Quantifying Safety under Uncertainty using Occupation Measures

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Control Seminars @UCI

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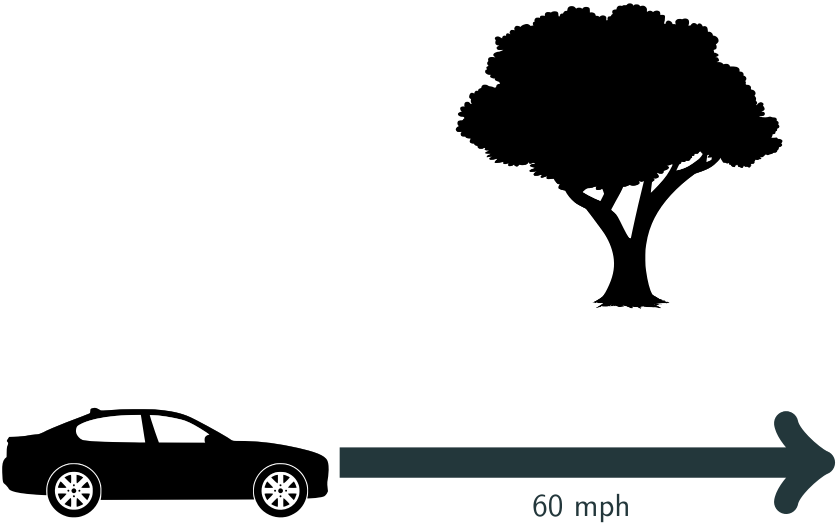
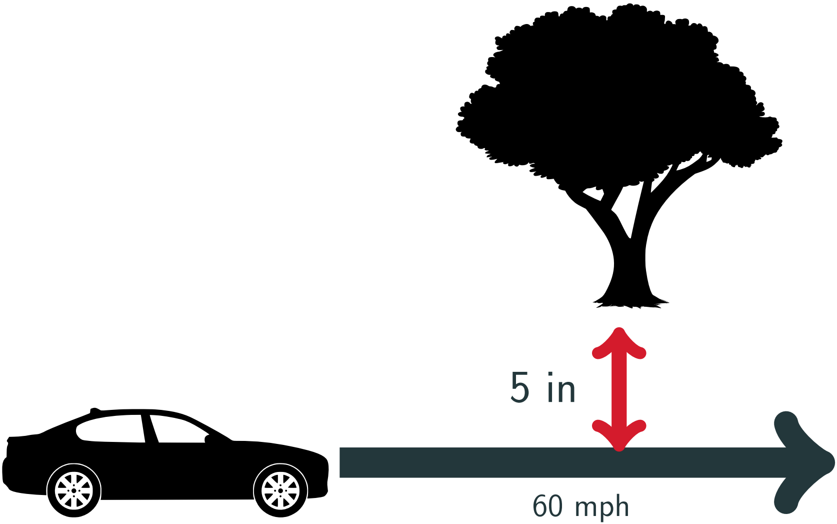
Safety Example



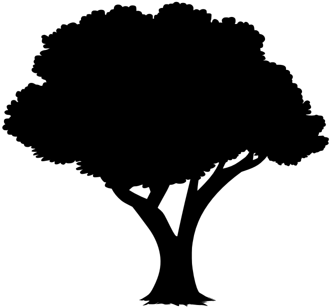
Safety Example (Barrier/Density Function)

Safety Example (Distance Estimate)

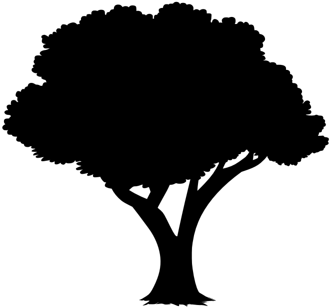


Safety Example

Safety Quantification



Safety Example (Crash Control Effort)



Safety Quantification

Motivation: Epidemic

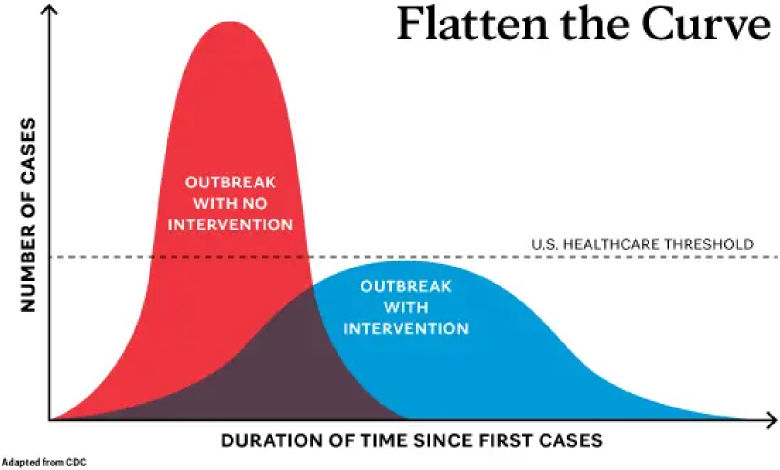


Image credit to Mayo Clinic News Network

Main Ideas

Pose safety quantification problems

Want convex, convergent, bisection-free algorithms

Formulate using convex programs in measures

Increasing-quality bounds using Semidefinite Programming

Overview of Presentation

Background: Peak estimation and Measures

Quantifications: Peak and Distance estimation

Dynamics with uncertainties:

