Bounding distances to unsafe sets

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SMAI-MODE: Optimal Transport 2

Northeastern University

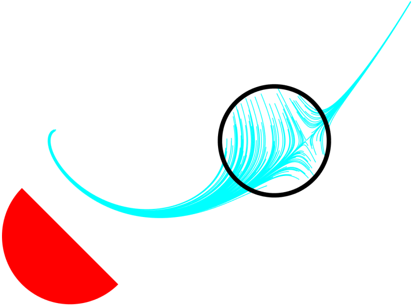
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Relax distance using optimal transport theory

Develop occupation measure programs to bound distance

Flow System Setting



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$
$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2,$$
$$\sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$$

Distance Function

Metric space (X, c) satisfying $\forall x, y \in X$:

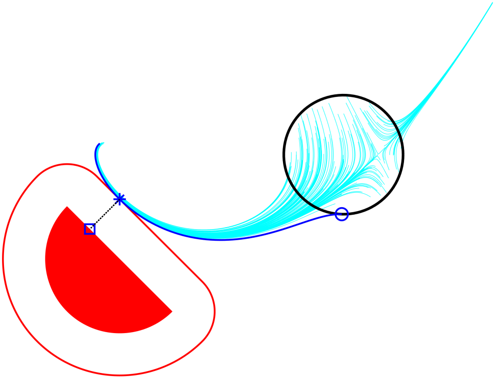
$$\begin{aligned} c(x,y) &> 0 && x \neq y \\ c(x,x) &= 0 \\ c(x,y) &= c(y,x) \\ c(x,y) &\leq c(x,z) + c(z,y) && \forall z \in X \end{aligned}$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x, y)$

Distance Estimation Problem

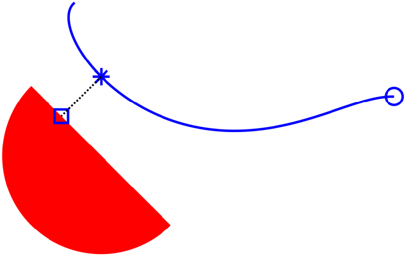
$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*\ x_0^*,\ t_p^*)$:

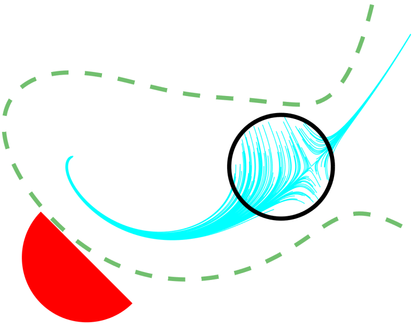
x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Safety Background

Barrier Program

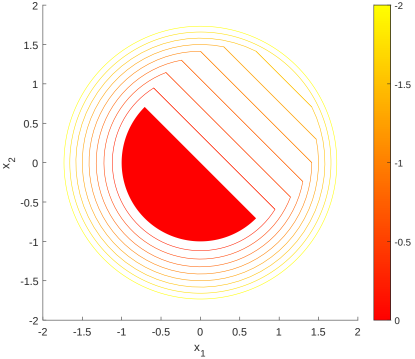
Barrier function $B : X \to \mathbb{R}$ indicates safety

$$\begin{aligned} B(x) &\leq 0 && \forall x \in X_u \\ B(x) &> 0 && \forall x \in X_0 \\ f(x) \cdot \frac{\partial B}{\partial x}(x) &\geq 0 && \forall x \in X \end{aligned}$$



