On the optimization of actuator saturation limits for LTI systems: an LMI-based invariant ellipsoid approach

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Abstract: This paper considers the problem of optimal actuator dimensioning for LTI systems, in the sense of choosing appropriate saturation limits for a given set of admissible initial conditions and for a predefined integral state-feedback control law. By using an invariant ellipsoid argument, it is shown that this problem can be described as a linear matrix inequality (LMI)-based optimization that can be solved efficiently. Moreover, the paper shows that the optimal actuator dimensioning is connected to the choice of the initial conditions of the integral states of the controller, which can be included in the overall optimization to improve further the results. Two different methods are described and analyzed by means of numerical simulation.

Keywords: Actuator optimization, Actuator saturation, LMI, Invariant ellipsoid, Integral state reset

1. INTRODUCTION

Actuator saturation is a nonlinearity that affects every practical control system. If not taken properly into account, it can lead to undesired performance degradation and even to instability of the closed-loop response. For this reason, actuator saturation has attracted a strong attention by several researchers, which is demonstrated by the large number of books on the topic, see e.g. Kapila and Grigoriadis (2002), Tarbouriech et al. (2011), Corradini et al. (2012). The developed approaches can be divided into two main categories. Some of them handle the saturation constraints by using *anti-windup compensators*, which are added to a control system previously designed without taking into account the saturation (Grimm et al., 2003, Yang et al., 2016). On the other hand, in *direct design* approaches, the input constraints are considered at the controller design stage (Da Silva and Tarbouriech, 2001, Ruiz et al., 2019).

Another problem of interest in control system design is *optimal component placement*, in which the best possible choice of actuators (sensors) which make the system controllable (observable) is sought (Casillas et al., 2013, Chanekar et al., 2017). This problem is usually motivated by energetic or economical considerations, which allow defining some kind of metrics in order to avoid non-optimal selection of the components. For instance, Johnson (1969) was among the earliest works to present on optimal actuator selection based on an energetic perspective. Müller and Weber (1972) proposed to use the determinant, trace and maximal eigenvalue of the inverse controllability grammian to quantify controllability for actuator selection purposes. Münz et al. (2014) developed an algorithm for actuator placement in linear systems based on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimization. Further related methods involve some kind of iterative (Dhingra et al., 2014, Tzoumas et al., 2015) or greedy heuristic (Olshevsky, 2014) procedures, among others.

The present work aims at considering the problem of optimal actuator selection from an alternative viewpoint. In particular, the problem under consideration is as follows: given a predefined control law and a set of initial conditions, optimize the deliverable actuator action, i.e. choose appropriate actuator saturation limits, needed to regulate to the origin any state trajectory starting from the specified set. For the sake of simplicity, the considered class of plants is the one of LTI systems subject to actuator saturation, controlled via a static state-feedback with integral action. A first contribution of this work is to show that this problem can be described as a linear matrix inequality (LMI)-based optimization, so that a solution can be found efficiently using available solvers. A second contribution of this work is to show that in a control loop that contains an integral action, the optimal actuator dimensioning is connected to the choice of the initial conditions of the integral states, which can be included in the overall optimization to improve further the results. It is worth noting that the proposed results are related to invariant ellipsoid methods for saturated systems, but with a different twist where instead of designing a controller for a system with a priori known saturation bounds, the saturation bounds are designed *a posteriori*. Also, it must be highlighted

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that to the best of the authors' knowledge, the problem of optimal actuator design has been considered so far by just a scarce number of works, such as Messine et al. (1998), which showed the advantage of a deterministic global-optimization method in the optimal design of electromechanical actuators.

The remaining of the paper is structured as follows. The problem formulation is provided in Section 2. Two different solutions are provided in Section 3 and Section 4, respectively. Section 5 is devoted to simulation results. Finally, the main conclusions and perspectives about future research are outlined in Section 6.

Notation: The notation is fairly standard. Given a symmetric matrix $P \in \mathbb{R}^n$, $P \succ 0$ ($P \prec 0$) stands for positive (negative) definiteness.