1. Robust (compact-valued)
2. Stochastic (noncompact, value-at-risk)

Wrap-up

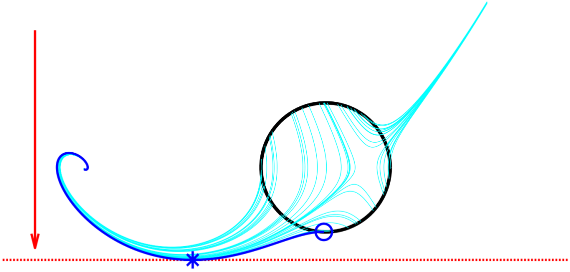
Peak Estimation Background

Peak Estimation Background

Find extreme value of $p(x)$ along trajectories

$$P^* = \sup_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \qquad x(0) = x_0$$



$$p(x) = -x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

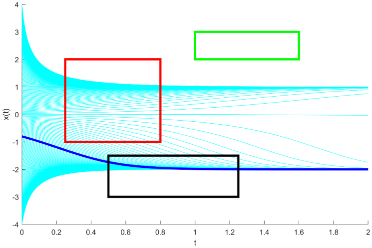
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

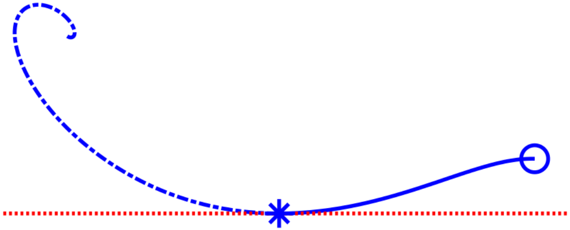
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t,x) \in C([0,T] \times X)$

$$
\begin{aligned}
\mu_0^* : \quad & \langle v(0,x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : \quad & \langle v(t,x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : \quad & \langle v(t,x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*)) dt
\end{aligned}
$$

Liouville Equation

Lie derivative (instantaneous change along f) $\forall v \in C^1$:

$$\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x) \tag{1a}$$

Conservation law: final = initial + accumulated change

$$\langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \tag{1b}$$
$$\mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \tag{1c}$$

Liouville 'represents' dynamics $\dot{x}(t) = f(t, x(t))$

Measures for Peak Estimation

Infinite-dimensional Linear Program (Cho, Stockbridge, 2002)

$$
\begin{aligned}
p^* = \sup \quad & \langle p(x), \mu_p \rangle && \text{(2a)} \\
& \langle 1, \mu_0 \rangle = 1 && \text{(2b)} \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(2c)} \\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(2d)} \\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(2e)}
\end{aligned}
$$

Instance of Optimal Control Program (Lewis and Vinter, 1980)

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle \leq p^*$

$P^* = p^*$ if compactness, Lipschitz properties hold

Moments for Peak Estimation

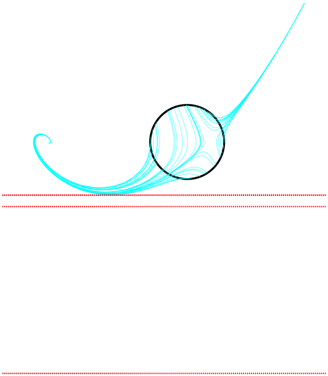
Moment: $y_\alpha = \langle x^\alpha, \nu \rangle \ \forall \alpha \in \mathbb{N}^n$

Moment matrix $\mathbb{M}[y]_{\alpha\beta} = y_{\alpha+\beta}$ is PSD (dual to SOS)

$$\mathbb{M}_2[y] = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{11} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0$$

Liouville induces affine relation in $(\mu^0, \mu^p, \mu) \to (y^0, y^p, y)$

Peak Estimation Example Bounds



Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)

Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

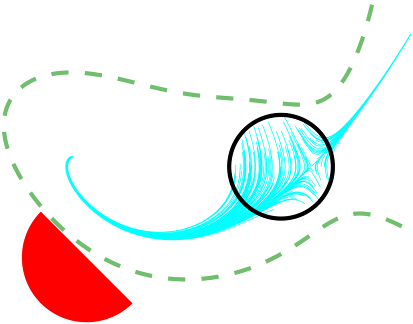
Max. PSD size: $\binom{(n+1)+(d+\lfloor \deg f/2 \rfloor)}{n+1}$ (Fantuzzi, Goluskin, 2020)

