# Limited Information Strategies and Discrete Selectivity

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## 1 Definitions

**Definition 1.** The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

**Definition 2.** The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

**Definition 3.** Let $\mathcal{O}_X$ be the collection of open covers for a topological space $X$. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for $X$ ($M$ for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

**Definition 4.** An *$\omega$-cover* $\mathcal{U}$ for a topological space $X$ is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

**Definition 5.** Let $\Omega_X$ be the collection of $\omega$-covers for a topological space $X$. Then $S_{fin}(\Omega_X, \Omega_X)$ is the *$\Omega$-Menger property* for $X$ ($\Omega M$ for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the *$\Omega$-Menger game*.

**Definition 6.** Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \operatorname{cl} A$. (Call $A$ a *blade* of $x$.) Then $S_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness property* for $X$ at $x$ ($CFT_x$ for short), and $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness game* for $X$ at $x$.

**Definition 7.** A space $X$ has *countable fan tightness* ($CFT$ for short) if it has countable fan tightness at each point $x \in X$.

**Definition 8.** Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space $X$. Then $S_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness property* for $X$ at $x$ ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness game* for $X$ at $x$.

**Definition 9.** A space $X$ has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

**Definition 10.** $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for $X$ ($SS$ for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for $X$.

**Definition 11.** A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function $\sigma$ satisfying $\sigma(\langle A_0, \ldots, A_n\rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \ldots, A_n\rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \ldots, A_n\rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

**Definition 12.** A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function $\sigma$ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write II $\underset{\text{mark}}{\uparrow}$ $G_{fin}(\mathcal{A}, \mathcal{B})$.

**Definition 13.** In some instances, player I will be able to win a game regardless of what II is playing. In this case, it is possible to have a strategy for I which depends only on the round of the game. We say I has a *pre-determined strategy* and write I $\underset{\text{pre}}{\uparrow}$ $G$.

**Notation 14.** If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property $P$, then we say II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^+$ ("strategically $P$"), and II $\underset{\text{mark}}{\uparrow}$ $G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^{+\text{mark}}$ ("Markov $P$"). Of course, $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$.

**Notation 15.** Let $S_1(\mathcal{A}, \mathcal{B}), G_1(\mathcal{A}, \mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A}, \mathcal{B}), G_{fin}(\mathcal{A}, \mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the "strong Menger" property is commonly known as "Rothberger". We will thus call "strong $\Omega$-Menger" "$\Omega$-Rothberger" and shorten it with $\Omega R$, and otherwise attach the prefix "s" when abbreviating to all other strong variants.

In addition to pure selection games, we also will be playing various point-picking games.

**Definition 16.** Set $T(X)$ to be the non-empty open subsets of $X$. The *point-open game* for $X$, denoted $PO(X)$, is played as follows. Each round, player I plays a point $x_n \in X$ and player II plays an open sets $U_n$ with the property that $x_n \in U_n$. I wins the play of the game if $X = \bigcup_n U_n$.

The *finite-open game* for $X$, denoted $FO(x)$, is played similarly, except that I now plays finite subsets of $X$, and II's open sets must cover I's finite set. Note that $PO(X)$ is just $sFO(S)$. $\Omega FO(X)$ and $\Omega PO(X)$ are defined according to convention: I now wins if $\{U_n : n \in \omega\}$ forms an $\omega$-cover of $X$.

**Definition 17.** Let $x \in X$. *Gruenhage's $W$-game* for $x$, denoted $Gru_{\vec{O},P}(X, x)$, is played as follows. Each round, player I plays an open set $U_n$ with the property that $x \in U_n$ and player II plays a point $x_n \in U_n$. I wins if $x_n \to x$.

The *closure game* for $x$, denoted $CL(X, x)$, is played the same as Gruenhage's $W$-game, but now I wins if $x \in \overline{\{x_n : n \in \omega\}}$. Note that this is $G_1(T(X), \Omega_{X,x})$.

The *discrete selectivity game*, denoted $CD(X)$, is also played the same as Gruenhage's $W$-game, but now II wins if $\{x_n : n \in \omega\}$ is closed and discrete. Note that this is $G_1(T(X), \mathcal{CD})$ if we let $\mathcal{CD}$ denoted the closed discrete subsets of $X$.

**Definition 18.** A space $X$ is *discretely selective* iff

# 2 2-marks in CD(X)

Let $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$.

**Game 19.** Let $G$ be the following game. During round $n$, player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

For $f \in \omega^\alpha$, let $f^\leftarrow[n] = \{\beta < \alpha : f(\beta) < n\}$.

