Math 21: Spring 2014
Final Exam

NAME: SOLUTIONS

LECTURE:

Time: **180 minutes**

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.
Failure to follow these instructions will constitute a breach of the Stanford Honor Code:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason.
- You are required to sit in your assigned seat.
- You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	7	
2	6	
3	6	
4	12	
5	8	
6	11	
7	9	
8	6	
9	11	
10	16	
11	8	
TOTAL	100	

Be of good cheer. [...] You have set yourselves a difficult task, but you will succeed if you persevere; and you will find a joy in overcoming obstacles. Remember, no effort that we make to attain something beautiful is ever lost. – Helen Keller

Problem 1 : (7 points) For this question, suppose that f is a smooth function at $x = 0$. In other words, all of the derivatives of f exist at $x = 0$.

a) (3 points) Write down the definition of the Maclaurin series of f (aka the Taylor series centered at $a = 0$).

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

b) (2 points) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = 5x^2 + 3x - 4$. Compute the Maclaurin series of f.

$f(0) = -4$

$f'(x) = 10x + 3$

$f'(0) = 3$

$f''(x) = 10$

$f''(0) = 10$

$f^{(n)}(x) = 0$ for $n \geq 3$

$$-4 \cdot \frac{x^0}{0!} + 3\,\frac{x^1}{1!} + 10 \cdot \frac{x^2}{2!} = -4 + 3x + 5x^2$$

(polynomials have themselves as their Maclaurin series)

c) (2 points) Consider now the function $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = 4x^{76} - 2x^{53} + 32x^{17} + 5x^{12} - 3x^2 + 5x + 1$. What is the Maclaurin series of g?

Same as g

$$4x^{76} - 2x^{53} + 32x^{17} + 5x^{12} - 3x^2 + 5x + 1$$

Problem 2 : (6 points) For each of the following statement, decide if it is TRUE or FALSE. You do not need to show your work.

a) If f is a solution to the differential equation

$$y'' - 3y = 0$$

then $3f$ is also a solution to this differential equation.

True, if $f'' - 3f = 0$

$(3f)'' = 3f''$ so $3f'' - 3(3f) = 3(f'' - 3f) = 0$

b) Every solution to the differential equation

$$y'' + 4y = 0$$

is of the form

$$y = C_1 \cos 2x + C_2 \sin 2x$$

for some choice of the constants C_1 and C_2.

True. 2^{nd} order eq. so expect 2-dim family

$y'' = -4C_1 \cos 2x - 4C_2 \sin 2x$ so $y'' + 4y = 0$

c) If f and g are solutions to the differential equation

$$y''' + 2y' + y = \sin x$$

then $f + g$ is also a solution.

False. $f''' + 2f' + f = \sin x$

$g''' + 2g' + g = \sin x$

$(f+g)''' + 2(f' + g') + (f+g) = 2\sin x$

however, $\frac{1}{2}(f+g)$ would work!

Problem 3 : (6 points) Find the sum of the following series, if it exists. Justify your answer.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+2}}{3^{2n}}$

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+2}}{3^{2n}} = \sum_{n=0}^{\infty} (-2)^2 \frac{(-2)^n}{9^n} = 4\sum_{n=0}^{\infty} \left(\frac{-2}{9}\right)^n$$

Geometric series with $r = -\frac{2}{9}$

$$= 4\left(\frac{1}{1+\frac{2}{9}}\right) = \frac{36}{11}$$

b) $2\pi - \dfrac{2^3\pi}{3!} + \dfrac{2^5\pi}{5!} - \dfrac{2^7\pi}{7!} + \ldots$

$$= \pi\left(\frac{2}{1!} - \frac{2^3}{3!} + \cdots\right)$$

$$= \pi\left(\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}(-1)^n\right)$$ Taylor series for $\sin x$ at $x = 2$

$$= \pi \sin 2$$

Problem 4 : (12 points) Decide whether the following series converge or diverge. Justify your answer.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(-4)^n n}$

Converges by ratio test since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(-4)^{n+1}} \frac{1}{n+1}}{\frac{(-1)^n}{(-4)^n} \frac{1}{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{4} \frac{n}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{1}{4} \frac{1}{1+1/n}$$

$$= \frac{1}{4} < 1$$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{n-1}{3n-1}$

