Analysis and Control of Time-Delay Systems using Polynomial Optimization

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Time-Delay Examples

Delay between state/input change and its effect on system

System	**Delay**
Epidemic	Incubation Period
Population	Gestation Time
Traffic	Reaction Time
Congestion	Queue Time
Fluid Flow	Moving in Pipe

Functional Differential Equation

Dynamics Model

Delay Differential Equation (DDE) for history $x_h(t)$

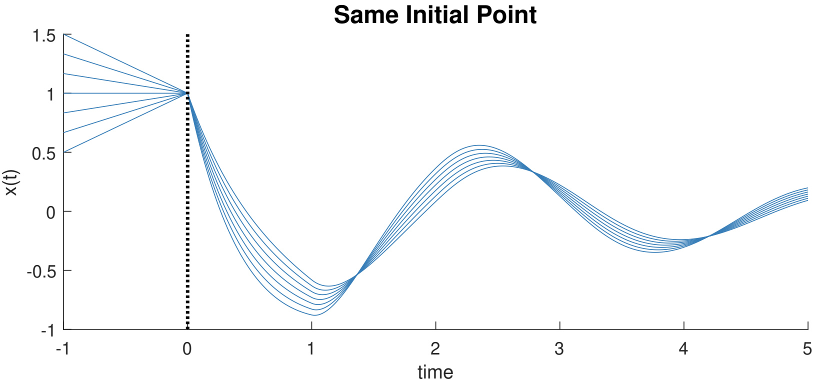
$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau)) \\ x(t) &= x_h(t) && \forall t \in [-\tau, 0] \end{aligned}$$

History $x_h(t)$ does not have to obey dynamics

Can be extended to multiple delays $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_r$

Others: proportional $x(\kappa t)$, distributed $\int_{-\tau_r}^{0} g(\tau')x(t+\tau')d\tau'$

Dependence on History

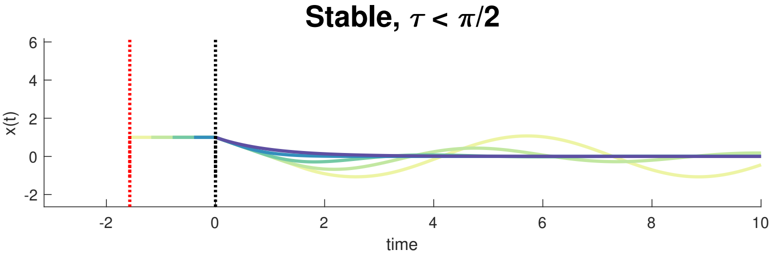


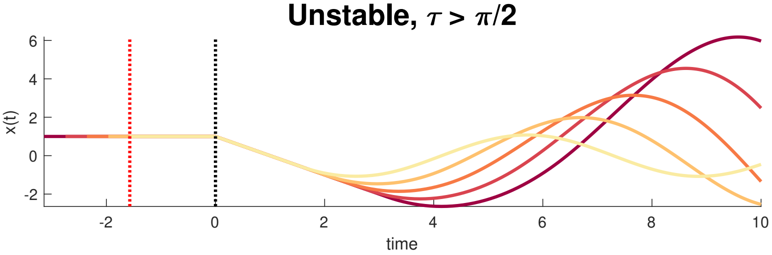
$$x'(t) = -2x(t) - 2x(t-1)$$

All trajectories pass through $(t, x) = (0, 1)$

Initial history determines behavior, not just initial point

Delay Bifurcation Example





$$x'(t) = -x(t - \tau) \qquad \text{(Fridman 2014)}$$

Flow of Presentation

Take an ODE problem with infinite-dimensional LP

- Peak estimation
- Optimal Control Bounds

Form a DDE analogue to ODE LP

Solve using polynomial optimization (moment-SOS)

Watch out for hazards (conservatism)

Existing Methods (very brief)

Certificates of Stability

- Lyapunov-Krasovskii
- Razumikhin
- ODE-Transport PDE

Relaxed control (Warga 1974, Vinter and Rosenblueth 1991-2)

Lyapunov (Papachristodoulou, 2004) (Pap., Peet, Lall 2009)

Barrier (Papachristodoulou and Peet, 2010)

Fixed-terminal-time OCP with gridding (Barati 2012)

Riesz Operators (Magron and Prieur, 2020)

