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Frequency Domain Identification via Sum-of-Rational Optimization

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1 Introduction

Goal Parameter estimation of rational canonical LTI TF model through global optimization, in a prediction-error framework

Challenge Non-convexity: most available techniques only guarantee local solutions, and require good initialization (see [4, Ch.10] and [5, Ch.9])

Proposed approach

- Problem formulated in freq. domain as a non-convex rational optimization problem.
- Equivalently written as an infinite-dimensional LP in the space of finite nonnegative Borel measures.
- Moment-SOS hierarchy used to get *finite* SDPs.
- Global optimally certified a-posteriori: checking a rank condition.

2 Problem Overview

Model: $y_t = G(q;\theta)u_t + e_t, \qquad G(q;\theta) = \frac{B(q;b)}{A(q;a)} = \frac{b_1 q^{-1} + \dots b_n q^n}{1 + a_1 q^{-1} + \dots a_n q^n}$

Parameters: $\theta = [a, b] \quad \in \Theta$

Data set: $D_N = \{\omega_f, \hat{G}_f\}_{f=1}^N$ frequency-domain data; e.g. $\hat{G}_f \equiv$ ETFE

Estimator:

$$\hat{\theta}_N := \arg\min_{\theta\in\Theta} \sum_{f=1}^N \left| W_f \left(\hat{G}_f - G(e^{j\omega_f};\theta) \right) \right|^2 \quad = \quad \arg\min_{\theta\in\Theta} \sum_{f=1}^N \frac{p_f(\theta)}{q_f(a)}$$

$W_f \neq 0$ are given weights,

$$p_f(\theta) = |W_f \hat{G}_f (1 + A(e^{-j\omega_f}; a)) - W_f B(e^{-j\omega_f}; b)|^2,$$
$$q_f(a) = |1 + A(e^{-j\omega_f}; a)|^2,$$

Degree-2 polynomials in θ

3 An Equivalent Infinite-dimensional LP

Sum-of-Rational optimization problem defining $\hat{\theta}_N$ is reformulated as LP on finite nonnegative Borel measures $\mu, \nu_1, \dots, \nu_N$ [1]

$$\begin{aligned} p^* = \inf & \textstyle\sum_{f=1}^{N_f} \langle p_f(\theta), \nu_f(\theta) \rangle \\ & \langle 1, \mu \rangle = 1, \\ & \mu, \nu_1, \dots, \nu_N \in \mathcal{M}_+(\Theta) \\ & \forall (\alpha, \beta) \in \mathbb{N}^{2n}, \; f = 1, \dots, N \\ & \quad \langle a^\alpha b^\beta q_f(a), \nu_f(\theta) \rangle = \langle a^\alpha b^\beta, \mu(\theta) \rangle. \end{aligned}$$

If Θ guarantees stability of $G(g;\theta)$, then $\forall (a,b) \in \Theta: \; q_f(a) > 0$,

$$p^* = \min_{\theta\in\Theta} \sum_{f=1}^N \left| W_f \left(\hat{G} - G(e^{j\omega_f};\theta) \right) \right|^2$$

Such a Θ is defined via a polynomial matrix inequality in a (see [2]).

