Bounding the Distance to Unsafe Sets with Convex Optimization

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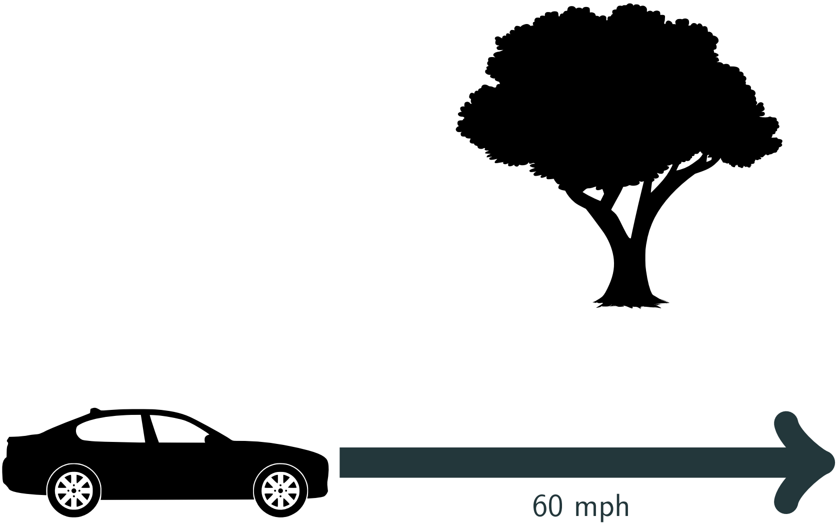
DSCD Rising Stars (Robotics)

Northeastern University

Safety Example



Safety Example

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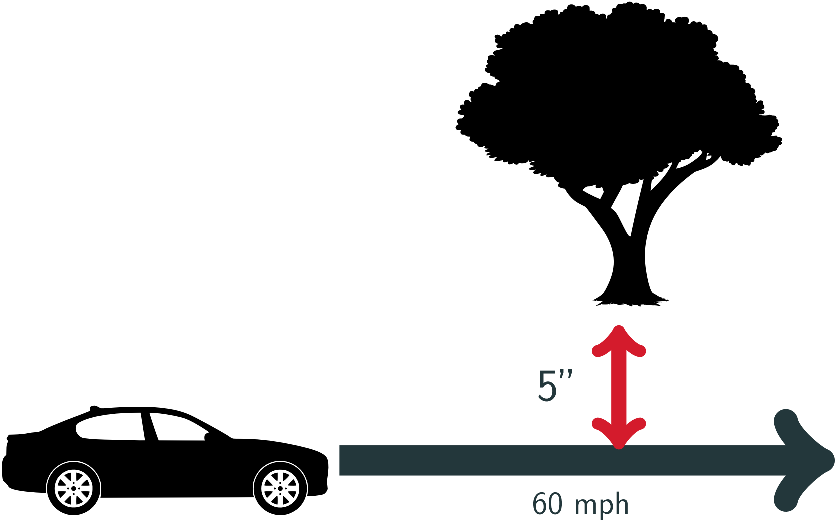


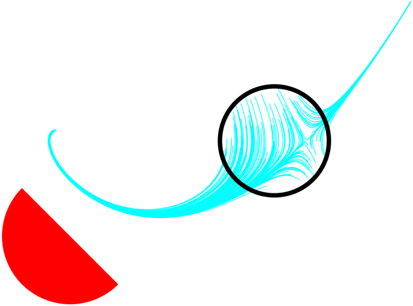
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Create linear program to bound distance

Solve using Semidefinite Programming

Flow System Setting



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$
$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2, \\ \sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$$

