Safety Analysis using Distance Estimation and Measures

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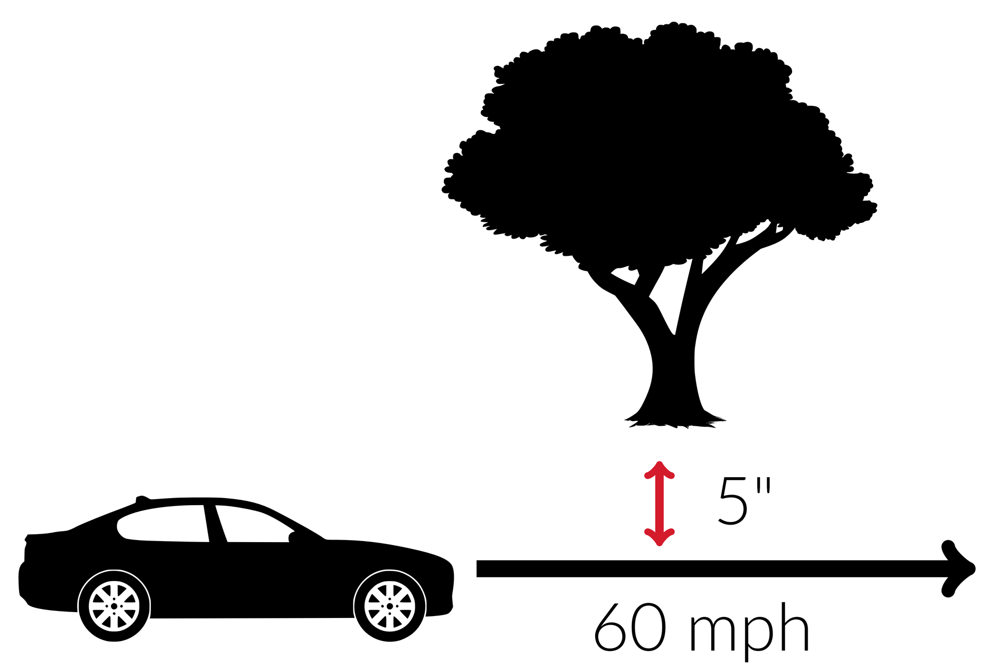
Motivation

Quantify trajectory safety (starting from X_0) by its **distance of closest approach** to the unsafe set X_u.



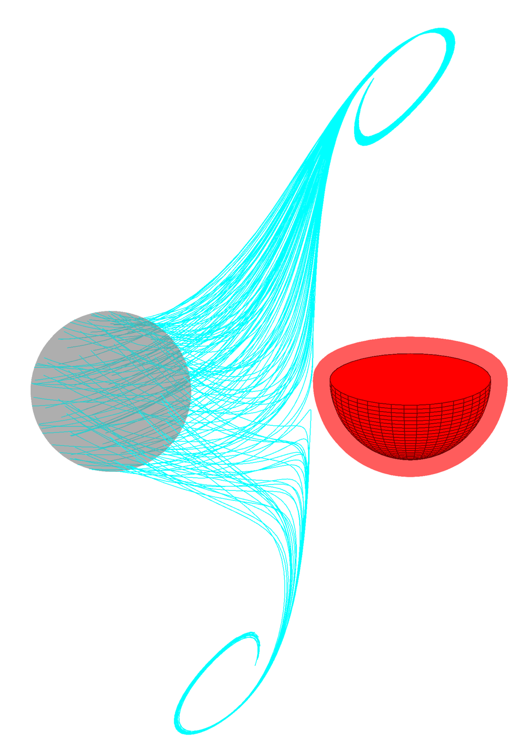
Clearance is important when planning control policies

Use convex optimization to compute converging lower-bounds

Distance Estimation

Distance $c(x, y)$ (e.g. $L_2: \|x-y\|_2$)

$$\begin{aligned} P^* = \inf_{t,\, x_0,\, y} \ & c(x(t \mid x_0), y) \\ & \dot{x}(t) = f(t, x), \quad \forall t \in [0, T] \\ & x_0 \in X_0, \ y \in X_u \end{aligned}$$



