Bounding distances to unsafe sets

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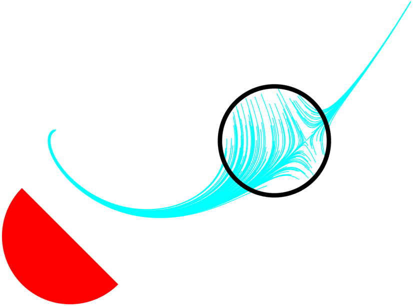
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Relax distance using optimal transport theory

Develop occupation measure programs to bound distance

Flow System Setting



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$
$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2, \quad \sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$$

Distance Function

Metric space (X, c) satisfying $\forall x, y \in X$:

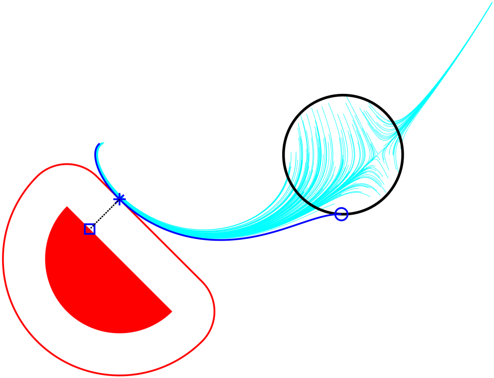
$$
\begin{aligned}
&c(x, y) > 0 && x \neq y \\
&c(x, x) = 0 && \\
&c(x, y) = c(y, x) && \\
&c(x, y) \leq c(x, z) + c(z, y) && \forall z \in X
\end{aligned}
$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x, y)$

Distance Estimation Problem (Nonconvex)

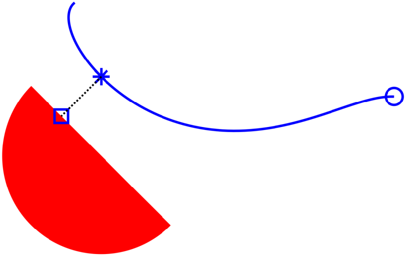
$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*\ x_0^*,\ t_p^*)$:

x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Safety Background

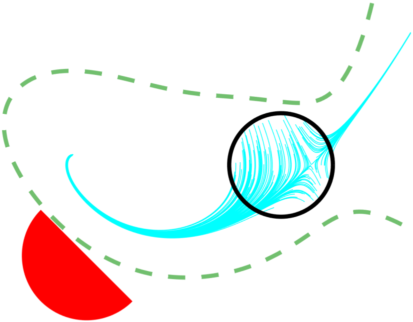
Barrier Program

Barrier function $B : X \to \mathbb{R}$ indicates safety

$$B(x) \leq 0 \qquad \forall x \in X_u$$

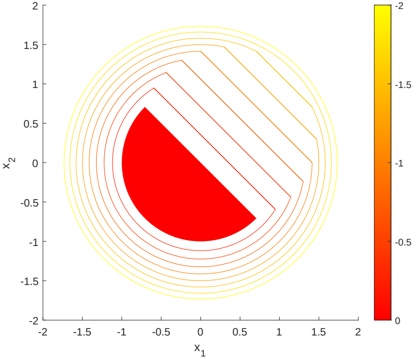
$$B(x) > 0 \qquad \forall x \in X_0$$

$$f(x) \cdot \frac{\partial B}{\partial x}(x) \geq 0 \qquad \forall x \in X$$



Half-circle Contours

Unsafe set $X_u = \{x \mid 1 - x_1^2 - x_2^2 \geq 0, -x_1 - x_2 \geq 0\}$



$q \leq 1 - x_1^2 - x_2^2$ $\qquad$ $q \leq -x_1 - x_2$

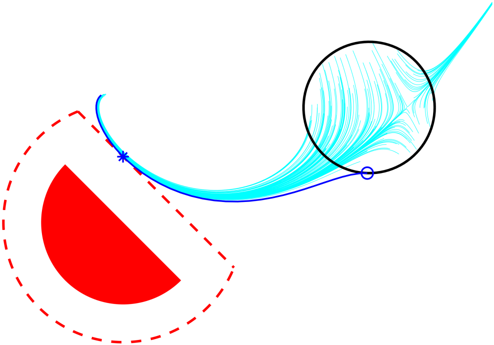
$$q = -0.25, -0.5, \ldots, -2$$

Safety Margin (2020)