**Proposition 20.** II $\underset{2\text{-}mark}{\uparrow} G$.

*Proof.* Let $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark $\sigma$ for II by

$$\sigma(\langle\alpha\rangle, 0) = \emptyset$$

and

$$\sigma(\langle\alpha, \beta\rangle, n+1) \ = f_\beta^{\leftarrow}[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let $\nu$ be an attack by I against $\sigma$, and let $\gamma < \nu(n)$ for $N \leq n < \omega$. If $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N \leq n < \omega$, then $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for infinitely-many $N \leq n < \omega$. Otherwise $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$ for cofinitely-many $N \leq n < \omega$, so $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for cofinitely-many $N \leq n < \omega$. Therefore $\sigma$ is a winning 2-mark. □

**Theorem 21.** I $\underset{2\text{-}mark}{\uparrow} CD(C_p(\omega_1 + 1))$

*Proof.* Let $\sigma$ be a winning 2-mark for II in $G$.

Given a point $f \in C_p(\omega_1 + 1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f\rangle, 1) = [\mathbf{0}; \sigma(\langle\alpha_f\rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g\rangle, n+2) = [\mathbf{0}; \sigma(\langle\alpha_f, \alpha_g\rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let $\nu$ be a legal attack by II against $\sigma$. For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle\alpha_{\nu(n)}, \alpha_{\nu(n+1)}\rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \mathrm{cl}\{\nu(n) : n < \omega\}$. □

# 3 Combining game results

**Theorem 22.** *The following are equivalent for $T_{3.5}$ spaces $X$.*

*a)* $X$ *is* $R^+$, *that is,* II $\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$.

*b)* I $\uparrow PO(X)$.

*c)* I $\uparrow FO(X)$.

*d)* I $\uparrow \Omega FO(x)$.

*e)* I $\uparrow Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$.

*f)* I $\uparrow Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.

*g)* I $\uparrow CL(C_p(X), \mathbf{0})$.

*h)* I $\uparrow CD(C_p(X))$.

*i)* $X$ *is* $\Omega R^+$, *that is,* II $\uparrow G_1(\Omega_X, \Omega_X)$.

*j)* $C_p(X)$ *is* $sCFT^+$, *that is,* II $\uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

*k)* $C_p(X)$ *is* $sCDFT^+$, *that is,* II $\uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

*Proof.* (a) $\Leftrightarrow$ (b) is a well-known result of Galvin.

(b) $\Leftrightarrow$ (c) is 4.3 of [Telgarksy 1975].

(d) $\Leftrightarrow$ (c) is clear, but we need to show that (a) $\Leftrightarrow$ (d). So assume $X$ is $R^+$, which is equivalent to $\Omega R^+$. Let $\sigma$ be a winning strategy for II in $G_1(\Omega_X, \Omega_X)$. Let $T(X)$ be the non-empty open sets of $X$, and let $s \in T(X)^{<\omega}$. Assume $\tau(t) \in [X]^{<\omega}$ is defined for all $t < s$, and $\mathcal{U}_t \in \Omega_X$ is defined for all $\emptyset < t \leq s$.

Suppose that for all $F \in [X]^{<\omega}$, there existed $U_F \in T(X)$ containing $F$ such that for all $\mathcal{U} \in \Omega_X$, $U_F \neq \sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, U_s, \mathcal{U}\rangle)$. Let $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$. Then $\sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, U_s, \mathcal{U}\rangle)$ must equal some $U_F$, demonstrating a contradiction.

So there exists $\tau(s) \in [X]^{<\omega}$ such that for all $U \in T(X)$ containing $\tau(s)$, there exists $\mathcal{U}_{s^\frown\langle U\rangle} \in \Omega_X$ such that $U = \sigma(\langle \mathcal{U}_{s\restriction 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle U\rangle}\rangle)$. (To complete the induction, $\mathcal{U}_{s^\frown\langle U\rangle}$ may be chosen arbitrarily for all other $U \in T(X)$.)

