On the Computation of the Peak of the Impulse Response of LTI Systems

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ABSTRACT
This paper addresses the problem of determining the peak of the impulse response of continuous-time linear time-invariant (LTI) systems, a classical problem in systems theory. A novel condition for establishing upper bounds of the sought peak is proposed in terms of feasibility of a system of linear matrix inequalities (LMIs). This condition is obtained by describing the trajectory of the system through the level set of a polynomial, by using projection techniques for evaluating the position of the level set, and by exploiting the Gram matrix method. The proposed condition is sufficient for any degree of the polynomial, moreover, the conservatism can be decreased by increasing this degree. As shown by some examples, the proposed condition provides significantly less conservative results than the existing methods.

CCS Concepts
• **Computing methodologies → Computational control theory.**

Keywords
LTI; Impulse response; LMI.

1. INTRODUCTION
Input-output relationships of linear time-invariant (LTI) systems can be characterized by various indexes, in particular the H-infinity norm (i.e., the maximum two norm of the frequency response) and the H-2 norm (i.e., the square root of the sum of the energies of the impulse responses). It is well-known that these indexes can play key roles in the analysis and synthesis of LTI systems, therefore, methods for determining and controlling these indexes have been studied and developed since long time. See for instance [6], [1], [9].

Another important index is the peak of the impulse response, i.e., the maximum amplitude of the output of a system in response to an impulse applied to one of its input channels. Indeed, the knowledge of this index may be required in order to verify whether amplitude constraints are satisfied, and the manipulation of this index may be required in order to ensure the satisfaction of such constraints.

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Unfortunately, even the mere determination of the peak of the impulse response is still an open problem. Indeed, classical methods based on set invariance of quadratic Lyapunov functions are generally conservative for determining the peak of the impulse response of an LTI system, see for instance the linear matrix inequality (LMI) methods [2], [10]. This is sharply in contrast with the fact that there exist nonconservative LMI methods for determining other indexes such as the H-infinity norm and the H-2 norm of an LTI system.

This paper addresses the problem of determining the peak of the impulse response of continuous-time LTI systems. A novel condition for establishing upper bounds of the sought peak is proposed in terms of feasibility of a system of LMIs. This condition is obtained by describing the trajectory of the system through the level set of a polynomial, by using projection techniques for evaluating the position of the level set, and by exploiting the Gram matrix method. The proposed condition is sufficient for any degree of the polynomial, moreover, the conservatism can be decreased by increasing this degree. As shown by some examples, the proposed condition provides significantly less conservative results than the existing methods.

The paper is organized as follows. Section 2 provides the preliminaries. Section 3 describes the proposed approach. Section 4 presents the examples. Lastly, Section 5 reports the conclusions and future works.

2. PRELIMINARIES
In this section we provide the preliminaries. Specifically, Section 2.1 introduces the problem formulation, and Section 2.2 presents a brief review of the Gram matrix method.

2.1 Problem Formulation
The notation adopted in the paper is as follows. The natural numbers set (including zero) and the real numbers set are denoted by $\mathbb{N}$ and $\mathbb{R}$. The symbol 0 denotes the null matrix of size specified by the context. The Euclidean norm and the infinity norm are denoted by $\|\cdot\|_2$ and $\|\cdot\|_\infty$. The notation $A\otimes B$ denotes the Kronecker product of matrices A and B. The transpose of a matrix A is denoted by A'. The notation $A>0$ (respectively, $A\geq 0$) denotes a symmetric positive definite (respectively, semidefinite) matrix A.

We consider the continuous-time LTI system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $t\in\mathbb{R}$ is the time, $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the input, $y(t)\in\mathbb{R}^p$ is the output, and A, B, C are matrices of suitable sizes. Let us introduce the following definition.

Definition 1: The impulse response of the system (1) with respect to the i-th input channel is the function $y_{IR}^{(i)}(t)$, defined as the solution $y(t)$ for initial condition $x(0^-) = 0$ and input $u(t) = \delta(t)E_m(i)$, where $\delta(t)$ is the Dirac distribution and $E_m(i)$ is the i-th column of the $m \times m$ identity matrix.