Diverges by test for divergence since,

$$\lim_{n\to\infty} \frac{n-1}{3n-1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{3-3/n} = \frac{1}{3} \neq 0$$

c) $\sum_{n=0}^{\infty} ne^{-n^2}$

Converges by ratio test since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)e^{-(n+1)^2}}{n e^{-(n)^2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n} e^{\cancel{-n^2}-2n-1+\cancel{n^2}}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n} e^{-2n-1}$$

$$= \lim_{n\to\infty}\left(e^{-2n-1} + \frac{1}{n}e^{-2n-1}\right)$$

$$= 0 < 1$$

Could also use Integral Test:

replace n by x: $f(x) = xe^{-x^2}$, f is positive, continuous, decreasing

$\sum_{n=0}^{\infty} ne^{-n^2}$ converges if and only if $\int_0^{\infty} xe^{-x^2}\,dx$ converges.

$$\int_0^{\infty} xe^{-x^2}\,dx = \int_0^{\infty}\frac{1}{2}e^{-u}\,du = \lim_{b\to\infty}\int_0^b \frac{1}{2}e^{-u}\,du$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \lim_{b\to\infty} -\frac{1}{2}e^{-u}\Big]_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}e^{-b} + \frac{1}{2}e^0$$

$= \frac{1}{2}$ so both converge.

Problem 5 : (8 points) Solve the following differential equations and initial value problems.

a) $xy' = y + x^2 \sin x, \qquad y(\pi) = 0.$

We can make this a first order linear diff eq.

$xy' - y = x^2 \sin x$

$y' - \frac{1}{x}y = x \sin x$ Now in correct $y' + P(x)y = Q(x)$ form

· Use integrating factor $I(x) = e^{\int P(x)\,dx}$

$= e^{\int -\frac{1}{x}dx} = e^{-\ln x}$

$= e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

if $x > 0$

· Multiply by $I(x)$

$\frac{1}{x}y' - \frac{1}{x^2}y = \sin x$

$\left(\frac{1}{x}y\right)' = \sin x$

Now Integrate

$\frac{1}{x}y + C_1 = \int \sin x\,dx$

$\frac{1}{x}y = -\cos x + C_2 - C_1$

$y = -x\cos x + Cx$ (let $C = C_2 - C_1$)

Don't forget the initial condition!

$0 = -\pi\cos\pi + C\pi$

$0 = \pi + C\pi$ 8

$C = -1$

$y = -x\cos x - x$

domain: $(0, \infty)$
codomain: $\mathbb{R}$

b) $\dfrac{dy}{dx} = 2 + 2y + x + xy, \qquad y > -1.$

We can write this as a first order linear diff eq.

$$y' - 2y - xy = 2 + x$$

$$y' + -(x+2)y = 2 + x$$

$$I(x) = e^{\int -(x+2)\,dx} = e^{-\frac{1}{2}x^2 - 2x}$$

$$\left(e^{-\frac{1}{2}x^2-2x}\, y\right)' = (x+2)e^{-\frac{1}{2}x^2-2x}$$

$$e^{-\frac{1}{2}x^2-2x}\, y + C_1 = \int (x+2)e^{-\frac{1}{2}x^2-2x}\,dx \qquad u = -\tfrac{1}{2}x^2 - 2x, \quad du = (-x-2)\,dx$$

$$= \int -e^u\,du$$

$$= -e^u + C_2$$

$$= -e^{-\frac{1}{2}x^2-2x} + C_2$$

So $y = -1 + (C_2 - C_1)e^{\frac{1}{2}x^2+2x} = -1 + Ce^{\frac{1}{2}x^2+2x}$

Much better if you realize its **separable**!

same answer

$$\frac{dy}{dx} = (2+x) + y(2+x) = (1+y)(2+x)$$

$$\int \frac{1}{1+y}\,dy = \int 2 + x\,dx$$

so $\boxed{y = -1 + De^{\frac{1}{2}x^2+2x}}$

$$\ln|1+y| + C_1 = 2x + \tfrac{1}{2}x^2 + C_2$$

$$\ln|1+y| = 2x + \tfrac{1}{2}x^2 + C_3$$

$$|1+y| = e^{2x + \frac{1}{2}x^2 + C_3}$$

$y > -1$ so can remove $|\ |$'s

$$1 + y = De^{2x + \frac{1}{2}x^2}$$

Problem 6 : (11 points) Consider the differential equation

$$y''' + 2y'' + y' + 2y = 0.$$

a) (3 points) Of the following functions, circle **all** of the ones that satisfy the differential equation. Show your work.