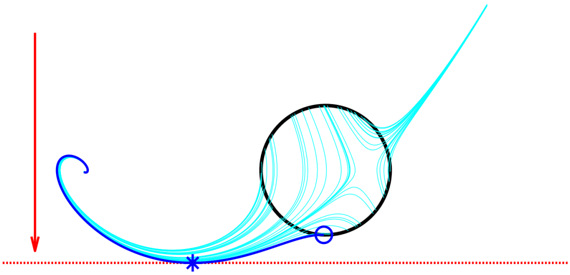
Peak Estimation (ODE)

Peak Estimation Background

Find supremal value of $p(x)$ along ODE trajectories

$$P^* = \sup_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \qquad x(0) = x_0$$



$$p(x) = x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

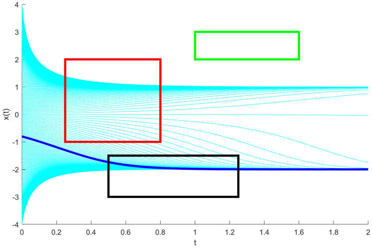
Occupation Measures

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

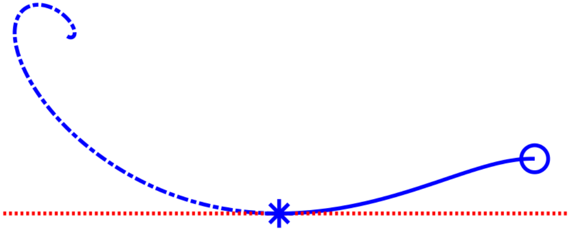
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t, x) \in C([0, T] \times X)$

$$
\begin{aligned}
\mu_0^* : \qquad & \langle v(0,x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : \qquad & \langle v(t,x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : \qquad & \langle v(t,x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt
\end{aligned}
$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

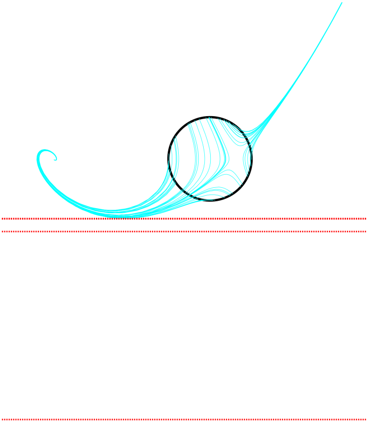
$$
\begin{aligned}
p^* = \ \sup \quad & \langle p(x), \mu_p \rangle && \text{(1a)}\\
& \langle 1, \mu_0 \rangle = 1 && \text{(1b)}\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) && \text{(1d)}\\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
$X = [-2.5, 2.5]$, time $t \in [0, 5]$ (Fantuzzi, Goluskin, 2020)

Peak Estimation (Delayed)

Peak Estimation

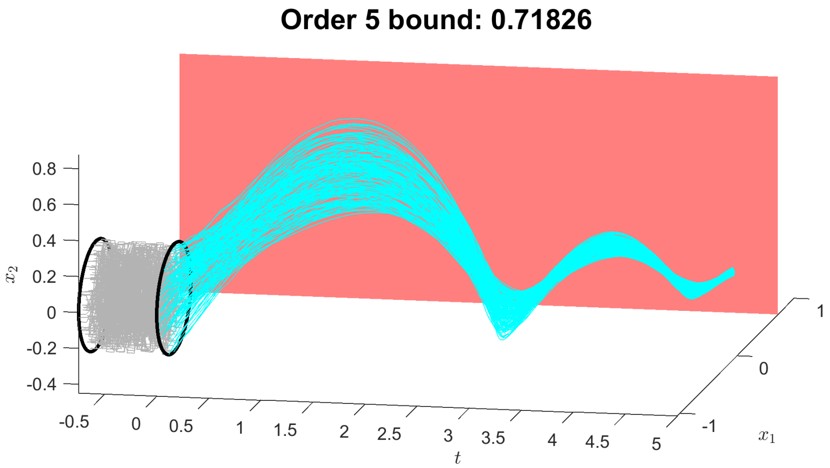
History $x_h(t)$ resides in a class of functions $\mathcal{H}$

Graph-constrained $\mathcal{H}$: $(t, x_h(t))$ contained in $H_0 \subset [-\tau, 0] \times X$

$$\begin{aligned} P^* = \sup_{t^*,\, x_h} \ & p(x(t^*)) \\ & \dot{x} = f(t, x(t), x(t-\tau)) & & t \in [0, t^*] \\ & x(t) = x_h(t) & & t \in [-\tau, 0] \\ & x_h(\cdot) \in \mathcal{H} \end{aligned}$$

