Math 20, Spring 2016 — Schaeffer
Midterm Exam 2 (May 18th, 2016)

Last/Family Name	First/Given Name

Failure to follow the instructions below will constitute a breach of the Stanford Honor Code:

- *You may not use a calculator or any notes or book during the exam.*
- *You may not access your cell phone or any other electronics during the exam for any reason.*
- *You may not sit directly adjacent to any other student.*
- *You are bound by the Stanford Honor Code, which stipulates among other things that you may not communicate with anyone other than the instructor or his designee(s) during the exam, or look at anyone else's solutions.*
- **Additionally, you may not discuss or communicate directly or indirectly the contents of this exam with ANY other students until noon today.**

I understand and accept these instructions, and I certify by my signature below that I have followed them.

Signature: ____

Remember to show your work and justify your answer if required (additional tips are on the next page). Present all solutions in as organized a manner as possible.

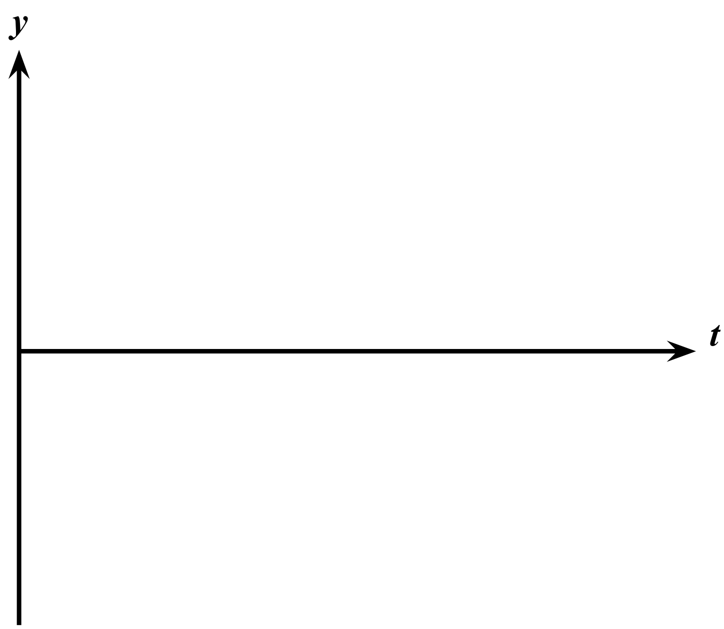
GOOD LUCK!

Problem		**Grade**	**Problem**		**Grade**
1.			6.		
2.			7.		
3.			8.		
4.			9.		
5.			**Total**	—	

Here are some tips:

- If you have time, it's always a good idea to *check your work*.
- If you get the wrong answer for an integral but *show your work*, chances are good that we can award you partial credit.
- *DO NOT attempt to estimate any of your answers as decimals.* For example, $1 - \frac{1}{\pi}$ is a much better answer than 0.682, because it is *exact*.
- The boxes at the end of each topic are for grading purposes only. *Do not touch or look at these boxes.* Pretend they are not there.
- The last page of the exam is blank, and can be used for extra work. If you think it would help for us to look at this work, you should indicate that CLEARLY on the problem's page.

Blank ty-plane below is provided for optional work for Problem 4.



1. Clover, our dog friend, has finally completed building his rocket ship, and will now return to his home planet. His ship's acceleration is given by

$$a(t) = 10000e^{-2t} \quad \text{(in miles per second per second)}$$

   a. Assuming Clover's initial position and initial velocity are both zero, solve for $p(t)$, the function that gives Clover's position (in miles) t seconds after lift-off.
   Draw a box around your final answer.
   We have
   $$v(t) = \int a(t)\,dt = \int 10000e^{-2t}\,dt = -5000e^{-2t} + C_1$$
   Plugging in $t = 0$ above and using $v(0) = 0$, we see that $C_1 = 5000$. Next,
   $$p(t) = \int v(t)\,dt = \int (-5000e^{-2t} + 5000)\,dt = 2500e^{-2t} + 5000t + C_2$$
   Plugging in $t = 0$ above and using $p(0) = 0$, we see that $C_2 = -2500$. So:
   $$p(t) = 2500e^{-2t} + 5000t - 2500$$
   is the final answer.

