, $\text{I} \uparrow_{pre} G_1(\Omega_X, \Omega_X)$
- e) $C_p(X)$ is not $sCFT$, that is, $\text{I} \uparrow_{pre} G_1(\Omega_{C_p(X), \mathbf{0}}, \Omega_{C_p(X), \mathbf{0}})$.
- f) $\text{I} \uparrow G_1(\Omega_{C_p(X), \mathbf{0}}, \Omega_{C_p(X), \mathbf{0}})$.
- g) $C_p(X)$ is not $sCDFT$, that is, $\text{I} \uparrow_{pre} G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X), \mathbf{0}})$.
- h) $\text{I} \uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X), \mathbf{0}})$.

Proof. (a)-(d) were just shown. The equivalence of (d),(e),(g) was shown by Sakai. The equivalence of (e) and (f) is given in 4.37 of [Comb. of Open Covers in Gen Prog Top III].

Of course (g) implies (h). And since $\mathcal{D}_{C_p(X)} \subseteq \Omega_{C_p(X), \mathbf{0}}$, any winning strategy for I in $G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X), \mathbf{0}})$ is a winning strategy for I in $G_1(\Omega_{C_p(X), \mathbf{0}}, \Omega_{C_p(X), \mathbf{0}})$, so (h) implies (f). \square

Theorem 30. *The following properties imply lower properites for all spaces X .*

- a) $I \uparrow G_1(\mathcal{D}_X, \Omega_{X,x})$
- b) $II \uparrow CL(X, x)$
- c) $II \uparrow Gru_{\tilde{O}, P}^{\omega}(X, x)$
- d) $I \uparrow G_1(\Omega_{X,x}, \Omega_{X,x})$

Proof. Begin by letting σ be a winning strategy for I in $G_1(\mathcal{D}_X, \Omega_{X,x})$. For $s \in T_X^{<\omega}$, assume $\tau(s \upharpoonright i+1)$ is defined for $i < |s|$, defining $s' \in X^{|s|}$ by $s'(i) = \tau(s \upharpoonright i+1)$, and let $\tau(s \frown \langle U \rangle) \in \sigma(s') \cap U$. So τ is a strategy for II in $CL(X, x)$. Then for any attack f against τ , an attack f' against σ is defined by $f'(i) = \tau(f \upharpoonright i+1)$. It follows that $\{f'(i) : i < \omega\} = \{\tau(f \upharpoonright i+1) : i < \omega\} \notin \Omega_{X,x}$, so τ is a winning strategy, witnessing (a) implies (b).

Let σ be a winning strategy for II in $CL(X, x)$. Then σ is also a winning strategy for II in $Gru_{\tilde{O}, P}^{\omega}(X, x)$, so (b) implies (c).

Given a winning strategy σ for II in $Gru_{\tilde{O}, P}^{\omega}(X, x)$, let $s \in T_{X,x}^{<\omega}$ and suppose and $B_t \in \Omega_{X,x}$ is defined for all $t < s$. Then let $B_s = \{\sigma(s \frown \langle U \rangle) : U \in T_{X,x}\}$; it's clear that $B_s \in \Omega_{X,x}$. Define τ for I in $G_1(\Omega_{X,x}, \Omega_{X,x})$ by $\tau(r) = B_{r'}$ where $r' \in T_{X,x}^{|r|}$ satisfies $r(i) = \sigma(r' \upharpoonright i+1)$ for all $i < |r|$. Then an attack f against τ yields an attack f' against σ such that $f(i) = \sigma(f' \upharpoonright i+1)$ for all $i < \omega$. Since σ is a winning strategy, it follows that $\{f(i) : i < \omega\} = \{\sigma(f' \upharpoonright i+1) : i < \omega\} \notin \Omega_{X,x}$. This verifies (c) implies (d). \square

Theorem 31. *The following properties imply lower properites for all spaces X .*

- a) $I \uparrow_{pre} G_1(\mathcal{D}_X, \Omega_{X,x})$
- b) $II \uparrow_{mark} CL(X, x)$
- c) $II \uparrow_{mark} Gru_{\tilde{O}, P}^{\omega}(X, x)$
- d) $I \uparrow_{pre} G_1(\Omega_{X,x}, \Omega_{X,x})$

Proof. Begin by letting σ be a winning predetermined strategy for I in $G_1(\mathcal{D}_X, \Omega_{X,x})$. Define the Markov strategy τ for II in $CL(X, x)$ by $\tau(U, n) \in \sigma(n) \cap U$. Since $\tau(U, n) \in \sigma(n)$ for all $n < \omega$, it's clear that $\{\tau(U, n) : n < \omega\} \notin \Omega_{X,x}$, making τ a winning strategy, witnessing (a) implies (b).

