Math 19: Fall 2014
Final Exam

NAME: SOLUTIONS

LECTURE:

Time: **3 hours**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Please turn to the next page for the statement on the Stanford Honor Code, which you must sign.

Problem	Value	Score
1	4	
2	2	
3	2	
4	5	
5	6	
6	12	
7	18	
8	8	
9	7	
10	8	
11	5	
12	16	
13	7	
TOTAL	100	

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are required to sit in your assigned seat.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

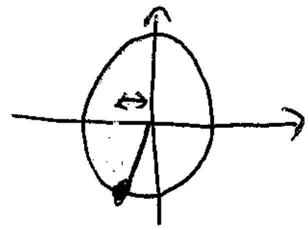
Signature: ____

Some derivatives

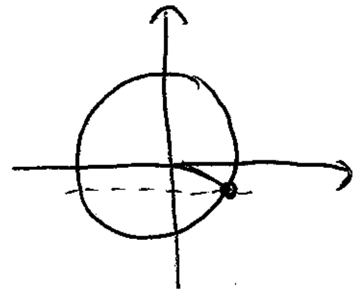
- $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arccos x = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arctan x = \dfrac{1}{1-x^2}$

Problem 1 : (4 points) Simplify each of the following numbers completely.

a) $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$



b) $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



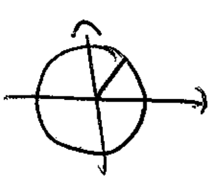
c) $\log_2 36 + \log_2 \frac{1}{15} - \log_2 \frac{3}{10}$

$$= \log_2\left(36 \cdot \frac{1}{15} \div \frac{3}{10}\right)$$

$$= \log_2\left(\frac{36 \cdot \cancel{10}^{2}}{3 \cdot \cancel{15}_{3}}\right) = \log_2 4 = \log_2 2^2 = 2$$

d) $\cos^2\left(\frac{\pi}{8}\right)$

$$= \frac{1 + \cos\left(2 \cdot \frac{\pi}{8}\right)}{2} = \frac{1 + \cos\frac{\pi}{4}}{2} = \frac{1 + \frac{\sqrt{2}}{2}}{2}$$



$$= \frac{1}{2}\left(\frac{2+\sqrt{2}}{2}\right) = \frac{2+\sqrt{2}}{4}$$

Problem 1 : (4 points) Simplify each of the following numbers completely.

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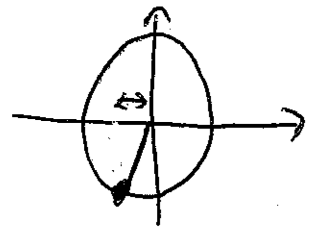

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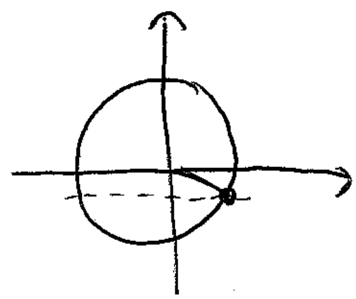

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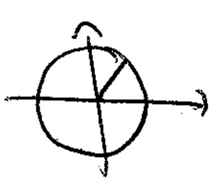


$$= \frac{1}{2}\left(\frac{2+\sqrt{2}}{2}\right) = \frac{2+\sqrt{2}}{4}$$

Problem 4 : (5 points) Decompose the following rational expression using partial fraction decomposition.

$$\frac{2x-1}{x^2-2x+1}$$

→ no need for long division

① Factor denominator: $x^2-2x+1 = (x-1)(x-1)$

② This is a repeated linear factor, so the form of the decomposition is

$$\frac{2x-1}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

③ Solve for A and B:

$$\frac{2x-1}{x^2-2x+1} = \frac{A(x-1)+B}{(x-1)^2} = \frac{Ax-A+B}{(x-1)^2}$$

If $2x-1 = Ax-A+B$, then $A=2$

and $-A+B=-1$

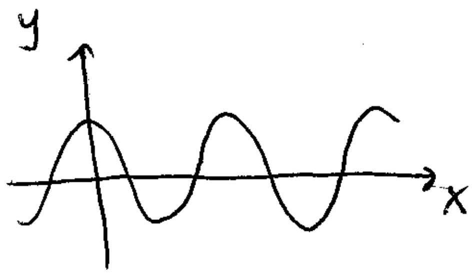
$-2+B=-1$

$B=1$

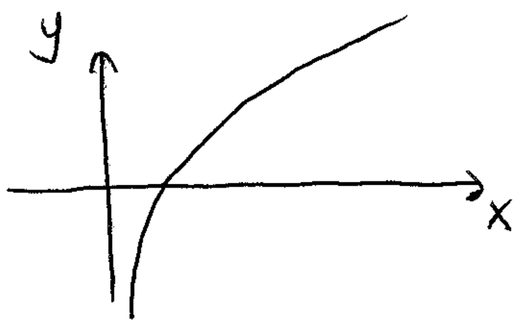
So $$\frac{2x-1}{x^2-2x+1} = \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