Distance and Safety

Barrier Program (Safety)

Barrier function $B : X \to \mathbb{R}$ indicates safety, binary certificate

$$\begin{aligned} &B(x) \leq 0 && \forall x \in X_u \\ &B(x) > 0 && \forall x \in X_0 \\ &f(x) \cdot \frac{\partial B}{\partial x}(x) + \phi(B(x)) \geq 0 && \forall x \in X \end{aligned}$$

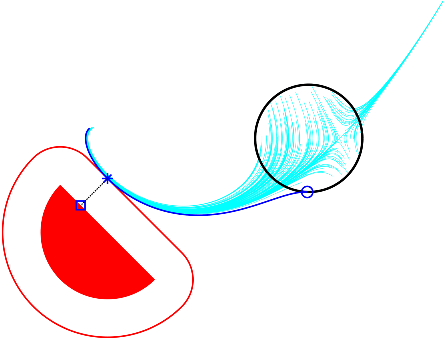


Distance Estimation Problem

Unsafe set X_u, point-set distance $c(x; X_u) = \inf_{y \in X_u} c(x, y)$

$$P^* = \inf_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in [0, T], \quad x(0) = x_0$$



L_2 bound of 0.2831

Distance Program (Measures)

Introduce joint $\eta(x, y)$ (from optimal transport)

Infinite Dimensional Linear Program (Convergent)

$$
\begin{aligned}
p^* = \ \inf \quad & \langle c(x,y), \eta(x,y) \rangle && \text{(3a)} \\
& \langle 1, \mu_0 \rangle = 1 && \text{(3b)} \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(3c)} \\
& \pi^x_\# \eta(x,y) = \pi^x_\# \mu_p(t,x) && \text{(3d)} \\
& \eta \in \mathcal{M}_+(X \times X_u) && \text{(3e)} \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) && \text{(3f)} \\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(3g)}
\end{aligned}
$$

Near-optimal trajectories if moment-matrix $\approx$ rank-1

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

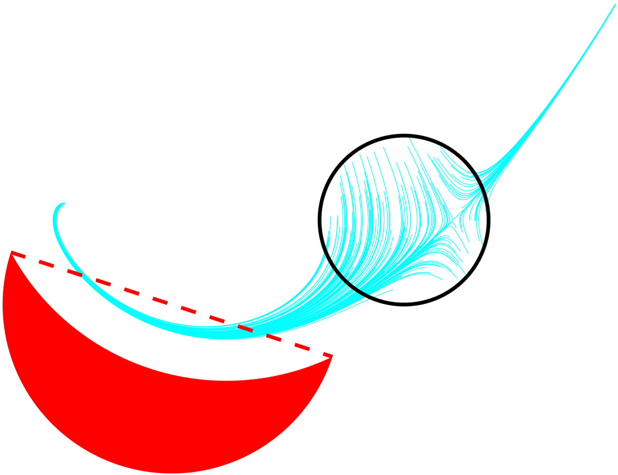
Degree d, dynamics degree $\tilde{d} = d + \lfloor \deg(f)/2 \rfloor - 1$

Bounds: $p^*_d \leq p^*_{d+1} \leq \ldots \leq p^* = P^*$

Measure	$\mu_0(x)$	$\mu_p(t,x)$	$\mu(t,x)$	$\eta(x,y)$
PSD Size	$\binom{n+d}{d}$	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$	$\binom{2n+d}{d}$

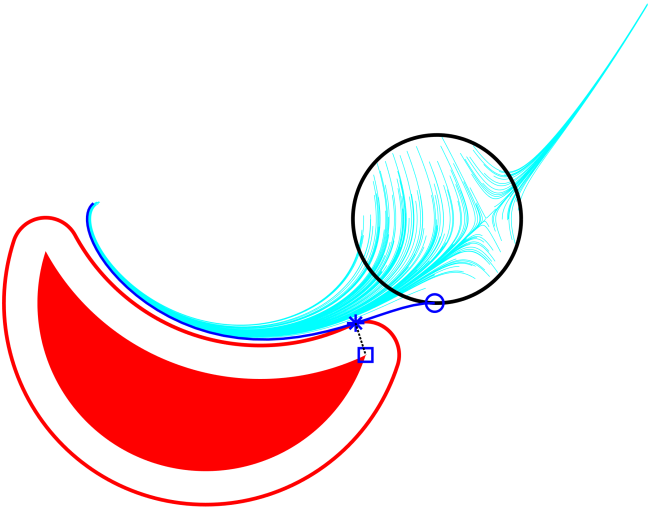
Timing scales approximately as $\max((1+n)^{6\tilde{d}}, (2n)^{6d})$

