

# Limited Information Strategies and Discrete Selectivity

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## 1 Definitions

**Definition 1.** The *selection principle*  $S_{fin}(\mathcal{A}, \mathcal{B})$  states that given  $A_n \in \mathcal{A}$  for  $n < \omega$ , there exist  $B_n \in [A_n]^{<\omega}$  such that  $\bigcup_{n < \omega} B_n \in \mathcal{B}$ .

**Definition 2.** The *selection game*  $G_{fin}(\mathcal{A}, \mathcal{B})$  is the analogous game to  $S_{fin}(\mathcal{A}, \mathcal{B})$ , where during each round  $n < \omega$ , Player I first chooses  $A_n \in \mathcal{A}$ , and then Player II chooses  $B_n \in [A_n]^{<\omega}$ . Player II wins in the case that  $\bigcup_{n < \omega} B_n \in \mathcal{B}$ , and Player I wins otherwise.

**Definition 3.** Let  $\mathcal{O}_X$  be the collection of open covers for a topological space  $X$ . Then  $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$  is the well-known *Menger property* for  $X$  ( $M$  for short), and  $G_{fin}(\mathcal{O}_X, \mathcal{O}_X)$  is the well-known *Menger game*.

**Definition 4.** An  $\omega$ -cover  $\mathcal{U}$  for a topological space  $X$  is an open cover such that for every  $F \in [X]^{<\omega}$ , there exists some  $U \in \mathcal{U}$  such that  $F \subseteq U$ .

**Definition 5.** Let  $\Omega_X$  be the collection of  $\omega$ -covers for a topological space  $X$ . Then  $S_{fin}(\Omega_X, \Omega_X)$  is the  $\Omega$ -Menger property for  $X$  ( $\Omega M$  for short), and  $G_{fin}(\Omega_X, \Omega_X)$  is the  $\Omega$ -Menger game.

**Definition 6.** Let  $\Omega_{X,x}$  be the collection of subsets  $A \subset X$  where  $x \in \text{cl } A$ . (Call  $A$  a *blade* of  $x$ .) Then  $S_{fin}(\Omega_{X,x}, \Omega_{X,x})$  is the *countable fan tightness property* for  $X$  at  $x$  ( $CFT_x$  for short), and  $G_{fin}(\Omega_{X,x}, \Omega_{X,x})$  is the *countable fan tightness game* for  $X$  at  $x$ .

**Definition 7.** A space  $X$  has *countable fan tightness* ( $CFT$  for short) if it has countable fan tightness at each point  $x \in X$ .

**Definition 8.** Let  $\mathcal{D}_X$  be the collection of dense subsets of a topological space  $X$ . Then  $S_{fin}(\mathcal{D}_X, \Omega_{X,x})$  is the *countable dense fan tightness property* for  $X$  at  $x$  ( $CDFT_x$  for short), and  $G_{fin}(\mathcal{D}_X, \Omega_{X,x})$  is the *countable dense fan tightness game* for  $X$  at  $x$ .

**Definition 9.** A space  $X$  has *countable dense fan tightness* ( $CDFT$  for short) if it has countable dense fan tightness at each point  $x \in X$ .

**Definition 10.**  $S_{fin}(\mathcal{D}_X, \mathcal{D}_X)$  is the *selective separability property* for  $X$  ( $SS$  for short), and  $G_{fin}(\mathcal{D}_X, \mathcal{D}_X)$  is the *selective separability game* for  $X$ .

**Definition 11.** A *strategy* for II in the game  $G_{fin}(\mathcal{A}, \mathcal{B})$  is a function  $\sigma$  satisfying  $\sigma(\langle A_0, \dots, A_n \rangle) \in [A_n]^{<\omega}$  for  $\langle A_0, \dots, A_n \rangle \in \mathcal{A}^{n+1}$ . We say this strategy is *winning* if whenever I plays  $A_n \in \mathcal{A}$  during each round  $n < \omega$ , II wins the game by playing  $\sigma(\langle A_0, \dots, A_n \rangle)$  during each round  $n < \omega$ . If a winning strategy exists, then we write  $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$ .