Distance Function

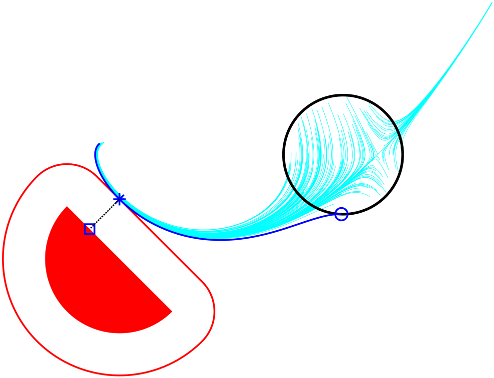
Metric space (X, c) satisfying $\forall x, y \in X$:

$$\begin{aligned} &c(x, y) > 0 && x \neq y \\ &c(x, x) = 0 && \\ &c(x, y) = c(y, x) && \\ &c(x, y) \leq c(x, z) + c(z, y) && \forall z \in X \end{aligned}$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x, y)$

Distance Estimation Problem (Nonconvex)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$
$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

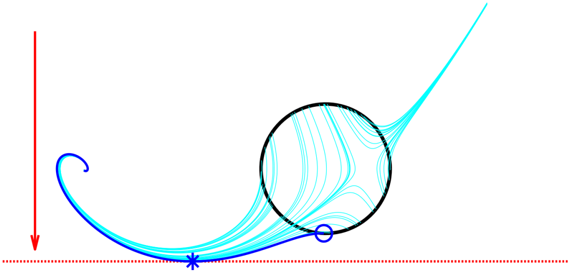
Peak Estimation

Peak Estimation Background

Find minimum value of $p(x)$ along trajectories

$$P^* = \min_{t,\, x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$p(x) = x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Peak Function Program

Infinite dimensional linear program (Fantuzzi, Goluskin, 2020)

Uses auxiliary function $v(t, x)$

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma && && \text{(1a)}\\
& v(t,x) \leq p(x) && \forall (t,x) \in [0,T] \times X && \text{(1b)}\\
& (\partial_t + f \cdot \nabla_x) v(t,x) \geq 0 && \forall (t,x) \in [0,T] \times X && \text{(1c)}\\
& \gamma \leq v(0,x) && \forall x \in X_0 && \text{(1d)}\\
& v \in C^1([0,T] \times X) && && \text{(1e)}
\end{aligned}
$$

$P^* = d^*$ holds if $[0, T] \times X$ is compact, f Lipschitz

Sum-of-Squares Programming

Nonnegativity imposition $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$

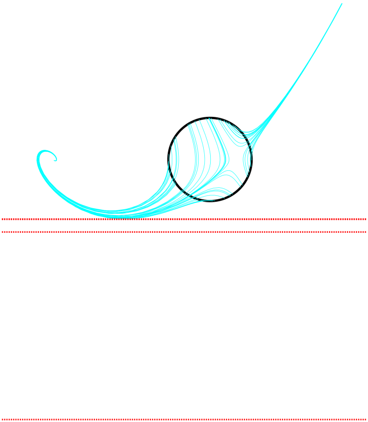
Sum-of-squares (SOS) $p(x) = \sum_i q_i(x)^2$ always nonnegative

SDP-representable: $p(x) = m(x)^T Q m(x)$ for $Q \succeq 0$

Putinar Psatz: Positivity over $\{x \mid g_k(x) \geq 0\}$,

$$p(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x)$$
$$\exists \sigma_0(x) \in \Sigma[x], \quad \sigma(x) \in (\Sigma[x])^{N_g}$$

Peak Estimation Example Bounds



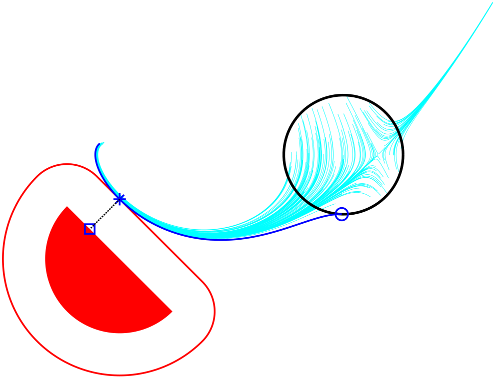
Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Connection to Peak Estimation