This paper aims at proposing a novel approach for computing the peak of the impulse response of the system (1). To this end, we formulate the following problem.

Problem 1: Given $c \in (0, \infty)$, establish whether c is an upper bound of the peak of the impulse response of the system (1) with respect to all input channels, i.e.,

$$\left\| y_{IR}^{(i)}(t) \right\|_\infty < c \quad \forall t \geq 0 \ \ \forall i = 1, \cdots, m. \tag{2}$$

2.2 Gram Matrix Method

Here we report a brief review of the Gram matrix method that will be exploited in the next section, see for instance [5], [3], [8] and references therein for more information.

Let $z: \mathbb{R}^{w} \to \mathbb{R}$ be a polynomial of degree not greater than $2d, d \in \mathbb{N}$. Then, $z(a), a \in \mathbb{R}^{w}$, can be expressed through the Gram matrix method as

$$z(a) = b(a)'\big(Z + L(\alpha)\big)b(a) \tag{3}$$

where $b(a)$ is a vector whose entries define a basis for the polynomials in a of degree not greater than d, Z is a symmetric matrix, $L(\alpha)$ is a linear matrix function that parameterizes the linear set

$$\mathcal{L} = \{\tilde{L} = \tilde{L}': b(a)'\tilde{L}b(a) = 0\}, \tag{4}$$

and α is a free vector. The Gram matrix method is useful to establish whether a polynomial is a sum of squares of polynomials (SOS). Let us denote with Σ the set of SOS polynomials. It turns out that $z(\cdot) \in \Sigma$ if and only if there exists α that satisfies the LMI

$$Z + L(\alpha) \geq 0. \tag{5}$$

Let us observe that $b(a)$ can be simply chosen as a vector whose entries are all the monomials in a of degree not greater than d. Moreover, in some cases, one can use a reduced vector $b(a)$ without losing the necessary and sufficient LMI condition (5). This is useful because, by reducing the vector $b(a)$, one also reduces the numerical complexity of the LMI (5). Some cases of interest are as follows:

- Case I: the first case considers that $z(a)$ is a homogeneous polynomial, i.e., a polynomial having all monomials with the same degree. In this case, let $2d$ be the degree of $z(a)$ (observe that there is not loss of generality in considering that the degree is even because a polynomial in $\sum$ cannot have odd degree). Then, $b(a)$ can be reduced without losing the necessary and sufficient LMI condition (5) by keeping only the monomials of degree d.

- Case II: the second case considers that z(a) is a polynomial satisfying

$$\begin{cases} z(0) = 0 \\ \nabla z(0) = 0. \end{cases} \tag{6}$$

  In this case, the vector $b(a)$ can be reduced without losing the necessary and sufficient LMI condition (5) by removing the monomial of degree zero.

The numerical complexity of the LMI feasibility test (5) is mainly dependent on the number of scalar variables in this LMI, which is the length of the vector α. See for instance [4] for formulas about this length, in the general case, and in cases where reductions are possible.

3. PROPOSED APPROACH

Let $f: \mathbb{R}^s \to \mathbb{R}$ be a polynomial of degree not greater than $d, d \in \mathbb{N}$. Let us express $z(a), a \in \mathbb{R}^{w}$, as

$$z(a) = \sum_{\substack{k \in \mathbb{N}^{w} \\ k_1 + \cdots + k_{w} \leq d}} c_k a^k \tag{7}$$

where $c_k \in \mathbb{R}$. For $g \in \mathbb{R}^{w}$ let us define

$$h(a) = \sum_{\substack{k \in \mathbb{N}^{w} \\ k_1 + \cdots + k_{w} \leq d}} c_k a^k \, (g'a)^{\, d - k_1 - \cdots - k_{w}}. \tag{8}$$

It follows that, if $g \neq 0$, $h(a)$ is a homogeneous polynomial in a of degree d. Moreover, the coefficients of $h(a)$ are linear functions of the coefficients of $z(a)$. Let us denote the definition of $h(a)$ from $z(a)$ and g as

$$h(a) = \Phi(z(a), a, g). \tag{9}$$

Let us express the matrices B and C as

$$\begin{cases} B = (B^{(1)} \ \cdots \ B^{(m)}) \\ C = (C^{(1)} \ \cdots \ C^{(p)})' \end{cases} \tag{10}$$

where $B^{(1)}, \cdots, B^{(m)}, C^{(1)}, \cdots, C^{(p)} \in \mathbb{R}^n$. The following theorem provides a solution for Problem 1.

