Limited Information Strategies and Discrete Selectivity

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1 Definitions

Definition 1. The *selection principle* $S_{fin}(\mathcal{A},\mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n<\omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A},\mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A},\mathcal{B})$, where during each round $n<\omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

Definition 3. Let $\mathcal{O}_X$ be the collection of open covers for a topological space X. Then $S_{fin}(\mathcal{O}_X,\mathcal{O}_X)$ is the well-known *Menger property* for X (M for short), and $G_{fin}(\mathcal{O}_X,\mathcal{O}_X)$ is the well-known *Menger game*.

Definition 4. An ω*-cover* $\mathcal{U}$ for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

Definition 5. Let Ω_X be the collection of ω-covers for a topological space X. Then $S_{fin}(\Omega_X,\Omega_X)$ is the Ω*-Menger property* for X (ΩM for short), and $G_{fin}(\Omega_X,\Omega_X)$ is the Ω*-Menger game*.

Definition 6. Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \operatorname{cl} A$. (Call A a *blade* of x.) Then $S_{fin}(\Omega_{X,x},\Omega_{X,x})$ is the *countable fan tightness property* for X at x (CFT_x for short), and $G_{fin}(\Omega_{X,x},\Omega_{X,x})$ is the *countable fan tightness game* for X at x.

Definition 7. A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.

Definition 8. Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space X. Then $S_{fin}(\mathcal{D}_X,\Omega_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X,\Omega_{X,x})$ is the *countable dense fan tightness game* for X at x.

Definition 9. A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Definition 10. $S_{fin}(\mathcal{D}_X,\mathcal{D}_X)$ is the *selective separability property* for X (SS for short), and $G_{fin}(\mathcal{D}_X,\mathcal{D}_X)$ is the *selective separability game* for X.

Definition 11. A *strategy* for II in the game $G_{fin}(\mathcal{A},\mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0,\dots,A_n\rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n\rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n<\omega$, II wins the game by playing $\sigma(\langle A_0,\dots,A_n\rangle)$ during each round $n<\omega$. If a winning strategy exists, then we write II $\uparrow G_{fin}(\mathcal{A},\mathcal{B})$.

Definition 12. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A},\mathcal{B})$ is a function σ satisfying $\sigma(A,n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write II $\underset{\text{mark}}{\uparrow}$ $G_{fin}(\mathcal{A},\mathcal{B})$.

Definition 13. In some instances, player I will be able to win a game regardless of what II is playing. In this case, it is possible to have a strategy for I which depends only on the round of the game. We say I has a *pre-determined strategy* and write I $\underset{\text{pre}}{\uparrow}$ G.

Notation 14. If $S_{fin}(\mathcal{A},\mathcal{B})$ characterizes the property P, then we say II $\uparrow G_{fin}(\mathcal{A},\mathcal{B})$ characterizes P^+ ("strategically P"), and II $\underset{\text{mark}}{\uparrow}$ $G_{fin}(\mathcal{A},\mathcal{B})$ characterizes $P^{+\text{mark}}$ ("Markov P"). Of course, $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$.

Notation 15. Let $S_1(\mathcal{A},\mathcal{B}), G_1(\mathcal{A},\mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A},\mathcal{B}), G_{fin}(\mathcal{A},\mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the "strong Menger" property is commonly known as "Rothberger". We will thus call "strong Ω-Menger" "Ω-Rothberger" and shorten it with ΩR, and otherwise attach the prefix "s" when abbreviating to all other strong variants.

In addition to pure selection games, we also will be playing various point-picking games.

Definition 16. Set $T(X)$ to be the non-empty open subsets of X. The *point-open game* for X, denoted $PO(X)$, is played as follows. Each round, player I plays a point $x_n \in X$ and player II plays an open sets U_n with the property that $x_n \in U_n$. I wins the play of the game if $X = \bigcup_n U_n$.

The *finite-open game* for X, denoted $FO(x)$, is played similarly, except that I now plays finite subsets of X, and II's open sets must cover I's finite set. Note that $PO(X)$ is just $sFO(S)$. $\Omega FO(X)$ and $\Omega PO(X)$ are defined according to convention: I now wins if $\{U_n : n \in \omega\}$ forms an ω-cover of X.

Definition 17. Let $x \in X$. *Gruenhage's W-game* for x, denoted $Gru_{\vec{O},P}(X,x)$, is played as follows. Each round, player I plays an open set U_n with the property that $x \in U_n$ and player II plays a point $x_n \in U_n$. I wins if $x_n \to x$.