Specific form of problem

$$p(x) = c(x; X_u)$$

Moment-SOS hierarchy requires polynomial data

Function $c(x; X_u)$ generally non-polynomial

$$\min_{y \in [-1,1]} \|x - y\|_2 = \begin{cases} 0 & x \in [-1, 1] \\ |x - \mathrm{sign}(x)| & \text{else} \end{cases}$$

Distance Program (Functions)

Auxiliary $v(t, x)$, point-set proxy $w(x) \leq c(x; X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma && && \text{(2a)} \\
& v(0, x) \geq \gamma && \forall x \in X_0 && \text{(2b)} \\
& w(x) \geq v(t, x) && \forall (t, x) \in [0, T] \times X && \text{(2c)} \\
& c(x, y) \geq w(x) && \forall (x, y) \in X \times X_u && \text{(2d)} \\
& (\partial_t + f \cdot \nabla_x) v(t, x) \geq 0 && \forall (t, x) \in [0, T] \times X && \text{(2e)} \\
& v \in C^1([0, T] \times X) && && \text{(2f)} \\
& w \in C(X) && && \text{(2g)}
\end{aligned}
$$

Chain $\forall (t, x, y) \in [0, T] \times X \times X_u : \; c(x, y) \geq w(x) \geq v(t, x)$

Computational Complexity

Use moment-SOS hierarchy

Degree d, dynamics degree $\tilde{d} = d + \lceil \deg(f)/2 \rceil - 1$

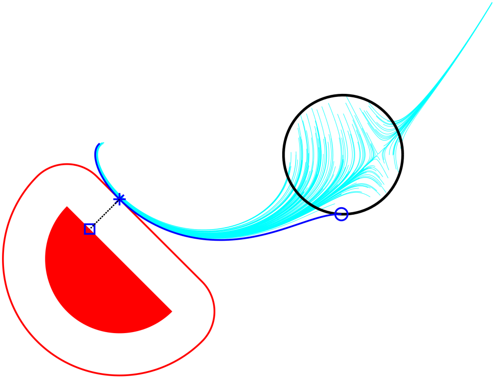
Largest PSD matrix size $\max\left[\binom{1+n+\tilde{d}}{\tilde{d}}, \binom{2n+d}{d}\right]$

Timing scales approximately as $\max\left[(1+n)^{6\tilde{d}}, (2n)^{6d}\right]$

Approximation and Recovery

Attempt recovery if SDP (dual) solution has low rank

Related to optima extraction in polynomial optimization



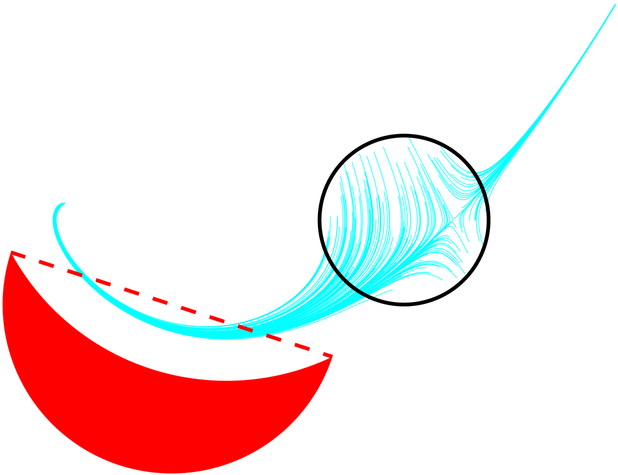
L_2 bound of 0.2831

More Examples

Moon L2 Contours

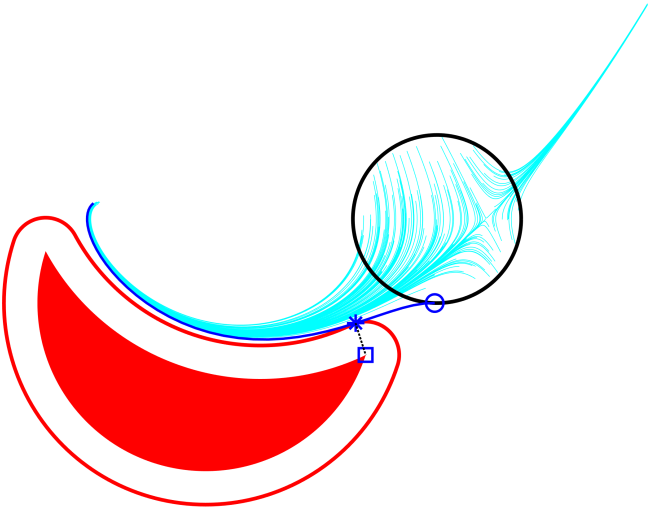
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