Theorem 1: Let $i \in \{1, \cdots, m\}$ and $c \in (0, \infty)$. Assume without loss of generality that

$$\left\| CB^{(i)} \right\|_\infty < c. \tag{11}$$

For the system (1) one has

$$\left\| y_{IR}^{(i)}(t) \right\|_\infty < c \quad \forall t \geq 0 \tag{12}$$

if there exist a polynomial $v: \mathbb{R}^n \to \mathbb{R}$ of degree not greater than $2d, d \in \mathbb{N}$, and a scalar $\varepsilon \in \mathbb{R}$ such that

$$\begin{cases} v(0) = 0 \\ \nabla v(0) = 0 \\ v\big(B^{(i)}\big) = 1 \\ \varepsilon > 0 \end{cases} \tag{13}$$

and

$$f(\cdot), h_{j,k}(\cdot) \in \Sigma \quad \forall \begin{cases} j = 0,1 \\ k = 1, \cdots, p \end{cases} \tag{14}$$

where

$$\begin{cases} f(x) = -\nabla v(x) A(x) \\ h_{j,k}(x) = \Phi\left(v(x) - 1, x, \frac{(-1)^j}{c} C^{(k)}\right) - \varepsilon \|x\|_2^{2d}. \end{cases} \tag{15}$$

Proof. Suppose that there exist $v(x)$ of degree not greater than $2d$ and a scalar ε such that (13)-(14) hold. This implies that $f(x)$ and $h_{j,k}(x)$ are nonnegative. Let us consider the unitary level set of $v(x)$:

$$\mathcal{V}_1 = \{x \in \mathbb{R}^n: \ v(x) = 1\}.$$

From the nonnegativity of $f(x)$, one has that the time derivative of $v(x)$ is nonpositive. This implies that any trajectory of the system (1) starting in $\mathcal{V}_1$ remains in the unitary sublevel set of $v(x)$:

$$\mathcal{V}_2 = \{x \in \mathbb{R}^n \colon\ v(x) \le 1\}.$$

Applying the Dirac distribution to the i-th input channel with initial condition $x(0^-) = 0$ has the effect to move the initial condition to $x(0) = B^{(i)}$. Hence, there is not loss of generality in assuming that (11) holds because (11) is equivalent to

$$\left\| y_{IR}^{(i)}(0) \right\|_\infty < c.$$

From the third constraint in (13) one has that this initial condition belongs to $\mathcal{V}_1$. From (15) and the definition (7)-(9), one has that

$$h_{j,k}(x) = v(x) - 1 - \varepsilon \|x\|_2^{2d} \quad \forall x \in \mathcal{T}_{j,k}(c)$$

where $\mathcal{T}_{j,k}(c)$ is the level set

$$\mathcal{T}_{j,k}(c) = \left\{x \in \mathbb{R}^n \colon\ x' C^{(k)} = (-1)^j c\right\}.$$

Since $h_{j,k}(x)$ is nonnegative and ε is positive, it follows that

$$v(x) > 1 \quad \forall x \in \mathcal{T}_{j,k}(c).$$

This implies that

$$\mathcal{V}_2 \cap \mathcal{T}_{j,k}(c) = \emptyset.$$

By observing that this holds for all $j = 0,1$ and $k = 1, \cdots, p$, it follows that

$$\mathcal{V}_2 \cap \mathcal{T}(c) = \emptyset$$

where

$$\mathcal{T}(c) = \{x \in \mathbb{R}^n \colon\ \|Cx\|_\infty = c\}.$$

Hence, we conclude that $\mathcal{V}_2$ does not intersect the set of states for which the output has infinity norm equal to c, and that the trajectory of the system (1) corresponding to the impulse response with respect to the i-th input channel remains in $\mathcal{V}_2$. Since this trajectory starts from $x(0) = B^{(i)}$ for which the output has infinity norm less than c due to (11), and since this trajectory is continuous, it follows that (12) holds. □