2. PROBLEM FORMULATION

Let us consider the following LTI system subject to actuator saturations:

$$\dot{x}(t) = Ax(t) + B\mathrm{sat}(u(t)) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, A, B are matrices of appropriate dimensions and $\mathrm{sat} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the saturation function, defined as:

$$\mathrm{sat}(u_i(t)) = \begin{cases} \overline{u}_i, & \text{if } u_i(t) > \overline{u}_i \\ u_i(t), & \text{if } |u_i(t)| \leq \overline{u}_i \\ -\overline{u}_i, & \text{if } u_i(t) < -\overline{u}_i \end{cases} \tag{2}$$

for $i = 1, \ldots, m$, with $\overline{u}_i > 0$. To enforce the closed loop robustness with respect to constant exogenous disturbances, it is common to introduce integral states in the control loop, according to the internal model principle (Francis and Wonham, 1976). To this end, we define first a performance output $y = Ex$, where $y \in \mathbb{R}^p$ with $p \leq m$ and E is a matrix of appropriate dimensions. Therefore, we introduce

$$\dot{x}_I(t) = Ex(t) \tag{3}$$

where $x_I \in \mathbb{R}^p$ denotes auxiliary integral states.

Then, one can feed back $x_I(t)$ along with $x(t)$ as part of the controller equations, for example by computing the control law by means of a state-feedback strategy, as follows:

$$u(t) = \begin{bmatrix} K_P & K_I \end{bmatrix} \begin{bmatrix} x(t) \\ x_I(t) \end{bmatrix} = \bar{K}\bar{x}(t) \tag{4}$$

where $\bar{x}(t) = \begin{bmatrix} x(t)^T & x_I(t)^T \end{bmatrix}^T$, K_P and K_I denote gains of appropriate dimensions, and $\bar{K}$ denotes the overall controller gain. Then, the augmented system with state vector $\bar{x}(t)$ would evolve according to the following:

$$\dot{\bar{x}}(t) = \begin{bmatrix} A & 0 \\ E & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathrm{sat}\left(\bar{K}\bar{x}(t)\right) \tag{5}$$

We assume that a stabilizing controller $\bar{K}$ is already given for the unsaturated system (5). The design of such a $\bar{K}$ can be achieved based on any of the methods available for LTI systems. The problem considered in this paper is the one of choosing adequate actuators, which means designing appropriate saturation limits $\overline{u}_i$, $i = 1, \ldots, m$, such that, given a polytopic region described by vertices $x_{(1)}, \ldots, x_{(h)}$:

$$\mathcal{P} = \mathrm{Co}\{x_{(1)}, x_{(2)}, \ldots, x_{(h)}\} \tag{6}$$

then the system trajectory belongs to the region of linearity (i.e., $|u_i(t)| \leq \bar{u}_i \; \forall i = 1, \ldots, m, \forall t \geq 0$) for any initial condition $x(0) \in \mathcal{P}$. Moreover, convergence of the state trajectory to the origin of the state space must be guaranteed.

Due to economic reasons, it makes sense to seek for the smallest values of $\overline{u}_i$ which make the above specifications hold true. In this way, smaller and cheaper actuators can be used without losing the benefits of working under a linear setting. The above requirement can be formally defined, e.g., as the minimization of an appropriate cost function, defined as follows:

$$J = \gamma_1 \overline{u}_1^2 + \gamma_2 \overline{u}_2^2 + \ldots + \gamma_m \overline{u}_m^2 \tag{7}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_m$ are positive weighting coefficients.

3. FIRST APPROACH

If $\bar{x}(t)$ is such that all inputs are working in the linearity region, i.e. $|u_i(t)| \leq \bar{u}_i \; \forall i = 1, \ldots, m, \forall t \geq 0$, then (5) becomes:

$$\dot{\bar{x}}(t) = \begin{bmatrix} A + BK_P & BK_I \\ E & 0 \end{bmatrix} \bar{x}(t) = \bar{A}\bar{x}(t) \tag{8}$$

