Risk Analysis of Stochastic Processes using Polynomial Optimization

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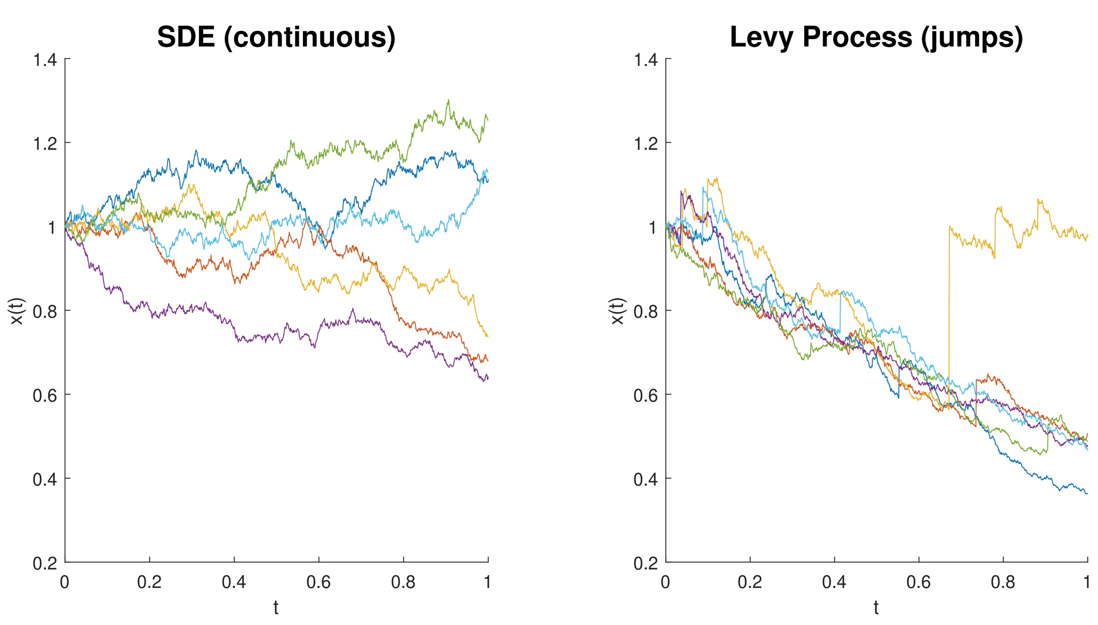
Stochastic Processes

Time-t-indexed prob. dists. $x(t)$



Uniquely described by generator $\mathcal{L}$

E.g., wind when flying, thermal noise in circuits, molecular interactions

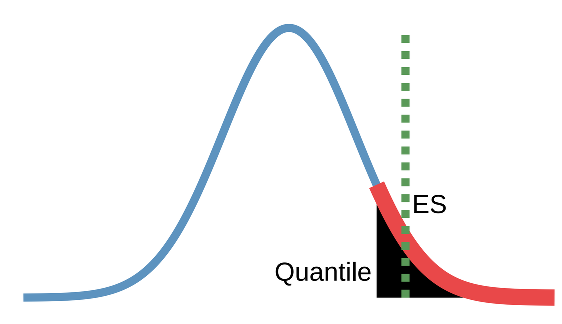
Expected Shortfall (ES)

ES: Average value above ϵ-quantile

Given state function $p(x)$ (e.g., height)

Want to find maximal ES of $p(x(t))$ along $\mathcal{L}$ trajectories (with $x(0) \sim \mu_0$)

$$\begin{aligned} ES_\epsilon(\nu) = \sup\ & \mathrm{mean}(\psi) \\ & \epsilon\psi + \hat{\psi} = \nu \\ & \langle 1, \psi\rangle = 1 \\ & \psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R}) \end{aligned}$$



