Limited Information Strategies and Discrete Selectivity

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1 Definitions

Definition 1. The *selection principle* $S_{fin}(\mathcal{A}, \mathcal{B})$ states that given $A_n \in \mathcal{A}$ for $n < \omega$, there exist $B_n \in [A_n]^{<\omega}$ such that $\bigcup_{n<\omega} B_n \in \mathcal{B}$.

Definition 2. The *selection game* $G_{fin}(\mathcal{A}, \mathcal{B})$ is the analogous game to $S_{fin}(\mathcal{A}, \mathcal{B})$, where during each round $n < \omega$, Player I first chooses $A_n \in \mathcal{A}$, and then Player II chooses $B_n \in [A_n]^{<\omega}$. Player II wins in the case that $\bigcup_{n<\omega} B_n \in \mathcal{B}$, and Player I wins otherwise.

Definition 3. Let $\mathcal{O}_X$ be the collection of open covers for a topological space X. Then $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger property* for X (M for short), and $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ is the well-known *Menger game*.

Definition 4. An *ω-cover* $\mathcal{U}$ for a topological space X is an open cover such that for every $F \in [X]^{<\omega}$, there exists some $U \in \mathcal{U}$ such that $F \subseteq U$.

Definition 5. Let Ω_X be the collection of ω-covers for a topological space X. Then $S_{fin}(\Omega_X, \Omega_X)$ is the *Ω-Menger property* for X (ΩM for short), and $G_{fin}(\Omega_X, \Omega_X)$ is the *Ω-Menger game*.

Definition 6. Let $\Omega_{X,x}$ be the collection of subsets $A \subset X$ where $x \in \operatorname{cl} A$. (Call A a *blade* of x.) Then $S_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness property* for X at x (CFT_x for short), and $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$ is the *countable fan tightness game* for X at x.

Definition 7. A space X has *countable fan tightness* (CFT for short) if it has countable fan tightness at each point $x \in X$.

Definition 8. Let $\mathcal{D}_X$ be the collection of dense subsets of a topological space X. Then $S_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness property* for X at x ($CDFT_x$ for short), and $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$ is the *countable dense fan tightness game* for X at x.

Definition 9. A space X has *countable dense fan tightness* ($CDFT$ for short) if it has countable dense fan tightness at each point $x \in X$.

Definition 10. $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability property* for X (SS for short), and $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ is the *selective separability game* for X.

Definition 11. A *strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0, \ldots, A_n\rangle) \in [A_n]^{<\omega}$ for $\langle A_0 \ldots, A_n\rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(\langle A_0, \ldots, A_n\rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 12. A *Markov strategy* for II in the game $G_{fin}(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(A, n) \in [A_n]^{<\omega}$ for $A \in \mathcal{A}$ and $n < \omega$. We say this Markov strategy is *winning* if whenever I plays $A_n \in \mathcal{A}$ during each round $n < \omega$, II wins the game by playing $\sigma(A_n, n)$ during each round $n < \omega$. If a winning Markov strategy exists, then we write II $\underset{\text{mark}}{\uparrow}$ $G_{fin}(\mathcal{A}, \mathcal{B})$.

Definition 13. In some instances, player I will be able to win a game regardless of what II is playing. In this case, it is possible to have a strategy for I which depends only on the round of the game. We say I has a *pre-determined strategy* and write I $\underset{\text{pre}}{\uparrow}$ G.

Notation 14. If $S_{fin}(\mathcal{A}, \mathcal{B})$ characterizes the property P, then we say II $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes P^+ ("strategically P"), and II $\underset{\text{mark}}{\uparrow}$ $G_{fin}(\mathcal{A}, \mathcal{B})$ characterizes $P^{+\text{mark}}$ ("Markov P"). Of course, $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$.