Half-circle Contours

Unsafe set $X_u = \{x \mid 1 - x_1^2 - x_2^2 \geq 0, -x_1 - x_2 \geq 0\}$



$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq -x_1 - x_2$$

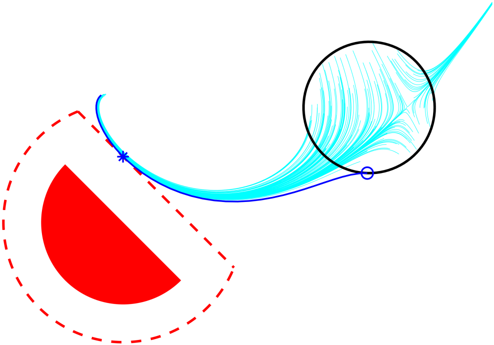
$$q = -0.25, -0.5, \dots, -2$$

Safety Margin

Unsafe set $X_u = \{x \mid p_i(x) \geq 0 \;\forall i = 1 \ldots N_u\}$

Safety margin $p^* = \max \min_i p_i(x)$ along trajectories

If $p^* < 0$, no trajectories enter X_u (safe)

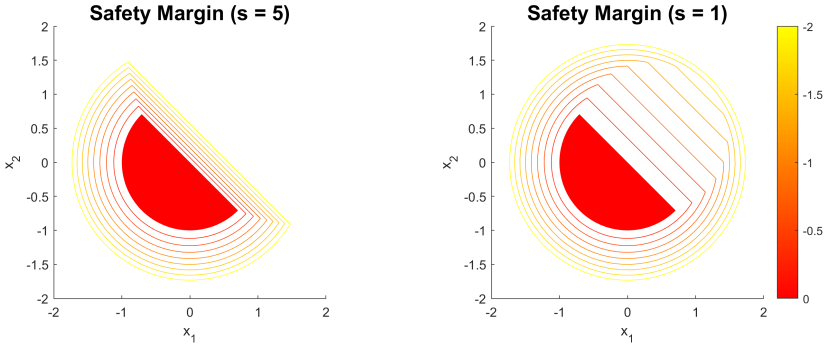


safe: $p^* \leq -0.2831$

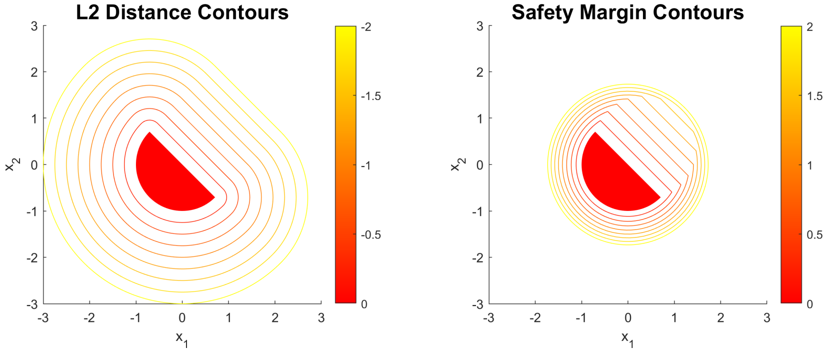
Safety Margin Scaling

Scale factor in constraints

$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq s(-x_1 - x_2)$$



Distance vs. Safety Margin



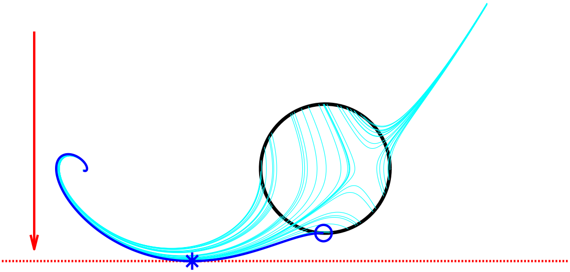
Peak Estimation

Peak Estimation Background

Find maximum value of $p(x)$ along trajectories

$$P^* = \max_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

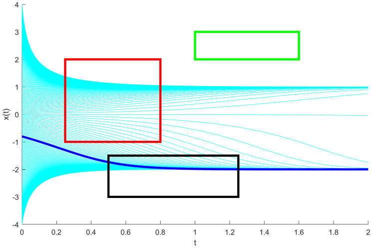
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