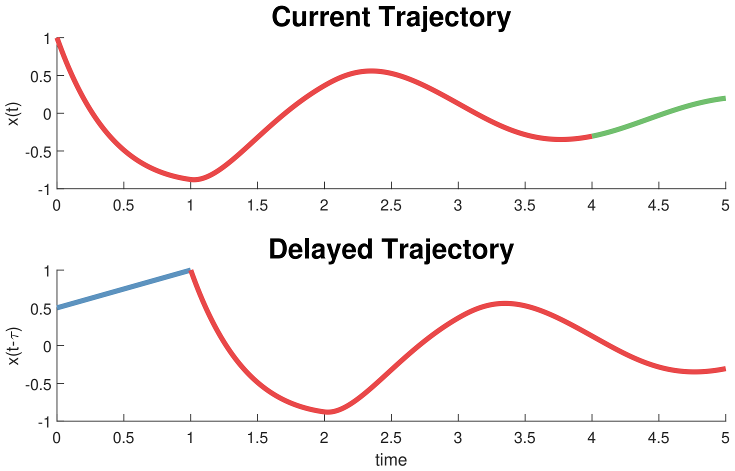
Represent $x(t \mid x_h) : t \in [-\tau, t^*]$ as occupation measure

Time-Varying System



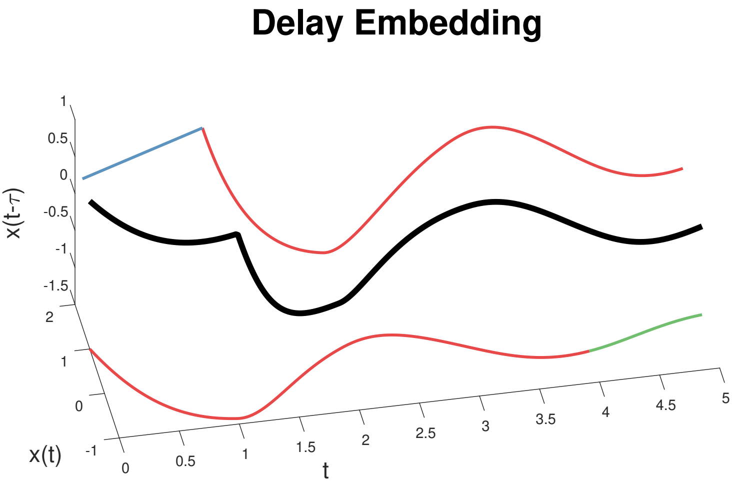
Maximize x_1 on $\dot{x}(t) = \begin{bmatrix} x_2(t)t - 0.1x_1(t) - x_1(t-\tau)x_2(t-\tau) \\ -x_1(t)t - x_2(t) + x_1(t)x_1(t-\tau) \end{bmatrix}$

Time-Delay Visualization



$$x(t) = -2x(t) - 2x(t-1), \qquad x_h(t) = 1 - t/2$$

Time-Delay Embedding



Black curve: $(t, x(t), x(t-\tau))$

Measure-Valued Solution

Tuple of measures for the delayed case

History	$\mu_{\boldsymbol{h}} \in \mathcal{M}_+(H_0)$
Initial	$\mu_0 \in \mathcal{M}_+(X_0)$
Peak	$\mu_{\boldsymbol{p}} \in \mathcal{M}_+([0, T] \times X)$
Occupation Start	$\bar{\mu}_0 \in \mathcal{M}_+([0, T - \tau] \times X^2)$
Occupation End	$\bar{\mu}_1 \in \mathcal{M}_+([T - \tau, T] \times X^2)$
Time-Slack	$\nu \in \mathcal{M}_+([0, T] \times X)$

