Necessary and Sufficient LMI Conditions for Establishing the Response Peak of LTI and Polytopic LTV Systems

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Abstract

This paper addresses the problem of determining the peak of the response of dynamical systems, a classical problem in control systems. Specifically, the paper starts by considering the impulse response of linear time-invariant (LTI) systems. A necessary and sufficient condition for establishing upper bounds of the peak of this response is proposed in terms of feasibility of a system of linear matrix inequalities (LMIs) by embedding the trajectory onto the level set of a polynomial and by introducing a projection technique for evaluating the extension of this set. Then, this condition is extended to address the impulse response of polytopic linear time-varying (LTV) systems, in particular, linear systems affected linearly by structured time-varying uncertainty constrained into a polytope. Lastly, generalizations to various responses and specializations to some structures are also presented. As shown by several examples, which include randomly generated systems and real physical systems, the proposed conditions provide significantly less conservative results than the existing methods, which provide sufficient conditions only.

Key words: LMI, LTI, LTV, Polytopic uncertainty, Response peak.

1 Introduction

Input-output relationships of linear time-invariant (LTI) systems can be characterized by various indexes, in particular the H-infinity norm (i.e., the maximum amplitude gain of the frequency response) and the H-2 norm (i.e., the square root of the sum of the energies of the impulse responses). It is well-known that these indexes can play key roles in the analysis and synthesis of LTI systems, therefore, methods for determining and controlling these indexes have been studied and developed since long time. See for instance [1, 14] and references therein.

Another important index of LTI systems is the peak of the impulse response. Indeed, this index is important because it provides the maximum amplitude of the output in response to an impulse applied to one of its input channels. Hence, the knowledge of this index may be required in order to verify whether amplitude constraints are satisfied, and the manipulation of this index may be required in order to ensure the satisfaction of such constraints.

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Unfortunately, the determination of this index is still an open problem. Indeed, the classical and pioneering linear matrix inequality (LMI) methods based on set invariance of quadratic Lyapunov functions are generally conservative for determining the peak of the impulse response of an LTI system, see [4, 17] for details. More specifically, it is conjectured in [4] that the smallest upper bound guaranteed by these methods is up to $2n-1$ times the actual peak, where n is the order of the system. This is sharply in contrast with the fact that the methodology of these methods is, however, nonconservative for determining other indexes such as the H-infinity norm and the H-2 norm of an LTI system.

It is useful to observe that the computation of the peak of the impulse response of LTI systems can be approached also with other techniques that do not involve LMIs. For instance, an approach consists of computing the impulse response through numerical integration and extracting the peak from the available samples. Another approach consists of building the analytical expression of the first derivative of the impulse response and searching for its zeros. In this paper, we focus on approaches based on LMIs for various reasons, in particular, because these approaches can be extended to deal with more complex

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classes of systems (for instance, uncertain rather than certain systems) and more complex tasks (for instance, synthesis rather than analysis tasks).

Clearly, the problem is even more challenging if one leaves the realm of the LTI systems. In particular, when considering uncertain linear time-varying (LTV) systems, one has to face additional issues generated by the fact that, not only there exists a family of models for the system (instead of one model only), but also the model that describes the system changes in this family with the time. See for instance [5, 7, 13, 19, 20] and references therein among the numerous contributions to the robust analysis of these systems. As it is well-known, this dependence on the time may generate instability even if all models in the family are stable, see for instance [3, 8]. In this case, the smallest upper bound guaranteed by classical LMI methods based on set invariance of quadratic Lyapunov functions may be arbitrarily conservative.

This paper addresses the problem of determining the peak of the response of dynamical systems. Specifically, the paper starts by considering the impulse response of LTI systems. A novel condition condition for establishing upper bounds of the sought peak is proposed in terms of feasibility of a system of LMIs by embedding the trajectory onto the level set of a polynomial and by introducing a projection technique for evaluating the extension of this set. It is shown that this condition, where the polynomial describing the level set is the only decision variable, is necessary and sufficient. Then, the proposed condition is extended to address the case of polytopic LTV systems, in particular, linear systems affected linearly by structured time-varying uncertainty constrained into a polytope. Lastly, generalizations and specializations of the proposed results are also presented. In particular, it is shown how the proposed result may be used for establishing the peak of the response to other input signals, and it is shown how the proposed results may be simplified depending on the system structure. As shown by several examples, which include randomly generated systems and real physical systems, the proposed conditions provide significantly less conservative results than the existing methods, which provide sufficient conditions only.

The paper is organized as follows. Section 2 provides the preliminaries. Section 3 describes the proposed approach for LTI systems. Section 4 describes the proposed approach for polytopic LTV systems. Section 5 discusses the generalizations and specializations. Section 6 presents the examples. Lastly, Section 7 reports the conclusions and future works. This paper extends the preliminary conference versions [9, 10] where only the sufficiency of the proposed conditions is stated, and where the generalizations and specializations discussed in Section 5 are not reported.

2 Preliminaries

In this section we provide the preliminaries. Specifically, Section 2.1 introduces the notation and the formulation of the main problems, and Section 2.2 presents a brief review of the Gram matrix method.

2.1 Formulation of the Main Problems

The notation adopted in the paper is as follows. The set of natural numbers (including zero) and the set of real numbers set are denoted by $\mathbb{N}$ and $\mathbb{R}$. The null matrix with size specified by the context is denoted by 0. The notation I_n denotes the $n \times n$ identity matrix. The Euclidean norm and the infinity norm of a matrix A are denoted by $\|A\|_2$ and $\|A\|_\infty$. The notation $A \otimes B$ denotes the Kronecker product of two matrices A and B. The transpose of a matrix A is denoted by A'. The notation $A > 0$ (respectively, $A \geq 0$) denotes a symmetric positive definite (respectively, semidefinite) matrix A. The degree of a polynomial $a(b)$ in the variable $b \in \mathbb{R}^n$ is denoted by $\deg(a(b))$. A square matrix A is said to be Hurwitz if all its eigenvalues have negative real parts.

Let us start by considering the linear time-invariant (LTI) system

$$\begin{cases} \dot{x}(t) \,=\, Ax(t) + Bu(t) \\ y(t) \,=\, Cx(t) \end{cases} \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, and A, B, C are given real matrices of suitable sizes. Let us introduce the definition of impulse response.

Definition 1 *The impulse response of the system* (1) *with respect to the i-th input channel is the function $Y_i(t)$, defined as the solution $y(t)$ for initial condition $x(0^-) = 0$ and input $u(t) = \delta(t)I_m^{(i)}$, where $\delta(t)$ is the Dirac distribution and $I_m^{(i)}$ is the i-th column of I_m.* □

The first problem addressed in this paper is as follows.

Problem 1 *Given $c \in (0, \infty)$, establish whether c is an upper bound of the peak of the impulse response of the system* (1) *with respect to all input channels, i.e.,*

$$\|Y_i(t)\|_\infty < c \;\; \forall t \geq 0 \; \forall i = 1, \ldots, m. \tag{2}$$

□

Next, let us consider the system (1) affected by time-

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varying uncertainty, specifically the system

$$\begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) = C(\theta(t))x(t) \\ \theta(\cdot) \in \Theta \end{cases} \quad (3)$$

where $\theta(t) \in \mathbb{R}^q$ is the time-varying uncertainty, $\Theta = \{\theta : \mathbb{R} \to \mathbb{R}^q\}$ is the set of admissible uncertainties, and $A(\theta(t)), B(\theta(t)), C(\theta(t))$ are given real matrix functions of suitable sizes. In particular, we consider the case where:

- the time-varying uncertainty is constrained into a polytope, i.e.,

$$\Theta = \{\theta : \mathbb{R} \to \mathcal{P}\} \quad (4)$$

where $\mathcal{P}$ is the polytope

$$\mathcal{P} = \mathrm{conv}\{P_1, \ldots, P_r\} \quad (5)$$

and $P_1, \ldots, P_r \in \mathbb{R}^q$ are given vectors, being $\mathrm{conv}\{\cdot\}$ the convex hull defined as

$$\mathrm{conv}\{P_1, \ldots, P_r\} = \left\{ \sum_{i=1}^{r} s_i P_i : \ s_i \geq 0, \ \sum_{i=1}^{r} s_i = 1 \right\}; \quad (6)$$

- the matrix functions $A(\theta(t)), B(\theta(t)), C(\theta(t))$ are affine linear in $\theta(t)$, i.e., for $A(\theta(t))$ one can write

$$A(\theta(t)) = A_0 + \sum_{i=1}^{q} \theta_i(t) A_i \quad (7)$$

where $A_0, \ldots, A_q$ are given real matrices of suitable sizes, and similarly for $B(\theta(t))$ and $C(\theta(t))$.