The *closure game* for x, denoted $CL(X,x)$, is played the same as Gruenhage's W-game, but now I wins if $x \in \overline{\{x_n : n \in \omega\}}$. Note that this is $G_1(T(X), \Omega_{X,x})$.

The *discrete selectivity game*, denoted $CD(X)$, is also played the same as Gruenhage's W-game, but now II wins if $\{x_n : n \in \omega\}$ is closed and discrete. Note that this is $G_1(T(X), \mathcal{CD})$ if we let $\mathcal{CD}$ denoted the closed discrete subsets of X.

Definition 18. A space X is *discretely selective* iff

2 2-marks in CD(X)

Let $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$.

Game 19. Let G be the following game. During round n, player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

For $f \in \omega^\alpha$, let $f^{\leftarrow}[n] = \{\beta < \alpha : f(\beta) < n\}$.

Proposition 20. II $\underset{2\text{-}mark}{\uparrow}$ G.

Proof. Let $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark σ for II by

$$\sigma(\langle\alpha\rangle, 0) = \emptyset$$

and

$$\sigma(\langle\alpha,\beta\rangle, n+1) \ = f_\beta^\leftarrow[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let ν be an attack by I against σ, and let $\gamma < \nu(n)$ for $N \leq n < \omega$. If $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N \leq n < \omega$, then $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for infinitely-many $N \leq n < \omega$. Otherwise $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$ for cofinitely-many $N \leq n < \omega$, so $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for cofinitely-many $N \leq n < \omega$. Therefore σ is a winning 2-mark. □

Theorem 21. I $\underset{2\text{-}mark}{\uparrow}$ $CD(C_p(\omega_1 + 1))$

Proof. Let σ be a winning 2-mark for II in G.

Given a point $f \in C_p(\omega_1 + 1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f\rangle, 1) = [\mathbf{0}; \sigma(\langle\alpha_f\rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g\rangle, n+2) = [\mathbf{0}; \sigma(\langle\alpha_f, \alpha_g\rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let ν be a legal attack by II against σ. For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle\alpha_{\nu(n)}, \alpha_{\nu(n+1)}\rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \mathrm{cl}\{\nu(n) : n < \omega\}$. □

3 Combining game results

Theorem 22. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X *is* R^+*, that is,* II $\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$.

b) I $\uparrow PO(X)$.

c) I $\uparrow FO(X)$.

d) I $\uparrow \Omega FO(x)$.

e) I $\uparrow Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$.

f) I $\uparrow Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.

g) I $\uparrow CL(C_p(X), \mathbf{0})$.

h) I $\uparrow CD(C_p(X))$.

i) X *is* ΩR^+*, that is,* II $\uparrow G_1(\Omega_X, \Omega_X)$.

j) $C_p(X)$ *is* $sCFT^+$*, that is,* II $\uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

k) $C_p(X)$ *is* $sCDFT^+$*, that is,* II $\uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

Proof. (a) $\Leftrightarrow$ (b) is a well-known result of Galvin.

(b) $\Leftrightarrow$ (c) is 4.3 of [Telgarksy 1975].

(d) $\Leftrightarrow$ (c) is clear, but we need to show that (a) $\Leftrightarrow$ (d). So assume X is R^+, which is equivalent to ΩR^+. Let σ be a winning strategy for II in $G_1(\Omega_X, \Omega_X)$. Let $T(X)$ be the non-empty open sets of X, and let $s \in T(X)^{<\omega}$. Assume $\tau(t) \in [X]^{<\omega}$ is defined for all $t < s$, and $\mathcal{U}_t \in \Omega_X$ is defined for all $\emptyset < t \leq s$.

Suppose that for all $F \in [X]^{<\omega}$, there existed $U_F \in T(X)$ containing F such that for all $\mathcal{U} \in \Omega_X$, $U_F \neq \sigma(\langle \mathcal{U}_{s\upharpoonright 1}, \ldots, U_s, \mathcal{U}\rangle)$. Let $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$. Then $\sigma(\langle \mathcal{U}_{s\upharpoonright 1}, \ldots, U_s, \mathcal{U}\rangle)$ must equal some U_F, demonstrating a contradiction.

