Math 19: Fall 2013
Midterm 1

NAME:

Solutions.

LECTURE:

Time: **75 minutes**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: _____

Problem	Value	Score
1	24	
2	6	
3	16	
4	18	
5	8	
6	10	
7	18	
TOTAL	100	

4 points each

Problem 1 : (24 points)

a) **[Section 1.5 # 4]** Simplify: $\dfrac{x^{2n} \cdot x^{3n-1}}{x^{n+2}}$.

$$= \frac{x^{5n-1}}{x^{n+2}}$$

$$= x^{4n-3}$$

b) Simplify: $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{3} + \frac{1}{4}}$.

$$= \frac{\dfrac{3}{6} - \dfrac{2}{6}}{\dfrac{4}{12} + \dfrac{3}{12}}$$

$$= \frac{1}{6} \div \frac{7}{12}$$

$$= \frac{1}{6} \cdot \frac{12}{7}$$

$$= \frac{2}{7}$$

Alternate Solution:

$$\frac{\left(\frac{1}{2} - \frac{1}{3}\right)12}{\left(\frac{1}{3} + \frac{1}{4}\right)12} = \frac{6-4}{4+3}$$

$$= \frac{2}{7}$$

2

c) [**HW 1 Trig problem # 4**] Find the exact value: $\sin^2(\pi/8)$.

$$\sin^2\left(\frac{\pi}{8}\right) = \frac{1 - \cos\left(2\frac{\pi}{8}\right)}{2}$$

$$= \frac{1 - \cos\left(\frac{\pi}{4}\right)}{2}$$

$$= \frac{1 - \frac{\sqrt{2}}{2}}{2} = \frac{2 - \sqrt{2}}{2} \div 2 = \frac{2 - \sqrt{2}}{4}$$

d) [**Section 1.6 # 37**] Find the exact value: $\log_2 6 - \log_2 15 + \log_2 20$.

$$= \log_2 \frac{6 \cdot 20^4}{15} = \log_2 8 = 3$$

3

e) [**Section 1.5 # 23**] If $f(x) = 5^x$, simplify the expression $\dfrac{f(x+h) - f(x)}{h}$.

$$= \frac{5^{x+h} - 5^x}{h} = 5^x \frac{(5^h - 1)}{h}$$

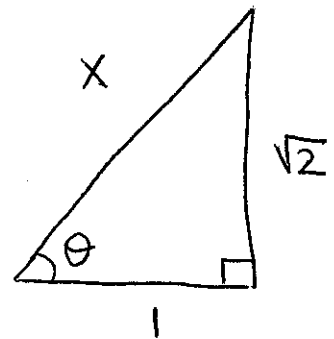f) [**Section 3.6 # 7**] Find the exact value: $\sin(2\tan^{-1}\sqrt{2})$.

Let $\theta = \tan^{-1}\sqrt{2}$ so $\tan\theta = \sqrt{2}$

$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$

$\qquad = 2\sin\theta\cos\theta$

$\qquad = 2 \dfrac{\sqrt{2}}{\sqrt{3}} \dfrac{1}{\sqrt{3}}$
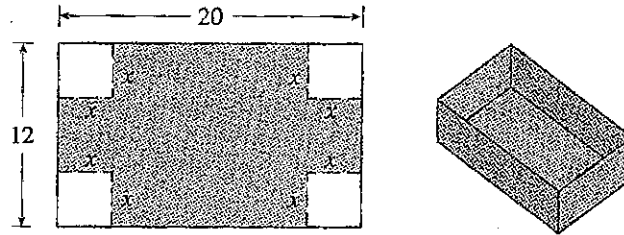
$\qquad = \dfrac{2\sqrt{2}}{3}$

