Robotics Report Lab 1

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Direct Kinematics

For the creation of the Direct Kinematics we decided to use the Denavit-Hartenberg convention to represent the robot arm. In this convention, the following formulas apply:

- $a_i \equiv$ Distance from Z_i to Z_{i+1} along X_i
- $\alpha_i \equiv$ Angle between Z_i and Z_{i+1} around X_i
- $d_i \equiv$ Distance from X_{i1} to X_i along Z_i
- $\theta_i \equiv$ Angle between X_{i1} and X_i around Z_i

Below, a real life and simulated example is giving of the used angles.

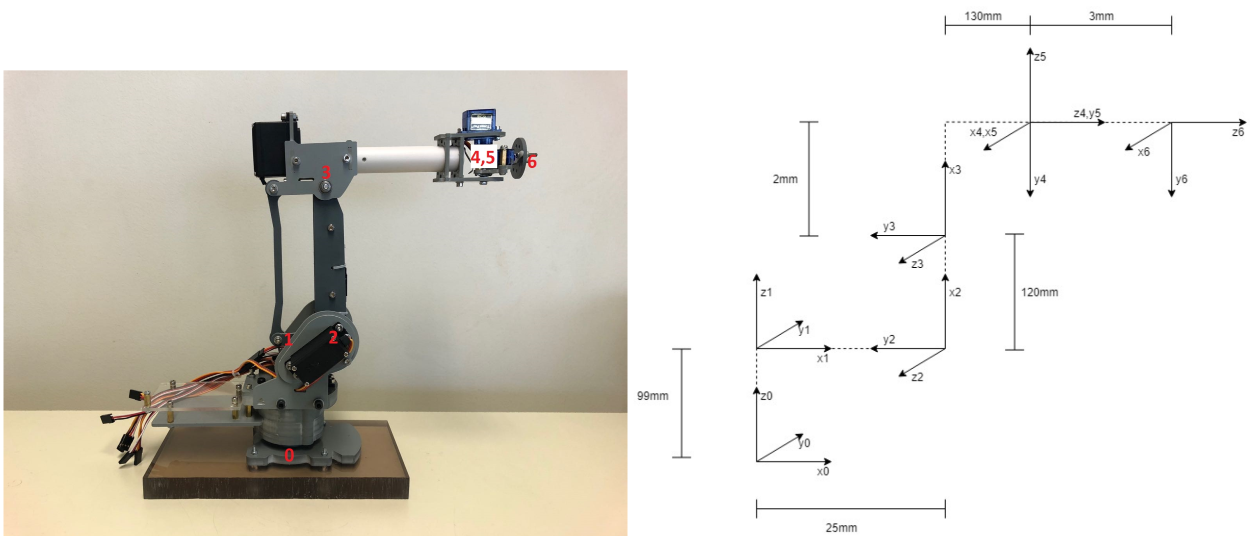

Figure 1: Robot initial position

This is the resting position that we choose for robot arm, this is a neutral position which has several of the angles at 90 degrees, which gives us a cleaner start to work from. Using the DH convention and our measurements of the robot arm in the lab we were able to fill the table below with the values shown in Figure 1.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	99	θ_1
2	$\frac{\pi}{2}$	25	0	$\theta_2+\frac{\pi}{2}$
3	0	120	0	θ_3
4	$\frac{\pi}{2}$	2	130	$\theta_4+\frac{\pi}{2}$
5	$\frac{\pi}{2}$	0	0	θ_5
6	$-\frac{\pi}{2}$	0	3	θ_6

Table 1: Robot Arm Measurements

$$T_i^{i-1} = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) & 0 & a_{i-1} \\ sin(\theta_i)\cos(\alpha_{i-1}) & cos(\theta_i)\cos(\alpha_{i-1}) & -sin(\alpha_{i-1}) & -sin(\alpha_{i-1})\mathrm{d}_i \\ sin(\theta_i)\sin(\alpha_{i-1}) & cos(\theta_i)\sin(\alpha_{i-1}) & cos(\alpha_{i-1}) & cos(\alpha_{i-1})\mathrm{d}_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 2: The link transformation matrix

To calculate the link transformation matrix (Fig 2), we used the values from Table 1. After calculating all 6 matrices, where each matrix T_i^{i+1} allows us to transform points from referential i to referential i-1, which allows us to calculate $T_0^6 = T_0^1 * T_1^2 * T_2^3 * T_3^4 * T_4^5 * T_5^6$. Since the Direct Kinematics supplies an angle for every joint calculating the above transformation result in the T_0^6 transformation matrix.

