LIMITED INFORMATION STRATEGIES AND DISCRETE SELECTIVITY

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Abstract. We relate the property of discrete selectivity and its corresponding game, both recently introduced by V.V¿ Tkachuck, to a variety of selection principles and point picking games. In particular we show that II can win the discrete selection game on $C_p(X)$ if and only if II can win a variant of the point open game on X. We also show that the existence of limited information starategies in the discrete selection game on $C_p(X)$ for either player are equivalent to other well-known topological properties.

1. Introduction

In the course of studying the strong domination of function spaces by second countable spaces and countable spaces, G. Sanchez and Tkachuk isolated the topological property of discrete selectivity. A space is discretely selective if for every sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of the space, there are points $x_n \in U_n$ so that $\{x_n : n \in \omega\}$ is closed discrete. In subsequent work, V.V. Tkachuk showed that for $T_{3.5}$-spaces, $C_p(X)$ is discretely selective if and only if X is uncountable.

Discrete selectivity naturally generates a game, in which player I plays open sets, player II responds with points from those open sets, and player II wins if the points form a closed discrete set. Tkachuk explored what happens when player I perfect information strategy in this game, showing that player I has a strategy in this game on $C_p(X)$ is equivalent to player I having a strategy for Gruenhage's W-game on $C_p(X, \mathbf{0})$ and is also equivalent to player I having a strategy for the point-open game on X. Tkachuk also showed that if player II has a strategy in the point-open game on X, then player II has a strategy in the discrete selection game on $C_p(X)$. Tkachuk hypothesized that the implication reverses for player II, and posed this problem as an open question.

By considering lower information strategies and other topological games, we were able to answer Tkachuk's question and uncover a number of interesting connections between the discrete selection game and other topological properties. Classic works by Telgarksy and Galvin show that the point open game is dual to the Rothberger game. Clontz, in work prior to this, established the equivalence of the existence of strategies for the Rothberger game and variants of the Rothberger game for X to the existence of strategies in games related to countable fan tightness for $C_p(X)$. Clontz did this both for strategies of perfect information and for limited information strategies. Starting with these results, we are able to relate a host of games on $C_p(X)$ and X for strategies of limited information and perfect information. As a result we answer Tkachuk's question: player II has a strategy for the discrete selection game on $C_p(X)$ if and only if player II has a strategy for the ω-cover variant of the finite-open game. The ω-cover variant of the finite-open game is closely related to the point open game, but it is consistent that they are different. Tkachuk referred to a strategy for this variant for player II as an almost winning strategy. So in Tkachuk's terminology, player II has a strategy for the discrete selection game on $C_p(X)$ if and only if player II has an almost winning strategy for the point-open game. Moreover, we answer the implied question "what topological property does a strategy for player II for the discrete selection game on $C_p(X)$ correspond to?" We show that player II has a strategy for the discrete selection game on $C_p(X)$ if and only if X is not Rothberger with respect to ω-covers. This in turn is true if and only if some finite power of X is not Lindelof.

2. Definitions

We will be using a number of definitions. These are broken up into three main categories: labeling schema, topological notions, and games.

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2.1. **Labeling Schema.**

Definition 1. The *selection principle* $S_{fin}(\mathcal{A},\mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A},\mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A},\mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

Definition 3. A *strategy* for II in the game $G_{fin}(\mathcal{A},\mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0,\dots,A_n\rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \dots, A_n\rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0,\dots,A_n\rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write $\text{II} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$.

Definition 4. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A},\mathcal{B})$ is a function σ satisfying $\sigma(A,n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n,n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.

Definition 5. In some instances, player I will be able to win a game regardless of what II is playing. In this case, it is possible to have a strategy for I which depends only on the round of the game. We say I has a *pre-determined strategy* and write $\text{I} \underset{\text{pre}}{\uparrow} G$.

Notation 6. If $S_{fin}(\mathcal{A},\mathcal{B})$ characterizes the property P, then we say $\text{II} \uparrow G_{fin}(\mathcal{A},\mathcal{B})$ characterizes P^+ ("strategically P"), and $\text{II} \underset{\text{mark}}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$ characterizes $P^{+\text{mark}}$ ("Markov P"). Of course, $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$.

