Chapter 6

Inner product spaces

Contents

6A Inner Products and Norms

1 Prove or give a counterexample: If $v_1, \dots, v_m \in V$, then

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j, v_k\rangle \geq 0.$$

Solution:

We can use additivity in both slots of the inner product to show that the sum in question is a squared norm of a vector:

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle v_j, v_k\rangle = \sum_{j=1}^{m}\langle v_j, \sum_{k=1}^{m} v_k\rangle = \langle\sum_{j=1}^{m} v_j, \sum_{k=1}^{m} v_k\rangle = \left\|\sum_{j=1}^{m} v_j\right\|^2 \geq 0 \quad \square$$

2 Suppose $S \in \mathcal{L}(V)$. Define $\langle\cdot,\cdot\rangle_1$ by

$$\langle u, v\rangle_1 = \langle Su, Sv\rangle$$

for $u, v \in V$. Show that $\langle\cdot,\cdot\rangle_1$ is an inner product on V if and only if S is injective.

Solution:

$\longrightarrow$ Suppose $\langle\cdot,\cdot\rangle_1$ is an inner product. Then let $v \in V$ be a vector such that $Sv = 0$. Then examine the following:

$$\langle v, v\rangle_1 = \langle Sv, Sv\rangle = \langle 0, 0\rangle = 0$$

By the definiteness of the inner product, $\langle v, v\rangle_1 = 0$ if and only if $v = 0$. This requires that $\operatorname{null} S = \{0\}$, which is equivalent to S being injective.

$\longleftarrow$ Suppose S is injective. Positivity and definiteness of $\langle\cdot,\cdot\rangle_1$ arise directly from fact that $\langle\cdot,\cdot\rangle$ is positive and definite:

$\langle Su, Su\rangle = 0$ if and only if $Su = 0$; S is injective, hence $Su = 0 \Leftrightarrow u = 0$.

Additivity in the first slot:

$$\begin{aligned}\langle u + v, w\rangle_1 = \langle S(u + v), Sw\rangle &= \langle Su + Sv, Sw\rangle \\ &= \langle Su, Sw\rangle + \langle Sv, Sw\rangle = \langle u, w\rangle_1 + \langle v, w\rangle_1\end{aligned}$$

Homogeneity in the first slot:

$$\langle \lambda u, v\rangle_1 = \langle S(\lambda u), Sv\rangle = \langle \lambda Su, Sv\rangle = \lambda \langle Su, Sv\rangle = \lambda\langle u, v\rangle_1$$

Conjugate symmetry:

$$\langle u, v\rangle_1 = \langle Su, Sv\rangle = \overline{\langle Sv, Su\rangle} = \overline{\langle v, u\rangle_1}$$

Thus, $\langle\cdot,\cdot\rangle_1$ is an inner product. $\square$

3 (a) Show that the function taking an ordered pair $((x_1, x_2), (y_1, y_2))$ of elements of $\mathbb{R}^2$ to $|x_1y_1| + |x_2y_2|$ is not an inner product on $\mathbb{R}^2$.

(b) Show that the function taking an ordered pair $((x_1, x_2, x_3), (y_1, y_2, y_3))$ of elements of $\mathbb{R}^3$ to $x_1y_1 + x_3y_3$ is not an inner product on $\mathbb{R}^3$.

Solution:

(a) Let us test the "additivity in first slot" property:

$$\begin{aligned}\langle (x_1 + y_1, x_2 + y_2), (z_1, z_2)\rangle &= |(x_1 + y_1)z_1| + |(x_2 + y_2)z_2| \\ &= |x_1z_1 + y_1z_1| + |x_2z_2 + y_2z_2|\end{aligned} \tag{6.1}$$

$$\langle (x_1, x_2), (z_1, z_2) \rangle + \langle (y_1, y_2), (z_1, z_2) \rangle = |x_1 z_1| + |y_1 z_1| + |x_2 z_2| + |y_2 z_2| \quad (6.2)$$

For a given function to be an inner product, the right hand sides of equations (1) and (2) must be equal. In fact, they aren't equal in general case, as only *inequality* $|a + b| \leq |a| + |b|$ holds. □

(b) Note that $((0, 1, 0), (0, 1, 0))$ maps to zero. Hence, the definiteness property is not satisfied and this function is not an inner product. □

4 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is injective.

