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Scattered data interpolation through B-spline wavelets and the Elastic Net

Jared Miller, Mario Sznaier

INTRODUCTION

Scattered Data is a class of data whose locations have no structure or order. This type of data has applications in the fields of medical imaging, GIS, plasma physics, photolithography, etc. Most conventional interpolation algorithms rely on the input data having a standard gridded structure, such as in images. Using these standard methods with scattered data forms undesirable distortions and artifacts. Waveletfit is an algorithm specifically geared for scattered data interpolation, extrapolation, and denoising.

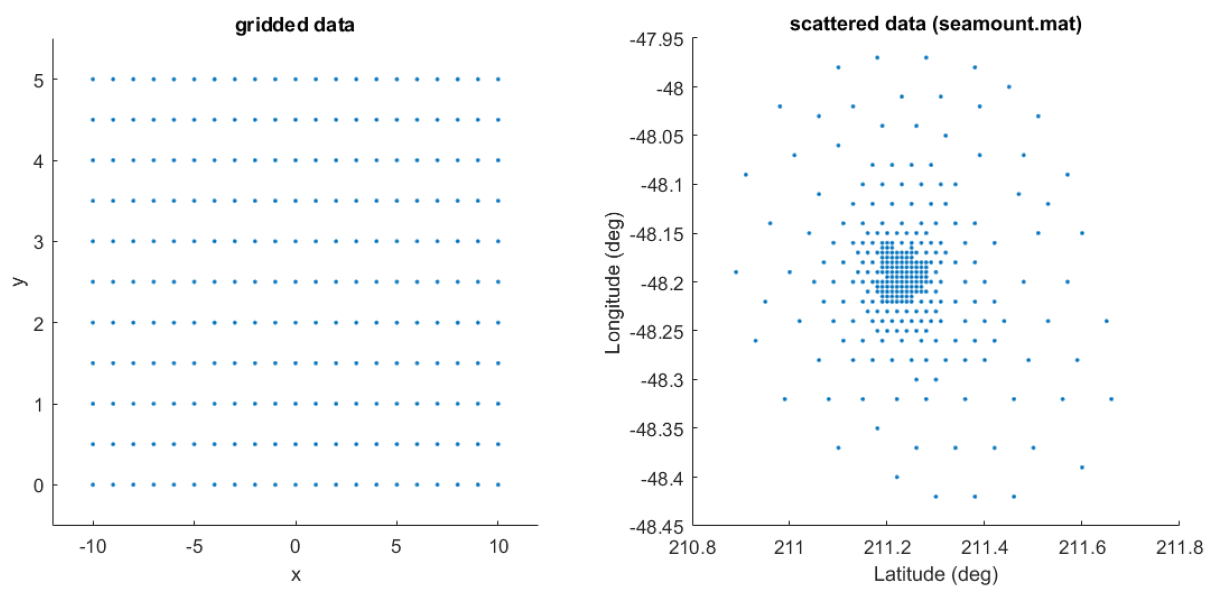


Figure 1: Gridded vs. Scattered data

Waveletfit uses a collection of B-spline wavelet basis function, and smoothly fits this set to the input data to provide an interpolation. Sparsity is leveraged in two ways: the compactness of the wavelet functions and the L1-regularization penalty. Waveletfit offers competitive performance in time and increased denoising capacity as compared to RBF, Sinusoidal, and Triangulation scattered data interpolation algorithms.

FUNCTIONAL FIT

The aim of interpolation is to predict the values of a function between given datapoints. This can be done by breaking down a function into a weighted sum of (usually orthogonal) basis functions. Once the weights are obtained through a regularized fit, any other point can be predicted by evaluating all the basis functions and adding up the weighted sum.

Figure 2: Data flow of a functional fit. Bases are sampled at locations, fitted, and then interpolated.

