Math 21: Spring 2014
Midterm 1

NAME:

LECTURE: BLUE VERSION, SOLUTIONS

Time: **75 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are required to sit in your assigned seat.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	8	
2	10	
3	5	
4	10	
5	17	
6	8	
7	12	
8	12	
9	6	
10	12	
TOTAL	100	

Problem 1 : (8 points) For this question, suppose that f is a smooth function at $x = 0$. In other words, all of the derivatives of f exist at $x = 0$.

a) (4 points) Give the definition of the Taylor polynomial of degree N of the function f.

$$\sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

b) (4 points) Give the definition of the Nth Remainder function of the function f.

$$R_N(x) = f(x) - T_N(x)$$

Problem 2 : (10 points) By whatever means you wish, compute the Taylor polynomial of degree 3 centered at $a = 0$ of the function $f\colon (-1/2, 1/2) \to \mathbb{R}$ given by the rule

$$f(x) = \frac{1}{1-2x}.$$

You must show your work to obtain credit for this question.

The Taylor polynomial of degree 3 of $\frac{1}{1-x}$ is $1+x+x^2+x^3$

Substituting $2x$ for x, we get

$$T_3(x) = 1+(2x)+(2x)^2+(2x)^3$$
$$= 1+2x+4x^2+8x^3$$

Problem 3 : (5 points) Suppose that the power series

$$\sum_{n=1}^{\infty} c_n x^n,$$

converges when $x = -4$ and diverges when $x = 8$.
Determine whether the following statements are **T**rue, **F**alse, or **M**aybe true. For each question, circle the correct choice. You do not need to show your work.

a) T/F/M The series converges when $x = -15$.

b) T/F/M The series converges when $x = -8$.

c) T/F/M The series diverges when $x = -8$.

d) T/F/M The series converges when $x = 4$.

e) T/F/M The series converges when $x = 2$.

Problem 4 : (10 points) Find the sum of each of the following series, if it exists. Justify your answer.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+5^n}{2^n}$ $= \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{5^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$

The first series is geometric with $r = \frac{1}{2}$, so it converges, but the second series is geometric with $r = \frac{5}{2} > 1$ so it diverges. Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1+5^n}{2^n}$ diverges.

b) $\pi - \dfrac{\pi^3}{3!} + \dfrac{\pi^5}{5!} - \dfrac{\pi^7}{7!} + \ldots$

$= \sin \pi = 0$

Problem 5 : (17 points) Determine if the following series converge or diverge. Justify your answer.

a) $\sum_{n=1}^{\infty} \cos(\pi n)$

Test for divergence:

$\lim_{n \to \infty} \cos(\pi n)$ does not exist since

$\cos(\pi n)$ oscillates between $+1$ and -1 as $n \to \infty$. Therefore the series diverges.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$

Alternating series test:

$b_n = \frac{1}{n+\sqrt{n}}$

- $b_n > 0$
- b_n are decreasing
- $\lim_{n \to \infty} \frac{1}{n+\sqrt{n}} = 0$

The series converges.

c) $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

Integral Test.

Let $f(x) = \frac{x}{x^2+4}$, on $[1, \infty)$,

f is
- positive
- continuous
- decreasing.

$$\int_1^{\infty} \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+4}\,dx \qquad u = x^2+4 \quad du = 2x\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2} \int_5^{b^2+4} \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \frac{1}{2} \left[\ln|u| \,\Big|_5^{b^2+4} \right.$$

$$= \lim_{b\to\infty} \frac{1}{2} \left[\ln(b^2+4) - \ln 5 \right]$$

$$= \infty$$

The series diverges.

Problem 6 : (8 points) Consider the function $f\colon [-1,1] \to [-\pi/2, \pi/2]$ given by the rule

$$f(x) = \arcsin x.$$

Recall that we have

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}.$$

a) (3 points) What is $f(0)$?

$\arcsin 0 = 0$

b) (5 points) Compute the Taylor polynomial of degree 2 of f.