**Definition 12.** A *Markov strategy* for II in the game  $G_{fin}(\mathcal{A}, \mathcal{B})$  is a function  $\sigma$  satisfying  $\sigma(A, n) \in [A_n]^{<\omega}$  for  $A \in \mathcal{A}$  and  $n < \omega$ . We say this Markov strategy is *winning* if whenever I plays  $A_n \in \mathcal{A}$  during each round  $n < \omega$ , II wins the game by playing  $\sigma(A_n, n)$  during each round  $n < \omega$ . If a winning Markov strategy exists, then we write  $\text{II} \uparrow_{\text{mark}} G_{fin}(\mathcal{A}, \mathcal{B})$ .

**Definition 13.** In some instances, player I will be able to win a game regardless of what II is playing. In this case, it is possible to have a strategy for I which depends only on the round of the game. We say I has a *pre-determined strategy* and write  $\text{I} \uparrow_{\text{pre}} G$ .

**Notation 14.** If  $S_{fin}(\mathcal{A}, \mathcal{B})$  characterizes the property  $P$ , then we say  $\text{II} \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$  characterizes  $P^+$  (“strategically  $P$ ”), and  $\text{II} \uparrow_{\text{mark}} G_{fin}(\mathcal{A}, \mathcal{B})$  characterizes  $P^{+\text{mark}}$  (“Markov  $P$ ”). Of course,  $P^{+\text{mark}} \Rightarrow P^+ \Rightarrow P$ .

**Notation 15.** Let  $S_1(\mathcal{A}, \mathcal{B}), G_1(\mathcal{A}, \mathcal{B})$  be the natural variants of  $S_{fin}(\mathcal{A}, \mathcal{B}), G_{fin}(\mathcal{A}, \mathcal{B})$  where each choice by II must either be a single element or singleton (whichever is more convenient for the proof at hand), rather than a finite set. Convention calls for denoting these as *strong* versions of the corresponding selection principles and games, although the “strong Menger” property is commonly known as “Rothberger”. We will thus call “strong  $\Omega$ -Menger” “ $\Omega$ -Rothberger” and shorten it with  $\Omega R$ , and otherwise attach the prefix “s” when abbreviating to all other strong variants.

In addition to pure selection games, we also will be playing various point-picking games.

**Definition 16.** Set  $T(X)$  to be the non-empty open subsets of  $X$ . The *point-open game* for  $X$ , denoted  $PO(X)$ , is played as follows. Each round, player I plays a point  $x_n \in X$  and player II plays an open sets  $U_n$  with the property that  $x_n \in U_n$ . I wins the play of the game if  $X = \bigcup_n U_n$ .

The *finite-open game* for  $X$ , denoted  $FO(X)$ , is played similarly, except that I now plays finite subsets of  $X$ , and II’s open sets must cover I’s finite set. Note that  $PO(X)$  is just  $sFO(X)$ .  $\Omega FO(X)$  and  $\Omega PO(X)$  are defined according to convention: I now wins if  $\{U_n : n \in \omega\}$  forms an  $\omega$ -cover of  $X$ .

**Definition 17.** Let  $x \in X$ . *Gruenhage’s  $W$ -game* for  $x$ , denoted  $Gru_{\vec{O}, P}(X, x)$ , is played as follows. Each round, player I plays an open set  $U_n$  with the property that  $x \in U_n$  and player II plays a point  $x_n \in U_n$ . I wins if  $x_n \rightarrow x$ .

The *closure game* for  $x$ , denoted  $CL(X, x)$ , is played the same as Gruenhage’s  $W$ -game, but now I wins if  $x \in \overline{\{x_n : n \in \omega\}}$ . Note that this is  $G_1(T(X), \Omega_{X, x})$ .

The *discrete selectivity game*, denoted  $CD(X)$ , is also played the same as Gruenhage’s  $W$ -game, but now II wins if  $\{x_n : n \in \omega\}$  is closed and discrete. Note that this is  $G_1(T(X), \mathcal{CD})$  if we let  $\mathcal{CD}$  denoted the closed discrete subsets of  $X$ .

