Math 21: Spring 2013
Final Exam

NAME: SOLUTIONS

LECTURE:

Time: **3 hours**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

I understand and accept the provisions of the Stanford Honor Code.

Signature: ______________________________

Problem	Value	Score
1	7	
2	4	
3	9	
4	15	
5	12	
6	12	
7	16	
8	10	
9	15	
TOTAL	100	

May your trails be crooked, winding, lonesome, dangerous, leading to the most amazing view.
– Edward Abbey

Problem 1 : (7 points)

a) (5 points) Write down the limit definition of the sum of the series

$$\sum_{n=0}^{\infty} a_n.$$

This series is defined to be

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n$$

b) (2 points) Explain in **one** concise sentence what the following equality means:

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \frac{1}{2}$$

The limit of the partial sums $\frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^N} = \frac{1-(1/3)^{N+1}}{1-1/3}$ converges as a limit to $\frac{1}{2}$ as N goes to infinity.

<u>OR</u> as we add more and more terms the sum gets closer and closer to $\frac{1}{2}$

Problem 2 : (4 points) Compute the sum of the following series, if it exists:

$$\frac{\pi^0}{1} - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \cdots$$

$0!$ $2!$ $4!$ $6!$

$$= \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n)!} (-1)^n = \cos \pi = -1$$

Problem 3 : (9 points) For each of the following statement, decide if it is TRUE or FALSE. You do not need to show your work.

a) If f is a solution of the differential equation

$$y' + \sin y = 0,$$

then $2f$ is also a solution of this differential equation.

False $2y' + \sin 2y \neq 0$ necessarily

Since this is not linear and homogeneous

b) Every solution of the differential equation

$$y''' - 5y'' + 9y' - 5y = 0$$

is of the form

$$y = C_1 e^{2x} \cos x + C_2 e^x$$

for some choice of the constants C_1 and C_2.

False 3rd order should have 3-dimensional answer

c) If f and g are solutions of the differential equation

$$y'' + 3y'' - 2y' + 5y = e^x,$$

then $f + g$ is also a solution of this differential equation.

False would get RHS $= 2e^x$ not e^x

since this equation is not homogeneous

Problem 4 : (15 points) Decide whether the following series converge or diverge.

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{1}{\ln(n+4)}$

Converges by Alternating Series Test

- Series is alternating, i.e.
  $b_n = \frac{1}{\ln(n+4)} \geq 0$
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln(n+4)} = 0$
- Decreasing $\frac{1}{\ln(n+5)} < \frac{1}{\ln(n+4)}$

b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{n^2}{n^2+1}$

Diverges by test for divergence.

$\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{1}{1+\frac{1}{n^2}}$ DNE

(oscillates btwn close to 1, -1 for n large)

c) $\sum_{n=0}^{\infty} \frac{n}{2^n}$

Converges by Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{2} \right|$$

$$= \lim_{n\to\infty} \frac{1}{2} \frac{1+\frac{1}{n}}{1} = \frac{1}{2} < 1$$

Problem 5 : (12 points) Solve the following differential equations.

a) $y' - e^x y = e^x$.

This is first order linear with $P(x) = -e^x$

Use integrating factor $I(x) = e^{\int -e^x dx} = e^{-e^x}$

$(e^{-e^x} y)' = e^{-e^x} e^x$

$e^{-e^x} y + C_1 = \int e^{-e^x} e^x \, dx \quad u = e^x \quad du = e^x dx$

$= \int e^{-u} du$

$= -e^{-u} + C_2 = e^{e^x} + C_2$

$e^{-e^x} y = -e^{-e^x} + C_3$

$y = -1 + \frac{C_3}{e^{-e^x}} = -1 + C_3 e^{e^x}$

Alternatively,

$y' = e^x + e^x y = (1+y)e^x$

$\int \frac{1}{1+y} dy = \int e^x dx$

$\ln|1+y| + C_1 = e^x + C_2$

$\ln|1+y| = e^x + C$

$|1+y| = e^{e^x + C}$

$1+y = De^{e^x}$

$y = -1 + De^{e^x}$

b) $\dfrac{dy}{dx} = \dfrac{x^2+3x+2}{2y}, \qquad y > 0, \quad x > 0.$

$\int 2y \, dy = \int (x^2+3x+2) dx$

$y^2 + C_1 = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C_2$

$y^2 = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$

$y = \pm \sqrt{\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C}$

Since $y > 0$ take positive root

$y = \sqrt{\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C}$

Problem 6 : (12 points) In this problem we will approximate the numbers

$$e = 2.71828182845904523536028747135266249775724709369995957496696 7\ldots$$

and

$$e^2 = 7.38905609893065022723042746057500781318031557055184732408 71\ldots$$

You may use the following values to help with computations:

$$\frac{1}{3} \approx 0.3333, \qquad \frac{2}{3} \approx 0.6666, \qquad \frac{1}{6} \approx 0.1666, \qquad \frac{8}{3} \approx 2.6666$$

$$\left(\frac{1}{6}\right)^2 \approx 0.0277, \qquad \left(\frac{8}{3}\right)^2 \approx 7.1111, \qquad \frac{8}{6} = \frac{4}{3} \approx 1.3333, \qquad \frac{19}{3} \approx 6.3333$$

a) (2 points) Write down the Maclaurin series (aka the Taylor series centered at $a = 0$) of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = e^x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

b) (2 points) Write down the Taylor polynomial of degree 3 of f.

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

c) (2 points) Use the Taylor polynomial of degree 3 of f to approximate the number e.

$$e = e^1 \approx T_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} \approx 2.6666$$

d) (2 points) Square your answer to part c) to obtain an approximation for the number e^2.

$$e^2 = (e)^2 \approx \left(2 + \tfrac{1}{2} + \tfrac{1}{6}\right)^2 = \left(\tfrac{16}{6}\right)^2 = \left(\tfrac{8}{3}\right)^2 = \tfrac{64}{9} \approx 7.1111$$

e) (2 points) Use the Taylor polynomial of degree 3 of f to approximate e^2.

$$e^2 \approx T_3(2) = 1 + 2 + \frac{4}{2} + \frac{8}{6} = 6 + \frac{1}{3} = \frac{19}{3} \approx 6.3333$$

f) (2 points) Which approximation is closer to the true value of e^2?

The one in (d).

Problem 7 : (16 points) Consider the differential equation

$$y''' - y'' + 4y' - 4y = 0.$$

a) (10 points) Circle all of the functions which satisfy this differential equation. Show your work. You may assume that the five functions below are linearly independent (because they are).

i. $f_1 \colon \mathbb{R} \to \mathbb{R}$, $f_1(x) = e^x$

~~ii. $f_2 \colon \mathbb{R} \to \mathbb{R}$, $f_2(x) = e^{-x}$~~

iii. $f_3 \colon \mathbb{R} \to \mathbb{R}$, $f_3(x) = \sin(2x)$

iv. $f_4 \colon \mathbb{R} \to \mathbb{R}$, $f_4(x) = \cos(2x)$

v. $f_5 \colon \mathbb{R} \to \mathbb{R}$, $f_5(x) = e^{2x}\sin x$

$f_1 = e^x$

$f_1' = e^x$

$\vdots$

$= f_1''' = e^x$

$e^x - e^x + 4e^x - 4e^x = 0$ ✓

$f_4 = \cos 2x$

$f_4' = -2\sin 2x$

$f_4'' = -4\cos 2x$

$f_4''' = 8\sin 2x$

$8\sin 2x + 4\cos 2x - 8\sin 2x - 4\cos 2x = 0$ ✓

$f_2 = e^{-x}$

$f_2' = -e^{-x}$

$f_2'' = e^{-x}$

$f_2''' = -e^{-x}$

$e^x + e^x + 4e^x + 4e^x \neq 0$

f_5 Don't have to check - can only have 3 families of solutions for 3rd order D.E.

$f_3 = \sin(2x)$

$f_3' = 2\cos(2x)$

$f_3'' = 4\sin(2x)$

$f_3''' = 8\cos(2x)$

$-8\cancel{\cos}(2x) + 4\sin(2x) + 8\cancel{\cos} 2x - 4\sin(2x) = 0$

b) (6 points) Using your work from part a), write down the general solution to the differential equation

$$y''' - y'' + 4y' - 4y = 0.$$

$$C_1 e^x + C_2 \cos 2x + C_3 \sin 2x$$

Problem 8 : (10 points) Use a power series to solve the following differential equation:

$$y'' + y = 0.$$

Hint: The solution is made up from functions you know. If you recognize them, you can check your work.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n$$ Reindex so same n value has same x^n degree.

$$y'' + y = \sum_{n=0}^{\infty} \left(a_n + a_{n+2}(n+2)(n+1)\right) x^n = 0$$

so $a_n + a_{n+2}(n+2)(n+1) = 0$

so $a_{n+2} = \frac{-1}{(n+2)(n+1)}$

If a_0, a_1 fixed $\quad a_2 = a_0 \frac{-1}{1\cdot 2} \quad a_4 = a_0 \cdot \frac{1}{1\cdot 2\cdot 3\cdot 4} \quad a_{2n} = a_0 \frac{(-1)^n}{(2n)!}$