$f(x) = \arcsin x \qquad f(0) = 0$

$f'(x) = (1-x^2)^{-1/2} \qquad f'(0) = 1$

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2} \cdot (-2x) \qquad f''(0) = 0$

$= \dfrac{x}{(1-x^2)^{3/2}}$

So $T_3(x) = x$

Problem 7 : (12 points) Let f be the function given by the rule

$$f(x) = x\ln(x+2).$$

a) (7 points) By whatever means you wish, compute the Taylor series of f centered at $a = 0$. Show your work, and simplify your answer.

$$x\ln(x+2) = x \cdot \ln\left(2\left(\frac{x}{2}+1\right)\right)$$

$$= x \cdot \left[\ln 2 + \ln\left(1-\left(-\frac{x}{2}\right)\right)\right]$$

Taylor series for $\ln(1-x)$: $-\sum_{n=1}^{\infty} \frac{x^n}{n}$

Substitute $-\frac{x}{2}$ for x: $-\sum_{n=1}^{\infty} \frac{\left(-\frac{x}{2}\right)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n \cdot n}$

Add $\ln 2$: $\ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n \cdot n}$

Multiply by x: $x\ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{2^n \cdot n}$

Recall that you computed the Taylor series of f centered at $a = 0$, for

$$f(x) = x \ln(x + 2).$$

b) (5 points) What is the interval of convergence of the Taylor series of f?

The interval of convergence for the Taylor series of $\ln(1-x)$ is

$$-1 \leq x < 1$$

Substitute $-\frac{x}{2}$ for x;

$$-1 \leq -\frac{x}{2} < 1$$

$$2 \geq x > -2$$

Adding $\ln 2$ and multiplying by x does not change the interval of convergence, so

$$\boxed{-2 < x \leq 2}$$

Problem 8 : (12 points) In this problem we will compute (almost) by hand the value of the constant e. For reference, using a calculator, we have that

$$e = 2.71828\ldots$$

a) (3 points) Write down the Taylor polynomial of degree 4 of the function given by the rule e^x.

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

For this next part, you may need to use the following values:

$$\frac{1}{2} = 0.5 \qquad \frac{1}{3} \approx 0.3333 \qquad \frac{1}{4} = 0.25 \qquad \frac{1}{5} = 0.2$$

$$\frac{1}{6} \approx 0.1667 \qquad \frac{1}{8} = 0.125 \qquad \frac{1}{24} \approx 0.0417 \qquad \frac{1}{120} \approx 0.0083$$

b) (3 points) Use the Taylor polynomial of degree 4 of e^x to approximate the number e.

$e = e^1$ so $x = 1$

$$T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

$$= 2.5 + 0.1667 + 0.0417$$

$$= 2.7084$$

$$\begin{array}{r} 2.5 \\ 0.1667 \\ 0.0417 \\ \hline 2.7084 \end{array}$$

c) (6 points) Use Taylor's Inequality to bound the error in your approximation.

$N=4 \quad |R_4(x)| \le \dfrac{K \cdot |x|^5}{5!}$

Here $x=1$. Also, $f^{(5)}(x) = e^x$.

on $0 \le x \le 1$, $e^x \le e^1 \le 3$, so we can take $K=3$.

So $|R_4(1)| \le \dfrac{3 \cdot 1^5}{120} = \dfrac{1}{40} = 0.025$

($3 \cdot 0.0083 = 0.0249$ is also acceptable)

Remark: We really are within this much of the "true" value of e!

Problem 9 : (6 points) Suppose that you need to compute $\cos(0.1)$ and make an error that is less than 0.01. Give the degree N of the Taylor polynomial of $\cos x$ that you should use to be guaranteed to obtain this level of accuracy.

$$|R_N(x)| \leq \frac{K\,|x|^{N+1}}{(N+1)!}$$

Here $x = 0.1$.

For $f(x) = \cos x$, $\quad f^{(N+1)}(x) = \pm \sin x$ or $\pm \cos x$

so we can take $K = 1$.

So $\quad |R_N(0.1)| \leq \dfrac{1 \cdot (0.1)^{N+1}}{(N+1)!}$

try $N=1$: $\quad |R_1(0.1)| \leq \dfrac{(0.1)^2}{2!} = \dfrac{0.01}{2} = 0.005$

This is less than 0.01, so $N=1$ is good enough.