Definition 7. Let $S_1(\mathcal{A},\mathcal{B}), G_1(\mathcal{A},\mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A},\mathcal{B}), G_{fin}(\mathcal{A},\mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, denoted here as sP for property P, with a few exceptions for properties which already have their own names.

Definition 8. We will use the following shorthand for various special colletions of subsets of X.

- Let $\mathcal{O}_X$ be the collection of open covers for a topological space X.
- An *ω-cover* $\mathcal{U}$ for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$. Let Ω_X be the collection of ω-covers for a topological space X.
- Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \operatorname{cl} A$. (Call A a *blade* of x.)
- Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space X.
- Set $T(X)$ to be the non-empty open subsets of X.

2.2. **Topological Notions.**

Definition 9. Using the notation just established, we can record a number of topological properties.

- $S_{fin}(\mathcal{O}_X,\mathcal{O}_X)$ is the well-known *Menger property* for X (M for short).
- $S_1(\mathcal{O}_x,\mathcal{O}_X)$ is then just the *Rothberger property* for X (R for short). We say this instead of sM.
- $S_{fin}(\Omega_X,\Omega_X)$ is the *Ω-Menger property* for X (ΩM for short).
- $S_1(\Omega_X,\Omega_X)$ is the *Ω-Rothberger property* for X (ΩR for short). We say this instead of $s\Omega M$.
- $S_{fin}(\Omega_{X,x},\Omega_{X,x})$ is the *countable fan tightness property* for X at x (CFT_x for short). A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.
- Then $S_{fin}(\mathcal{D}_X,\Omega_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short). A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Tkachuk isolated the following the notion.

Definition 10. A space X is *discretely selective* if whenever $\{U_n : n \in \omega\}$ is a sequence of open subsets of X, there are points $x_n \in U_n$ so that $\{x_n : n \in \omega\}$ is closed discrete.

We will use the following notation when working with $C_p(X)$.

Definition 11. Suppose X is $T_{3.5}$. Basic open subsets of $C_p(X)$ will be written as

$$[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$$

where $f \in C_p(X)$, F is a finite subset of X and $\epsilon > 0$ is a real number. F is called the *support* of $[f, F, \epsilon]$. We can extend the notion of support to all open $U \subseteq C_p(X)$, label it $\mathrm{supp}(U)$.

2.3. **Topological Games.**

Definition 12. The following selection games will be played in this paper.

- $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.
- $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is the *Rothberger game*.
- $G_{fin}(\Omega_X, \Omega_X)$ is the Ω*-Menger game*.
- $G_1(\Omega_X, \Omega_X)$ is the Ω*-Rothberger game*.
- $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness game* for X at x.
- $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness game* for X at x.

Definition 13. The following point-picking games will also be played in this paper.

- The *point-open game* for X, denoted $PO(X)$, is played as follows. Each round, player I plays a point $x_n \in X$ and player II plays an open sets U_n with the property that $x_n \in U_n$. I wins the play of the game if $X = \bigcup_n U_n$.
- The *finite-open game* for X, denoted $FO(x)$, is played similarly, except that I now plays finite subsets of X, and II's open sets must cover I's finite set.
- $\Omega FO(X)$ and $\Omega PO(X)$ are defined according to convention: I now wins if $\{U_n : n \in \omega\}$ forms an ω-cover of X.
- Fix $x \in X$. *Gruenhage's W-game* for x, denoted $Gru_{\vec{O},P}(X, x)$, is played as follows. Each round, player I plays an open set U_n with the property that $x \in U_n$ and player II plays a point $x_n \in U_n$. I wins if $x_n \to x$.
- Fix $x \in X$. *Gruenhage's clustering-game* for x, denoted $Gru_{\tilde{\vec{O}},P}(X, x)$, is played the same as $Gru_{\vec{O},P}(X, x)$, except that I wins if x is a cluster point of $\{x_n : n \in \omega\}$.
- Fix $x \in X$. The *closure game* for x, denoted $CL(X, x)$, is played the same as $Gru_{\vec{O},P}(X, x)$, except that I wins if $x \in \overline{\{x_n : n \in \omega\}}$.
- The *discrete selectivity game*, denoted $CD(X)$, is also played the same as $Gru_{\vec{O},P}(X, x)$, but now II wins if $\{x_n : n \in \omega\}$ is closed and discrete.

