Methods of Data-Driven Reachability Analysis

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Introduction

Reachability analysis is an effective method to guarantee the safety of power systems, safety-critical robots, and other nonlinear systems in the face of uncertainty. Traditionally, approaches to reachability analysis involve complex algorithms that obtain formal guarantees of the reachable set. The problem with these methods is that they require full system information, but most systems of practical interest do not come in a form that is easily analyzable, as they are often high-dimensional and with imperfect system information.

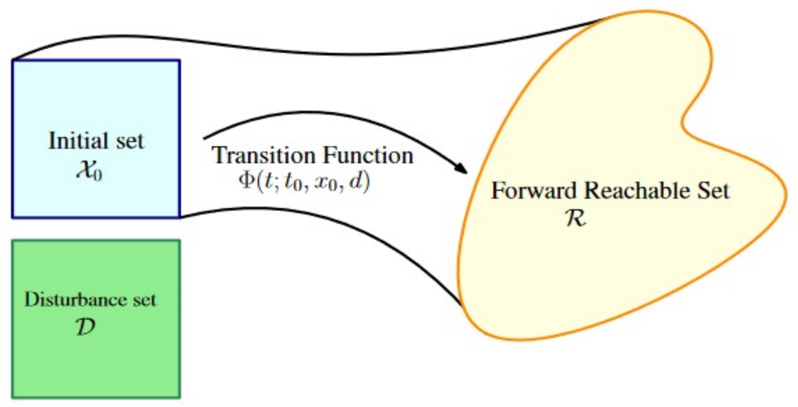
For data-driven reachability analysis, rather than obtaining a formal guarantee of the reachable set, data is acquired from experiments and simulations in order to estimate the reachable set with a probabilistic guarantee. The benefit of this approach is that virtually any system whose behavior can be simulated or measured experimentally can be evaluated with data-driven reachability analysis.

Problem Formulation

In the data-driven approach to reachability analysis, we consider a dynamical system with an initial set $X_0 \subseteq \mathbb{R}^{n_x}$, a set of disturbances D where $d: [t_0, t_1] \to \mathbb{R}^{n_d}$ and $d \in D$, and a state transition function $\Phi: X_0 \times \mathcal{D} \to \mathbb{R}^{n_x}$. The goal is to estimate the image of this state transition function defined on an initial set and acted upon by disturbance signals. That is, we would like to obtain an estimate of the forward reachable set

$$\mathcal{R} = \{\Phi(t_1; t_0, x_0, d) : x_0 \in X_0, d \in \mathcal{D}\},$$

all possible evolutions of a given system in the time range $[t_0, t_1]$.



As long as the behavior of the system can be simulated or measured experimentally, it can be treated as a black-box model within the context of data-driven reachability analysis.

Rather than obtaining formal guarantees of the reachable set, data-driven approaches yield probabilistic guarantees of estimates of the reachable set. Samples $r_i = \phi(t_1; t_0, x_{0i}, d_i), i = 1, \dots, N$ are drawn, where $x_{01}, \dots, x_{0N} \overset{\text{i.i.d.}}{\sim} X_0, d_1, \dots, d_N \overset{\text{i.i.d.}}{\sim} D$. Let $C \subseteq 2^{\mathbb{R}^{n_x}}$ denote the class of admissible set estimators and P_R the probability measure with respect to R. Then, a compact estimate $\hat{R} \in C$ of the reachable is computed such that $P_R^N\left(P_R(\hat{R}) \geq 1 - \epsilon\right) \geq 1 - \delta$, where P_R^N is the product measure of N copies of P_R. $\epsilon \in (0,1)$ is the accuracy parameter and $\delta \in (0,1)$ is the confidence parameter. The double inequality, a special case of the bound used in the Probably Approximately Correct (PAC) framework of statistical learning theory provides two assertions. First, the inner equality $\left(P_R(\hat{R}) \geq 1 - \epsilon\right)$ asserts that $\hat{R}$ attains a probability mass of at least $1 - \epsilon$ under P_R. Second, the outer inequality $P_R^N\left(P_R(\hat{R}) \geq 1 - \epsilon\right) \geq 1 - \delta$ asserts that $\hat{R}$ attains a $1 - \epsilon$ accuracy with probability $1 - \delta$ with respect to the samples $r_1, \dots, r_N$.

(Devonport and Arcak, 2021)

Main Contribution

Most of the current existing tools for reachability analysis employ the traditional approaches, making them impractical for analyzing real-world complex systems. As a solution, we propose **DaDRA**, a Python library built specifically for data-driven reachability analysis.

The library allows users much of the same functionality as traditional reachability analysis tools, but the nature of the data-driven methods allows for analysis of far more complex and realistic systems. Furthermore, the tool provides the ability to estimate reachable sets with arbitrary desired probabilistic guarantees while taking advantage of parallelizability to accelerate computation and allowing for insightful visualizations.

