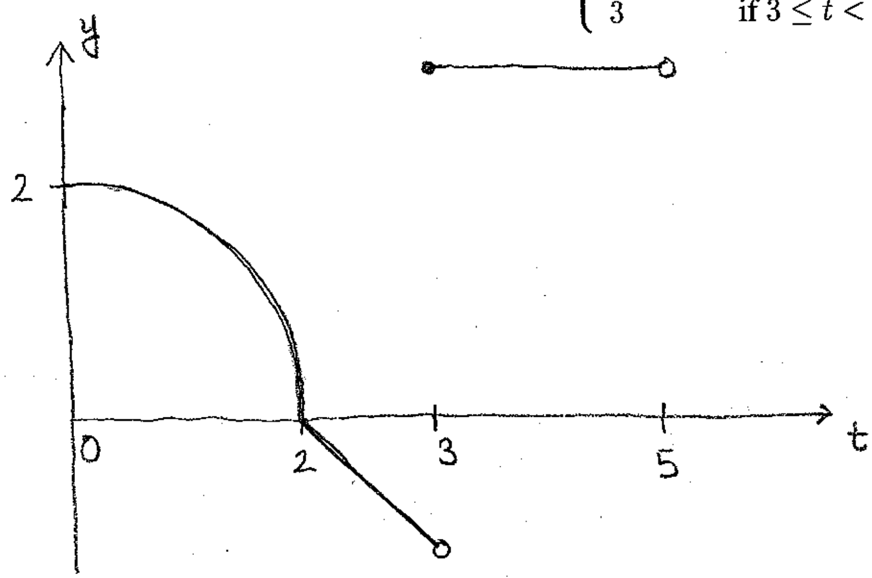
Unless otherwise specified, no justification or explanation is necessary for problems 1–6.

1. Below is the graph of $y = f(t)$ where $f(x) = \begin{cases} \sqrt{4-t^2} & \text{if } 0 \le t < 2 \\ -t+2 & \text{if } 2 \le t < 3 \\ 3 & \text{if } 3 \le t < 5 \end{cases}$



Let $F(x) = \int_0^x f(t)\,dt$.

a. What is the value of $F(2)$? (Hint: Use the picture, not a formula.)
Draw a box around your answer. You do not need to justify your answer.

$= \frac{1}{4}\pi r^2$ w/ $r = 2$ so $\boxed{\pi}$

b. Which of the following quantities is positive? *Circle all correct answers.*

$\boxed{F(2) + F(3)}$ $\qquad$ $F(3) - F(2)$

c. On which of the following intervals is $F(x)$ decreasing? *Circle all correct answers.*

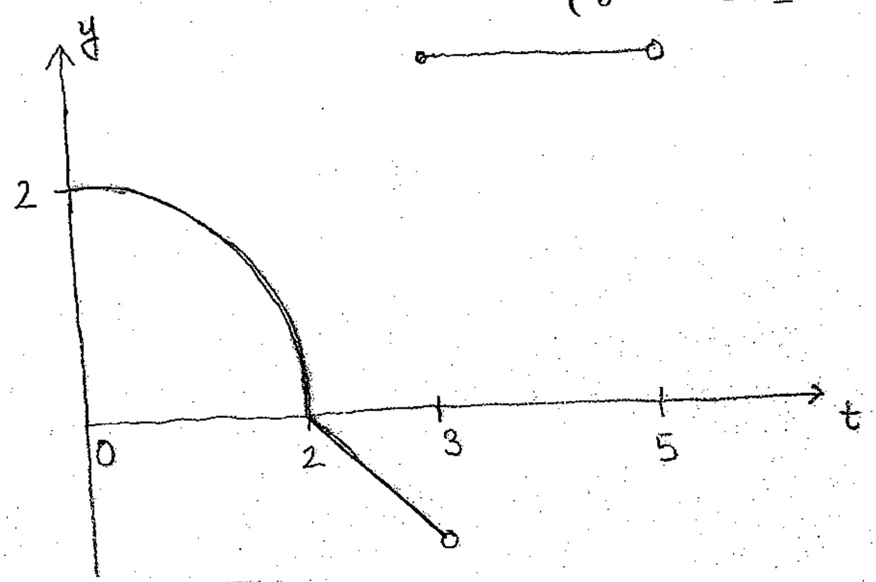
$(0, 2)$ $\qquad$ $\boxed{(2, 3)}$ $\qquad$ $(3, 5)$

d. How many points $a \in [0, 5]$ are there where $F'(a) = 0$? *Circle the correct answer.*

- i. $\boxed{\text{Just one.}}$
- ii. Two or three.
- iii. Infinitely many.
- iv. There aren't any.