Unsafe set $X_u = \{x \mid p_i(x) \geq 0 \; \forall i = 1 \ldots N_u\}$

Safety margin $p^* = \max_{t,x_0} \min_i p_i(x(t \mid x_0))$

If $p^* < 0$, no trajectories enter X_u (safe)

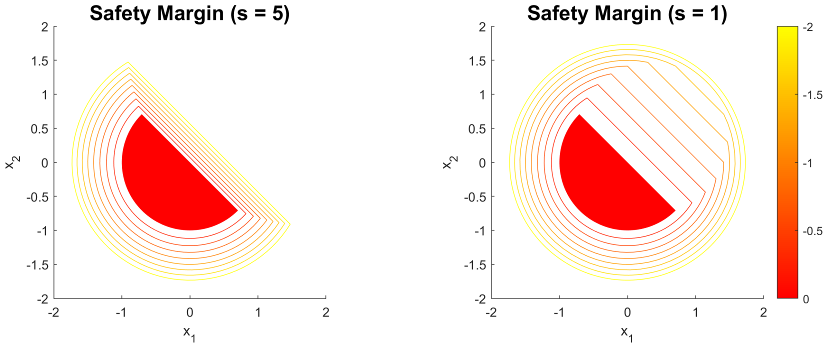


safe: $p^* \leq -0.2831$

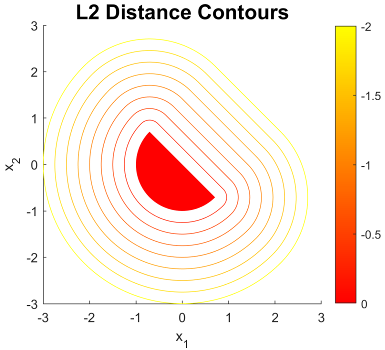
Safety Margin Scaling

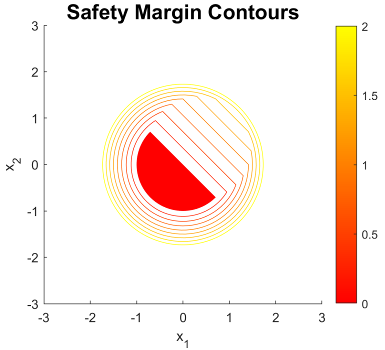
Scale factor in constraints

$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq s(-x_1 - x_2)$$



Distance vs. Safety Margin





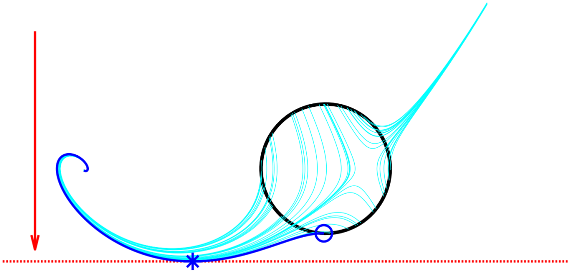
Peak Estimation

Peak Estimation Background

Find minimum value of $p(x)$ along trajectories

$$P^* = \min_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$p(x) = x_2, \qquad \dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Measures

Nonnegative Borel Measure μ

Assigns each set $A \subseteq X$ a 'size' $\mu(A)$ (Measure)

Mass $\mu(X) = \langle 1, \mu \rangle = 1$: Probability distribution

$\mu \in \mathcal{M}_+(X)$: space of measures on X

$f \in C(X)$: continuous function on X

Pairing by Lebesgue integration $\langle f, \mu \rangle = \int_X f(x) d\mu(x)$

Dirac Delta Measure

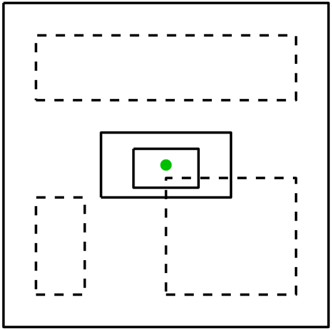
Dirac delta $\delta_{x'}(A) = \begin{cases} 1 & x' \in A \\ 0 & x' \notin A \end{cases}$