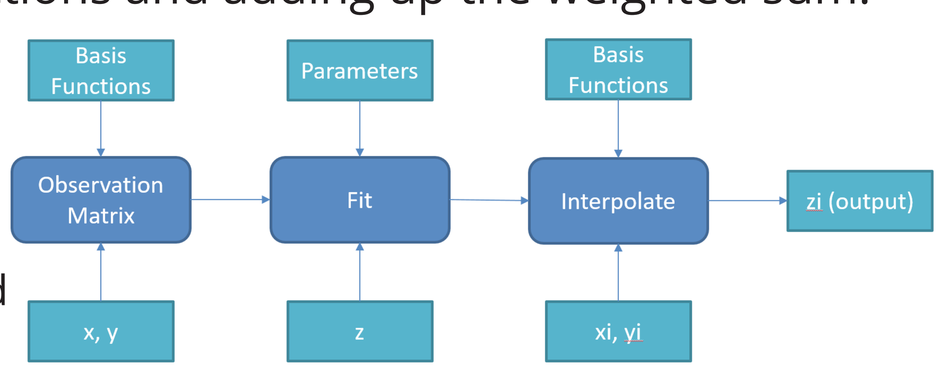
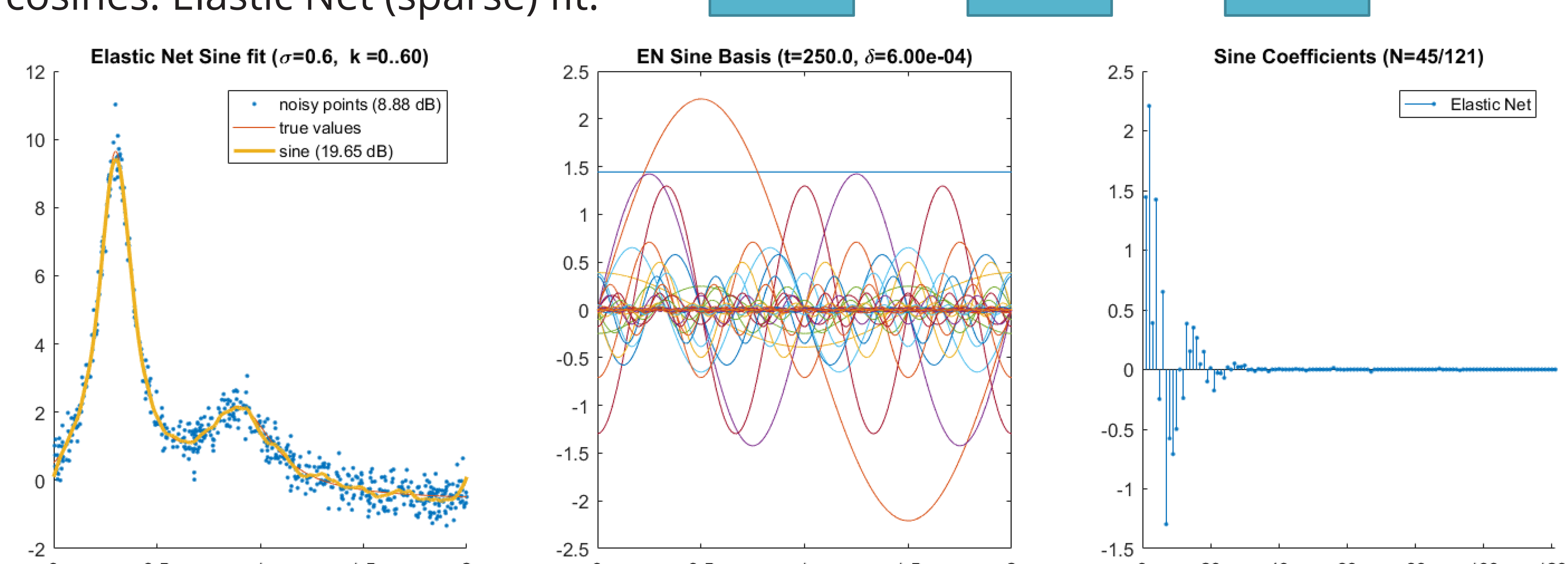


Figure 3: Functional fit of the 'humps' function through sines and cosines. Elastic Net (sparse) fit.



1D INTERPOLATION

Spline Wavelets

Waveletfit uses a collection of spline wavelets as its basis functions. Spline wavelets are the only family of wavelets with explicit equations. The scaling (approximation) functions are the B-spline basis functions, and the wavelet (detail) functions are a weighted shifted sum of the scaling functions. Their collective translations and dilations form a multiresolution analysis (MRA).