Unless otherwise specified, no justification or explanation is necessary for problems 1–6.

1. Below is the graph of $y = f(t)$ where $f(x) = \begin{cases} \sqrt{4-t^2} & \text{if } 0 \le t < 2 \\ -t+2 & \text{if } 2 \le t < 3 \\ 3 & \text{if } 3 \le t < 5 \end{cases}$



Let $F(x) = \int_0^x f(t)\,dt$.

a. What is the value of $F(2)$? (Hint: Use the picture, not a formula.)
Draw a box around your answer. You do not need to justify your answer.

$= \frac{1}{4}$ area of circle w/ rad. $2 = \frac{1}{4} \cdot \pi r^2$

$= \frac{1}{4}\pi(2)^2 = \boxed{\pi}$

b. Which of the following quantities is <u>negative</u>? *Circle all correct answers.*

$F(2) + F(3)$ $\qquad$ $\boxed{F(3) - F(2)}$

c. On which of the following intervals is $F(x)$ <u>increasing</u>? *Circle all correct answers.*

$\boxed{(0,2)}$ $\qquad$ $(2,3)$ $\qquad$ $\boxed{(3,5)}$

d. How many points $a \in [0,5]$ are there where $F'(a) = 0$? *Circle the correct answer.*

i. There aren't any.
$\boxed{\text{ii. Just one.}}$
iii. Two or three.
iv. Infinitely many.

2. What is the *name* of the formal mathematical proposition asserting an inverse relationship between integration and differentiation?

The Fundamental Theorem of Calculus.

3. Isaac and Gottfried are testing each other's integration mettle. The first challenge question in their contest is a rather simple one,

Find an antiderivative for $(x+1)^3$.

Here's how these two intellectual gladiators solved this problem:

ISAAC	GOTTFRIED
Expanded $(x+1)^3$ and obtained x^3+3x^2+3x+1. Integrated term-by-term, and chose $C=0$ to obtain $\frac{1}{4}x^4+x^3+\frac{3}{2}x^2+x$.	Substituted $u=x+1$ and $du=dx$. Evaluated $\int u^3\,du=\frac{1}{4}u^4+C$. Re-substituted $u=x+1$, expanded $\frac{1}{4}(x+1)^4+C$, and chose $C=0$ to get $\frac{1}{4}x^4+x^3+\frac{3}{2}x^2+x+\frac{1}{4}$.

Their answers are different: Gottfried's includes an extra term of $\frac{1}{4}$. How is this possible?

Circle the true statement below.

i. Gottfried is incorrect.

ii. Isaac is incorrect.

iii. They are both correct.

Briefly explain the answer you circled above.

Antiderivatives may differ by a constant. Both given antiderivatives have derivative

$$x^3+3x^2+3x+1$$

$$=(x+1)^3.$$

2. What is the *name* of the formal mathematical proposition asserting an inverse relationship between integration and differentiation?

The Fundamental Theorem of Calculus

3. Isaac and Gottfried are testing each other's integration mettle. The first challenge question in their contest is a rather simple one,

Find an antiderivative for $(x+1)^3$.

Here's how these two intellectual gladiators solved this problem:

ISAAC	GOTTFRIED
Expanded $(x+1)^3$ and obtained x^3+3x^2+3x+1. Integrated term-by-term, and chose $C=0$ to obtain $\frac{1}{4}x^4+x^3+\frac{3}{2}x^2+x$.	Substituted $u=x+1$ and $du=dx$. Evaluated $\int u^3\,du=\frac{1}{4}u^4+C$. Re-substituted $u=x+1$, expanded $\frac{1}{4}(x+1)^4+C$, and chose $C=0$ to get $\frac{1}{4}x^4+x^3+\frac{3}{2}x^2+x+\frac{1}{4}$.