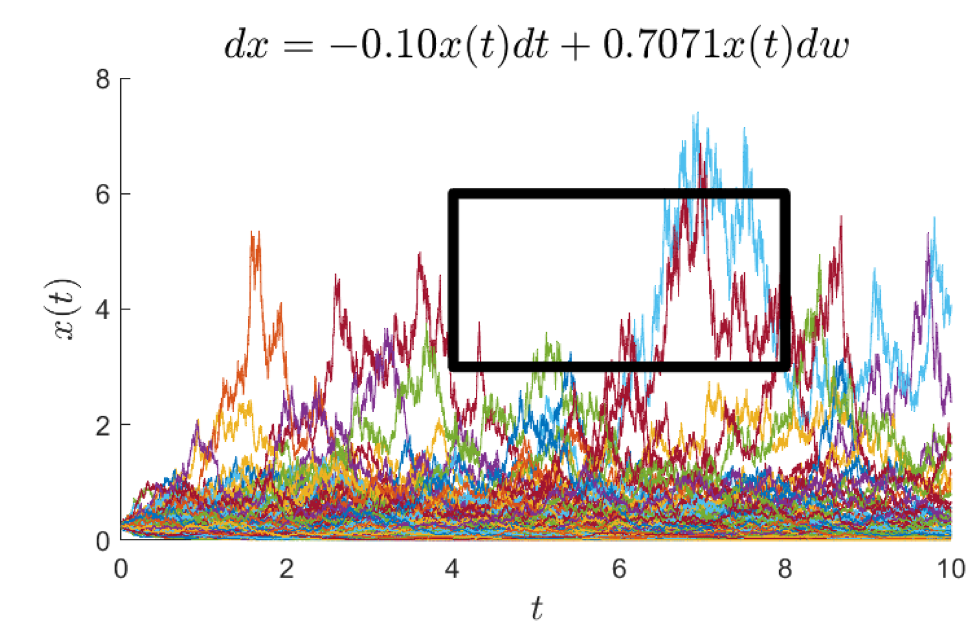
Top $\epsilon = 10\%$ ES, Quantile

Coherent risk measure, tail-sensitive

Occupation Measures

Borel measure $\mu(A, B)$ given dynamics, distribution of initial conditions

Average time trajectories spend in each time-space set $A \times B \in [0, T] \times X$



Chance-Peak Convex Linear Programs (LPs)

Primal-dual infinite-dimensional LPs: $P^* = p_c^* = d_c^*$ (compactness and regularity)

Occ. meas. μ, terminal μ_τ, ES $\psi, \hat{\psi}$

$$\begin{aligned} p_c^* = \sup\ & \mathrm{mean}(\psi) && \text{(1a)} \\ & \mu_\tau = \delta_0 \otimes \mu_0 + \mathcal{L}^\dagger \mu && \text{(1b)} \\ & \langle 1, \psi\rangle = 1 && \text{(1c)} \\ & \epsilon\psi + \hat{\psi} = p_\# \mu_\tau && \text{(1d)} \\ & \mu, \mu_\tau \in \mathcal{M}_+([0,T] \times X) \\ & \psi, \hat{\psi} \in \mathcal{M}_+(\mathbb{R}) \end{aligned}$$

$\mathcal{L}$-Martingale law (1b), ES (1c)-(1d)

Auxiliary func. $v(t, x)$, map w, scalar u

$$\begin{aligned} d_c^* = \inf\ & u + \langle v(0, \bullet), \mu_0\rangle && \text{(2a)} \\ & \mathcal{L}v \leq 0 && \text{(2b)} \\ & v \geq w \circ p && \text{(2c)} \\ & u + \epsilon\, w \geq \mathrm{id}_\mathbb{R} && \text{(2d)} \\ & w \geq 0 && \text{(2e)} \\ & u \in \mathbb{R}, v \in \mathrm{dom}(\mathcal{L}), w \in C(\mathbb{R}) \end{aligned}$$

$\mathbb{E}[v(t)]$ falls (2b), v sits above ES (2c)-(2e)

Complexity

Sum-of-Squares (SOS) truncations to (2), hierarchy of Semidefinite Programs

Degree $k \in \mathbb{N}$, and $\Delta = \lfloor k / \deg p(x) \rfloor$
Restrict to $v \in \mathbb{R}[t, x]_{\leq 2k},\ w \in \mathbb{R}[z]_{\leq 2\Delta}$

Dynamics degree D:

$$\deg x^\alpha t^\beta \leq 2k \implies \deg \mathcal{L} x^\alpha t^\beta \leq 2D$$

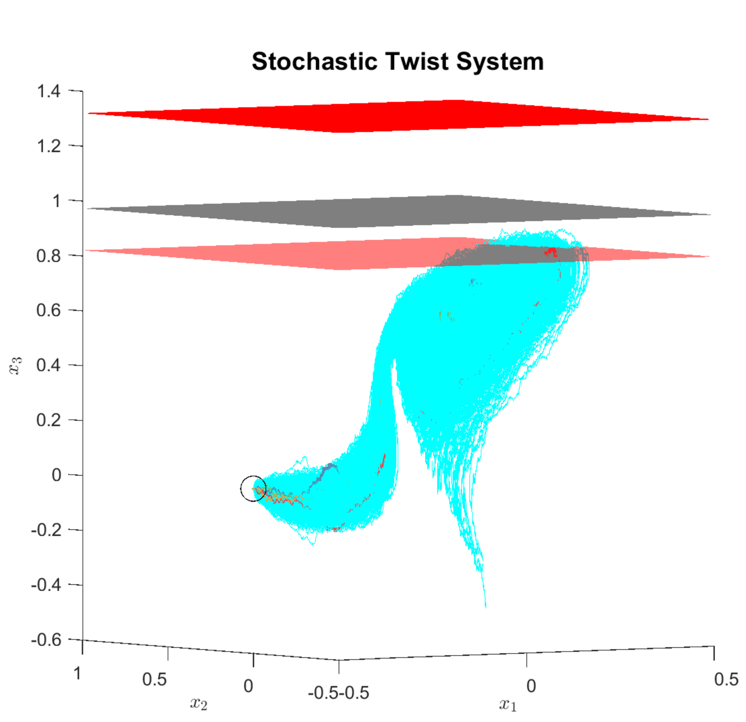
Largest PSD Matrix constraints $\binom{1+n+D}{D}$

