Math 21: Spring 2013
Midterm 2

NAME: SOLUTIONS

LECTURE:

Time: **75 minutes**

This is a closed book and closed notes exam. Calculators and any electronic aid are not allowed.

For each problem, you should write down all of your work carefully and legibly to receive full credit. When asked to justify your answer, you should use theorems and/or mathematical reasoning to support your answer, as appropriate.

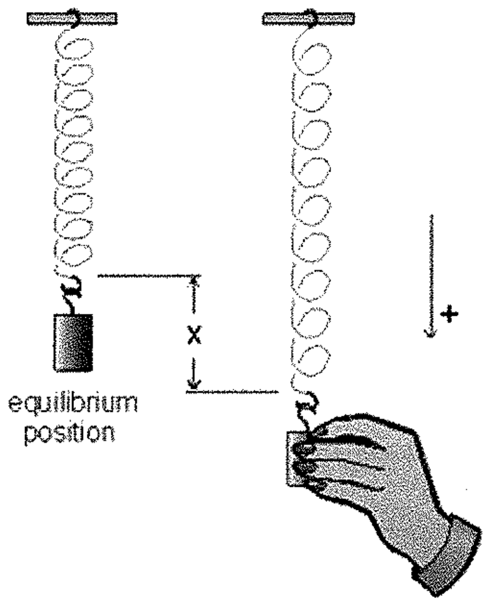
I understand and accept the provisions of the Stanford Honor Code.

Signature: ______________________________

Problem	Value	Score
1	10	
2	15	
3	18	
4	10	
5	25	
6	22	
TOTAL	100	

Be of good cheer. [...] You have set yourselves a difficult task, but you will succeed if you persevere; and you will find a joy in overcoming obstacles. Remember, no effort that we make to attain something beautiful is ever lost. – Helen Keller

Problem 1 : (10 points) Consider the following situation: A small weight is attached to the end of a spring which is hanging from the ceiling. When the weight is left alone, it pulls the spring a certain amount and settles into an equilibrium position.



When one pulls on the weight, the equilibrium is broken and the weight starts oscillating up and down. As in the picture above, let t denote time and x denote the displacement of the weight from its equilibrium position. Then x is a function of t. The motion of the weight can be described in the following manner:

The **acceleration** of the weight is **proportional** to the **negative** of the **displacement** of the weight.

Write down a differential equation that describes the motion of the weight.

X = displacement, X'' = acceleration

So $X'' = -KX$ for some constant $K > 0$.

Problem 2 : (15 points) Consider the differential equation

$$y''' + y'' - 2y = 0.$$

Of the following functions, circle **all** of the ones that satisfy the differential equation. Show your work.

a) $a\colon \mathbb{R} \to \mathbb{R}$ given by $a(x) = e^x$.

b) $b\colon \mathbb{R} \to \mathbb{R}$ given by $b(x) = e^{-x}$.

c) $c\colon \mathbb{R} \to \mathbb{R}$ given by $c(x) = e^{-x}\cos(x)$.

$a''' = a'' = a' = a = e^x$

So $a''' + a'' - 2a = e^x + e^x - 2e^x = 0.$

(b) $b'''(x) = b'(x) = -e^{-x}$

$b''(x) = b(x) = e^{x}$

So $b''' + b'' - 2b = -e^{-x} + e^{x} - 2e^{-x}$

$= -2e^{-x} \neq 0.$

(c) $c'(x) = -e^{-x}\sin(x) - e^{-x}\cos(x) = -e^{-x}\sin(x) - c(x)$

$c''(x) = -e^{-x}\cos(x) + e^{-x}\sin(x) - c'(x)$

$= -e^{-x}\cos(x) + e^{-x}\sin(x) + e^{-x}\sin(x) + e^{-x}\cos(x)$

$= 2e^{-x}\sin(x)$

$c'''(x) = 2e^{-x}\cos(x) - 2e^{-x}\sin(x)$

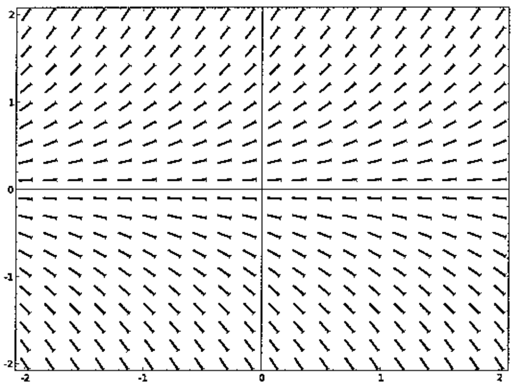
So $c''' + c'' - 2c = 2e^{-x}\cos(x) - 2e^{-x}\sin(x)$