It is possible to assess the stability of (8) by using Lyapunov function $V(\bar{x}(t)) = \bar{x}(t)^T W^{-1} \bar{x}(t)$, with $W \succ 0$, thus obtaining the following condition corresponding to $\dot{V}(\bar{x}(t)) < 0$:

$$\bar{A}^T W^{-1} + W^{-1}\bar{A} \prec 0 \tag{9}$$

which, pre- and post-multiplied by W, leads to:

$$W\bar{A}^T + \bar{A}W \prec 0 \tag{10}$$

As stated in the previous section, it is assumed that control gains K_P and K_I are already given in such a way that (10) holds true for a suitable choice of W. In order to ensure that the state trajectory does not leave the actuators linearity region, the following set of LMIs can be used (Nguyen and Jabbari, 2000):

$$\begin{bmatrix} W & W\bar{K}_i^T \\ \bar{K}_i W & \overline{u}_i^2 \end{bmatrix} \succ 0 \qquad i = 1, \ldots, m \tag{11}$$

where $\bar{K}_i$ denotes the i-th row of $\bar{K}_i$, which enforces $|\bar{K}_i\bar{x}| \leq \overline{u}_i \; \forall i = 1, \ldots, m, \forall \bar{x} \in \{\bar{x} : \bar{x}^T W^{-1} \bar{x} \leq 1\}$.

The requirement that the convergence to the origin is ensured for any $x(0) \in \mathcal{P}$ can be expressed as:

$$\bar{x}(0) = \begin{bmatrix} x(0) \\ x_I(0) \end{bmatrix} \in \{\bar{x} : \bar{x}^T W^{-1} \bar{x} \leq 1\} \quad \forall x(0) \in \mathcal{P} \tag{12}$$

which can be rewritten as:

$$1 - \begin{bmatrix} x(0)^T & x_I(0)^T \end{bmatrix} W^{-1} \begin{bmatrix} x(0) \\ x_I(0) \end{bmatrix} \geq 0 \quad \forall x(0) \in \mathcal{P} \tag{13}$$

and, through Schur complement:

$$\begin{bmatrix} W & \begin{bmatrix} x(0) \\ x_I(0) \end{bmatrix} \\ \begin{bmatrix} x(0)^T & x_I(0)^T \end{bmatrix} & 1 \end{bmatrix} \succeq 0 \quad \forall x(0) \in \mathcal{P} \tag{14}$$

In order to obtain a finite number of conditions from (14), let us choose $x_I(0)$ such that:

$$x(0) = \sum_{j=1}^{h} \alpha_j x_{(j)} \quad \Rightarrow \quad x_I(0) = \sum_{j=1}^{h} \alpha_j x_{I(j)} \tag{15}$$

with:

$$\sum_{j=1}^{h} \alpha_j = 1, \; \alpha_j \geq 0 \qquad \forall j = 1, \ldots, h \tag{16}$$

Then, (14) becomes equivalent to:

$$\begin{bmatrix} W & \begin{bmatrix} x_{(j)} \\ x_{I(j)} \end{bmatrix} \\ \begin{bmatrix} x_{(j)}^T & x_{I(j)}^T \end{bmatrix} & 1 \end{bmatrix} \succeq 0 \qquad j = 1, \ldots, h \tag{17}$$

Hence, the problem formulated in Section 2 can be cast as the minimization of J in (7) subject to the LMI constraints (10), (11) and (17), where the decision variables are W, $\overline{u}_i^2$ and $x_{I(j)}$, with $i = 1, \ldots, m$ and $j = 1, \ldots, h$. Note that, once $x(0)$ is known, coefficients α_j can be obtained from (15)-(16) and used to compute the corresponding integrator initial condition $x_I(0)$.

4. SECOND APPROACH

Let us define a new augmented state variable as follows:

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ \dot{x}_I(t) \\ u(t) \end{bmatrix} \tag{18}$$

Then, by differentiating (18), and taking into account (4) and (8), one can write the dynamics of $z(t)$ in a compact form as:

$$\dot{z}(t) = \tilde{A} z(t) \tag{19}$$

with:

$$\tilde{A} = \begin{bmatrix} A + BK_P & BK_I & 0 \\ E & 0 & 0 \\ K_P & K_I & 0 \end{bmatrix} \tag{20}$$

Note that z_1 and z_2 can be obtained from x and x_I through the following linear state transformation

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} A + BK_P & BK_I \\ E & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_I(t) \end{bmatrix} \tag{21}$$

The stability of the subsystem which describes the evolution of $z_1(t)$ and $z_2(t)$ can be assessed using the condition (10). Note that this condition results into the convergence of z_1 and z_2, i.e., $\dot{x}$ and $\dot{x}_I$. On the other hand, due to the fact that $\bar{A}$ is Hurwitz, equation (21) implies that the convergence of z_1 and z_2 results into the convergence of x and x_I as well.