Probability measure: $\delta_{x'}(X) = 1$

$\mu(A) = 1$: Solid Box

$\mu(A) = 0$: Dashed Box

$\langle f(x), \delta_{x'} \rangle = f(x')$



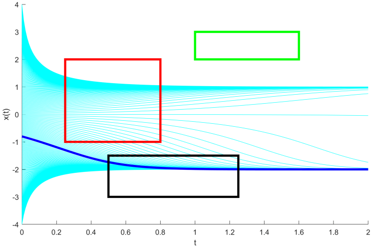
Occupation Measure

Time trajectories spend in set

Test function
$v(t, x) \in C([0, T] \times X)$

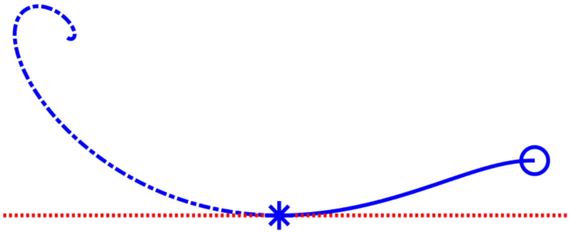
Single trajectory:
$\langle v, \mu \rangle = \int_0^T v(t, x(t \mid x_0))dt$

Averaged trajectory: $\langle v, \mu \rangle = \int_X \left(\int_0^T v(t, x)dt \right) d\mu_0(x)$



$x' = -x(x+2)(x-1)$

Connection to Measures



Measures: Initial μ_0, Peak μ_p, Occupation μ

For all functions $v(t,x) \in C([0,T] \times X)$

$$\begin{aligned}
\mu_0^* : \quad & \langle v(0,x), \mu_0^* \rangle = v(0, x_0^*) \\
\mu_p^* : \quad & \langle v(t,x), \mu_p^* \rangle = v(t_p^*, x_p^*) \\
\mu^* : \quad & \langle v(t,x), \mu^* \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*)) dt
\end{aligned}$$

Measures for Peak Estimation

Infinite dimensional linear program (Cho, Stockbridge, 2002)

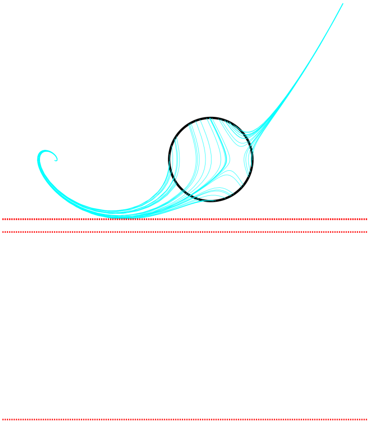
$$
\begin{aligned}
p^* = \ & \min \quad \langle p(x), \mu_p \rangle && \text{(1a)}\\
& \langle 1, \mu_0 \rangle = 1 && \text{(1b)}\\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \quad \forall v && \text{(1c)}\\
& \mu, \mu_p \in \mathcal{M}_+([0,T] \times X) && \text{(1d)}\\
& \mu_0 \in \mathcal{M}_+(X_0) && \text{(1e)}
\end{aligned}
$$

Test functions $v(t,x) \in C^1([0,T] \times X)$

Lie derivative $\mathcal{L}_f v = \partial_t v(t,x) + f(t,x) \cdot \nabla_x v(t,x)$

$(\mu_0^*, \mu_p^*, \mu^*)$ is feasible with $P^* = \langle p(x), \mu_p^* \rangle$

Peak Estimation Example Bounds

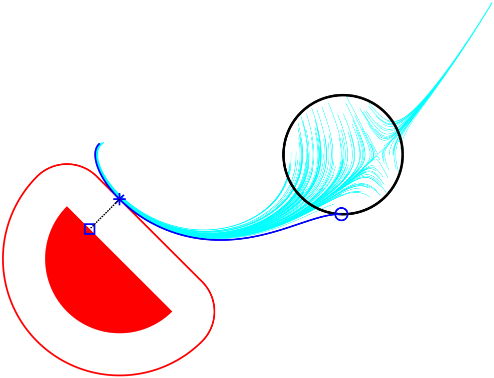


Converging bounds to min. $x_2 = -0.5734$ (moment-SOS)
Box region $X = [-2.5, 2.5]$, time $t \in [0, 5]$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$
$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Connection to Peak Estimation

Specific form of problem

$$p(x) = c(x; X_u)$$

Moment-SOS hierarchy requires polynomial data

Function $c(x; X_u)$ generally non-polynomial

$$\min_{y \in [-1,1]} \|x - y\|_2 = \begin{cases} 0 & x \in [-1, 1] \\ |x - \mathrm{sign}(x)| & \text{else} \end{cases}$$