Theorem 1 states that an upper bound on the peak of the impulse response of the system (1) with respect to the i-th channel can be established if there exist a polynomial (i.e., $v(x)$) and a scalar (i.e., ε) that satisfy (13)-(14). Let us observe that (13) is a set of linear equalities and inequalities on the coefficients of $v(x)$ and on ε. Also, let us observe that (14) is equivalent to a system of LMIs because the coefficients of the polynomials $f(x)$ and $h_{j,k}(x)$ depend affine linearly on the coefficients of $v(x)$ and on ε and because the condition that any of these polynomials is in Σ can be equivalently reformulated as an LMI feasibility test as explained in Section 2.2. In conclusion, the condition of Theorem 1 is equivalent to establish feasibility of a system of LMIs.

Problem 1 can be addressed by repeating the condition of Theorem 1 for all channels, i.e., for all $i = 1, \cdots, m$. That is, (2) holds if, for all $i = 1, \cdots, m$, there exist a polynomial $v_i \colon \mathbb{R}^n \to \mathbb{R}$ of degree not greater than $2d_i$, $d_i \in \mathbb{N}$, and a scalar $\varepsilon_i \in \mathbb{R}$ such that (13)-(14) hold with $v(x)$, ε and d replaced by $v_i(x)$, ε_i and d_i.

It is interesting to observe that the condition of Theorem 1 is obtained by evaluating the projection of $v(x) - 1$ onto the set of states for which the output has infinity norm equal to c. This is realized by the function in (7)-(9) and has the benefit of avoiding the introduction of multipliers.

Let us observe that the polynomial $f(x)$ satisfies the condition (6) (with $z(a) = f(x)$) since $v(x)$ has to satisfy (13). This means that the LMI needed to establish whether $f(x)$ is in Σ can be reduced in size and number of variables by selecting the vector of monomials as explained in Case II of Section 2.2. Also, the polynomials $h_{j,k}(x)$ are homogeneous due to the definition (7)-(9). This means that the LMIs needed to establish whether $h_{j,k}(x)$ are in Σ can be reduced in size and number of variables by selecting the vector of monomials as explained in Case I of Section 2.2 (with $z(a) = h_{j,k}(x)$).

4. EXAMPLES

In this section we present some examples for illustrating the use of the condition proposed in Theorem 1. The toolbox SeDuMi [11] for Matlab is adopted to test this condition and existing LMI conditions on a standard computer with Windows 10, Intel Core i7, 3.4 GHz, 8 GB RAM. The computational time for testing the condition proposed in Theorem 1, i.e., feasibility of the system of LMIs (13)-(14), is less than one second for all examples.

4.1 Example 1

Let us consider the LTI system

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -0.5 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t). \end{cases}$$

We want to determine the peak of the impulse response of the system. To this end, we exploit the condition proposed in Theorem 1 by performing a bisection search on c.

By choosing $d = 1$ (quadratic Lyapunov functions), we obtain the upper bound $c_1 = 0.8284$. The number of LMI scalar variables in (13)-(14) is 3. The found polynomial $v(x)$ is

$$v(x) = 1.707x_1^2 + x_1x_2 + x_2^2.$$

This upper bound can be improved by increasing d. Indeed, for $d = 2$ (quartic Lyapunov functions), we obtain the upper bound $c_2 = 0.6448$. The number of LMI scalar variables in (13)-(14) is 17. The found polynomial $v(x)$ is

$$\begin{aligned} v(x) = 35.418x_1^4 + 141.998x_1^3x_2 - 42.972x_1^3 + 243.745x_1^2x_2^2 \\ - 183.422x_1^2x_2 + 15.388x_1^2 + 204.460x_1x_2^3 \\ - 260.977x_1x_2^2 + 59.233x_1x_2 + 73.207x_2^4 \\ - 145.697x_2^3 + 73.490x_2^2. \end{aligned}$$

The upper bound found for $d = 2$ is tight. Indeed, as shown by Figure 1, the state trajectory corresponding to the impulse response (found by solving the differential equation) touches the set $\|Cx\|_\infty = c_2$.

