Math 21: Spring 2013
Midterm 1

NAME: SOLUTIONS

LECTURE:

Time: **75 minutes**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ______________________________

Problem	Value	Score
1	10	
2	20	
3	5	
4	10	
5	15	
6	30	
7	10	
TOTAL	100	

Problem 1 : (10 points)

a) Write down the limit definition of the sum of a series.

The series $\sum_{n=0}^{\infty} a_n$ exists if and only if

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n \quad \text{exists.}$$

b) Write down the definition of the Maclaurin series of a function f.

The MacLaurin series for f is

$$\sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!} .$$

Problem 2 : (20 points) Decide whether the following series converge or diverge.

a) $\sum_{n=0}^{\infty}(-1)^n \frac{2n-1}{3n+1}$

$\lim_{n\to\infty} (-1)^n \frac{2n-1}{3n+1}$ does not exist.

So, by the test for divergence, the series does not converge.

b) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

~~We~~ Use the integral test with $f:[2,\infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x\ln(x)}$. So

$$f(n) = \frac{1}{n\ln(n)}.$$

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\ln(x)}\,dx =$$

$$= \lim_{N\to\infty} \int_2^N \frac{1}{x\ln(x)}\,dx \qquad u = \ln(x), \quad du = \frac{dx}{x}$$

So the series diverges.

$$= \lim_{N\to\infty} \int_{\ln(2)}^{\ln(N)} \frac{du}{u} = \lim_{N\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(N)}$$

$$= \lim_{N\to\infty} \Big(\ln(\ln(N)) - \ln(\ln(2))\Big)$$ does not exist.

c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Again, use the integral test with $f:[2,\infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x\ln(x)^2}$

So $\int_2^\infty f(x)dx = \lim_{N\to\infty} \int_2^N \frac{1}{x\ln(x)^2} dx = \lim_{N\to\infty} \int_{\ln(2)}^{\ln(N)} \frac{du}{u^2}$

$= \lim_{N\to\infty} -\frac{1}{u}\Big|_{\ln(2)}^{\ln(N)} = \lim_{N\to\infty} \left(\frac{1}{\ln(2)} - \frac{1}{\ln(N)} \right)$

$= \frac{1}{\ln(2)}$

So the series converges.

(but it is <u>not</u> true that it converges to $\frac{1}{\ln 2}$!)

d) $\sum_{n=0}^{\infty} \frac{2^n(n+1)}{n!}$

Let $a_n = \frac{2^n(n+1)}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{2^{n+1}(n+2)}{(n+1)!} \cdot \frac{n!}{2^n(n+1)} \right)$$

$$= \lim_{n\to\infty} \frac{2(n+2)}{(n+1)^2} = \lim_{n\to\infty} \frac{2n+4}{n^2+2n+1}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{4}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{0+0}{1+0+0} = 0.$$

So, by the Ratio Test the series converges.

Problem 3 : (5 points) Compute the sum of the following series, if it exists:

$$\sum_{n=0}^{\infty} \frac{2^n}{3}.$$

The series does not exist because

$\lim_{n \to \infty} \frac{2^n}{3} = +\infty$. So the test for

divergence implies the series does not exist.

Problem 4 : (10 points)

a) Write down the Maclaurin series of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \sin x$.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x.$$

b) Suppose that you want to compute $\sin(0.1)$ and make an error that is less than 0.001. Use Taylor's Inequality to find the degree of a Taylor polynomial which is guaranteed to estimate $\sin(0.1)$ to this level of accuracy. As always, justify your answer.

Taylor's inequality ~~shows~~ says

$$|R_n(x)| = |f(x) - T_n(x)| \leq K \cdot \frac{|x|^{n+1}}{(n+1)!}$$

for any K with $K \geq |f^{(n+1)}(x)|$

Since all derivatives of $\sin(x) = f(x)$ are $\pm\sin(x)$ or $\pm\cos(x)$ we have $|f^{(n+1)}(x)| \leq 1$ for all n and all x.

We need to find an n such that $1 \cdot \frac{(0.1)^{n+1}}{(n+1)!} < 0.001$.

Since $(0.1)^3 = 0.001$, we can be sure that $\frac{(0.1)^3}{3!} < 0.001$. So $n = 2$ will have $|T_2(0.1) - \sin(0.1)|$ less than 0.001.

Problem 5 : (15 points) Let $f\colon(-1,1)\to\mathbb{R}$ be given by the rule $f(x)=\frac{1}{1-x}$.

a) Compute the Maclaurin series of the function f'.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)=\sum_{n=0}^{\infty}nx^{n-1}$$

$$=\sum_{n=0}^{\infty}(n+1)x^n.$$

b) What are all of the values of x such that the Maclaurin series of f' converges? In other words, compute the interval of convergence the Maclaurin series of f'.

