1. Suppose we have a large tank initially containing 500 gallons of pure water. We begin adding in an alcohol-water mix at a rate of 2 gallons per minute. This alcohol-water mix being added is 43% alcohol. At the same time, the mixture in the tank is drained at a rate of 2 gallons per minute. Throughout the process, the mixture in the tank is thoroughly and uniformly mixed.

A) Derive the differential equation for A(t).

To start off I will list all the information I know.
t: time
Alcohol percentage: 43%
Size of Water tank: 500 gal
A(t): Amount of alcohol after time t
A(0) = 0
Rate in: 2 gal of 43 % mix per minute
Rate out: 2 gal of mix per minute
Now I will start constructing the differential equation by identifying the rates of alcohol entering the tank, and the rate of school leaving the tank.

$$\frac{dA(t)}{dt} = rate\ in - rate\ out$$

$$\frac{dA(t)}{dt} = \left(\frac{2\ gal\ alchohol\ mix}{minute} * \frac{43\ gal\ alcohol}{100\ gal\ mix}\right) - \left(\frac{2\ gal\ tank\ mix}{minute} * \frac{A(t)}{500}\right)$$

Simplifying this equation yields a differential equation for A(t).

$$\frac{dA(t)}{dt} = \left(\frac{1}{minute} * \frac{43\ gal\ alcohol}{50\ gal\ mix}\right) - \left(\frac{1}{minute} * \frac{A(t)}{250}\right)$$

$$\frac{dA(t)}{dt} = \left(\frac{43}{50} * \frac{gal\ alcohol}{minute}\right) - \left(\frac{A(t)}{250} * \frac{gal\ alcohol}{minute}\right)$$

$$\frac{dA(t)}{dt} = \left(\frac{43}{50}\right) - \left(\frac{A(t)}{250}\right)$$

$$\frac{dA(t)}{dt} = \left(\frac{215}{250}\right) - \left(\frac{A(t)}{250}\right)$$

$$\boxed{\frac{dA(t)}{dt} = \left(\frac{1}{250}\right)(215 - A(t))}$$

This is a first order linear separable ordinary differential equation.

B) Solve the differential equation to obtain a formula for A(t)

First I will separate the equation.

$$\frac{dA(t)}{dt} = \left(\frac{1}{250}\right)(215 - A(t))$$

$$\frac{1}{(215 - A(t))}\frac{dA(t)}{dt} = \frac{1}{250}$$

Next I will integrate both sides with respect to t

$$\int \frac{1}{(215 - A(t))}\frac{dA(t)}{dt}dt = \frac{1}{250}\int dt$$

$$\int \frac{1}{(215 - A(t))}dA(t) = \frac{1}{250}\int dt$$

$$-\ln(215 - A(t)) = \frac{1}{250}t + c$$

Exponentiating both sides and simplifying yields a formula for A(t).

$$e^{\ln(215 - A(t))} = e^{\frac{-t}{250}+c}$$

$$215 - A(t) = e^{\frac{-t}{250}}e^{c}$$

$$215 - A(t) = ke^{\frac{-t}{250}}$$

$$-A(t) = ke^{\frac{-t}{250}} - 215$$

$$A(t) = -ke^{\frac{-t}{250}} + 215$$

$$A(t) = 215 - ke^{\frac{-t}{250}}$$

This is the general solution. In order to identify the value of k, I will plug in the initial value condition and solve for k.

$$A(0) = 215 - ke^{\frac{0}{250}} = 0$$
$$215 - ke^{0} = 0$$
$$215 - k(1) = 0$$
$$-k = -215$$
$$k = 215$$

Therefore, the specific solution is going to be.

$$\boxed{A(t) = 215 - 215e^{\frac{-t}{250}}}$$

C) Approximately how many gallons of alcohol are in the tank after 1 hour?

$$A(60) = 215 - 215e^{\frac{-60}{250}}$$
$$A(60) = 215 - 215e^{\frac{-12}{50}}$$
$$A(60) = 215 - 215(.78662)$$
$$A(60) = 215 - 169.125$$

$$\boxed{A(60) = 45.875}$$

D) When will the mixture in the tank be exactly half alcohol?

$$A(t) = 215 - 215e^{\frac{-t}{250}} = 250$$

$$-215e^{\frac{-t}{250}} = 250 - 215$$

$$e^{\frac{-t}{250}} = \frac{35}{-215}$$

$$e^{\frac{-t}{250}} = \frac{35}{-215}$$

$$\ln(e^{\frac{-t}{250}}) = \ln\left(\frac{35}{-215}\right)$$

$$\frac{-t}{250} = \ln(35) - \ln(-215)$$

Notice that the natural log of a negative number is undefined. Therefore, I must conclude that the mixture will never actually reach an exact 50/50 alcohol to water mixture. By running the model created in previous steps with a value of 150 days as the time. I get a result of approximately 215 gallons of alcohol present. This suggests to me that the equilibrium of the system is 215 gallons of alcohol for every 500 gallons of mixture, or 43% just like the alcohol mixture being added.