DaDRA Methods of Estimation

The DaDRA library incorporates two data-driven methods: a scenario approach to chance-constrained convex optimization with p-norm balls and an empirical risk minimization approach using a class of polynomials called empirical inverse Christoffel functions. Both approaches are accompanied by a known lower bound for the number of samples N in order to satisfy the specified probabilistic parameters ϵ and δ.

For state dimension n_x, the number of samples required to meet the probabilistic guarantees with the algorithm using p-norm balls is

$$N = \left\lceil \frac{1}{\epsilon} \frac{e}{e-1} \left(\log \frac{1}{\delta} + \frac{1}{2}(n_x^2 + 3n_x) \right) \right\rceil$$

and for the Christoffel function method, the number of samples required is

$$N = \left\lceil \frac{5}{\epsilon} \left(\log \frac{4}{\delta} + \binom{n_x + 2k}{n_x} \log \frac{40}{\epsilon} \right) \right\rceil$$

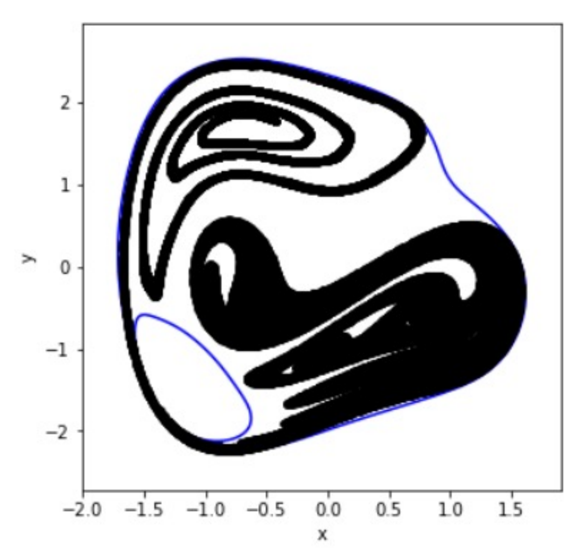
where k is the order of the Christoffel function.

(Devonport and Arcak, 2021)

Chaotic System Example

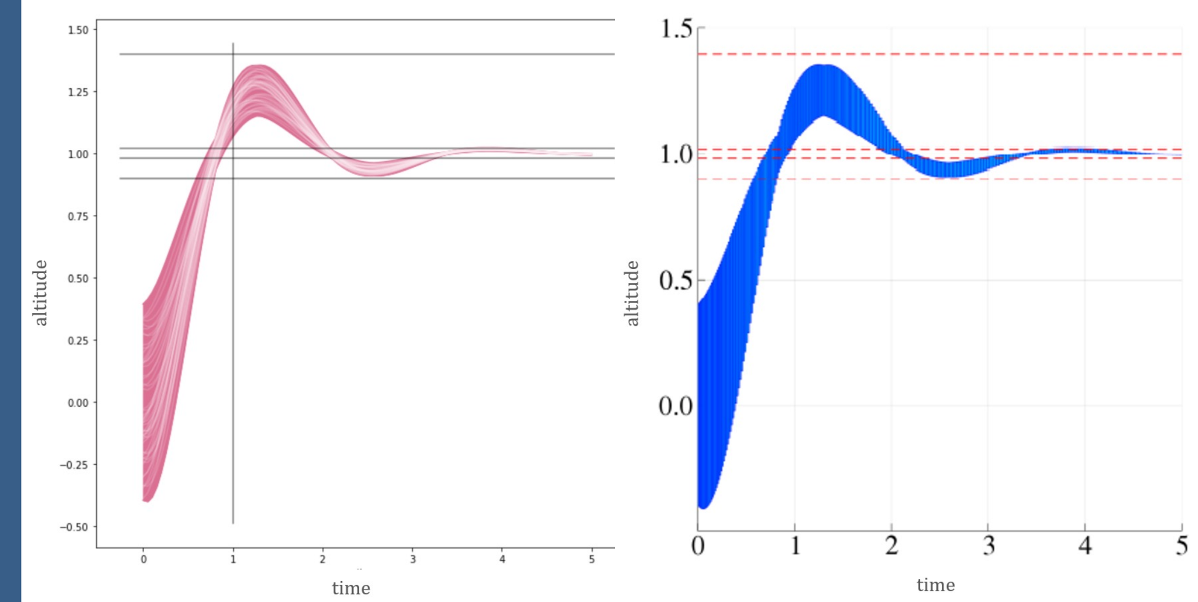
To demonstrate the utility of the DaDRA library we perform reachability analysis on a Duffing oscillator, a kind of chaotic system. While traditional approaches tend to find difficulty performing reachability analysis on chaotic systems, we can do so easily using DaDRA due to the nature of its data-driven approach.

Shown here are the samples from the chaotic system in black plotted along with the computed reachable set estimate in blue. In this case, the Christoffel function method was used with probabilistic parameters $\epsilon = 0.05$ and $\delta = 10^{-9}$, corresponding to a 95% accuracy with a 1 in a billion chance of failure.