3. Strategies for Player I for the Discrete Selection Game on $C_p(X)$

We begin by extending theorem 3.8 of Tkachuk to equate the existence of strategies for 11 games.

Theorem 14. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X *is* R^+.
b) X *is* ΩR^+.
c) $\mathrm{I} \uparrow PO(X)$.
d) $\mathrm{I} \uparrow FO(X)$.
e) $\mathrm{I} \uparrow \Omega FO(x)$.
f) $\mathrm{I} \uparrow Gru_{\vec{O},P}(C_p(X), \mathbf{0})$.
g) $\mathrm{I} \uparrow Gru_{\tilde{\vec{O}},P}(C_p(X), \mathbf{0})$.
h) $\mathrm{I} \uparrow CL(C_p(X), \mathbf{0})$.
i) $\mathrm{I} \uparrow CD(C_p(X))$.
j) $C_p(X)$ *is* $sCFT^+$.
k) $C_p(X)$ *is* $sCDFT^+$.

Proof. (a) $\Leftrightarrow$ (b) is true by [citation needed].
(a) $\Leftrightarrow$ (c) is a well-known result of Galvin.
(c) $\Leftrightarrow$ (d) is 4.3 of [Telgarksy 1975].

(e) $\Rightarrow$ (d) is clear, but we need to show that (b) $\Rightarrow$ (e). So assume X is ΩR^+. Let σ be a winning strategy for II in $G_1(\Omega_X, \Omega_X)$. Let $T(X)$ be the non-empty open sets of X, and let $s \in T(X)^{<\omega}$. Assume $\tau(t) \in [X]^{<\omega}$ is defined for all $t < s$, and $\mathcal{U}_t \in \Omega_X$ is defined for all $\emptyset < t \leq s$.

Suppose that for all $F \in [X]^{<\omega}$, there existed $U_F \in T(X)$ containing F such that for all $\mathcal{U} \in \Omega_X$, $U_F \neq \sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, U_s, \mathcal{U}\rangle)$. Let $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$. Then $\sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, U_s, \mathcal{U}\rangle)$ must equal some U_F, demonstrating a contradiction.

So there exists $\tau(s) \in [X]^{<\omega}$ such that for all $U \in T(X)$ containing $\tau(s)$, there exists $\mathcal{U}_{s^\frown\langle U\rangle} \in \Omega_X$ such that $U = \sigma(\langle \mathcal{U}_{s\restriction 1}, \ldots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle U\rangle}\rangle)$. (To complete the induction, $\mathcal{U}_{s^\frown\langle U\rangle}$ may be chosen arbitrarily for all other $U \in T(X)$.)

So τ is a strategy for I in $\Omega FO(X)$. Let ν legally attack τ, so $\tau(\nu \restriction n) \subseteq \nu(n)$ for all $n < \omega$. It follows that $\nu(n) = \sigma(\langle \mathcal{U}_{\nu\restriction 1}, \ldots, \mathcal{U}_{\nu\restriction n}, \mathcal{U}_{\nu\restriction n+1}\rangle)$. Since $\langle \mathcal{U}_{\nu\restriction 1}, \mathcal{U}_{n\restriction 2}, \ldots\rangle$ is a legal attack against σ, it follows that $\{\sigma(\langle \mathcal{U}_{\nu\restriction 1}, \ldots, \mathcal{U}_{\nu\restriction n+1}\rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$ is an ω-cover. Therefore τ is a winning strategy, verifying $\mathrm{I} \uparrow \Omega FO(X)$.

The equivalence of (c), (f), (h), and (i) are given as 3.8 of [Tkachuk 2017].

The equivalence of (f) and (g) are given in [https://doi.org/10.1016/0016-660X(76)90024-6].

The equivalence of (b), (j), and (k) are due to Clontz's Menger preprint.

(k) $\Leftrightarrow$ (h) follows from 3.18a of [Tkachuk 2017]. Tkachuk referes to the $sCDFT$ game at a point p and $CLD(X, p)$. $\square$

If we require the strategies for I for $CD(C_p(X))$ to be low information, then it must be that X is countable, and so $C_p(X)$ is first countable. Combining with several other results in the literature, we observe this behavior for a variety of games.

