XIII.1A Differential Equations

a. Solve the following initial value problem for $y(t)$: $y''(t) = t^2 - \sin t$ with $y'(0) = 1$ and $y(0) = 4$. *Show your work and draw a box around your final answer.*

b. The half-life of Thorium-232 is 14.05 billion years. That is, $P(14.05) = \frac{1}{2}P_0$ where $P(t)$ represents the amount of uranium present in the sample after t billions of years, and $P_0 = P(0)$ represents the amount of uranium initially present.

$P(t)$ is known to satisfy the differential equation $P' = rP$ for some constant r. Use the information given to solve for r. *Show your work and draw a box around your final answer.*

c. Suppose that when the Earth formed, the amounts of Thorium-232 and Thorium-229 in the crust were about equal. If the half-life of Thorium-229 is 75405 years (0.000075405 billions of years) and R is the ratio of Thorium-229 / Thorium-232 in the earth's crust after 50 billion years pass (assuming Thorium-232 is still present), then which of the following qualitative statements is true? *Circle it.*

i. R is very close to 0.

ii. R is very close to 1.

iii. R is somewhere between 1 and ∞.

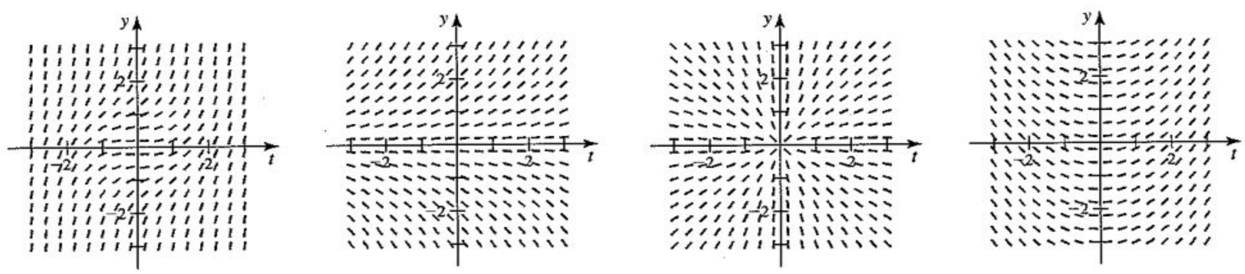
iv. None of the above.

XIII.1B Differential Equations

a. Below are 4 differential equations:

A. $y'(t) = \dfrac{t^2 + y^2}{2}$ B. $y'(t) = \dfrac{y}{2}$ C. $y'(t) = \dfrac{y}{t}$ D. $y'(t) = \dfrac{t}{2}$

and 4 direction fields:



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Under each direction field , write the letter A–D corresponding to that direction field.

b. Suppose the population of a certain shell-dwelling cichlid in Lake Taganyika follow a logistic model with $r = 0.105$ and $K = 500000$. Which of these gives the population in terms of t (time) and P_0 (the initial population)? *Circle it.*

i. $P(t) = P_0 e^{0.105 \cdot 500000 t}$.

ii. $P(t) = \dfrac{500000 P_0}{P_0 + (500000 - P_0)e^{-0.105 \cdot t}}$

iii. $P(t) = 0.105 \cdot P_0 \left(1 - \dfrac{t}{500000}\right)$

iv. $P(t) = 0.105^{t/500000}$

v. $P'(t) = 1/P(t)$

c. With the set-up of (b.): If $P_0 > 0$, then what is $\lim_{t \to \infty} P(t)$?
Draw a box around your answer.

XIII.1A Differential Equations

a. Solve the following initial value problem for $y(t)$: $y''(t) = 1 - e^{-t}$ with $y'(0) = 1$ and $y(0) = 4$.

b. The half-life of Uranium-232 is 68.9 years. That is, $P(68.9) = \frac{1}{2}P_0$ where $P(t)$ represents the amount of uranium present in the sample after t years, and $P_0 = P(0)$ represents the amount of uranium initially present.

   $P(t)$ is known to satisfy an exponential model $P'(t) = rP(t)$. Use the information given to solve for r.

XIII.1B Differential Equations

a. Suppose a function $y(t)$ satisfies the differential equation $y' = (y-1)(y-2)$ and the initial condition $y(0) = 1/2$.

   What is $\lim_{t\to\infty} y(t)$ equal to?

b. A rare fish species population is observed to grow like $P(t) = P_0 \cdot 5^{t/100}$ (t in years) when resources are unlimited.

   i. If $P_0 = 10$ and resources are unlimited, then how many fish will you expect there to be after 200 years pass?

   ii. If $P_0 = 25$ and there are only enough resources in the environment to support 100 individuals, then how many fish will you expect there to be after 100 years pass? (You may leave your answer as a fraction.)

II.1 Definite integrals and area

a. Evaluate the definite integral $\int_{-1}^{1} (x^5 + \sqrt{1 - x^2})\, dx$.

   Do not attempt to find an antiderivative of $\sqrt{1 - x^2}$.

   Draw a box around your final answer.

b. Let R be the region bounded by $y = 6x(1 - x)$ (above), $y = 2x$ (below), $x = 0$ (left), and $x = 1/2$ (right). What is the area of the region R?

   Draw a box around your final answer.

V.1 Volume,

a. Let R_1 be the region bounded above by $y = x^3$ (above), $y = 0$ (below), $x = -1$ (left), and $x = 1$ (right) and let S_1 be the solid obtained by revolving R_1 around the $\underline{x\text{-axis}}$. What is the volume of S_1?

Draw a box around your final answer.

b. Let R_2 be the region bounded by $y = 1$ (above), $y = \sqrt[3]{x}$ (below), $x = 0$ (left), and $x = 1$ (right). If S_2 is the solid obtained by revolving R_2 around the $\underline{y\text{-axis}}$, then how do the volumes of S_1 (as defined in part a.) and S_2 compare?

Circle the correct answer.

i. The volume of S_2 is equal to the volume of S_1. ($V_2 = V_1$)

ii. The volume of S_2 is twice the volume of S_1. ($V_2 = 2V_1$)

iii. The volume of S_2 is half the volume of S_2. ($V_2 = \frac{1}{2}V_1$)

iv. None of the above.

It is not necessary to justify your answer. If you would like to evaluate a second volume integral for your own confirmation, there is some room on the next page. Again, it is not necessary to justify your answer above.

I.5 Volume,

Let $f(x) = 4x^3$, let ℓ be the curve traced by the graph of $y = f(x)$ from $x = 0$ to $x = 1$, and let R be the region bounded by ℓ (above), the x-axis (below), $x = 0$ (left), and $x = 1$ (right).

a. If S is the solid obtained by revolving R around the x-axis, what is the volume of S?

b. Same problem but revolving around the y-axis.

c. Same problem, revolving around the y-axis, but using y=4x as the lower boundary.