**Definition 18.** A space  $X$  is *discretely selective* iff

## 2 2-marks in $\mathcal{CD}(X)$

Let  $[f, F, \epsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in F\}$ .

**Game 19.** Let  $G$  be the following game. During round  $n$ , player I chooses  $\beta_n < \omega_1$ , and player II chooses  $F_n \in [\omega_1]^{<\aleph_0}$ . II wins if whenever  $\gamma < \beta_n$  for co-finitely many  $n < \omega$ ,  $\gamma \in F_n$  for infinitely many  $n < \omega$ .

For  $f \in \omega^\alpha$ , let  $f^\leftarrow[n] = \{\beta < \alpha : f(\beta) < n\}$ .

**Proposition 20.**  $\text{II} \uparrow_{2\text{-mark}} G.$

*Proof.* Let  $\{f_\alpha \in \omega^\alpha : \alpha < \omega_1\}$  be a collection of pairwise almost-compatible finite-to-one functions. Define a 2-mark  $\sigma$  for II by

$$\sigma(\langle \alpha \rangle, 0) = \emptyset$$

and

$$\sigma(\langle \alpha, \beta \rangle, n+1) = f_\beta^\leftarrow[n] \cup \{\gamma < \alpha \cap \beta : f_\alpha(\gamma) \neq f_\beta(\gamma)\}.$$

Let  $\nu$  be an attack by I against  $\sigma$ , and let  $\gamma < \nu(n)$  for  $N \leq n < \omega$ . If  $f_{\nu(n)}(\gamma) \neq f_{\nu(n+1)}(\gamma)$  for infinitely-many  $N \leq n < \omega$ , then  $\gamma \in \sigma(\langle \nu(n), \nu(n+1) \rangle, n+1)$  for infinitely-many  $N \leq n < \omega$ . Otherwise  $f_{\nu(n)}(\gamma) = f_{\nu(n+1)}(\gamma) = M$  for cofinitely-many  $N \leq n < \omega$ , so  $\gamma \in \sigma(\langle \nu(n), \nu(n+1) \rangle, n+1)$  for cofinitely-many  $N \leq n < \omega$ . Therefore  $\sigma$  is a winning 2-mark.  $\square$

**Theorem 21.**  $\text{I} \uparrow_{2\text{-mark}} CD(C_p(\omega_1 + 1))$

*Proof.* Let  $\sigma$  be a winning 2-mark for II in  $G$ .

Given a point  $f \in C_p(\omega_1 + 1)$ , let  $\alpha_f < \omega_1$  satisfy  $f(\beta) = f(\gamma)$  for all  $\alpha_f \leq \beta \leq \gamma \leq \omega_1$ .

Let  $\tau(\emptyset, 0) = [\mathbf{0}, \{\omega_1\}, 4]$ ,  $\tau(\langle f \rangle, 1) = [\mathbf{0}; \sigma(\langle \alpha_f \rangle, 0) \cup \{\omega_1\}; 2]$ , and

$$\tau(\langle f, g \rangle, n+2) = [\mathbf{0}; \sigma(\langle \alpha_f, \alpha_g \rangle, n+1) \cup \{\omega_1\}; 2^{-n}].$$

Let  $\nu$  be a legal attack by II against  $\sigma$ . For  $\beta \leq \omega_1$ , if  $\beta < \alpha_{\nu(n)}$  for co-finitely many  $n < \omega$ , then  $\beta \in \sigma(\langle \alpha_{\nu(n)}, \alpha_{\nu(n+1)} \rangle)$  for infinitely-many  $n < \omega$ , and thus  $0 \in \text{cl}\{\nu(n)(\beta) : n < \omega\}$ . Otherwise  $\beta \geq \alpha_{\nu(n)}$  for infinitely many  $n < \omega$ , and thus  $0 \in \text{cl}\{\nu(n)(\beta) : n < \omega\}$  as well. Thus  $\mathbf{0} \in \text{cl}\{\nu(n) : n < \omega\}$ .  $\square$

