Math 21: Spring 2013
Midterm 1

NAME:

LECTURE:

Time: **75 minutes**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ____

Problem	Value	Score
1	10	
2	20	
3	5	
4	10	
5	15	
6	30	
7	10	
TOTAL	100	

Problem 1 : (10 points)

a) Write down the limit definition of the sum of a series.

b) Write down the definition of the Maclaurin series of a function f.

Problem 2 : (20 points) Decide whether the following series converge or diverge.

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2n-1}{3n+1}$

b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\,\ln n}$

c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n\,(n+1)}{n!}$

Problem 3 : (5 points) Compute the sum of the following series, if it exists:

$$\sum_{n=0}^{\infty} \frac{2^n}{3}.$$

Problem 4 : (10 points)

a) Write down the Maclaurin series of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \sin x$.

b) Suppose that you want to compute $\sin(0.1)$ and make an error that is less than 0.001. Use Taylor's Inequality to find the degree of a Taylor polynomial which is guaranteed to estimate $\sin(0.1)$ to this level of accuracy. As always, justify your answer.

Problem 5 : (15 points) Let $f\colon (-1,1) \to \mathbb{R}$ be given by the rule $f(x) = \frac{1}{1-x}$.

a) Compute the Maclaurin series of the function f'.

b) What are all of the values of x such that the Maclaurin series of f' converges? In other words, compute the interval of convergence the Maclaurin series of f'.

c) What is the infinite sum

$$\sum_{n=0}^{\infty} \frac{n}{2^{n-1}}$$

equal to?

Problem 6 : (30 points) In this problem, we will compute some digits of the number π.

a) Compute the Maclaurin series of the function $f\colon (-1,1) \to \mathbb{R}$ given by the rule $f(x) = \frac{1}{1+x^2}$. Simplify your answer.

b) What are all of the values of x such that the Maclaurin series of f converges? In other words, compute the interval of convergence the Maclaurin series of f.

c) Use the fact that

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

and the fact that $\arctan 0 = 0$ to compute the Maclaurin series of the function whose rule is $g(x) = \arctan x$.

d) What is the radius of convergence of the Maclaurin series of g?

e) What are all of the values of x such that the Maclaurin series of g converges? In other words, compute the interval of convergence the Maclaurin series of g.

f) Write down the Taylor polynomial of degree 3 of the function g.

g) It is a fact that

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}.$$

Therefore,

$$4\left(\arctan\frac{1}{2}+\arctan\frac{1}{3}\right)=\pi.$$

For the following questions, you may use the following values:

$$\frac{1}{2}=0.5 \qquad \frac{1}{3}\approx 0.333 \qquad \frac{1}{24}\approx 0.042 \qquad \frac{1}{81}\approx 0.012$$

Hint: If you get any denominator that is not listed here, then you are doing the problem wrong.

i. Use the Taylor polynomial of degree 3 of g to estimate $\arctan\frac{1}{2}$.

ii. Use the Taylor polynomial of degree 3 of g to estimate $\arctan\frac{1}{3}$.

iii. Add these two numbers together and multiply by 4 to get an estimate for π.

Problem 7 : (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = x \sin x^2$.

a) Write down the Maclaurin series of f. Simplify your answer.

b) What is $f^{(100)}(0)$?