Their answers are different: Gottfried's includes an extra term of $\frac{1}{4}$. How is this possible? *Circle the true statement below.*

i. They are both correct.
ii. Gottfried is correct.
iii. Isaac is incorrect.

Briefly explain the answer you circled above.

Both proposed antiderivatives have derivative

$= x^3+3x^2+3x+1 = (x+1)^3$

Antiderivatives of the same function may differ by a constant

4. Which table entry below would be useful for evaluating the following integral?

$$\int (7 - 3x - 2x^2)^{1/2}\, dx$$

Circle the correct answer.

i. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cdots$

ii. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \cdots$

iii. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \cdots$

iv. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \cdots$

(For brevity we have only provided the left-hand side of the table entry.)

(You do NOT have to evaluate the integral above.)

5. Under each of the integrals below, write the letter (A.–G.) corresponding to its solution.

$\int \sec^2 u\, du$	$\int \frac{du}{u^2 + a^2}$	$\int \frac{du}{\sqrt{a^2 - u^2}}$	$\int \frac{u}{u^2 + a^2}\, du$
G	C	D	B

A. $\frac{1}{3}\sec^3 u + C$
B. $\frac{1}{2}\ln|u^2 + a^2| + C$
C. $\frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
D. $\arcsin\left(\frac{u}{a}\right) + C$
E. $\ln|\sec x| + C$
F. $\frac{1}{a}\arccos\left(\frac{u}{a}\right) + C$
G. $\tan u + C$

6. Fill in the blank:

$$\int 2x \qquad dx = x^2 + C$$

4. Which table entry below would be useful for evaluating the following integral?

$$\int (7 - 3x - 2x^2)^{-1/2}\, dx$$

Circle the correct answer.

i. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cdots$

ii. $\boxed{\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \cdots}$

iii. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \cdots$

iv. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \cdots$

(For brevity we have only provided the left-hand side of the table entry.)

(You do NOT have to evaluate the integral above.)

5. Under each of the integrals below, write the letter (A.–G.) corresponding to its solution.

$\displaystyle\int \sec^2 u\, du$	$\displaystyle\int \frac{du}{u^2 + a^2}$	$\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}$	$\displaystyle\int \frac{u}{u^2 + a^2}\, du$
D	A	E	G

A. $\frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$
B. $\ln|\sec x| + C$
C. $\frac{1}{3} \sec^3 u + C$
D. $\tan u + C$
E. $\arcsin\left(\frac{u}{a}\right) + C$
F. $\frac{1}{a} \arccos\left(\frac{u}{a}\right) + C$
G. $\frac{1}{2} \ln|u^2 + a^2| + C$

6. Fill in the blank:

$$\int 3x^2 \quad dx = x^3 + C$$

For problems 7–11, show your work. If you use u-substitution, clearly indicate your choices of u and du. If you use integration by parts, clearly indicate your choices of u, du, v, dv.

If you need a table entry to evaluate a given integral, please refer to the second page of the exam or the appropriate answer in Problem 5.

7. Evaluate the integral $\int e^{-x}\,dx$. *Draw a box around your final answer.*

Sub $u = -x$ $du = -dx$

$$\rightarrow -\int e^{u}\,du = -e^{u} + C$$
$$= \boxed{-e^{-x} + C}$$

8. Evaluate the integral $\int \frac{x^2+4}{\sqrt{x}}\,dx$. *Draw a box around your final answer.*

$$= \int \frac{x^2}{\sqrt{x}}\,dx + \int \frac{4}{\sqrt{x}}\,dx$$
$$= \int x^{3/2}\,dx + 4\int x^{-1/2}\,dx$$
$$= \frac{2}{5}x^{5/2} + 4\cdot 2x^{1/2} + C$$
$$= \boxed{\frac{2}{5}x^{5/2} + 8\sqrt{x} + C}$$

For problems 7–11, show your work. If you use u-substitution, clearly indicate your choices of u and du. If you use integration by parts, clearly indicate your choices of u, du, v, dv.

