Bounding distances to unsafe sets

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IfA Coffee Talks

Northeastern University

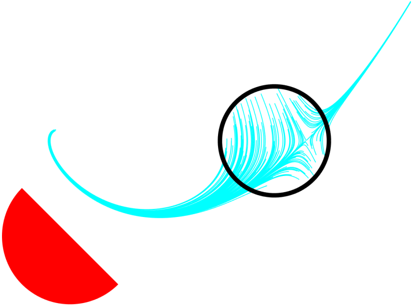
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Relax distance using optimal transport theory

Develop occupation measure programs to bound distance

Flow System Setting



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \le 0.4^2\}$$
$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \le 0.5^2,$$
$$\sqrt{2}/2(x_1 + x_2 - 0.7) \le 0\}$$

Distance Function

Metric space (X, c) satisfying $\forall x, y \in X$:

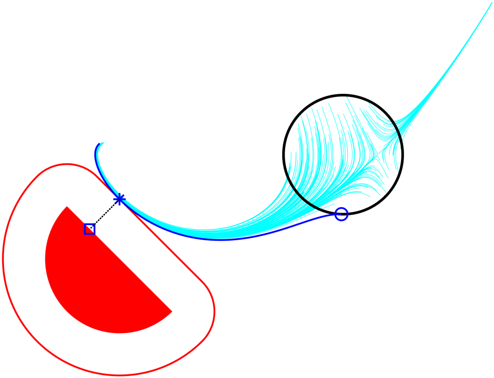
$$\begin{aligned} &c(x,y) > 0 && x \neq y \\ &c(x,x) = 0 && \\ &c(x,y) = c(y,x) && \\ &c(x,y) \leq c(x,z) + c(z,y) && \forall z \in X \end{aligned}$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x, y)$

Distance Estimation Problem (Nonconvex)

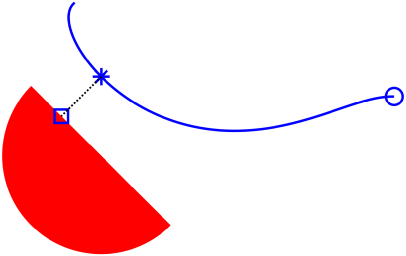
$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*\ x_0^*,\ t_p^*)$:

x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Safety Background

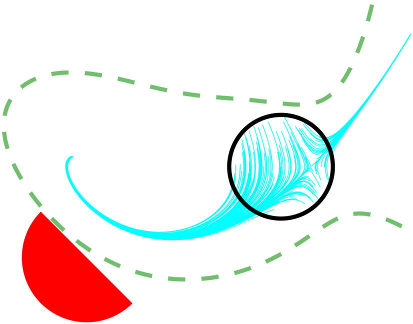
Barrier Program

Barrier function $B : X \to \mathbb{R}$ indicates safety

$$B(x) \leq 0 \qquad \forall x \in X_u$$

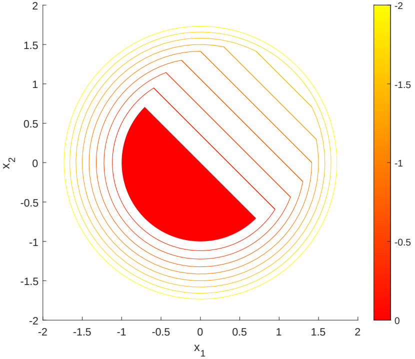
$$B(x) > 0 \qquad \forall x \in X_0$$

$$f(x) \cdot \frac{\partial B}{\partial x}(x) \geq 0 \qquad \forall x \in X$$



Half-circle Contours

Unsafe set $X_u = \{x \mid 1 - x_1^2 - x_2^2 \geq 0, -x_1 - x_2 \geq 0\}$



$q \leq 1 - x_1^2 - x_2^2$ $\qquad$ $q \leq -x_1 - x_2$

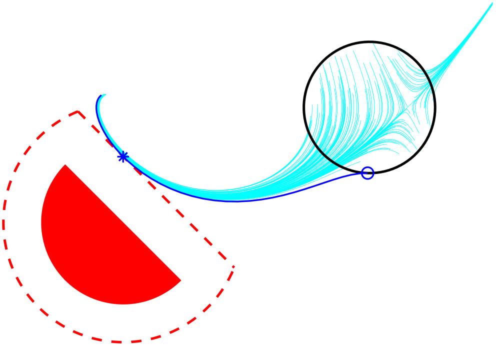
$$q = -0.25, -0.5, \ldots, -2$$

Safety Margin (2020)

Unsafe set $X_u = \{x \mid p_i(x) \geq 0 \; \forall i = 1 \dots N_u\}$

Safety margin $p^* = \max_{t,x_0} \min_i p_i(x(t \mid x_0))$

If $p^* < 0$, no trajectories enter X_u (safe)

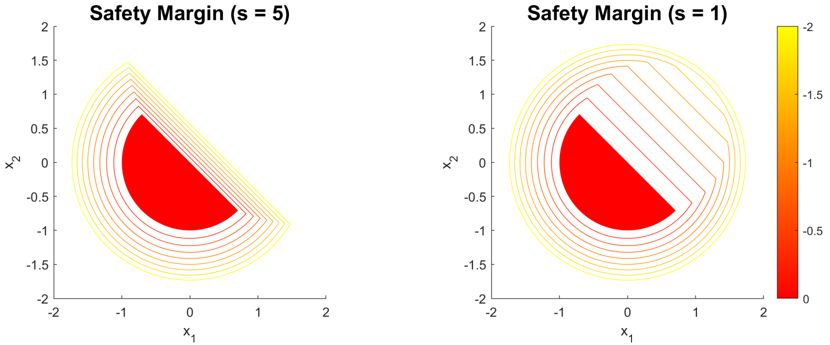


safe: $p^* \leq -0.2831$

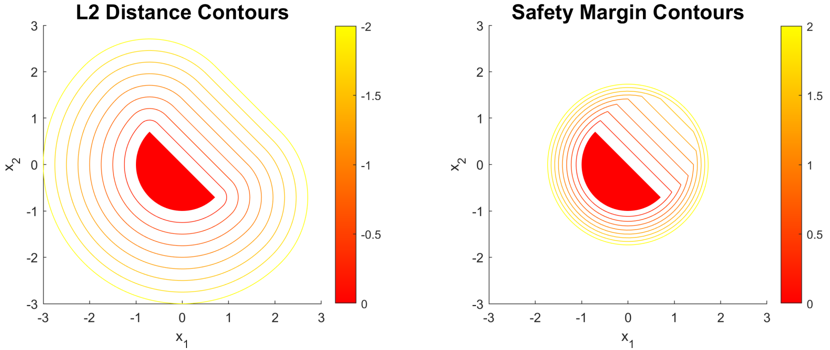
Safety Margin Scaling

Scale factor in constraints

$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq s(-x_1 - x_2)$$



Distance vs. Safety Margin



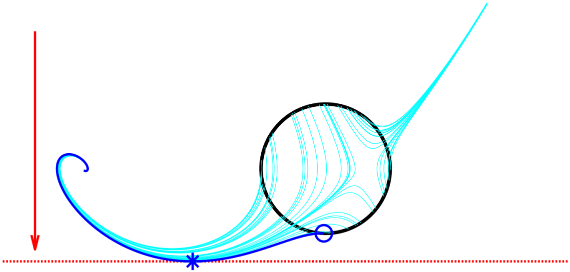
Peak Estimation

Peak Estimation Background

Find minimum value of $p(x)$ along trajectories

$$P^* = \min_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$p(x) = x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Measures

Nonnegative Borel Measure μ

Assigns each set $A \subseteq X$ a 'size' $\mu(A)$ (Measure)