~~i. $f_1 : \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = e^x$.~~

~~ii.~~ $f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_2(x) = e^{-2x}$.

~~iii. $f_3 : \mathbb{R} \to \mathbb{R}$ given by $f_3(x) = \sin(2x)$.~~

$f_1 = e^x \quad f_1' = e^x \quad f_1'' = e^x \quad f_1''' = e^x$

$e^x + 2e^x + e^x + 2e^x \neq 0$ NOT f_1

$f_2 = e^{-2x} \quad f_2' = -2e^{-2x} \quad f_2'' = 4e^{-2x} \quad f_2''' = -8e^{-2x}$

$-8e^{-2x} + 2(4e^{-2x}) + -2e^{-2x} + 2e^{-2x} = 0$ ✓

$f_3 = \sin(2x) \quad f_3' = 2\cos(2x) \quad f_3'' = -4\sin(2x) \quad f_3''' = -8\cos(2x)$

$-8\cos(2x) + 2(-4\sin(2x)) + 2\cos(2x) + 2\sin 2x \neq 0$

b) (3 points) There are exactly two numbers b such that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \cos(bx)$$

is a solution of the differential equation

$$y''' + 2y'' + y' + 2y = 0.$$

What are these two numbers?

$f(x) = \cos bx$

$f'(x) = -b \sin bx$

$f''(x) = -b^2 \cos bx$

$f'''(x) = b^3 \sin bx$

$b^3 \sin bx + 2(-b^2 \cos bx) + (-b \sin bx) + 2 \cos bx = 0$

$(b^3 - b) \sin bx + (-2b^2 + 2) \cos bx = 0$

$b^3 - b = 0$ $\qquad$ $-2b^2 + 2 = 0$

$b(b^2-1) = b(b-1)(b+1) = 0$ $\qquad$ $-2(b^2-1) = -2(b-1)(b+1) = 0$

Only common solutions

$\boxed{b = 1,\ b = -1}$

(note: $\cos(-x) = \cos(x)$ since cosine is even)
So this is only one family of solutions

c) (3 points) Of the two numbers that you obtained in part b), one should be positive, and one should be negative. Keep the positive one. Assume that for this positive value of b, the function $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$b=1$

$$g(x) = \sin bx$$

is also solution of the differential equation

$$y''' + 2y'' + y' + 2y = 0.$$

You may further assume that f, g, f_1, f_2, and f_3 (these are all of the functions that have appeared in this problem) are all linearly independent. Using you work so far, write down the general solution of the differential equation

$$y''' + 2y'' + y' + 2y = 0.$$

$$c_1 e^{-2x} + C_2 \sin x + C_3 \cos x$$

d) (2 points) True, False, or Maybe true: Any solution of the differential equation is given by your solution in c), when a choice is made for the arbitrary constant(s).

True. A 3rd order D-E has a 3-dim family of solutions

Problem 7 : (9 points) Use a power series to solve the differential equation

$$(x-3)y' + 2y = 0.$$

Simplify your answer.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(x-3)y' + 2y = xy' - 3y' + 2y$$

$$= \sum_{n=0}^{\infty} n a_n x^n - 3(n+1)a_{n+1}x^n + 2a_n x^n$$

$$= \sum_{n=0}^{\infty} (n a_n - 3(n+1)a_{n+1} + 2a_n)x^n$$

So $\quad n a_n - 3(n+1)a_{n+1} + 2a_n = 0$

$$(n+2)a_n - 3(n+1)a_{n+1} = 0$$

$$\frac{(n+2)}{3(n+1)} a_n = a_{n+1}$$

If $a_0 = C$ $\quad a_n = C \cdot \frac{2}{3\cdot 1} \cdot \frac{3}{3(2)} \cdot \frac{4}{3\cdot 3} \cdots = C \cdot \left(\frac{1}{3}\right)^n \cdot (n+1)$

$$\boxed{y = C \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n (n+1) x^n}$$

this is enough for full credit

$$= C \frac{d}{dx}\left(\sum_{n=0}^{\infty} 3\left(\frac{x}{3}\right)^{n+1}\right)$$

$$= 3C \left(\frac{d}{dx}\left(\frac{1}{1-\frac{x}{3}} - 1\right)\right)$$

$$= 3C \frac{-1/3}{\left(1-\frac{x}{3}\right)^2} = \frac{C_2}{(X-3)^2}$$

We can check $y' = \frac{-2C_2}{(x-3)^3}$ so $\quad (x-3)\left(\frac{-2C}{(x-3)^3}\right) + 2\frac{C_2}{(x-3)^2} = 0$

so this is indeed a solution.