Let σ be a winning Markov strategy for II in $CL(X, x)$. Then σ is also a winning Markov strategy for II in $Gru_{\tilde{O}, P}^{\omega}(X, x)$, so (b) implies (c).

Given a winning Markov strategy σ for II in $Gru_{\tilde{O}, P}^{\omega}(X, x)$, let $\tau(n) = \{\sigma(U, n) : U \in T_{X,x}\}$. Then τ is a predetermined strategy for I in $G_1(\Omega_{X,x}, \Omega_{X,x})$. For any attack f against τ , $f(n) = \sigma(g(n), n)$ for some $g(n) \in T_{X,x}$. But then g is an attack against σ , and thus $\{f(n) : n < \omega\} = \{\sigma(g(n), n) : n < \omega\} \notin \Omega_{X,x}$, so we have (c) implies (d). \square

Note that for $C_p(X)$, (a)-(d) in both of the above theorems are actually equivalent.

Corollary 32. *The following are equivalent for all $T_{3.5}$ spaces.*

- a) $\text{II} \uparrow \Omega FO(X)$
- b) $\text{II} \underset{\text{mark}}{\uparrow} \Omega FO(X)$
- c) $\text{I} \uparrow G_1(\Omega_X, \Omega_X)$
- d) X is not ΩR , that is, $\text{I} \underset{\text{pre}}{\uparrow} G_1(\Omega_X, \Omega_X)$
- e) $C_p(X)$ is not $sCFT$, that is, $\text{I} \underset{\text{pre}}{\uparrow} G_1(\Omega_{C_p(X)}, \Omega_{C_p(X)}, \mathbf{0})$.
- f) $\text{I} \uparrow G_1(\Omega_{C_p(X)}, \Omega_{C_p(X)}, \mathbf{0})$
- g) $C_p(X)$ is not $sCDFT$, that is, $\text{I} \underset{\text{pre}}{\uparrow} G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X)}, \mathbf{0})$.
- h) $\text{I} \uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X)}, \mathbf{0})$
- i) $\text{II} \underset{\text{mark}}{\uparrow} \text{Gru}_{\vec{O}, P}^{\omega}(C_p(X), \mathbf{0})$
- j) $\text{II} \uparrow \text{Gru}_{\vec{O}, P}^{\omega}(C_p(X), \mathbf{0})$
- k) $\text{II} \underset{\text{mark}}{\uparrow} CL(C_p(X), \mathbf{0})$
- l) $\text{II} \uparrow CL(C_p(X), \mathbf{0})$
- m) $\text{II} \underset{\text{mark}}{\uparrow} CD(C_p(X))$
- n) $\text{II} \uparrow CD(C_p(X))$

Proof. We've established (a)-(l) are equivalent. Assuming (b), we adapt Proposition 3.9 of [Tkachuk Two Point-Picking Games] as follows. Let σ be a winning Markov strategy for II in $\Omega FO(X)$. Then for $U = [\mathbf{x}(U), \text{supp}(U), \epsilon(U)] \in T_{C_p(X)}$, let $\tau(U, n) \in C_p(X)$ satisfy $\tau(U, n)(x) = \mathbf{x}(U)(x)$ for $x \in F$ and $\tau(U, n)(x) = n$ for $x \in X \setminus \sigma(U, n)$. Then τ is a Markov strategy for II, and when it is attacked by f , we note that $\{\sigma(\text{supp}(f(n)), n) : n < \omega\}$ is not an ω -cover. So choose $G \in [X]^{<\omega}$ such that $G \not\subseteq \sigma(\text{supp}(f(n)), n)$ for all $n < \omega$. Then for $\mathbf{y} \in C_p(X)$, choose m such that $\mathbf{y}(x) < m$ for all $x \in G$. Note then that for $n \geq m$, there exists $x \in G \setminus \sigma(f(n), n)$ such that $\tau(f(n), n)(x) = n \geq m$. Then $\{\mathbf{z} \in C_p(X) : \mathbf{z}(x) < m \text{ for all } x \in G\}$ is an open neighborhood of \mathbf{y} that misses $\tau(f(n), n)$ for all $n \geq m$, so it follows that $\{\tau(f(n), n) : n < \omega\}$ is closed and discrete in $C_p(X)$. Therefore τ is a winning Markov strategy, verifying (b) implies (m).

It's clear that (m) implies (n), so finally note that a winning strategy for II in $CD(C_p(X))$ is also a winning strategy for II in $CL(C_p(X), \mathbf{0})$, so (n) implies (l). \square

The equivalence of (a) and (n) answers Quesiton 4.6 of Tkachuk in [Two Point-picking games].