Math 19: Fall 2013
Final Exam

NAME:

LECTURE:

Time: **3 hours**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ____

Problem	Value	Score
1	6	
2	12	
3	8	
4	5	
5	5	
6	5	
7	5	
8	10	
9	6	
10	6	
11	4	
12	11	
13	17	
TOTAL	100	

Limit laws

Throughout, let a and c be real numbers, and suppose that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$, for n a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$, for n a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, for n a positive integer, and provided that $a \geq 0$ if n is even
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, for n a positive integer, and provided that $\lim_{x\to a} f(x) \geq 0$ if n is even

Some derivatives

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arctan x = \dfrac{1}{1-x^2}$

Problem 1 : (6 points) Evaluate the following limits, if they exist. At every step, justify your work with a limit rule or a theorem.

a) $\lim\limits_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

b) $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x}{3x^2 + 500}$

Problem 2 : (12 points) For each of the following expressions, compute $\frac{dy}{dx}$. You do not need to simplify your answer, but you do need to solve for $\frac{dy}{dx}$.

a) **[Section 3.2 # 27]** $y = \dfrac{x^2}{1+2x}$

b) **[Section 3.5 # 13]** $e^{x/y} = x - y$

c) **[Section 3.6 # 21]** $y = \sqrt{1-x^2}\,\arccos(x)$

d) **[Section 3.7 # 39]** $y = (\cos x)^x$

Problem 3 : (8 points) Sketch a picture of the graph and use it to evaluate the following limits. NOTE: You must sketch the graph to receive credit.

a) $\lim\limits_{x \to -\infty} (e^x + 1)$

b) $\lim\limits_{x \to 1^+} \ln(x - 1)$

c) $\lim_{x \to 0^-} \left(\frac{1}{x} + 2 \right)$

d) $\lim_{x \to \infty} \left(\frac{1}{x} + 2 \right)$

Problem 4 : (5 points)

a) (2 points) Write down the definition for the following statement: The function f is differentiable at $x = a$.

b) (3 points) Use your definition above to determine if $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is differentiable at $x = 2$.

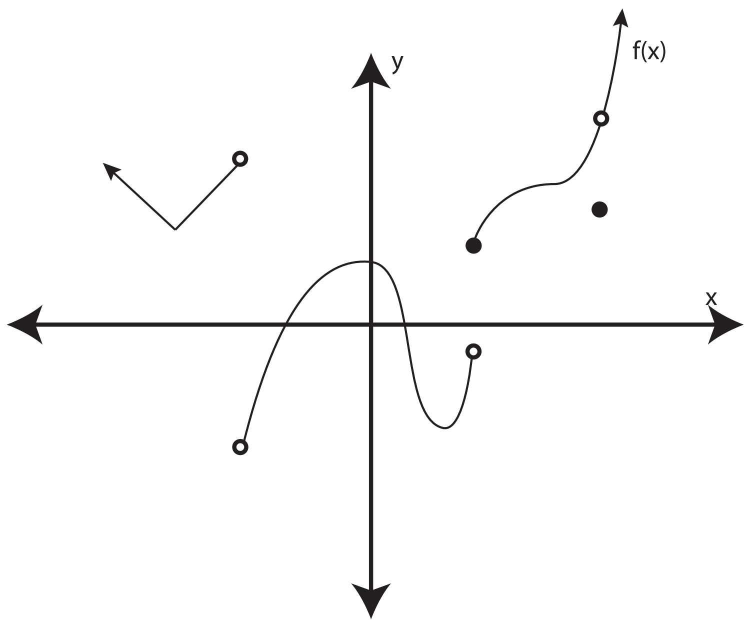
Problem 5 : (5 points) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x^2 + 2x + 3 & \text{if } x \leq 1, \\ 5x + 1 & \text{if } x > 1. \end{cases}$$

a) (3 points) Is f continuous at $x = 1$? Justify using the definition of continuity.

b) (2 points) Is f differentiable at $x = 1$? Justify using the definition of differentiability.

Problem 6 : (5 points) Given this graph of f sketch a plausible graph for f'



Problem 7 : (5 points) Suppose that a certain commodity has revenue function

$$R\colon [0,4] \to \mathbb{R}, \quad R(x) = -x^3 + 3x^2 + 9x.$$

What is the maximum revenue?

Problem 8 : (10 points) Let $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. We have that

$$f(3) = \sqrt{3} \approx 1.73205080757.$$

a) (3 points) Compute the equation of the line tangent to f at $x = 4$.

b) (2 points) Graph f along with its tangent line at $x = 4$.

c) (2 points) The tangent line you found in a) is the graph of a function L. Give the **function** L.

d) (3 points) When x is near 4, $L(x)$ is near $f(x)$. In particular,

$$L(3) \approx f(3) = \sqrt{3}.$$

Compute $L(3)$, and see how well it agrees with the value of $\sqrt{3}$ above.

Recall that

$$\frac{d}{dx}2^x = \ln 2 \cdot 2^x.$$

Problem 9 : (6 points) Let $y = \log_2(x)$. Compute $\dfrac{dy}{dx}$ using implicit differentiation.

Problem 10 : (6 points)
One day Dr. Campisi and Dr. Vincent went out for a drive. The FasTrak in their car registered them crossing the Richmond-San Rafael bridge at 12:05pm and it registered them crossing the Golden Gate bridge, 20 miles away, at 12:20pm. Assume that the position function of the car is differentiable at each time during the drive. Two weeks later they received a ticket in the mail, citing them for exceeding the speed limit of 60 miles per hour.

a) (2 points) State the Mean Value Theorem.

b) (4 points) Show that Dr. Campisi and Dr. Vincent did in fact exceed the speed limit at some point during their drive. Be sure to clearly define any variable or functions that you use.

Problem 11 : (4 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two differentiable functions such that

$$g(0) = \pi, \quad g'(0) = 3, \quad f(0) = 0, \quad \text{and} \quad f'(0) = 0.$$

Let B be a function with rule

$$B(x) = \cos(g(\sin(f(x)))).$$

Compute $B'(0)$.

Problem 12 : (11 points) Consider the function

$$f\colon (-\infty, 0] \to \mathbb{R}, \qquad f(x) = x^2.$$

a) (1 point) What is the domain of f?

b) (2 points) Sketch the graph of f.

c) (1 point) Is f one-to-one? Please simply answer yes or no.

d) (2 points) Sketch the graph of f^{-1}.

e) (2 points) Give the rule of the function f^{-1}. Hint: Its graph is a transformation of the graph of a function you know.

f) (1 point) Let $x < 0$. Compute $f^{-1}(f(x))$.

g) (1 point) Let $x > 0$. Compute $f^{-1}(f(x))$.

h) (1 point) Compute $f(f^{-1}(x))$.

Problem 13 : (17 points) Let $f(x) = x^3 - 3x^2$.

a) (1 point) State the domain of f.

b) (2 points) Determine all x- and y-intercepts.

c) (2 points) Find the equations for all vertical and horizontal asymptotes. If there are none, please write “none.”

d) (3 points) Find all values for which f is increasing and decreasing.

e) (1 point) Determine all maximum and minimum values of f.

f) (3 points) Find all values for which f is concave up and concave down.

g) (1 point) Determine all inflections points.

h) (4 points) Neatly sketch the graph of f. Label all asymptotes with their equation, and label all special points (intercepts, holes, maxima, minima, points of inflection) with their coordinates.