If you need a table entry to evaluate a given integral, please refer to the second page of the exam or the appropriate answer in Problem 5.

7. Evaluate the integral $\int e^{3x}\,dx$. *Draw a box around your final answer.*

Sub $u = 3x$ $du = 3dx$ (so $dx = \frac{1}{3}du$)

$$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C$$

$$= \boxed{\frac{1}{3}e^{3x} + C}$$

8. Evaluate the integral $\int \frac{4x^2+1}{\sqrt{x}}\,dx$. *Draw a box around your final answer.*

$$= \int \frac{4x^2}{\sqrt{x}}\,dx + \int \frac{1}{\sqrt{x}}\,dx$$

$$= \int 4x^{3/2}\,dx + \int x^{-1/2}\,dx$$

$$= 4\int x^{3/2}\,dx + \int x^{-1/2}\,dx$$

$$= 4\cdot\frac{2}{5}x^{5/2} + 2x^{1/2} + C$$

$$= \boxed{\frac{8}{5}x^{5/2} + 2\sqrt{x} + C}$$

9. Evaluate the integral $\int x \ln x \, dx$. *Draw a box around your final answer.*

Parts

$u = \ln x$	$dv = x\,dx$
$du = \frac{1}{x}\,dx$	$v = \frac{1}{2}x^2$

$$\int = \frac{1}{2}x^2 \ln x - \int \left(\frac{1}{2}x^2 \cdot \frac{1}{x}\right) dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$$

$$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

10. Evaluate the integral $\int \frac{dx}{4x^2+4x-3}$. *Draw a box around your final answer.*

(Hint: What is $(2x+1)^2$?)

$(2x+1)^2 = 4x^2+4x+1$

Complete the □

$$4x^2+4x-3 = (2x+1)^2 - 4$$

$$\int = \int \frac{dx}{(2x+1)^2-4}$$

Let $u = 2x-1$

$du = 2dx$

$a = 2$

$$= \frac{1}{2}\int \frac{du}{u^2-a^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

$$= \boxed{\frac{1}{8}\ln\left|\frac{2x-3}{2x+1}\right| + C}$$

9. Evaluate the integral $\int x^2 \ln x\, dx$. *Draw a box around your final answer.*

Parts:

$u = \ln x$	$dv = x^2\, dx$
$du = \frac{1}{x}\, dx$	$v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 + C$$

$$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

10. Evaluate the integral $\int \frac{dx}{4x^2 - 4x + 5}$. *Draw a box around your final answer.*
(Hint: What is $(2x-1)^2$?) $(2x-1)^2 = 4x^2 - 4x + 1$

Completing the □:

$$4x^2 - 4x + 5 = (2x-1)^2 + 4$$

So

$$= \int \frac{dx}{(2x-1)^2 + 4}$$

Let $u = 2x-1$
$du = 2\,dx$
$a = 2$

Table entry from p. 5

$$= \frac{1}{2}\int \frac{du}{u^2 + a^2}$$

$$= \frac{1}{2}\left(\frac{1}{a}\arctan\left(\frac{u}{a}\right)\right) + C$$

$$= \boxed{\frac{1}{4}\arctan\left(\frac{2x-1}{2}\right) + C}$$

11. Evaluate the integral $\int \sin^2 x\, dx$. *Draw a box around your final answer.*

(Hint: Start by applying one of the trig identities on the second page.)

Use $\sin^2 x = \dfrac{1-\cos 2x}{2}$

so $\int = \int \dfrac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$ $\quad u=2x$, $du=2dx$

$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$

11. Evaluate the integral $\int \cos^2 x\, dx$. *Draw a box around your final answer.*

(Hint: Start by applying one of the trig identities on the second page.)

$$\int \cos^2 x\, dx = \int \frac{1 + \cos(2x)}{2} dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\, dx$$

$\downarrow u = 2x, \ du = 2dx$

$$= \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + C$$

$$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}$$