So $\tau$ is a strategy for I in $\Omega FO(X)$. Let $\nu$ legally attack $\tau$, so $\tau(\nu \restriction n) \subseteq \nu(n)$ for all $n < \omega$. It follows that $\nu(n) = \sigma(\langle \mathcal{U}_{\nu\restriction 1}, \dots, \mathcal{U}_{\nu\restriction n}, \mathcal{U}_{\nu\restriction n+1}\rangle)$. Since $\langle \mathcal{U}_{\nu\restriction 1}, \mathcal{U}_{n\restriction 2}, \dots\rangle$ is a legal attack against $\sigma$, it follows that $\{\sigma(\langle \mathcal{U}_{\nu\restriction 1}, \dots, \mathcal{U}_{\nu\restriction n+1}\rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$ is an $\omega$-cover. Therefore $\tau$ is a winning strategy, verifying $\mathrm{I} \uparrow \Omega FO(X)$.

The equivalence of (b), (e), (g), and (h) are given as 3.8 of [Tkachuk 2017].

The equivalence of (e) is (f) are given in [https://doi.org/10.1016/0016-660X(76)90024-6].

The equivalence of (i), (j), and (k) are due to Clontz's Menger preprint.

(k) $\Leftrightarrow$ (g) follows from 3.18a of [Tkachuk 2017]. □

Tkachuk showed the following in [CLOSEDDISCRETESELECTIONS].

**Theorem 23.** *The following are equivalent for $T_{3.5}$ spaces $X$.*

*a) $X$ is uncountable.*

*b) $C_p(X)$ has discrete selectivity, that is,* $\mathrm{I} \underset{pre}{\not\uparrow} CD(C_p(X))$.

Clontz came across these in grad school (didn't make it into the dissertation):

**Theorem 24.** $\mathrm{I} \underset{pre}{\uparrow} PO(X)$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

*Proof.* Let $\sigma$ be a winning Markov strategy for II in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood $U_x$ of $x$ where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. Then $\sigma(\{U_x : x \in X\}, n) \notin \{U_x : x \in X\}$, a contradiction.

So for each $n < \omega$, there exists $\tau(n) \in X$ such that for any open neighborhood $U$ of $\tau(n)$, there exists an open cover $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = U$. Then $\tau$ is a predetermined strategy for I in $PO(X)$.

It is also winning: for every attack $f$ against $\tau$, note that $f(n)$ is an open neighborhood of $\tau(n)$, so choose $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = f(n)$. Then since $\langle \mathcal{U}_0, \mathcal{U}_1, \dots\rangle$ is a legal attack against $\sigma$, it follows that $\{f(n) : n < \omega\}$ is an open cover of $X$. Therefore $\tau$ is a winning predetermined strategy.

Now let $\sigma$ be a winning predetermined strategy for I in $PO(X)$. For an open cover $\mathcal{U}$ of $X$ and $n < \omega$, let $\tau(\mathcal{U}, n)$ be any open set in $\mathcal{U}$ containing $\sigma(n)$. It follows that $\tau$ is a winning Markov strategy for II in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. □

**Theorem 25.** $\mathrm{II} \underset{mark}{\uparrow} PO(X)$ *if and only if* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

*Proof.* Let $\sigma$ be a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. For $x \in X$ and $n < \omega$, let $\tau(x, n)$ be any open set in $\sigma(n)$ containing $x$. It follows that $\tau$ is a winning Markov strategy for II in $PO(X)$.

Now let $\sigma$ be a winning Markov strategy for II in $PO(X)$. We may defined the open cover $\tau(n) = \{\sigma(x, n) : x \in X\}$ of $X$. It follows that $\tau$ is a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. □

Combining with several other results in the literature, we may observe the following.

**Theorem 26.** *The following are equivalent for $T_{3.5}$ spaces $X$.*

*a)* $X$ *is countable.*

*b)* $X$ *is* $R^{+mark}$.

*c)* $\mathrm{I} \underset{pre}{\uparrow} PO(X)$.

*d)* $\mathrm{I} \underset{pre}{\uparrow} FO(X)$.

*e)* $\mathrm{I} \underset{pre}{\uparrow} \Omega FO(x)$.

*f)* $C_p(X)$ *is first-countable.*

*g)* $\mathrm{I} \underset{pre}{\uparrow} Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$.

*h)* $\mathrm{I} \underset{pre}{\uparrow} Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.

*i)* $\mathrm{I} \underset{pre}{\uparrow} CL(C_p(X), \mathbf{0})$.

*j)* $\mathrm{I} \underset{pre}{\uparrow} CD(C_p(X))$.

*k)* $X$ *is* $\Omega R^{+mark}$.

*l)* $C_p(X)$ *is* $sCFT^{+mark}$.

*m)* $C_p(X)$ *is* $sCDFT^{+mark}$.