Solution:

$\|Tv\| \leqslant \|v\| = \sqrt{\langle v, v \rangle} < \sqrt{2}\sqrt{\langle v, v \rangle} = \sqrt{2\langle v, v \rangle} = \sqrt{\langle \sqrt{2}v, \sqrt{2}, v \rangle} = \|\sqrt{2}v\|$

Hence $\|Tv\| < \|\sqrt{2}v\|$

Suppose $T - \sqrt{2}I$ is not invertible Then $\exists v \in V$, $v \neq 0$ such that :

$(T - \sqrt{2}I)v = 0 \Rightarrow Tv = \sqrt{2}v$ which must mean $\|Tv\| = \|\sqrt{2}v\|$

which is not true, as we have shown earlier.

Thus $T - \sqrt{2}I$ is invertible and hence is injective. □

5 Suppose V is a real inner product space.

(a) Show that $\langle u + v, u - v \rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

(b) Show that if $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$.

(c) Use part (b) to show that the diagonals of a rhombus are perpendicular to each other.

Solution:

$$\begin{aligned}
\text{(a) } \langle u + v, u - v \rangle &= \langle u, u \rangle - \langle u, v \rangle + \langle v, u \rangle - \langle v, v \rangle \\
&= \{\langle v, u \rangle = \langle u, v \rangle \text{ for real inner product spaces}\} \\
&= \langle u, u \rangle - \langle v, v \rangle = \|u\|^2 - \|v\|^2
\end{aligned}$$

(b) If $\|u\| = \|v\|$, then $\langle u + v, u - v \rangle = \|u\|^2 - \|v\|^2 = 0$, so $u + v$ is orthogonal to $u - v$.

(c) Rhombus is a parralelogram with equal sides. If vectors v and $u \in \mathbb{R}^2$ define sides of rhombus, then diagonals are defined by $v + u$ and $v - u$. From (b) follows that $v + u$ and $v - u$ are orthogonal, i.e. diagonals of rhombus are perpendicular. □

6 Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0 \Leftrightarrow \|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$.
Solution:

$$\langle u + av, u + av\rangle = \langle u, u\rangle + \langle u, av\rangle + \langle av, u\rangle + |a|^2\langle v, v\rangle$$

$\longrightarrow$ If $\langle v, u\rangle = 0$:

$$\langle u+av, u+av\rangle = \langle u, u\rangle + \bar{a}\overline{\langle v, u\rangle} + a\langle v, u\rangle + |a|^2\langle v, u\rangle = \langle u, u\rangle + |a|^2\langle v, v\rangle \geq \langle u, u\rangle$$

Hence $\|u + av\| \geq \|u\|$

$\longleftarrow$ If $\|u\| \leq \|u + av\|$

Let $a = \varepsilon \quad (\varepsilon \in \mathbb{R}, \varepsilon > 0)$. Then:

$$\langle u + av, u + av\rangle = \langle u, u\rangle + \varepsilon\overline{\langle v, u\rangle} + \varepsilon\langle v, u\rangle + \varepsilon^2\langle v, v\rangle$$
$$2\varepsilon\Re\langle v, u\rangle = \langle u + av, u + av\rangle - \langle u, u\rangle - \varepsilon^2\langle v, v\rangle \geq -\varepsilon^2\langle v, v\rangle$$

$$\Re\langle v, u\rangle \geq -\frac{\varepsilon}{2}\langle v, v\rangle \tag{6.1}$$

Let $a = -\varepsilon$. Then:

$$2\varepsilon\Re(\langle v, u\rangle) = \langle u, u\rangle + \varepsilon^2\langle v, v\rangle - \langle u + av, u + av\rangle \leq \varepsilon^2\langle v, v\rangle$$

$$\Re\langle v, u\rangle \leq \frac{\varepsilon}{2}\langle v, v\rangle \tag{6.2}$$

Note, that $\langle v, v\rangle$ and ε are greater than zero. Both (1) and (2) can hold simultaneously only if $\Re\langle v, u\rangle = 0$.