The specification that the actuators work in their region of linearity can be enforced by requiring that the state $z_3(t)$ does not leave an ellipsoidal invariant set contained within the saturation limits. First of all, let us consider the following LMI:

$$\tilde{W}^{-1}\tilde{A} + \tilde{A}^T\tilde{W}^{-1} \preceq 0 \tag{22}$$

with $\tilde{W} \succ 0$, which ensures that $\mathcal{Z} = \{z : z^T\tilde{W}^{-1}z \leq 1\}$ is an invariant set (Boyd et al., 1994). By pre- and post-multiplying (22) by $\tilde{W}$, one gets:

$$\tilde{A}\tilde{W} + \tilde{W}\tilde{A}^T \preceq 0 \tag{23}$$

Next, let us consider the following lemma, which can be derived from some related results presented in Pope (2008).

Lemma 1. Given an ellipsoid $\mathcal{Z} = \{z : z^T\tilde{W}^{-1}z \leq 1\}$, $\tilde{W} \succ 0$, and a line $\mathcal{L}$ parameterized by s, and defined by $\mathcal{L} = \{z : z = sv\}$, where v is a given non-zero vector with unit Euclidean norm ($v^T v = 1$), the projection of $\mathcal{Z}$ onto $\mathcal{L}$ corresponds to the interval $[-\sqrt{v^T\tilde{W}v}, \sqrt{v^T\tilde{W}v}]$ in s.

Proof: According to Pope (2008), the projection of $\mathcal{Z}$ onto $\mathcal{L}$ corresponds to the interval $[-\bar{s}, \bar{s}]$ in s, where $\bar{s}$ is the Euclidean norm of the vector $\tilde{W}^{1/2}v$. Hence:

$$\bar{s} = \|\tilde{W}^{1/2}v\| = \sqrt{v^T\tilde{W}^{1/2}\tilde{W}^{1/2}v} = \sqrt{v^T\tilde{W}v} \tag{24}$$

In order for the control input not to saturate, we require that the projection of $\mathcal{Z}$ onto each of the directions of the state $z_3(t) = u(t)$ is contained within the saturation limits. Let us refer to $v_1, v_2, \ldots, v_m$ as the natural basis which generates the subspace $z(t) = [0, 0, z_3(t)]^T$. Hence, to ensure that $u(t)$ remains inside the saturation bounds, the following is required:

$$\sqrt{v_i^T\tilde{W}v_i} \leq \overline{u}_i \qquad i = 1, \ldots, m \tag{25}$$

which is equivalent to:

$$v_i^T\tilde{W}v_i \leq \overline{u}_i^2 \qquad i = 1, \ldots, m \tag{26}$$

that is:

$$\tilde{W}_{n+p+i,n+p+i} \leq \overline{u}_i^2 \qquad i = 1, \ldots, m \tag{27}$$

where $\tilde{W}_{n+p+i,n+p+i}$ denotes the $n+p+i-th$ element of the diagonal of $\tilde{W}$.

Note that at time $t = 0$, the following holds:

$$z(0) = \begin{bmatrix} \dot{x}(0) \\ \dot{x}_I(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} (A + BK_P)x(0) + BK_I x_I(0) \\ Ex(0) \\ K_P x(0) + K_I x_I(0) \end{bmatrix} \tag{28}$$

so that the requirement that convergence to the origin is ensured for any $x(0) \in \mathcal{P}$ can be expressed as:

$$z(0) \in \{z : z^T\tilde{W}^{-1}z \leq 1\} \qquad \forall x(0) \in \mathcal{P} \tag{29}$$

which means that:

$$1 - z(0)^T\tilde{W}^{-1}z(0) \geq 1 \qquad \forall x(0) \in \mathcal{P} \tag{30}$$

and, by Schur complements:

$$\begin{bmatrix} \tilde{W} & \begin{bmatrix} (A + BK_P)x(0) + BK_I x_I(0) \\ Ex(0) \\ K_P x(0) + K_I x_I(0) \end{bmatrix} \\ \star & 1 \end{bmatrix} \succeq 0 \quad \forall x(0) \in \mathcal{P} \tag{31}$$