The impulse response of the system (3) is defined analogously to the one of the system (1), and depends on the time-varying uncertainty $\theta(t)$. The second problem addressed in this paper is as follows.

Problem 2 *Given $c \in (0, \infty)$, establish whether c is an upper bound of the peak of the impulse response of the system* (3) *with respect to all input channels for all admissible uncertainties, i.e.,*

$$\|Y_i(t)\|_\infty < c \ \ \forall t \geq 0 \ \forall i = 1, \ldots, m \ \forall \theta(\cdot) \in \Theta. \quad (8)$$

□

The dependence on the time of the various quantities will be omitted in the sequel for ease of notation unless specified otherwise.

2.2 Gram Matrix Method

Here we report a brief review of the Gram matrix method that will be exploited in the next sections, see for instance [6, 11] and references therein.

Let $z : \mathbb{R}^w \to \mathbb{R}$ be a polynomial of even degree. Then, $z(a)$, $a \in \mathbb{R}^w$, can be expressed through the Gram matrix method as

$$z(a) = b(a)' \left(Z + L(\alpha) \right) b(a) \quad (9)$$

where $b(a)$ is a vector whose entries define a basis for the polynomials in a of degree not greater than $\deg(z(a))/2$, Z is a symmetric matrix, and $L(\alpha)$ is a linear matrix function, where α is a free vector, that parameterizes the linear set

$$\mathcal{L} = \left\{ \tilde{L} = \tilde{L}' : \ b(a)' \tilde{L} b(a) = 0 \right\}. \quad (10)$$

The Gram matrix method is useful to establish whether a polynomial is a finite sum of squares of polynomials. Let us define the set of such polynomials as

$$\Sigma = \left\{ z : \mathbb{R}^w \to \mathbb{R}, \ z(a) = \sum_{i=1}^{k} z_i(a)^2, \ z_i(a) \text{ polynomial} \right\}. \quad (11)$$

It turns out that $z(x)$ is in Σ if and only if there exists α that satisfies the linear matrix inequality (LMI)

$$Z + L(\alpha) \geq 0. \quad (12)$$

Let us observe that $b(a)$ can be simply chosen as a vector of monomials. Moreover, in some cases, one can use a reduced vector $b(a)$ without losing the necessary and sufficient LMI condition (12). This is useful because, by reducing the vector $b(a)$, one also reduces the numerical complexity of the LMI (12). The procedure for obtaining this reduction exploits the Newton polytope, see for instance [15]. This procedure requires to solve some linear programs in order to establish whether some points belong to a polytope. In some cases of interest as those reported below and needed for this paper, this reduction is a priori known.

- Case I: the first case considers that $z(a)$ is homogeneous, i.e., all its monomials have the same degree. Then, $b(a)$ can be reduced without losing the necessary and sufficient LMI condition (12) by keeping only the monomials of maximum degree.
- Case II: the second case considers that $z(a)$ is locally quadratic, i.e., does not contain monomials of degree smaller than two. In this case, the vector $b(a)$ can be reduced without losing the necessary and sufficient LMI condition (12) by removing the constant monomials.

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- Case III: the last case considers that $z(a)$ has only monomials of the form

$$\bar{m}(\bar{a})\tilde{m}(\tilde{a}) : \ \bar{m}(\bar{a}) \in \bar{\mathcal{C}}, \ \tilde{m}(\tilde{a}) \in \tilde{\mathcal{C}} \tag{13}$$

where $\bar{a}$ and $\tilde{a}$ are sub-vectors of a, $\bar{m}(\bar{a})$ and $\tilde{m}(\tilde{a})$ are monomials in $\bar{a}$ and $\tilde{a}$, and $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are sets for such monomials. In this case, the vector $b(a)$ can be reduced without losing the necessary and sufficient LMI condition (12) by defining

$$b(a) = \bar{b}(\bar{a}) \otimes \tilde{b}(\tilde{a}) \tag{14}$$

where $\bar{b}(\bar{a})$ and $\tilde{b}(\tilde{a})$ are vectors whose entries define bases for the polynomials with monomials in $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$.

3 Solution for LTI Systems

Let $f : \mathbb{R}^s \to \mathbb{R}$ be a polynomial of degree not greater than d, $d \in \mathbb{N}$. Let us express $f(a)$, $a \in \mathbb{R}^s$, as

$$f(a) = \sum_{\substack{k \in \mathbb{N}^s \\ k_1 + \ldots + k_s \le d}} c_k a^k \tag{15}$$

where $c_k \in \mathbb{R}$. For $g \in \mathbb{R}^s$ let us define

$$h(a) = \sum_{\substack{k \in \mathbb{N}^s \\ k_1 + \ldots + k_s \le d}} c_k a^k (g'a)^{d-k_1-\ldots-k_s}. \tag{16}$$

Whenever $g \neq 0$, $h(a)$ is a homogeneous polynomial in a of degree d. Moreover, the coefficients of $h(a)$ are linear functions of the coefficients of $f(a)$. Let us denote the definition of $h(a)$ from $f(a)$ and g as

$$h(a) = \Phi(f(a), a, g). \tag{17}$$

Let us express the matrices B and C as

$$\begin{cases} B = \begin{pmatrix} B^{(1)} & \ldots & B^{(m)} \end{pmatrix} \\ C = \begin{pmatrix} C^{(1)} & \ldots & C^{(p)} \end{pmatrix}' \end{cases} \tag{18}$$

where $B^{(1)}, \ldots, B^{(m)}, C^{(1)}, \ldots, C^{(p)} \in \mathbb{R}^n$. Lastly, let us introduce the definition of asymptotical stability for the system (1), which will be exploited in the result of this section.

Definition 2 *The system* (1) *is said to be asymptotically stable if*

$$\begin{cases} \forall \varepsilon > 0 \ \exists \delta > 0 : \ \|x(0)\|_2 < \delta \ \Rightarrow \ \|x(t)\|_2 < \varepsilon \ \forall t \ge 0 \\ \lim_{t\to\infty} x(t) = 0. \end{cases} \tag{19}$$

□

As it is well-known, the system (1) is asymptotically stable if and only if the matrix A is Hurwitz. The following theorem provides a solution for Problem 1.