Let $a = i\varepsilon$. Then:

$$\begin{aligned}\langle u + av, u + av\rangle &= \langle u, u\rangle - i\varepsilon\overline{\langle v, u\rangle} + i\varepsilon\langle v, u\rangle + \varepsilon^2\langle v, v\rangle \\ &= \langle u, u\rangle + \varepsilon^2\langle v, v\rangle - 2\varepsilon\Im\langle v, u\rangle\end{aligned}$$

$$2\varepsilon\Im\langle v, u\rangle = \langle u, u\rangle - \langle u + av, u + av\rangle + \varepsilon^2\langle v, v\rangle \leq \varepsilon^2\langle v, v\rangle$$
$$\Im\langle v, u\rangle \leq \frac{\varepsilon}{2}\langle v, v\rangle \tag{6.3}$$

Let $a = -i\varepsilon$. Then:

$$\langle u + av, u + av\rangle = \langle u, u\rangle + \varepsilon^2\langle v, v\rangle + 2\varepsilon\Im\langle v, u\rangle$$
$$2\varepsilon\Im\langle v, u\rangle = \langle u + av, u + av\rangle - \langle u, u\rangle - \varepsilon^2\langle v, v\rangle \geq -\varepsilon^2\langle v, v\rangle$$

$$\Im\langle v, u\rangle \geq -\frac{\varepsilon}{2}\langle v, v\rangle \tag{6.4}$$

As before, (3) and (4) must be valid for all ε, thus we conclude $\Im\langle v, u\rangle = 0$.

That means $\langle v, u\rangle = 0$, as desired. $\square$

7 Suppose $u, v \in V$. Prove that $\|au+bv\| = \|bu+av\|$ for all $a, b \in \mathbb{R}$ if and only if $\|u\| = \|v\|$.

Solution:

$$\langle au+bv, au+bv\rangle = a^2\langle u,u\rangle + b^2\langle v,v\rangle + ab\left(\langle v,u\rangle + \langle u,v\rangle\right)$$

$\longleftarrow$ If $\|u\| = \|v\|$, then $\langle u,u\rangle = \langle v,v\rangle$

$$\|au+bv\|^2 = a^2\langle u,u\rangle + b^2\langle v,v\rangle + ab\left(\langle v,u\rangle + \langle u,v\rangle\right) = \\ a^2\langle v,v\rangle + 2ab\,\Re\langle v,u\rangle + b^2\langle u,u\rangle = \langle av+bu, av+bu\rangle$$

Hence $\|au+bv\| = \|bu+av\|$ for all $a, b \in \mathbb{R}$

$\longrightarrow$ If $\|au+bu\| = \|bu+av\|$ for all $a, b \in \mathbb{R}$

$$\|bu+av\|^2 = b^2\langle u,u\rangle + 2ab\,\Re\langle v,u\rangle + a^2\langle v,v\rangle$$

$$\|au+bv\|^2 - \|bu+av\|^2 = 0 = \left(a^2-b^2\right)\langle u,u\rangle + \left(b^2-a^2\right)\langle v,v\rangle \\ = \left(a^2-b^2\right)\left(\|u\|^2 - \|v\|^2\right)$$

The last equation is valid for all $a, b \in \mathbb{R}$. That means $\|u\|^2 - \|v\|^2 = 0 \Rightarrow \|u\| = \|v\|$ $\square$

8 Suppose $a, b, c, x, y \in \mathbb{R}$ and $a^2 + b^2 + c^2 + x^2 + y^2 \leq 1$. Prove that $a + b + c + 4x + 9y \leq 10$.

Solution:

Let $v, u \in \mathbb{R}^5$: $v = (a, b, c, x, y)$ and $u = (1, 1, 1, 4, 9)$. Then:

$$\|v\|^2 = a^2 + b^2 + c^2 + x^2 + y^2 \leq 1 \quad \text{and} \quad \|u\|^2 = 1^2 + 1^2 + 1^2 + 4^2 + 9^2 = 100$$

Now we use Cauchy-Schwarz inequality:

$$a + b + c + 4x + 9y = \langle v,u\rangle \leq \|v\|\|u\| \leq 1 \cdot 10 = 10 \quad \square$$

9 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u,v\rangle = 1$. Prove that $u = v$.