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

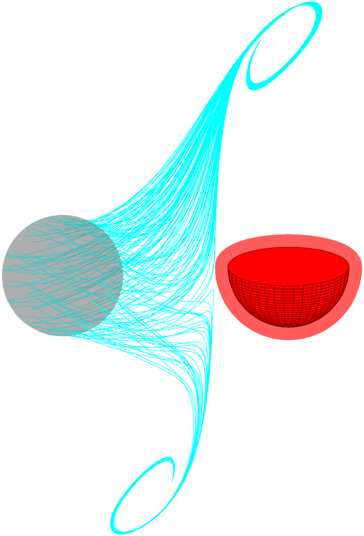
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

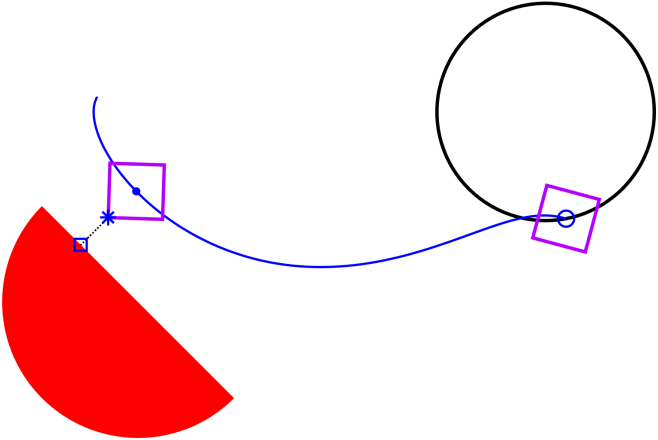
$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

Safety of Shapes

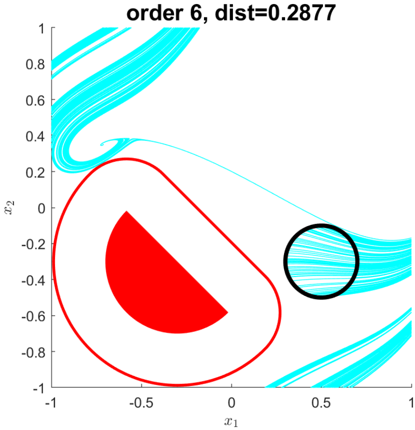
Points on shape S with orientation ω (e.g., rigid body motion)



L_2 bound of 0.1465, rotating square

Hybrid Systems

Continuous dynamics with discrete jumps/transitions



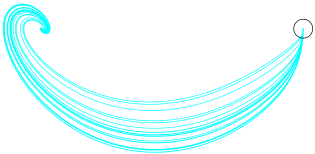
$$R_{\text{left}\to\text{bottom}} = [1 - x_2; x_1], \qquad R_{\text{right}\to\text{top}} = [x_2; x_1]$$

Compact-Valued Uncertainty

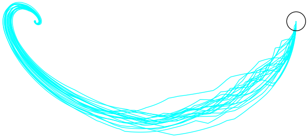
with D. Henrion, M. Korda

System with Uncertainty Example

Time-Independent Uncertainty



Time-Dependent Uncertainty



$$\dot{x} = [x_2, -x_1 w - x_2 + x_1^3/3]$$

$$w \in [0.5, 1.5],\ \ x_0 = [1; 0]$$

Peak Estimation with Uncertainty

Time independent $\theta \in \Theta$

Time dependent $w(t) \in W, \ \forall t \in [0, T]$

$$P^* = \sup_{t \in [0, T], \, x_0 \in X_0, \, \theta \in \Theta, \, w(t)} p(x(t \mid x_0, \theta, w(t)))$$

$$\dot{x}(t) = f(t, x(t), \theta, w(t)), \quad w(t) \in W \quad \forall t \in [0, T]$$

Adversarial optimal control problem with $(\theta, w(\cdot))$

Uncertain Peak Measure Program

$P^* = p^*$ when f Lipschitz, $[0, T] \times X \times \Theta \times W$ compact

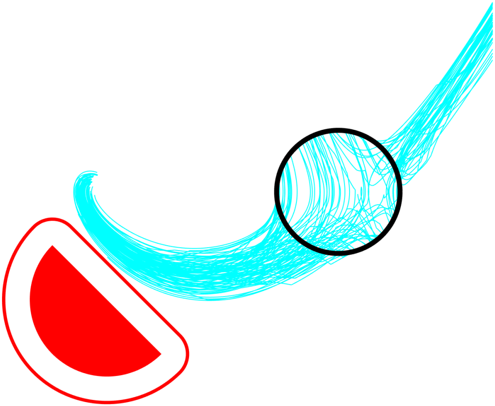
$$
\begin{aligned}
p^* = \ \sup \quad & \langle p(x), \mu_p \rangle \\
& \langle v(t, x, \theta), \mu_p \rangle = \langle v(0, x, \theta), \mu_0 \rangle + \langle \mathcal{L}_f v(t, x, \theta), \mu \rangle \quad \forall v \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu \in \mathcal{M}_+([0, T] \times X \times \Theta \times W) \\
& \mu_p \in \mathcal{M}_+([0, T] \times X \times \Theta) \\
& \mu_0 \in \mathcal{M}_+(X_0 \times \Theta)
\end{aligned}
$$

