Math 21: Spring 2014
Midterm 2

NAME: SOLUTIONS (BLUE)

LECTURE:

Time: **75 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are required to sit in your assigned seat.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	10	
2	5	
3	12	
4	10	
5	20	
6	18	
7	25	
TOTAL	100	

Problem 1 : (10 points) Suppose you have just poured a cup of freshly brewed coffee. Newton's Law of Cooling states:

> The rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

In this problem, let T denote the temperature of the coffee and R denote the temperature of the room, which we assume to be constant.

a) Suppose that T is larger than R. Is $\frac{dT}{dt}$ positive or negative?

negative. If $T \geq R$, the coffee is hotter than the room and it will cool down. The temperature will decrease.

b) Write down a differential equation that expresses Newton's Law of Cooling in this situation.

$$\frac{dT}{dt} = -k(T-R), \quad \text{for } k > 0$$

(or $\frac{dT}{dt} = k(T-R)$ for $k<0$)

Problem 2 : (5 points) Find all values of k for which the function

$$y\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad y = e^{kx}$$

satisfies the differential equation

$$y'' - 5y' - 6y = 0.$$

if $y = e^{kx}$ then $y' = ke^{kx}$

$y'' = k^2 e^{kx}$

so $k^2 e^{kx} - 5ke^{kx} - 6e^{kx} = 0$

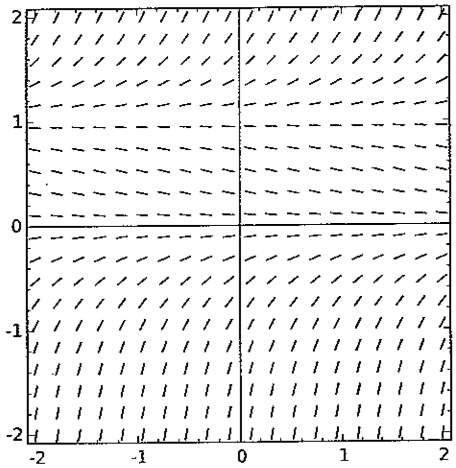
$e^{kx}(k^2 - 5k - 6) = 0$

since $e^{kx} \neq 0$ $k^2 - 5k - 6 = 0$

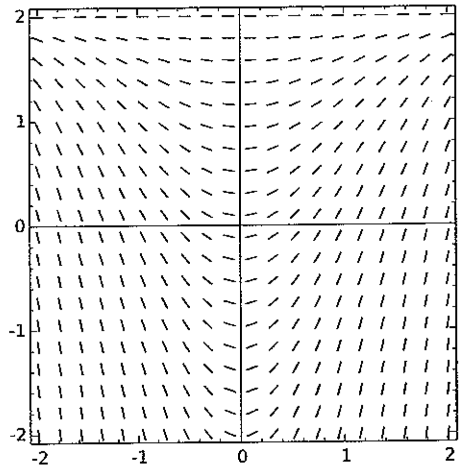
$(k-6)(k+1) = 0$

$k = 6, \ k = -1.$

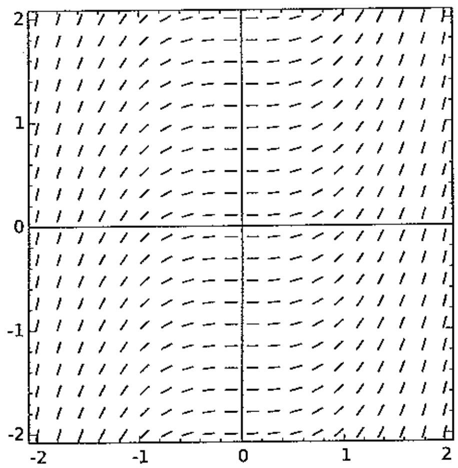
Problem 3 : (12 points) Match the direction fields below with their differential equations. Also indicate which two equations do not have matches.



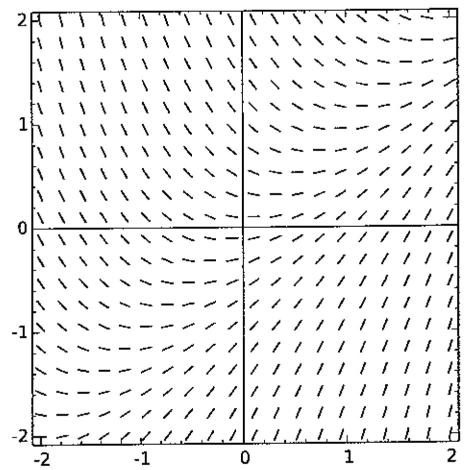
I.



III.



II.



IV.