> Therefore, the answer to part D is never.

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2. Given that $y_1 = \frac{1}{x}$ is a solution to the differential equation, $x^2y'' + 3xy' + y = 3x$

To start I will confirm that $y_1 = \frac{1}{x}$ is in fact a solution to the homogeneous equation $x^2y'' + 3xy' + y = 0$

$$y_1 = \frac{1}{x}, \qquad y_1' = -\frac{1}{x^2}, \qquad y_1'' = \frac{2}{x^3}$$

By inserting these values into the homogeneous equation I can see that $y_1 = \frac{1}{x}$ is in fact a solution.

$$x^2\left(\frac{2}{x^3}\right) + 3x\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x}\right) = 0$$

$$\frac{2x^2}{x^3} - \frac{3x}{x^2} + \frac{1}{x} = 0$$

$$\frac{2}{x} - \frac{3}{x} + \frac{1}{x} = 0$$

$$\frac{2-3+1}{x}=0$$

$$\frac{0}{x}=0$$

$$0=0$$

Therefore I know that a reduction of order is possible. I will first multiply the solution by a u(x) and find the derivatives.

$$y_1=\frac{1}{x}u\,,\qquad y_1'=-\frac{1}{x^2}u+\frac{1}{x}u',\qquad y_1''=\frac{2}{x^3}u-\frac{1}{x^2}u'-\frac{1}{x^2}u'+\frac{1}{x}u''=\frac{2}{x^3}u-2\frac{1}{x^2}u'+\frac{1}{x}u''$$

Now I will insert these values into the original equation.

$$x^2(\frac{2}{x^3}u-2\frac{1}{x^2}u'+\frac{1}{x}u'')+3x(-\frac{1}{x^2}u+\frac{1}{x}u')+\frac{1}{x}u=3x$$

Now I will multiply out the terms and simplify.

$$\frac{2x^2}{x^3}u-\frac{2x^2}{x^2}u'+\frac{x^2}{x}u''-\frac{3x}{x^2}u+\frac{3x}{x}u'+\frac{1}{x}u=3x$$

$$\frac{2}{x}u-\frac{2}{1}u'+\frac{x}{1}u''-\frac{3}{x}u+\frac{3}{1}u'+\frac{1}{x}u=3x$$

$$\frac{2}{x}u-2u'+\ xu''-\frac{3}{x}u+\ 3u'+\frac{1}{x}u=3x$$

$$xu''-2u'+\ 3u'+\frac{2}{x}u-\frac{3}{x}u+\frac{1}{x}u=3x$$

$$xu''+u'+\frac{2-3+1}{x}u=3x$$

$$xu''+u'+\frac{0}{x}u=3x$$

$$xu''+u'+0(u)=3x$$

$$xu''+u'=3x$$

This has removed the u terms allowing us to make the substitution $v=u',\ v'=u''$ to reduce the order by one.

$$xv'+v=3x$$

Rearranging this into standard form will allow us to solve the reduced differential equation.

$$xv'+v=3x$$

$$eq\ 1)\ v'+\frac{1}{x}v=3$$

I will now find the integrating factor µ.

$$\mu=\ e^{\int\frac{1}{x}dx}=\ e^{\ln|x|}=x$$

I will now multiply eq 1 by the integrating factor µ.

$$xv'+\frac{x}{x}v=3x$$

$$xv'+v=3x$$

Notice that $xv'+v=\frac{d}{dx}(xv)$, substituting this into our equation yields.

$$\frac{d}{dx}(xv)=3x$$

Now I will integrate both sides with respect to x.

$$\int \frac{d}{dx}(xv)dx = 3\int x\,dx$$

$$xv = 3\frac{x^2}{2}$$

$$v = 3\frac{x^2}{2x}$$

$$v = \frac{3}{2}x$$

I will now back substitute the v with u' and integrate once more.

$$u' = \frac{3}{2}x$$

$$\int \frac{du}{dx}dx = \frac{3}{2}\int x\,dx$$

$$\int du = \frac{3}{2}\frac{x^2}{2}$$

$$eq\ 2\ )\ u = \frac{3x^2}{4}$$

Returning all the way back to the beginning, recall that $y_1 = \frac{1}{x}u$. Therefore multiplying eq 2 by $\frac{1}{x}$ will yield the general solution.

$$y = \frac{1}{x}\frac{3x^2}{4}$$

$$y = \frac{3x^2}{4x}$$

$$\boxed{y = \frac{3x}{4}}$$

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3. Solve the initial value problem.

$$5y'' - 4y' - y = 0\ with\ y(0) = 9\ and\ y'(0) = 3$$

First off I will try an exponential solution for y.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2e^{rx}$$

By substituting these values into the original equation we get.

$$5r^2e^{rx} - 4re^{rx} - e^{rx} = 0$$

I will then simplify by factoring out the e^{rt} term and dividing it over to the other side of the equation.