Theorem 1 *Let $i \in \{1, \ldots, m\}$ and $c \in (0, \infty)$. Assume without loss of generality that*

$$\|CB^{(i)}\|_\infty < c. \tag{20}$$

For the system (1) *one has*

$$\|Y_i(t)\|_\infty < c \ \ \forall t \ge 0 \tag{21}$$

if there exist a polynomial $v : \mathbb{R}^n \to \mathbb{R}$ of even degree and a scalar $\varepsilon \in \mathbb{R}$ such that

$$\begin{cases} v(0) = 0 \\ \nabla v(0) = 0 \\ v(B^{(i)}) = 1 \\ \varepsilon > 0 \end{cases} \tag{22}$$

and

$$f(\cdot), h_{j,k}(\cdot) \in \Sigma \ \forall \begin{cases} j = 0, 1 \\ k = 1, \ldots, p \end{cases} \tag{23}$$

where

$$\begin{cases} f(x) = -\nabla v(x) A x \\ h_{j,k}(x) = \Phi\left(v(x) - 1, x, \dfrac{(-1)^j}{c} C^{(k)}\right) - \varepsilon \|x\|_2^{\deg(v(x))}. \end{cases} \tag{24}$$

Moreover, if the system (1) *is asymptotically stable, the condition is not only sufficient but also necessary, and $v(x)$ can be chosen homogeneous.*

Proof. "Sufficiency". Let us suppose that there exist a polynomial $v(x)$ and a scalar ε such that (22)–(23) hold. This implies that $f(x)$ and $h_{j,k}(x)$ are nonnegative. Let us consider the unitary level set of $v(x)$:

$$\mathcal{V}_1 = \{x \in \mathbb{R}^n : \ v(x) = 1\}.$$

From the nonnegativity of $f(x)$, one has that the time derivative of $v(x)$ is nonpositive. This implies that any trajectory of the system (1) starting in $\mathcal{V}_1$ remains in the unitary sublevel set of $v(x)$:

$$\mathcal{V}_2 = \{x \in \mathbb{R}^n : \ v(x) \le 1\}.$$

Applying the Dirac distribution to the i-th input channel with initial condition $x(0^-) = 0$ has the effect to move the initial condition to $x(0) = B^{(i)}$. Hence, there is not

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loss of generality in assuming that (20) holds because (20) is equivalent to

$$\|Y_i(0)\|_\infty < c.$$

From the third constraint in (22) one has that this initial condition belongs to $\mathcal{V}_1$. From (24) and the definition (15)–(17), one has that

$$h_{j,k}(x) = v(x) - 1 - \varepsilon \|x\|_2^{\deg(v(x))} \ \ \forall x \in \mathcal{T}_{j,k}(c)$$

where $\mathcal{T}_{j,k}(c)$ is the level set

$$\mathcal{T}_{j,k}(c) = \left\{ x \in \mathbb{R}^n : \ x' C^{(k)} = (-1)^j c \right\}.$$

Since $h_{j,k}(x)$ is nonnegative and ε is positive, it follows that

$$v(x) > 1 \ \ \forall x \in \mathcal{T}_{j,k}(c).$$

This implies that

$$\mathcal{V}_2 \cap \mathcal{T}_{j,k}(c) = \emptyset.$$

By observing that this holds for all $j = 0, 1$ and $k = 1, \ldots, p$, it follows that

$$\mathcal{V}_2 \cap \mathcal{T}(c) = \emptyset$$

where

$$\mathcal{T}(c) = \left\{ x \in \mathbb{R}^n : \ \|Cx\|_\infty = c \right\}.$$

Hence, we conclude that $\mathcal{V}_2$ does not intersect the set of states for which the output has infinity norm equal to c, and that the trajectory of the system (1) corresponding to the impulse response with respect to the i-th input channel remains in $\mathcal{V}_2$. Since this trajectory starts from $x(0) = B^{(i)}$ for which the output has infinity norm less than c due to (20), and since this trajectory is continuous, it follows that (21) holds.

"Necessity". Let us suppose that (21) holds and that the system (1) is asymptotically stable. It follows that A is Hurwitz, and the Lyapunov equation

$$A'\Xi + \Xi A + I = 0$$

has a unique matrix solution $\Xi = \Xi'$ which is positive definite. Let us define the quadratic function

$$\xi(x) = x' \Xi x.$$

Let us define the set of states

$$\mathcal{X} = \mathcal{X}_1 \cap \mathcal{X}_2$$

where

$$\mathcal{X}_1 = \left\{ x \in \mathbb{R}^n : \ \left\| C e^{At} x \right\|_\infty < c \ \forall t \geq 0 \right\}$$

and

$$\mathcal{X}_2 = \left\{ x \in \mathbb{R}^n : \ \xi(x) \leq \xi(B^{(i)}) \right\}.$$

Let us observe that both $\mathcal{X}_1$ and $\mathcal{X}_2$ are invariant sets for the system (1), hence implying that also $\mathcal{X}$ is invariant. Moreover, $\mathcal{X}$ is bounded, and has the property

$$\alpha x \in \mathcal{X} \ \ \forall x \in \mathcal{X}, \alpha \in [0, 1].$$

Hence, $\mathcal{X}$ can be approximated arbitrarily well by the sublevel set of a positive definite homogeneous polynomial. Let $\tilde{v}(x)$ be such a homogeneous polynomial, and let us denote its sublevel set of interest as

$$\tilde{\mathcal{V}}_2 = \left\{ x \in \mathbb{R}^n : \ \tilde{v}(x) \leq 1 \right\}.$$

Since $\tilde{\mathcal{V}}_2$ is arbitrary close to $\mathcal{X}$, we choose $\tilde{v}(x)$ such that

$$\begin{cases} \tilde{\mathcal{V}}_2 \subseteq \mathcal{X} \\ B^{(i)} \in \tilde{\mathcal{V}}_2. \end{cases}$$

For $k \in \mathbb{N}$, let us define the polynomial

$$\begin{cases} v(x) = \hat{v}(x)^k \\ \hat{v}(x) = \left(\dfrac{\tilde{v}(x)}{\tilde{v}(B^{(i)})} \right)^2. \end{cases}$$

It follows that $v(B^{(i)}) = 1$. Let us observe that

$$\begin{aligned} f(x) &= -\nabla v(x) A x \\ &= -2k \hat{v}(x)^{k-1} \tilde{v}(x) \frac{\nabla \tilde{v}(x) A x}{\tilde{v}(B^{(i)})^2}. \end{aligned}$$

Since $\tilde{v}(x)$ is positive definite and $\nabla \tilde{v}(x) A x$ is negative definite, there exists k such that $f(x)$ is in Σ, see [16]. Let $\tilde{d}$ be the degree of $\tilde{v}(x)$, and let us define

$$g(x) = \frac{(-1)^j}{c} C^{(k)} x.$$

Let us observe that

$$\begin{aligned} \hat{h}_{j,k}(x) &= \Phi\left(v(x) - 1, x, \tfrac{(-1)^j}{c} C^{(k)} \right) \\ &= \hat{v}(x)^k - g(x)^{2\tilde{d}k} \\ &= \hat{v}(x)^k + w_1(x) - w_2(x)^k \\ &= w_1(x) + (\hat{v}(x) - w_2(x))(\hat{v}(x)^{k-1} \\ &\quad + \hat{v}(x)^{k-2} w_2(x) + \ldots + w_2(x)^{k-1}) \end{aligned}$$

where

$$\begin{cases} w_1(x) = w_2(x)^k - g(x)^{2\tilde{d}k} \\ w_2(x) = \alpha (x'x)^{\tilde{d}} + g(x)^{2\tilde{d}} \end{cases}$$

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and $\alpha \in \mathbb{R}$ is an auxiliary quantity. For some positive and sufficiently small α, one has that $w_2(x)$ and $\hat{v}(x) - w_2(x)$ are positive definite. Hence, there exists k such that $\hat{h}_{j,k}(x)$ is in Σ similarly to $f(x)$. Therefore, (22)–(23) hold. □

Theorem 1 provides a necessary and sufficient condition for establishing an upper bound on the peak of the impulse response of the system (1) with respect to the i-th channel. Let us observe that (22) is a set of linear equalities and inequalities on the coefficients of $v(x)$ and ε. Also, let us observe that (23) is equivalent to a set of LMIs because the coefficients of the polynomials $f(x)$ and $h_{j,k}(x)$ depend affine linearly on the coefficients of $v(x)$ and ε, and because the condition that any of these polynomials is in Σ can be equivalently reformulated as an LMI feasibility test as explained in Section 2.2. Hence, the condition of Theorem 1 is equivalent to establish feasibility of a system of LMIs.