Mass $\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution

$\mu \in \mathcal{M}_+(X)$: space of measures on X

$f \in C(X)$: continuous function on X

Pairing by Lebesgue integration $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$

Dirac Delta Measure

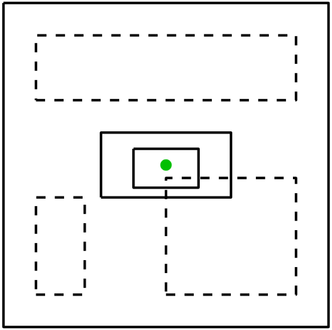
Dirac delta $\delta_{x'}(A) = \begin{cases} 1 & x' \in A \\ 0 & x' \notin A \end{cases}$

Probability measure: $\delta_{x'}(X) = 1$

$\mu(A) = 1$: Solid Box

$\mu(A) = 0$: Dashed Box

$\langle f(x), \delta_{x'} \rangle = f(x')$



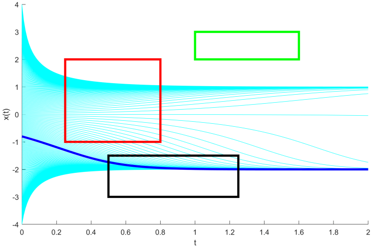
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

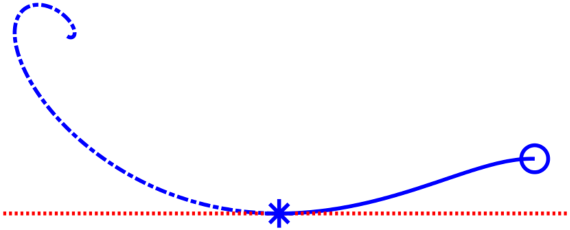
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$\begin{aligned}
\mu_0^* : &\qquad \langle v(0, x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : &\qquad \langle v(t, x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : &\qquad \langle v(t, x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt
\end{aligned}$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

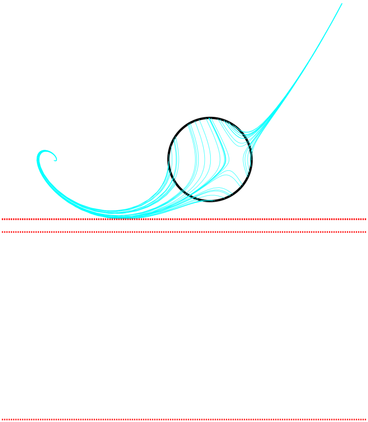
$$
\begin{aligned}
p^* = \ \min \quad & \langle p(x), \mu_p \rangle && \text{(1a)}\\
& \langle 1, \mu_0 \rangle = 1 && \text{(1b)}\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(1d)}\\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



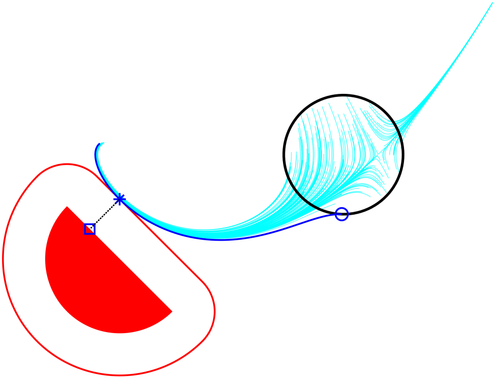
Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Connection to Peak Estimation

Specific form of problem

$$p(x) = c(x; X_u)$$

Moment-SOS hierarchy requires polynomial data

Function $c(x; X_u)$ generally non-polynomial

$$\min_{y \in [-1,1]} \|x - y\|_2 = \begin{cases} 0 & x \in [-1, 1] \\ |x - \mathrm{sign}(x)| & \text{else} \end{cases}$$