Notice that e^{rt} is never 0.

$$e^{rx}(5r^2 - 4r - 1) = 0$$

$$5r^2 - 4r - 1 = \frac{0}{e^{rx}}$$

$$5r^2 - 4r - 1 = 0$$

By using the quadratic equation, I will then find the values of r.

$$r = -\frac{(-4) \pm \sqrt{(-4^2) - 4(5)(-1)}}{2(5)}$$

$$r = -\frac{-4 \pm \sqrt{16+20}}{10}$$

$$r = \frac{4 \pm \sqrt{36}}{10}$$

$$r = \frac{4 \pm 6}{10}, \qquad r_+ = \frac{4+6}{10} = \frac{10}{10} = 1, \qquad r_- = \frac{4-6}{10} = \frac{-2}{10} = -\frac{1}{5}$$

Therefore, r has two distinct real roots.

$$r_+ = 1 \; and \; r_- = -\frac{1}{5}$$

By inserting these values into the linear combination general solution format, the general solution and its derivative become the equations

$$y = c_1 e^{-x} + c_2 e^{\frac{1}{5}x} \; and \; y' = -c_1 e^{-x} + \frac{c_2}{5} e^{\frac{1}{5}x}$$

I will now solve for the specific solution by plugging in our initial value conditions.

$$y(0) = c_1 e^{-0} + c_2 e^{\frac{1}{5}0} = 9$$

$$eq\ 1)\ y(0) = c_1 + c_2 = 9$$

$$y'(0) = -c_1 e^{-0} + \frac{c_2}{5} e^{\frac{1}{5}0} = 3$$

$$eq\ 2)\ y'(0) = -c_1 + \frac{c_2}{5} = 3$$

Adding eq 1 and eq 2 together yields

$$c_1 + c_2 = 9$$

$$+$$

$$-c_1 + \frac{c_2}{5} = 3$$

$$=$$

$$\frac{5}{5}c_2 + \frac{1}{5}c_2 = 12$$

$$\frac{6}{5}c_2 = 12$$

$$6c_2 = 60$$

$$c_2 = 10$$

By substituting this back into eq 1 and eq 2 I get the value of c_1

$$c_1 + 10 = 9 \;\Rightarrow\; c_1 = -1$$

$$-c_1 + \frac{10}{5} = 3 \;\Rightarrow\; -c_1 + 2 = 3 \;\Rightarrow\; -c_1 = 1 \;\Rightarrow\; c_1 = -1$$

Therefore, $c_1 = -1$ and $c_2 = 10$, making our specific solution to this initial value problem the equation

$$\boxed{y = -e^{-x} + 10e^{\frac{1}{5}x}}$$

4. Find the general solution to the equation.

$$2y'' - 6y' + 7y = 0$$

First off I will try an exponential solution for y.

$$y = e^{rx}, \qquad y' = re^{rx}, \qquad y'' = r^2e^{rx}$$

By substituting these values into the original equation we get.

$$2r^2e^{rx} - 6re^{rx} + 7e^{rx} = 0$$

I will then simplify by factoring out the e^{rt} term and dividing it over to the other side of the equation.

Notice that e^{rt} is never 0.

$$e^{rx}(2r^2 - 6r + 7) = 0$$

$$2r^2 - 6r + 7 = \frac{0}{e^{rx}}$$

$$2r^2 - 6r + 7 = 0$$

By using the quadratic equation, I will then find the values of r.

$$r = -\frac{(-6) \pm \sqrt{(-6^2) - 4(2)(7)}}{2(2)}$$

$$r = -\frac{-6 \pm \sqrt{36 - 56}}{4}$$

$$r = \frac{6 \pm \sqrt{-20}}{4}$$

Notice that the discriminant is negative so the values of r will be two complex conjugate roots.

$$r = \frac{3 \pm i\sqrt{20}}{2}, \qquad r_+ = \frac{3}{2} + i\sqrt{20}, \qquad r_- = \frac{3}{2} - i\sqrt{20}$$

Therefore, two solutions to the original differential equation are as follows.

$$y_1 = e^{\left(\frac{3}{2}+i\sqrt{20}\right)x}, \qquad y_2 = e^{\left(\frac{3}{2}-i\sqrt{20}\right)x}$$

By using the Euler formula can rewrite these solutions as the equations

$$y_1 = e^{\frac{3}{2}x}\cos\left(\sqrt{20}x\right), \qquad y_2 = e^{\frac{3}{2}x}\sin\left(\sqrt{20}x\right)$$

Therefore, the general solution becomes any linear combination of these two solutions.

$$\boxed{y = c_1e^{\frac{3}{2}x}\cos\left(\sqrt{20}x\right) + c_2e^{\frac{3}{2}x}\sin\left(\sqrt{20}x\right)}$$