Problem 5 : (6 points) Sketch a picture of the graph and use it to evaluate the following limits. NOTE: You must sketch the graph to receive credit.

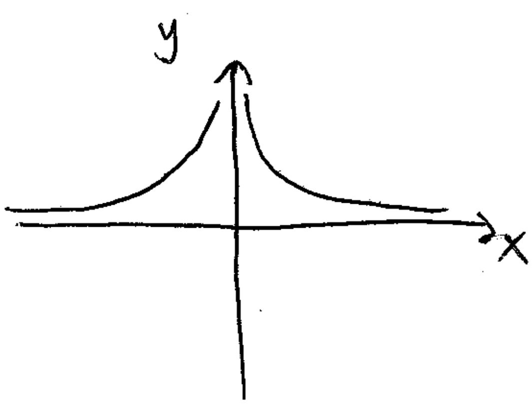
a) $\lim_{x \to \infty} \cos x$ does not exist (oscillates infinitely often as $x \to \infty$)



b) $\lim_{x \to \infty} \ln x = +\infty$



c) $\lim_{x \to 0^-} \frac{1}{x^2} = +\infty$



Problem 6 : (12 points) Evaluate the following limits, if they exist. You must show your work to receive credit.

a) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

$\frac{0}{0}$ indeterminate form

$$= \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \frac{(\sqrt{x} + 1)}{(\sqrt{x} + 1)}$$

$$= \lim_{x \to 1} \frac{x - 1}{(x-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

b) $\lim_{x \to 2^-} \frac{x^2 + 3x + 2}{x^2 - x - 2}$

$$\frac{4+6+2}{4-2-2} = \frac{12}{0} \rightsquigarrow \pm\infty$$

$$= \lim_{x \to 2^-} \frac{x^2+3x+2}{(x-2)(x+1)} = \frac{12}{0^- \cdot 3} = -\infty$$

c) $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 1}$ $= \frac{9-6-3}{9-1} = \frac{0}{8} = 0$

d) $\lim_{x \to \infty} \frac{x+1}{x^2 + x + 1}$ $= \lim_{x \to \infty} \frac{\frac{1}{x^2}(x+1)}{\frac{1}{x^2}(x^2+x+1)}$

$= \lim_{x \to \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}} = \frac{0}{1} = 0$

Problem 7 : (18 points) For each of the following, compute $\frac{dy}{dx}$.
There is no need to give the domain and codomain for the rule you obtain. You do not need to simplify your answer. Whenever necessary, solve for $\frac{dy}{dx}$.

a) $y = x^2 e^x$

$$\frac{dy}{dx} = 2xe^x + x^2e^x$$

b) $y = \ln(\sin x) - \ln(\cos x)$

$$\frac{dy}{dx} = \frac{1}{\sin x}\cos x - \frac{1}{\cos x}(-\sin x)$$

c) $y = \cos^{-1}(\tan x)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(\tan^2 x)}} \cdot \underbrace{\sec^2 x}_{\frac{d}{dx}\tan x = \sec^2 x}$$

d) $y = \dfrac{(2x-2)(x^3+4)^7}{(x+1)^3(x^2-6)^4}$

$$\ln y = \ln(2x-2) + 7\ln(x^3+4) - 3\ln(x+1) - 4\ln(x^2-6)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x-2} \cdot 2 + 7 \cdot \frac{1}{x^3+4} \cdot 3x^2 - 3\frac{1}{x+1}$$

$$- 4 \cdot \frac{1}{x^2-6} \cdot 2x$$

$$\frac{dy}{dx} = \underbrace{\frac{(2x-2)(x^3+4)^7}{(x+1)^3(x^2-6)^4}}_{y}\left[\frac{2}{2x-2} + \frac{21x^2}{x^3+4} - \frac{3}{x+1} - \frac{8x}{x^2-6}\right]$$

e) $y^3 + xy = 8$

$$3y^2 \frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$\left(3y^2 + x\right)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{3y^2 + x}$$

f) $y = (\sin x)^{x^2}$

$$\ln y = \ln\left[(\sin x)^{x^2}\right] = x^2 \cdot \ln(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln(\sin x) + x^2 \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\frac{dy}{dx} = \underbrace{(\sin x)^{x^2}}_{y}\left[2x\ln(\sin x) + \frac{x^2\cos x}{\sin x}\right]$$