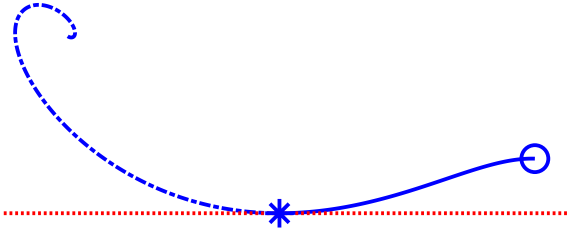
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x + 2)(x - 1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t,x) \in C([0,T] \times X)$

$$\begin{aligned}
\mu_0^* : &\qquad \langle v(0,x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : &\qquad \langle v(t,x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : &\qquad \langle v(t,x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt
\end{aligned}$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

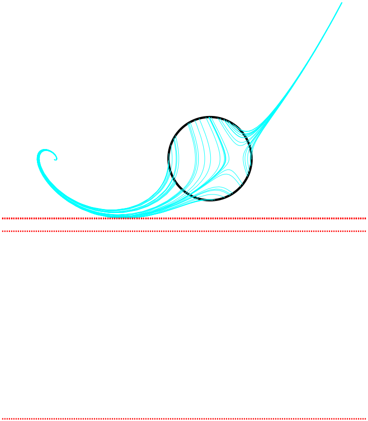
$$
\begin{aligned}
p^* = \ \max \quad & \langle p(x), \mu_p \rangle && \text{(1a)} \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1b)} \\
& \langle 1, \mu_0 \rangle = 1 && \text{(1c)} \\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) && \text{(1d)} \\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0, T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)

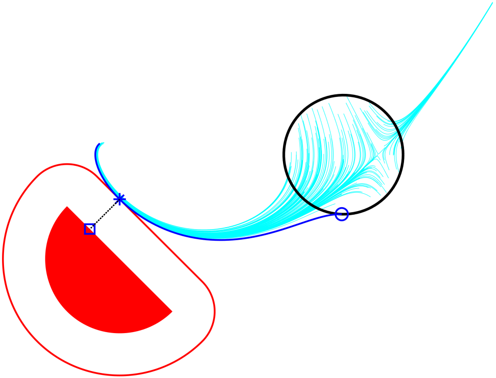
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Distance Relaxation

Distance in points $\rightarrow$ Earth-Mover distance

$$
\begin{array}{lcl}
c(x,y) & & \langle c(x,y), \eta \rangle \\
x \in X & \rightarrow & \langle 1, \eta \rangle = 1 \\
y \in X_u & & \eta \in \mathcal{M}_+(X \times X_u)
\end{array}
$$

Joint (Wasserstein) probability measure η

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\mu_0^* : \qquad \delta_{x=x_0^*}$$

$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$

$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0,T] \times X)$

$$\mu^* : \qquad \langle v(t,x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Distance Program (Measures)

Infinite Dimensional Linear Program (Convergent)

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \langle w(x), \eta(x,y) \rangle = \langle w(x), \mu_p(t,x) \rangle & \forall w \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle & \forall v \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Prob. Measures: $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1$

Approximation and Recovery

Use moment-SOS hierarchy (Archimedean assumption)

Bounds: $p^*_d \leq p^*_{d+1} \leq \ldots \leq p^* = P^*$

Attempt recovery if LMI solution has low rank

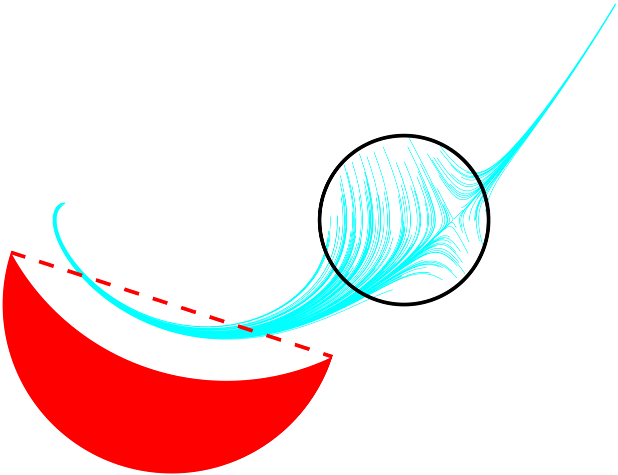
Moment matrices for (μ_0, μ_p, η) are rank-1