Let us observe that this condition is guaranteed to be necessary under the assumption that the system (1) is asymptotically stable, i.e., whenever the matrix A is Hurwitz. This is a mild assumption since the impulse response is generally unbounded whenever this assumption does not hold, due to the presence of eigenvalues with positive real part or eigenvalues on the imaginary axis with different geometric and algebraic multiplicity.

Problem 1 can be addressed by repeating the condition of Theorem 1 for all channels, i.e, for all $i = 1, \ldots, m$. That is, (2) holds if and only if, for all $i = 1, \ldots, m$ there exist a polynomial $v_i : \mathbb{R}^n \to \mathbb{R}$ of even degree and a scalar $\varepsilon_i \in \mathbb{R}$ such that (22)–(23) hold with $v(x)$ and ε replaced by $v_i(x)$ and ε_i.

It is interesting to observe that the condition of Theorem 1 is obtained by evaluating the projection of $v(x) - 1$ onto the set of states for which the output has infinity norm equal to c. This is realized by the function in (15)–(17) and has the benefit of avoiding the introduction of multipliers.

Another remark concerns the possible presence of null columns in B and null rows in C. These columns and rows do not need to be considered for Problem 1 because any of them lead to a constantly null response. Hence, the condition of Theorem 1 should be tested only for the values of i for which the i-th column of B is not null. Moreover, one should consider in (23) only the polynomials $h_{j,k}(x)$ for which the k-th row of C is not null.

Let us observe that the polynomial $f(x)$ is locally quadratic since $v(x)$ has to satisfy (22). This means that the LMI needed to establish whether $f(x)$ is in Σ can be reduced in size and number of variables by selecting the vector of monomials as explained in Case II of Section 2.2. Also, the polynomials $h_{j,k}(x)$ are homogeneous due to the definition (15)–(17). This means that the LMIs needed to establish whether $h_{j,k}(x)$ are in Σ can be reduced in size and number of variables by selecting the vector of monomials as explained in Case I of Section 2.2 (with $z(a) = h_{j,k}(x)$).

Lastly, let us define the smallest upper bound guaranteed by Theorem 1 on the peak of the impulse response of the system (1) as

$$\begin{aligned} c^* = & \inf_{c, v_i(x), \varepsilon} c \\ & \text{s.t. } (20),\ (22)\text{–}(23)\ \ \forall i = 1, \ldots, m. \end{aligned} \tag{25}$$

The upper bound c^* can be found through a bisection search on c where, for any fixed value of c, one tests the condition provided by Theorem 1. Let us observe that this search can be simplified as follows: if the condition provided by Theorem 1 holds for some $c = \bar{c}$ and $i = \bar{i}$, then one does not need to consider in the search $i = \bar{i}$ for all $c \geq \bar{c}$.

4 Solution for Polytopic LTV Systems

Let $s \in \mathbb{R}^r$ and define $\zeta : \mathbb{R}^r \to \mathbb{R}$ as

$$\zeta(s) = \sum_{i=1}^{r} s_i. \tag{26}$$

Also, let us define $P : \mathbb{R}^r \to \mathbb{R}^q$ as

$$P(s) = \sum_{i=1}^{r} s_i P_i \tag{27}$$

and $\text{sq} : \mathbb{R}^r \to \mathbb{R}^r$ as

$$\text{sq}(s) = \begin{pmatrix} s_1^2 & \ldots & s_r^2 \end{pmatrix}'. \tag{28}$$

Let us express the matrices $B(\theta(t))$ and $C(\theta(t))$ as

$$\begin{cases} B(\theta(t)) = \begin{pmatrix} B^{(1)}(\theta(t)) & \ldots & B^{(m)}(\theta(t)) \end{pmatrix} \\ C(\theta(t)) = \begin{pmatrix} C^{(1)}(\theta(t)) & \ldots & C^{(p)}(\theta(t)) \end{pmatrix}' \end{cases} \tag{29}$$

where $B^{(1)}, \ldots, B^{(m)}, C^{(1)}, \ldots, C^{(p)} : \mathbb{R}^q \to \mathbb{R}^n$. Lastly, let us introduce the definition of robust asymptotical stability for the system (3).

Definition 3 *The system* (3) *is said to be robustly asymptotically stable if its state* $x(t)$ *satisfies* (19) *for all* $\theta(\cdot) \in \Theta$. □

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As it is well-known, robust asymptotical stability of the system (3) is not equivalent to having the matrix $A(t)$ Hurwitz for all admissible uncertainties, but to having a common Lyapunov function, see for instance [3, 8] and references therein. The following theorem provides a solution for Problem 2.

Theorem 2 *Let $i \in \{1, \dots, m\}$ and $c \in (0, \infty)$. Assume without loss of generality that*

$$\|C(\theta(0))B^{(i)}(\theta(0))\|_\infty < c \quad \forall \theta(0) \in \mathcal{P}. \tag{30}$$

For the system (3) one has

$$\|Y_i(t)\|_\infty < c \quad \forall t \geq 0 \; \forall \theta(\cdot) \in \Theta \tag{31}$$

if there exist a polynomial $v : \mathbb{R}^n \to \mathbb{R}$ of even degree and a scalar $\varepsilon \in \mathbb{R}$ such that

$$\begin{cases} v(0) \;=\; 0 \\ \nabla v(0) \;=\; 0 \\ \varepsilon \;>\; 0 \end{cases} \tag{32}$$

and

$$f_l(\cdot), g_2(\cdot), h_{3,j,k}(\cdot) \in \Sigma \quad \forall \begin{cases} j = 0, 1 \\ k = 1, \dots, p \\ l = 1, \dots, r \end{cases} \tag{33}$$

where

$$f_l(x) = -\nabla v(x) A(P_l) x \tag{34}$$

and

$$\begin{cases} g_1(s) \;=\; \Phi\left(1 - v(B^{(i)}(P(s))), s, \zeta(s)\right) \\ g_2(s) \;=\; g_1(\mathrm{sq}(s)) \end{cases} \tag{35}$$

and

$$\begin{cases} h_{1,j,k}(x,s) \;=\; \Phi\left(v(x) - 1, x, \dfrac{(-1)^j}{c} C^{(k)}(P(s))\right) \\ \qquad\qquad\qquad -\varepsilon \|x\|_2^{\deg(v(x))} \\ h_{2,j,k}(x,s) \;=\; \Phi\left(h_{1,j,k}(x,s), s, \zeta(s)\right) \\ h_{3,j,k}(x,s) \;=\; h_{2,j,k}(x, \mathrm{sq}(s)). \end{cases} \tag{36}$$

Moreover, if the system (3) is robustly asymptotically stable, the condition is not only sufficient but also necessary, and $v(x)$ can be chosen homogeneous.