$+ 2e^{-x}\sin(x)$

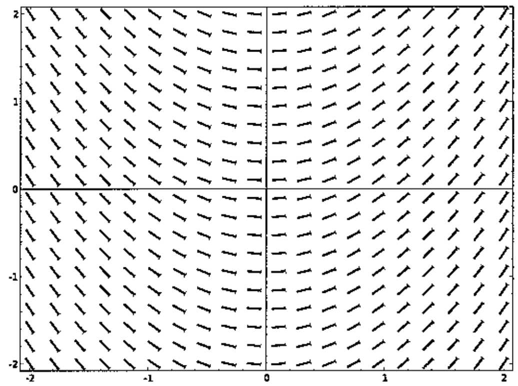
$- 2e^{-x}\cos(x)$

$= 0.$

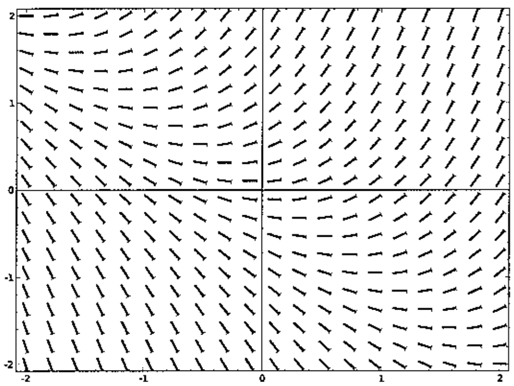
Problem 3 : (18 points) Match the direction fields below with their differential equations. Also indicate which two equations do not have matches.



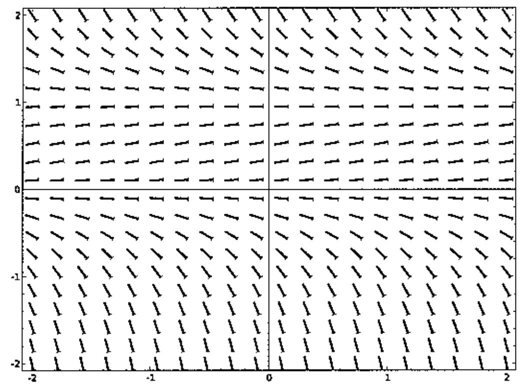
I.



III.



II.



IV.

Equation	I, II, III, IV, V, VI, or "none"	Equation	I, II, III, IV, V, VI, or "none"
$y' = \sin(x+y)$	None	$y' = x$	III
$y' = y$	I	$y' = x + y$	II
$y' = y(1-y)$	IV	$y' = y - x$	None

Problem 4 : (10 points) Consider the initial value problem

$$y' = x + y, \quad y(0) = 1$$

Use Euler's method with step size $h = 1$ to estimate $y(2)$, where y is the solution to the initial value problem above.

Let $y_0 = y(0) = 1$ and $y_{n+1} = y_n + \underset{\substack{\uparrow \\ \text{step size}}}{h} \cdot \underset{\substack{\uparrow \\ y'}}{F(x_n, y_n)}$

$x_0 = 0$

For us $h = 1$ and $F(x, y) = x + y$.

So $y(2) \approx y_2$

$$y_1 = y_0 + 1 \cdot F(x_0, y_0) = 1 + 1 \cdot (0 + 1)$$
$$= 2$$

$$y_2 = y_1 + 1 \cdot F(x_1, y_1)$$
$$= y_1 + 1(x_1 + y_1)$$
$$= 2 + 1(1 + 2) = 5.$$

Problem 5 : (25 points) Solve the following differential equations and initial value problems.

a) (8 points) $xy' + y = x, \qquad x > 0$

Note $\frac{d}{dx}(xy) = x \cdot y' + y$

So $xy' + y = x \Rightarrow \frac{d}{dx}(xy) = x$

$\Rightarrow xy = \frac{1}{2}x^2 + C$

$\Rightarrow y = \frac{1}{2}x + \frac{C}{x}$. since $x>0$.

b) (8 points) $y' + e^{x+y} = 0$

$\frac{dy}{dx} = -e^x \cdot e^y$

$\Rightarrow \frac{dy}{e^y} = -e^x dx$

$\Rightarrow e^{-y} dy = -e^x dx$

$\Rightarrow -e^{-y} = -e^x + C$

$\Rightarrow e^{-y} = e^x - C$ (need >0)

$\Rightarrow -y = \ln(e^x - c)$

$\Rightarrow y = -\ln(e^x - c)$.

c) (9 points) $y' + 2y = 5e^x, \quad y(0) = \dfrac{2}{3}$

Introduce the integration factor of

$$e^{\int 2\,dx} = e^{2x}$$

$$\Rightarrow \quad e^{2x}y' + 2e^{2x}y = 5e^{3x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(e^{2x}y\right) = 5e^{3x}$$

$$\Rightarrow \quad e^{2x}y = \frac{5}{3}e^{3x} + C$$

$$\Rightarrow \quad y = \frac{5}{3}e^{x} + Ce^{-2x}$$

Now $\quad y(0) = \dfrac{5}{3} + C \overset{\text{need}}{=\!=} \dfrac{2}{3}$

$$\Longrightarrow \quad C = -1.$$

So $\quad y(x) = \dfrac{5}{3}e^{x} - e^{-2x}.$

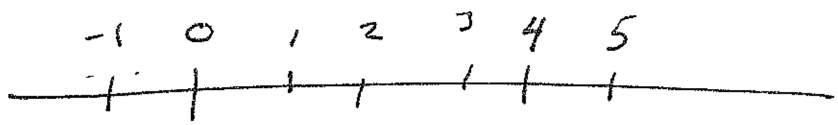
Problem 6 : (22 points) Consider the differential equation

$$\frac{dP}{dt} = -P\left(1 - \frac{P}{4}\right),$$

for a function $P\colon [0,\infty) \to [0,\infty)$.