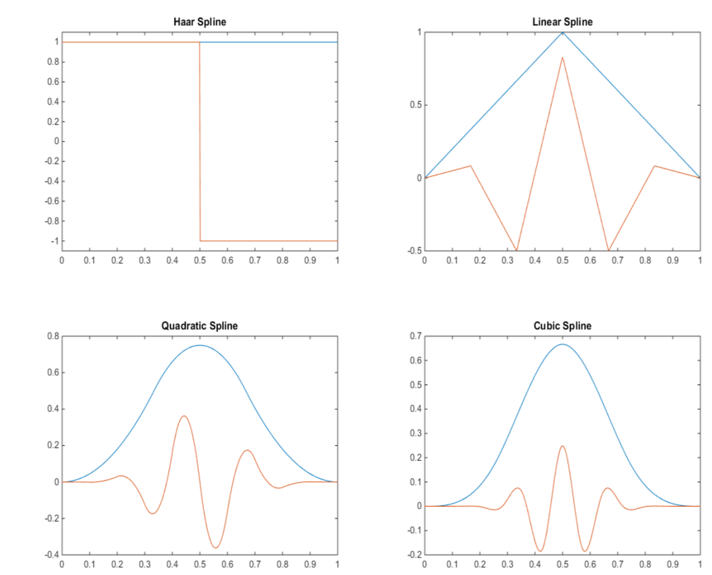


Figure 4: B-spline Scaling (blue) and Wavelet (orange) functions. Implemented for orders 0-3.

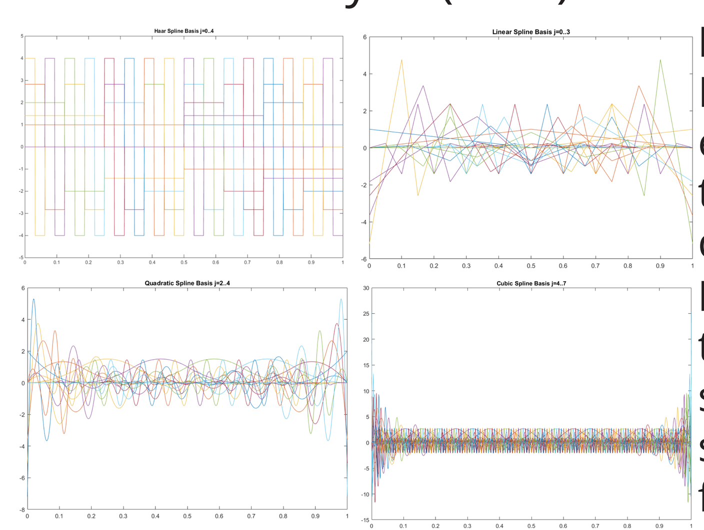
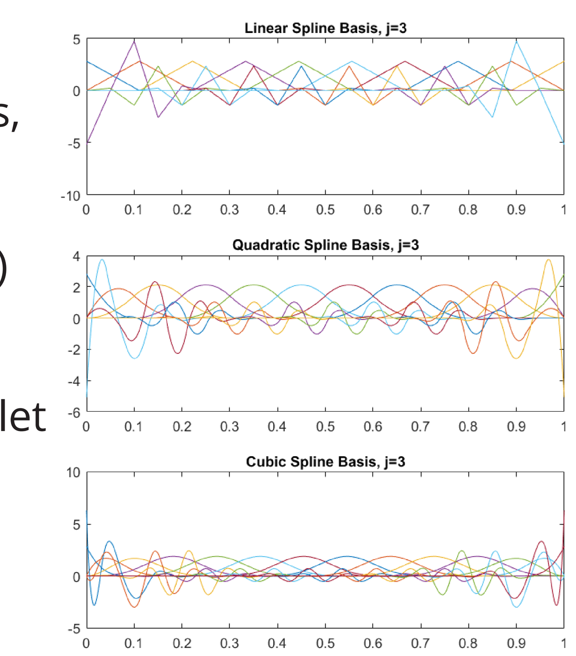


Figure 5: (Left) Full wavelet bases, all boundaries, translations, and dilations.