### 3 Combining game results

**Theorem 22.** *The following are equivalent for  $T_{3,5}$  spaces  $X$ .*

- a)  $X$  is  $R^+$ , that is,  $\text{II} \uparrow G_1(\mathcal{O}_X, \mathcal{O}_X)$ .
- b)  $\text{I} \uparrow PO(X)$ .
- c)  $\text{I} \uparrow FO(X)$ .
- d)  $\text{I} \uparrow \Omega FO(X)$ .
- e)  $\text{I} \uparrow \text{Gru}_{\vec{\mathcal{O}}, P}(C_p(X), \mathbf{0})$ .
- f)  $\text{I} \uparrow CL(C_p(X), \mathbf{0})$ .
- g)  $\text{I} \uparrow CD(C_p(X))$ .
- h)  $X$  is  $\Omega R^+$ , that is,  $\text{II} \uparrow G_1(\Omega_X, \Omega_X)$ .
- i)  $C_p(X)$  is  $sCFT^+$ , that is,  $\text{II} \uparrow G_1(\Omega_{C_p(X), \mathbf{0}}, \Omega_{C_p(X), \mathbf{0}})$ .
- j)  $C_p(X)$  is  $sCDFT^+$ , that is,  $\text{II} \uparrow G_1(\mathcal{D}_{C_p(X)}, \Omega_{C_p(X), \mathbf{0}})$ .

*Proof.* (a)  $\Leftrightarrow$  (b) is a well-known result of Galvin.

(b)  $\Leftrightarrow$  (c) is 4.3 of [Telgarksy 1975].

(d)  $\Leftrightarrow$  (c) is clear, but we need to show that (a)  $\Leftrightarrow$  (d). So assume  $X$  is  $R^+$ , which is equivalent to  $\Omega R^+$ . Let  $\sigma$  be a winning strategy for II in  $G_1(\Omega_X, \Omega_X)$ . Let  $T(X)$  be the non-empty open sets of  $X$ , and let  $s \in T(X)^{<\omega}$ . Assume  $\tau(t) \in [X]^{<\omega}$  is defined for all  $t < s$ , and  $\mathcal{U}_t \in \Omega_X$  is defined for all  $\emptyset < t \leq s$ .

Suppose that for all  $F \in [X]^{<\omega}$ , there existed  $U_F \in T(X)$  containing  $F$  such that for all  $\mathcal{U} \in \Omega_X$ ,  $U_F \neq \sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, U_s, \mathcal{U} \rangle)$ . Let  $\mathcal{U} = \{U_F : F \in [X]^{<\omega}\} \in \Omega_X$ . Then  $\sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, U_s, \mathcal{U} \rangle)$  must equal some  $U_F$ , demonstrating a contradiction.

So there exists  $\tau(s) \in [X]^{<\omega}$  such that for all  $U \in T(X)$  containing  $\tau(s)$ , there exists  $\mathcal{U}_{s \smallfrown \langle U \rangle} \in \Omega_X$  such that  $U = \sigma(\langle \mathcal{U}_{s \upharpoonright 1}, \dots, \mathcal{U}_s, \mathcal{U}_{s \smallfrown \langle U \rangle} \rangle)$ . (To complete the induction,  $\mathcal{U}_{s \smallfrown \langle U \rangle}$  may be chosen arbitrarily for all other  $U \in T(X)$ .)

