Student Seminar in Algebraic Geometry

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Chapter 1

Lecture 1

1.1 Motivation

Let $\mathcal{A}$ be an abelian category (generally this will either be R-mod for a commutative ring R or $Q - Coh_X$ or $\mathcal{O}_X$-mod for a scheme X). Loosely speaking, the phisosophy of derived categories is the following: Before doing anything to an object, first pass to a projective/injective resolution of the object. Thus the derived category of an object is the "correct" category for doing homological algebra and derived functors on it.

Derived categories are also an algebraic analogy to homotopy-categories for topological spaces.

Example 1.1.1. If $\mathcal{F}$ is a coherent sheaf on a scheme X there exists a length $\dim X$ resolution of $\mathcal{F}$ by locally free sheaves, that is there is an exact sequence

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where each $\mathcal{E}_i$ is a locally free sheaf on X. Thus studying coherent sheaves on X is equivalent to studying complexes of locally free sheaves on X. $\heartsuit$

1.2 Some homological algebra

We here recall some (hopefully) well-known results and definitions from homological algebra which we will need.

The category $\mathrm{Kom}(\mathcal{A})$ has objects cochain complexes, meaning complexes $\cdots \to A^i \xrightarrow{d^i} A^{i+1} \to \cdots$ with $d^{i+1} \circ d^i = 0$. A morphism $X \to Y$ is a collection of maps $f^n : X^n \to Y^n$ which commute with the differentials.

We have the following list of properties/defintions:

1. $\mathrm{Kom}(\mathcal{A})$ is an abelian category.
2. There exists functors $H^i : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}$ given by taking cohomology of an object.
3. If $f : X \to Y$, we define f to be a quasi-isomorphism if $H^i(f)$ is an isomorphism for all i.

4. There exists shift-functors $[n] : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$, we write $X[n]$ for the object $[n](X)$, defined by $X[n]^i = X^{i-n}$ and $d^i_{X[n]} = (-1)^n d^{i-n}_X$.
5. Two morphisms $f, g : X \to Y$ are defined to be chain-homotopic (we write $f \simeq g$) if there exists a collection of functions $s^n : X^n \to Y^{n-1}$ such that $f^n - g^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n$.
6. Chain-homotopy is an equivalence relation on $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}$ which respects $+$ and composition.
7. If $f \simeq g$ then $H^i(f) = H^i(g)$ for all i.
8. $f : X \to Y$ is defined to be a chain-homotopy-equivalence if there exists a $g : Y \to X$ such that $f \circ g \simeq id$ and $g \circ f \simeq id$.
9. Given $f : X \to Y$ there exists an object $\mathrm{Cone}(f)$ in $\mathrm{Kom}(\mathcal{A})$ defined as follows: $\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n$, $d^n = \begin{bmatrix} -d^{n+1}_X & 0 \\ -f^{n+1} & d^n_Y \end{bmatrix}$.
10. There is a short exact sequence $0 \to Y \to \mathrm{Cone}(f) \to X[-1] \to 0$ given by the inclusion $Y^n \to \mathrm{Cone}(f)^n$ and the projection $\mathrm{Cone}(f)^n \to X[-1]^n$.
11. The sequence above induces a long exact sequence in cohomology, and by using $H^i(X[-1]) = H^{i+1}(X)$ we get a long exact sequence

$$\cdots \to H^i(X) \to H^i(Y) \to H^i(\mathrm{Cone}(f)) \to H^{i+1}(X) \to \dots$$

The connecting homomorphism $H^i(X[-1]) = H^{i+1}(X) \to H^{i+1}(Y)$ is exactly $H^{i+1}(f)$.

1.3 Derived categories

Informally we want the derived category $D(\mathcal{A})$ to be $\mathrm{Kom}(\mathcal{A})$ with quasi-isomorphisms inverted. The way we will do this is by localizing the category at Q, where Q is the set of quasi-isomorphisms. Details of this construction will be done in the next lecture.

The objects of $D(\mathcal{A})$ are defined to be the objects of $\mathrm{Kom}(\mathcal{A})$, while a morphim from $X \to Y$ is an equivalence class of diagrams of the form

$$X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

where X_1 is some object and s is a quasi-isomorphism. We call this a (left) fraction fs^{-1}.