Theorem 15. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X *is countable.*
b) X *is* R^{+mark}.
c) X *is* ΩR^{+mark}.
d) $\mathrm{I} \underset{pre}{\uparrow} PO(X)$.
e) $\mathrm{I} \underset{pre}{\uparrow} FO(X)$.
f) $\mathrm{I} \underset{pre}{\uparrow} \Omega FO(x)$.
g) $C_p(X)$ *is first-countable.*
h) $\mathrm{I} \underset{pre}{\uparrow} Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$.
i) $\mathrm{I} \underset{pre}{\uparrow} Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.
j) $\mathrm{I} \underset{pre}{\uparrow} CL(C_p(X), \mathbf{0})$.
k) $\mathrm{I} \underset{pre}{\uparrow} CD(C_p(X))$.
l) $C_p(X)$ *is* $sCFT^{+mark}$.
m) $C_p(X)$ *is* $sCDFT^{+mark}$.

Proof. (a) $\Rightarrow$ (d) is straight-forward. So let σ be a predetermined strategy for I in $PO(X)$. If $x \notin \{\sigma(n) : n < \omega\}$, let $f(n) = X \setminus \{x\}$ for all $n < \omega$. It follows that f is a legal counter-attack for II defeating σ. Thus not (a) implies not (d).

The equivalence of (b) and (d) was shown by Clontz in unpublished work. We provide a proof here. Let σ be a winning Markov strategy for II in $Roth_{C,S}(X)$. Let $n < \omega$. Suppose that for each $x \in X$, there was an open neighborhood U_x of x where for every open cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_x$. Then $\sigma(\{U_x : x \in X\}, n) \notin \{U_x : x \in X\}$, a contradiction.

So for each $n < \omega$, there exists $\tau(n) \in X$ such that for any open neighborhood U of $\tau(n)$, there exists an open cover $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = U$. Then τ is a predetermined strategy for I in $PO(X)$.

It is also winning: for every attack f against τ, note that $f(n)$ is an open neighborhood of $\tau(n)$, so choose $\mathcal{U}_n$ such that $\sigma(\mathcal{U}_n, n) = f(n)$. Then since $\langle \mathcal{U}_0, \mathcal{U}_1, \ldots\rangle$ is a legal attack against σ, it follows that $\{f(n) : n < \omega\}$ is an open cover of X. Therefore τ is a winning predetermined strategy.

Now let σ be a winning predetermined strategy for I in $PO(X)$. For an open cover $\mathcal{U}$ of X and $n < \omega$, let $\tau(\mathcal{U}, n)$ be any open set in $\mathcal{U}$ containing $\sigma(n)$. It follows that τ is a winning Markov strategy for II in $Roth_{C,S}(X)$.

Clearly (d) implies (e), so we will see that (e) implies (a). Let $\sigma(n)$ be a pre-determined strategy for I for $FO(X)$. Towards a contradiction, suppose that there is some $x \in X \setminus \bigcup_n \sigma(n)$. II could then play $FO(X)$ as follows. At round n II can play an open set U_n which contains $\sigma(n)$ but excludes x. Then $x \notin \bigcup_n U_n$, and so I has lost. This is a contradiction. So $X = \bigcup_n \sigma(n)$, which means it is countable.

It also clear that (f) implies (e), we will show that (a) implies (f). If X is countable, then so is $[X]^{<\omega}$, enumerate it as $\{s_n : n \in \omega\}$. I's pre-determined strategy for $\Omega FO(X)$ is to play s_n are round n. Clearly whatever II plays will be an ω-cover. Thus (a), (c), (d), (e), and (f) are equivalent.

It is well-known and easy to see that (a) is equivalent to (g).

To see that (g) implies (h), note that we can find a sequence of open sets U_n so that $\mathbf{0} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$ for all n. I simply plays U_n are turn n, and whatever x_n are played by II must converge to x.

Clearly (h) implies (j) which in turn implies (k), which is equivalent to (a) by [CLOSEDDISCRETESELECTIONS]

Clontz showed (h) and (i) are equivalent in his dissertation; it's not hard to prove this.

Clontz showed that (c), (l), and (m) are equivalent in his Menger/CFT preprint.