Proof. "Sufficiency". Suppose that there exist a polynomial $v(x)$ and a scalar ε such that (32)–(33) hold. Analogously to the proof of Theorem 1, the nonnegativity of $f_l(x)$ implies that any trajectory of the system (3) starting in the unitary sublevel set $\mathcal{V}_2$ of $v(x)$ remains in $\mathcal{V}_2$ for all uncertain vectors $\theta(t)$ fixed at the vertices of $\mathcal{P}$, i.e., $P_1, \dots, P_l$. Since $A(\theta(t))$ is affine linear in $\theta(t)$, any trajectory of the system (3) starting in $\mathcal{V}_2$ remains in $\mathcal{V}_2$ for all admissible uncertainties. Applying the Dirac distribution to the i-th input channel with initial condition $x(0^-) = 0$ has the effect to move the initial condition to $x(0) = B^{(i)}(\theta(0))$. Hence, there is not loss of generality in assuming that (30) holds because (30) is equivalent to

$$\|Y_i(0)\|_\infty < c \quad \forall \theta(\cdot) \in \Theta.$$

From (35) and the definition (15)–(17), one has that

$$g_1(s) = 1 - v(B^{(i)}(P(s))) \quad \forall s \in \mathcal{S}$$

where $\mathcal{S}$ is the simplex

$$\mathcal{S} = \{s \in \mathbb{R}^r : \; \zeta(s) = 1, \; s_i \geq 0 \; \forall i = 1, \dots, r\}.$$

Moreover, the nonnegativity of $g_2(s)$ is equivalent to

$$g_1(s) \geq 0 \quad \forall s \in \mathcal{S}.$$

This implies that $x(0) \in \mathcal{V}_2$ for all admissible uncertainties. Similarly, one has that

$$h_{2,j,k}(x,s) = v(x) - 1 - \varepsilon \|x\|_2^{\deg(v(x))} \; \forall x \in \mathcal{T}_{j,k}(c,s) \, \forall s \in \mathcal{S}$$

where $\mathcal{T}_{j,k}(c,s)$ is the level set

$$\mathcal{T}_{j,k}(c,s) = \left\{x \in \mathbb{R}^n : \; x' C^{(k)}(P(s)) = (-1)^j c\right\}.$$

From the nonnegativity of $h_{3,j,k}(x,s)$, one has that

$$v(x) > 1 \quad \forall x \in \mathcal{T}_{j,k}(c,s) \; \forall s \in \mathcal{S}.$$

Hence, we conclude that $\mathcal{V}_2$ does not intersect the set of states for which the output has infinity norm equal to c for all admissible uncertainties, and that the trajectory of the system (1) corresponding to the impulse response with respect to the i-th input channel remains in $\mathcal{V}_2$ for all admissible uncertainties. Since this trajectory starts from $x(0) = B^{(i)}(\theta(0))$ for which the output has infinity norm less than c due to (30), and since this trajectory is continuous, it follows that (31) holds.

"Necessity". Let us suppose that (31) holds and that the system (3) is robustly asymptotically stable. From [2], there exists a homogeneous polynomial Lyapunov function $\xi(x)$ proving robust asymptotical stability. Let us define the set of states

$$\mathcal{X} = \mathcal{X}_1 \cap \mathcal{X}_2$$

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where

$$\mathcal{X}_1 = \left\{ x_0 \in \mathbb{R}^n : \sup_{t \geq 0,\ \theta(\cdot) \in \Theta} \left\| y(t)|_{x(0)=x_0,\ u(t)=0} \right\|_\infty < c \right\}$$

and

$$\mathcal{X}_2 = \left\{ x \in \mathbb{R}^n : \xi(x) \leq \max_{s \in \mathcal{S}} \xi(B^{(i)}(P(s))) \right\}.$$

Let us observe that both $\mathcal{X}_1$ and $\mathcal{X}_2$ are robustly invariant sets for the system (3), hence implying that also $\mathcal{X}$ is robustly invariant. Analogously to the proof of Theorem 1, we choose a homogeneous polynomial $\tilde{v}(x)$ such that its sublevel set

$$\begin{cases} \tilde{\mathcal{V}}_2 \subseteq \mathcal{X} \\ B^{(i)}(P(s)) \in \tilde{\mathcal{V}}_2 \ \ \forall s \in \mathcal{S} \\ \tilde{\mathcal{V}}_2 = \{ x \in \mathbb{R}^n : \tilde{v}(x) \leq 1 \}. \end{cases}$$

For $k \in \mathbb{N}$, let us define the polynomial

$$\begin{cases} v(x) = \hat{v}(x)^k \\ \hat{v}(x) = \tilde{v}(x)^2. \end{cases}$$

Let us observe that the matrices $A(P_l)$, $l = 1, \ldots, r$, are Hurwitz. Hence, analogously to the proof of Theorem 1, there exists k such that each $f_l(x)$ is in Σ. Also, let us observe that $1 - v(B^{(i)}(P(s))) \geq 0$ for all $s \in \mathcal{S}$. Let $\tilde{d}$ be the degree of $\tilde{v}(x)$. It follows that

$$\begin{aligned} g_1(s) &= \Phi\left(1 - v(B^{(i)}(P(s))), s, \zeta(s)\right) \\ &= \zeta(s)^{2\tilde{d}k} - \tilde{v}(B^{(i)}(P(s)))^k \\ &= \left(\zeta(s)^{2\tilde{d}} - \tilde{v}(B^{(i)}(P(s)))\right) \\ &\quad \cdot \sum_{l=0}^{k-1} \zeta(s)^{2\tilde{d}(k-1-l)} \tilde{v}(B^{(i)}(P(s)))^l. \end{aligned}$$

Since $\zeta(s)^{2\tilde{d}} - \tilde{v}(B^{(i)}(P(s)))$ is positive over $\mathcal{S}$, there exists k such that $g_1(s)$ has all positive coefficients, and, hence, $g_2(s)$ is in Σ. Lastly, analogously to the proof of Theorem 1, there exists k such that each $h_{1,j,k}(x, s)$ is in Σ for any frozen $s \in \mathcal{S}$. Moreover, as for $g_1(s)$, there exists k such that $h_{2,j,k}(x, s)$ is the sum of squares of polynomials in x multiplied by monomials in s with positive coefficients. Hence, $h_{3,j,k}(x, s)$ is in Σ. □

Theorem 2 provides a necessary and sufficient condition for establishing an upper bound on the peak of the impulse response of the system (3) with respect to the i-th channel for all admissible uncertainties. Analogously to the case addressed in Section 3, one has that:

- this condition is equivalent to establishing feasibility of a system of LMIs;
- Problem 2 can be addressed by repeating the condition of Theorem 2 for all channels, i.e, for all $i = 1, \ldots, m$;
- null columns in B and null rows in C do not need to be considered.

Also, it is useful to observe that the LMIs needed to establish whether the polynomials $f_l(x)$, $g_2(s)$ and $h_{3,j,k}(x, s)$ are in Σ can be reduced in size and number as explained in Section 2.2. Specifically:

- the polynomials $f_l(x)$ are locally quadratic and, hence, they belong to Case II;
- the polynomial $g_2(s)$ is homogeneous due to the definition (15)–(17) and, hence, it belongs to Case I;
- the polynomials $h_{3,j,k}(x, s)$ have monomials of the form (13) where $\bar{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ are the sets of homogeneous monomials in x and s, respectively, of suitable degree. Hence, these polynomials belong to Case III.

Let us define the smallest upper bound guaranteed by Theorem 2 on the peak of the impulse response of the system (3) for all admissible uncertainties as

$$\begin{aligned} c^* = & \inf_{c, v_i(x), \varepsilon} \ c \\ & \text{s.t. } (30),\ (32)\text{–}(33) \ \ \forall i = 1, \ldots, m. \end{aligned} \tag{37}$$

Also in this case, the upper bound c^* can be found through a bisection search on c. Moreover, this search can be simplified by avoiding to consider pairs $(\bar{i}, c)$ if the condition provided by Theorem 1 holds for a pair $(\bar{i}, \bar{c})$ with $c \geq \bar{c}$.

Lastly, let us observe that the condition (30) can be trivially checked if $C(\theta(0))B^{(i)}(\theta(0))$ is independent on $\theta(0)$. Also, if $C(\theta(0))B^{(i)}(\theta(0))$ is a linear function of $\theta(0)$, then the condition (30) can be easily checked by considering only the vertices of $\mathcal{P}$. Otherwise, $C(\theta(0))B^{(i)}(\theta(0))$ is a quadratic function of $\theta(0)$. For this or the previous cases, one can use the following result, where a necessary and sufficient condition is proposed via an LMI feasibility test.