So there exists $\tau(s) \in [X]^{<\omega}$ such that for all $U \in T(X)$ containing $\tau(s)$, there exists $\mathcal{U}_{s^\frown\langle U\rangle} \in \Omega_X$ such that $U = \sigma(\langle \mathcal{U}_{s\upharpoonright 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle U\rangle}\rangle)$. (To complete the induction, $\mathcal{U}_{s^\frown\langle U\rangle}$ may be chosen arbitrarily for all other $U \in T(X)$.)

So τ is a strategy for I in $\Omega FO(X)$. Let ν legally attack τ, so $\tau(\nu \upharpoonright n) \subseteq \nu(n)$ for all $n < \omega$. It follows that $\nu(n) = \sigma(\langle \mathcal{U}_{\nu\upharpoonright 1}, \ldots, \mathcal{U}_{\nu\upharpoonright n}, \mathcal{U}_{\nu\upharpoonright n+1}\rangle)$. Since $\langle \mathcal{U}_{\nu\upharpoonright 1}, \mathcal{U}_{n\upharpoonright 2}, \ldots\rangle$ is a legal attack against σ, it follows that $\{\sigma(\langle \mathcal{U}_{\nu\upharpoonright 1}, \ldots, \mathcal{U}_{\nu\upharpoonright n+1}\rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$ is an ω-cover. Therefore τ is a winning strategy, verifying $\text{I} \uparrow \Omega FO(X)$.

The equivalence of (b), (e), (g), and (h) are given as 3.8 of [Tkachuk 2017].

The equivalence of (e) is (f) are given in [https://doi.org/10.1016/0016-660X(76)90024-6].

The equivalence of (i), (j), and (k) are due to Clontz's Menger preprint.

(k) $\Leftrightarrow$ (g) follows from 3.18a of [Tkachuk 2017]. $\square$

Tkachuk showed the following in [CLOSEDDISCRETESELECTIONS].

Theorem 23. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X is uncountable.

b) $C_p(X)$ has discrete selectivity, that is, $\text{I} \underset{pre}{\not\uparrow} CD(C_p(X))$.

Clontz came across these in grad school (didn't make it into the dissertation):

Theorem 24. $\text{I} \underset{pre}{\uparrow} PO(X)$ *if and only if* $\text{II} \underset{mark}{\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Let σ be a winning Markov strategy for II in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. Then $\sigma(\{U_x : x \in X\}, n) \notin \{U_x : x \in X\}$, a contradiction.

So for each $n < \omega$, there exists $\tau(n) \in X$ such that for any open neighborhood U of $\tau(n)$, there exists an open cover $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = U$. Then τ is a predetermined strategy for I in $PO(X)$.

It is also winning: for every attack f against τ, note that $f(n)$ is an open neighborhood of $\tau(n)$, so choose $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = f(n)$. Then since $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots\rangle$ is a legal attack against σ, it follows that $\{f(n) : n < \omega\}$ is an open cover of X. Therefore τ is a winning predetermined strategy.

Now let σ be a winning predetermined strategy for I in $PO(X)$. For an open cover $\mathcal{U}$ of X and $n < \omega$, let $\tau(\mathcal{U}, n)$ be any open set in $\mathcal{U}$ containing $\sigma(n)$. It follows that τ is a winning Markov strategy for II in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. $\square$

Theorem 25. $\text{II} \underset{mark}{\uparrow} PO(X)$ *if and only if* $\text{I} \underset{pre}{\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. Let σ be a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. For $x \in X$ and $n < \omega$, let $\tau(x,n)$ be any open set in $\sigma(n)$ containing x. It follows that τ is a winning Markov strategy for II in $PO(X)$.

Now let σ be a winning Markov strategy for II in $PO(X)$. We may defined the open cover $\tau(n) = \{\sigma(x,n) : x \in X\}$ of X. It follows that τ is a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. $\square$

Combining with several other results in the literature, we may observe the following.

Theorem 26. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X *is countable.*

b) X *is* R^{+mark}.

c) $\mathrm{I} \underset{pre}{\uparrow} PO(X)$.

d) $\mathrm{I} \underset{pre}{\uparrow} FO(X)$.

e) $\mathrm{I} \underset{pre}{\uparrow} \Omega FO(x)$.

f) $C_p(X)$ *is first-countable.*

g) $\mathrm{I} \underset{pre}{\uparrow} Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$.

h) $\mathrm{I} \underset{pre}{\uparrow} Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.

i) $\mathrm{I} \underset{pre}{\uparrow} CL(C_p(X), \mathbf{0})$.

j) $\mathrm{I} \underset{pre}{\uparrow} CD(C_p(X))$.

k) X *is* ΩR^{+mark}.

l) $C_p(X)$ *is* $sCFT^{+mark}$.

m) $C_p(X)$ *is* $sCDFT^{+mark}$.

