

Student Seminar in Algebraic Geometry

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Chapter 1

Lecture 1

1.1 Motivation

Let \mathcal{A} be an abelian category (generally this will either be $R\text{-mod}$ for a commutative ring R or $Q\text{-Coh}_X$ or $\mathcal{O}_X\text{-mod}$ for a scheme X). Loosely speaking, the philosophy of derived categories is the following: Before doing anything to an object, first pass to a projective/injective resolution of the object. Thus the derived category of an object is the “correct” category for doing homological algebra and derived functors on it.

Derived categories are also an algebraic analogy to homotopy-categories for topological spaces.

Example 1.1.1. If \mathcal{F} is a coherent sheaf on a scheme X there exists a length $\dim X$ resolution of \mathcal{F} by locally free sheaves, that is there is an exact sequence

$$0 \rightarrow \mathcal{E}_n \rightarrow \mathcal{E}_{n-1} \rightarrow \cdots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$$

where each \mathcal{E}_i is a locally free sheaf on X . Thus studying coherent sheaves on X is equivalent to studying complexes of locally free sheaves on X . \heartsuit

1.2 Some homological algebra

We here recall some (hopefully) well-known results and definitions from homological algebra which we will need.

The category $\text{Kom}(\mathcal{A})$ has objects cochain complexes, meaning complexes $\cdots \rightarrow A^i \xrightarrow{d^i} A^{i+1} \rightarrow \cdots$ with $d^{i+1} \circ d^i = 0$. A morphism $X \rightarrow Y$ is a collection of maps $f^n : X^n \rightarrow Y^n$ which commute with the differentials.

We have the following list of properties/definitions:

1. $\text{Kom}(\mathcal{A})$ is an abelian category.
2. There exists functors $H^i : \text{Kom}(\mathcal{A}) \rightarrow \mathcal{A}$ given by taking cohomology of an object.
3. If $f : X \rightarrow Y$, we define f to be a quasi-isomorphism if $H^i(f)$ is an isomorphism for all i .

4. There exists shift-functors $[n] : \text{Kom}(\mathcal{A}) \rightarrow \text{Kom}(\mathcal{A})$, we write $X[n]$ for the object $[n](X)$, defined by $X[n]^i = X^{i-n}$ and $d_{X[n]}^i = (-1)^n d_X^{i-n}$.
5. Two morphisms $f, g : X \rightarrow Y$ are defined to be chain-homotopic (we write $f \simeq g$) if there exists a collection of functions $S^n : X^n \rightarrow Y^{n-1}$ such that $f^n - g^n = S^{n+1} \circ d^n + d^{n-1} \circ S^n$.
6. Chain-homotopy is an equivalence relation which respects $+$ and composition.
7. $f \simeq g$ implies that $H^i(f) = H^i(g)$ for all i .
8. $f : X \rightarrow Y$ is defined to be a chain-homotopy-equivalence if there exists a $g : Y \rightarrow X$ such that $f \circ g \simeq \text{id}$ and $g \circ f \simeq \text{id}$.
9. Given $f : X \rightarrow Y$ there exists an object $\text{Cone}(f)$ in $\text{Kom}(\mathcal{A})$ defined as follows: $\text{Cone}(f)^n = X^{n+1} \oplus Y^n$, $d^n = \begin{bmatrix} -d_X^{n+1} & 0 \\ -f^{n+1} & d_Y^n \end{bmatrix}$.
10. There is a short exact sequence $0 \rightarrow Y \rightarrow \text{Cone}(f) \rightarrow X[-1] \rightarrow 0$ given by the inclusion $Y^n \rightarrow \text{Cone}(f)^n$ and the projection $\text{Cone}(f)^n \rightarrow X[-1]^n$.
11. The sequence above induces a long exact sequence in cohomology, and by using $H^i(X[-1]) = H^{i+1}(X)$ we get a long exact sequence

$$\cdots \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H^i(\text{Cone}(f)) \rightarrow H^{i+1}(X) \rightarrow \cdots$$

Then the connecting homomorphism $H^i(X[-1]) = H^{i+1} \rightarrow H^{i+1}(Y)$ is exactly $H^{i+1}(f)$.

1.3 Derived categories

Informally we want the derived category $D(\mathcal{A})$ to be $\text{Kom}(\mathcal{A})$ with quasi-isomorphisms inverted. The way we will do this is by localizing the category at Q , where Q is the set of quasi-isomorphisms. Details of this construction will be done in the next lecture.