4 Hierarchy of SDPs

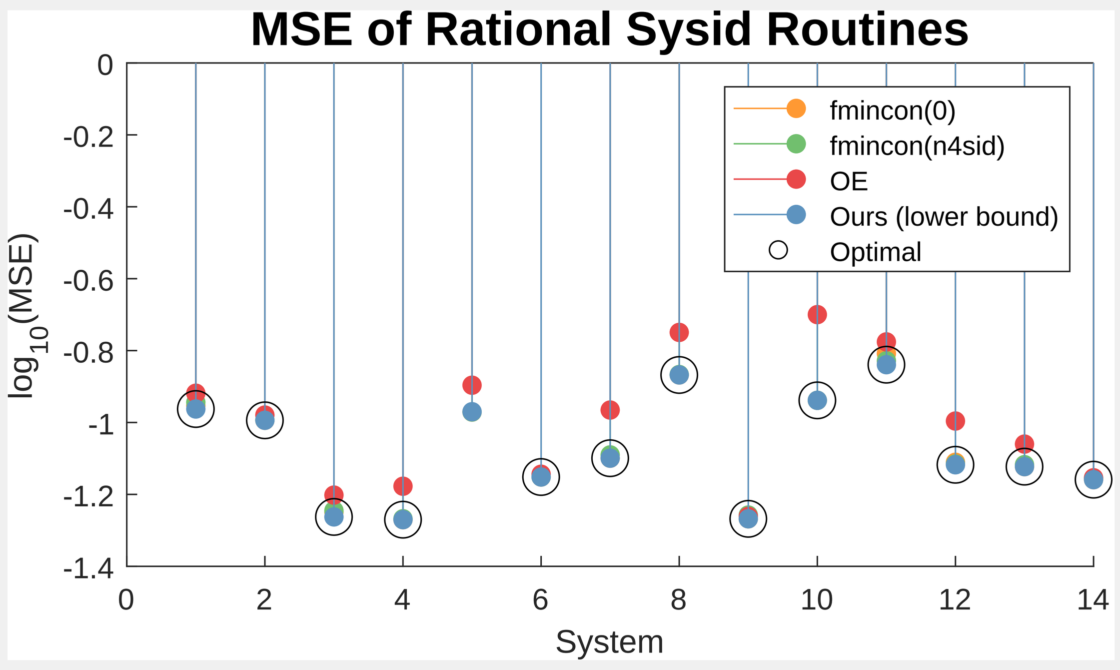
When Θ is a compact basic semi-algebraic set, a truncation of degree d using the Moment-SOS approach [3], gives an SDP

$$\begin{aligned} p_d^* = \inf_{y, y^f} & \sum_{f=1}^N \mathbb{L}_{y^f}(p_f(\theta)) \\ & y_0 = 1 \\ & \forall (\alpha, \beta) \in \mathbb{N}_{2d}^n, f = 1, \dots, N \\ & \quad \mathbb{L}_{y^f}(a^\alpha b^\beta q_f(a)) = \mathbb{L}_y(a^\alpha b^\beta) \\ & \mathbb{M}_d[\Theta y], \forall f: \mathbb{M}_{d+1}[\Theta y^f] \succeq 0 \end{aligned}$$

- Size of the largest LMI is independent of N;
- $p_d^* \to p^*$ as $d \to \infty$ (if Θ has the stability constraint)
- Exploit *term-sparsity*: q_f does not depend on b
- Solution recovery $\hat{\theta}_N$ under rank-condition on $\mathbb{M}_d[\Theta y]$

5 Preliminary Results

Monte Carlo: sampled gains $\hat{G}_f = G(e^{j\omega};\theta_\circ) + E_f, \quad E_f \overset{\text{iid}}{\sim} \mathcal{CN}(0, 0.1^2)$



6 Extensions

- *Continuous-time models:* with band-limited inputs by minimizing $\sum_{f=1}^N |W_f \left(\hat{G} - G(\Omega_f;\theta) \right)|^2$; includes fractional-order models.
- *Closed-loop identification:* by modifying p_f and q_f using a known controller and reference signal.
- *Min-Max estimation* by solving $\min_{\theta\in\Theta} \max_{f\in 1..N_f} |W_f \left(G_f - G(e^{j\omega_f}) \right)|^2$

7 Conclusion

- Frequency-domain formulation $\equiv$ global rational optimization problem that can be solved via a multi-measure approach
- Several extensions possible

References

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[4] L. Ljung. *System Identification: Theory for the User*. Pearson Education, 1999.

[5] R. Pintelon and J. Schoukens. *System identification: a frequency domain approach*. John Wiley & Sons, 2012.