Choosing $x_I(0)$ according to (15)-(16) allows transforming (31) into:

$$\begin{bmatrix} \tilde{W} & \begin{bmatrix} (A + BK_P)x_{(j)} + BK_I x_{I(j)} \\ Ex_{(j)} \\ K_P x_{(j)} + K_I x_{I(j)} \end{bmatrix} \\ \star & 1 \end{bmatrix} \succeq 0 \quad j = 1, \ldots, h \tag{32}$$

so that the problem formulated in Section 2 can be cast as the minimization of J in (7) subject to the LMI constraints (23), (27) and (32), where the decision variables are $\tilde{W}$, $\overline{u}_i^2$ and $x_{I(j)}$, with $i = 1, \ldots, m$ and $j = 1, \ldots, h$.

4.1 Additional considerations

Due to the special structure of $\tilde{A}$ in (20), it is possible to transform the semi-definiteness constraint in (23) into a definiteness one due to the equivalence stated in the following proposition.

Proposition 1. Let matrices $\tilde{A}$ and $\tilde{W}$, with $\tilde{W}$ symmetric, be partitioned as follows:

$$\tilde{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} \qquad \tilde{W} = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} \tag{33}$$

with A_{11} Hurwitz. Given a matrix W_{11} such that:

$$A_{11}W_{11} + W_{11}A_{11}^T \prec 0 \tag{34}$$

then (23) holds if and only if:

$$W_{12} = W_{11}A_{11}^{-T}A_{21}^T \tag{35}$$

Proof: (Sufficiency) Taking into account (33), then (23) can be written as follows:

$$\begin{bmatrix} A_{11}W_{11} & A_{11}W_{12} \\ A_{21}W_{11} & A_{21}W_{12} \end{bmatrix} + \star \preceq 0 \tag{36}$$

where $\star$ denotes the term induced by symmetry.

By replacing (35) into (36), the following is obtained:

$$\begin{bmatrix} A_{11}W_{11} & A_{11}W_{11}A_{11}^{-T}A_{21}^{T} \\ A_{21}W_{11} & A_{21}W_{11}A_{11}^{-T}A_{21}^{T} \end{bmatrix} + \star \preceq 0 \tag{37}$$

By pre- and post-multiplying (37) by:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \tag{38}$$

and its transpose, respectively, the following is obtained:

$$\begin{bmatrix} A_{11}W_{11} + W_{11}A_{11}^{T} & 0 \\ 0 & 0 \end{bmatrix} \preceq 0 \tag{39}$$

which holds if (34) holds.

(Necessity) Let us consider a perturbed matrix $\tilde{W}$, as follows:

$$\tilde{W} = \begin{bmatrix} W_{11} & W_{12} + \Delta \\ W_{12}^{T} + \Delta^{T} & W_{22} \end{bmatrix} \tag{40}$$

where Δ is any matrix of appropriate dimensions. Following the previous reasoning, (37) is replaced by:

$$\begin{bmatrix} A_{11}W_{11} & A_{11}W_{11}A_{11}^{-T}A_{21}^{T} + A_{11}\Delta \\ A_{21}W_{11} & A_{21}W_{11}A_{11}^{-T}A_{21}^{T} + A_{21}\Delta \end{bmatrix} + \star \preceq 0 \tag{41}$$

and, through pre- and post-multiplication by (38) and its transpose, respectively, the following is obtained:

$$\begin{bmatrix} A_{11}W_{11} + W_{11}A_{11}^{T} & A_{11}\Delta \\ \Delta^{T}A_{11}^{T} & 0 \end{bmatrix} \preceq 0 \tag{42}$$

Noting that (42) is associated to the quadratic form:

$$2x_1^T\left(A_{11}W_{11}x_1 + A_{11}\Delta x_2\right) \leq 0 \tag{43}$$

it is clear that if $\Delta \neq 0$, one can always find a state x_2 such that (43) does not hold, hence the matrix in (42) would be not definite. Hence, $\Delta = 0$, which proves the necessity, thus completing the proof. □