For comparison, we test the classical LMI condition based on invariant ellipsoids [2], [10], finding that the upper bound provided by this condition coincides with c_1.

4.2 Example 2

In this second example we consider the model of a DC motor, specifically (see for instance [7])

$$\begin{cases} J_m \ddot{\psi}_m(t) + b_m \dot{\psi}_m(t) = K_t i_a(t) \\ L_a \dot{i}_m(t) + R_a i_a(t) = -K_e \dot{\psi}_m(t) + v_a(t) \end{cases}$$

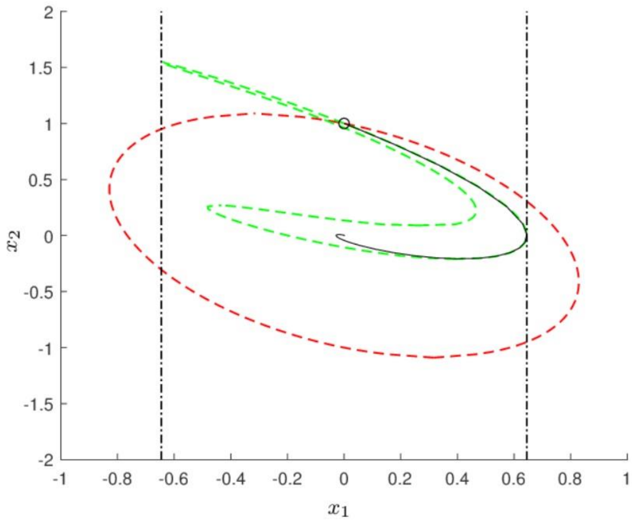

Figure 1. Example 1: set $\|Cx\|_\infty = c_2$ (black dash-dot line), impulse response (black solid line), and level sets of the found Lyapunov functions for $d = 1$ and $d = 2$ (red and green dashed lines).

where $\psi_m(t)$ is the angle, $i_a(t)$ is the current, $v_a(t)$ is the voltage, and J_m, b_m, K_t, L_a, R_a and K_e are parameters. Let us define

$$\begin{cases} x(t) = (\psi_m(t), \dot{\psi}_m(t), i_a(t))' \\ u(t) = v_a(t) \\ y(t) = \psi_m(t). \end{cases}$$

It follows that the model can be rewritten as

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{b_m}{J_m} & -\frac{K_t}{J_m} \\ 0 & -\frac{K_e}{L_a} & -\frac{R_a}{L_a} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{pmatrix} u(t) \\ y(t) = (1 \quad 0 \quad 0) x(t). \end{cases}$$

Let us choose the plausible values

$$J_m = 1, b_m = 0.2, K_t = 1, L_a = 0.5, R_a = 1, K_e = 0.5.$$

We want to determine the peak of the impulse response of the system. To this end, we exploit the condition proposed in Theorem 1 for different values of d. In particular, for $d = 1,2,3$, we obtain the upper bounds $c_1 = 2.8568, c_2 = 1.5995$ *and* $c_3 = 1.4483$. The number of LMI scalar variables in (13)-(14) is, respectively, $6, 57, 244$.

For comparison, we test the classical LMI condition based on invariant ellipsoids [2], [10], finding that the upper bound provided by this condition coincides with c_1.

5. CONCLUSIONS

A novel condition for establishing upper bounds of the peak of the impulse response of continuous-time LTI systems has been proposed in terms of feasibility of a system of LMIs. The proposed condition is sufficient for any degree of a polynomial used to describe the trajectory of the system, moreover, the conservatism can be decreased by increasing this degree. As shown by some examples, the proposed condition provides significantly less conservative results than the existing methods. Several directions can be considered in future work, such as the extension of the proposed condition to non-LTI systems and to the design of feedback controllers for ensuring desired upper bounds on the peak of the impulse response.

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