Distance Relaxation

Distance in points $\rightarrow$ Expectation of distance

$$\begin{array}{lcl} c(x,y) & & \langle c(x,y), \eta \rangle \\ x \in X & \rightarrow & \langle 1, \eta \rangle = 1 \\ y \in X_u & & \eta \in \mathcal{M}_+(X \times X_u) \end{array}$$

Joint probability measure η

Inspired by Optimal Transport

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\mu_0^* : \qquad \delta_{x=x_0^*}$$

$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$

$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t,x) \in C([0,T] \times X)$

$$\mu^* : \qquad \langle v(t,x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Distance Program (Measures)

Infinite Dimensional Linear Program (Convergent)

$$p^* = \min \quad \langle c(x,y), \eta \rangle \tag{2a}$$

$$\langle 1, \mu_0 \rangle = 1 \tag{2b}$$

$$\langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle \qquad \forall v \tag{2c}$$

$$\langle w(x), \eta(x,y) \rangle = \langle w(x), \mu_p(t,x) \rangle \qquad \forall w \tag{2d}$$

$$\eta \in \mathcal{M}_+(X \times X_u) \tag{2e}$$

$$\mu_p, \ \mu \in \mathcal{M}_+([0,T] \times X) \tag{2f}$$

$$\mu_0 \in \mathcal{M}_+(X_0) \tag{2g}$$

Prob. Measures: $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1$

Computational Complexity

Use moment-SOS hierarchy (Archimedean assumption)

Degree d, dynamics degree $\tilde{d} = d + \lceil \deg(f)/2 \rceil - 1$

Bounds: $p^*_d \leq p^*_{d+1} \leq \ldots \leq p^* = P^*$

Measure	$\mu_0(x)$	$\mu_p(t, x)$	$\mu(t, x)$	$\eta(x, y)$
PSD Size	$\binom{n+d}{d}$	$\binom{1+n+d}{d}$	$\binom{1+n+\tilde{d}}{\tilde{d}}$	$\binom{2n+d}{d}$

Timing scales approximately as $\max((1+n)^{6\tilde{d}}, (2n)^{6d})$

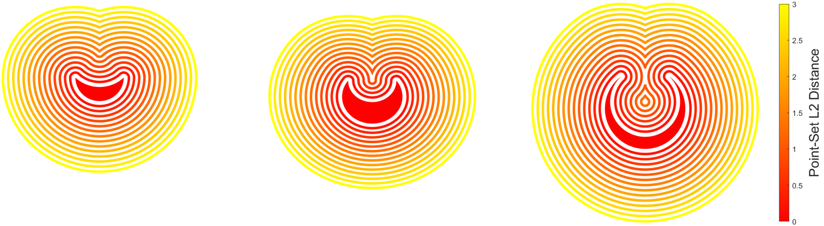
Approximation and Recovery

Attempt recovery if LMI solution has low rank

Moment matrices for $(\mu_0, \mu_{\boldsymbol{p}}, \eta)$ are rank-1

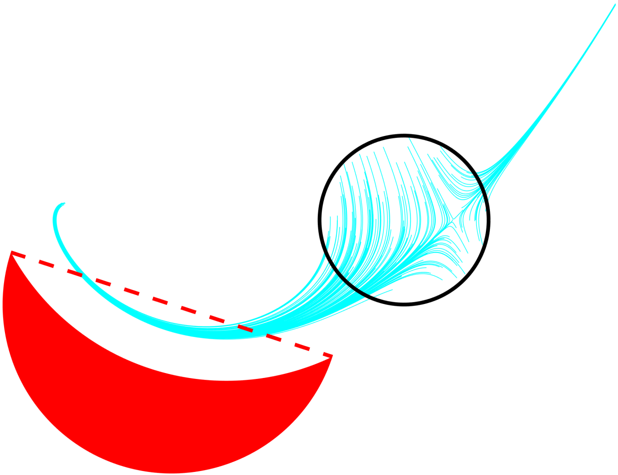
Related to optima extraction in polynomial optimization