Two such diagrams $X \xleftarrow{s} X_1 \xrightarrow{f} Y$ and $X \xleftarrow{s'} X_2 \xrightarrow{f'} Y$ are said to be equivalent if there exists a third fraction $X \xleftarrow{s''} X_3 \xrightarrow{f''} Y$, and maps $X_3 \to X_1$ and $X_3 \to X_2$, such that the diagram commutes:

$$\begin{tikzcd}
& X_2 \arrow[dl] \arrow[dr] & \\
X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\
& X_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

More formally the derived category of $\mathcal{A}$ is a category $D(\mathcal{A})$ and a functor $Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ which maps quasi-isomorphisms to isomorphisms, with the following universal property: Assume we have a functor $F : \mathrm{Kom}(\mathcal{A}) \to D$ which sends quasi-isomorphisms to isomorphisms. Then there exists a unique functor $\widetilde{F} : D(\mathcal{A}) \to D$ such that we get a commutative diagram

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & D \\ D(\mathcal{A}) \arrow[ur, dashed, "\exists! \widetilde{F}"'] & \end{tikzcd}$$

To construct $D(\mathcal{A})$ we will first go to another category $K(\mathcal{A})$ defined as follows: The objects are the objects of $\mathrm{Kom}(\mathcal{A})$ while $\mathrm{Hom}_{K(\mathcal{A})}(X, Y) = \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(X, Y)/\simeq$ (where the equivalence relation is chain-homotopy as defined above). We will check that this is a triangulated category and that it satisfies some desireable properties, before using this to construct $D(\mathcal{A})$.

1.4 Triangulated categories

A (pre)triangulated category consists of a pair (K, T) where K is an additive category and T is an automorphism together with a collection of triangles, called exact triangles or distinguised triangles satisfying the (5) 6 axioms below. A triangle is a triple of maps (u, v, w) in K, where the maps are of the form $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$. The exact triangles is a subclass of the class of triangles, and is part of the defining data of a triangulated category. A morphism of triangles $(u, v, w) \to (u', v', w')$ is defined as a commutative diagram of the form:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & TA \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

1. Any morphism is part of an exact triangle
2. $A \xrightarrow{id} A \to 0 \to TA$ is exact.
3. Any triangle isomorphic to an exact triangle is exact.
4. Consider the diagram $T^{-1}C \xrightarrow{T^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{Tu} TB$. If (u, v, w) is exact then $(v, w, -Tu)$ and $(-T^{-1}w, u, v)$ are exact.
5. Given maps $A \to A'$, $B \to B'$ such that the diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & TA \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

Then there exists an (not necessarily unique) arrow $C \to C'$ such that the diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & TA \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA' \end{tikzcd}$$

6. This axiom is called the octahedron axiom and is a pain to write out and will almost never be used, thus we skip it. Without this axiom the category is called a pre-triangulated category.

In our setting of $Kom(\mathcal{A})$ or $D(\mathcal{A})$ an exact triangle is any triangle isomorphic to a triangle of the form $X \to Y \to \mathrm{Cone}(f) \to X[-1]$.

Proposition 1.4.1. *$K(\mathcal{A})$ is a triangulated category.*

Proof. Axiom 1 follows by construction of the mapping cone.

Cone(id) $\simeq 0$, thus we have axiom 2.

Axiom 3 follows by definition of triangles in our categrory.

Axiom 5 is true since the cone construction is functorial.

Axiom 4 requires a little work: Assume $X \to Y \to \mathrm{Cone}(f) \to X[-1]$ is an exact triangle. We must show that $Y \xrightarrow{v} \mathrm{Cone}(f) \to X[-1] \xrightarrow{-f[-1]} Y[-1]$ is an exact triangle. We will construct an explicit isomorphism $X[-1] \simeq \mathrm{Cone}(v)$ in $K(A)$ giving an isomorphism of triangles:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathrm{Cone}(f) \arrow[r] \arrow[d] & X[-1] \arrow[r] \arrow[d] & Y[-1] \arrow[d] \\
Y' \arrow[r] & \mathrm{Cone}(f) \arrow[r] & \mathrm{Cone}(v) \arrow[r] & Y[-1]
\end{tikzcd}$$

Recall $\mathrm{Cone}(v)^n = Y^{n+1} \oplus \mathrm{Cone}(f)^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$ with differential

$$d^n = \begin{bmatrix} -d_Y^{n+1} & 0 \\ -v^{n+1} & d^n_{\mathrm{Cone}(f)} \end{bmatrix} = \begin{bmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ -id_Y & -f^{n+1} & d_Y^n \end{bmatrix}$$