Degree-k SOS-truncations converge from above to P^* as $k \to \infty$

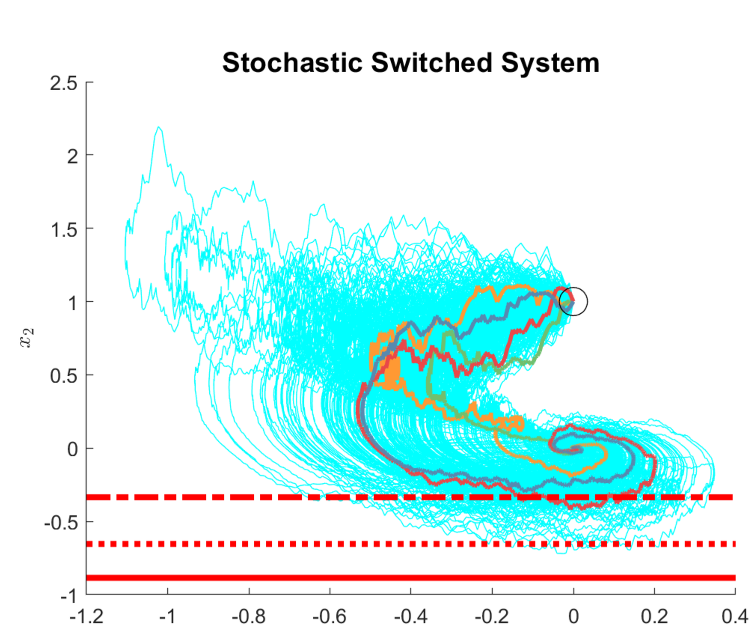
Numerical Examples

Stochastic processes with polynomial dynamics: $\mathcal{L}$ sends $\mathbb{R}[t, x]$ to $\mathbb{R}[t, x]$

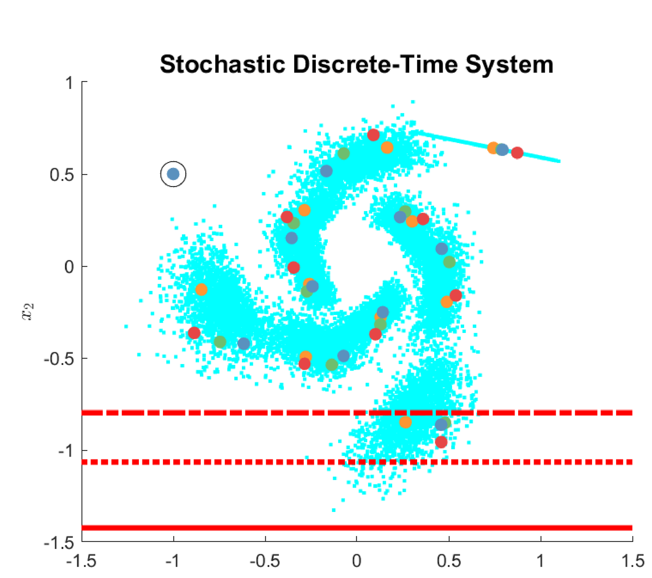
Max Mean $\leq$ **Max ES** (gray/dotted: ours) $\leq$ Max concentration-bound (previous)



$p(x) = x_3$



$p(x) = -x_2$



$p(x) = -x_2$

All SOS truncations are bounds computed at degree $k = 6$.

Certifiable analysis of stochastic trajectory behavior

Conclusion

Formed convex LPs to bound ES of stochastic processes

Polynomial Optimization (discretization) will converge to supremal ES

Technique Extensions:

- Distance Estimation
- Maximin Objectives
- Stochastic Hybrid Systems
- Non-Markovian processes

arxiv:2303.16064