Complexity: μ has maximal PSD size $\binom{n+\tilde{d}+N_\theta+N_w}{\tilde{d}}$

Distance Uncertainty

Time dependent uncertainty $w(t) \in W \ \forall t \in [0, T]$

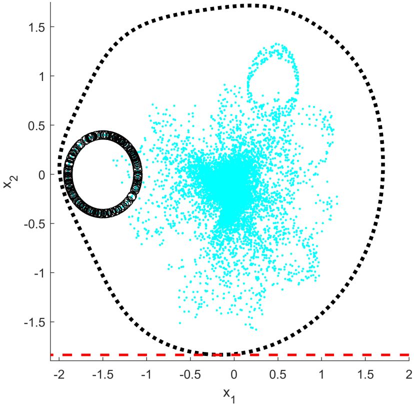
Uncertainty changes Liouville, Distance changes cost



L_2 bound of 0.1691, $w(t) \in [-1, 1]$

Other types of Uncertainty Structures

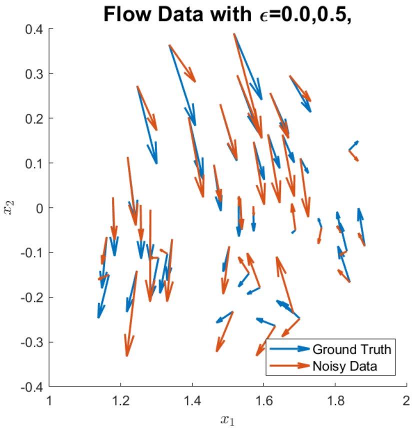
- Switching Uncertainty
- Polytoptic Uncertainty
- Lipschitz Uncertainty
- Discrete-Time



Discrete dynamics with switching and time-dependent uncertainty

Data-Driven Sampling: Flow System

Data $\mathcal{D} = \{(t_j, x_j, \dot{x}_j)\}_j$ under mixed L_∞-bounded noise



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Dynamics Model

Given data $\mathcal{D}$, budget ϵ, system model $\{f_0, f_\ell\}$

Parameterize ground truth F by functions in dictionary

$$\dot{x}(t) = f(t, x, w) = f_0(t, x) + \sum_{\ell=1}^{L} w_\ell f_\ell(t, x)$$

Ground truth satisfies corruption $J(w^*) \leq \epsilon$

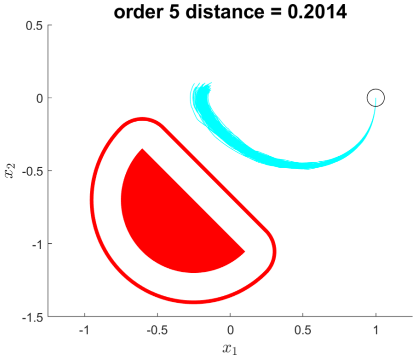
L_∞ example: $J(w) = \max_j \|f(t_j, x_j, w) - \dot{x}_j\|_\infty$

Distance Estimation Example (Flow)

Input-affine + Semidefinite Representable uncertainty

$$\mathcal{L}_f v(t,x,w) \leq 0 \qquad \forall (t,x,w) \in [0,T] \times X \times W$$

PSD Size 8568 $\to$ 56 ($L = 10$) using robust counterparts



$$\dot{x} = [x_2,\ \text{cubic}(x_1, x_2)]$$

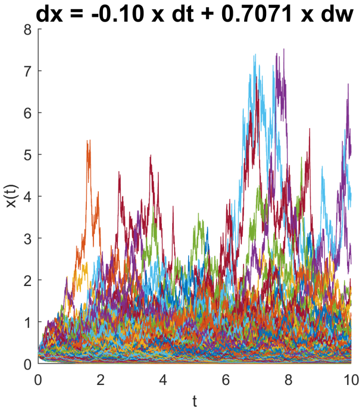
Peak Value-at-Risk Estimation

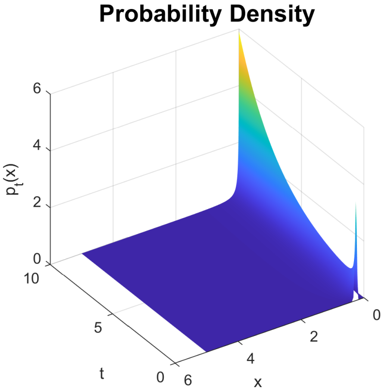
with M. Tacchi, A. Jasour

Stochastic Differential Equation

Multivariate SDE $dx = f(t,x)dt + g(t,x)dw$ (Itô)