Problem 8 : (8 points) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 4x-3 & \text{if } x \leq 2, \\ x^2-1 & \text{if } x > 2. \end{cases}$$

$f(2) = 8-3 = 5$

a) (4 points) Is f differentiable at $x = 2$? Justify using the definition of differentiability.

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{(2+h)^2-1-5}{h}$$

$$= \lim_{h\to 0^+} \frac{4+4h+h^2-6}{h}$$

$$= \lim_{h\to 0^+} \frac{-2+4h+h^2}{h} = \frac{-2}{0^+} = -\infty$$

This limit doesn't exist therefore $f'(2)$ does not exist, and f is not differentiable at $x=2$.

For your convenience, here is the function f again: $f\colon \mathbb{R} \to \mathbb{R}$ is given by the rule

$$f(x) = \begin{cases} 4x-3 & \text{if } x \leq 2, \\ x^2-1 & \text{if } x > 2. \end{cases}$$

b) (4 points) Is f continuous at $x = 2$? Justify using the definition of continuity.

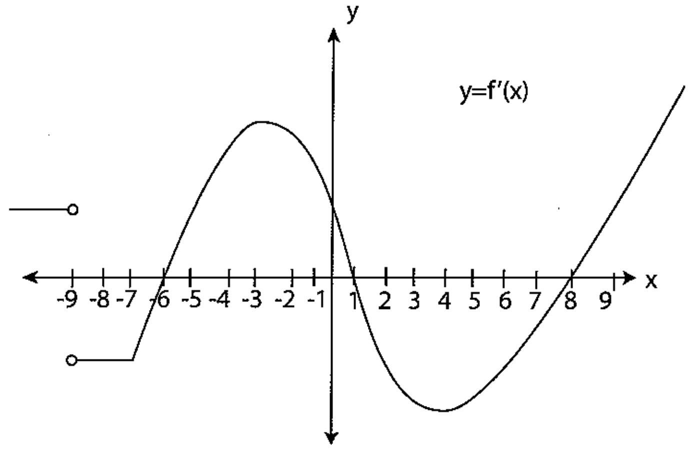
$f(2) = 5$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-1) = 4-1=3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4x-3) = 5$$

$\lim_{x\to 2} f(x)$ does not exist so f is not continuous at $x=2$.

Problem 9 : (5 points) Let f be a continuous function whose domain is all real numbers. The graph below is the graph of its derivative, f'.



a) List the x-values(s) for which f has a local maximum. $\rightsquigarrow$ $\nearrow \searrow$ so $f'>0$ then $f'<0$.

$x=-9$, $x=1$

b) List the x-values(s) for which f has a local minimum. $\rightsquigarrow$ $\searrow \nearrow$ so $f'<0$ then $f'>0$

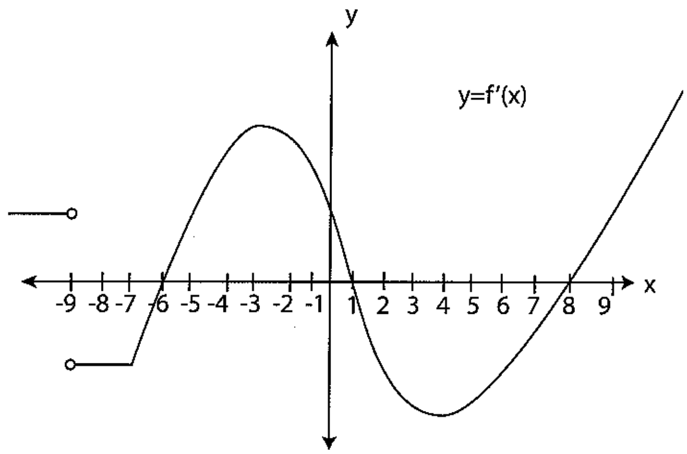
$x=-6$, $x=8$

c) List the x-values(s) for which f has a point of inflection.