Distance Relaxation

Distance in points $\rightarrow$ Expectation of distance

$$\begin{array}{lcl} c(x,y) & & \langle c(x,y), \eta \rangle \\ x \in X & \rightarrow & \langle 1, \eta \rangle = 1 \\ y \in X_u & & \eta \in \mathcal{M}_+(X \times X_u) \end{array}$$

Joint probability measure η

Inspired by Optimal Transport

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\mu_0^* : \qquad \delta_{x=x_0^*}$$

$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$

$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0, T] \times X)$

$$\mu^* : \qquad \langle v(t,x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Distance Program (Measures)

Infinite Dimensional Linear Program (Convergent)

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x, y), \eta \rangle && \text{(2a)} \\
& \langle 1, \mu_0 \rangle = 1 && \text{(2b)} \\
& \langle v(t, x), \mu_p \rangle = \langle v(0, x), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x), \mu \rangle \quad && \forall v \quad \text{(2c)} \\
& \langle w(x), \eta(x, y) \rangle = \langle w(x), \mu_p(t, x) \rangle && \forall w \quad \text{(2d)} \\
& \eta \in \mathcal{M}_+(X \times X_u) && \text{(2e)} \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) && \text{(2f)} \\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(2g)}
\end{aligned}
$$

Prob. Measures: $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1$

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lceil \deg(f)/2 \rceil - 1$

Bounds: $p^*_d \leq p^*_{d+1} \leq \ldots \leq p^* = P^*$

Measure	$\mu_0(x)$	$\mu_p(t,x)$	$\mu(t,x)$	$\eta(x,y)$
PSD Size	$\binom{n+d}{d}$	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$	$\binom{2n+d}{d}$

Timing scales approximately as $\max((1+n)^{6\tilde{d}}, (2n)^{6d})$

Approximation and Recovery

Attempt recovery if LMI solution has low rank

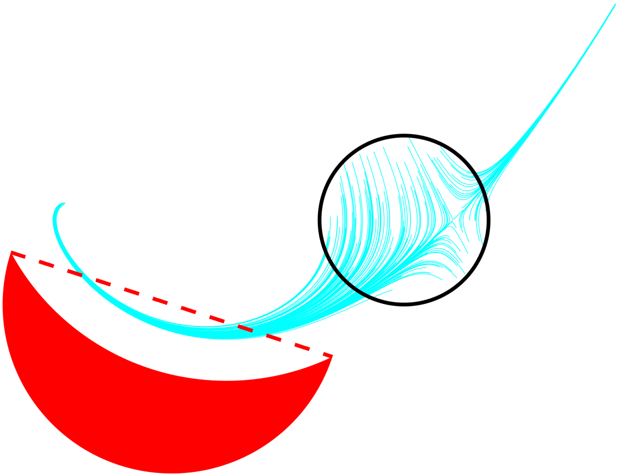
Moment matrices for $(\mu_0, \mu_{\boldsymbol{p}}, \eta)$ are rank-1

Related to optima extraction in polynomial optimization

Moon L2 Contours

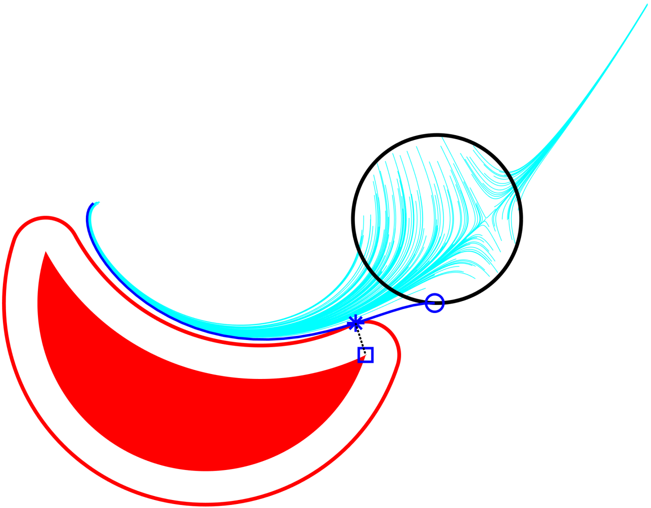
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