Note that (c) implies (b) implies (a). So the last thing we need to show is that (f) implies (c). Let $\sigma(n)$ be a pre-determined strategy for I for $\Omega FO(X)$. We define a Markov strategy, $\tau(\mathcal{U}, n)$ for II for ΩR as follows. At round n suppose I has played $\mathcal{U}$. As $\mathcal{U}$ must be an ω-cover, there is a $U \in \mathcal{U}$ so that $\sigma(n) \subseteq U$. II plays such a U_n. Now suppose this game has been played according to τ, and that I has played $\mathcal{U}_n$ for $n < \omega$. Then the sequence of open sets $\tau(\mathcal{U}_n, n)$ forms a legal play against σ for $\Omega FO(X)$. Thus $\{\tau(\mathcal{U}_n, n) : n \in \omega\}$ is an ω-cover of X and so τ is a winning Markov strategy. □

In [CLOSEDDISCRETESELECTIONS], Tkachuk characterizes $\text{II} \uparrow \Omega FO(X)$ as the second player having an "almost winning strategy" (II can prevent I from constructing an ω-cover but perhaps not an arbitrary open cover) in $PO(X)$, which he conflates with $FO(X)$ as they are equivalent for "completely" winning perfect information strategies.

But they cannot be interchanged in general. Note that $\text{II} \underset{\text{tact}}{\uparrow} \Omega PO(2)$, where 2 is the two-point discrete space: let $\sigma(\langle x \rangle) = \{x\}$. Since every ω-cover of 2 includes 2, and σ never produces 2, this is a winning tactic. But since 2 is countable, 2 is ΩR^{+mark}. So $\Omega PO(X)$ is a very different game than those described previously.

In [CLOSEDDISCRETESELECTIONS], Tkachuk showed that for $T_{3.5}$ spaces X, X is uncountable if and only if $C_p(X)$ is discretely selective. Phrased in terms of games, Tkachuk showed the following.

Theorem 16. *For $T_{3.5}$ spaces X, X is uncountable if and only* $\text{I} \underset{pre}{\not\uparrow} CD(C_p(X))$.

4. Strategies for player II for the Discrete Selection Game on $C_p(X)$

Now we turn our attention to the opponent. We will begin by analyzing these games not just for $T_{3.5}$ spaces X or on $C_p(X)$, but in general. First off we will look at games related to open covers.

Proposition 17. *The following are equivalent for all spaces X.*

a) $\text{II} \uparrow PO(X)$.
b) $\text{II} \underset{mark}{\uparrow} PO(X)$.
c) $\text{II} \uparrow FO(X)$.
d) $\text{II} \underset{mark}{\uparrow} FO(X)$.
e) $\text{I} \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$.
f) X is not R, that is, $\text{I} \underset{pre}{\uparrow} G_1(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. (a) ⇔ (c) is 4.4 of [Telgarksy 1975].

The duality of $PO(X)$ and $G_1(\mathcal{O}_X, \mathcal{O}_X)$ for both players when considering perfect information is a well-known result of Galvin. So (a) is equivalent to (e).

The equivalence of (e) and (f) is just a restatement of Pawlikowski's result that the Rothberger selection principle is equivalent to I $\not\uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$, since the Rothberger selection principle is equivalent to I $\underset{\text{pre}}{\not\uparrow}$ $G_1(\mathcal{O}_X, \mathcal{O}_X)$.

(f) and (b) were shown to be equivalent by Clontz in unpublished work. We provide a proof here. Let σ be a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$. For $x \in X$ and $n < \omega$, let $\tau(x, n)$ be any open set in $\sigma(n)$ containing x. It follows that τ is a winning Markov strategy for II in $PO(X)$.

Now let σ be a winning Markov strategy for II in $PO(X)$. We may defined the open cover $\tau(n) = \{\sigma(x, n) : x \in X\}$ of X. It follows that τ is a winning predetermined strategy for I in $G_1(\mathcal{O}_X, \mathcal{O}_X)$.