Problem 8 : (6 points) You arrive to tutoring shortly after Aparna has finished solving a problem on Euler's method. On the board you see the differential equation $y' = y + 2x$ and the table below:

n	x_n	y_n
0	1	0
1	1.5	1
2	2	3
3	2.5	6.5
4	3	12.25
5	3.5	21.3750
6	4	35.5625

a) (2 points) Write down the initial value problem that Aparna was working on.

Approximate $y(4)$ given $y(1)=0$ $y'=y+2x$
with step size .5

b) (2 points) What is the step size h in this application of Euler's method?

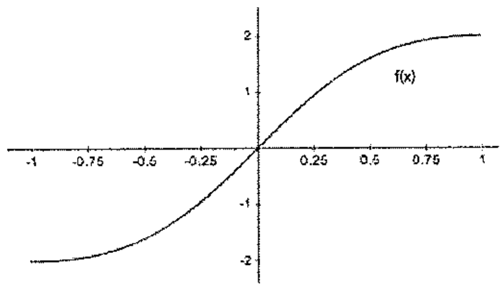
$.5 = h$

c) (2 points) Use Aparna's work to find an estimate for $y(3)$ where y is a solution to the initial value problem you wrote in part a).

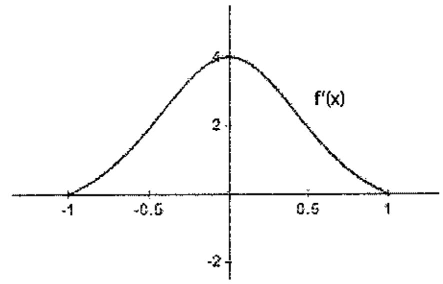
$y(3) \approx 12.25$

Problem 9 : (11 points) An unknown function $f\colon [-1,1] \to \mathbb{R}$ and several of its derivatives are pictured below.

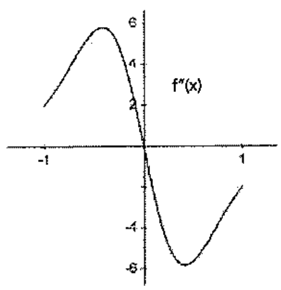
Graph of f:



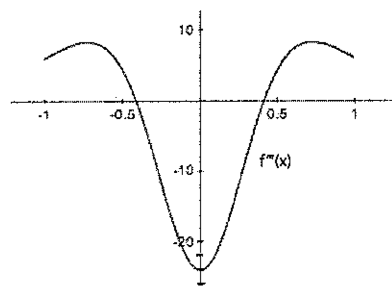
Graph of f':



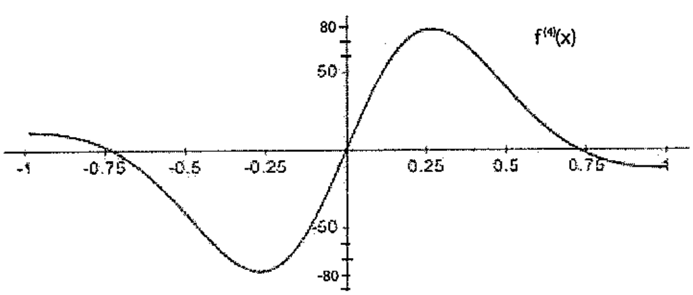
Graph of f'':



Graph of f''':



Graph of $f^{(4)}$:



a) (2 points) Find the following:

i. $f(0) =$ 0

ii. $f'(0) =$ 4

iii. $f''(0) =$ 0

iv. $f'''(0) =$ -22

v. $f^{(4)}(0) =$ 0

b) (3 points) Write down the Taylor polynomial of degree 3 of f.

$$T_3(x) = 0 + 4x + 0 \cdot \frac{x^2}{2} + (-22)\frac{x^3}{6}$$

c) (3 points) Use your answer from part b) to approximate $f(0.1)$.

$$f(.1) \approx T_3(.1) = 4(.1) + (-22)\frac{(.1)^3}{6}$$

$$= .4 - \tfrac{11}{3}(.001)$$

$$= .4 - 3(.001) + \tfrac{2}{3}(.001)$$

$$= .403\overline{6}$$

d) (3 points) Use Taylor's inequality to find an upper bound for the error in your approximation in part c).
Hint: Use the pictures on the previous page to help you find K.