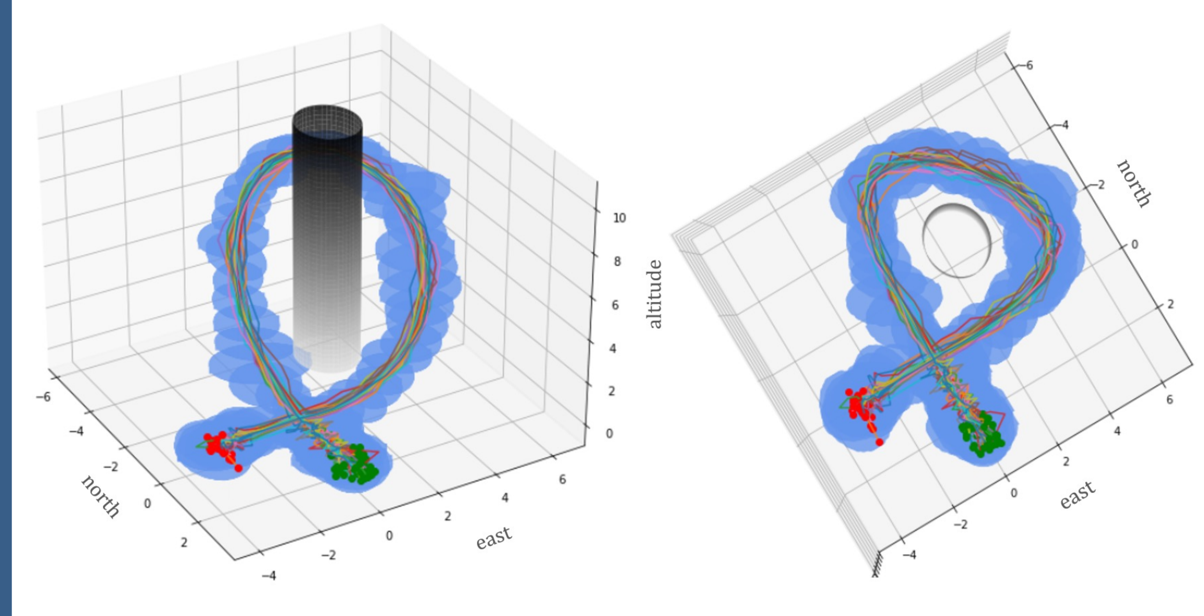


Quadrotor Demonstration

In accordance with the quadrotor benchmark from (Immler, et al., 2019) we compare the results of the data-driven methods built into DaDRA with the results of previous tools using traditional approaches to reachability analysis. The 12-state quadrotor benchmark is meant to check control specifications for stabilization using PD controllers for height, roll, and pitch. The objective is to control the quadrotor to reach and remain in the goal region of height [0.98, 1.02] within 5 seconds. Below, on the left is the computed reachable set estimate from DaDRA and on the right is the computed reachable set from another tool called JuliaReach using traditional algorithms. As we can see, the results are similar, serving as evidence that the data-driven methods of DaDRA are as effective as the traditional methods when applied to simplified systems.



The benchmark quadrotor consists of an undisturbed system with a simple controller with the system's goal region residing in a single dimension. To demonstrate the effectiveness of DaDRA, we perform reachability analysis on a 12-state quadrotor with added disturbance by a modified version of a military-specified wind turbulence model in order to tune a complex controller to perform a difficult flight maneuver while avoiding an unsafe region.



The aim was to use DaDRA to iteratively tune a controller to perform a clover-leaf maneuver, starting at the green points and ending at the red, without reaching the unsafe set, the black cylindrical region. As can be seen in the 3-dimensional figures, the reachable set estimate, computed with probabilistic parameters $\epsilon = 0.05$ and $\delta = 10^{-9}$ corresponding to a 95% accuracy with a 1 in a billion chance of failure, outlines the desired trajectory while remaining outside of the unsafe region.

Conclusion

Though traditional approaches to reachability analysis have the advantage of providing formal guarantees of the reachable set, they tend to be impractical for use in real world settings, as most systems of interest possess high-dimensional, analytically intractable, and possibly unknown dynamics. Applying data-driven methods allows for reachability analysis with arbitrarily robust probabilistic guarantees, so long as the system of interest is capable of being simulated or sampled from.

DaDRA provides an easy-to-use alternative to libraries implementing traditional algorithms for reachability analysis and takes advantage of data-driven methods. We demonstrate the practical functionality of DaDRA initially on a chaotic system and subsequently on a realistic system acting under complex disturbance signals and controlled with an intricate controller across multiple dimensions. The examples outline the utility of the library on analytically intractable systems for the purpose of safety verification and controller design.

Future Work

To improve the tool, additional data-driven methods besides the scenario approach with p-norm balls and the Christoffel function method could be implemented. Additionally, further support for systems that cannot be modeled with ordinary differential equations, such as hybrid models, could be provided.

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