Solution:

Note, that $\langle u,v\rangle = 1 = \|u\| \cdot \|v\|$. By the Cauchy-Schwarz inequality u is a scalar multiple of v: $u = \alpha v$.

$$1 = \langle u,v\rangle = \langle \alpha v, v\rangle = \alpha\langle v,v\rangle = \alpha\|v\|^2 = \alpha \cdot 1 = \alpha \quad \Rightarrow \quad \alpha = 1$$

Hence, $v = u$ as desired $\square$

10 Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \leq 1 - |\langle u, v\rangle|$$

Solution:

$$\begin{aligned}(1 - \|u\|^2)(1 - \|v\|^2) &= (1 - \|u\|)(1 - \|u\|)(1 - \|v\|)(1 - \|v\|) \\ &= (1 - \|u\| + \|v\| - \|u\|\|v\|)(1 + \|u\| - \|v\| - \|u\|\|v\|) \\ &= (1 - \|u\|\|v\|)^2 - (\|u\| - \|v\|)^2 \leq (1 - \|u\|\|v\|)^2\end{aligned}$$

Taking the square root on both sides we get:

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \leq 1 - \|u\| \cdot \|v\| \leq 1 - |\langle u, v\rangle| \quad \square$$

11 Find vectors u, $v \in \mathbb{R}^2$ such that u is a scalar multiple of $(1, 3)$, v is orthogonal to $(1, 3)$, and $(1, 2) = u + v$.

Solution:

This problem is an orthogonal decomposition problem. Let $v = \lambda \cdot (1, 3)$, then:

$$(1, 2) = \lambda \cdot (1, 3) + v$$

Thus we can find λ as:

$$\lambda = \frac{\langle (1, 2), (1, 3)\rangle}{\langle (1, 3), (1, 3)\rangle} = \frac{1 \cdot 1 + 2 \cdot 3}{1 \cdot 1 + 3 \cdot 3} = \frac{7}{10} = 0.7$$

and v as:

$$v = (1, 2) - 0.7 \cdot (1, 3) = (0.3, -0.1)$$

The answer is:

$$u = (0.7, 2.1), \qquad v = (0.3, -0.1)$$

12 Suppose a, b, c, d are positive numbers.

(a) Prove that $(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16$.

(b) For which positive numbers a, b, c, d is the inequality above an equality?

Solution:

(a) As a, b, c, d are all positive we can represent sums in both brackets as squared norms of vectors $v = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ and $u = (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}})$.

$$a + b + c + d = \|v\|^2 \qquad 1/a + 1/b + 1/c + 1/d = \|u\|^2$$

By the Cauchy-Schwarz inequality:

$$\|v\|^2\|u\|^2 \geq |\langle v, u\rangle|^2 = \left|\sqrt{a}\cdot\frac{1}{\sqrt{a}} + \sqrt{b}\cdot\frac{1}{\sqrt{b}} + \sqrt{c}\cdot\frac{1}{\sqrt{c}} + \sqrt{d}\cdot\frac{1}{\sqrt{d}}\right|^2 =$$
$$= |1+1+1+1|^2 = 16$$

Thus indeed:

$$16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \quad \square$$

(b) The inequality becomes equality when $a = b = c = d = 1$.

13 Show that the square of an average is less than or equal to the average of the squares. More precisely, show that if $a_1, \ldots, a_n \in \mathbb{R}$, then the square of the average of $a_1, \ldots, a_n$ is less than or equal to the average of $a_1^2, \ldots, a_n^2$.

Solution:

We need to prove the inequality:

$$\left(\frac{1}{n}(a_1 + a_2 + \cdots + a_n)\right)^2 \leq \frac{1}{n}(a_1^2 + \cdots + a_n^2)$$

which can be rearranged into:

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$$

thus we will try to derive the last inequality.