Equation	I, II, III, IV, or "none"	Equation	I, II, III, IV, or "none"
$y' = y(1-y)$	none	$y' = x + y$	none
$y' = -y(1-y)$	I	$y' = x - y$	IV
$y' = x^2$	II	$y' = x(2-y)$	III

Problem 4 : (10 points) Consider the initial value problem

$$y' = x + y + xy, \quad y(0) = 2.$$

Use Euler's method with step size $h = 1$ to estimate $y(2)$, where y is the solution to the initial value problem above.

n	(x_n, y_n)	$F(x_n, y_n)$	$y_{n+1} = y_n + hF(x_n, y_n)$
0	$(0,2)$	$0+2+0\cdot 2 = 2$	$y_1 = 2 + 1\cdot 2 = 4$
1	$(1,4)$	$1+4+1\cdot 4 = 9$	$y_2 = 4 + 1\cdot 9 = 13$
2	$(2,13)$		

↑ this is the x-value we are interested in.

$$y(2) \approx 13$$

Problem 5 : (20 points) Solve the following differential equations and initial value problems.

a) $y' = xy + x, \quad y(0) = 1$

This is both separable and linear.

Separable: $\frac{dy}{dx} = x(y+1) \rightsquigarrow \frac{1}{y+1}\,dy = x\,dx$

$$\int \frac{1}{y+1}\,dy = \int \frac{1}{u}\,du = \ln|u| + C_1 = \ln|y+1| + C_1$$

$u = y+1 \quad du = dy$

$$\int x\,dx = \frac{x^2}{2} + C_2$$

so $\ln|y+1| = \frac{x^2}{2} + C$

$|y+1| = e^{x^2/2 + C} = Ce^{x^2/2}$ (new C)

$y+1 = Ce^{x^2/2}$ (new C again)

$y = -1 + Ce^{x^2/2}$

$y(0) = 1 \Rightarrow C = 2$ (see green solutions)

$$\boxed{y: \mathbb{R} \to \mathbb{R}, \quad y(x) = 2e^{x^2/2} - 1}$$

For linear solutions please see green exam solutions.

b) $\frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2$

This is separable (but not linear)

$$y\,dy = \frac{\ln x}{x}\,dx$$

$$\int y\,dy = \frac{y^2}{2} + C_1$$

$u = \ln x \quad du = \frac{1}{x}dx$

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + C_2$$

$$= \frac{(\ln x)^2}{2} + C_2$$

so $\quad \frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$

$$y^2 = (\ln x)^2 + C \qquad \text{(new } c\text{)}$$

$y(1)=2$: $\quad 2^2 = (\ln 1)^2 + C$

$$4 = C$$

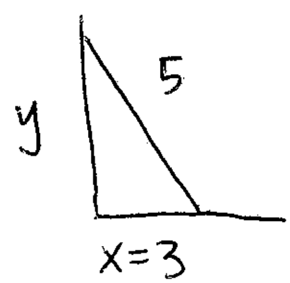
We can take positive square root since the initial value is positive. We also need $(\ln x)^2 + 4 \geq 0$ which is always true (if $\ln x$ is defined)

$$\boxed{\begin{array}{l} y:(0,\infty) \to \mathbb{R} \\ y(x) = \sqrt{(\ln x)^2 + 4} \end{array}}$$

Problem 6 : (18 points) A 5-foot long ladder rests against a wall in such a way that the foot of the ladder is 3 feet away from the wall. Then at time $t = 0$, the foot of the ladder starts slipping away from the wall at a speed of 1 foot per second.

a) (3 points) Write an equation relating x, the distance between the foot of the ladder and the wall, and y, the distance between the top of the ladder and the floor. Note that both x and y are functions of t here, where t is measured in seconds.

At $t=0$



$$x^2+y^2=25$$

b) (3 points) Write an initial value problem satisfied by x as a function of t.

$$\frac{dx}{dt}=1 \qquad x(0)=3$$

c) (4 points) Solve the initial value problem you wrote to find the **function** x.

$$\int dx = \int dt$$

$$x = t + C$$

$$3 = 0 + C$$

$$C = 3$$

So $x = t+3$. Situation ends when $x=5$, so $t=2$

$$\boxed{\begin{array}{c} x: [0,2] \to \mathbb{R} \\ x(t) = t+3 \end{array}}$$

d) (4points) Write down the initial value problem satisfied by y. Hint: You will have to differentiate the equation you got in part (a), and use information you already know about x.

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(25)$$

$$3^2+y^2=25$$

$$y^2=25-9=16$$

$$y=4$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\boxed{(t+3)\cdot 1+y\frac{dy}{dt}=0 \qquad y(0)=4}$$

e) (4 points) Solve the initial value problem you wrote to find the **function** y.