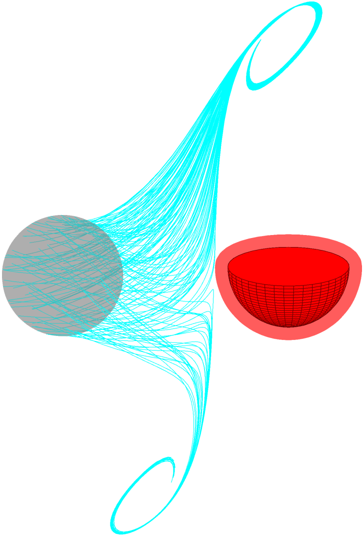
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

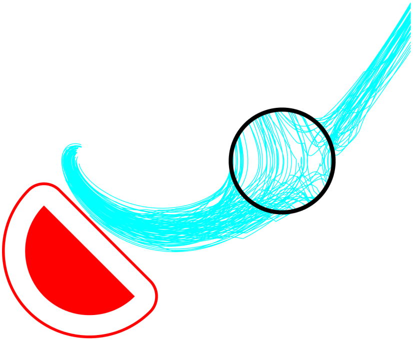
Distance Variations

Distance Uncertainty

Time dependent (bounded) uncertainty $w(t) \in W \;\forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

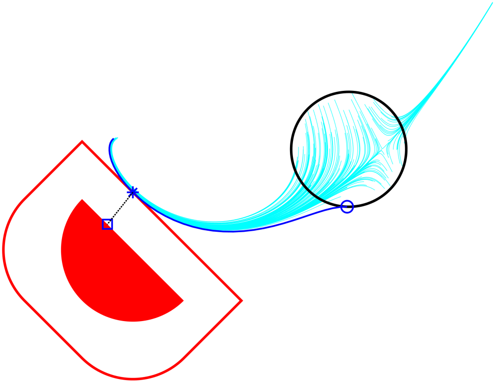
Young measure $\mu(t, x, w)$, Liouville term $\langle \mathcal{L}_f v(t, x, w), \mu \rangle$



L_2 bound of 0.1691, $w(t) \in [-1, 1]$

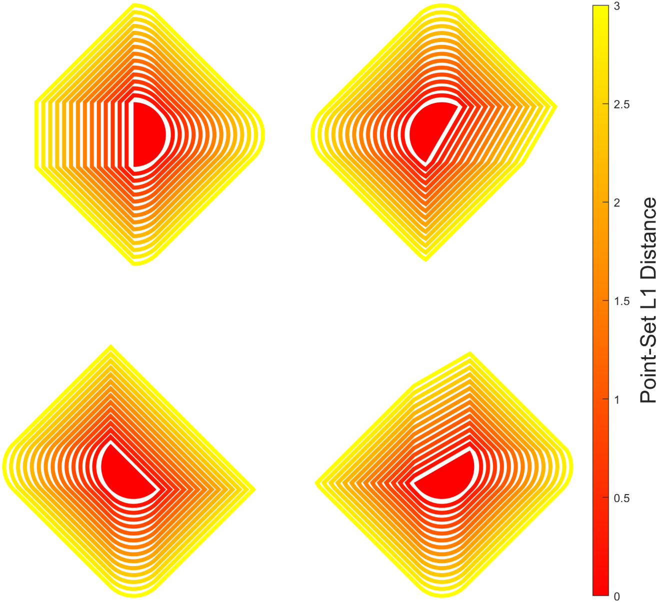
Lifted Distance (L1) Example

$$\|x - y\|_1 = \sum_i |x_i - y_i| \quad \Longrightarrow \quad \begin{aligned} &\min \ \sum_i q_i \\ -q_i \leq \langle x_i &- y_i, \eta \rangle \leq q_i \end{aligned}$$



