Math 18.100B
Final Exam
Spring 2002

Name: ________________________

e-mail: ________________________

If you want your course grade posted outside my office door, write down a secret code.

secret code: ________________________

1.		5.	
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		Total	

Make sure your name is written on every page of this exam.

Each problem is worth 25 points.

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1. (a) Suppose that X, Y, and Z are metric spaces and that $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Show that

$$g \circ f : X \to Z$$

is continuous.

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(b) Does there exist a continuous function function $f : [0,1] \to \mathbb{R}$ with the property that $f(s) = 0$ for countably many values of s? If so, give an example. If not explain why not.

(c) Does there exist a differentiable function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$, $f(1) = 1$, and $f'(x) > 2$ for all x? If so, give an example. If not explain why not.

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2. Suppose that $\alpha : [a, b] \to \mathbb{R}$ is a monotone increasing function, and that $f : [a, b] \to \mathbb{R}$ is bounded.

(a) Give the definitions of

$$\underline{\int_a^b} f \, d\alpha \quad \text{and} \quad \overline{\int_a^b} f \, d\alpha,$$

the *lower* and *upper* Riemann-Stieltjes integrals of f with respect to α.

(b) What does it mean to say $f \in \mathcal{R}(\alpha)$ on $[a, b]$?

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3. Fill in the blanks.

(a) Suppose that X and Y are metric spaces, and that S is a collection of functions from $X \to Y$. The collection S is said to be ____________________ if given $\epsilon > 0$ there exists δ such that for all $f \in S$, and all $x, y \in X$,

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

(b) A function $f : X \to Y$ is said to be ____________________ if given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

(c) A sequence of functions $\{f_n\}$ is said to ____________________ to a function f if given $\epsilon > 0$ there exists N such that for all x

$$n > N \implies d(f_n(x), f(x)) < \epsilon.$$

(d) The ____________________ says that a sequence of functions

$$f_n : X \to \mathbb{R}$$

____________________ if and only if given $\epsilon > 0$ there exists N such for all x

$$m, n > N \implies |f_n(x) - f_m(x)| < \epsilon.$$

4. Which of the following are true of a continuous function $f : K \to \mathbb{R}$ from a compact metric space K to $\mathbb{R}$:

(a) f is bounded;
(b) f is uniformly continuous;
(c) f satisfies the intermediate value theorem;
(d) There exists a point $x \in K$ at which f attains its maximum.

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5. Suppose that K is a compact metric space and $\{f_n : K \to \mathbb{R}\}$ is an equicontinuous sequence of continuous functions. Show that if $\{f_n\}$ converges pointwise then it converges uniformly.

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6. Suppose that $\alpha : [a, b] \to \mathbb{R}$ is monotone increasing, and $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ is a sequence of functions which are integrable (with respect to α) on $[a, b]$. Suppose that the sequence f_n converges uniformly to a function f.

(a) Show that given ϵ, there exists N so that $n > N$ implies

$$\int_a^b (f_n - \epsilon)\, d\alpha \le \underline{\int} f\, d\alpha \le \overline{\int} f\, d\alpha \le \int_a^b (f_n + \epsilon)\, d\alpha.$$

(b) Conclude that $f \in \mathcal{R}(\alpha)$ and that

$$\lim_{n\to\infty} \int_a^b f_n\, d\alpha = \int_a^b f\, d\alpha.$$

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7. Consider the sequences of functions $\{f_n(x)\}$ *on the open interval* $(0,1)$, with f_n given by:

(i)	$\sin(x/n)$	(ii)	$\sin(nx)$
(iii)	$n^2 x^n$	(iv)	$\dfrac{1}{x^n}$
(v)	$\displaystyle\sum_{k=1}^{n} \frac{1}{1+k^2x}$	(vi)	$n(1-x)$

(a) Which sequences are pointwise bounded?

(b) Which sequences are uniformly bounded?

(c) Which sequences converge uniformly?

(d) Which sequences form an equicontinuous family?

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8. (a) Suppose that E is a metric space, and that $\{f_n\}$ and $\{g_n\}$ are sequences of *bounded* real-valued functions on E. Show that if $\{f_n\}$ and $\{g_n\}$ converge uniformly then so does $\{f_n g_n\}$.

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(b) Does the result of part (a) remain true if we drop the assumption that the sequences are bounded? Give a reason for your answer.

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