$$R_n = |f(.1) - T_n(.1)| \le \frac{K}{(n+1)!}|.1|^{n+1} \quad \text{where } K > f^{(n+1)}(x) \text{ on } [-.1, .1]$$

$n = 3$

$$R_3 = |f(.1) - T_n(.1)| \le \frac{K}{4!}(.1)^4 \qquad f^{(4)} \le 80 \text{ on } [-.25, .25]$$

so also on $[-.1, .1]$

$$\le \frac{80}{4!}(.1)^4$$

$$= \frac{1}{3}(.1)^3$$

$$= .000\overline{3}$$

Problem 10 : (16 points) In this problem, we will compute some digits of the number $\pi = 3.14159265\ldots$.

a) (2 points) Compute the Maclaurin series of the function $f\colon (-1,1) \to \mathbb{R}$ where f is given by the rule $f(x) = \dfrac{1}{1+x^2}$. Simplify your answer.

Start with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

to get $f(x)$
replace x with $-x^2$

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

b) (2 points) What is the interval of convergence the Maclaurin series of f, where $f(x) = \dfrac{1}{1+x^2}$?

From ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)}}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty} x^2 = x^2 < 1 \text{ for } |x|<1$$

so $\boxed{\text{Radius } 1}$

Test end points:

$x=-1$

$\sum_{n=0}^{\infty} (-1)^n (-1)^{2n} = \sum_{n=0}^{\infty} (-1)^n$ diverges by test for divergence since $\lim_{n\to\infty} (-1)^n$ DNE

$x=1$

$\sum_{n=0}^{\infty} (-1)^n (1)^{2n} = \sum_{n=0}^{\infty} (-1)^n$

so $\boxed{\text{Interval of Conv } (-1,1)}$

c) (3 points) Use the fact that

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

and the fact that $\arctan 0 = 0$ to compute the Maclaurin series of the function whose rule is $g(x) = \arctan x$.

$$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$g(x) = \int f(x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} + C$$

since $g(0) = C = 0$

$$\boxed{g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}}$$

d) (2 points) What is the radius of convergence of the Maclaurin series of g, where $g(x) = \arctan x$?

Same as f : 1

Radius not changed by integration

e) (2 points) What is the interval of convergence the Maclaurin series of g, where $g(x) = \arctan x$?

Test end points:

$\underline{x=1}$: $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (1)^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

This is alternating: $a_n = (-1)^n b_n$, $b_n = \frac{1}{2n+1}$

we have: $\cdot\ b_n > 0$

$\cdot$ The b_n's are decreasing

$\cdot\ \lim_{n\to\infty} b_n = 0$

$\rightarrow$ converges by Alternating Series Test

$\underline{x=-1}$ $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (-1)^{2n+1} = -\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

This also converges. The interval of convergence is $[-1, 1]$.

f) (2 points) Write down the Taylor polynomial of degree 3 of the function g.

$$T_3(x) = \frac{1}{2\cdot 0+1} \cdot x^{0+1} - \frac{1}{2\cdot 1+1} x^{2+1}$$

$$= x - \frac{1}{3} x^3$$

$\uparrow$

Stop at $\underline{\underline{x^3}}$

g) (3 points) It is a fact that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

Therefore,

$$4\left(\arctan\frac{1}{2} + \arctan\frac{1}{3}\right) = \pi.$$

For the following questions, you may use the following values:

$$\frac{1}{2} = 0.5 \qquad \frac{1}{3} \approx 0.333 \qquad \frac{1}{24} \approx 0.042 \qquad \frac{1}{81} \approx 0.012$$

Hint: If you get any denominator that is not listed here, then you are doing the problem wrong.

i. Use the Taylor polynomial of degree 3 of g to approximate $\arctan\frac{1}{2}$.

$$\arctan\left(\tfrac{1}{2}\right) \approx T_3\left(\tfrac{1}{2}\right) = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3$$

ii. Use the Taylor polynomial of degree 3 of g to approximate $\arctan\frac{1}{3}$.

$$\arctan\left(\tfrac{1}{3}\right) \approx T_3\left(\tfrac{1}{3}\right) = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3$$

iii. Add these two numbers together and multiply by 4 to get an approximation for π.