We define maps $X[-1] \xrightarrow{g} \mathrm{Cone}(v) \xrightarrow{h} X[-1]$ by

$$X^{n+1} \xrightarrow{g^n} Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

$$x \mapsto (-f^{n+1}(x), x, 0)$$

$$Y^{n+1} \oplus X^{n+1} \oplus Y^n \xrightarrow{h^n} X^{n+1}$$

$$(y, x, z) \mapsto x$$

Then $h \circ g = id$ and $g \circ h$ are homotopic to the identity via the map

$$S^n : Y^{n+1} \oplus X^{n+1} \oplus Y^n \to Y^n \oplus X^n \oplus Y^{n+1}$$

$$(y, x, z) \mapsto (z, 0, 0)$$

Thus we are done. The proof of the second requirement of axiom 4 is similar.

Next we will see some useful properties of triangulated categories.

{compzero}

Proposition 1.4.2. *Assume $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is an exact triangle. Then $v \circ u = 0$.*

Proof. Applying axiom 5 to the following diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & TX \\
X \arrow[u, "id"] \arrow[r, "id"] & X \arrow[u, "u"] \arrow[r] & 0 \arrow[r] & TX \arrow[u]
\end{tikzcd}$$

yields a morphism $0 \to Z$, making the diagram commutative, hence $v \circ u = 0$.

Proposition 1.4.3 (5-lemma)**.** *Assume we have a commutative diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"] & Y \arrow[r] \arrow[d, "\cong"] & Z \arrow[r] \arrow[d, dashed, "\exists h"] & TX \arrow[d, "\cong"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX'
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Then the morphism h induced by axiom 4 is also an isomorphism.

Proof. We reduce to the case where the vertical arrows are identities:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "id"] & Y \arrow[r, "v"] \arrow[d, "id"] & Z \arrow[r] \arrow[d, dashed, "\exists h"] & TX \arrow[d, "id"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & TX
\end{tikzcd}$$

We wish to show that h is an isomorphism by showing that it is surjective and injective. To check that h is surjective it suffices to show that for any $f : Z \to D$ such that $f \circ h = 0$, we have $f = 0$ (remember K is additive).

From the diagram we have that $h \circ v = v$ which implies that $0 = f \circ h \circ v = f \circ v$. This means we have a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "id"] & Y \arrow[r, "v"] \arrow[d, "id"] & Z \arrow[r] \arrow[d, "h"] & TX \arrow[d, "id"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d, "f"] & TX \\
 & 0 \arrow[r] & D \arrow[r, "id"] & D
\end{tikzcd}$$

By axiom 5 there exists a $g : TX \to D$ such that $f = g \circ w$. But then $0 = f \circ h = g \circ w \circ h = g \circ w = f$. Thus h is surjective. Checking that h is injective is by a similar argument.

Definition 1.4.4. A cohomological functor $f : K \to \mathcal{A}$ from a triangulated category K to an abelian category $\mathcal{A}$ is a functor which takes exact triangles to long exact sequences, that is any exact triangle $X \to Y \to Z \to TX$ induces a long exact sequence

$$\cdots \to FT^iX \to FT^iY \to FT^iZ \to FT^{i+1}X \to \cdots$$

For short we write F^iX for FT^iX.

Lemma 1.4.5. $\mathrm{Hom}(A,-)$ *and* $\mathrm{Hom}(-,A)$ *are cohomological functors.*

Proof. Assume we have an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$. After applying $\mathrm{Hom}(A,-)$ we get maps

$$\cdots \to \mathrm{Hom}(A,X) \xrightarrow{u^*} \mathrm{Hom}(A,Y) \xrightarrow{v^*} \mathrm{Hom}(A,Z) \to \cdots$$

We will check that this is exact at $\mathrm{Hom}(A,Y)$, then the general case follows by shifting.

Since $v \circ u = 0$ by 1.4.2 we have that $\operatorname{im} u^* \subset \ker v^*$.

Assume now that $f \in \mathrm{Hom}(A,Y)$ such that $v^*(f) = f \circ v = 0$. Thus we have a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "id"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & TA \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & TX
\end{tikzcd}$$

By axiom 5 there exists a $g : A \to X$ such that $u \circ g = u^*(g) = f$, hence $\ker v^* \subset \operatorname{im} u^*$.

The proof for $\mathrm{Hom}(-,A)$ is similar. ∎