Figure 6: (Right) translations of scale j=3 b-spline scaling and wavelet functions



Waveletfit

This algorithm is the combination of two techniques: B-spline wavelets and Elastic Net regularization. The compact B-splines offer sparsity in space, and the L2 penalty of Elastic Net adds additional stability while the L1 term forces coefficient sparsity. The Elastic Net problem can be solved through CVX, SVEN, Atomic Norm, ALM/ADMM, etc. Cross validation is used to find the hyperparameters δ (L2 penalty), t (L1 budget), and α (scale penalty).

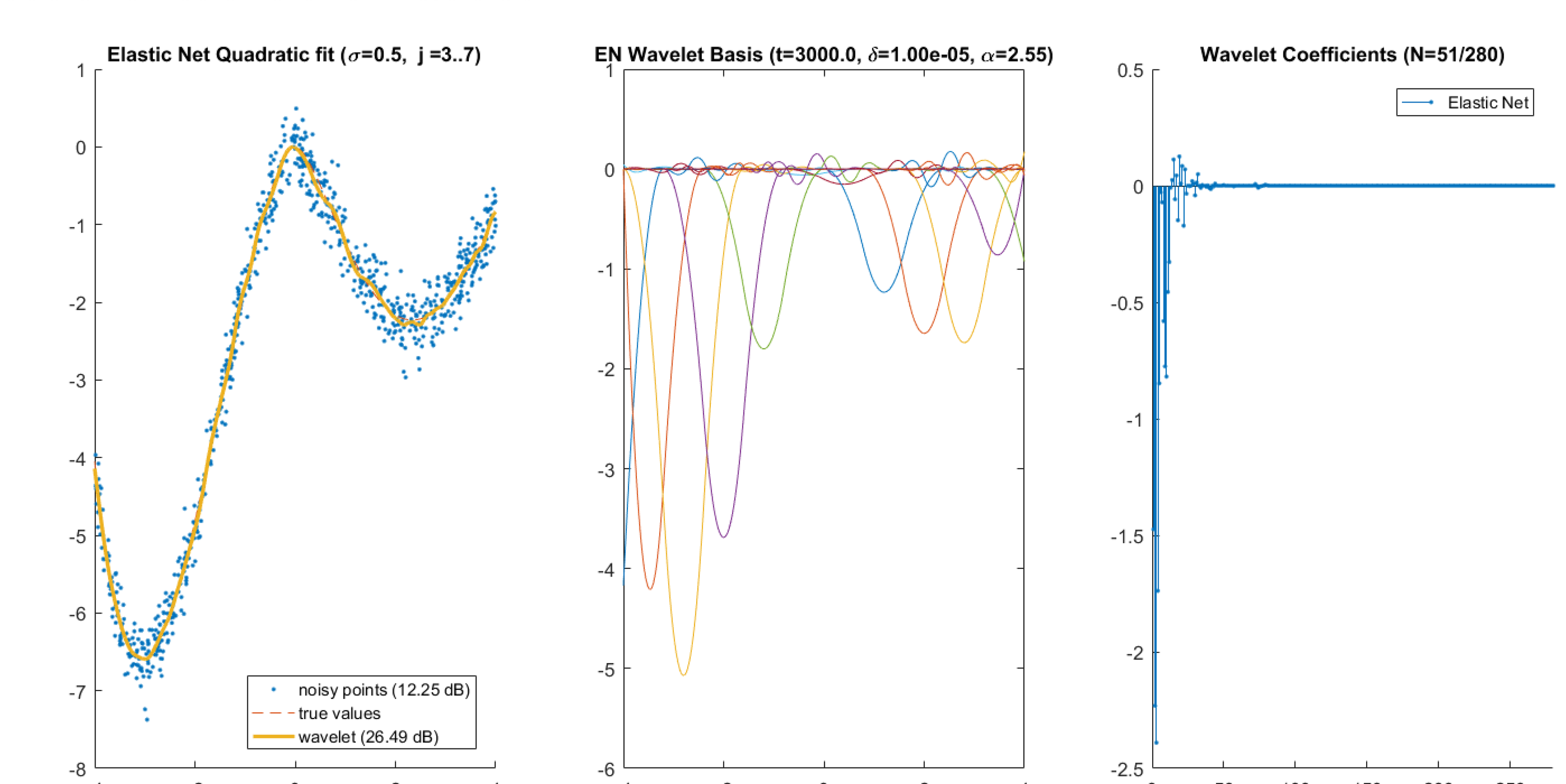
$$\min_x \|Ax - b\|_2^2 + \delta\|Lx\|_2^2$$