$a_3 = a_1 \frac{-1}{2\cdot 3} \quad a_5 = a_0 \cdot \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} \quad a_{2n+1} = a_1 \frac{(-1)^n}{(2n+1)!}$

$$y = \sum_{n=0}^{\infty} a_0 \frac{(-1)^n}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} a_1 \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= a_0 \cos x + a_1 \sin x$$

Problem 9 : (15 points) Consider the following initial value problem:

$$(-16x^2 + x)y'' + (-32x + 1)y' - 4y = 0, \qquad y(0) = 1.$$

Throughout, suppose that this initial value problem has a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The rest of the page is intentionally left blank for work on the differential equation.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} a_{n+1}(n+1)n x^{n-1}$$

Plug these in for y, y', y'' so get x^n at n term

$$\sum_{n=0}^{\infty} \left(-16a_n n(n-1) + a_{n+1}(n+1)n + -32a_n n + a_{n+1}(n+1) - 4a_n \right) x^n = 0$$

$$-16a_n n(n-1) + a_{n+1}(n+1)n + -32a_n n + a_{n+1}(n+1) - 4a_n = 0$$

$$a_{n+1}(n+1)^2 = a_n\left(16n^2 - 16n + 32n + 4\right)$$

$$a_{n+1}(n+1)^2 = a_n\left(16n^2 + 16n + 4\right)$$

$$= a_n(4n+2)^2$$

$$a_{n+1} = a_n \frac{(4n+2)^2}{(n+1)^2}$$

a) i. (2 points) What is a_0?

$$a_0 = y(0) = 1$$

ii. (2 points) What is a_1?

$$a_1 = \frac{(4\cdot 0+2)^2}{(0+1)^2}\cdot a_0 = 4a_0 = 4$$

iii. (2 points) What is a_2?

$$a_2 = \frac{(4\cdot 1+2)^2}{(1+1)^2} = 9a_1 = 36$$

iv. (2 points) What is a_3?

$$a_3 = \frac{(4\cdot 2+2)^2}{(2+1)^2}\, a_2 = \frac{100}{9}\cdot 36 = 400$$

b) (2 points) Circle the expression that gives a formula for a_n:

i. $\dfrac{(n+1)!}{(n!)^2((n-1)!)^2}$

ii. $\dfrac{(2n)!}{n!}$

iii. $\dfrac{((2n)!)^2}{(n!)^4}$

iv. $\dfrac{((2n+1)!)^2}{(n!)^4}$

Can just look for pattern 1, 4, 36, 100

i $n=0$

$a_0 = \dfrac{1!}{0!((-1)!)^2} = ?$

$a_1 = \dfrac{2!}{1!^2\, 0!^2} = 2 \neq 4$

no

iv $a_0 = \dfrac{1!^2}{0!} = 1$

$a_1 = \dfrac{3!^2}{1!^4} = 36 \neq 4$

no

ii $a_0 = \dfrac{0!}{0!} = 1$

$a_1 = \dfrac{2!}{1!} = 2$ no

iii $a_0 = \dfrac{0!^2}{0!^4} = 1$

$a_1 = \dfrac{2!^2}{1!^4} = 4$

$a_2 = \dfrac{(4!)^2}{(2!)^4} = 36$

$a_3 =$

Can explicitly compute

$a_0 = 1$

$a_{n+1} = \dfrac{(4n+2)^2}{(n+1)^2} a_n$

$= \dfrac{2^2(2n+1)^2}{(n+1)^2} a_n$

$= \dfrac{(2n+2)^2(2n+1)^2}{(n+1)^4} a_n$

so if $a_n = \dfrac{(2n)!^2}{n!^4}$

$a_{n+1} = \dfrac{(2(n+1))!^2}{(n+1)!^4}$ so the pattern fits by induction

c) (5 points) Using your answer from part b) to give you a formula for a_n, compute the radius of convergence of the series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

$$y = \sum_{n=0}^{\infty} \frac{(2n!)^2}{n!^4} x^n$$

Use Ratio Test :

$$\lim_{n\to\infty} \left| \frac{\frac{((2(n+1))!)^2}{(n+1)!^4} x^{n+1}}{\frac{((2n)!)^2}{n!^4} x^n} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)^2 (2n+1)^2}{(n+1)^4} x \right|$$

$$= \lim_{n\to\infty} |x| 16 \cdot \frac{(1+\frac{1}{n})^2 (1+\frac{1}{2n})^2}{(1+\frac{1}{n})^4} = 16|x|$$

$16|x| < 1$

So radius of conv. $\frac{1}{16}$