Finally, (d) implies (b) is obvious. We therefore finish the proof by showing that (b) implies (d). Let $b : \omega^2 \to \omega$ be a bijection. Given a winning Markov strategy σ for II in $PO(X)$, define $\tau(F_n, n) = \bigcup\{\sigma(x(i, n), b(i, n)) : i < \omega\}$ where $F_n = \{x(i, n) : i < \omega\}$ (this indexing will cause at least one point to be repeated infinitely often, but this won't be a problem). So given an attack $\langle F_0, F_1, \ldots\rangle$ against τ, consider the attack g against σ, where $g(n) = x_{b^{\leftarrow}(n)}$. It follows that

$$X \neq \bigcup\{\sigma(g(n), n) : n < \omega\} = \bigcup\{\sigma(x(i, n), b(i, n)) : i, n < \omega\} = \bigcup\{\tau(F_n, n) : n < \omega\}$$

and therefore τ is a winning Markov strategy for II. Thus (b) implies (d). $\square$

Next we will consider games related to ω-covers.

Proposition 18. *The following are equivalent for all spaces X.*

a) II $\uparrow \Omega FO(X)$.
b) II $\underset{mark}{\uparrow} \Omega FO(X)$.
c) I $\uparrow G_1(\Omega_X, \Omega_X)$.
d) *X is not ΩR, that is,* I $\underset{pre}{\uparrow} G_1(\Omega_X, \Omega_X)$.

Proof. Let σ be a winning strategy for II in $\Omega FO(X)$. For $s \in ([X]^{<\omega})^{<\omega}$, let $\mathcal{U}_s = \{\sigma(s^\frown\langle F\rangle : F \in [X]^{<\omega}\}$. Then define the strategy τ for I by $\tau(s) = \sigma(\langle \mathcal{U}_{s\restriction 0}, \ldots, \mathcal{U}_s\rangle)$. Then every attack f against τ yields $g \in ([X]^{<\omega})^\omega$ such that $f(n) = \sigma(g \restriction n+1)$. Thus $\{f(n) : n < \omega\} = \{\sigma(g \restriction n+1) : n < \omega\}$ is not an ω-cover, so τ is a winning strategy, verifying that (a) implies (c).

The equivalence of (c) and (d) is given by Thm2 of [http://eudml.org/doc/212209].

Let σ be a winning predetermined strategy for I in $G_1(\Omega_x, \Omega_x)$. For $F \in [X]^{<\omega}$ and $n < \omega$, let $\tau(F, n)$ be any open set in $\sigma(n)$ containing F. It follows that τ is a winning Markov strategy for II in $\Omega FO(X)$, verifying that (c) implies (b).

(b) implies (a) is trivial, so the proof is complete. $\square$

ΩR is equivalent to all finite powers being R: Thm3 of [http://eudml.org/doc/212209]. It is consistent that these notions do not coincide: see Thm9 of [http://dx.doi.org/10.1016/j.topol.2013.07.022] for a consistent example of a $T_{3.5}$ R space whose square is not R, so therefore not ΩR. Note the distinction with strategies for the opponent, as R^+ is equivalent to ΩR^+ and R^{+mark} is equivalent to ΩR^{+mark}.

Finally we will examine the point-picking games.

Proposition 19. *The following properties imply lower properites for all spaces X.*

a) I $\uparrow G_1(\mathcal{D}_X, \Omega_{X,x})$.
b) II $\uparrow CL(X, x)$.
c) II $\uparrow Gru^{\rightsquigarrow}_{O,P}(X, x)$.
d) I $\uparrow G_1(\Omega_{X,x}, \Omega_{X,x})$.

Proof. Begin by letting σ be a winning strategy for I in $G_1(\mathcal{D}_X, \Omega_{X,x})$. For $s \in T_X^{<\omega}$, assume $\tau(s \restriction i+1)$ is defined for $i < |s|$, defining $s' \in X^{|s|}$ by $s'(i) = \tau(s \restriction i+1)$, and let $\tau(s^\frown\langle U\rangle) \in \sigma(s') \cap U$. So τ is a strategy for II in $CL(X, x)$. Then for any attack f against τ, an attack f' against σ is defined by $f'(i) = \tau(f \restriction i+1)$. It follows that $\{f'(i) : i < \omega\} = \{\tau(f \restriction i+1) : i < \omega\} \notin \Omega_{X,x}$, so τ is a winning strategy, witnessing (a) implies (b).

Let σ be a winning strategy for II in $CL(X, x)$. Then σ is also a winning strategy for II in $Gru^{\rightsquigarrow}_{O,P}(X, x)$, so (b) implies (c).