A relevant consequence of Proposition 1 is that it limits the applicability of the second approach when linear parameter varying (LPV) of the form (Mohammadpour and Scherer, 2012):

$$\dot{x}(t) = A\left(\theta(t)\right)x(t) + B\mathrm{sat}\left(u(t)\right) \tag{44}$$

are considered, where $\theta \in \Theta$ defines some time-varying parameters. In fact, in this case it follows from (35) that a necessary condition for the applicability of the approach, which is based on a Lyapunov function with constant matrix, is that:

$$A_{21}(\theta)A_{11}^{-1}(\theta) = const. \quad \forall\theta \in \Theta \tag{45}$$

5. SIMULATION RESULTS

This section presents an extensive numerical validation of the methods presented above. To compare the different approaches in a meaningful way, a large number of systems is considered. In particular, for each system the entries of matrices A and B are chosen as randomly generated numbers distributed uniformly in the interval [0, 1], while E is always chosen as $E = [I \ 0]$. The stabilizing controller for the linear dynamics is designed via an LQR strategy, with Q and R both chosen as identity matrices of appropriate dimensions. Additionally, the convex set of initial conditions is selected as the unit cross-polytope defined by $\mathcal{P} = \mathrm{Co}\{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$, where the generic e_k represents the k-th natural basis vector of $\mathbb{R}^n$. Several simulations are performed by considering systems of different orders, i.e., $n = 1, 2, 3, 4$. Only SISO models are considered, i.e., $m = 1$ and $p = 1$. For each value of n, 1000 models are generated randomly based on the criteria described above. For each plant, an LQR controller is designed first, and the saturation limits $\bar{u}_i$ which minimize the cost function J given by (7) is subsequently evaluated. Four different methods are used for evaluating the optimal $\bar{u}_i$, i.e., the approach in section 3 with optimal reset of $x_I(0)$ (denoted as 1R), the approach in section 4 with optimal reset of $x_I(0)$ (denoted as 2R), and the same methods with no reset of $x_I(0)$ (denoted as 1NR and 2NR, respectively). The optimal integral state reset is implemented by letting variables $x_{I(j)}$ in (17) and (32) be free decision variables for the LMI solver, while for the no-reset methods $x_{I(j)} = 0$ is considered $\forall j$. In this way, the effectiveness of each method as well as the amount of control effort reduction due smart integral reset can be assessed.

The simulation studies are conducted in *MATLAB* environment, by implementing the LMI problems via the *YALMIP* toolbox (Löfberg, 2004) using the *SeDuMi* solver (Sturm, 1999). The results are shown in Figs. 1-4 for $n = 1, 2, 3, 4$, respectively. The left-hand side of each figure shows the optimal saturation value $\bar{u}_i$ obtained for each plant, which is denoted as a blue circle for 1R, as a red cross for 2R, as a cyan diamond for 1NR, and as a magenta plus for 2NR. At the same time, the right-hand side plots illustrate the ratio between no-reset and reset values of optimal $\bar{u}_i$, computed for both approach 1 (blue circle) and approach 2 (red cross). For better illustration purpose, the results are sorted such that $\bar{u}_i$ appears as monotonically decreasing for 1NR. It can be noticed immediately that approaches 1 and 2 provide practically the same results, with just a few observed differences (see Figs. 3-4) which might be due to numerical issues. Therefore, we deduce that the two approaches appear to be numerically equivalent for the considered systems. This holds true even if models of higher order or with more inputs and outputs are considered, although this is not shown in this paper due to space limitations. The other interesting result which can be observed is that the optimal reset of $x_I(0)$ permits to achieve a remarkable reduction of the saturation limit $\bar{u}_i$. In particular, a reduction up to 15 times can be observed on the right-hand side of Figs. 1-4. Interestingly enough, the plots for $n = 2, 3, 4$ show similar results with most of the data concentrated in the range from 2 to 5, while for $n = 1$ a value of about 2.4 is obtained in practically all cases. As a result, we conclude that the improvement in control effort reduction is more remarkable for plants of order $n \geq 2$.