Since $\lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|=|x|$, the Ratio Test implies the series converges for $|x|<1$, ~~Test test for divergence~~ and diverges for $|x|>1$. The test for divergence implies $\sum_{n=0}^{\infty}(n+1)$ and $\sum_{n=0}^{\infty}(n+1)(-1)^n$ both diverge.

c) What is the infinite sum

$$\sum_{n=0}^{\infty} \frac{n}{2^{n-1}}$$

equal to?

$$\sum_{n=0}^{\infty} n x^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \quad \text{and } \left|\frac{1}{2}\right| < 1.$$

$$\text{So} \quad \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$$

Problem 6 : (30 points) In this problem, we will compute some digits of the number π.

a) Compute the Maclaurin series of the function $f\colon (-1,1) \to \mathbb{R}$ given by the rule $f(x) = \frac{1}{1+x^2}$. Simplify your answer.

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$ by the Geometric series

when $|x^2| < 1, \Leftrightarrow |x| < 1.$

b) What are all of the values of x such that the Maclaurin series of f converges? In other words, compute the interval of convergence the Maclaurin series of f.

The Ratio Test shows $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = |x|^2$

that the series converges for $-1 < x < 1$ and diverges for $x > 1$ and $x < -1$.

$\sum_{n=0}^{\infty} (-1)^n$ diverges. So the series diverges for $x = \pm 1$.

c) Use the fact that

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

and the fact that $\arctan 0 = 0$ to compute the Maclaurin series of the function whose rule is $g(x) = \arctan x$.

$$\arctan(x) + C = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

But $\left.\begin{array}{l} \arctan(0) + C = C \\ \text{and} \\ \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1} = 0 \end{array}\right\} \Rightarrow C = 0.$

So $\arctan(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ whenever the series converges.

d) What is the radius of convergence of the Maclaurin series of g?

The ratio test shows the radius of convergence is 1.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right|$$

$$= |x|^2 \lim_{n\to\infty} \left(\frac{2n+1}{2n+3} \right) = |x|^2.$$

e) What are all of the values of x such that the Maclaurin series of g converges? In other words, compute the interval of convergence the Maclaurin series of g.

The series diverges for $|x|^2 > 1 \iff x < -1$ and $x > 1$

and The series converges for $|x|^2 < 1 \iff -1 < x < 1$.

by the previous part.

$x = 1$ $\quad \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ which converges by the alternating series test

Since ① $\frac{1}{2n+1} > 0$ for all $n \geq 0$

② $\frac{1}{2n+3} < \frac{1}{2n+1}$ for all $n \geq 0$

③ $\lim_{n\to\infty} \frac{1}{2n+1} = 0$.

$x = -1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} = -1 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which converges as in the $x = 1$ case

So the series converges for $x \in [-1, 1]$ and diverges otherwise.

f) Write down the Taylor polynomial of degree 3 of the function g.

Since $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

$$T_3(x) = x - \frac{x^3}{3}.$$

g) It is a fact that

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}.$$

Therefore,

$$4\left(\arctan\frac{1}{2}+\arctan\frac{1}{3}\right)=\pi.$$

For the following questions, you may use the following values:

$$\frac{1}{2}=0.5 \qquad \frac{1}{3}\approx 0.333 \qquad \frac{1}{24}\approx 0.042 \qquad \frac{1}{81}\approx 0.012$$

Hint: If you get any denominator that is not listed here, then you are doing the problem wrong.

i. Use the Taylor polynomial of degree 3 of g to estimate $\arctan\frac{1}{2}$.

$$T_3\left(\tfrac{1}{2}\right)=\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{2^3}=\frac{1}{2}-\frac{1}{24}\approx 0.5-0.042$$
$$=0.458.$$

ii. Use the Taylor polynomial of degree 3 of g to estimate $\arctan\frac{1}{3}$.

$$T_3\left(\tfrac{1}{3}\right)=\frac{1}{3}-\frac{1}{3^4}\approx 0.333-0.012$$
$$=0.321$$

iii. Add these two numbers together and multiply by 4 to get an estimate for π.

$$4\cdot\left(T_3\left(\tfrac{1}{2}\right)+T_3\left(\tfrac{1}{3}\right)\right)\approx 4\cdot(0.458+0.321)$$
$$\approx 4(0.779)$$
$$=3.116$$

$$\begin{array}{r} {}^{3\,3}\\ 779\\ \times\quad 4\\ \hline 3116\end{array}$$

Recall $\pi\approx 3.14$

Problem 7 : (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = x \sin x^2$.

a) Write down the Maclaurin series of f. Simplify your answer.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$x \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!} = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)\, x^m}{m!}$$

So $\frac{f^{(100)}(0)}{(100)!}$ is the coefficient of x^{100}.

Since $100 = 4n+3 \Rightarrow n = \frac{97}{4} = 24 + \frac{1}{4}$ is not an integer we must have $f^{(100)}(0) = 0$.

b) What is $f^{(100)}(0)$?