   b. If Clover's home planet is 7 million miles away from Earth, briefly explain how you would figure out when he arrives. (Do not attempt to solve this problem.)
   Solve $p(t) = 7000000$ for t. If you're curious, the answer is $\approx$ 1400.5 seconds. So Clover will arrive in 23 minutes. Better than my commute! :P

2. Solve the differential equation $y' = \dfrac{\sin t}{e^y}$ for the *general* solution $y(t)$.

   We have
   $$y'(t)e^{y(t)} = \sin t$$
   Integrating both sides by t,
   $$\begin{aligned}\int y'(t)e^{y(t)}\,dt &= \int \sin t\,dt\\ e^{y(t)} &= -\cos t + C\\ y(t) &= \ln(-\cos t + C)\end{aligned}$$
   (where for the left integral we use $u = y(t)$ and $du = y'(t)\,dt$).

3. For each of the following situations, circle the differential equation that we studied that is relevant in that particular situation.

   a. Neptunium-235 decays into Proactinium-231 and Uranium-235 with a half-life of 396.1 days. You want a function $P(t)$ that expresses how much Neptunium-235 is present in a sample after t days.

   $P' = rP$ with $r > 0$ $\qquad$ $P' = rP$ with $r < 0$ $\qquad$ $P' = rP(1 - P/K)$

b. You are depositing money in a savings account that continuously compounds (positive) interest. You want a function $P(t)$ that expresses how much money will be in the account after t years.

$P' = rP$ with $r > 0$ $\qquad$ $P' = rP$ with $r < 0$ $\qquad$ $P' = rP(1 - P/K)$

c. Based on previous research, you know that the population of the cyanobacterium *Synechocystis* is approximately 12 hours. You are growing a colony of *Synechosystis* in a test tube and you want a function $P(t)$ that models the population after t hours.

$P' = rP$ with $r > 0$ $\qquad$ $P' = rP$ with $r < 0$ $\qquad$ $P' = rP(1 - P/K)$

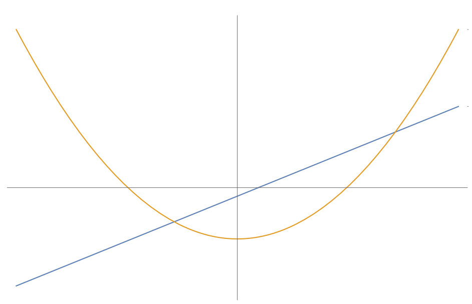
4. Consider the differential equation $y' = (y + 1)(2y - 6)(y - 5)$.

   a. What are the equilibria of this system?
   The equilibria are the y values at which y' is equal to zero. Here, $y = -1$, $y = 3$, and $y = 5$ are the equilibria.

   b. Which of the equilibria you found are stable? (It may help you to draw a diagram on the ty-plane provided on the 2nd page of the exam, but you do not have to for credit.)
   $y' > 0$ when $-1 < y < 3$ and when $y > 5$. On the other hand, $y' < 0$ when $y < -1$ and when $3 < y < 5$. Drawing the direction field shows us that 3 is a stable equilibrium.

   c. If $y(t)$ is a solution to the diff. eq. above, and $y(0) = 1$, then what is $\lim_{t\to\infty} y(t)$?
   Since $y'(0) > 1$, y will grow towards the stable equilibrium at $y = 3$. The limit is 3.

5. Let R be the region bounded above by the line $y = 3x - 2$ and the x-axis, and below by the parabola $x^2 - 12$.



Your friend Mip wants to know the area of this region. Mip has set up the integrals:

$$A = \int_a^b [(3x - 2) - (x^2 - 12)]\,dx + \int_b^c [-(x^2 - 12)]\,dx$$

but hasn't figured out where the endpoints of integration should be. Help Mip out by finding the values for a, b, c. *Draw a box around your final answer(s).*

DO NOT ATTEMPT TO COMPUTE A, JUST FIND THE ENDPOINTS.