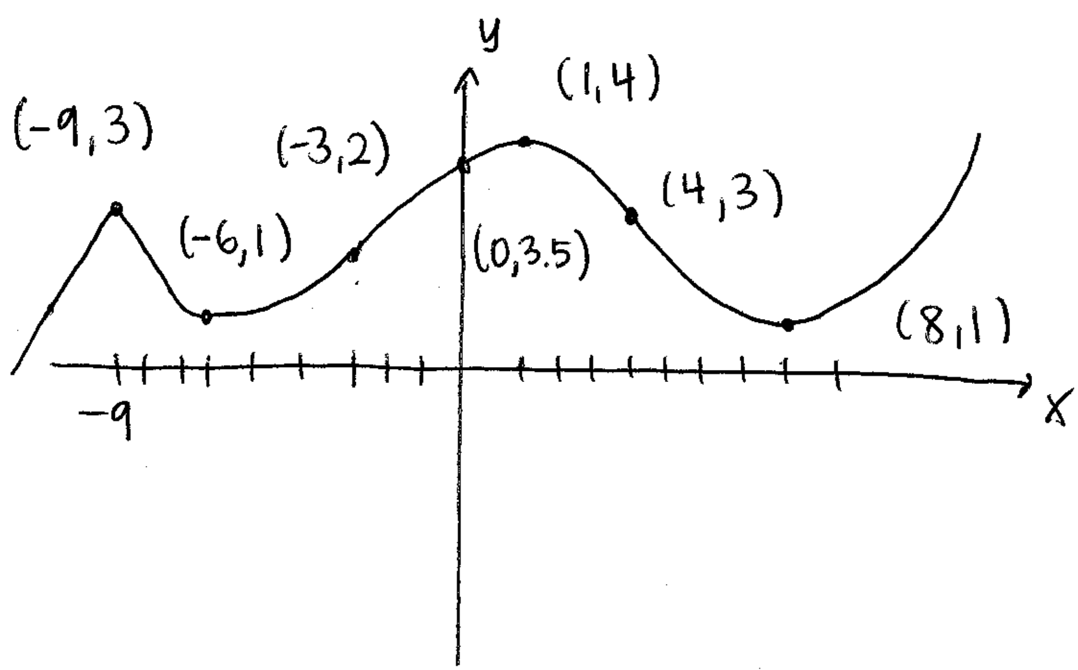
This is when f'' changes sign, i.e. when f' goes from inc to dec OR dec to inc.

$x=-3$, $x=4$

For your convenience, here is the graph of f' again:



d) Sketch a plausible graph for f. Label all special points (intercepts, holes, maxima, minima, points of inflection) with their coordinates.



For this problem you might need the following formulae from geometry:
Circumference of a circle of radius r:

$$C = 2\pi r$$

Area of a circle of radius r:

$$A = \pi r^2$$

Volume of a cylinder with radius r and height h:

$$V = \pi r^2 h$$

Problem 10 : (8 points) You are asked to construct a cylindrical can with a bottom and no top, and which has volume 8π in^3.

a) (2 points) Write an expression for the height of the cylinder in terms of the radius.

If $8\pi = \pi r^2 h$, then $h = \frac{8}{r^2}$

b) (2 points) Write the surface area of the can **as a function** of the radius.

A = can area + bottom area

$A = 2\pi r h + \pi r^2$

$A = 2\pi r \cdot \frac{8}{r^2} + \pi r^2$

$A = \frac{16\pi}{r} + \pi r^2$

$A : (0, \infty) \to \mathbb{R}$

$A(r) = \frac{16\pi}{r} + \pi r^2$

c) (4 points) What radius r and height h will result in a can of minimum surface area? Be sure to prove that you have found a minimum using the first or second derivative test.

$$A'(r) = 16\pi(-1)r^{-2} + \pi\, 2r$$

$$= \frac{-16\pi}{r^2} + 2\pi r = \frac{2\pi r^3 - 16\pi}{r^2}$$

critical points: $2\pi r^3 - 16\pi = 2\pi(r^3 - 8) = 0$ if $r = 2$

$r^2 = 0$ if $r = 0$ but this is not in the domain.

So $r=2$ is our candidate to be a minimum.