So  $\tau$  is a strategy for I in  $\Omega FO(X)$ . Let  $\nu$  legally attack  $\tau$ , so  $\tau(\nu \upharpoonright n) \subseteq \nu(n)$  for all  $n < \omega$ . It follows that  $\nu(n) = \sigma(\langle \mathcal{U}_{\nu \upharpoonright 1}, \dots, \mathcal{U}_{\nu \upharpoonright n}, \mathcal{U}_{\nu \upharpoonright n+1} \rangle)$ . Since  $\langle \mathcal{U}_{\nu \upharpoonright 1}, \mathcal{U}_{\nu \upharpoonright 2}, \dots \rangle$  is a legal attack against  $\sigma$ , it follows that  $\{\sigma(\langle \mathcal{U}_{\nu \upharpoonright 1}, \dots, \mathcal{U}_{\nu \upharpoonright n+1} \rangle) : n < \omega\} = \{\nu(n) : n < \omega\}$  is an  $\omega$ -cover. Therefore  $\tau$  is a winning strategy, verifying  $I \uparrow \Omega FO(X)$ .

The equivalence of (b), (e), (f), and (g) are given as 3.8 of [Tkachuk 2017].

The equivalence of (h), (i), and (j) are due to Clontz.

(j)  $\Leftrightarrow$  (f) follows from 3.18a of [Tkachuk 2017]. □

Tkachuk showed the following in [CLOSEDDISCRETESELECTIONS].

**Theorem 23.** *The following are equivalent for  $T_{3.5}$  spaces  $X$ .*

- a)  $X$  is uncountable.
- b)  $C_p(X)$  has discrete selectivity, that is,  $I \nuparrow_{pre} CD(C_p(X))$ .

Combining with several other results in the literature, we may observe the following.

**Theorem 24.** *The following are equivalent for  $T_{3.5}$  spaces  $X$ .*

- a)  $X$  is countable.
- b)  $X$  is  $R^{+mark}$ .
- c)  $I \uparrow_{pre} PO(X)$ .
- d)  $I \uparrow_{pre} FO(X)$ .
- e)  $I \uparrow_{pre} \Omega FO(x)$ .
- f)  $C_p(X)$  is first-countable.
- g)  $I \uparrow_{pre} Gru_{O,P}^{\rightarrow}(C_p(X), \mathbf{0})$ .
- h)  $I \uparrow_{pre} CL(C_p(X), \mathbf{0})$ .
- i)  $I \uparrow_{pre} CD(C_p(X))$ .

j)  $X$  is  $\Omega R^{+mark}$ .

k)  $C_p(X)$  is  $sCFT^{+mark}$ .

l)  $C_p(X)$  is  $sCDFT^{+mark}$ .

*Proof.* The equivalence of (a) - (c) is contained in Clontz' dissertation.

Clearly (c) implies (d), so we will see that (d) implies (a). Let  $\sigma(n)$  be a pre-determined strategy for I for  $FO(X)$ . Towards a contradiction, suppose that there is some  $x \in X \setminus \bigcup_n \sigma(n)$ . II could then play  $FO(X)$  as follows. At round  $n$  II can play an open set  $U_n$  which contains  $\sigma(n)$  but excludes  $x$ . Then  $x \notin \bigcup_n U_n$ , and so I has lost. This is a contradiction. So  $X = \bigcup_n \sigma(n)$ , which means it is countable.

It also clear that (e) implies (d), we will show that (a) implies (e). If  $X$  is countable, then so is  $[X]^{<\omega}$ , enumerate it as  $\{s_n : n \in \omega\}$ . I's pre-determined strategy for  $\Omega FO(X)$  is to play  $s_n$  around  $n$ . Clearly whatever II plays will be an  $\omega$ -cover. Thus (a) - (e) are equivalent.

It is well-known and easy to see that (a) is equivalent to (f).

To see that (f) implies (g), note that we can find a sequence of open sets  $U_n$  so that  $\mathbf{0} \in U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$  for all  $n$ . I simply plays  $U_n$  are turn  $n$ , and whatever  $x_n$  are played by II must converge to  $x$ .

Clearly (g) implies (h) which in turn implies (i), which is equivalent to (a) by [CLOSEDDISCRETESELECTIONS]

Clontz showed that (j), (k), and (l) are equivalent.