Proof. (a) implies (c) is straight-forward. So let σ be a predetermined strategy for I in $PO(X)$. If $x \notin \{\sigma(n) : n < \omega\}$, let $f(n) = X \setminus \{x\}$ for all $n < \omega$. It follows that f is a legal counter-attack for II defeating σ. Thus not (a) implies not (c).

The equivalence of (b) and (c) was shown above.

Clearly (c) implies (d), so we will see that (d) implies (a). Let $\sigma(n)$ be a pre-determined strategy for I for $FO(X)$. Towards a contradiction, suppose that there is some $x \in X \smallsetminus \bigcup_n \sigma(n)$. II could then play $FO(X)$ as follows. At round n II can play an open set U_n which contains $\sigma(n)$ but excludes x. Then $x \notin \bigcup_n U_n$, and so I has lost. This is a contradiction. So $X = \bigcup_n \sigma(n)$, which means it is countable.

It also clear that (e) implies (d), we will show that (a) implies (e). If X is countable, then so is $[X]^{<\omega}$, enumerate it as $\{s_n : n \in \omega\}$. I's pre-determined strategy for $\Omega FO(X)$ is to play s_n are round n. Clearly whatever II plays will be an ω-cover. Thus (a) - (e) are equivalent.

It is well-known and easy to see that (a) is equivalent to (f).

To see that (f) implies (g), note that we can find a sequence of open sets U_n so that $\mathbf{0} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$ for all n. I simply plays U_n are turn n, and whatever x_n are played by II must converge to x.

Clearly (g) implies (i) which in turn implies (j), which is equivalent to (a) by [CLOSEDDISCRETESELECTIONS]

Clontz showed (g) and (h) are equivalent in his dissertation; it's not hard to prove this.

Clontz showed that (k), (l), and (m) are equivalent in his Menger/CFT preprint.

Note that (k) implies (b) implies (a). So the last thing we need to show is that (e) implies (k). Let $\sigma(n)$ be a pre-determined strategy for I for $\Omega FO(X)$. We define a Markov strategy, $\tau(\mathcal{U}, n)$ for II for ΩR as follows. At round n suppose I has played $\mathcal{U}$. As $\mathcal{U}$ must be an ω-cover, there is a $U \in \mathcal{U}$ so that $\sigma(n) \subseteq U$. II plays such a U_n. Now suppose this game has been played according to τ, and that I has played $\mathcal{U}_n$ for $n < \omega$. Then the sequence of open sets $\tau(\mathcal{U}_n, n)$ forms a legal play against σ for $\Omega FO(X)$. Thus $\{\tau(\mathcal{U}_n, n) : n \in \omega\}$ is an ω-cover of X and so τ is a winning Markov strategy. $\square$

In that paper, Tkachuk characterizes II $\uparrow \Omega FO(X)$ as the second player having an "almost winning strategy" (II can prevent I from constructing an ω-cover but perhaps not an arbitrary open cover) in $PO(X)$, which he conflates with $FO(X)$ as they are equivalent for "completely" winning perfect information strategies.

But they cannot be interchanged in general. Note that II $\underset{\text{tact}}{\uparrow} \Omega PO(2)$, where 2 is the two-point discrete space: let $\sigma(\langle x \rangle) = \{x\}$. Since every ω-cover of 2 includes 2, and σ never produces 2, this is a winning tactic. But since 2 is countable, 2 is ΩR^{+mark}. So $\Omega PO(X)$ is a very different game than those described previously.

Now we turn our attention to the opponent.

Theorem 27. *The following are equivalent for all spaces X.*

a) II $\uparrow FO(X)$

b) II $\uparrow PO(X)$

c) I $\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$

d) I $\underset{pre}{\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$

e) II $\underset{mark}{\uparrow} PO(X)$

f) II $\underset{mark}{\uparrow} FO(X)$

In particular, these are all equivalent to X not being R.

Proof. (a) $\Leftrightarrow$ (b) is 4.4 of [Telgarksy 1975].

The duality of $PO(X)$ and $G_1(\mathcal{O}_X, \mathcal{O}_X)$ for both players when considering perfect information is a well-known result of Galvin. So (b) is equivalent to (c).