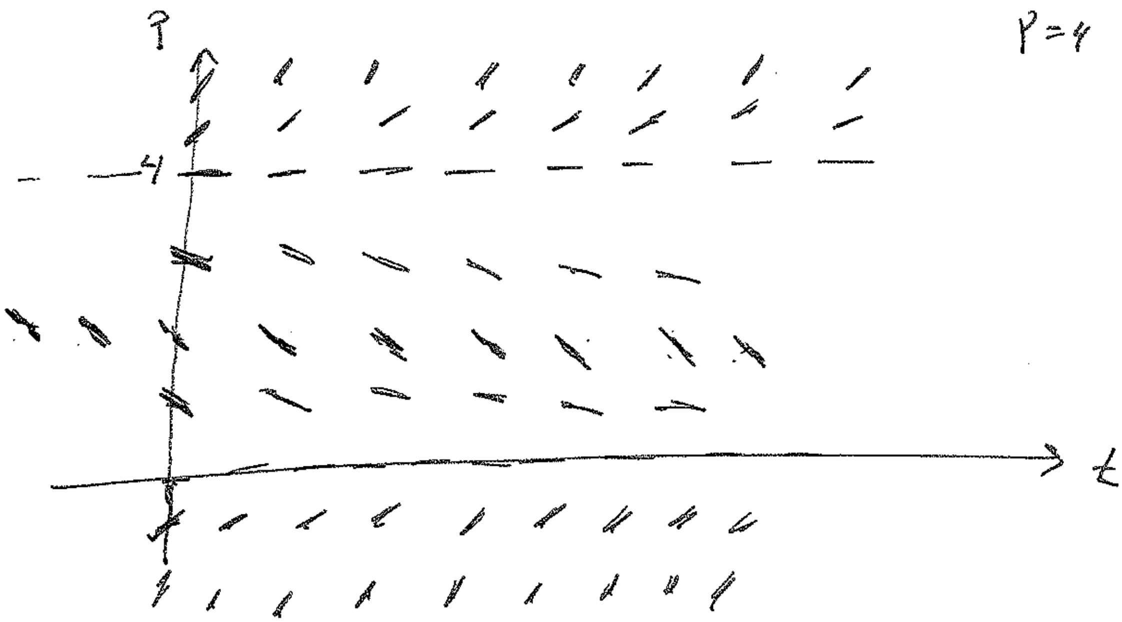
a) (5 points) Find all critical points of the differential equation, and classify them as stable, unstable, or semi-stable.

The critical points are all points with $P = 0$ or $P = 4$.



P' $\quad +5/4 \quad 0 \quad -3/4 \quad -1 \quad -3/4 \quad 0 \quad +5/4$

So the direction field is roughly



$P = 0$ is a stable critical pt.

$P = 4$ is an unstable critical pt.

b) (10 points) Give a general solution for the differential equation

$$\frac{dP}{dt} = -P\left(1 - \frac{P}{4}\right), \quad P \geq 0, \quad t \geq 0$$

Seperate and use continued fractions

$$\frac{1}{P\left(1-\frac{P}{4}\right)} = \frac{1}{P} + \frac{1}{4-P}$$

So $$\frac{dP}{P\left(1-\frac{P}{4}\right)} = \frac{dP}{P} + \frac{dP}{4-P} = -dt$$

$$\Rightarrow \int \frac{dP}{P} + \int \frac{dP}{4-P} = \int -dt = -t + C$$ for some constant C.

$$\Rightarrow \ln|P| - \ln|4-P| = -t + c$$

$$\Rightarrow \ln\left|\frac{P}{4-P}\right| = -t + c$$

$$\Rightarrow \left|\frac{P}{4-P}\right| = e^{-t+c}$$

If $0 < P < 4$ then $$\frac{P}{4-P} = e^{-t+c}$$

$$\Rightarrow P = 4e^{-t+c} - Pe^{-t+c}$$

$$\Rightarrow P = \frac{4e^{-t+c}}{1+e^{-t+c}} = \frac{4}{1+e^{t-c}}$$

If $P > 4$ then $$\frac{P}{P-4} = e^{-t+c} \Rightarrow P = \frac{-4e^{-t+c}}{1-e^{-t+c}}$$

$$= \frac{4}{1-e^{t-c}}$$

If $P = 4$ at $t = 0$, then $P(t) = 0$ for all $t \geq 0$.

c) (7 points) Sketch, **and label**, on one graph, the 3 solutions of the differential equation

$$\frac{dP}{dt} = -P\left(1 - \frac{P}{4}\right), \quad P \geq 0, \quad t \geq 0$$

that have the following initial values:

i. $P(0) = 1$

ii. $P(0) = 3$

iii. $P(0) = 5$