$1^2 + (\sqrt{2})^2 = x^2$

$3 = x^2$

$x = \sqrt{3}$

4

Problem 2 : (6 points) [Section 1.1 # 59] A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a **function** of x.
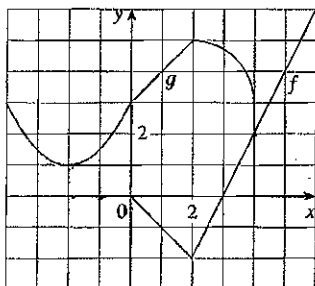

Volume $= \ell b h$

here, $h = x$

$b = 12 - 2x$

$\ell = 20 - 2x$

also, $0 \leq x \leq 6$ ← cannot have corners wider

↖ cannot have negative length   than $\dfrac{12}{2}$

So $V : [0, 6] \to \mathbb{R}$

$V(x) = (20 - 2x)(12 - 2x) \, x$

Problem 3 : (16 points) [Section 1.3 # 51] Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

2pts

a) $f(g(2))$ $= f(5) = 4$

2pts

b) $g(f(0))$ $= g(0) = 3$

3pts

c) $(f \circ g)(0)$ $= f(g(0)) = f(3) = 0$

6

3pts

d) $(g \circ f)(6) = g(f(6)) = g(6)$ is undefined

since 6 is not in the domain
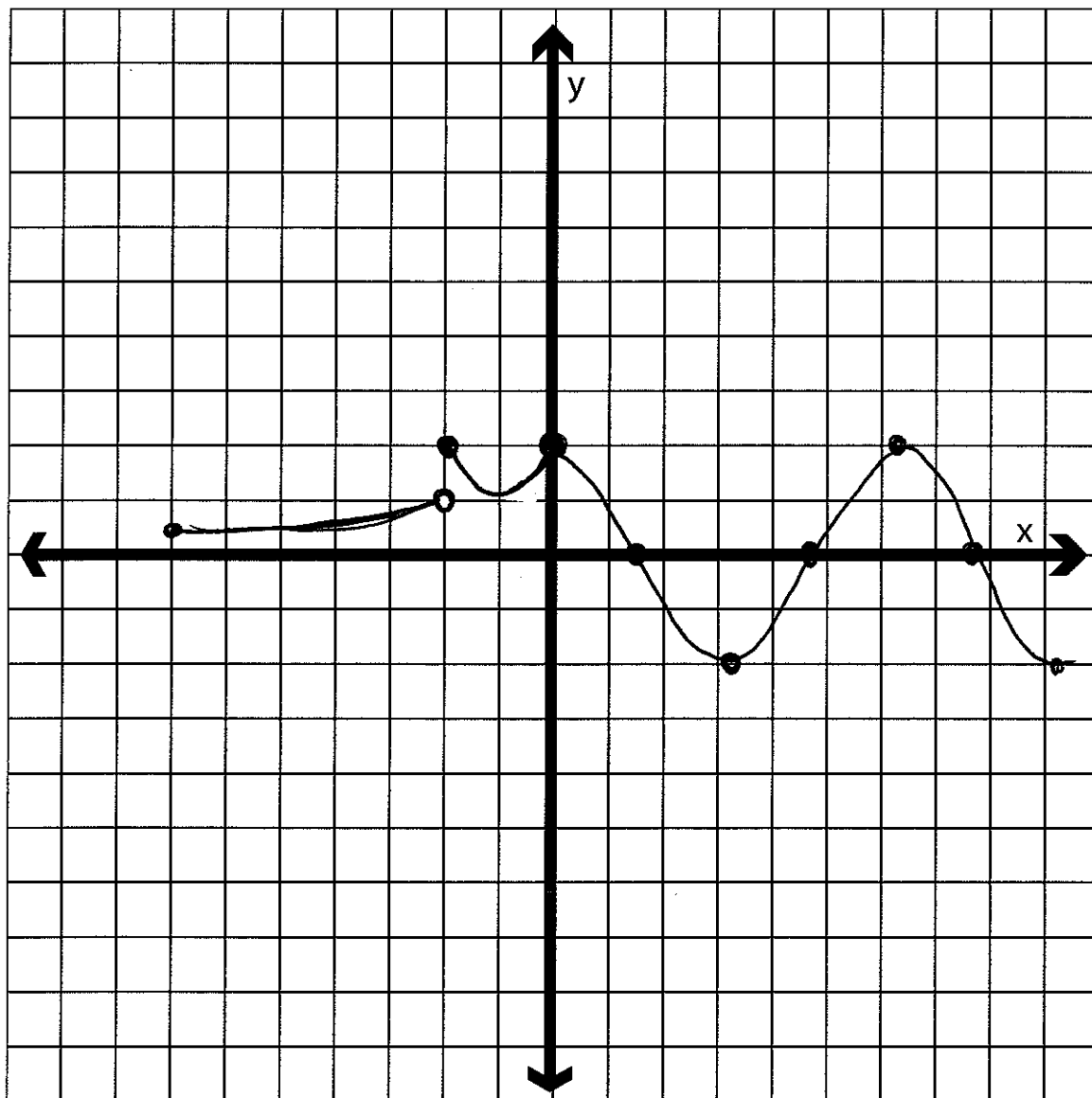
of g

3pts

e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

3pts

f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

Problem 4 : (18 points) Consider the function $f: [-7, \infty) \to \mathbb{R}$ given by the rule

$$f(x) = \begin{cases} e^{x+2} & \text{if } -7 \leq x < -2, \\ (x+1)^2 + 1 & \text{if } -2 \leq x \leq 0, \\ 2\cos x & \text{if } 0 < x. \end{cases}$$

a) (10 points) Graph f.

b) (2 points) Is f a one-to-one function?

NO

c) (3 points) List all of f's y-intercepts, if any.

$$f(0) = (0+1)^2 + 1 = 2$$