L_1 bound of 0.4003

Half-Circle L1 Contours



Sparsity

Separable $c(x, y) = \sum_k c_k(x_k, y_k)$

Use correlative sparsity with measures and cliques

$$\eta_k : \quad I_i = (x_k : x_n,\ y_1 : y_k) \qquad \forall k = 1, \ldots, n$$

Reduced computational complexity

	Measure	PSD size	Multiplicity
Dense	η	$\binom{2n+d}{d}$	1
Sparse	$\{\eta_k\}_{k=1}^{n}$	$\binom{n+1+d}{d}$	n

Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

$$A : S \times \Omega \to X \qquad (s, \omega) \mapsto A(s; \omega)$$

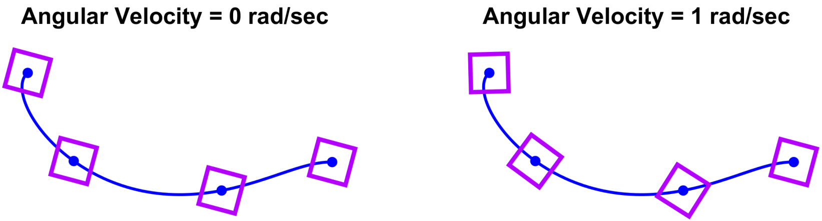


Figure 1: Shape translating and (possibly) rotating

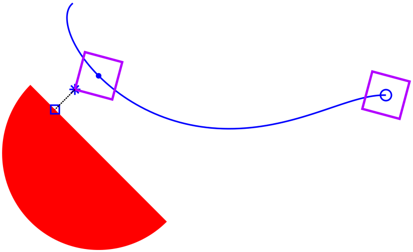
Set-Set Distance Problem

Set-Set distance between $A(S\ ; \omega(t))$ and X_u given t

$$P^* = \min_{t,\ \omega_0 \in \Omega_0,\ s \in S} c(x(t); X_u)$$

$$x(t) = A(s;\ \omega(t \mid \omega_0)) \quad \forall t \in [0, T]$$

$$\dot{\omega}(t) = f(t, \omega) \quad \forall t \in [0, T]$$



L_2 bound of 0.1465

Take-aways

Conclusion

Distance Estimation with occupation measures

Approximate recovery if moment matrices are low-rank

Extend to uncertain, lifted, set-set scenarios

Future Work

- Distance-Maximizing Control
- Chance-constrained distance
- Further Sparsity
- Efficient Computation
- Other nonnegativity cones and proofs

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Thank you for your attention

arxiv:2110.14047

`http://github.com/jarmill/distance`

Other Projects

Peak-related

- Occupation measures for time-delay systems
- Peak/Distance for hybrid systems
- Exploit polytopic input-affine structure
- Value-at-risk (probabilistic) peak

Non-Peak

- Data-driven stabilization under measurement noise
- Frank-Wolfe for LTI MIMO SysID
- Disconnectedness of sets
- Improve SDP approximation quality
- Orthogonal Ribosomes

Bonus Material and Ideas

Distance Program (Functions)

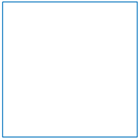
Auxiliary $v(t,x)$, point-set proxy $w(x) \leq c(x; X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t,x) && \forall (t,x) \in [0,T] \times X \\
& c(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& \mathcal{L}_f v(t,x) \geq 0 && \forall (t,x) \in [0,T] \times X \\
& v \in C^1([0,T] \times X) \\
& w \in C(X)
\end{aligned}
$$

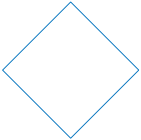
Chain $\forall (t,x,y) \in [0,T] \times X \times X_u: \; c(x,y) \geq w(x) \geq v(t,x)$