The matrices T_0^6 transforms the points from referential 6 to referential 0. From T_0^6 we can directly remove the (x, y, z) coordinates, these are the position of the end factor in the referential 0.

We choose the moving axis solution, using Euler angles Z-Y-Z convention. From the above calculated matrices the Euler angles can be obtained such that the α β γ can be obtained.

$$\gamma = atan2(T_0^6[3,2], -T_0^6[3,1])\ \ if s_\beta > 0$$

$$\gamma = atan2(-T_0^6[3,2], T_0^6[3,1])\ \ if s_\beta < 0$$

$$\beta = atan2(\pm\sqrt{(T_0^6[1,3])^2 + (T_0^6[2,3])^2}, T_0^6[3,3])$$

$$\alpha = atan2(T_0^6[2,3], T_0^6[1,3])\ \ if s_\beta > 0$$

$$\alpha = atan2(-T_0^6[2,3], -T_0^6[1,3])\ if s_\beta < 0$$

Figure 3: Angle Formulas

First we calculate β by the equation shown is in figure 3.
In the particular case where sin(β) is zero, it means we choose α equal to zero and calculate γ in a different way depending if cos(β) is either 1 or -1:

If cos(β) = 1 then $\gamma = atan2(T_0^6[2,1], T_0^6[1,1])$

Else if cos(β) = -1 then $\gamma = -atan2(-T_0^6[2,1], -T_0^6[1,1])$

For all other cases we find the γ and α by the equations shown is in figure 3.

$$T_1^0 = \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 & 0 \\ sin(\theta_1) & cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 99 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} cos(\theta_2 + \frac{\pi}{2}) & -sin(\theta_2 + \frac{\pi}{2}) & 0 & 25 \\ 0 & 0 & -1 & 0 \\ sin(\theta_2 + \frac{\pi}{2}) & cos(\theta_2 + \frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^2 = \begin{bmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 120 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^3 = \begin{bmatrix} cos(\theta_4 + \frac{\pi}{2}) & -sin(\theta_4 + \frac{\pi}{2}) & 0 & 2 \\ 0 & 0 & -1 & -130 \\ sin(\theta_4 + \frac{\pi}{2}) & cos(\theta_4 + \frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_5^4 = \begin{bmatrix} cos(\theta_5) & -sin(\theta_5) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ sin(\theta_5) & cos(\theta_5) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^5 = \begin{bmatrix} cos(\theta_6) & -sin(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & 3 \\ -sin(\theta_6) & -cos(\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All necessary angles are now obtained from the above calculation. The $\alpha - \beta - \gamma$ angles represent $Z - Y - Z$ rotations respectively.

In conclusion, through the above calculations, we obtained x, y, z and the $\alpha - \beta - \gamma$ rotations by using the given joint angles θ_1 to θ_6.

Inverse Kinematics

For the Inverse Kinematics problem, the (x, y, z) coordinates of the end factor are known as well as the rotation around Z-Y-Z. This means the $\alpha - \beta - \gamma$ are known, then it becomes possible to calculate the transformation matrix as show in figure 4.

$$\begin{bmatrix} \cos(\alpha)cos(\beta)cos(\gamma) - sin(\alpha)sin(\gamma) & -\cos(\alpha)cos(\beta)sin(\gamma) - sin(\alpha)cos(\gamma) & \cos(\alpha)sin(\beta) & x \\ \sin(\alpha)cos(\beta)cos(\gamma) + cos(\alpha)sin(\gamma) & -\sin(\alpha)cos(\beta)sin(\gamma) + cos(\alpha)cos(\gamma) & \sin(\alpha)sin(\beta) & y \\ -\sin(\beta)cos(\gamma) & -\sin(\beta)sin(\gamma) & \cos(\beta) & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 4: Transformation Matrix T_0^6