Corollary 1 *The condition* (30) *holds if and only if there exists a nonnegative integer l and a scalar $\varepsilon \in \mathbb{R}$ such that*

$$\begin{cases} m_{3,j,k}(\cdot) \in \Sigma \ \ \forall j = 0, 1 \\ \varepsilon > 0 \end{cases} \tag{38}$$

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where

$$\begin{cases} m_{1,j,k}(s) = \left(c-(-1)^j C^{(k)}(P(s))B^{(i)}(P(s))) - \varepsilon\right) \\ \qquad \cdot\zeta(s)^l \\ m_{2,j,k}(s) = \Phi\left(m_{1,j,k}(s), s, \zeta(s)\right) \\ m_{3,j,k}(s) = m_{2,j,k}(\mathrm{sq}(s)) \end{cases} \tag{39}$$

Proof. Analogous to the proof of Theorem 2. □

5 Generalizations and Specializations

This section explains how one can consider more general problems than Problems 1-2 (see Sections 5.1 and 5.2), and how one can simplify in some cases the conditions provided in Theorems 1–2 (see Sections 5.3, 5.4 and 5.5).

5.1 Generalizations: Case I

Here we consider the response of an LTI system to a constant signal with possibly nonzero initial condition. Specifically, we consider the system (1) in the case

$$\begin{cases} u(t) = \bar{u} \ \ \forall t \geq 0 \\ x(0) = x_0 \end{cases} \tag{40}$$

where $\bar{u} \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$. The problem consists of establishing whether a given quantity $c \in (0,\infty)$ is an upper bound of the peak of the output, i.e.,

$$\|y(t)\|_\infty < c \ \ \forall t \geq 0. \tag{41}$$

This problem can be addressed through Theorem 1 by introducing the following changes whenever A is Hurwitz. First, let $\bar{x} \in \mathbb{R}^n$ the equilibrium point corresponding to $\bar{u}$, which is given by

$$\bar{x} = -A^{-1}B\bar{u}. \tag{42}$$

Second, let us introduce the change of variable

$$\tilde{x} = x - \bar{x}. \tag{43}$$

It follows that the system (1) can be equivalently rewritten as

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) \\ y(t) = C(\tilde{x}(t) + \bar{x}(t)) \\ \tilde{x}(0) = x_0 - \bar{x}. \end{cases} \tag{44}$$

Hence, Theorem 1 can be used by redefining B as

$$B \ \rightarrow \ x_0 - \bar{x} \tag{45}$$

and the homogeneous polynomial $h_{j,k}(x)$ as

$$h_{j,k}(x) \ \rightarrow \ \Phi\left(v(x-\bar{x}) - 1, x, \frac{(-1)^j}{c}C^{(k)}\right) - \varepsilon\|x\|_2^{\deg(v(x))}. \tag{46}$$

5.2 Generalizations: Case II

Here we consider the response of an LTI system or a polytopic LTV system to a signal obtainable as impulse response of an LTI system, with possibly nonzero initial condition. Specifically, we consider the system (1) or (3) in the case

$$\begin{cases} u(t) = \tilde{Y}(t) \ \ \forall t \geq 0 \\ x(0) = x_0 \end{cases} \tag{47}$$

where $x_0 \in \mathbb{R}^n$, and $\tilde{Y}(t) \in \mathbb{R}^m$ is the impulse response of the auxiliary single-input system

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t) \end{cases} \tag{48}$$

where $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ is the state, $\tilde{u}(t) \in \mathbb{R}$ is the input, $\tilde{y}(t) \in \mathbb{R}^m$ is the output, and $\tilde{A}, \tilde{B}, \tilde{C}$ are given real matrices of suitable sizes. The problem consists of establishing whether a given quantity $c \in (0,\infty)$ is an upper bound of the peak of the output.

This problem can be addressed through Theorems 1 or 2 by including the auxiliary system into the system (1) or (3). Specifically, for the system (1), this can be done by redefining A, B and C as

$$A \rightarrow \begin{pmatrix} A & B\tilde{C} \\ 0 & \tilde{A} \end{pmatrix}, \ B \rightarrow \begin{pmatrix} x_0 \\ \tilde{B} \end{pmatrix}, \ C \rightarrow \begin{pmatrix} C \\ 0 \end{pmatrix}. \tag{49}$$

The system (3) can be analogously addressed, and it is omitted for brevity.

5.3 Specializations: Case I

Here we consider the case where the system (1) or (3) has order two, i.e., $n = 2$. In this case, the conditions provided by Theorems 1 or (2) can be simplified. The following corollary explains how this can be done for Theorem 1.

Corollary 2 *Suppose that $n = 2$. Theorem 1 still holds by removing in (23) the polynomials $h_{j,k}(x)$ for which j*

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and k satisfy

$$(-1)^j C^{(k)} A B^{(i)} < 0. \quad (50)$$

Proof. Suppose that (2) holds. This implies that the scalar product between the vector field at $B^{(i)}$ and the cost vector $(-1)^j C^{(k)}$ is negative. Since $n = 2$, it follows that the supremum of $|C^{(k)}x|$ along the trajectory, whenever finite, is achieved for $x = B^{(i)}$ or for x on the positive level sets of $-(-1)^j C^{(k)}x$. Hence, $h_{j,k}(x)$ can be neglected. □

The same idea can be used for Theorem 2 as explained in the following corollary.

Corollary 3 *Suppose $n = 2$. Theorem 2 still holds by removing in (33) the polynomials $h^3_{j,k}(x,s)$ for which j and k satisfy*

$$(-1)^j C^{(k)}(\theta(0)) A(\theta(0)) B^{(i)}(\theta(0)) < 0 \quad \theta(0) \in \mathcal{P}. \quad (51)$$

Proof. Analogous to the proof of Corollary 2. □

5.4 Specializations: Case II

Here we consider the case where the polynomial $v(x)$ is chosen homogeneous. In this case, the smallest upper bound c^* guaranteed by Theorem 1 or 2 can be found without performing a bisection search on c. Moreover, the conditions provided by these theorems can be simplified analogously to the case of second order systems discussed in Section 5.3. The following corollary explains how this can be done for Theorem 1.

Corollary 4 *The smallest upper bound for which the condition of Theorem 1 holds with $v(x)$ homogeneous is*

$$c^* = (\beta^*)^{-\frac{1}{\deg(v(x))}} \quad (52)$$

where β^* is the solution of the semidefinite program (SDP)

$$\begin{array}{rl} \beta^* = & \sup\limits_{\beta, v(x), \varepsilon} \beta \\ & s.t.\ (20),\ (22)\text{–}(23) \end{array} \quad (53)$$

and the polynomials $h_{j,k}(x)$, $j = 0,1$, are replaced by the polynomials $h_k(x)$ (independent on j)

$$h_k(x) = v(x) - \beta \left(C^{(k)}x\right)^{\deg(v(x))} - \varepsilon \|x\|_2^{\deg(v(x))}. \quad (54)$$

Proof. Suppose that $v(x)$ is homogeneous. It follows that $h_{j,k}(x)$ in (24) is given by

$$\begin{aligned} h_{j,k}(x) &= \Phi\left(v(x) - 1, x, \frac{(-1)^j}{c} C^{(k)}\right) - \varepsilon \|x\|_2^{\deg(v(x))} \\ &= v(x) - \left(\frac{(-1)^j}{c} C^{(k)}\right)^{\deg(v(x))} - \varepsilon \|x\|_2^{\deg(v(x))} \\ &= v(x) - \left(\frac{1}{c} C^{(k)}\right)^{\deg(v(x))} - \varepsilon \|x\|_2^{\deg(v(x))}. \end{aligned}$$

By defining

$$\beta = c^{-\deg(v(x))}$$

and by observing that maximizing β is equivalent to minimize c, the result follows. □

The same idea can be used for Theorem 2 as explained in the following corollary.