Red corona: certified L_2 clearance of ≥ 0.0425

Relax to Linear Program (LP) in Measures

Finite degree-d LMIs, $\tilde{d} = d + \lceil \deg f/2 \rceil - 1$

Name	Measure	PSD Size
Initial	$\mu_0(x) \in \mathcal{M}_+(X_0)$	$\binom{n+d}{n}$
Peak	$\mu_p(t, x) \in \mathcal{M}_+([0, T] \times X)$	$\binom{n+1+d}{d}$
Occ.	$\mu(t, x) \in \mathcal{M}_+([0, T] \times X)$	$\binom{n+1+\tilde{d}}{\tilde{d}}$
Joint	$\eta(x, y) \in \mathcal{M}_+(X \times X_u)$	$\binom{2n+d}{d}$

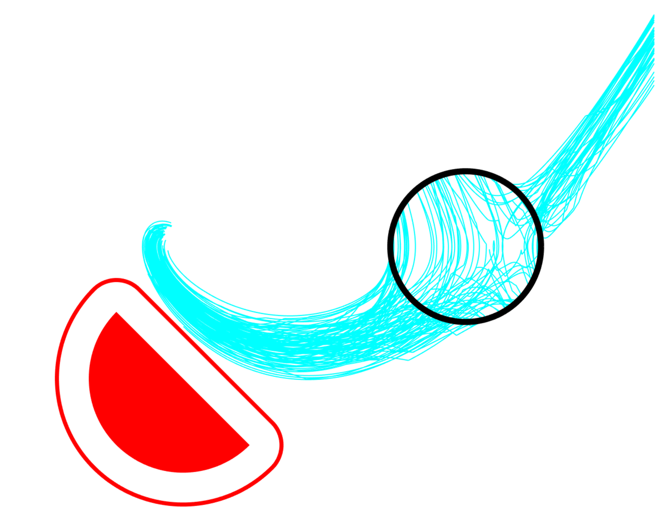
LP Objective and Constraints

Distance	$p^* = \inf \langle c(x, y), \eta(x, y) \rangle$
Probability	$\mu_0(X_0) = 1$
Liouville	$\mu_p = \delta_{t=0} \otimes \mu_0 + \mathcal{L}_f^\dagger \mu$
x-marginals	$\pi^x_\# \mu_p = \pi^x_\# \eta$

Uncertainty

Dynamics $\dot{x}(t) = f(t, x(t), h(t))$ with compactly-supported $h(t) \in H$

$$\mu_p = \delta_{t=0} \otimes \mu_0 + \pi^{tx}_\# \mathcal{L}^\dagger_{f(t,x,h)} \bar{\mu}$$
$$\bar{\mu} \in \mathcal{M}_+([0, T] \times X \times H)$$



Time-dependent uncertainty in vertical coordinate

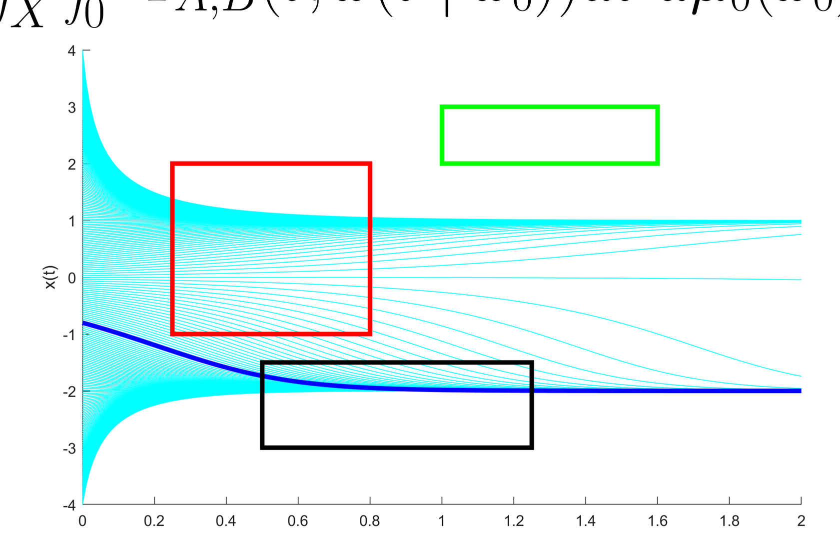
Time-independent uncertainty: new state $\theta' = 0, \ \theta \in \Theta, \ \dot{x} = f(t, x(t), \theta)$