The objects of $D(\mathcal{A})$ is defined to be the objects of $\text{Kom}(\mathcal{A})$, while a morphism from $X \rightarrow Y$ is an equivalence class of diagrams of the form

$$X \xleftarrow{f} X_1 \xrightarrow{g} Y$$

where X_1 is some object and f is a quasi-isomorphism. We call this the fraction gf^{-1} .

Two such diagrams $X \xleftarrow{f} X_1 \xrightarrow{g} Y$ and $X \xleftarrow{f'} X_2 \xrightarrow{g'} Y$ are said to be equivalent if there exists a third diagram $X \xleftarrow{f''} X_3 \xrightarrow{g''} Y$ such that the diagram commutes:

$$\begin{array}{ccccc} & & X_2 & & \\ & \swarrow & \uparrow & \searrow & \\ X & \xleftarrow{\quad} & X_3 & \xrightarrow{\quad} & Y \\ & \swarrow & \downarrow & \searrow & \\ & & X_1 & & \end{array}$$

To construct $D(\mathcal{A})$ we will first go to another category $K(\mathcal{A})$ defined as follows: The objects are the objects of $\text{Kom}(\mathcal{A})$ while $\text{Hom}_{K(\mathcal{A})}(X, Y) = \text{Hom}_{\text{Kom}(\mathcal{A})}(X, Y) / \simeq$ (where the equivalence relation is chain-homotopy as defined above). We will check that this is a triangulated category and that it satisfies some desirable properties, before using this to construct $D(\mathcal{A})$.

We will want the functor $Q : \text{Kom}(\mathcal{A}) \rightarrow D(\mathcal{A})$ to satisfy the following universal property: Assume we have a functor $F : \text{Kom}(\mathcal{A}) \rightarrow D$ which sends quasi-isomorphisms to isomorphisms. Then there exists a unique functor $\tilde{F} : D(\mathcal{A}) \rightarrow D$ such that we have an isomorphism of functors $\tilde{F} \circ Q \simeq F$:

$$\begin{array}{ccc} \text{Kom}(\mathcal{A}) & & \\ \downarrow Q & \searrow F & \\ D(\mathcal{A}) & \xrightarrow{\exists! \tilde{F}} & D \end{array}$$

1.4 Triangulated categories

A (pre)triangulated category consists of a pair (K, T) where K is an additive category and T is an automorphism together with a collection of exact triangles of the form $A \rightarrow B \rightarrow C \rightarrow TA$ satisfying the (5) 6 axioms below.

A morphism of triangles is defined as a commutative diagram of the form:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

1. Any morphism is part of an exact triangle
2. $A \xrightarrow{id} A \rightarrow 0 \rightarrow TA$ is exact.
3. Any triangle isomorphic to an exact triangle is exact.
4. Consider the diagram $T^{-1}C \xrightarrow{T^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{Tu} TB$. If (u, v, w) is exact then $(v, w, v - Tu)$ and $(-T^{-1}w, u, v)$ are exact.
5. Given maps $A \rightarrow A'$, $B \rightarrow B'$ such that the diagram commutes:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow & & \downarrow & & & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

Then there exists an (not necessarily unique) arrow $C \rightarrow C'$ such that the diagram commutes:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

6. This is called the octahedron axiom and is a pain to write out and will almost never be used, thus we skip it.

In our setting of $Kom(\mathcal{A})$ or $D(\mathcal{A})$ an exact triangle is any triangle isomorphic to a triangle of the form $X \rightarrow Y \rightarrow \text{Cone}(f) \rightarrow X[-1]$.

Proposition 1.4.1. *$Kom(\mathcal{A})$ is a triangulated category.*

Proof. Axiom 1 follows by construction of the mapping cone.

$\text{Cone}(\text{id}) \simeq 0$, thus we have axiom 2.

Axiom 3 follows by definition of triangles in our category.

Axiom 5 is true since the cone construction is functorial.

Axiom 4 requires a little work:

Assume $X \rightarrow Y \rightarrow \text{Cone}(f) \rightarrow X[-1]$ is an exact triangle. We must show that $Y \xrightarrow{v} \text{Cone}(f) \rightarrow X[-1] \rightarrow Y[-1]$ is an exact triangle. We will construct an explicit isomorphism $X[-1] \simeq \text{Cone}(v)$ giving an isomorphism of triangles:

$$\begin{array}{ccccccc} Y & \longrightarrow & \text{Cone}(f) & \longrightarrow & X[-1] & \longrightarrow & Y[-1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & \text{Cone}(f) & \longrightarrow & \text{Cone}(v) & \longrightarrow & Y[-1] \end{array}$$