Types of Constraints

History-Validity: initial conditions

Liouville: Dynamics

Consistency: Time-delay overlaps

History Validity

History $(t, x_h(t))$ defines a curve $[-\tau, 0]$, point at $x_h(0)$

Point evaluation $\langle 1, \mu_0 \rangle = 1$

t-marginal of μ_h should be the Lebesgue measure in $[-\tau, 0]$

Liouville

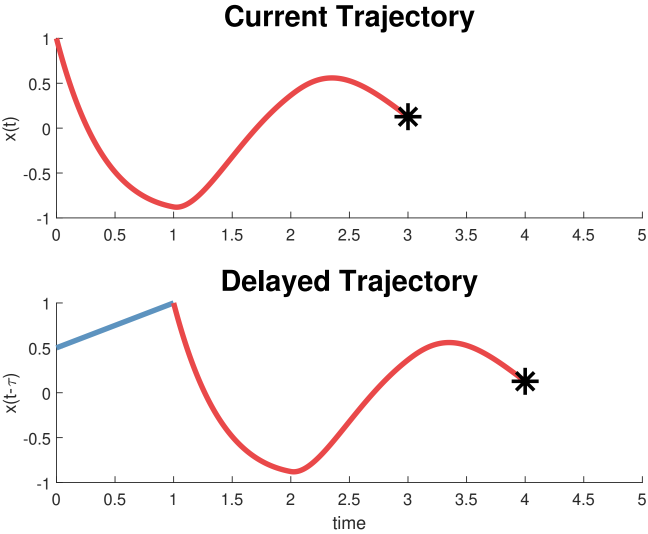
Sum $\bar{\mu} = \bar{\mu}_0 + \bar{\mu}_1$ is a relaxed occupation measure of the delay embedding $(t, x(t), x(t-\tau))$

For all test functions $v \in C^1([0, T] \times X)$:

$$\langle v, \mu_p \rangle = \langle v(0, x), \mu_0(x) \rangle + \langle \mathcal{L}_f v(t, x_0, x_1), [\bar{\mu}_0 + \bar{\mu}_1](t, x_0, x_1) \rangle$$

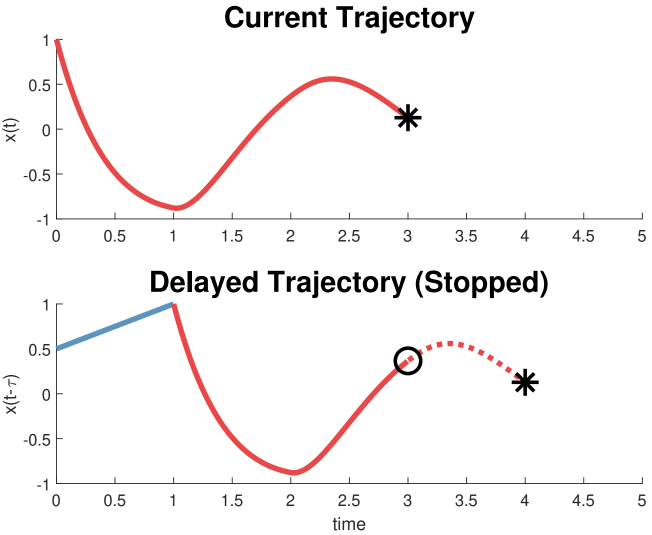
Specific realization $x_0 = x(t),\ x_1 = x(t-\tau)$

Consistency Issue



Inconsistent elapsed times

Consistency Fix



Early stopping in delayed time

Consistency Constraint

Inspired by changing limits of integrals

$$\left(\int_0^{t^*} + \int_{t^*}^{\min(T, t^*+\tau)}\right) \phi(t, x(t-\tau))dt$$
$$= \left(\int_{-\tau}^{0} + \int_{0}^{\min(T-\tau, t^*)}\right) \phi(t'+\tau, x(t'))dt'$$

Shift-push $S^\tau_\#$ with $\langle \phi, S^\tau_\# \mu\rangle = \langle S^\tau \phi, \mu\rangle = \langle \phi(t+\tau, x), \mu\rangle$

Consistency constraint with time-slack ν

$$\pi^{tx_1}_\#(\bar{\mu}_0 + \bar{\mu}_1) + \nu = S^\tau_\#(\mu_h + \pi^{tx_0}_\# \bar{\mu}_0)$$

Measure Linear Program

Linear program for DDE peak estimation with $p^* \geq P^*$

$$
\begin{aligned}
p^* = \ \sup \quad & \langle p, \mu_p \rangle && \text{(2a)} \\
& \text{History-Validity}(\mu_0, \mu_h) && \text{(2b)} \\
& \text{Liouville}(\mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1) && \text{(2c)} \\
& \text{Consistency}(\bar{\mu}_h, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(2d)} \\
& \text{Measure Definitions for } (\mu_h, \mu_0, \mu_p, \bar{\mu}_0, \bar{\mu}_1, \nu) && \text{(2e)}
\end{aligned}
$$

Largest PSD matrices $\binom{2n+1+d+\lfloor \deg f/2 \rfloor}{d+\lfloor \deg f/2 \rfloor}$ from $\bar{\mu}_0, \bar{\mu}_1$