Corollary 5 *The smallest upper bound for which the condition of Theorem 2 holds with $v(x)$ homogeneous is*

$$c^* = (\beta^*)^{-\frac{1}{\deg(v(x))}} \quad (55)$$

where β^* is the solution of the SDP

$$\begin{array}{rl} \beta^* = & \sup\limits_{\beta, v(x), \varepsilon} \beta \\ & s.t.\ (30),\ (32)\text{–}(33) \end{array} \quad (56)$$

and the polynomials $h_{a,j,k}(x,s)$, $a = 1,2,3$ and $j = 0,1$, are replaced by the polynomials $h_{a,k}(x,s)$ (independent on j)

$$\begin{cases} h_{1,k}(x,s) = v(x) - \beta \left(C^{(k)}(P(s))x\right)^{\deg(v(x))} - \varepsilon \|x\|_2^{\deg(v(x))} \\ h_{2,k}(x,s) = \Phi\left(h_{1,k}(x,s), s, \zeta(s)\right) \\ h_{3,k}(x,s) = h_{2,k}(x, \mathrm{sq}(s)). \end{cases} \quad (57)$$

Proof. Analogous to the proof of Corollary 4. □

5.5 Specializations: Case III

Here we consider the cases where the matrix $B(\theta(t))$ or $C(\theta(t))$ in the system (3) is independent on $\theta(t)$. In these cases, the condition of Theorem (2) can be simplified as follows.

If the matrix $B(\theta(t))$ is independent on $\theta(t)$, then Theorem 2 still holds by removing the polynomial $g_2(s)$ in

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(33), and by imposing the third constraint in (22). This is advantageous since, instead of imposing whether a polynomial is in Σ, one simply imposes a linear inequality.

Similarly, if the matrix $C(\theta(t))$ is independent on $\theta(t)$, then Theorem 2 still holds by removing the polynomials $h_{3,j,k}(x,s)$ in (33), and by imposing that $h_{j,k}(x)$ defined in (24) is in Σ. This is advantageous since $h_{j,k}(x)$ has a smaller degree and number of variables than $h_{3,j,k}(x,s)$.

6 Examples

In this section we present some examples. The toolbox SeDuMi [18] for Matlab is adopted to test all the mentioned LMI conditions. The computed quantities are rounded to the third decimal digit.

6.1 Example 1

In this first example we consider the LTI system

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -0.5 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t). \end{cases}$$

First, we want to establish whether the peak of the impulse response is smaller than $c = 0.8$. In order to solve this problem, we exploit Theorem 1. Let us observe that the condition of this theorem can be simplified as explained in Corollary 2 in this case. We have that (20) is verified since $0 = \|CB^{(i)}\|_\infty < c$. By choosing $\deg(v(x)) = 2$, the condition proposed in Theorem 1 does not hold. However, this condition holds by choosing $\deg(v(x)) = 4$, hence implying that the peak of the impulse response is smaller than c.

Next, we consider the problem of determining the peak of the impulse response. To this end, we compute the smallest upper bound guaranteed by Theorem 1, i.e., c^* in (25). With $\deg(v(x)) = 2$, we obtain the upper bound $c_2^* = 0.828$. This upper bound can be improved by increasing $\deg(v(x))$. Indeed, with $\deg(v(x)) = 4$, we obtain the upper bound $c_4^* = 0.645$.

It turns out that the upper bound c_4^* is tight. Indeed, as shown by Figure 1, the state trajectory that corresponds to the impulse response (found by solving the differential equation) touches the level set $\|Cx\|_\infty = c_4^*$.

It is interesting to observe that the level set of the found polynomial $v(x)$ for $\deg(v(x)) = 4$ is not included in the sublevel set $\|Cx\|_\infty \leq c_4^*$. This is due to the fact that the condition proposed in Theorem 1 has been simplified as explained in Corollary 2.

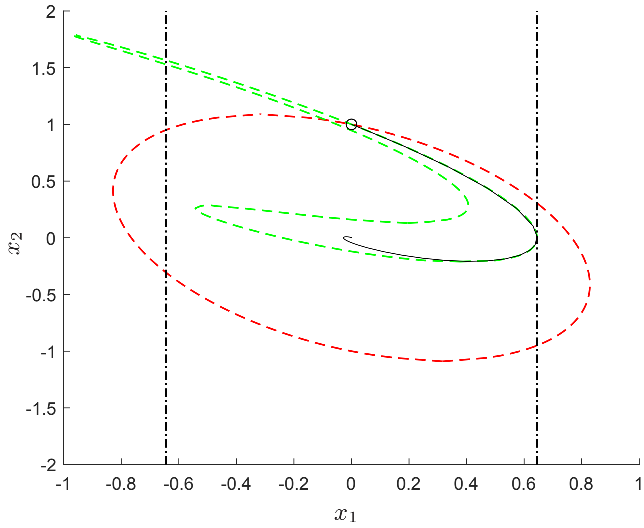

Fig. 1. Example 1: level set $\|Cx\|_\infty = c_4^*$ (black dash-dot line), impulse response (black solid line), starting point of the impulse response (black circle), and level sets of the found polynomial $v(x)$ for $\deg(v(x)) = 2, 4$ (red and green dashed lines).

For comparison, we test the classical LMI condition based on invariant ellipsoids [4, 17], finding that the upper bound provided by this condition coincides with c_2^*.

We conclude this example by considering the case where the polynomial $v(x)$ is constrained to be homogeneous (in fact, as stated in Theorem 1, $v(x)$ can be assumed to be homogeneous). In such a case, the peak 0.645 previously found with $\deg(v(x)) = 4$ can be now guaranteed if one increases the degree of $v(x)$ till 16. Clearly, this does not seem advantageous, however, one can avoid the bisection search according to Corollary 4 whenever $v(x)$ is homogeneous. More specifically:

- with $\deg(v(x)) = 4$ and $v(x)$ not homogeneous, a bisection search is needed to solve (25). At each step of this search, one has to establish feasibility of a system of LMIs with 16 scalar variables;
- with $\deg(v(x)) = 16$ and $v(x)$ homogeneous, only an SDP is needed to solve (25) according to Corollary 4, and this SDP has 73 scalar variables.

6.2 Example 2

In this second example we consider the polytopic LTV system

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} \theta(t) & 2 \\ -1-2\theta(t) & -1-\theta(t) \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 1 & 3 \end{pmatrix} x(t) \\ \Theta = \{\theta : \mathbb{R} \to [0,1]\}. \end{cases}$$

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First, we want to establish whether the peak of the impulse response is smaller than $c = 5$ for all admissible uncertainties. In order to solve this problem, we exploit Theorem 2. Let us observe that the condition of this theorem can be simplified as explained in Corollary 3 in this case. We have that (30) is verified since $4 = \|C(\theta(0))B^{(i)}(\theta(0))\|_\infty < c$ for all $\theta(0) \in \mathcal{P}$. By choosing $\deg(v(x)) = 2$, the condition proposed in Theorem 2 does not hold. However, this condition holds by choosing $\deg(v(x)) = 4$, hence implying that the peak of the impulse response is smaller than c for all admissible uncertainties.

Next, we consider the problem of determining the peak of the impulse response for all admissible uncertainties. To this end, we compute the smallest upper bound guaranteed by Theorem 2, i.e., c^* in (37). With $\deg(v(x)) = 2$, we obtain the upper bound $c_2^* = \infty$, i.e., no upper bound. With $\deg(v(x)) = 4, 6, 8$, we obtain the upper bounds $c_4^* = 4.751$, $c_6^* = 4.280$ and $c_8^* = 4.221$.

Figure 2 shows the state trajectories that correspond to the impulse response for some admissible uncertainties randomly generated (found by solving the differential equations). As we can see, there are trajectories that almost touches the level set $\|Cx\|_\infty = c_8^*$, which means that the upper bound c_8^* is close to the sought value.