Drift f and Diffusion g



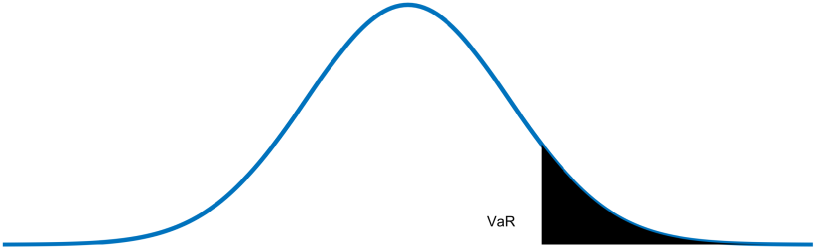


Geometric Brownian Motion

Value-at-Risk (Quantile)

ϵ-VaR of univariate measure $\xi(\omega)$ is unique number with

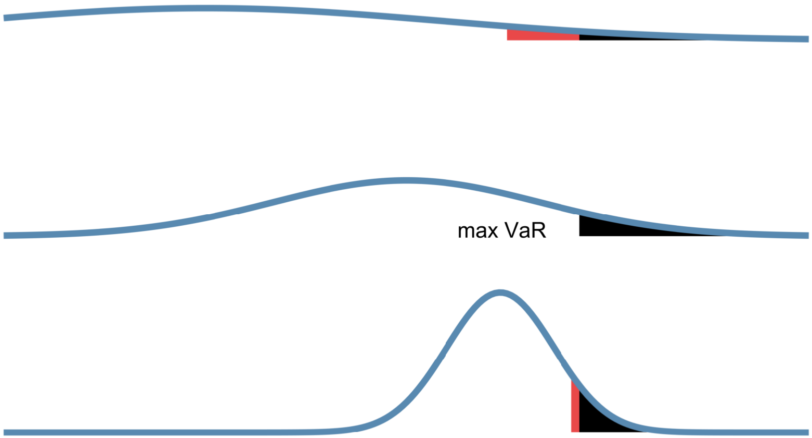
$$\text{Prob}_{\xi}(\omega \geq VaR_{\epsilon}(\xi)) = \epsilon$$



$VaR = 1.282$ for unit normal distribution at $\epsilon = 10\%$

Maximal Value at Risk

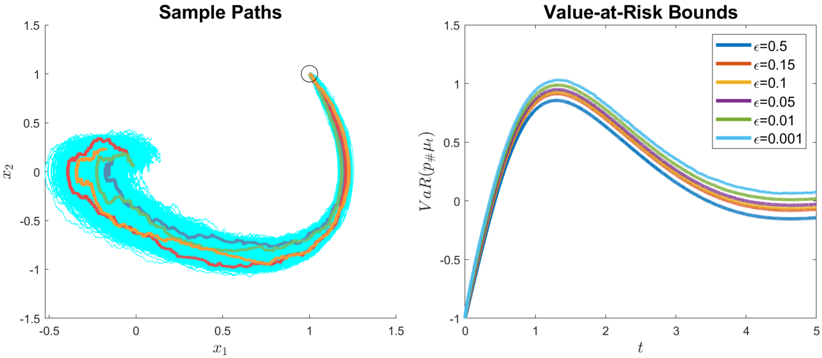
Maximize ϵ-VaR among multiple distributions



Red + Black areas = 10% probability

Value-at-Risk Example (Monte Carlo)

50,000 samples with $T = 5,\ \Delta t = 10^{-3}$



VaR of $p = -x_2$ along $dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$

Chance-Peak Problem

Maximize VaR of $p(x)$ along SDE trajectories

$p_{\#}\mu_{t^*}$: distribution of $p(x(t))$ at time t^*

$$P^* = \sup_{t^* \in [0, T]} VaR_\epsilon(p_{\#}\mu_{t^*}) \tag{4a}$$

$$dx = f(t, x)dt + g(t, x)dw \tag{4b}$$

$$\text{stopping time of } \min(t^*, \text{exit from } X) \tag{4c}$$

$$x(0) \sim \mu_0 \tag{4d}$$

Value-at-Risk Bounds

Concentration inequalities can upper-bound VaR

$$VaR_\epsilon(\xi) \leq \mathrm{stdev}(\xi) r + \mathrm{mean}(\xi)$$

Name	r	Condition
Cantelli	$\sqrt{1/(\epsilon) - 1}$	ξ probability distribution
VP	$\sqrt{4/(9\epsilon) - 1}$	ξ unimodal, $\epsilon < 1/6$