Given a winning strategy σ for II in $Gru_{O,P}^{\rightsquigarrow}(X,x)$, let $s \in {T_{X,x}}^{<\omega}$ and suppose and $B_t \in \Omega_{X,x}$ is defined for all $t < s$. Then let $B_s = \{\sigma(s^\frown\langle U\rangle) : U \in T_{X,x}\}$; it's clear that $B_s \in \Omega_{X,x}$. Define τ for I in $G_1(\Omega_{X,x}, \Omega_{X,x})$ by $\tau(r) = B_{r'}$ where $r' \in {T_{X,x}}^{|r|}$ satisfies $r(i) = \sigma(r' \restriction i+1)$ for all $i < |r|$. Then an attack f against τ yields an attack f' against σ such that $f(i) = \sigma(f' \restriction i+1)$ for all $i < \omega$. Since σ is a winning strategy, it follows that $\{f(i) : i < \omega\} = \{\sigma(f' \restriction i+1) : i < \omega\} \notin \Omega_{X,x}$. This verifies (c) implies (d). □

Proposition 20. *The following properties imply lower properites for all spaces X.*

a) I $\underset{pre}{\uparrow}$ $G_1(\mathcal{D}_X, \Omega_{X,x})$.
b) II $\underset{mark}{\uparrow}$ $CL(X,x)$.
c) II $\underset{mark}{\uparrow}$ $Gru_{O,P}^{\rightsquigarrow}(X,x)$.
d) I $\underset{pre}{\uparrow}$ $G_1(\Omega_{X,x}, \Omega_{X,x})$.

Proof. Begin by letting σ be a winning predetermined strategy for I in $G_1(\mathcal{D}_X, \Omega_{X,x})$. Define the Markov strategy τ for II in $CL(X,x)$ by $\tau(U,n) \in \sigma(n) \cap U$. Since $\tau(U,n) \in \sigma(n)$ for all $n < \omega$, it's clear that $\{\tau(U,n) : n < \omega\} \notin \Omega_{X,x}$, making τ a winning strategy, witnessing (a) implies (b).

Let σ be a winning Markov strategy for II in $CL(X,x)$. Then σ is also a winning Markov strategy for II in $Gru_{O,P}^{\rightsquigarrow}(X,x)$, so (b) implies (c).

Given a winning Markov strategy σ for II in $Gru_{O,P}^{\rightsquigarrow}(X,x)$, let $\tau(n) = \{\sigma(U,n) : U \in T_{X,x}$. Then τ is a predetermined strategy for I in $G_1(\Omega_{X,x}, \Omega_{X,x})$. For any attack f against τ, $f(n) = \sigma(g(n),n)$ for some $g(n) \in T_{X,x}$. But then g is an attack against σ, and thus $\{f(n) : n < \omega\} = \{\sigma(g(n),n) : n < \omega\} \notin \Omega_{X,x}$, so we have (c) implies (d). □

Note that for $C_p(X)$ with X $T_{3.5}$, (a)-(d) in both of the above propositions are actually equivalent.

Theorem 21. *The following are equivalent for all $T_{3.5}$ spaces.*

a) II $\uparrow \Omega FO(X)$.
b) II $\underset{mark}{\uparrow}$ $\Omega FO(X)$.
c) I $\uparrow G_1(\Omega_X, \Omega_X)$.
d) *X is not ΩR, that is,* I $\underset{pre}{\uparrow}$ $G_1(\Omega_X, \Omega_X)$.
e) I $\uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.
f) *$C_p(X)$ is not $sCFT$, that is,* I $\underset{pre}{\uparrow}$ $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.
g) I $\uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.
h) *$C_p(X)$ is not $sCDFT$, that is,* I $\underset{pre}{\uparrow}$ $G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.
i) II $\uparrow Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.
j) II $\underset{mark}{\uparrow}$ $Gru_{O,P}^{\rightsquigarrow}(C_p(X), \mathbf{0})$.
k) II $\uparrow CL(C_p(X), \mathbf{0})$.
l) II $\underset{mark}{\uparrow}$ $CL(C_p(X), \mathbf{0})$.
m) II $\uparrow CD(C_p(X))$.
n) II $\underset{mark}{\uparrow}$ $CD(C_p(X))$.