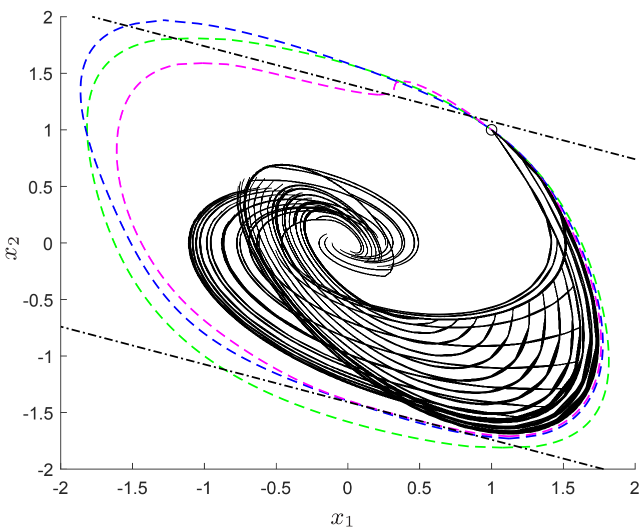

Fig. 2. Example 2: level set $\|Cx\|_\infty = c_8^*$ (black dash-dot line), impulse response for some admissible uncertainties randomly generated (black solid lines), starting point of the impulse responses (black circle), and level sets of the found polynomial $v(x)$ for $\deg(v(x)) = 4, 6, 8$ (green, blue and magenta dashed lines).

For comparison, we test an extension of the classical LMI condition based on invariant ellipsoids [4,17] to deal with the case of polytopic LTV systems. In particular, this extension is obtained by repeating the LMI involving the matrix A at all vertices of the polytope $\mathcal{P}$ (the condition is still sufficient since this LMI depends linearly on the matrix A, and since $\mathcal{P}$ is convex). We find that no upper bound is guaranteed by this condition.

We conclude this example by considering the case where the polynomial $v(x)$ is constrained to be homogeneous (in fact, as stated in Theorem 2, $v(x)$ can be assumed to be homogeneous). In such a case, the upper bound 4.221 previously found with $\deg(v(x)) = 8$ can be now guaranteed if one increases the degree of $v(x)$ till 14. In particular, with $\deg(v(x)) = 14$ and $v(x)$ homogeneous, we find $c_{14}^* = 4.216$. It is interesting to observe that, contrary to Example 1, the computation of this new upper bound is cheaper from the complexity point of view. Indeed:

- with $\deg(v(x)) = 8$ and $v(x)$ not homogeneous, a bisection search is needed to solve (37). At each step of this search, one has to establish feasibility of a system of LMIs with 174 scalar variables;
- with $\deg(v(x)) = 14$ and $v(x)$ homogeneous, only an SDP is needed to solve (37) according to Corollary 5, and this SDP has 78 scalar variables.

6.3 Example 3

In this last example we consider the model of a DC motor, specifically (see for instance [12])

$$\begin{cases} J_m\ddot{\psi}_m(t) + b_m\dot{\psi}_m(t) = K_t i_a(t) \\ L_a\dot{i}_a(t) + R_a i_a(t) = -K_e\dot{\psi}_m(t) + v_a(t) \end{cases}$$

where $\psi_m(t)$ is the angle, $i_a(t)$ is the current, $v_a(t)$ is the voltage, and J_m, b_m, K_t, L_a, R_a and K_e are parameters. Let us define

$$\begin{cases} x(t) = (\psi_m(t), \dot{\psi}_m(t), i_a(t))' \\ u(t) = v_a(t) \\ y(t) = \psi_m(t). \end{cases}$$

It follows that the model can be rewritten as

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{b_m}{J_m} & \dfrac{K_t}{J_m} \\ 0 & -\dfrac{K_e}{L_a} & -\dfrac{R_a}{L_a} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{L_a} \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t). \end{cases}$$

Let us choose the plausible values

$$b_m = 0.2,\ K_t = 1,\ L_a = 0.5,\ R_a = 1,\ K_e = 0.5.$$

For the parameter J_m we consider the following two scenarios.

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6.3.1 *Scenario 1: constant J_m.*

In the first scenario we consider that the parameter J_m is constant, in particular $J_m = 1$. The problem is to determine the peak of the impulse response. To this end, we compute the smallest upper bound guaranteed by Theorem 1, i.e., c^* in (25). With $\deg(v(x)) = 2, 4, 6, 8$, we obtain the upper bounds $c_2^* = 2.857$, $c_4^* = 1.602$, $c_6^* = 1.450$ and $c_8^* = 1.443$.

For comparison, we test the classical LMI condition based on invariant ellipsoids [4, 17], finding that the upper bound provided by this condition coincides with c_2^*.

6.3.2 *Scenario 2: time-varying J_m.*

In the second scenario we consider that the parameter J_m is a time-varying uncertainty in the interval $[1, 3]$. This situation can be considered with the system (3) by defining

$$\begin{cases} \theta(t) = \dfrac{1}{J_m} \\ \Theta = \{\theta : \mathbb{R} \to [1/3, 1]\}. \end{cases}$$

The problem is to determine the peak of the impulse response for all admissible uncertainties. To this end, we compute the smallest upper bound guaranteed by Theorem 2, i.e., c^* in (37). With $\deg(v(x)) = 2, 4, 6, 8$, we obtain the upper bounds $c_2^* = \infty$, $c_4^* = 13.349$, $c_6^* = 6.183$ and $c_8^* = 4.648$. Figure 3 shows the level set of the polynomial found for $\deg(v(x)) = 8$.

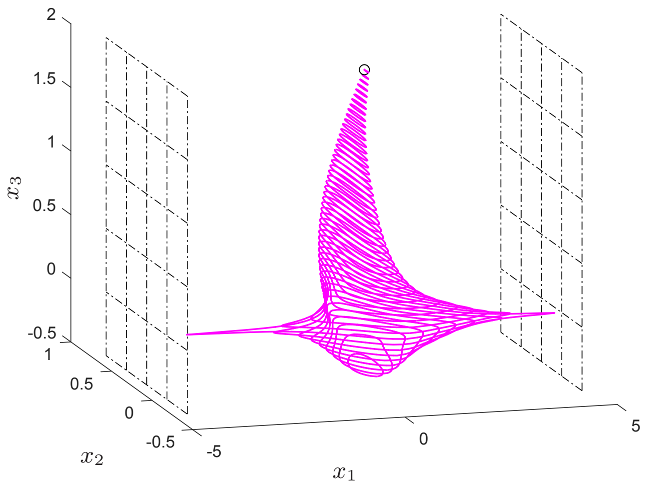

Fig. 3. Example 3 (Scenario 2): level set $\|Cx\|_\infty = c_8^*$ (black grid), starting point of the impulse response (black circle), and level set of the found polynomial $v(x)$ for $\deg(v(x)) = 8$ (magenta surface).

For comparison, we test an extension of the classical LMI condition based on invariant ellipsoids [4, 17] to deal with the case of polytopic LTV systems as done in Example 2. We find that no upper bound is guaranteed by this condition.

7 Conclusions

This paper has addressed the problem of determining the peak of the response of LTI and polytopic LTV systems. Necessary and sufficient conditions for establishing upper bounds of the sought peak have been proposed in terms of feasibility of a system of LMIs. As shown by several examples, which include randomly generated systems and real physical systems, the proposed conditions provide significantly less conservative results than the existing methods, which provide sufficient conditions only.

Various directions can be considered in future work. In particular, it would be interesting to investigate the extension of the proposed conditions to the design of feedback controllers for ensuring desired upper bounds on the peak of the response. Another interesting direction could be the extension of the proposed conditions to systems with nonlinear dynamics.

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