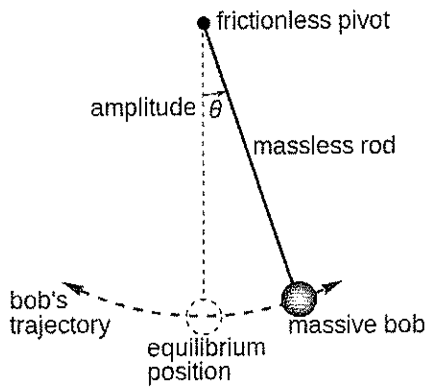
$$4\left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3} - \frac{1}{3^4}\right)$$

$$= 4(.5 - .042 + .333 - .012)$$

$$= 4(.832 - .054)$$

$$= 4(.768) = 3.072$$

Problem 11 : (8 points) The simple gravity pendulum is an idealized model of a pendulum in which a weight (also called a bob in this context) swings on the end of a massless cord from a pivot, without any effects from friction or air drag, as in the picture below.



In this context, the angle θ between the cord's actual position and its equilibrium position is a function of t satisfying the following initial value problem:

$$\theta'' = -k\sin\theta, \qquad \theta(0) = a, \quad \theta'(0) = 0.$$

for some positive constants a and k. For simplicity, let us use the values $a = 1/4$ and $k = 1$ in this problem, so that θ satisfies the initial value problem

$$\theta'' = -\sin\theta, \qquad \theta(0) = 1/4, \quad \theta'(0) = 0.$$

a) (1 point) Is the differential equation $\theta'' = -\sin\theta$, where θ is a function of t, a linear differential equation?

No $\sin\theta$ is not linear

b) (1 point) For small values of θ, we have that $\sin\theta \approx \theta$. Making this substitution we now have the differential equation

$$\theta'' = -\theta.$$

Is this a linear differential equation?

Yes, $-\theta = c\theta$ for $c = -1$

c) (4 points) Use a power series to solve the initial value problem

$$\theta'' = -\theta, \qquad \theta(0) = 1/4, \quad \theta'(0) = 0.$$

Please note that you **must** use a power series to solve this problem to obtain full credit.

Hint: You should be able to recognize your solution; if you can recognize your solution you can check your work.

$$\theta = \sum_{n=0}^{\infty} a_n x^n$$

$$\theta' = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$\theta'' = \sum_{n=0}^{\infty} a_n (n)(n-1) x^{n-2} = 0 + 0 + a_2 \cdot 2 \cdot 1 \cdot x^0 + a_3 \cdot 3 \cdot 2 \cdot x^1 + \ldots$$

Re-index so $n=0$ is the constant term

$$\theta'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

$\theta'' = -\theta$ so

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = -\sum_{n=0}^{\infty} a_n x^n$$

Therefore $a_{n+2}(n+2)(n+1) = -a_n$ each n.

$$a_{n+2} = a_n \cdot \frac{-1}{(n+1)(n+2)}$$

Now from initial conditions $a_0 = \theta(0) = 1/4$, $a_1 = \theta'(0) = 0$

$a_2 = a_0 \cdot \frac{-1}{1 \cdot 2} = \frac{-1}{1 \cdot 2} \cdot \frac{1}{4}$, $a_3 = 0$

$a_4 = a_2 \cdot \frac{-1}{3 \cdot 4} = \frac{1}{4} \cdot \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$, $a_5 = 0$

So $a_{2n} = (-1)^n \cdot \frac{1}{4} \cdot \frac{1}{(2n)!}$

$a_{2n+1} = 0$

This is $\frac{1}{4}\cos x = \theta = \sum_{n=0}^{\infty} (-1)^n \frac{1}{4} \frac{1}{(2n)!} x^{2n} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

d) (2 points) With a set-up as above, we will have that

$$-1/4 \leq \theta \leq 1/4$$

for all values of t. Use Taylor's Inequality to compute how much of an error in the value of θ is introduced by making the substitution $\sin\theta \approx \theta$.

Hint: There is a function here which is approximated by its Taylor polynomial. What is the function? What is the independent variable? What is the degree of this Taylor polynomial?

We approximated $\sin\theta$ by $\theta = T_1(\theta)$

$R_2(\theta) = |\sin\theta - \theta| \leq \frac{K}{2!}|1/4|^2$ where $K > |(\sin\theta)''|$ on $[-1/4, 1/4]$

$(\sin\theta)'' = -\sin\theta$, so $|(\sin\theta)''| = |-\sin\theta|$

$\leq \frac{1}{2} \cdot \frac{1}{16} = \boxed{\frac{1}{32}}$ can take $K=1$