*Proof.* (a) implies (c) is straight-forward. So let $\sigma$ be a predetermined strategy for I in $PO(X)$. If $x \notin \{\sigma(n) : n < \omega\}$, let $f(n) = X \setminus \{x\}$ for all $n < \omega$. It follows that $f$ is a legal counter-attack for II defeating $\sigma$. Thus not (a) implies not (c).

The equivalence of (b) and (c) was shown above.

Clearly (c) implies (d), so we will see that (d) implies (a). Let $\sigma(n)$ be a pre-determined strategy for I for $FO(X)$. Towards a contradiction, suppose that there is some $x \in X \setminus \bigcup_n \sigma(n)$. II could then play $FO(X)$ as follows. At round $n$ II can play an open set $U_n$ which contains $\sigma(n)$ but excludes $x$. Then $x \notin \bigcup_n U_n$, and so I has lost. This is a contradiction. So $X = \bigcup_n \sigma(n)$, which means it is countable.

It also clear that (e) implies (d), we will show that (a) implies (e). If $X$ is countable, then so is $[X]^{<\omega}$, enumerate it as $\{s_n : n \in \omega\}$. I's pre-determined strategy for $\Omega FO(X)$ is to play $s_n$ are round $n$. Clearly whatever II plays will be an $\omega$-cover. Thus (a) - (e) are equivalent.

It is well-known and easy to see that (a) is equivalent to (f).

To see that (f) implies (g), note that we can find a sequence of open sets $U_n$ so that $\mathbf{0} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$ for all $n$. I simply plays $U_n$ are turn $n$, and whatever $x_n$ are played by II must converge to $x$.

Clearly (g) implies (i) which in turn implies (j), which is equivalent to (a) by [CLOSEDDISCRETESELECTIONS]

Clontz showed (g) and (h) are equivalent in his dissertation; it's not hard to prove this.

Clontz showed that (k), (l), and (m) are equivalent in his Menger/CFT preprint.

Note that (k) implies (b) implies (a). So the last thing we need to show is that (e) implies (k). Let $\sigma(n)$ be a pre-determined strategy for I for $\Omega FO(X)$. We define a Markov strategy, $\tau(\mathcal{U}, n)$ for II for $\Omega R$ as follows. At round $n$ suppose I has played $\mathcal{U}$. As $\mathcal{U}$ must be an $\omega$-cover, there is a $U \in \mathcal{U}$ so that $\sigma(n) \subseteq U$. II plays such a $U_n$. Now suppose this game has been played according to $\tau$, and that I has played $\mathcal{U}_n$ for $n < \omega$. Then the sequence of open sets $\tau(\mathcal{U}_n, n)$ forms a legal play against $\sigma$ for $\Omega FO(X)$. Thus $\{\tau(\mathcal{U}_n, n) : n \in \omega\}$ is an $\omega$-cover of $X$ and so $\tau$ is a winning Markov strategy. $\square$

In that paper, Tkachuk characterizes II $\uparrow \Omega FO(X)$ as the second player having an "almost winning strategy" (II can prevent I from constructing an $\omega$-cover but perhaps not an arbitrary open cover) in $PO(X)$, which he conflates with $FO(X)$ as they are equivalent for "completely" winning perfect information strategies.

But they cannot be interchanged in general. Note that II $\underset{\text{tact}}{\uparrow}$ $\Omega PO(2)$, where 2 is the two-point discrete space: let $\sigma(\langle x \rangle) = \{x\}$. Since every $\omega$-cover of 2 includes 2, and $\sigma$ never produces 2, this is a winning tactic. But since 2 is countable, 2 is $\Omega R^{+mark}$. So $\Omega PO(X)$ is a very different game than those described previously.

Now we turn our attention to the opponent.

**Theorem 27.** *The following are equivalent for all spaces $X$.*

*a)* II $\uparrow FO(X)$

*b)* II $\uparrow PO(X)$

*c)* I $\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$

*d)* I $\underset{pre}{\uparrow}$ $G_1(\mathcal{O}_X, \mathcal{O}_X)$

*e)* II $\underset{mark}{\uparrow}$ $PO(X)$

*f)* II $\underset{mark}{\uparrow}$ $FO(X)$

*In particular, these are all equivalent to $X$ not being $R$.*

*Proof.* (a) $\Leftrightarrow$ (b) is 4.4 of [Telgarksy 1975].

The duality of $PO(X)$ and $G_1(\mathcal{O}_X, \mathcal{O}_X)$ for both players when considering perfect information is a well-known result of Galvin. So (b) is equivalent to (c).