Related to optima extraction in polynomial optimization

Moon L2 Contours

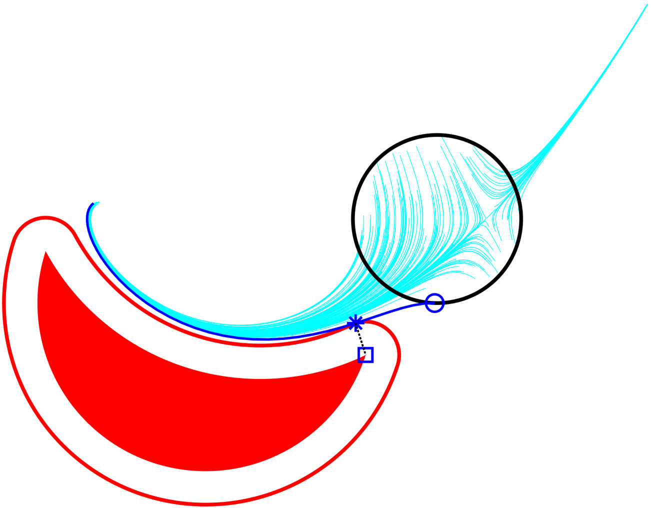
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

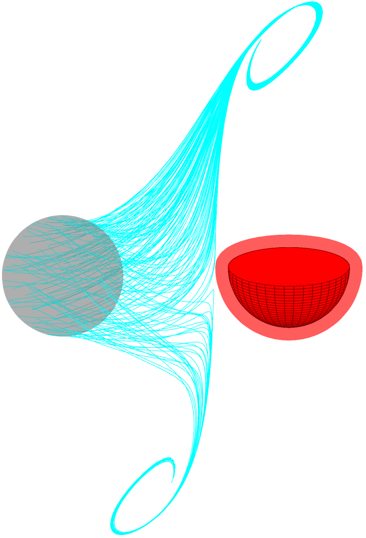
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

Distance Variations

Distance Variations

Uncertainty in dynamics

Lifted distances (with absolute values)

Sparsity

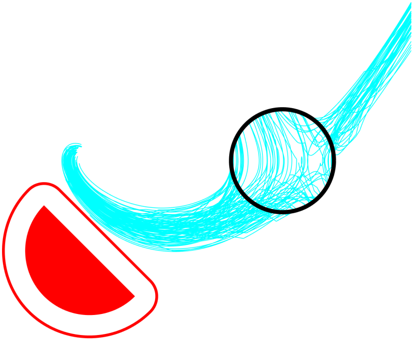
Set-Set distances for shape safety

Distance Uncertainty

Time dependent uncertainty $w(t) \in W \; \forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

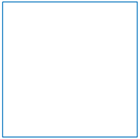
Young measure $\mu(t, x, w)$, Liouville term $\langle \mathcal{L}_{f(t,x,w)} v(t, x), \mu \rangle$



L_2 bound of 0.1691

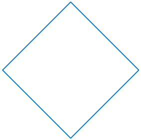
Lifted Distance

LP lifts to deal with absolute values



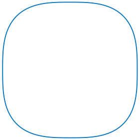
$\|x-y\|_\infty$

$$\begin{aligned} &\min \quad q \\ -q \le \langle x_i - y_i, \eta \rangle \le q \quad &\forall i \end{aligned}$$