Moon L2 Contours



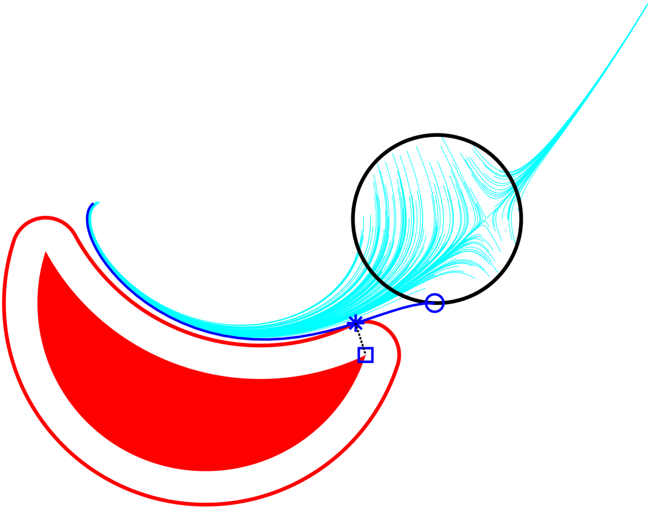
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)



L_2 bound of 0.1592

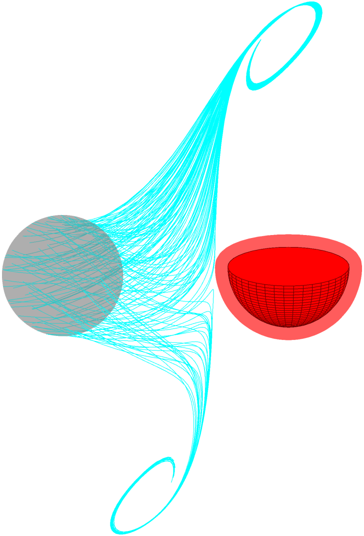
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$



L_2 bound of 0.0425

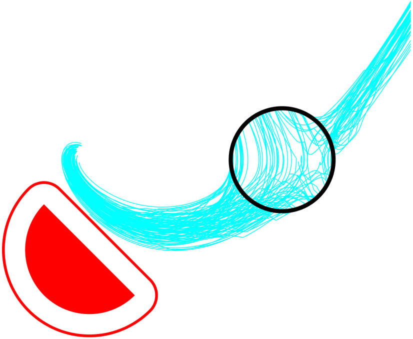
Distance Variations

Distance Uncertainty

Time dependent (bounded) uncertainty $w(t) \in W \; \forall t \in [0, T]$

Dynamics $\dot{x}(t) = f(t, x(t), w(t))$

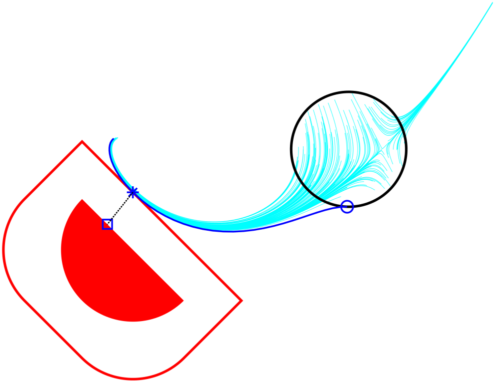
Young measure $\mu(t, x, w)$, Liouville term $\langle \mathcal{L}_f v(t, x, w), \mu \rangle$



L_2 bound of 0.1691, $w(t) \in [-1, 1]$

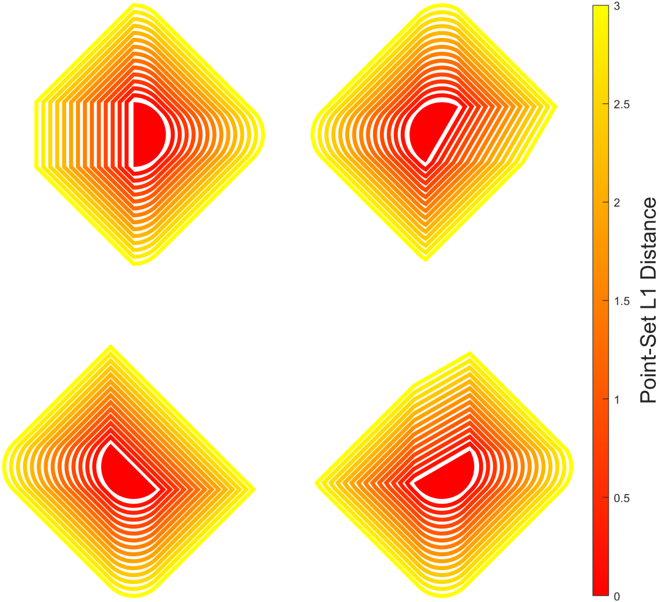
Lifted Distance (L1) Example

$$\|x - y\|_1 = \sum_i |x_i - y_i| \quad \implies \quad \begin{aligned} &\min \ \sum_i q_i \\ -q_i \leq \langle x_i &- y_i, \eta\rangle \leq q_i \end{aligned}$$