Notation 15. Let $S_1(\mathcal{A}, \mathcal{B}), G_1(\mathcal{A}, \mathcal{B})$ be the natural variants of $S_{fin}(\mathcal{A}, \mathcal{B}), G_{fin}(\mathcal{A}, \mathcal{B})$ where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the "strong Menger" property is commonly known as "Rothberger". We will thus call "strong Ω-Menger" "Ω-Rothberger" and shorten it with ΩR, and otherwise attach the prefix "s" when abbreviating to all other strong variants.

In addition to pure selection games, we also will be playing various point-picking games.

Definition 16. Set $T(X)$ to be the non-empty open subsets of X. The *point-open game* for X, denoted $PO(X)$, is played as follows. Each round, player I plays a point $x_n \in X$ and player II plays an open sets U_n with the property that $x_n \in U_n$. I wins the play of the game if $X = \bigcup_n U_n$.

The *finite-open game* for X, denoted $FO(x)$, is played similarly, except that I now plays finite subsets of X, and II's open sets must cover I's finite set. Note that $PO(X)$ is just $sFO(S)$. $\Omega FO(X)$ and $\Omega PO(X)$ are defined according to convention: I now wins if $\{U_n : n \in \omega\}$ forms an ω-cover of X.

Definition 17. Let $x \in X$. *Gruenhage's* W*-game* for x, denoted $Gru_{\vec{O},P}(X, x)$, is played as follows. Each round, player I plays an open set U_n with the property that $x \in U_n$ and player II plays a point $x_n \in U_n$. I wins if $x_n \to x$.

The *closure game* for x, denoted $CL(X, x)$, is played the same as Gruenhage's W-game, but now I wins if $x \in \overline{\{x_n : n \in \omega\}}$. Note that this is $G_1(T(X), \Omega_{X,x})$.

The *discrete selectivity game*, denoted $CD(X)$, is also played the same as Gruenhage's W-game, but now II wins if $\{x_n : n \in \omega\}$ is closed and discrete. Note that this is $G_1(T(X), \mathcal{CD})$ if we let $\mathcal{CD}$ denoted the closed discrete subsets of X.

Definition 18. A space X is *discretely selective* iff

2 2-marks in CD(X)

Let $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$.

Game 19. Let G be the following game. During round n, player I chooses $\beta_n < \omega_1$, and player II chooses $F_n \in [\omega_1]^{<\aleph_0}$. II wins if whenever $\gamma < \beta_n$ for co-finitely many $n < \omega$, $\gamma \in F_n$ for infinitely many $n < \omega$.

For $f \in \omega^\alpha$, let $f^{\leftarrow}[n] = \{\beta < \alpha : f(\beta) < n\}$.

Proposition 20. II $\underset{2\text{-}mark}{\uparrow}$ G.

Proof. Let $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$ be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark σ for II by

$$\sigma(\langle\alpha\rangle, 0) = \emptyset$$

and

$$\sigma(\langle\alpha, \beta\rangle, n+1) \ = f_\beta^{\leftarrow}[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let ν be an attack by I against σ, and let $\gamma < \nu(n)$ for $N \leq n < \omega$. If $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$ for infinitely-many $N \leq n < \omega$, then $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for infinitely-many $N \leq n < \omega$. Otherwise $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$ for cofinitely-many $N \leq n < \omega$, so $\gamma \in \sigma(\langle\nu(n), \nu(n+1)\rangle, n+1)$ for cofinitely-many $N \leq n < \omega$. Therefore σ is a winning 2-mark. □

Theorem 21. I $\underset{2\text{-}mark}{\uparrow}$ $CD(C_p(\omega_1+1))$

Proof. Let σ be a winning 2-mark for II in G.

Given a point $f \in C_p(\omega_1+1)$, let $\alpha_f < \omega_1$ satisfy $f(\beta) = f(\gamma)$ for all $\alpha_f \leq \beta \leq \gamma \leq \omega_1$.