1st der test:

r	1	2	3
A'	$-$		$+$
A	$\searrow$		$\nearrow$

so $r=2$ is a min

OR

2nd der test $A''(r) = \dfrac{6\pi r^2 \cdot r^2 - (2\pi r^3 - 16\pi)2r}{r^4}$

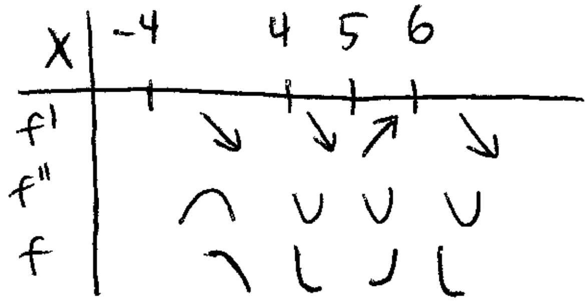
$$A''(2) = \frac{6\pi \cdot 16 - (16\pi - 16\pi)\cdot 4}{16} = 6\pi > 0$$

so A is concave up $\cup$ and $r=2$ is a min.

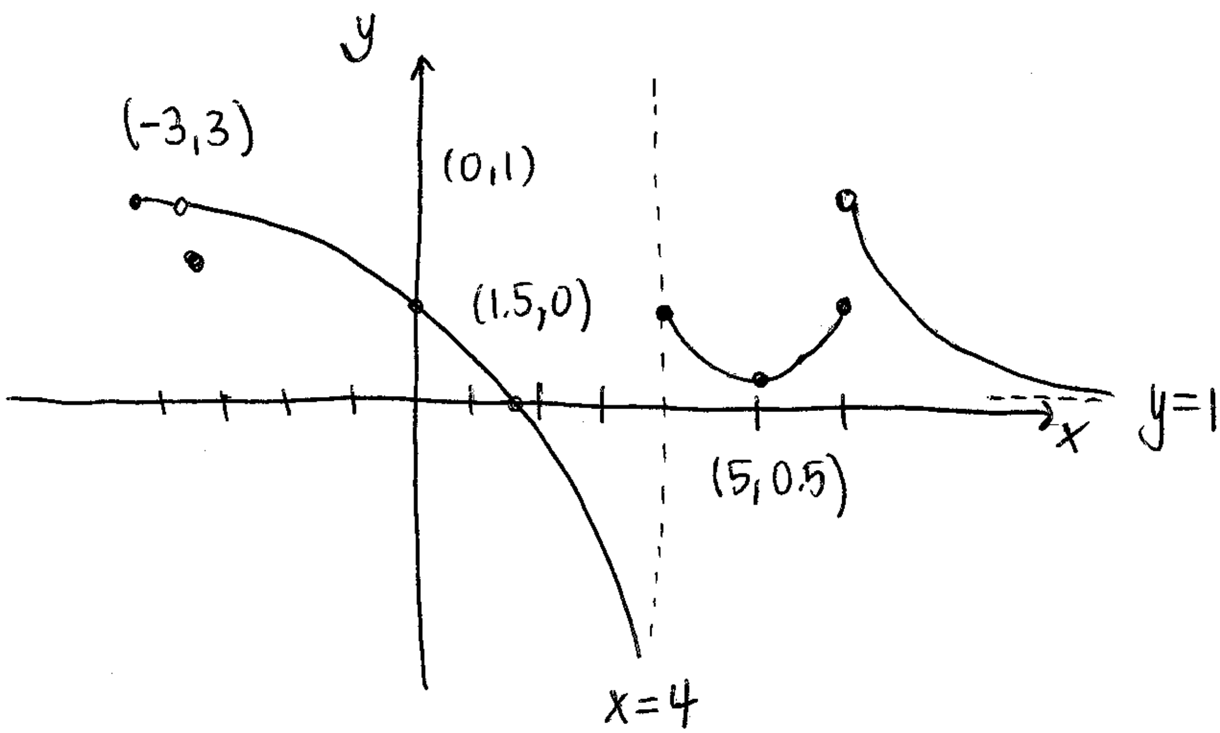
$\boxed{r=2,\ h=2}$

Problem 11 : (5 points) Sketch the graph of a function $f\colon [-4, \infty) \to \mathbb{R}$ that has the following properties:

- f has an infinite discontinuity at $x = 4$.
- f has a jump discontinuity at $x = 6$.
- f has a removable discontinuity at $x = -3$.
- $\lim_{x \to 4^+} f(x) = 1$.
- f has a horizontal asymptote at $y = 1$.
- $f'(x) > 0$ on the interval $(5, 6)$.
- $f'(x) < 0$ on the intervals $[-4, 4)$, $(4, 5)$ and $(6, \infty)$.
- $f''(x) > 0$ on the intervals $(4, 6)$ and $(6, \infty)$.
- $f''(x) < 0$ on the interval $[-4, 4)$.