Optimal Control (DDE)

Optimal Control Setup

Running cost $J(t, x, u)$, Terminal cost $J_T(x)$

Bounded control set U, Terminal set $X_T \subseteq X$

$$
\begin{aligned}
P^* = \min_{u(t)} \int_{t=0}^{T} & J(t, x(t), u(t))dt + J_T(x(T)) \\
& \dot{x} = f(t, x(t), x(t-\tau), u(t)) && \forall t \in [0, T] \\
& u(t) \in U && \forall t \in [0, T] \\
& x(t) = x_h(t) && \forall t \in [-\tau_r, 0] \\
& x(T) \in X_T
\end{aligned}
$$

Optimal control is (if it exists) $u^*(t)$

Measure-Valued Solution

Tuple of measures for the delayed case

Terminal	$\mu_T \in \mathcal{M}_+(X)$
Occupation Start	$\bar{\mu}_0 \in \mathcal{M}_+([0, T-\tau] \times X^2 \times U)$
Occupation End	$\bar{\mu}_1 \in \mathcal{M}_+([T-\tau, T] \times X^2 \times U)$

Modified terms (v.s. peak estimation)

Objective: $\langle J, \bar{\mu}_0 + \bar{\mu}_1 \rangle + \langle J_T, \mu_T \rangle$

Controlled Liouville: $\forall v \in C^1([0, T] \times X)$:

$$\begin{aligned}\langle v(T, x), \mu_T \rangle = & \langle v(0, x), \mu_0(x) \rangle \\ & + \langle \mathcal{L}_f v(t, x_0, x_1, u), [\bar{\mu}_0 + \bar{\mu}_1](t, x_0, x_1, u) \rangle\end{aligned}$$

History validity: $\mu_0 = \delta_{x_h(0)}$, μ_h occupation measure of $x_h(\cdot)$

Time-slack: $\nu = 0$ due to fixed-terminal-time

Subvalue Functional Structure

Choose $v \in C^1([0, T] \times X)$, $\phi \in C([0, T] \times X)$

Define functional that stops at time T

$$V(t, z, w(\cdot)) = v(t, z) + \int_t^{\min(t+\tau, T)} \phi(s, w(s - \tau))ds \quad (3)$$

Endpoint-evaluation $V(T, z, w(\cdot)) = v(t, z)$

Time-derivative at $(t, z, w(\cdot), u)$

$$\dot{V} = \mathcal{L}_f v(t, z, w, u) + I_{[0, T-\tau]}(t)\phi(t + \tau, z) - \phi(t, w(-\tau)) \quad (4)$$

Discontinuous $\dot{V}$ at time $t = T - \tau$

Subvalue Functional LP

HJB inequalities for subvalue: $J_T \geq v(T, x)$, $\dot{V} + J \geq 0$

Lower bound $d^* = p^* \leq P^*$ (strong duality)

$$
\begin{aligned}
d^* = \ \sup \quad & v(0, x_h(0)) + \textstyle\int_0^\tau \phi(t, x_h(t-\tau))dt && \text{(5a)}\\
& \forall x \in X_T: \quad J_T(x) - v(T, x) \geq 0 && \text{(5b)}\\
& \forall (t, x_0, x_1, u) \in [0, T-\tau] \times X^2 \times U: \\
& \qquad \mathcal{L}_f v + J(t, x_0, u) - \phi(t, x_1) + \phi(t+\tau, x_0) \geq 0 && \text{(5c)}\\
& \forall (t, x_0, x_1, u) \in [T-\tau, T] \times X^2 \times U: \\
& \qquad \mathcal{L}_f v + J(t, x_0, u) - \phi(t, x_1) \geq 0 && \text{(5d)}\\
& v \in C^1([0, T] \times X), \ \phi \in C([0, T] \times X) && \text{(5e)}
\end{aligned}
$$