Note that:

$$a_1^2 + \ldots a_n^2 = \|(a_1, \ldots, a_n)\|^2,$$
$$n = \|(1, \ldots, 1)\|^2$$

for $(a_1, \ldots, a_n), (1, \ldots, 1) \in \mathbb{R}^n$ with euclidean inner product. Also, note that:

$$a_1 + \cdots + a_n = \langle(1, \ldots, 1), (a_1, \ldots, a_n)\rangle$$

Setting $v_1 = (a_1, \ldots, a_n)$ and $v_2 = (1, \ldots, 1)$ and using the Cauchy-Schwarz inequality, we get:

$$|\langle v_1, v_2\rangle| \leq \|v_1\| \cdot \|v_2\| \quad \text{or} \quad |\langle v_1, v_2\rangle|^2 \leq \|v_1\|^2 \cdot \|v_2\|^2$$

Substituting v_1 and v_2 back we get:

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2) \quad \square$$

14 Suppose $v \in V$ and $v \neq 0$. Prove that $v/\|v\|$ is the unique closest element on the unit sphere of V to v. More precisely, prove that if $u \in V$ and $\|u\| = 1$, then

$$\left\| v - \frac{v}{\|v\|} \right\| \leq \|v - u\|,$$

with equality only if $u = v/\|v\|$.

Solution:

Firstly, we calculate the value of the suppposed "least norm":

$$\left\| v - \frac{v}{\|v\|} \right\| = \left| 1 - \frac{1}{\|v\|} \right| \|v\| = |\|v\| - 1|$$

Now, let us examine any u on the unit sphere of V.

$$\|v - u\|^2 = \|v\|^2 + \|u\|^2 - 2\Re\langle v, u\rangle \geq \|v\|^2 - 2\|u\|\|v\| + \|u\|^2$$
$$= \|v\|^2 - 2\|v\| + 1 = (\|v\| - 1)^2$$

Where we used Cauchy-Schwarz inequality. The sign is *greater than* because the inner product has minus sign in front of it. Thus indeed

$$\|v - u\| \geq \left\| v - \frac{v}{\|v\|} \right\|$$

The uniqueness is also guaranteed by the Cauchy-Schwarz inequality, as it becomes equality only when one of the vectors is a scalar multliple of the other. Here, there are two options: it is either $v/\|v\|$ or $-v/\|v\|$. For the second option the "distance" is obviously greater:

$$\left\| v + \frac{v}{\|v\|} \right\| = |1 + \|v\|| > |\|v\| - 1|$$

QED $\square$

15 Suppose u, v are nonzero vectors in $\mathbb{R}^2$. Prove that

$$\langle u, v\rangle = \|u\|\|v\| \cos\theta$$

where θ is the angle between u and v (thinking of u and v as arrows with initial point at the origin.

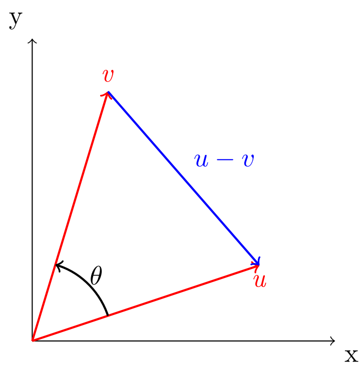

Figure 6.1: Illustration for *Problem 6A.15.*

Solution:

Let us write the law of cosines on the triangle shown in figure in the left part of the page:

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

At the same time:

$$\begin{aligned}\|u-v\|^2 &= \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \langle v,v\rangle - 2\langle u,v\rangle \\ &= \|u\|^2 + \|v\|^2 - 2\langle u,v\rangle\end{aligned}$$

Comparing right sides of these two expressions we conclude that:

$$\langle u,v\rangle = \|u\|\|v\|\cos\theta \quad \square$$

16 The angle between two vectors (thought of as arrows with initial point at the origin) in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be defined geometrically. However, geometry is not as clear in $\mathbb{R}^n$ for $n > 3$. Thus the angle between two nonzero vectors $x, y \in \mathbb{R}^n$ is defined to be

$$\arccos\frac{\langle x,y\rangle}{\|x\|\|y\|},$$

where the motivation for this definition comes from the previous exercise. Explain why the Cauchy-Schwarz inequality is needed to show that this definition makes sense.