$$\text{such that } \|Lx\|_1 < t_1$$

$$L = diag(2^{\alpha j})$$

Figure 7: Equation of Waveletfit. A is the observation matrix, x, the vector of coefficients, and L the smoothness penalty.

Figure 8: Using Waveletfit on the 'ugly' function. It is very sparse, as only 20/158 basis functions are included



2D INTERPOLATION

Tensor Product Basis

This interpolation method can be naturally extended to higher dimensions using tensor product basis functions. The complete set of basis functions are evaluated for each axis, and then multiplied together to form the final fit. The total penalty of each basis function (L) is the sum of each axis' penalty. 2D Fitting works the same way as 1D fitting, except the observation matrix is less sparse and the fit takes longer to calculate. This lack of spatial sparsity is due to the additional overlaps that occur between dimensions.

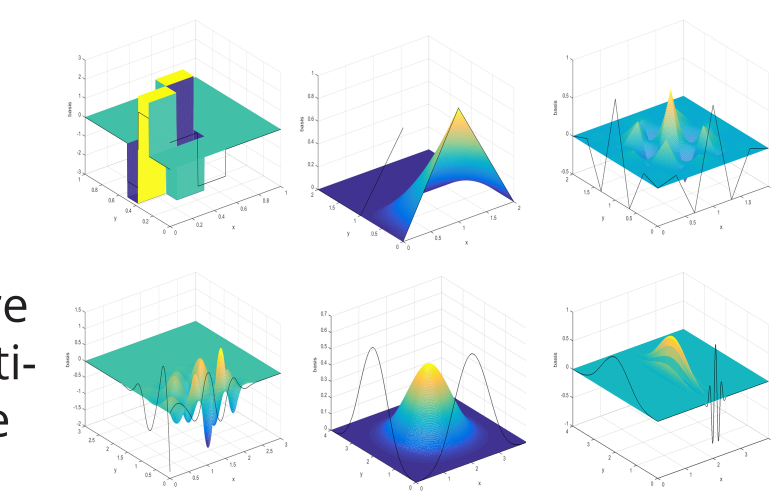
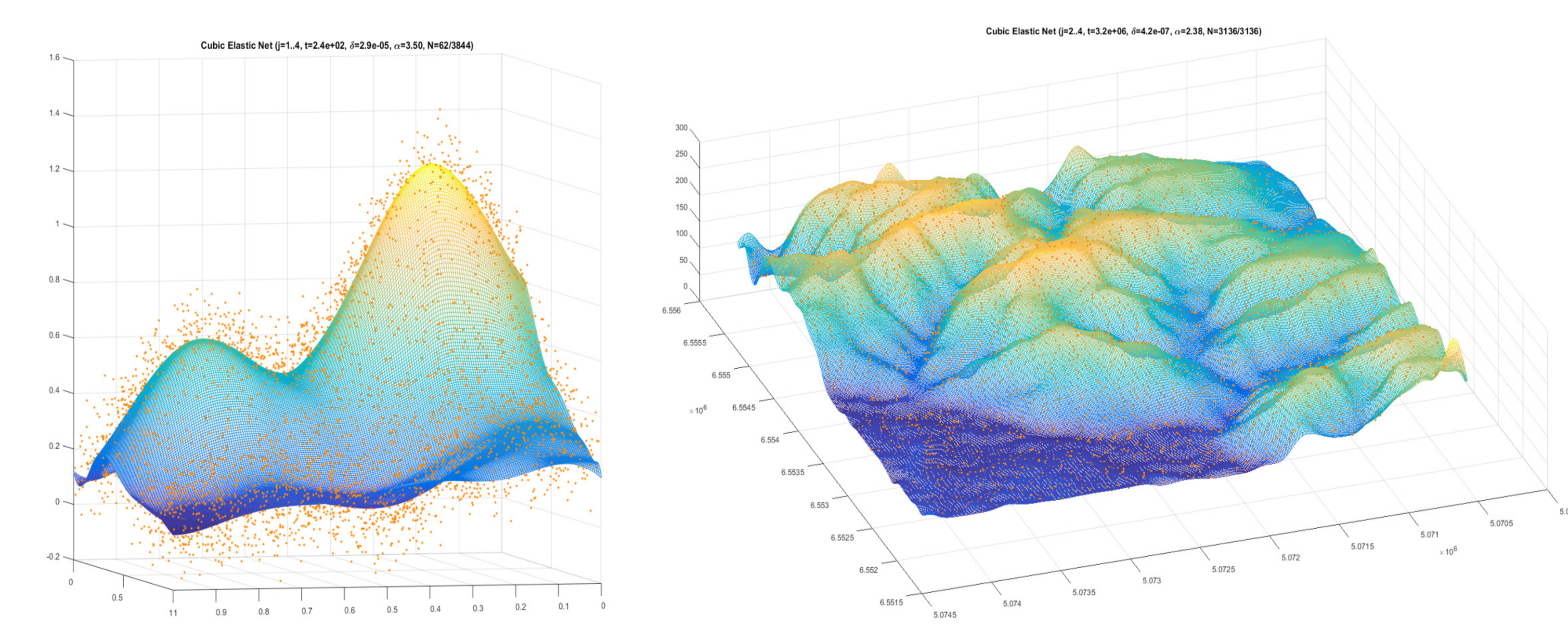