This matrix is obtain by multiplying three rotation matrices. The first and third one are a rotation on the Z axis and the second is a rotation on the Y axis.
We assume our point (0,0,0) in the World Coordinates is the point (0,0,0) of the referential 0. Making referential 0 our base for all other referentials.
From having T_0^6, the Transformation matrix from referential 6 to 0, we are able to calculate the center of our referential 5 in the coordinates of referential 0. This is because independently of the rotation on joint 5 (figure 1) the point (0,0,-3) in the referential 6 is always the center of the referential 5. For this reason we do the following:

$$C_5 = T_0^6 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_5 \\ y_5 \\ z_5 \\ 1 \end{bmatrix}$$

Figure 5: Center of referential 5 in World Coordinates

The coordinate x_5, y_5, z_5 are all known valuables with specific values.
As seen in figure 1, the referential 4 and 5 are overlapping, and have the same origin $C_4 \equiv C_5$. With this we use geometry to calculate the angle $\theta_1 = atan2(y_5, x_5)$. Because the arm can rotate $\theta_1 - \pi$ and obtain the same position and rotation of the end factor, we will calculate everything below for θ_1 and for $\theta_1 - \pi$.

Calculation to find θ_2 and θ_3

First we find the matrix $T_0^4 = T_0^1 T_1^2 T_2^3 T_3^4$. For T_0^1 we know the value of θ_1. Both T_1^2 and T_2^3 have the unknown variable we want, θ_2 and θ_3. Lastly for T_3^4 the value of the θ_4 is not relevant for the current calculations, for that reason we assume $\theta_4 = 0 + \pi/2$, putting it in the rest position.
By multiplying T_0^4 by the origin (0,0,0,1) we obtain the center of referential 4 in World Coordinates with the unknown variables of θ_2 and θ_3, than we can match the parts like so:

$$\begin{bmatrix} x_5 \\ y_5 \\ z_5 \\ 1 \end{bmatrix} = T_0^4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then using the *solve* function from matlab, we solve the equation and get two solutions for θ_2 and θ_3.

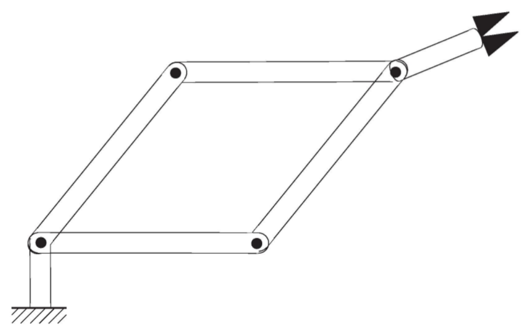

Figure 6: Two solutions problem

The figure above shows the reason for the two solutions for θ_2 and θ_3, where for the same position and orientations of the end factor we can have two different values for θ_2 and θ_3.

Calculation to find θ_4, θ_5 and θ_6

For each solution of θ_2 and θ_3 we calculate different values for θ_4, θ_5 and θ_6. Now that we have θ_1, θ_2 and θ_3 we calculate $T_0^3 = T_0^1 T_1^2 T_2^3$

$$[T_0^3]^{-1} \mathrm{T}_0^6 = \begin{bmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_3^4 T_4^5 T_5^6$$

Figure 7: Transformations matrices to calculate θ_5

All T_3^4, T_4^5 and T_5^6 have the unknown variables θ_4, θ_5 and θ_6. While $[T_0^3]^{-1}\mathrm{T}_0^6$ is a set matrix with specific known values.
By observation, we choose the equation of the elements r23 where $\theta_5 = \cos^{-1}(-r23)$. This equation gives two solutions: θ_5 and $-\theta_5$.
After, for each solution of θ_5 we do the fowling:

If $\sin(\theta_5)$=0 then we choose $\theta_4 = 0$. This is because in this case the joints θ_5 and θ_6 are align like in the rest position and so there are infinite possibilities for both θ_4 and θ_6, such that they all give the same final rotation. And so we choose $\theta_4 = 0$ and the calculate the correspondent θ_6.

Else we calculate $\theta_4 = \text{atan2}(\frac{-r33}{\sin(\theta_5)}, \frac{2-px}{3\sin(\theta_5)}) - \frac{\pi}{2}$, the case where there's only one solution for θ_4, in this case we also choose the best elements in figure 7 by observation.