Solution:

Note that $\arccos x$ is defined for $x \in [-1, 1]$. So for the definition above to make sense, it must always produce expression under inverse cosine function that lies within this range. Cauchy-Schwarz inequality does exactly that:

$$|\langle u,v\rangle| \leq \|u\|\|v\|$$

for any $u, v \in V$. So the expression:

$$\frac{|\langle u,v\rangle|}{\|u\|\|v\|}$$

is always less than or equal to 1. $\square$

17 Prove that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} k a_k^2\right)\left(\sum_{k=1}^{n} \frac{b_k^2}{k}\right)$$

for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

Solution:

Consider vectors $v_1, v_2 \in \mathbb{R}^n$ with Euclidean inner-product :

$$\begin{array}{ll} v_1 = \left(a_1, \sqrt{2}a_2, \cdots, \sqrt{n}a_n\right) & \text{with some } a_1, \ldots, a_n \\ v_2 = \left(b_1, \frac{b_2}{\sqrt{2}}, \cdots, \frac{b_n}{\sqrt{n}}\right) & \text{and } b_1, \ldots, b_n \in \mathbb{R} \end{array}$$

By Cauchy-Schwarz inequality

$$(\langle v_1, v_2\rangle)^2 \leqslant \|v_1\|^2 \|v_2\|^2$$

$$\|v_1\|^2 = a_1^2 + 2a_2^2 + \cdots + na_n^2 = \sum_{j=1}^{n} j a_j^2$$

$$\|v_2\|^2 = b_1^2 + \frac{b_2^2}{2} + \cdots + \frac{b_n^2}{n} = \sum_{j=1}^{n} \frac{b_j^2}{j}$$

$$\langle v_1, v_2\rangle = a_1 b_1 + \sqrt{2}a_2 \cdot \frac{b_2}{\sqrt{2}} + \ldots + \sqrt{n}a_n \cdot \frac{b_n}{\sqrt{n}}$$

$$= a_1 b_1 + \ldots + a_n b_n = \sum_{j=1}^{n} a_j b_j$$

Thus $\quad \left(\sum_{i=1}^{n} a_j b_j\right)^2 \leqslant \left(\sum_{i=1}^{n} j a_j\right)\left(\sum_{j=1}^{n} \frac{b_j}{j}\right)$ $\square$

18 (a) Suppose $f : [1, \infty) \to [0, \infty)$ is continuous. Show that

$$\left(\int_1^\infty f\right)^2 \leq \int_1^\infty x^2 (f(x))^2 \, dx.$$

(b) For which continuous function $f : [1, \infty) \to [0, \infty)$ is the inequality in (a) an equality with both sides finite?

Solution:

(a) To show that the inequality in this problem is true, we will first define an inner product on space V of continuous functions $f : [1, \infty) \to [0, \infty)$:

$$\langle f, g\rangle = \int_1^\infty x^2 f(x) g(x) dx$$

Indeed, this definition satisfies all properties of an inner product:

- Positivity: $f(x)$ and $g(x) \geq 0$ by definition and $x^2 > 0$, hence the intergral is non-negative. ✓
- Definiteness: the non-negativity of the expression under integral sign guarantees that the integral equals zero if and only if this expression is always zero. ✓
- Additivity in first slot: this is guaranteed by the properties of intergral

$$
\begin{aligned}
\langle f+h, g\rangle &= \int_1^\infty x^2\left(f(x)+h(x)\right)g(x)dx \\
&= \int_1^\infty x^2 f(x)g(x)dx + \int_1^\infty x^2 h(x)g(x)dx = \langle f, g\rangle + \langle h, g\rangle \quad ✓
\end{aligned}
$$

- Homogeneity in first slot:

$$\langle \lambda f, g\rangle = \int_1^\infty x^2 \lambda f(x)g(x)dx = \lambda \int_1^\infty x^2 f(x)g(x)dx = \lambda\langle f, g\rangle \quad ✓$$

- Conjugate symmetry is guaranteed by the fact the functions in question are real-valued. ✓