Lifted Distance

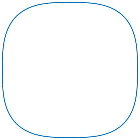
LP lifts to deal with absolute values



$$\|x - y\|_\infty \qquad \begin{aligned} &\min \quad q \\ -q \leq \langle x_i - y_i, \eta \rangle &\leq q \qquad \forall i \end{aligned}$$



$$\|x - y\|_1 \qquad \begin{aligned} &\min \quad \textstyle\sum_i q_i \\ -q_i \leq \langle x_i - y_i, \eta \rangle &\leq q_i \qquad \forall i \end{aligned}$$



$$\|x - y\|_3^3 \qquad \begin{aligned} &\min \quad \textstyle\sum_i q_i \\ -q_i \leq \langle (x_i - y_i)^3, \eta \rangle &\leq q_i \quad \forall i \end{aligned}$$

Lifted Distance Program (Measure)

New terms for lifted distance

$$
\begin{aligned}
p^* = \ \min \quad & \textstyle\sum_i q_i \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \pi^x_\# \eta = \pi^x_\# \mu_p \\
& \langle 1, \mu_0 \rangle = 1 \\
& -q_i \leq \langle c_{ij}(x, y), \eta \rangle \leq q_i & \forall i, j \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p,\ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Same process as maximin peak

Lifted Distance Program (Function)

New terms $\beta_i^\pm$ on costs

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t,x) && \forall (t,x) \in [0,T] \times X \\
& \textstyle\sum_{i,j} (\beta_{ij}^+ - \beta_{ij}) c_{ij}(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& \mathcal{L}_f v(t,x) \geq 0 && \forall (t,x) \in [0,T] \times X \\
& 1^T(\beta_i^+ + \beta_i^-) = 1, \ \beta_i^\pm \in \mathbb{R}_+^{n_i} && \forall i \\
& v \in C^1([0,T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Set-Set Program (Measures)

Add new 'shape' measure μ_s

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle && \forall v \\
& \langle w(x), \eta(x,y) \rangle = \langle w(A(s;\omega)), \mu_s(s,\omega) \rangle && \forall w \\
& \langle z(\omega), \mu_p(t,\omega) \rangle = \langle z(\omega), \mu_s \rangle && \forall z \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_s \in \mathcal{M}_+(\Omega \times S) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times \Omega) \\
& \mu_0 \in \mathcal{M}_+(\Omega_0)
\end{aligned}
$$

Set-Set Program (Function)

Set-Set distance proxy $z(\omega) \leq \max_{s \in S} c(A(s;\omega); X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,\omega) \geq \gamma && \forall x \in \Omega_0 \\
& c(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& w(A(s;\omega)) \geq z(\omega) && \forall (s,\omega) \in S \times \Omega \\
& z(\omega) \geq v(t,\omega) && \forall (t,\omega) \in [0,T] \times \Omega \\
& \mathcal{L}_f v(t,\omega) \geq 0 && \forall (t,\omega) \in [0,T] \times \Omega \\
& v \in C^1([0,T] \times X) \\
& w \in C(X), \ z \in C(\Omega)
\end{aligned}
$$

Sparsity (cont.)

Separable $c(x,y) = \sum_i c_i(x_i, y_i)$

Use correlative sparsity with measures and cliques

$$\eta_k : \quad I_k = (x_k : x_n,\ y_1 : y_k) \qquad \forall k = 1, \dots, n$$

Sparse decomposition of η:

$$\begin{aligned} &\min \textstyle\sum_i \langle c_i(x_i, y_i), \eta_i \rangle & \quad & \eta^1 \in \mathcal{M}_+(X \times \mathbb{R}) \\ &\pi_{\#}^{I_k \cap I_{k+1}} \eta_k = \pi_{\#}^{I_k \cap I_{k+1}} \eta_{k+1} & & \eta^k \in \mathcal{M}_+(\mathbb{R}^{n+1}) \\ &\pi_{\#}^{x} \mu^p = \pi_{\#}^{x} \eta_1 & & \eta^n \in \mathcal{M}_+(\mathbb{R} \times X_u) \end{aligned}$$