The point a is where the parabola and the slanted line intersect. To find this, we set them equal to each other and solve for x:

$$\begin{aligned} x^2 - 12 &= 3x - 2 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

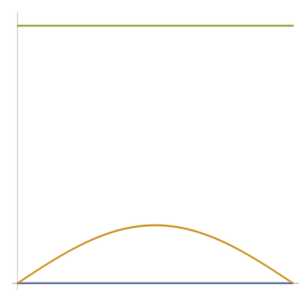
so either $x = -2$ or $x = 5$. Since the intersection point we care about is to the left of the y-axis, $a = -2$.

The point b is where the slanted line and the x-axis intersect. This is where $3x - 2 = 0$, so $b = 2/3$.

The point c is where the parabola and the x-axis intersect. There are two such points, $x = -\sqrt{12}$ and $x = \sqrt{12}$. Since c is to the right of the y-axis, $c = \sqrt{12}$.

6. Let R be the region bounded above by $y = 2\pi$, below by $y = \sqrt{1 - \cos x}$, on the left by $x = 0$, and on the right by $x = 2\pi$.

   If $\int_0^{2\pi} \sqrt{1 - \cos x}\, dx = 4\sqrt{2}$, then what is the area of R?



The area of R is the area of the square minus the bump created by the curve $y = \sqrt{1 - \cos x}$. Thus,

$$\begin{aligned} A(R) &= (2\pi)^2 - \int_0^{2\pi} \sqrt{1 - \cos x}\, dx \\ &= 4\pi^2 - 4\sqrt{2} \end{aligned}$$

7. Let R_1 be the region bounded above by $y = \sqrt{1 + \sin x \cos x}$, below by the x-axis, on the left by $x = 0$, and on the right by $x = \pi$. Let S_1 be the solid obtained by revolving R around the <u>x-axis</u>. <u>Write down a *definite integral* which is equal to the volume of S_1.</u>

   Using the washers formula,

$$V(S_1) = \int_0^{\pi} \pi(\sqrt{1 + \sin x \cos x})^2\, dx = \int_0^{\pi} \pi(1 + \sin x \cos x)\, dx$$

8. Let R_2 be the region bounded above by $y = e^{-x^2}$, below by the x-axis, on the left by $x = 0$, and on the right by $x = 1$. Let S_2 be the solid obtained by revolving R around the $\underline{y\text{-axis}}$. Write down a *definite integral* which is equal to the volume of S_2.

Using the shells formula

$$V(S_2) = \int_0^1 2\pi x e^{-x^2}\, dx$$

9. Find the volume of either S_1 (from Problem 7) or S_2 (from Problem 8). Your choice!

Circle which solid you've chosen: S_1 S_2 , then find its area.

Hint: Both integrals can be evaluated with u-substitution. Neither requires IbP.

Draw a box around your final answer.

For S_1,

$$\begin{aligned}\int_0^\pi \pi(1 + \sin x \cos x)\, dx &= \pi \int_0^\pi dx + \pi \int_0^\pi \sin x \cos x\, dx \\ &= \pi^2 + \pi \left[\frac{\sin^2 x}{2}\right]_0^\pi \\ &= \pi^2 + 0 \\ &= \pi^2\end{aligned}$$

where the for the third integral in the first line, we sub $u = \sin x$ (subbing $y = \cos x$ also works but it flips the sign because then $dy = -\sin x\, dx$).

For S_2, we sub $u = -x^2$ and $du = -2x\, dx$ to get

$$\begin{aligned}\int_0^1 2\pi x e^{-x^2}\, dx &= 2\pi \int_0^1 x e^{-x^2}\, dx \\ &= 2\pi \left[-\frac{e^{-x^2}}{2}\right]_0^1 \\ &= \pi(1 - e^{-1})\end{aligned}$$