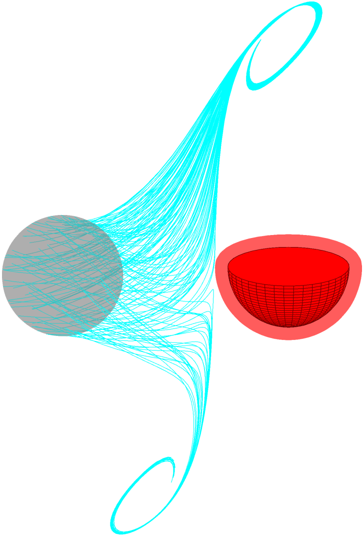
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

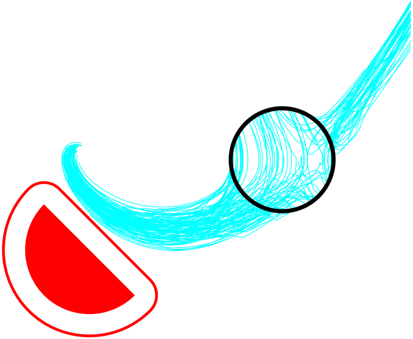
Distance Variations

Distance Uncertainty

Time dependent (bounded) uncertainty $w(t) \in W \ \forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

$(\partial_t + f(t, x, w) \cdot \nabla_x) v(t, x) \geq 0, \quad \forall (t, x, w) \in [0, T] \times X \times W$



L_2 bound of 0.1691, $w(t) \in [-1, 1]$

Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

$$A : S \times \Omega \to X \qquad (s, \omega) \mapsto A(s; \omega)$$

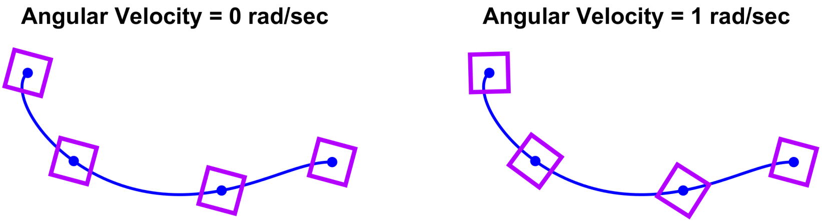


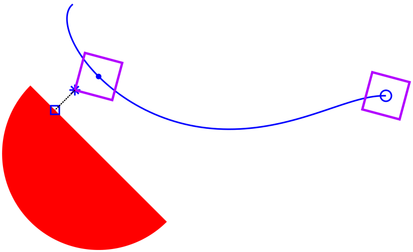
Figure 1: Shape translating and (possibly) rotating

Set-Set Distance Problem

Set-Set distance between $A(S\ ;\omega(t))$ and X_u given t

$$P^* = \min_{t,\, \omega_0 \in \Omega_0,\, s \in S} c(A(s;\, \omega(t \mid \omega_0)); X_u)$$

$$\dot{\omega}(t) = f(t, \omega) \qquad \forall t \in [0, T]$$



L_2 bound of 0.1465

Take-aways

Conclusion

Distance as a measure for safety

Distance estimation with polynomial optimization

Approximate recovery if moment matrices are low-rank

Extend to uncertain and set-set scenarios

Future Work

- Distance-Maximizing Control
- Chance-constrained distance
- Further Sparsity
- Efficient Computation
- Other nonnegativity cones and proofs

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Thank you for your attention

arxiv:2110.14047

`http://github.com/jarmill/distance`

Graduating May 2023, looking for postdocs