This is separable (but not linear)

$$y\,dy=-(t+3)\,dt$$

$$\int y\,dy=\frac{y^2}{2}+C_1 \qquad -\int(t+3)\,dt=-\left(\frac{t^2}{2}+3t\right)+C_2$$

so $\frac{y^2}{2}=-\frac{t^2}{2}-3t+C \rightsquigarrow y=\sqrt{t^2-6t+C}$ (new C)

(can take positive square root since $y\geq 0$)

$4=\sqrt{C}$ $C=16$

Note that y has same domain as x!

$$\boxed{\begin{array}{l} y:[0,2]\to\mathbb{R} \\ y(t)=\sqrt{-t^2-6t+16} \end{array}}$$

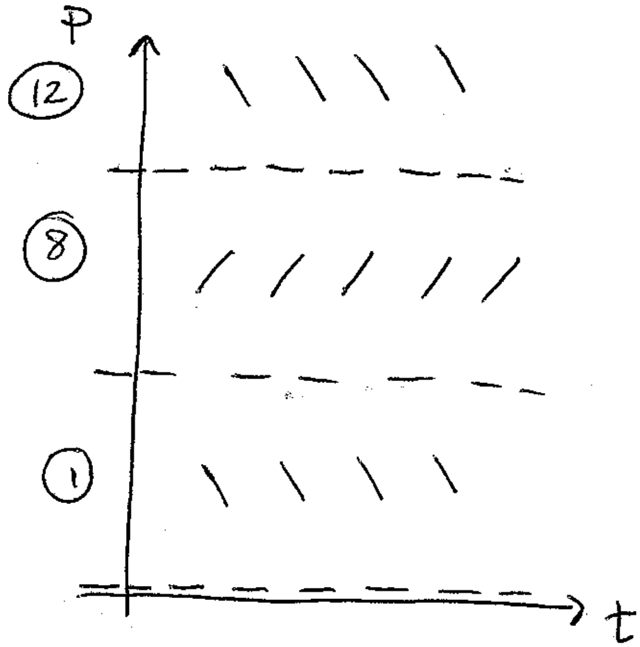
Problem 7 : (25 points) Consider the differential equation

$$\frac{dP}{dt} = -P\left(1 - \frac{P}{5}\right)\left(1 - \frac{P}{10}\right),$$

for a function $P\colon [0,\infty) \to [0,\infty)$.

a) (8 points) Find all critical points of the differential equation, and classify them as stable, unstable, or semi-stable.

critical points: $P=0$, $P=5$, $P=10$



$P=10$ stable

$P=5$ unstable

$P=0$ stable

Doctor Campisi and Doctor Vincent have computed the following partial fraction decomposition for you:

$$\frac{1}{P\left(1-\frac{P}{5}\right)\left(1-\frac{P}{10}\right)} = \frac{50}{P(5-P)(10-P)} = \frac{1}{P} + \frac{2}{5-P} - \frac{1}{10-P}.$$

b) (10 points) Give a general solution for the differential equation

$$\frac{dP}{dt} = -P\left(1-\frac{P}{5}\right)\left(1-\frac{P}{10}\right), \quad P \geq 0, \quad t \geq 0$$

You do not need to solve for P.

This equation is separable (but not linear)

$$\frac{1}{P(1-\frac{P}{5})(1-\frac{P}{10})}\, dP = -\, dt$$

$u = 5-P$, $du = -dP$ $\qquad$ $v = 10-P$, $dv = -dP$

$$\int \frac{1}{P(1-\frac{P}{5})(1-\frac{P}{10})}\, dP = \int \frac{1}{P}\, dP + 2\int \frac{1}{5-P}\, dP - \int \frac{1}{10-P}\, dP$$

$$= \ln|P| + 2\int -\frac{1}{u}\, du - \int -\frac{1}{v}\, dv$$

$$= \ln|P| - 2\ln|5-P| + \ln|10-P| + C_1$$

$$-\int dt = -t + C_2$$

So $\ln|P| - 2\ln|5-P| + \ln|10-P| = -t + C$

$$\ln\left|\frac{P(10-P)}{(5-P)^2}\right| = -t + C$$

$$\frac{P(10-P)}{(5-P)^2} = Ce^{-t} \qquad \text{(new } C\text{)}$$

you do not want to solve this for P...

c) (7 points) Sketch, **and label**, on one graph, the 4 solutions of the differential equation

$$\frac{dP}{dt} = -P\left(1 - \frac{P}{5}\right)\left(1 - \frac{P}{10}\right), \quad P \geq 0, \quad t \geq 0$$

that have the following initial values:

i. $P(0) = 4$
ii. $P(0) = 6$
iii. $P(0) = 9$
iv. $P(0) = 12$