Let $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$, $\tau(\langle f\rangle, 1) = [\mathbf{0}; \sigma(\langle\alpha_f\rangle, 0) \cup \{\omega_1\}; 2]$, and

$$\tau(\langle f, g\rangle, n+2) = [\mathbf{0}; \sigma(\langle\alpha_f, \alpha_g\rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let ν be a legal attack by II against σ. For $\beta \leq \omega_1$, if $\beta < \alpha_{\nu(n)}$ for co-finitely many $n < \omega$, then $\beta \in \sigma(\langle\alpha_{\nu(n)}, \alpha_{\nu(n+1)}\rangle)$ for infinitely-many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$. Otherwise $\beta \geq \alpha_{\nu(n)}$ for infinitely many $n < \omega$, and thus $0 \in \mathrm{cl}\{\nu(n)(\beta) : n < \omega\}$ as well. Thus $\mathbf{0} \in \mathrm{cl}\{\nu(n) : n < \omega\}$. □

3 Combining game results

Theorem 22. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X is uncountable.

b) $C_p(X)$ has discrete selectivity.

Theorem 23. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X is R^+, that is, $\mathrm{II} \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$.

b) $\mathrm{I} \uparrow PO(X)$.

c) $\mathrm{I} \uparrow FO(X)$.

d) $\mathrm{I} \uparrow \Omega FO(x)$.

e) $\mathrm{I} \uparrow Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$.

f) $\mathrm{I} \uparrow CL(C_p(X), \mathbf{0})$.

g) $\mathrm{I} \uparrow CD(C_p(X))$.

h) X is ΩR^+, that is, $\mathrm{II} \uparrow G_1(\Omega_X, \Omega_X)$.

i) $C_p(X)$ is $sCFT^+$, that is, $\mathrm{II} \uparrow G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

j) $C_p(X)$ is sCDFT$^+$, that is, $\text{II} \uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X),\mathbf{0}})$.

Proof. (a) $\Leftrightarrow$ (b) is a well-known result of Galvin.

(b) $\Leftrightarrow$ (c) is 4.3 of [Telgarksy 1975].

(d) $\Leftrightarrow$ (c) is clear, but we need to show that (a) $\Leftrightarrow$ (d). So assume X is R^+, which is equivalent to ΩR^+. Let σ be a winning strategy for II in $G_1(\Omega_X, \Omega_X)$. Let $T(X)$ be the non-empty open sets of X, and let $s \in T(X)^{<\omega}$. Assume $\tau(t) \in [X]^{<\omega}$ is defined for all $t < s$, and $\mathcal{U}_t \in \Omega_X$ is defined for all $\emptyset < t \leq s$.

Suppose that for all $F \in [X]^{<\omega}$, there existed $U_F \in T(X)$ containing F such that for all $\mathcal{U} \in \Omega_X$, $U_F \neq \sigma(\langle \mathcal{U}_{s\upharpoonright 1}, \dots, U_s, \mathcal{U} \rangle)$. Let $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$. Then $\sigma(\langle \mathcal{U}_{s\upharpoonright 1}, \dots, U_s, \mathcal{U} \rangle)$ must equal some U_F, demonstrating a contradiction.

So there exists $\tau(s) \in [X]^{<\omega}$ such that for all $U \in T(X)$ containing $\tau(s)$, there exists $\mathcal{U}_{s^\frown\langle U \rangle} \in \Omega_X$ such that $U = \sigma(\langle \mathcal{U}_{s\upharpoonright 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s^\frown\langle U \rangle} \rangle)$. (To complete the induction, $\mathcal{U}_{s^\frown\langle U \rangle}$ may be chosen arbitrarily for all other $U \in T(X)$.)