Peak Estimation Functional LP

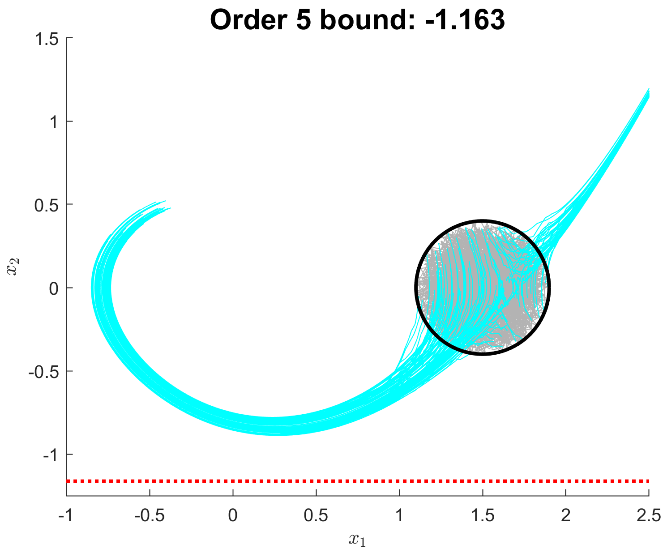
Strong duality $p^* = d^*$ with compactness, regularity

$$d^* = \inf_{\gamma \in \mathbb{R}} \ \gamma + \int_{-\tau}^{0} \xi(t) dt$$

$$\begin{aligned}
&\xi(t) + \phi(t+\tau, x) \geq 0 && \forall (t, x) \in H_0 \\
&\gamma \geq v(0, x) && \forall x \in X_0 \\
&v(t, x) \geq p(x) && \forall (t, x) \in [0, T] \times X \\
&\phi(t, x) \leq 0 && \forall [0, T] \times X \\
&\mathcal{L}_f v(t, x_0) + \phi(t, x_1) \leq \phi(t+\tau, x_0) && \forall [0, T] \times X \\
&\mathcal{L}_f v(t, x_0) + \phi(t, x_1) \leq 0 && \forall [T-\tau, T] \times X^2 \\
&\xi \in C([-\tau, 0]), \ \ \phi \in C([0, T] \times X) \\
&v \in C^1([0, T] \times X)
\end{aligned}$$

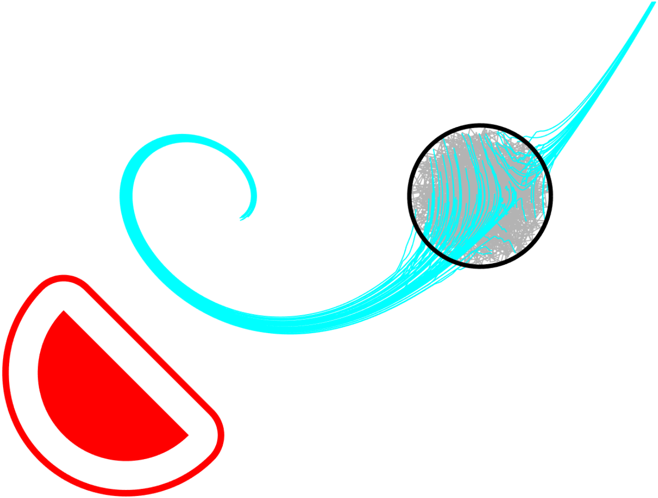
Examples

Peak Estimate with Multiple Histories



Minimize x_2 on the delayed Flow system

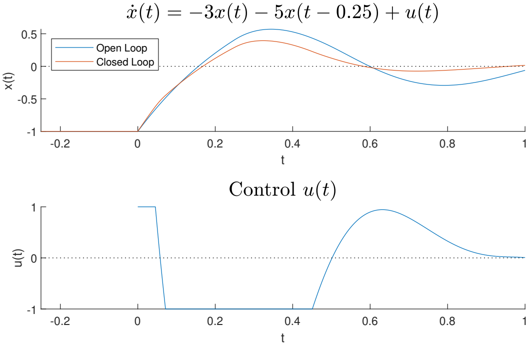
Distance Estimate with Multiple Histories



Min $c(x; X_u) \geq 0.1820$ on the delayed Flow system

Control Example: Trajectory

$$J(t,x,u) = 0.5x^2 + 0.5Ru^2 \qquad J_T(x) = 0$$



Order-5 lower bound 0.0391, executed cost 0.0394

$$u(t) \in \operatorname*{argmin}_{u \in U} \mathcal{L}_f v(t, x(t), x(t-\tau), u) + J(t, x(t), u)$$

Take-aways

Conclusion

Posed peak estimation and OCP for DDEs

Defined measure-valued solutions

Solved sequence of SDPs to get peak/OCP bounds

Future Work

- Conditions for no conservatism
- Improve scaling/computational complexity
- Better bounds and conditioning
- Interpretation of $\xi(t)$ in peak estimation LP
- Reachable set estimation

Thank you for your attention