We have that $\text{Cone}(v)^n = Y^{n+1} \oplus \text{Cone}(f)^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$ with differential

$$d^n = \begin{bmatrix} -d_Y^{n+1} & 0 \\ -v^{n+1} & d_{\text{Cone}(f)}^n \end{bmatrix} = \begin{bmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ -id_Y & -f^{n+1} & d_Y^n \end{bmatrix}$$

We define maps $X[-1] \xrightarrow{g} \text{Cone}(v) \xrightarrow{h} X[-1]$ by

$$X^{n+1} \xrightarrow{g^n} Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

$$x \mapsto (-f^{n+1}(x), x, 0)$$

$$Y^{n+1} \oplus X^{n+1} \oplus Y^n \xrightarrow{h^n} X^{n+1}$$

$$(y, x, z) \mapsto x$$

Then $h \circ g = id$ and $g \circ h$ are homotopic to the identity via the map

$$S^n : Y^{n+1} \oplus X^{n+1} \oplus Y^n \rightarrow Y^n \oplus X^n \oplus Y^{n+1}$$

$$(y, x, z) \mapsto (z, 0, 0)$$

Thus we are done. {compzero}

Next we will see some useful properties of triangulated categories.

Proposition 1.4.2. *Assume $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is an exact triangle. Then $v \circ u = 0$.*

Proof. Applying axiom 5 to the following diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \uparrow id & & \uparrow u & & \uparrow & & \uparrow \\ X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & TX \end{array}$$

yields a morphism $0 \rightarrow Z$ such that $v \circ u = 0$. {compzero}

Proposition 1.4.3 (5-lemma). Assume we have a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \exists h & & \downarrow \simeq \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \end{array}$$

where the vertical arrows are isomorphisms. Then the morphism h induced by axiom 4 is also an isomorphism.


Proof. We reduce to the case where the vertical arrows are identities:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & TX \\ \downarrow id & & \downarrow id & & \downarrow \exists h & & \downarrow id \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & TX \end{array}$$

We wish to show that h is an isomorphism by showing that it is surjective and injective. To check that h is surjective it suffices to show that for any $f : Z \rightarrow D$ such that $f \circ h = 0$, we have $f = 0$.


From the diagram we have that $h \circ v = v$ which implies that $0 = f \circ h \circ v = f \circ v$. This means we have a commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & TX \\ \downarrow id & & \downarrow id & & \downarrow h & & \downarrow id \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & TX \\ & & \downarrow & & \downarrow f & & \\ & & 0 & \longrightarrow & D & \xrightarrow{id} & D \end{array}$$

By axiom 5 there exists a $g : TX \rightarrow D$ such that $f = g \circ w$. But then $0 = f \circ h = g \circ w \circ h = g \circ w = f$. Thus h is surjective. Checking that h is injective is by a similar argument. 

Definition 1.4.4. A cohomological functor $f : K \rightarrow \mathcal{A}$ from a triangulated category K to an abelian category \mathcal{A} is a functor which takes exact triangles to long exact sequences, that is any exact triangle $X \rightarrow Y \rightarrow Z \rightarrow TX$ induces a long exact sequence

$$\dots \rightarrow FT^i X \rightarrow FT^i Y \rightarrow FT^i Z \rightarrow FT^{i+1} X \rightarrow \dots$$

For short we write $F^i X$ for $FT^i X$. 

Lemma 1.4.5. $\text{Hom}(A, -)$ and $\text{Hom}(-, A)$ are cohomological functors.

Proof. Assume we have an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$. After applying $\text{Hom}(A, -)$ we get maps

$$\dots \rightarrow \text{Hom}(A, X) \xrightarrow{u^*} \text{Hom}(A, Y) \xrightarrow{v^*} \text{Hom}(A, Z) \rightarrow \dots$$

We will check that this is exact at $\text{Hom}(A, Y)$, then the general case follows from the properties of triangulated categories.

Since $v \circ u = 0$ by 1.4.2 we have that $\text{im } u^* \subset \ker v^*$.

Assume now that $f \in \text{Hom}(A, Y)$ such that $v^*(f) = f \circ v = 0$. Thus we have a commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{id} & A & \longrightarrow & 0 & \longrightarrow & TA \\
& & \downarrow f & & \downarrow & & \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & TX
\end{array}$$

By axiom 5 there exists a $g : A \rightarrow X$ such that $u \circ g = u^*(g) = f$.
The proof for $\text{Hom}(-, A)$ is similar.



Bibliography

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