Note that (j) implies (b) implies (a). So the last thing we need to show is that (e) implies (j). Let  $\sigma(n)$  be a pre-determined strategy for I for  $\Omega FO(X)$ . We define a Markov strategy,  $\tau(\mathcal{U}, n)$  for II for  $\Omega R$  as follows. At round  $n$  suppose I has played  $\mathcal{U}$ . As  $\mathcal{U}$  must be an  $\omega$ -cover, there is a  $U \in \mathcal{U}$  so that  $\sigma(n) \subseteq U$ . II plays such a  $U_n$ . Now suppose this game has been played according to  $\tau$ , and that I has played  $\mathcal{U}_n$  for  $n < \omega$ . Then the sequence of open sets  $\tau(\mathcal{U}_n, n)$  forms a legal play against  $\sigma$  for  $\Omega FO(X)$ . Thus  $\{\tau(\mathcal{U}_n, n) : n \in \omega\}$  is an  $\omega$ -cover of  $X$  and so  $\tau$  is a winning Markov strategy.  $\square$

In that paper, Tkachuk characterizes  $\text{II} \uparrow \Omega FO(X)$  as the second player having an “almost winning strategy” (II can prevent I from constructing an  $\omega$ -cover but perhaps not an arbitrary open cover) in  $PO(X)$ , which he conflates with  $FO(X)$  as they are equivalent for “completely” winning perfect information strategies.

But they cannot be interchanged in general. Note that  $\text{II} \uparrow \Omega PO(2)$ , where 2 is the two-point discrete space: let  $\sigma(\langle x \rangle) = \{x\}$ . Since every  $\omega$ -cover of 2 includes 2, and  $\sigma$  never produces 2, this is a winning tactic. But for the same reason, 2 is  $R^+$ : it is legal to play 2 every round, which produces the  $\omega$ -cover  $\{2\}$ . So  $\Omega PO(X)$  is a very different game than those described previously.

**Theorem 25.** *Let  $X$  be  $T_{3.5}$ . The following are equivalent.*

a)  $\text{II} \uparrow PO(X)$

b)  $\text{II} \uparrow FO(X)$

c)  $\text{II} \uparrow \Omega FO(X)$

d)  $\text{II} \uparrow CD(C_p(X))$

*Proof.* (a)  $\Leftrightarrow$  (b) is due to Telgarksy [1975]. (b)  $\Rightarrow$  (c) is immediate, and (c)  $\Rightarrow$  (d) was shown in [Tkachuk 2017].

Assume (d), so let  $\sigma$  be a winning strategy for II in  $CD(C_p(X))$ . Let  $s \in ([X]^{<\omega})^{<\omega} \setminus \{\emptyset\}$ . Let  $s' \in (T(C_p(X)))^{<\omega} \setminus \{\emptyset\}$  be defined by

$$s'(n) = [\mathbf{0}, \bigcup \{s(m) : m \leq n\}, 1/2^{n+1}].$$

Let  $\tau$  be the strategy for II in  $FO(X)$  be defined by  $\tau(s) = \sigma(s')^{\leftarrow}(-1/2^{|s|}, 1/2^{|s|})$ . This is legal as  $\sigma(s') \in s'(|s| - 1) \subseteq [\mathbf{0}, s(|s| - 1), 1/2^{|s|}]$  implies  $s(|s| - 1) \subseteq \sigma(s')^{\leftarrow}(-1/2^{|s|}, 1/2^{|s|})$ .

Let  $\nu \in ([X]^{<\omega})^\omega$  attack  $\tau$ , and let

$$\nu'(n) = [\mathbf{0}, \bigcup \{\nu(m) : m \leq n\}, 1/2^{n+1}].$$

Let  $F \in [X]^{<\omega}$  and  $\epsilon > 0$ . If  $\{\tau(\nu \upharpoonright n + 1) : n < \omega\}$  is an open cover for  $X$ , choose  $\frac{1}{\epsilon} < N < \omega$  such that

$$F \subseteq \bigcup \{\tau(\nu \upharpoonright n + 1) : n \leq N\} = \bigcup \{\sigma(\nu' \upharpoonright n + 1)^{\leftarrow}(-1/2^{n+1}, 1/2^{n+1}) : n \leq N\}.$$

TODO: not quite there...

□