Math 21: Spring 2013
Final Exam

NAME:

LECTURE:

Time: **3 hours**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ____

Problem	Value	Score
1	7	
2	4	
3	9	
4	15	
5	12	
6	12	
7	16	
8	10	
9	15	
TOTAL	100	

May your trails be crooked, winding, lonesome, dangerous, leading to the most amazing view.

– Edward Abbey

Problem 1 : (7 points)

a) (5 points) Write down the limit definition of the sum of the series

$$\sum_{n=0}^{\infty} a_n.$$

b) (2 points) Explain in **one** concise sentence what the following equality means:

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \frac{1}{2}$$

Problem 2 : (4 points) Compute the sum of the following series, if it exists:

$$\frac{\pi^0}{1} - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \cdots$$

Problem 3 : (9 points) For each of the following statement, decide if it is TRUE or FALSE. You do not need to show your work.

a) If f is a solution of the differential equation

$$y' + \sin y = 0,$$

then $2f$ is also a solution of this differential equation.

b) Every solution of the differential equation

$$y''' - 5y'' + 9y' - 5y = 0$$

is of the form

$$y = C_1 e^{2x} \cos x + C_2 e^x$$

for some choice of the constants C_1 and C_2.

c) If f and g are solutions of the differential equation

$$y'' + 3y'' - 2y' + 5y = e^x,$$

then $f + g$ is also a solution of this differential equation.

Problem 4 : (15 points) Decide whether the following series converge or diverge.

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{1}{\ln(n+4)}$

b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{n^2}{n^2+1}$

c) $\sum_{n=0}^{\infty} \frac{n}{2^n}$

Problem 5 : (12 points) Solve the following differential equations.

a) $y' - e^x y = e^x$.

b) $\dfrac{dy}{dx} = \dfrac{x^2 + 3x + 2}{2y}, \qquad y > 0, \quad x > 0.$

Problem 6 : (12 points) In this problem we will approximate the numbers

$$e = 2.718281828459045235360287471352662497757247093699959574966967\ldots$$

and

$$e^2 = 7.389056098930650227230427460575007813180315570551847324087 1\ldots$$

You may use the following values to help with computations:

$$\frac{1}{3} \approx 0.3333, \qquad \frac{2}{3} \approx 0.6666, \qquad \frac{1}{6} \approx 0.1666, \qquad \frac{8}{3} \approx 2.6666$$

$$\left(\frac{1}{6}\right)^2 \approx 0.0277, \qquad \left(\frac{8}{3}\right)^2 \approx 7.1111, \qquad \frac{8}{6} = \frac{4}{3} \approx 1.3333, \qquad \frac{19}{3} \approx 6.3333$$

a) (2 points) Write down the Maclaurin series (aka the Taylor series centered at $a = 0$) of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = e^x$.

b) (2 points) Write down the Taylor polynomial of degree 3 of f.

c) (2 points) Use the Taylor polynomial of degree 3 of f to approximate the number e.

d) (2 points) Square your answer to part c) to obtain an approximation for the number e^2.

e) (2 points) Use the Taylor polynomial of degree 3 of f to approximate e^2.

f) (2 points) Which approximation is closer to the true value of e^2?

Problem 7 : (16 points) Consider the differential equation

$$y''' - y'' + 4y' - 4y = 0.$$

a) (10 points) Circle all of the functions which satisfy this differential equation. Show your work. You may assume that the five functions below are linearly independent (because they are).

   i. $f_1 \colon \mathbb{R} \to \mathbb{R}$, $f_1(x) = e^x$
   ii. $f_2 \colon \mathbb{R} \to \mathbb{R}$, $f_2(x) = e^{-x}$
   iii. $f_3 \colon \mathbb{R} \to \mathbb{R}$, $f_3(x) = \sin(2x)$
   iv. $f_4 \colon \mathbb{R} \to \mathbb{R}$, $f_4(x) = \cos(2x)$
   v. $f_5 \colon \mathbb{R} \to \mathbb{R}$, $f_5(x) = e^{2x} \sin x$

b) (6 points) Using your work from part a), write down the general solution to the differential equation

$$y''' - y'' + 4y' - 4y = 0.$$

Problem 8 : (10 points) Use a power series to solve the following differential equation:

$$y'' + y = 0.$$

Hint: The solution is made up from functions you know. If you recognize them, you can check your work.

Problem 9 : (15 points) Consider the following initial value problem:

$$(-16x^2 + x)y'' + (-32x + 1)y' - 4y = 0, \qquad y(0) = 1.$$

Throughout, suppose that this initial value problem has a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The rest of the page is intentionally left blank for work on the differential equation.

a) i. (2 points) What is a_0?

ii. (2 points) What is a_1?

iii. (2 points) What is a_2?

iv. (2 points) What is a_3?

b) (2 points) Circle the expression that gives a formula for a_n:

i. $\dfrac{(n+1)!}{(n!)^2((n-1)!)^2}$

ii. $\dfrac{(2n)!}{n!}$

iii. $\dfrac{((2n)!)^2}{(n!)^4}$

iv. $\dfrac{((2n+1)!)^2}{(n!)^4}$

c) (5 points) Using your answer from part b) to give you a formula for a_n, compute the radius of convergence of the series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$