Conditional Value at Risk (CVaR) can also bound VaR

Concentration-Bounded Chance-Peak

Apply concentration inequalities to get upper bound $P_r^* \geq P^*$

Objective upper-bounds VaR w.r.t. time-t^* distribution μ_{t^*}

$$P_r^* = \sup_{t^* \in [0,T]} r\sqrt{\langle p^2, \mu_{t^*}\rangle - \langle p, \mu_{t^*}\rangle^2} + \langle p, \mu_{t^*}\rangle \tag{5a}$$

$$dx = f(t,x)dt + g(t,x)dw \tag{5b}$$

$$\text{stopping time of } \min(t^*, \text{exit from } X) \tag{5c}$$

$$x(0) \sim \mu_0 \tag{5d}$$

Max-Mean: $\epsilon = 0.5, \ r = 0$ (Cho, Stockbridge, 2002)

Occupation Measure Formulation

Occupation measure μ, terminal measure μ_τ

Second-Order Cone Program in measures (3d SOC)

$$
\begin{aligned}
p_r^* = \sup r\sqrt{\langle p^2, \mu_\tau\rangle - \langle p, \mu_\tau\rangle^2} + \langle p, \mu_\tau\rangle \quad && \text{(6a)}\\
\mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu \quad && \text{(6b)}\\
\mu_\tau,\ \mu \in \mathcal{M}_+([0, T] \times X) \quad && \text{(6c)}
\end{aligned}
$$

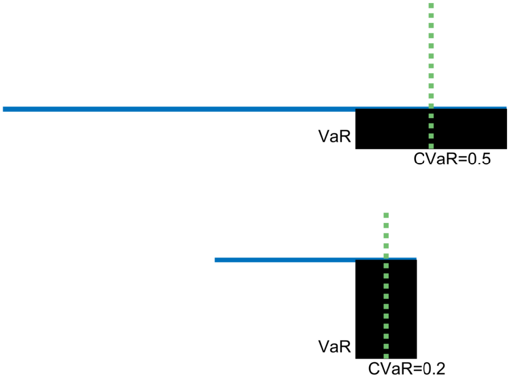
Generator $\mathcal{L}v = \partial_t v + f \cdot \nabla_x v + g^T(\nabla^2_{xx} v)g/2$ (Dynkin's)

Results in upper-bound $p_r^* \geq P_r^* \geq P^*$, use moments

Conditional Value-at-Risk

CVAR: Average quantity above the Value-at-Risk

$$CVaR_\epsilon(\xi(\omega)) = (1/\epsilon)\int_{\omega \geq VaR_\epsilon(\xi)} \omega d\xi(\omega)$$



Uniform distributions with same VaR, different CVaR (70%)

Properties of CVaR

CVaR is ‘Coherent Risk Measure’: convex, subadditive

Cantelli $=$ worst-case CVaR (Čerbáková, 2005)

$$\mathrm{stdev}(\xi)\sqrt{1/\epsilon - 1} + \mathrm{mean}(\xi) \geq \mathit{CVaR}_\epsilon(\xi) \geq \mathit{VaR}_\epsilon(\xi)$$

Scenario approach LP (Rockafeller, Ursayev, 2002):

$$\mathit{CVaR}_\epsilon(\xi) = \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{\epsilon} \int_{\omega > \alpha} (\omega - \alpha) d\xi(\omega).$$

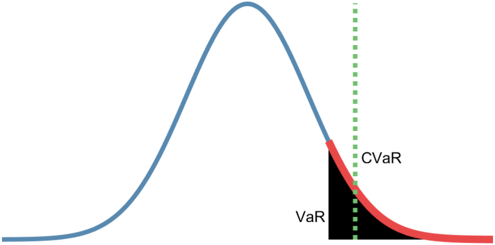
CVaR LP

Measure LP to compute CVaR

$$CVaR_\epsilon(\nu) = \sup_{\psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R})} \langle \omega, \psi \rangle \tag{7a}$$

$$\epsilon\psi + \hat{\psi} = \xi \tag{7b}$$

$$\langle 1, \psi \rangle = 1 \tag{7c}$$



VaR $= 1.2816$, CVaR$= 1.7550$, $\epsilon\psi \leq \xi$

CVaR Chance-Peak

Highest CVaR along SDE trajectories

$$P_c^* = \sup_{t^* \in [0,T]} CVaR_\epsilon(p_\# \mu_{t^*}) \tag{8a}$$
$$dx = f(t,x)dt + g(t,x)dw \tag{8b}$$
$$\text{stopping time of } \min(t^*, \text{ exit from } X) \tag{8c}$$
$$x(0) \sim \mu_0 \tag{8d}$$

Almost the same as VaR chance-peak, with $P_c^* \geq P^*$

CVaR Measure program

Add CVaR objective, constraints to chance-peak

$$
\begin{aligned}
p_c^* = \ \sup \quad & \langle \omega, \psi \rangle && \text{(9a)}\\
& \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu && \text{(9b)}\\
& \langle 1, \psi \rangle = 1 && \text{(9c)}\\
& \epsilon\psi + \hat{\psi} = p_\# \mu_\tau && \text{(9d)}\\
& \mu, \mu_\tau \in \mathcal{M}_+([0, T] \times X) && \text{(9e)}\\
& \psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R}) && \text{(9f)}
\end{aligned}
$$