The equivalence of (c) and (d) is just a restatement of Pawlikowski's result that the Rothberger selection principle is equivalent to I $\not\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$, since the Rothberger selection principle is equivalent to I $\underset{\text{pre}}{\not\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

(d) and (e) were shown to be equivalent above.

Finally, (f) implies (e) is obvious. Let $b : \omega^2 \to \omega$ be a bijection. Given a winning Markov strategy σ for II in $PO(X)$, define $\tau(F_n, n) = \bigcup\{\sigma(x(i,n), b(i,n)) : i < \omega\}$ where $F_n = \{x(i,n) : i < \omega\}$ (this indexing will cause at least one point to be repeated infinitely often, but this won't be a problem). So given an attack $\langle F_0, F_1, \dots\rangle$ against τ, consider the attack g against σ, where $g(n) = x_{b^{\leftarrow}(n)}$. It follows that

$$X \neq \bigcup\{\sigma(g(n), n) : n < \omega\} = \bigcup\{\sigma(x(i,n), b(i,n)) : i, n < \omega\} = \bigcup\{\tau(F_n, n) : n < \omega\}$$

and therefore τ is a winning Markov strategy for II. Thus (e) implies (f). $\square$

Theorem 28. *The following are equivalent for all spaces X.*

a) $\mathrm{II} \uparrow \Omega FO(X)$

b) $\mathrm{I} \uparrow G_1(\Omega_X, \Omega_X)$

c) $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_X, \Omega_X)$

d) $\mathrm{II} \underset{mark}{\uparrow} \Omega FO(X)$

In particular, these are all equivalent to X not being ΩR.

Proof. Let σ be a winning strategy for II in $\Omega FO(X)$. For $s \in ([X]^{<\omega})^{<\omega}$, let $\mathcal{U}_s = \{\sigma(s^\frown\langle F\rangle : F \in [X]^{<\omega}\}$. Then define the strategy τ for I by $\tau(s) = \sigma(\langle \mathcal{U}_{s\restriction 0}, \dots, \mathcal{U}_s\rangle)$. Then every attack f against τ yields $g \in ([X]^{<\omega})^\omega$ such that $f(n) = \sigma(g \restriction n+1)$. Thus $\{f(n) : n < \omega\} = \{\sigma(g \restriction n+1) : n < \omega\}$ is not an ω-cover, so τ is a winning strategy, verifying that (a) implies (b).

The equivalence of (b) and (c) is given by Thm2 of [http://eudml.org/doc/212209].

Let σ be a winning predetermined strategy for I in $G_1(\Omega_x, \Omega_x)$. For $F \in [X]^{<\omega}$ and $n < \omega$, let $\tau(F, n)$ be any open set in $\sigma(n)$ containing F. It follows that τ is a winning Markov strategy for II in $\Omega FO(X)$, verifying that (c) implies (d).

(d) implies (a) is trivial, so the proof is complete. $\square$

ΩR is equivalent to all finite powers being R: Thm3 of [http://eudml.org/doc/212209]. These notions cannot coincide: see Thm9 of [http://dx.doi.org/10.1016/j.topol.2013.07.022] for a consisent example of a R space whose square is not R, so therefore not ΩR. Note the distinction with strategies for the opponent, as R^+ is equivalent to ΩR^+ and R^{+mark} is equivalent to ΩR^{+mark}.

Corollary 29. *The following are equivalent for all $T_{3.5}$ spaces.*

a) X is not ΩR

b) $\mathrm{II} \uparrow \Omega FO(X)$

c) $\mathrm{II} \underset{mark}{\uparrow} \Omega FO(X)$

d) $\mathrm{I} \uparrow G_1(\Omega_X, \Omega_X)$

e) $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_X, \Omega_X)$

f) $C_p(X)$ is not sCFT, that is, $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

g) $C_p(X)$ is not sCDFT, that is, $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

Proof. (a)-(e) were just shown. The equivalence of (a),(f),(g) was shown by Clontz. $\square$

Theorem 30. $\mathrm{II} \uparrow Gru_{\vec{O},P}(X,x)$ *if and only if* $\mathrm{II} \underset{mark}{\uparrow} Gru_{\vec{O},P}(X,x)$

Proof. Essentially Theorem 1 of [https://doi.org/10.1016/0016-660X(78)90032-6]. $\square$

We adapt this result to show the equivalence of (a) and (b) in the following theorem. First, a lemma.