So τ is a strategy for I in $\Omega FO(X)$. Let ν legally attack τ, so $\tau(\nu \upharpoonright n) \subseteq \nu(n)$ for all $n < \omega$. It follows that $\nu(n) = \sigma(\langle \mathcal{U}_{\nu\upharpoonright 1}, \dots, \mathcal{U}_{\nu\upharpoonright n}, \mathcal{U}_{\nu\upharpoonright n+1} \rangle)$. Since $\langle \mathcal{U}_{\nu\upharpoonright 1}, \mathcal{U}_{n\upharpoonright 2}, \dots \rangle$ is a legal attack against σ, it follows that $\{\sigma(\langle \mathcal{U}_{\nu\upharpoonright 1}, \dots, \mathcal{U}_{\nu\upharpoonright n+1} \rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$ is an ω-cover. Therefore τ is a winning strategy, verifying $\text{I} \uparrow \Omega FO(X)$.

The equivalence of (b), (e), (f), and (g) are given as 3.8 of [Tkachuk 2017].

The equivalence of (h), (i), and (j) are due to Clontz.

(j) $\Leftrightarrow$ (f) follows from 3.18a of [Tkachuk 2017]. $\square$

Theorem 24. *The following are equivalent for $T_{3.5}$ spaces X.*

a) X is countable.

b) X is R^{+mark}.

c) $\text{I} \underset{pre}{\uparrow} PO(X)$.

d) $\text{I} \underset{pre}{\uparrow} FO(X)$.

e) $\text{I} \underset{pre}{\uparrow} \Omega FO(x)$.

f) $C_p(X)$ is first-countable.

g) $\text{I} \underset{pre}{\uparrow} Gru_{\overrightarrow{O,P}}(C_p(X), \mathbf{0})$.

h) $\text{I} \underset{pre}{\uparrow} CL(C_p(X), \mathbf{0})$.

i) $\text{I} \underset{pre}{\uparrow} CD(C_p(X))$.

j) X is ΩR^{+mark}.

k) $C_p(X)$ is sCFT^{+mark}.

l) $C_p(X)$ is sCDFT^{+mark}.

Proof. The equivalence of (a) - (c) is contained in Clontz' dissertation.

Clearly (c) implies (d), so we will see that (d) implies (a). Let $\sigma(n)$ be a pre-determined strategy for I for $FO(X)$. Towards a contradiction, suppose that there is some $x \in X \setminus \bigcup_n \sigma(n)$. II could then play $FO(X)$ as follows. At round n II can play an open set U_n which contains $\sigma(n)$ but excludes x. Then $x \notin \bigcup_n U_n$, and so I has lost. This is a contradiction. So $X = \bigcup_n \sigma(n)$, which means it is countable.

It also clear that (e) implies (d), we will show that (a) implies (e). If X is countable, then so is $[X]^{<\omega}$, enumerate it as $\{s_n : n \in \omega\}$. I's pre-determined strategy for $\Omega FO(X)$ is to play s_n are round n. Clearly whatever II plays will be an ω-cover. Thus (a) - (e) are equivalent.

It is well-known and easy to see that (a) is equivalent to (f).

To see that (f) implies (g), note that we can find a sequence of open sets U_n so that $\mathbf{0} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$ for all n. I simply plays U_n are turn n, and whatever x_n are played by II must converge to x.

Clearly (g) implies (h) which in turn implies (i).

We now show that (i) implies (a). Let $\sigma(n)$ be a pre-determined strategy for I for $CD(C_p(X))$. Towards a contradiction suppose X is uncountable. Without loss of generality we can assume that $\sigma(n)$ is a basic open set, $[f_n, F_n, \varepsilon_n]$. If I is not already playing basic open sets, we can choose basic open subsets of I's play, and the resulting game will still produce a a sequence $\{x_n : n \in \omega\}$ which is not closed discrete. Let $F = \bigcup_n F_n$. As X is uncountable, we can choose a point $x \in X \setminus F$. Then II can play against σ by playing a function g_n at round n with the properties that $g_n \in [f_n, F_n, \varepsilon_n]$ and $g_n(x) = n$. The collection $\{g_n : n \in \omega\}$ is then closed discrete, and so I has lost. This is a contradiction, so X must be countable. Thus (a) - (i) are equivalent.