L_1 bound of 0.4003

Half-Circle L1 Contours



Sparsity

Separable $c(x, y) = \sum_k c_k(x_k, y_k)$

Use correlative sparsity with measures and cliques

$$\eta_k : \quad I_i = (x_k : x_n,\ y_1 : y_k) \qquad \forall k = 1, \ldots, n$$

Reduced computational complexity

	Measure	PSD size	Multiplicity
Dense	η	$\binom{2n+d}{d}$	1
Sparse	$\{\eta_k\}_{k=1}^{n}$	$\binom{n+1+d}{d}$	n

Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

$$A : S \times \Omega \to X \qquad (s, \omega) \mapsto A(s; \omega)$$

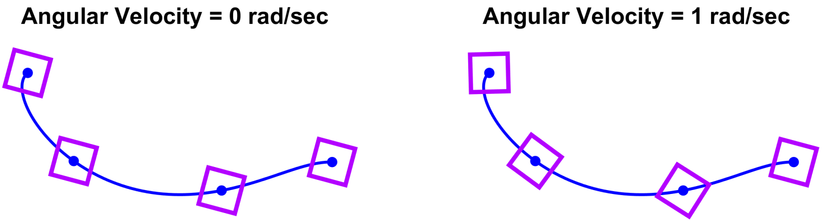


Figure 1: Shape translating and (possibly) rotating

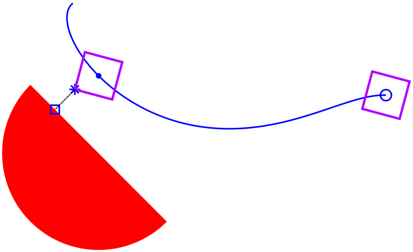
Set-Set Distance Problem

Set-Set distance between $A(S\ ;\omega(t))$ and X_u given t

$$P^* = \min_{t,\ \omega_0 \in \Omega_0,\ s \in S} c(x(t); X_u)$$

$$x(t) = A(s;\ \omega(t \mid \omega_0)) \quad \forall t \in [0, T]$$

$$\dot{\omega}(t) = f(t, \omega) \quad \forall t \in [0, T]$$



L_2 bound of 0.1465

Take-aways

Conclusion

Distance Estimation with occupation measures

Approximate recovery if moment matrices are low-rank

Extend to uncertain, lifted, set-set scenarios

Future Work

- Distance-Maximizing Control
- Chance-constrained distance
- Further Sparsity
- Efficient Computation
- Other nonnegativity cones and proofs

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Thank you for your attention

arxiv:2110.14047

`http://github.com/jarmill/distance`

Other Projects

Peak-related

- Occupation measures for time-delay systems
- Peak/Distance for hybrid systems
- Exploit polytopic input-affine structure
- Value-at-risk (probabilistic) peak

Non-Peak

- Data-driven stabilization under measurement noise
- Frank-Wolfe for LTI MIMO SysID
- Disconnectedness of sets
- Improve SDP approximation quality
- Orthogonal Ribosomes

Bonus Material and Ideas

Distance Program (Functions)

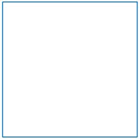
Auxiliary $v(t, x)$, point-set proxy $w(x) \leq c(x; X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0, x) \geq \gamma & & \forall x \in X_0 \\
& w(x) \geq v(t, x) & & \forall (t, x) \in [0, T] \times X \\
& c(x, y) \geq w(x) & & \forall (x, y) \in X \times X_u \\
& \mathcal{L}_f v(t, x) \geq 0 & & \forall (t, x) \in [0, T] \times X \\
& v \in C^1([0, T] \times X) \\
& w \in C(X)
\end{aligned}
$$

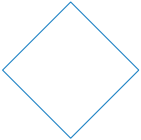
Chain $\forall (t, x, y) \in [0, T] \times X \times X_u : \; c(x, y) \geq w(x) \geq v(t, x)$