Lemma 31. *If* $\mathrm{II} \underset{mark}{\not\uparrow} Gru_{\overset{\leadsto}{O},P}(X,x)$*, then X is countably tight at x.*

Proof. Let $A \in \Omega_{X,x}$, and assume $\mathrm{II} \underset{\text{mark}}{\not\uparrow} Gru_{\overset{\leadsto}{O},P}(X,x)$. Let σ be a Markov strategy for II that chooses exclusively from A. Since σ is not winning, there exist open sets $U_n \in T_{X,x}$ such that $\{\sigma(\langle U_n\rangle, n) : n < \omega\} \in \Omega_{X,x}$. This shows that X is countably tight at x. $\square$

Theorem 32. *TODO The following are equivalent.*

a) $\mathrm{II} \uparrow Gru_{\overset{\leadsto}{O},P}(X,x)$

b) $\mathrm{II} \underset{mark}{\uparrow} Gru_{\overset{\leadsto}{O},P}(X,x)$

c) $\mathrm{I} \uparrow G_1(\Omega_{X,x}, \Omega_{X,x})$

d) $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_{X,x}, \Omega_{X,x})$

Proof. So assume (a). If X is not countably tight at x, then we already have (b) by the lemma. So assume X is countably tight at x.

Let τ be a winning strategy for II. Let $f(\emptyset) = \emptyset$. Suppose $f(s) \in T_{X,x}^{|s|}$ has been defined for $s \in \omega^{<\omega}$. Then the set $\{\tau(s^\frown\langle U\rangle) : U \in T_{X,x}\} \in \Omega_{X,x}$, so define $f(s^\frown\langle n\rangle) \in T_{X,x}^{|s|+1}$ for $n < \omega$ such that $\{\tau(f(s^\frown\langle n\rangle)) : n < \omega\} \in \Omega_{X,x}$.

Let $b : \omega \to \omega^\omega$, and let $\sigma(U,m) \in \{\tau(f(b(m)^\frown\langle n\rangle : n < \omega\} \cap U$. Then for any attack $a \in T_{X,x}^\omega$,

that for all $x_n \in A_n$, $\{x_n : n < \omega\} \notin \Omega_{X,x}$. So let $\sigma(U,n) \in A_n \cap U$. It follows that σ is a winning Markov strategy for II in $Gru_{\vec{O},P}(X,x)$, so (a) is equivalent to (b).

Furthermore, let $\tau(n) = A_n$. Since $x_n \not\to x$ implies $\{x_n : n < \omega\} \notin \Omega_{X,x}$ as X is F-U, it follows that τ is a winning predetermined strategy for I in $G_1(\Omega_{X,x}, \Omega_{X,x})$. Thus (a) implies (d) implies (c).

So finally, let σ be a winning strategy for I in $G_1(\Omega_{X,x}, \Omega_{X,x})$. For $s^\frown\langle U\rangle \in {T_{X,x}}^{<\omega} \setminus \{\emptyset\}$, let $\tau(s^\frown\langle U\rangle) \in \sigma(\langle\tau(s \restriction 1), \ldots, \tau(s)\rangle) \cap U$ be a strategy for II in $Gru_{\vec{O},P}(X,x)$. If f attacks τ, then $\langle\tau(f \restriction 1), \tau(f \restriction 2), \ldots\rangle$ attacks σ, and therefore $\{\tau(f \restriction n+1) : n < \omega\} \notin \Omega_{X,x}$. It follows that $\tau(f \restriction n+1) \not\to x$, so τ is a winning strategy for II, verifying that (c) implies (a). $\square$

Corollary 33. *The following are equivalent for all $T_{3.5}$ spaces.*

a) X is not ΩR

b) $\mathrm{II} \uparrow \Omega FO(X)$

c) $\mathrm{II} \underset{mark}{\uparrow} \Omega FO(X)$

d) $\mathrm{I} \uparrow G_1(\Omega_X, \Omega_X)$

e) $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_X, \Omega_X)$

f) $C_p(X)$ *is not sCFT, that is,* $\mathrm{I} \underset{pre}{\uparrow} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

g) $C_p(X)$ *is not sCDFT, that is,* $\mathrm{I} \underset{pre}{\uparrow} G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

h) $\mathrm{I} \uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$

i) $\mathrm{II} \uparrow Gru_{O,P}^{\leadsto}(C_p(X), \mathbf{0})$

j) $\mathrm{II} \underset{mark}{\uparrow} Gru_{O,P}^{\leadsto}(C_p(X), \mathbf{0})$