On your graph, label all asymptotes with their equation, and label all special points (intercepts, holes, maxima, minima, points of inflection) with their coordinates.



Problem 12 : (16 points) Let $f(x) = \dfrac{x^2+2x}{(x^2+1)(x+2)}$. $= \dfrac{x^2+2x}{x^3+2x^2+x+2} = \dfrac{x}{x^2+1}$ if $x \neq 2$

a) (2 points) State the domain of f.

$x \neq -2$

b) (2 points) Find any removable discontinuities. If there are none, please write "none." You do not need to compute any limits.

$x = -2$

c) (2 points) Find the equations for all vertical asymptotes. If there are none, please write "none." You do not need to compute any limits.

none

d) (2 points) Find the equations for all horizontal asymptotes. If there are none, please write "none." Support your work with one or more limits.

$$\lim_{x\to\infty} \frac{x^2+2x}{x^3+2x^2+x+2} = \lim_{x\to\infty} \frac{\frac{x^2}{x^3}+\frac{2x}{x^3}}{\frac{x^3}{x^3}+\frac{2x^2}{x^3}+\frac{x}{x^3}+\frac{2}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}+\frac{2}{x^3}}$$

$$= \frac{0}{1} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{2}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}+\frac{2}{x^3}} = 0$$

$\boxed{y=0}$ (on both sides)

e) (3 points) Find all x-values for which f is increasing and all x-values for which f is decreasing.

Since $f(x) = \dfrac{x}{x^2+1}$ when $x \neq -2$,

$$f'(x) = \frac{1(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} \quad \text{if } x \neq -2$$

$f'(x) = 0$ when $x = \pm 1$

x	-3	-2		-1	0	1	2
$f'(x)$	$-$				$+$		$-$
$f(x)$	$\searrow$	Rem. disc.	$\searrow$		$\nearrow$		$\searrow$

f increasing on $(-1, 1)$

f decreasing on $(-\infty, -2) \cup (-2, -1) \cup (1, \infty)$

f) (1 point) Determine all local maximum and minimum values of f.

local max when $x = 1$ $\quad (1, \frac{1}{2})$

local min when $x = -1$ $\quad (-1, -\frac{1}{2})$

g) (3 points) Find all x-values for which f is concave up and all x-values for which f is concave down.

Since $f'(x) = \dfrac{-x^2+1}{(x^2+1)^2}$ when $x \neq -2$

$$f''(x) = \frac{-2x(x^2+1)^2 - (-x^2+1)\,2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

$$= \frac{(x^2+1)\left[-2x^3-2x-(-4x^3+4x)\right]}{(x^2+1)^4}$$

$$= \frac{2x^3-6x}{(x^2+1)^3} \quad \text{if } x \neq 2 \qquad f''(x)=0 \text{ if } x = 0, \pm\sqrt{3}$$

since $2x^3-6x = 2x(x^2-3)$.

x	-3	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
$f''(x)$	$-$			$+$		$-$		$+$
$f(x)$	$\cap$	rem disc.	$\cap$	$\cup$		$\cap$		$\cup$

f is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

concave down on $(-\infty, -2) \cup (-2, -\sqrt{3}) \cup (0, \sqrt{3})$

h) (1 point) Determine all inflections points.

$x = -\sqrt{3}$ $\qquad$ $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

$x = 0$ $\qquad$ $(0, 0)$

$x = \sqrt{3}$ $\qquad$ $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

Problem 13 : (7 points)

a) (2 points) State the Intermediate Value Theorem.

Suppose that f is continuous on $[a,b]$.
Then for any N between $f(a)$ and $f(b)$,
there is c in $[a,b]$ such that $f(c)=N$.

b) (2 points) State the Mean Value Theorem.

Suppose that f is differentiable on $[a,b]$.
Then there is c in $[a,b]$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

c) (3 points) Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit – which is 55 miles per hour on this stretch of road – at some time during the four minutes.

You may assume that all functions mentioned in this problem are continuous and differentiable.
Remark: Mind your units!

Let s be the position function of the car

then $s(0)=0$

$s(\frac{1}{15})=5$

Since s is differentiable, there is c in $[0,\frac{1}{15}]$ such that

$$s'(c)=v(c)=\frac{s(\frac{1}{15})-s(0)}{\frac{1}{15}-0}=\frac{5-0}{\frac{1}{15}}=5\cdot 15$$

$$=75$$

At some point in the 4 minutes, the truck was going 75mph, so it exceeded the speed limit.