Can exploit polytopic and switching structure for simpler LMIs

Occupation Measures

Given initial distribution $\mu_0 \in \mathcal{M}_+(X_0)$, occupation measure $\mu(A \times B)$ for sets $A \subseteq [0, T]$, $B \subseteq X$ is,

$$\int_X \int_0^T I_{A,B}(t, x(t \mid x_0)) dt \, d\mu_0(x_0)$$



Average amount of time trajectories spend in set

Set	$x_0 = -0.8$	$x_0 \in [-4, 4]$
Black	> 0	> 0
Red	0	> 0
Green	0	0

LMIs

Liouville $\mathrm{Liou}_{\alpha\beta}$: $\alpha \in \mathbb{N}^n, \ \beta \in \mathbb{N}$

$\langle x^\alpha t^\beta, \mu_p \rangle - \langle x^\alpha, \mu_0 \rangle \delta_{\beta 0} - \langle \mathcal{L}_f(x^\alpha t^\beta), \mu \rangle$

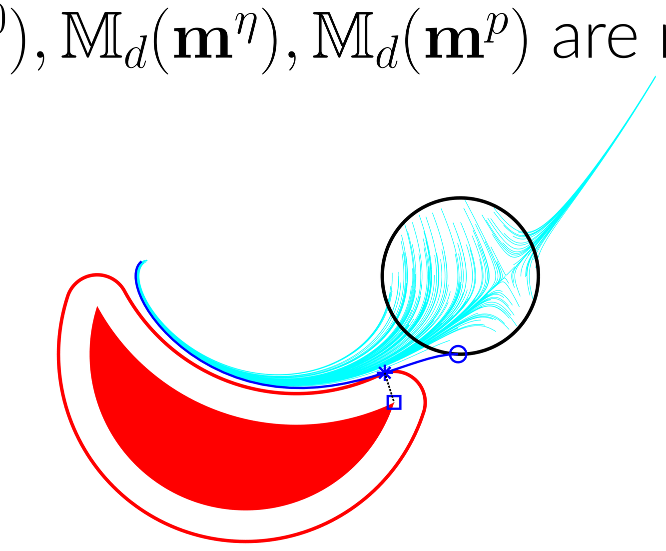
LMIs with $p^*_d \leq P^*$

$$\begin{aligned} p^*_d = \min \quad & \textstyle\sum_{\alpha,\gamma} c_{\alpha\gamma} \mathbf{m}^\eta_{\alpha\gamma}. \\ & \mathbf{m}^0_0 = 1 \\ & \forall (\alpha, \beta) \in \mathbb{N}^{n+1}_{\leq 2d}: \\ & \quad \mathrm{Liou}_{\alpha\beta}(\mathbf{m}^0, \mathbf{m}^p, \mathbf{m}) = 0 \\ & \forall \alpha \in \mathbb{N}^n_{\leq 2d}: \quad \mathbf{m}^\eta_{\alpha 0} = \mathbf{m}^p_{\alpha 0} \\ & \mathbb{M}_d(X_0 \mathbf{m}^0) \succeq 0 \\ & \mathbb{M}_d(([0, T] \times X)\mathbf{m}^p) \succeq 0 \\ & \mathbb{M}_{\tilde{d}}(([0, T] \times X)\mathbf{m}) \succeq 0 \\ & \mathbb{M}_d((X \times X_u)\mathbf{m}^\eta) \succeq 0. \end{aligned}$$