Clontz showed that (j), (k), and (l) are equivalent.

Note that (j) implies (b) implies (a). So the last thing we need to show is that (e) implies (j). Let $\sigma(n)$ be a pre-determined strategy for I for $\Omega FO(X)$. We define a Markov strategy, $\tau(\mathcal{U}, n)$ for II for ΩR as follows. At round n suppose I has played $\mathcal{U}$. As $\mathcal{U}$ must be an ω-cover, there is a $U \in \mathcal{U}$ so that $\sigma(n) \subseteq U$. II plays such a U_n. Now suppose this game has been played according to τ, and that I has played $\mathcal{U}_n$ for $n < \omega$. Then the sequence of open sets $\tau(\mathcal{U}_n, n)$ forms a legal play against σ for $\Omega FO(X)$. Thus $\{\tau(\mathcal{U}_n, n) : n \in \omega\}$ is an ω-cover of X and so τ is a winning Markov strategy. □

In that paper, Tkachuk characterizes II $\uparrow \Omega FO(X)$ as the second player having an "almost winning strategy" (II can prevent I from constructing an ω-cover but perhaps not an arbitrary open cover) in $PO(X)$, which he conflates with $FO(X)$ as they are equivalent for "completely" winning perfect information strategies.

But they cannot be interchanged in general. Note that II $\underset{\text{tact}}{\uparrow}$ $\Omega PO(2)$, where 2 is the two-point discrete space: let $\sigma(\langle x \rangle) = \{x\}$. Since every ω-cover of 2 includes 2, and σ never produces 2, this is a winning tactic. But for the same reason, 2 is R^+: it is legal to play 2 every round, which produces the ω-cover $\{2\}$. So $\Omega PO(X)$ is a very different game than those described previously.

Theorem 25. *Let X be $T_{3.5}$. The following are equivalent.*

a) II $\uparrow PO(X)$

b) II $\uparrow FO(X)$

c) II $\uparrow \Omega FO(X)$

d) II $\uparrow CD(C_p(X))$

Proof. (a) $\Leftrightarrow$ (b) is due to Telgarksy [1975]. (b) $\Rightarrow$ (c) is immediate, and (c) $\Rightarrow$ (d) was shown in [Tkachuk 2017].

Assume (d), so let σ be a winning strategy for II in $CD(C_p(X))$. Let $s \in ([X]^{<\omega})^{<\omega} \setminus \{\emptyset\}$. Let $s' \in (T(C_p(X)))^{<\omega} \setminus \{\emptyset\}$ be defined by

$$s'(n) = [\mathbf{0}, \bigcup\{s(m) : m \leq n\}, 1/2^{n+1}].$$

Let τ be the strategy for II in $FO(X)$ be defined by $\tau(s) = \sigma(s')^{\leftarrow}(-1/2^{|s|}, 1/2^{|s|})$. This is legal as $\sigma(s') \in s'(|s|-1) \subseteq [\mathbf{0}, s(|s|-1), 1/2^{|s|}]$ implies $s(|s|-1) \subseteq \sigma(s')^{\leftarrow}(-1/2^{|s|}, 1/2^{|s|})$.

Let $\nu \in ([X]^{<\omega})^{\omega}$ attack τ, and let

$$\nu'(n) = [\mathbf{0}, \bigcup\{\nu(m) : m \leq n\}, 1/2^{n+1}].$$

Let $F \in [X]^{<\omega}$ and $\epsilon > 0$. If $\{\tau(\nu \restriction n+1) : n < \omega\}$ is an open cover for X, choose $\frac{1}{\epsilon} < N < \omega$ such that

$$F \subseteq \bigcup\{\tau(\nu \restriction n+1) : n \leq N\} = \bigcup\{\sigma(\nu' \restriction n+1)^{\leftarrow}(-1/2^{n+1}, 1/2^{n+1}) : n \leq N\}.$$

TODO: not quite there...

$\square$