For further comparison, the statistical distribution corresponding to the previous simulations are also reported in Figs. 5-8 for $n = 1, 2, 3, 4$, respectively. In each plot, the left-hand side shows the probability distributions of 1R (solid blue line), 2R (dashed red line), 1NR (solid cyan line), and 2NR (dashed magenta line), while the right-hand side shows the corresponding probability distribution for the ratios 1NR/1R (solid blue line) and 2NR/2R (dashed red line). Trends which are similar to Figs. 1-4 can be observed. For a quantitative evaluation, mean and standard deviation are also computed for each distribution, and denoted as μ and σ in the corresponding legend entry. For $n = 2, 3, 4$, both mean and standard deviation of the absolute performance (left-hand side) increase with the plant order, while the corresponding values change only slightly in the relative performance plots (right-hand side). For such plots, the optimal integral reset leads to an average relative improvement between 3.6 and 3.9. The results are quite different for the case $n = 1$, for which the distributions show a significantly smaller spread, in agreement with Fig. 1. For this case, an average relative improvement of 2.36 is obtained. Finally, mean and standard deviation values for each

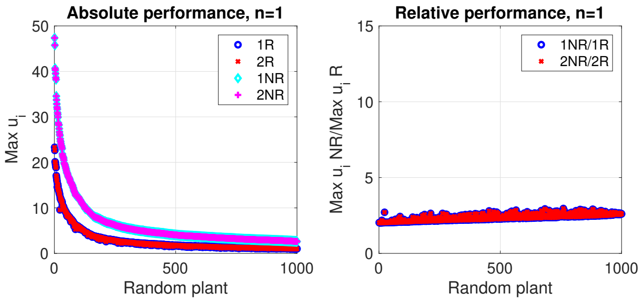

Fig. 1. Simulation results for $n = 1$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

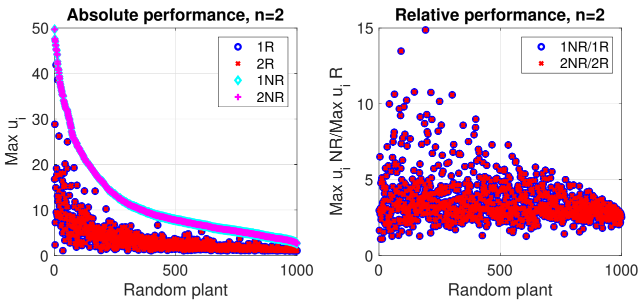

Fig. 2. Simulation results for $n = 2$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

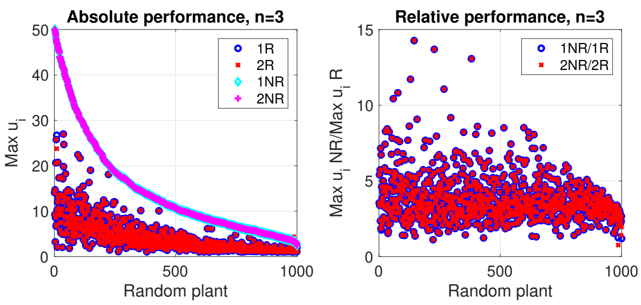

Fig. 3. Simulation results for $n = 3$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

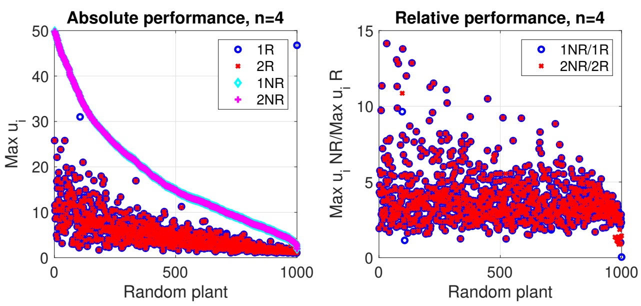

Fig. 4. Simulation results for $n = 4$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

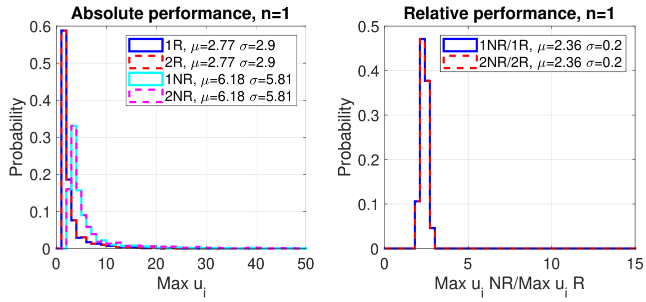

Fig. 5. Probability distributions for $n = 1$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