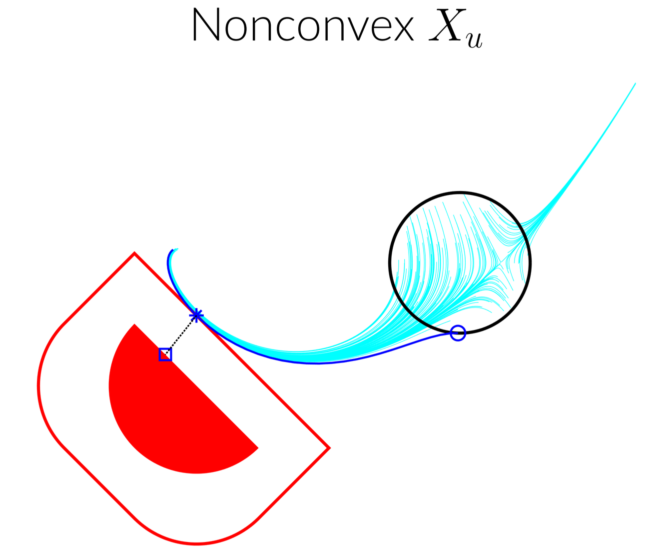
Under mild compactness and regularity conditions $\lim_{d \to \infty} p^*_d = P^*$.

Recovery

Approx. recovery possible when $\mathbb{M}_d(\mathbf{m}^0), \mathbb{M}_d(\mathbf{m}^\eta), \mathbb{M}_d(\mathbf{m}^p)$ are rank-1



Nonconvex X_u

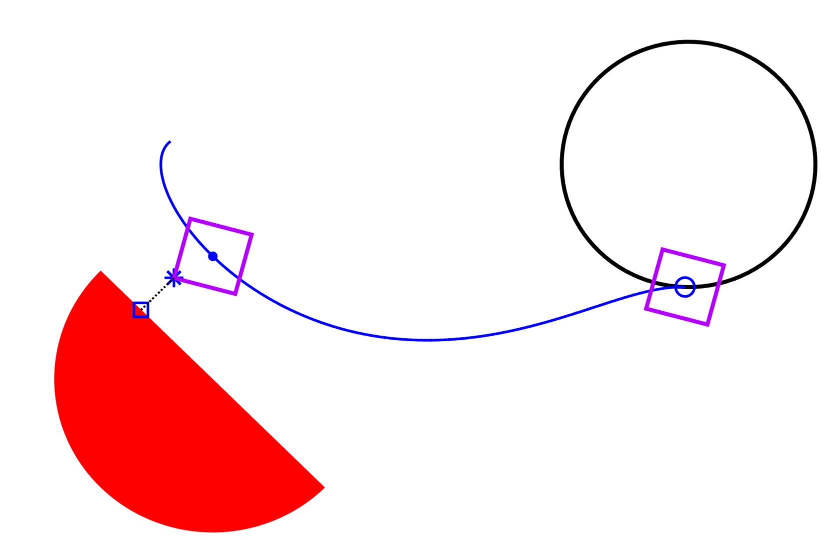


Piecewise $c(x, y) = \|x - y\|_1$

Shapes

Orientation ω, body coordinates s, coordinate transformation $x = A(s; \omega)$

$$\begin{aligned} P^* = \inf_{t, \omega_0, s, y} \ & c(A(s; \omega(t \mid \omega_0)), y) \\ & \dot{\omega}(t) = f(t, \omega) \qquad \forall t \in [0, T] \\ & \omega_0 \in \Omega_0, \ s \in S, \ y \in X_u \end{aligned}$$



Translating square: shape measure $\mu_s \in \mathcal{M}_+(S \times \Omega)$

`arxiv:2110.14047`