Problem 10 : (12 points) Consider the function given by the rule $f(x) = \sqrt{1+x}$. The Taylor series of this function centered at $a = 0$ is given by the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n} x^n.$$

a) (4 points) What is the radius of convergence of this power series?

Ratio Test! $a_n = \frac{(-1)^n (2n)! \, x^n}{(1-2n)\, n!\, n!\, 4^n}$

$$a_{n+1} = \frac{(-1)^{n+1} (2(n+1))! \; x^{n+1}}{(1-2(n+1))(n+1)!(n+1)!\, 4^{n+1}} = \frac{(-1)^{n+1} (2n+2)! \; x^{n+1}}{(-2n-1)(n+1)!(n+1)!\, 4^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (2n+2)! \; x^{n+1}}{(-2n-1)(n+1)!(n+1)!\,4^{n+1}} \cdot \frac{(1-2n)\, n!\, n!\, 4^n}{(-1)^n (2n)!\, x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{(2n+2)!}{(2n)!} \frac{(1-2n)}{(-2n-1)} \frac{n!}{(n+1)!} \frac{n!}{(n+1)!} \frac{4^n}{4^{n+1}} \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2n+2)(2n+1)}{(n+1)(n+1)\,4} \frac{(1-2n)}{(-2n-1)} \, x \right|$$

$$= \left| \frac{(-1)(2)(2)(-2)\,x}{1 \cdot 1 \cdot 4(-2)} \right| = |x|$$

Ratio test says this converges if $|x|<1$, so $R=1$

Recall that if f is given by the rule $f(x) = \sqrt{1+x}$, then its Taylor series centered at $a = 0$ is given by the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n} x^n.$$

b) (4 points) Write down the Taylor polynomial of degree 3 of f.

$$n=0 \quad a_0 = \frac{(-1)^0 (2\cdot 0)!}{(1-2\cdot 0)(0!)^2 4^0} x^0 = \frac{1\cdot 1}{1\cdot 1^2 \cdot 1} \cdot 1 = 1$$

$$n=1 \quad a_1 = \frac{(-1)^1 (2\cdot 1)!}{(1-2\cdot 1)(1!)^2 4^1} x^1 = \frac{-1\cdot 2}{(-1)\cdot 1^2\, 4} x = \frac{x}{2}$$

$$n=2 \quad a_2 = \frac{(-1)^2 (2\cdot 2)!}{(1-2\cdot 2)(2!)^2 4^2} x^2 = \frac{1\cdot 4!}{(-3)\, 2^2\, 16} x^2$$

$$= \frac{24}{-3\cdot 4\cdot 16} x^2 = -\frac{1}{8} x^2$$

$$n=3 \quad a_3 = \frac{(-1)^3 (2\cdot 3)!}{(1-2\cdot 3)(3!)^2 \cdot 4^3} x^3 = \frac{-1\cdot 1\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}\cdot \cancel{6}}{(\cancel{-5})\, \cancel{6}\cdot \cancel{6}\cdot \cancel{4}\cdot 16} = \frac{1}{16} x^3$$

$$T_3(x) = 1 + \frac{x}{2} - \frac{1}{8} x^2 + \frac{1}{16} x^3$$

For this last part, you may need to use the following values:

$\frac{1}{2} = 0.5$	$\frac{1}{3} \approx 0.3333$	$\frac{1}{4} = 0.25$	$\frac{1}{5} = 0.2$
$\frac{1}{8} = 0.125$	$\frac{1}{12} \approx 0.0833$	$\frac{1}{15} \approx 0.0667$	$\frac{1}{16} = 0.0625$

c) (1 point) What is $f(-1)$? (You should be able to compute its exact value.)

$$f(-1) = \sqrt{1-1} = 0$$

d) (2 points) Use the Taylor polynomial of degree 3 of f to approximate $f(-1)$.

$$T_3(-1) = 1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{16}$$

$$= 0.5 - 0.125 - 0.0625$$

$$= 0.3125$$

$$\begin{array}{r} 0.500 \\ -\,0.125 \\ \hline 0.375 \end{array}$$

$$\begin{array}{r} 0.3750 \\ 0.0625 \\ \hline 0.3125 \end{array}$$

e) (1 point) Is this a good or a bad approximation?

This is bad.