$$(0, 2)$$

d) (3 points) List all of f's x-intercepts, if any.

$f(x) = 0$: between -7 & -2 f has no x-intercept

between -2 & 0 f has no x-intercept.

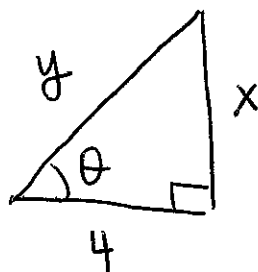when $x > 0$, $f(x) = 2\cos x = 0$

when $\cos x = 0$

$$x = \frac{\pi}{2} + n\pi, \quad n \text{ an integer}$$

OR $\left(\frac{\pi}{2} + n\pi, 0\right)$, n an integer

9

Problem 5 : (8 points) Let θ be an angle such that $\tan\theta = \frac{x}{4}$.

a) Write down an expression for $\cos\theta$ in terms of x.

3pts

$$y^2 = x^2 + 4^2$$

$$y = \sqrt{x^2 + 16}$$

$$\cos\theta = \frac{4}{\sqrt{x^2 + 16}}$$

2pts

b) Write down an expression for x in terms of θ

$$x = 4\tan\theta$$

3pts

c) Write down an expression for $x\sqrt{16 + x^2}$ in terms of θ.

we have $\quad \sec\theta = \dfrac{\sqrt{16 + x^2}}{4} \quad$ so $\quad \sqrt{16 + x^2} = 4\sec\theta$

also, $\quad x = 4\tan\theta$

So $\quad x\sqrt{16 + x^2} = 4\tan\theta \cdot 4\sec\theta = 16\tan\theta\sec\theta$

$$= \frac{16\sin\theta}{\cos^2\theta}$$

Problem 6 : (10 points) Consider the function given by the rule $F(x) = \sqrt{\tan x}$.

3pts a) There exist functions f and g such that $F = g \circ f$. Give the rule for f and for g.

$$f(x) = \tan x$$
$$g(x) = \sqrt{x}$$

7pts b) Among the values of x that are such that $0 \le x \le 4\pi$, which ones are in the domain of F?