Upper-bound $p_c^* \geq P_c^* \geq P^*$, LP in measures

Comparison of bounds

$P_r^* = p_r^*$ and $P_c^* = p_c^*$ if

1. SDE has unique solutions (Lipchitz, Growth)
2. $[0, T] \times X$ compact
3. $p(x)$ is continuous

$P^*_{\text{Cantelli}} \geq P_c^*$ always, but $(P_c^*,\ P^*_{\text{VP}})$ incomparable (so far)

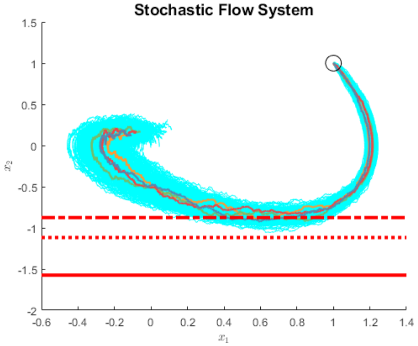
Empirically, degree-k moment LMIs satisfy $p^*_{\text{Cantelli},k} \geq p^*_{c,k}$

Chance-Peak Examples

Two-State

Stochastic Flow (Prajna, Rantzer) with $T = 5$, $p(x) = -x_2$

$$dx = \begin{bmatrix} x_2 \\ -x_1 - x_2 - \frac{1}{2}x_1^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} dw$$

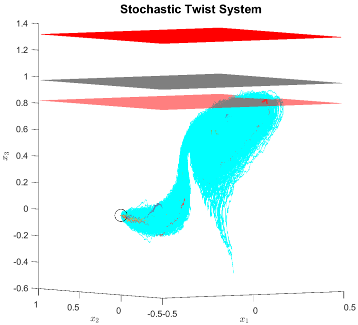


$d = 6$ (dash-dot=50%, dotted=85% CVAR, solid=85% VP)

Three-State

Stochasic Twist system with $T = 5$, $p(x) = x_3$

$$dx = \begin{bmatrix} -2.5x_1 + x_2 - 0.5x_3 + 2x_1^3 + 2x_3^3 \\ -x_1 + 1.5x_2 + 0.5x_3 - 2x_2^3 - 2x_3^3 \\ 1.5x_1 + 2.5x_2 - 2x_3 - 2x_1^3 - 2x_2^3 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} dw$$

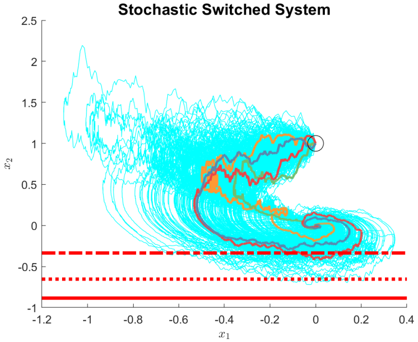


$d = 6$ (translucent=50%, gray=85% CVAR, solid=85% VP)

Two-State Switching

Switching subsystems at $T = 5$, $p(x) = -x_2$

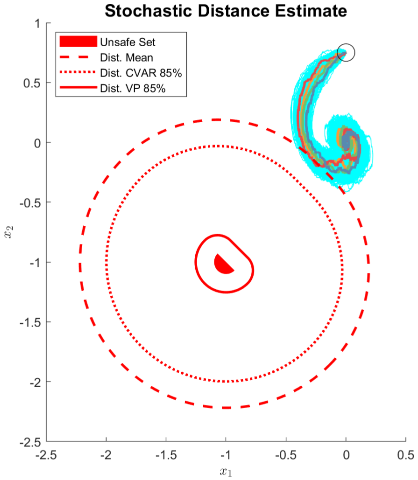
$$dx = \left\{ \begin{bmatrix} -2.5x_1 - 2x_2 \\ -0.5x_1 - x_2 \end{bmatrix}, \begin{bmatrix} -x_1 - 2x_2 \\ 2.5x_1 - x_2 \end{bmatrix} \right\} dt + \begin{bmatrix} 0 \\ 0.25x_2 \end{bmatrix} dw$$



$d = 6$ (dash-dot=50%, dotted=85% CVAR, solid=85% VP)

Two-State Distance

Maximize VaR of (negative) L_2 distance to X_u



$d = 6$ (dash-dot=50%, dotted=85% CVaR, solid=85% VP)

Take-aways

Summary

Noted importance of safety quantification

Extended occupation measure methods for peak estimation

Quantified safety in robust and stochastic settings

Safety is Important

Quantify using Peak Estimation