$\|x-y\|_1$

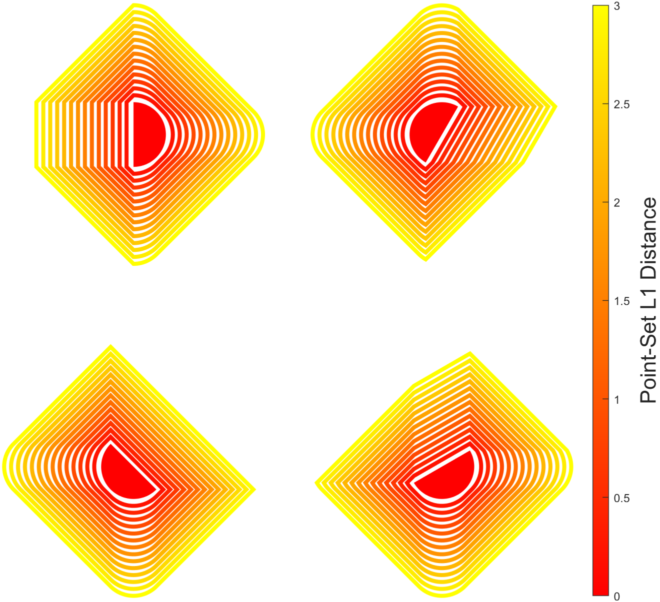
$$\begin{aligned} &\min \quad \textstyle\sum_i q_i \\ -q_i \le \langle x_i - y_i, \eta \rangle \le q_i \quad &\forall i \end{aligned}$$



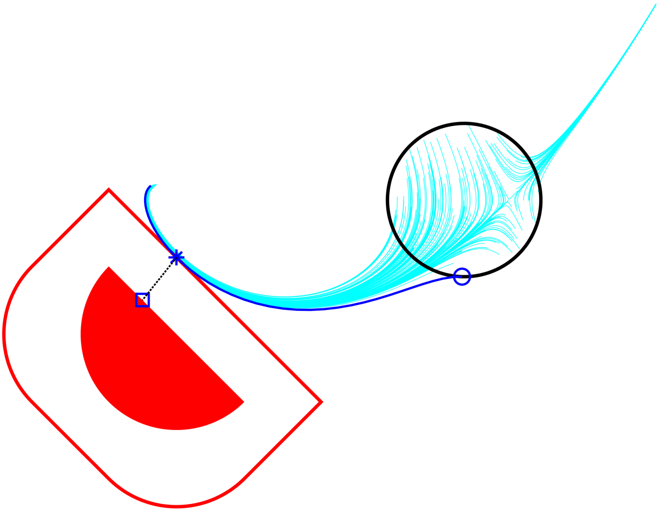
$\|x-y\|_3^3$

$$\begin{aligned} &\min \quad \textstyle\sum_i q_i \\ -q_i \le \langle (x_i - y_i)^3, \eta \rangle \le q_i \quad &\forall i \end{aligned}$$

Half-Circle L1 Contours



Lifted Distance (L1) Example



L_1 bound of 0.4003

Sparsity

Separable $c(x, y) = \sum_i c_i(x_i, y_i)$

Use correlative sparsity with measures and cliques

$$\eta_k : \quad I_k = (x_k : x_n,\ y_1 : y_k) \qquad \forall k = 1, \dots, n$$

Sparse decomposition of η:

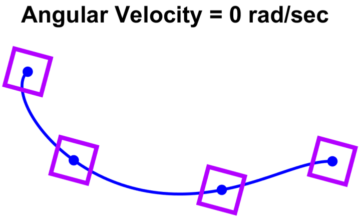
$$\begin{aligned} &\min \textstyle\sum_i \langle c_i(x_i, y_i), \eta_i \rangle && \eta^1 \in \mathcal{M}_+(X \times \mathbb{R}) \\ &\pi_{\#}^{I_k \cap I_{k+1}} \eta_k = \pi_{\#}^{I_k \cap I_{k+1}} \eta_{k+1} && \eta^k \in \mathcal{M}_+(\mathbb{R}^{n+1}) \\ &\pi_{\#}^{x} \mu^p = \pi_{\#}^{x} \eta_1 && \eta^n \in \mathcal{M}_+(\mathbb{R} \times X_u) \end{aligned}$$

Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

$$A : S \times \Omega \to X \qquad\qquad (s, \omega) \mapsto A(s; \omega)$$



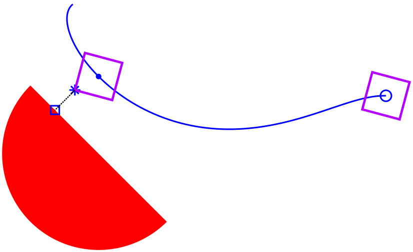
Set-Set Distance Problem

Set-Set distance between $A(\,\cdot\,;\omega) \circ S$ and X_u given ω

$$P^* = \min_{t,\, \omega_0 \in \Omega_0,\, s \in S} c(x(t); X_u)$$

$$x(t) = A(s;\, \omega(t \mid \omega_0)) \quad \forall t \in [0, T]$$

$$\dot{\omega}(t) = f(t, \omega) \quad \forall t \in [0, T]$$



L_2 bound of 0.1465

Set-Set Program (Measures)

Add new ‘shape’ measure μ_s

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle && \forall v \\
& \langle w(x), \eta(x,y) \rangle = \langle w(A(s;\omega)), \mu_s(s,\omega) \rangle && \forall w \\
& \langle z(\omega), \mu_p(t,\omega) \rangle = \langle z(\omega), \mu_s \rangle && \forall z \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_s \in \mathcal{M}_+(\Omega \times S) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times \Omega) \\
& \mu_0 \in \mathcal{M}_+(\Omega_0)
\end{aligned}
$$

Take-aways

Conclusion

Distance Estimation with occupation measures

Approximate recovery if moment matrices are low-rank

Extend to uncertain, lifted, set-set scenarios

Future Work

- Distance-Maximizing Control
- Further Sparsity
- Efficient Computation
- Other nonnegativity cones and proofs

Acknowledgements

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Thank you for your attention

arxiv:2110.14047

`http://github.com/jarmill/distance`

Graduating May 2023, looking for postdocs

Bonus Material and Ideas

Distance Program (Functions)

Auxiliary $v(t,x)$, point-set proxy $w(x) \leq c(x; X_u)$:

$$\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t,x) && \forall (t,x) \in [0,T] \times X \\
& c(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& \mathcal{L}_f v(t,x) \geq 0 && \forall (t,x) \in [0,T] \times X \\
& v \in C^1([0,T] \times X) \\
& w \in C(X)
\end{aligned}$$

Chain $\forall (t,x,y) \in [0,T] \times X \times X_u : \; c(x,y) \geq w(x) \geq v(t,x)$

Lifted Distance Program (Measure)

New terms for lifted distance

$$
\begin{aligned}
p^* = \ \min \quad & \textstyle\sum_i q_i \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \pi^x_\# \eta = \pi^x_\# \mu_p \\
& \langle 1, \mu_0 \rangle = 1 \\
& -q_i \leq \langle c_{ij}(x,y), \eta \rangle \leq q_i & \forall i, j \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Same process as maximin peak

Lifted Distance Program (Function)

New terms $\beta_i^{\pm}$ on costs

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0, x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t, x) && \forall (t, x) \in [0, T] \times X \\
& \textstyle\sum_{i,j} (\beta_{ij}^{+} - \beta_{ij}) c_{ij}(x, y) \geq w(x) && \forall (x, y) \in X \times X_u \\
& \mathcal{L}_f v(t, x) \geq 0 && \forall (t, x) \in [0, T] \times X \\
& 1^T(\beta_i^{+} + \beta_i^{-}) = 1, \ \beta_i^{\pm} \in \mathbb{R}_+^{n_i} && \forall i \\
& v \in C^1([0, T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Set-Set Program (Function)

Set-Set distance proxy $z(\omega) \leq \max_{s \in S} c(A(s;\omega); X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,\omega) \geq \gamma && \forall x \in \Omega_0 \\
& c(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& w(A(s;\omega)) \geq z(\omega) && \forall (s,\omega) \in S \times \Omega \\
& z(\omega) \geq v(t,\omega) && \forall (t,\omega) \in [0,T] \times \Omega \\
& \mathcal{L}_f v(t,\omega) \geq 0 && \forall (t,\omega) \in [0,T] \times \Omega \\
& v \in C^1([0,T] \times X) \\
& w \in C(X), \ z \in C(\Omega)
\end{aligned}
$$