For θ_6 we use the same strategy as before, calculating $T_0^5 = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5$ and then:

$$[T_0^5]^{-1}\mathrm{T}_0^6 = \begin{bmatrix} r11 & r12 & r13 & px \\ r21 & r22 & r23 & py \\ r31 & r32 & r33 & pz \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_5^6$$

Figure 8: Transformations matrices to calculate θ_6

And again by observation of figure 8 we are able to calculate θ_6 as follow : $\theta_6 = atan2(\frac{-r12}{r11})$
Now we have calculate all the solutions for θ_1 and $\theta_1 - \pi$, and for each solution of θ_2 and θ_3 we calculate the correspondent θ_5 that can also have two solutions, with which we calculate the correct θ_4 and θ_6. In the end, this leaves with 8 solutions except the particular cases where the robot arm is stretched out making impossible to rotate θ_1 and achieve the same position. In this cases we only have 4 solutions.

Tests

Direct Kinematics:

Test Number	Input (θ_1,θ_2,θ_3,θ_4,θ_5,θ_6)	Output (x,y,z,α,β,γ)
1	($\pi/4$,$\pi/7$,-$\pi/3$,-$\pi/5$,$\pi/2$,-$\pi/6$)	(55.19, 58.62, 134.08, 2.74, 2.08, -1.39)
2	(-$\pi/1.5$,$\pi/7$,-$\pi/3$,-$\pi/5$,$\pi/2$,-$\pi/6$)	(-38.14, -70.91, 134.08, -0.14, 2.08, -1.39)
3	(0,0,-π,$\pi/2$,0,0)	(-108, 0, 217, π, $\pi/2$, π)
4	(0,0,0,-$\pi/2$,$\pi/2$,0)	(155, 1.50, 218.40, $\pi/2$, 2.62, -$\pi/2$)
5	(0,0,0,0,0,0)	(158, 0, 221, 0, $\pi/2$, -$\pi/2$)
6	(π,0,0,0,0,0)	(-158, 0, 221, π, $\pi/2$, -$\pi/2$)
7	(-π,0,0,0,0,0)	(-158, 0, 221, π, $\pi/2$, -$\pi/2$)
8	(0,π,0,0,0,0)	(-108, 0, -23, π, $\pi/2$, $\pi/2$)
9	(0,-π,0,0,0,0)	(-108, 0, -23, π, $\pi/2$, $\pi/2$)
10	(0,$\pi/2$,0,0,0,0)	(-97, 0, 232, 0, 0, -$\pi/2$)
11	(0,0,π,0,0,0)	(-108, 0, 217, π, $\pi/2$, $\pi/2$)
12	(0,0,-π,0,0,0)	(-108, 0, 217, π, $\pi/2$, $\pi/2$)
13	(0,0,$\pi/2$,0,0,0)	(23, 0, 352, 0, 0, -$\pi/2$)
14	(0,0,-$\pi/2$,0,0,0)	(27, 0, 86, 0, π, -$\pi/2$)
15	(0,0,0,π,0,0)	(158, 0, 221, 0, $\pi/2$, $\pi/2$)
16	(0,0,0,-π,0,0)	(158, 0, 221, 0, $\pi/2$, $\pi/2$)
17	(0,0,0,$\pi/2$,0,0)	(158, 0, 221, 0, $\pi/2$, 0)
18	(0,0,0,-$\pi/2$,0,0)	(158, 0, 221, 0, $\pi/2$,π)
19	(0,0,0,0,π,0)	(152, 0, 221, π, $\pi/2$, -$\pi/2$)
20	(0,0,0,0,-π,0)	(152, 0, 221, π, $\pi/2$, -$\pi/2$)
21	(0,0,0,0,$\pi/2$,0)	(155, 3, 221, $\pi/2$, $\pi/2$, -$\pi/2$)
22	(0,0,0,0,-$\pi/2$,0)	(155, -3, 221, -$\pi/2$, $\pi/2$, -$\pi/2$)
23	(0,0,0,0,0,π)	(158, 0, 221, 0, $\pi/2$, $\pi/2$)
24	(0,0,0,0,0,-π)	(158, 0, 221, 0, $\pi/2$, $\pi/2$)
25	(0,0,0,0,0,$\pi/2$)	(158, 0, 221, 0, $\pi/2$, 0)
26	(0,0,0,0,0,-$\pi/2$)	(158, 0, 221, 0, $\pi/2$, π)

Inverse Kinematics:

Test Number	Input (x,y,z,α,β,γ)	Output (θ_1,θ_2,θ_3,θ_4,θ_5,θ_6)
1	(158, 0, 221, 0, $\pi/2$, -$\pi/2$)	(0, -1.63, 3.11, -$\pi/2$, 1.48 $\pi/2$), (0, -1.63 3.11 -4.71 -1.48 -$\pi/2$), (0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0), (-π, 1.51, 0.52, $\pi/2$, 1.10, $\pi/2$), (-π, 1.51, 0.52, -$\pi/2$, -1.10, -$\pi/2$), (-π, 0.44, 2.59, $\pi/2$, 0.12, $\pi/2$), (-π, 0.44, 2.59, -$\pi/2$, -0.12, -$\pi/2$)
2	(0, 158, 221, $\pi/2$, $\pi/2$, -$\pi/2$)	($\pi/2$, -1.63, 3.11, -$\pi/2$, 1.48, $\pi/2$), ($\pi/2$, -1.63, 3.11, $\pi/2$, -1.48, -$\pi/2$), ($\pi/2$, 0, 0, 0, 0, 0), ($\pi/2$, 0, 0, 0, 0, 0), (-$\pi/2$, 1.51, 0.52, $\pi/2$, 1.10, $\pi/2$), (-$\pi/2$, 1.51, 0.52, -$\pi/2$, -1.10, -$\pi/2$), (-$\pi/2$, 0.44, 2.59, $\pi/2$, 0.12, $\pi/2$), (-$\pi/2$, 0.44, 2.59, -$\pi/2$, -0.12, -$\pi/2$)
3	(50, 0, 0, 0, 0, 0)	(0, -1.67, -0.72, -1.57, 2.31, π), (0, -1.67, -0.72, -4.71, -2.31, 0), (0, -4.12, -2.44, 1.57, 1.86, 0), (0, -4.12, -2.44, -1.57, -1.86, -π), (-π, -2.66, -0.52, -1.57, 1.51, 0), (-π, -2.66, -0.52, -4.71, -1.51, π), (-π, 1.39, -2.64, -4.71, 2.81, π), (-π, 1.39, -2.64, -1.57, -2.81, 0)
4	(158, 0, 221, 0, $\pi/2$, 0)	(-π, 0.44, 2.59, -$\pi/2$, -0.12, 0) (0, -1.63, 3.11, -$\pi/2$, 1.48, -π) (0, -1.63, 3.11, -4.71, -1.48, 0) (0, 0, 0, 0, 0, $\pi/2$) (0, 0, 0, 0, 0, $\pi/2$) (-π, 1.51, 0.52, $\pi/2$, 1.10, π) (-π, 1.51, 0.52, -$\pi/2$, -1.10, 0) (-π, 0.44, 2.59, $\pi/2$, 0.12, -π)
5	(23, 0, 352, 0, 0, -$\pi/2$)	(0, 0.02, 1.54, $\pi/2$ 0.01, -$\pi/2$) (0, 0.02, 1.54, -$\pi/2$, -0.01, $\pi/2$) (0, 0, $\pi/2$, 0, 0, 0) (0, 0, $\pi/2$, 0, 0, 0)

The reason for some examples have repeated solutions, is that when the $\theta_5 = 0$, we still have solutions for $-\theta_5$ and θ_5 .

User Manual

To use the *DirectKinematics* function it receives 6 number that represent the angles of the 6 joints of the robot arm (θ_1, θ_2,θ_3,θ_4,θ_5, θ_6) and returns the positions and orientation of the end factor (x, y, z, α, β, γ).
The *InverseKinematics* receives the positions and orientation of the end factor(x, y, z, α, β, γ) and returns 8 solutions for the angles of the joints(θ_1, θ_2,θ_3,θ_4,θ_5, θ_6), all which gives the same positions and orientation for the end factor. The solution returned in *InverseKinematics* is a structure with 8 or 4 entries, and each entry is an array of 6 values.
Below there's an example how to use the M-functions:

```
function main(o1,o2,o3,o4,o5,o6)

[x,y,z,Alpha,Beta,Gamma] = DirectKinematics(o1,o2,o3,o4,o5,o6);

solutions = InverseKinematics(x,y,z,Alpha,Beta,Gamma);


end
```