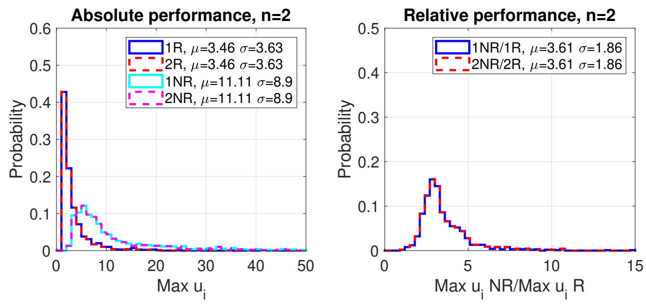

Fig. 6. Probability distributions for $n = 2$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

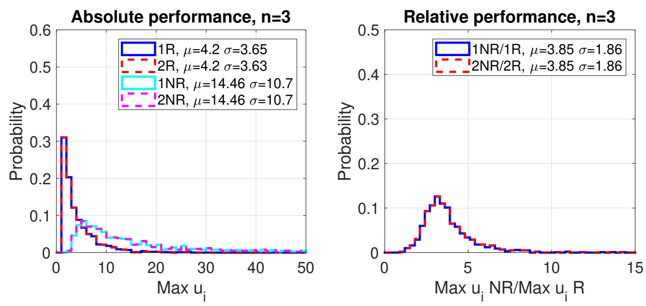

Fig. 7. Probability distributions for $n = 3$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

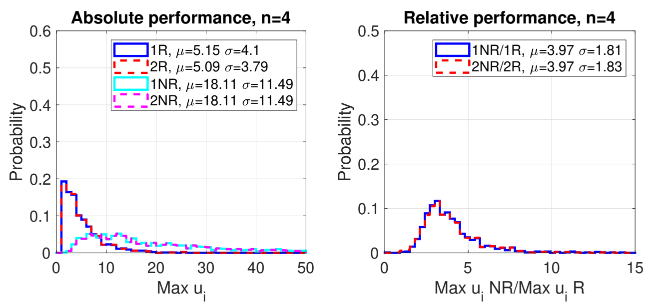

Fig. 8. Probability distributions for $n = 4$, maximum control input without and with integral reset (left-hand side), and ratio between maximum control inputs without and with integral reset (right-hand side).

Table 1. Mean μ and standard deviation σ values for the distributions reported in Figs. 5-8, given in the format (μ, σ)

Method	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1R	(2.77, 2.90)	(3.46, 3.63)	(4.20, 3.65)	(5.15, 4.10)
2R	(2.77, 2.90)	(3.46, 3.63)	(4.20, 3.63)	(5.09, 3.79)
1NR	(6.18, 5.81)	(11.11, 8.90)	(14.46, 10.70)	(18.11, 11.49)
2NR	(6.18, 5.81)	(11.11, 8.90)	(14.46, 10.70)	(18.11, 11.49)
1NR/1R	(2.36, 0.20)	(3.61, 1.86)	(3.85, 1.86)	(3.97, 1.81)
2NR/2R	(2.36, 0.20)	(3.61, 1.86)	(3.85, 1.86)	(3.97, 1.83)

experiment are summarized in Table 1.

6. CONCLUSIONS

This paper has presented two LMI strategies for optimal selection of actuator saturation limits in LTI systems. Both methods aim at finding the maximal control effort required by a given state feedback + integral controller to regulate a convex set of initial conditions to the origin of the state space. In addition, an optimal integral reset strategy has been also proposed to reduce further the amount of control effort. By means of an extensive simulation study, it has been found that: 1) both developed approaches provide nearly identical results from the numerical point of view; 2) the integral reset strategy permits a significant reduction of the required control input saturation value, by a mean factor of 2.36 for plants of order 1 and larger than 3.6 for plants of higher order; 3) the improvement in performance is relatively insensitive to system parameters and dynamic order, with the only exception given by first order models. Based on the developed results, optimal selection of cheap and saturation-avoiding actuator components can be carried out effectively.

Future research will aim at developing alternative LMI conditions to combine the actuator optimization with the controller design. In addition, further theoretical studies will be conducted to prove if the equivalence between both methods holds not only on a numerical, but also on an analytical point of view.

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