Lifted Distance

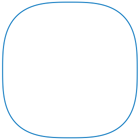
LP lifts to deal with absolute values



$$\|x - y\|_\infty \qquad \begin{aligned} &\min \quad q \\ -q \leq \langle x_i - y_i, \eta \rangle &\leq q \qquad \forall i \end{aligned}$$



$$\|x - y\|_1 \qquad \begin{aligned} &\min \quad \textstyle\sum_i q_i \\ -q_i \leq \langle x_i - y_i, \eta \rangle &\leq q_i \qquad \forall i \end{aligned}$$



$$\|x - y\|_3^3 \qquad \begin{aligned} &\min \quad \textstyle\sum_i q_i \\ -q_i \leq \langle (x_i - y_i)^3, \eta \rangle &\leq q_i \quad \forall i \end{aligned}$$

Lifted Distance Program (Measure)

New terms for lifted distance

$$
\begin{aligned}
p^* = \ \min \quad & \textstyle\sum_i q_i \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \pi^x_\# \eta = \pi^x_\# \mu_p \\
& \langle 1, \mu_0 \rangle = 1 \\
& -q_i \leq \langle c_{ij}(x, y), \eta \rangle \leq q_i & \forall i, j \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Same process as maximin peak

Lifted Distance Program (Function)

New terms $\beta_i^\pm$ on costs

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,x) \geq \gamma && \forall x \in X_0 \\
& w(x) \geq v(t,x) && \forall (t,x) \in [0,T] \times X \\
& \textstyle\sum_{i,j} (\beta_{ij}^+ - \beta_{ij}) c_{ij}(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& \mathcal{L}_f v(t,x) \geq 0 && \forall (t,x) \in [0,T] \times X \\
& 1^T(\beta_i^+ + \beta_i^-) = 1, \ \beta_i^\pm \in \mathbb{R}_+^{n_i} && \forall i \\
& v \in C^1([0,T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Set-Set Program (Measures)

Add new ‘shape’ measure μ_s

$$\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \langle v(t,x), \mu_p \rangle = \langle v(0,x), \mu_0 \rangle + \langle \mathcal{L}_f v(t,x), \mu \rangle & \forall v \\
& \langle w(x), \eta(x,y) \rangle = \langle w(A(s;\omega)), \mu_s(s,\omega) \rangle & \forall w \\
& \langle z(\omega), \mu_p(t,\omega) \rangle = \langle z(\omega), \mu_s \rangle & \forall z \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_s \in \mathcal{M}_+(\Omega \times S) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times \Omega) \\
& \mu_0 \in \mathcal{M}_+(\Omega_0)
\end{aligned}$$

Set-Set Program (Function)

Set-Set distance proxy $z(\omega) \leq \max_{s \in S} c(A(s;\omega); X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0,\omega) \geq \gamma && \forall x \in \Omega_0 \\
& c(x,y) \geq w(x) && \forall (x,y) \in X \times X_u \\
& w(A(s;\omega)) \geq z(\omega) && \forall (s,\omega) \in S \times \Omega \\
& z(\omega) \geq v(t,\omega) && \forall (t,\omega) \in [0,T] \times \Omega \\
& \mathcal{L}_f v(t,\omega) \geq 0 && \forall (t,\omega) \in [0,T] \times \Omega \\
& v \in C^1([0,T] \times X) \\
& w \in C(X), \ z \in C(\Omega)
\end{aligned}
$$

Sparsity (cont.)

Separable $c(x, y) = \sum_i c_i(x_i, y_i)$

Use correlative sparsity with measures and cliques

$$\eta_k : \quad I_k = (x_k : x_n,\ y_1 : y_k) \qquad \forall k = 1, \ldots, n$$

Sparse decomposition of η:

$$\begin{aligned} &\min \textstyle\sum_i \langle c_i(x_i, y_i), \eta_i \rangle && \eta^1 \in \mathcal{M}_+(X \times \mathbb{R}) \\ &\pi_\#^{I_k \cap I_{k+1}} \eta_k = \pi_\#^{I_k \cap I_{k+1}} \eta_{k+1} && \eta^k \in \mathcal{M}_+(\mathbb{R}^{n+1}) \\ &\pi_\#^x \mu^p = \pi_\#^x \eta_1 && \eta^n \in \mathcal{M}_+(\mathbb{R} \times X_u) \end{aligned}$$