Figure 9: Examples of 2d tensor product basis functions

Figure 10: Franke's Function and GIS data from the Villány mountains in Baranja, Croatia, denoised by Waveletfit



Polar Wavelets

Some sets of data are on a circle (wafer) rather than a rectangular region, and may exhibit circular symmetry. Polar coordinates can be used, with appropriate modifications. r can be kept the same, but θ requires periodic basis functions. Computation in polar coordinates introduces a singularity at r=0, which manifests as a spike in the center. The effect of the spike can be lessened through block-mirror centrosymmetric continuity (BMC), in which the interpolation grid of (r, θ) is glide reflected across the line r=0. More work needs to be done to properly reduce the effect of the spike.

Figure 11: An example of a Polar Basis function

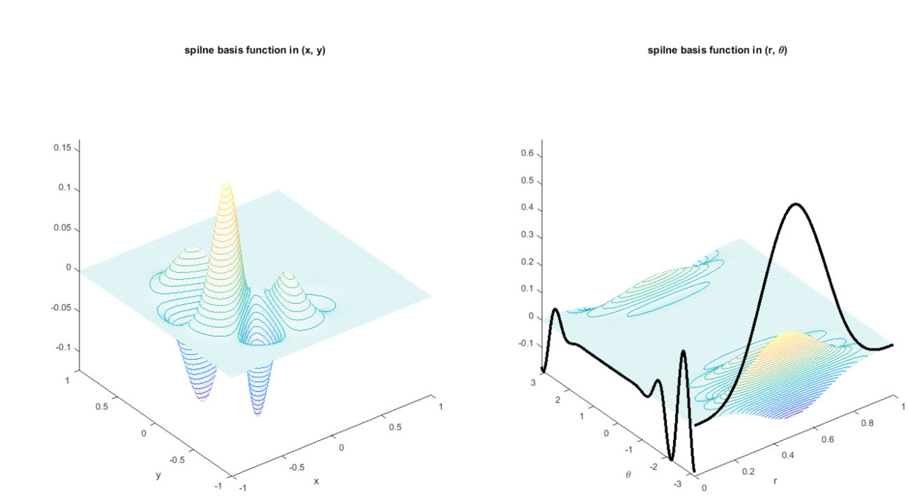


Figure 12: Interpolating coscos with polar. There is a spike in the middle.