We need $\tan x \ge 0$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\sin x$	0 $+$	$+$ 0	$-$	$-$ 0	$+$	$+$ 0	$-$	$-$	0
$\cos x$	$+$ 0	$-$	$-$ 0	$+$	$+$ 0	$-$	$-$ 0	$+$
$\tan x$	$0+$	$-$ 0 $+$	$-$ 0 $+$	$-$ 0 $+$	$-$

undefined   undefined

So $\tan x \ge 0$ when x is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \cup \left[2\pi, \frac{5\pi}{2}\right)$

$\cup \left[3\pi, \frac{7\pi}{2}\right)$

and also $x = 4\pi$

these are the values of x between 0 and 4π that are in the domain of F

11

Problem 7 : (18 points) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + 4$.
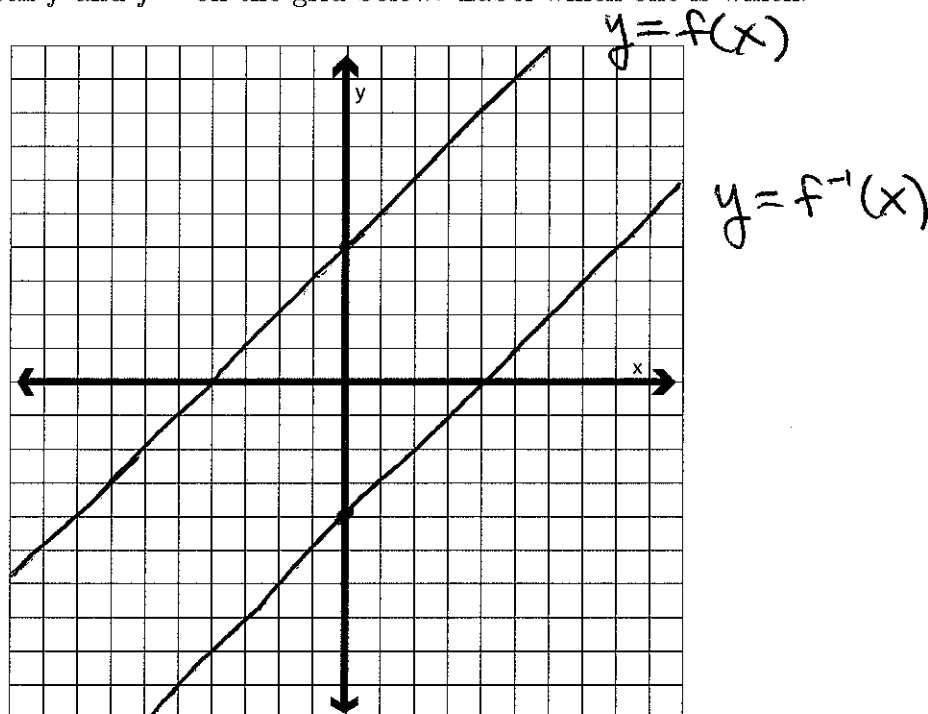
a) (2 points) Is f one-to-one?

yes

b) (5 points) Find an expression for the rule of the function f^{-1} and state its domain.

$$y = x+4 \rightarrow x = y+4$$
$$x-4 = y$$

domain: $\mathbb{R}$

c) (5 points) Sketch f and f^{-1} on the grid below. Label which one is which.

$y = f(x)$

$y = f^{-1}(x)$

d) (3 points) Show that $(f \circ f^{-1})(x) = x$.

$$f(f^{-1}(x)) = f(x-4)$$
$$= (x-4) + 4 = x \quad \checkmark$$

e) (3 points) Show that $(f^{-1} \circ f)(x) = x$.

$$f^{-1}(f(x)) = f^{-1}(x+4)$$
$$= (x+4) - 4$$
$$= x \quad \checkmark$$