Proof. (a)-(d) were shown in Proposition 18. The equivalence of (d), (f), and (h) was shown by Sakai. The equivalence of (f) and (e) is given in 4.37 of [Comb. of Open Covers in Gen Prog Top III].

Of course (h) implies (g). And since $\mathcal{D}_{C_p(X)} \subseteq \Omega_{C_p(X),\mathbf{0}}$, any winning strategy for I in $G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$ is a winning strategy for I in $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$, so (g) implies (e). We have so far shown that (a) - (h) are equivalent.

Proposition 19 established that (e), (g), (i), and (k) are equivalent. Proposition 20 established that (f), (h), (j), and (l) are equivalent. Thus (a) - (l) are equivalent.

Assuming (b), we adapt Proposition 3.9 of [Tkachuk Two Point-Picking Games] as follows. Let σ be a winning Markov strategy for II in $\Omega FO(X)$. Then for $U = [\mathbf{x}(U), supp(U), \epsilon(U)] \in T_{C_p(X)}$, let $\tau(U,n) \in C_p(X)$ satisfy $\tau(U,n)(x) = \mathbf{x}(U)(x)$ for $x \in F$ and $\tau(U,n)(x) = n$ for $x \in X \setminus \sigma(U,n)$. Then τ is a Markov strategy for II, and when it is attacked by f, we note that $\{\sigma(supp(f(n)), n) : n < \omega\}$ is not an ω-cover. So choose $G \in [X]^{<\omega}$ such that $G \not\subseteq \sigma(supp(f(n)), n)$ for all $n < \omega$. Then for $\mathbf{y} \in C_p(X)$, choose m such that $\mathbf{y}(x) < m$ for all $x \in G$. Note then that for $n \geq m$, there exists $x \in G \setminus \sigma(f(n), n)$ such that $\tau(f(n), n)(x) = n \geq m$. Then $\{\mathbf{z} \in C_p(X) : \mathbf{z}(x) < m \text{ for all } x \in G\}$ is an open neighborhood of $\mathbf{y}$ that misses $\tau(f(n), n)$ for all $n \geq m$, so it follows that $\{\tau(f(n), n) : n < \omega\}$ is closed and discrete in $C_p(X)$. Therefore τ is a winning Markov strategy, verifying (b) implies (n).

It's clear that (n) implies (m), so finally note that a winning strategy for II in $CD(C_p(X))$ is also a winning strategy for II in $CL(C_p(X), \mathbf{0})$, so (m) implies (k). □

The equivalence of (a) and (m) answers Quesiton 4.6 of Tkachuk in [Two Point-picking games].

5. The Discrete Selection Game on Function Spaces of Cardinals

Game 22. Let G be the following game. During round n, player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

For $f \in \omega^\alpha$, let $f^{\leftarrow}[n] = \{\beta < \alpha : f(\beta) < n\}$.

Proposition 23. II $\underset{2\text{-}mark}{\uparrow}$ G.

Proof. Let $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark σ for II by

$$\sigma(\langle\alpha\rangle, 0) = \emptyset$$

and

$$\sigma(\langle\alpha, \beta\rangle, n+1) \ = f_\beta^{\leftarrow}[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let ν be an attack by I against σ, and let $\gamma < \nu(n)$ for $N \leq n < \omega$. If $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N \leq n < \omega$, then $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for infinitely-many $N \leq n < \omega$. Otherwise $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$ for cofinitely-many $N \leq n < \omega$, so $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for cofinitely-many $N \leq n < \omega$. Therefore σ is a winning 2-mark. □

Theorem 24. I $\underset{2\text{-}mark}{\uparrow}$ $CD(C_p(\omega_1 + 1))$

Proof. Let σ be a winning 2-mark for II in G.

Given a point $f \in C_p(\omega_1 + 1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f\rangle, 1) = [\mathbf{0}; \sigma(\langle\alpha_f\rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g\rangle, n+2) = [\mathbf{0}; \sigma(\langle\alpha_f, \alpha_g\rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let ν be a legal attack by II against σ. For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle\alpha_{\nu(n)}, \alpha_{\nu(n+1)}\rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \mathrm{cl}\{\nu(n) : n < \omega\}$. □

6. Open Problems

Acknowledgements

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