The equivalence of (c) and (d) is just a restatement of Pawlikowski's result that the Rothberger selection principle is equivalent to I $\not\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$, since the Rothberger selection principle is equivalent to I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\mathcal{O}_X, \mathcal{O}_X)$.

(d) and (e) were shown to be equivalent above.

Finally, (f) implies (e) is obvious. Let $b : \omega^2 \to \omega$ be a bijection. Given a winning Markov strategy $\sigma$ for II in $PO(X)$, define $\tau(F_n, n) = \bigcup\{\sigma(x(i,n), b(i,n)) : i < \omega\}$ where $F_n = \{x(i,n) : i < \omega\}$ (this indexing will cause at least one point to be repeated infinitely often, but this won't be a problem). So given an attack $\langle F_0, F_1, \ldots\rangle$ against $\tau$, consider the attack $g$ against $\sigma$, where $g(n) = x_{b^{\leftarrow}(n)}$. It follows that

$$X \neq \bigcup\{\sigma(g(n), n) : n < \omega\} = \bigcup\{\sigma(x(i,n), b(i,n)) : i, n < \omega\} = \bigcup\{\tau(F_n, n) : n < \omega\}$$

and therefore $\tau$ is a winning Markov strategy for II. Thus (e) implies (f). □

**Theorem 28.** *The following are equivalent for all spaces $X$.*

*a)* $\mathrm{II} \uparrow \Omega FO(X)$

*b)* $\mathrm{I} \uparrow G_1(\Omega_X, \Omega_X)$

*c)* $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_X, \Omega_X)$

*d)* $\mathrm{II} \underset{mark}{\uparrow} \Omega FO(X)$

*In particular, these are all equivalent to $X$ not being $\Omega R$.*

*Proof.* Let $\sigma$ be a winning strategy for II in $\Omega FO(X)$. For $s \in ([X]^{<\omega})^{<\omega}$, let $\mathcal{U}_s = \{\sigma(s^\frown\langle F\rangle : F \in [X]^{<\omega}\}$. Then define the strategy $\tau$ for I by $\tau(s) = \sigma(\langle \mathcal{U}_{s\restriction 0}, \ldots, \mathcal{U}_s\rangle)$. Then every attack $f$ against $\tau$ yields $g \in ([X]^{<\omega})^\omega$ such that $f(n) = \sigma(g \restriction n+1)$. Thus $\{f(n) : n < \omega\} = \{\sigma(g \restriction n+1) : n < \omega\}$ is not an $\omega$-cover, so $\tau$ is a winning strategy, verifying that (a) implies (b).

The equivalence of (b) and (c) is given by Thm2 of [http://eudml.org/doc/212209].

Let $\sigma$ be a winning predetermined strategy for I in $G_1(\Omega_x, \Omega_x)$. For $F \in [X]^{<\omega}$ and $n < \omega$, let $\tau(F, n)$ be any open set in $\sigma(n)$ containing $F$. It follows that $\tau$ is a winning Markov strategy for II in $\Omega FO(X)$, verifying that (c) implies (d).

(d) implies (a) is trivial, so the proof is complete. □

$\Omega R$ is equivalent to all finite powers being $R$: Thm3 of [http://eudml.org/doc/212209]. These notions cannot coincide: see Thm9 of [http://dx.doi.org/10.1016/j.topol.2013.07.022] for a consisent example of a $R$ space whose square is not $R$, so therefore not $\Omega R$. Note the distinction with strategies for the opponent, as $R^+$ is equivalent to $\Omega R^+$ and $R^{+mark}$ is equivalent to $\Omega R^{+mark}$.

**Corollary 29.** *The following are equivalent for all $T_{3.5}$ spaces.*

*a) $X$ is not $\Omega R$*

*b)* $\mathrm{II} \uparrow \Omega FO(X)$

*c)* $\mathrm{II} \underset{mark}{\uparrow} \Omega FO(X)$

*d)* $\mathrm{I} \uparrow G_1(\Omega_X, \Omega_X)$

*e)* $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_X, \Omega_X)$

*f) $C_p(X)$ is not sCFT, that is,* $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

*g)* $\mathrm{I} \uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

*h)* $C_p(X)$ *is not sCDFT, that is,* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

*Proof.* (a)-(e) were just shown. The equivalence of (a),(f),(h) was shown by Sakai. The equivalence of (f) and (g) is given in 4.37 of [Comb. of Open Covers in Gen Prog Top III]. □