Now we note that

$$\int_1^\infty f dx = \int_1^\infty x^2 f(x)\frac{1}{x^2}dx = \langle f, \frac{1}{x^2}\rangle$$

Then we can use Cauchy-Schwarz inequality:

$$\langle f, \frac{1}{x^2}\rangle^2 \leq \langle f, f\rangle \cdot \langle \frac{1}{x^2}, \frac{1}{x^2}\rangle = \int_1^\infty x^2\left(f(x)\right)^2 dx \cdot \int_1^\infty x^2\left(\frac{1}{x^2}\right)^2 dx$$

Using the fact that

$$\int_1^\infty \frac{1}{x^2}dx = 1$$

we arrive at the conclusion that:

$$\left(\int_1^\infty f dx\right)^2 \leq \int_1^\infty x^2\left(f(x)\right)^2 dx \quad \square$$

(b) The inequality becomes equality for $f(x) = 1/x^2$.

19 Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda|^2 \leq \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{M}(T)_{j,k}|^2,$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

Solution:

For a given basis $v_1, \ldots, v_n$ of V we will define an isomorphic space $\mathbb{C}^n$ such that if $v \in V$ and $v = a_1 v_1 + \cdots + a_n v_n$ then the isomorphism is defined as:

$$x = S(v) = (a_1, \ldots, a_n) \in \mathbb{C}^n$$

For $\mathbb{C}^n$ we will use a Euclidean inner product.

Now let $v \in V$ be an eigenvector of T with eigenvalue λ and such that the corresponding vector $x \in \mathbb{C}^n$ has norm $\|x\| = 1$. A vector in $\mathbb{C}^n$ corresponding to Tv is $(\sum_j^n a_j \mathcal{M}(T)_{j,1}, \ldots, \sum_j^n a_j \mathcal{M}(T)_{j,n})$.

Then we have:

$$\langle S(Tv), S(Tv) \rangle = \langle \lambda x, \lambda x \rangle = |\lambda|^2$$

$$\begin{aligned}
\langle S(Tv), S(Tv) \rangle &= \sum_k^n \left| \sum_j^n a_j \mathcal{M}(T)_{j,k} \right|^2 = \sum_k^n |\langle x, (\mathcal{M}(T)_{1,k}, \ldots, \mathcal{M}(T)_{n,k}) \rangle|^2 \\
&\leq \sum_k^n \|x\|^2 \cdot \|(\mathcal{M}(T)_{1,k}, \ldots, \mathcal{M}(T)_{n,k})\|^2 \\
&= \sum_k^n \sum_j^n |\mathcal{M}(T)_{j,k}|^2
\end{aligned}$$

where we used Cauchy-Schwarz inequality. Thus

$$|\lambda|^2 \leq \sum_{j=1}^{n} \sum_{k=1}^{n} |\mathcal{M}(T)_{j,k}|^2 \quad \square$$

20 Prove that if $u, v \in V$, then $|\|u\| - \|v\|| \leq \|u - v\|$.

Solution:

$$\begin{aligned}\|u-v\|^2 &= \langle u-v, u-v\rangle \\ &= \langle u,u\rangle + \langle v,v\rangle - 2\Re\langle u,v\rangle \\ &\geq \|u\|^2 + \|v\|^2 - 2|\langle u,v\rangle| \\ &\geq \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \\ &= (\|u\| - \|v\|)^2\end{aligned}$$

Thus

$$|\|u\| - \|v\|| \leq \|u-v\| \quad \square$$

21 Suppose $u, v \in V$ are such that

$$\|u\| = 3, \quad \|u+v\| = 4, \quad \|u-v\| = 6$$

What number does $\|v\|$ equal?

Solution:

By parallelogram equality:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

$$\|v\| = +\left(\frac{1}{2}\left(\|u+v\|^2 + \|u-v\|^2\right) - \|u\|^2\right)^{\frac{1}{2}} =$$

$$= \left(\frac{1}{2}(16+36) - 9\right)^{1/2} = (8+18-9)^{1/2} = (8+9)^{1/2} = \sqrt{17}$$

22 Show that if $u, v \in V$, then

$$\|u+v\|\|u-v\| \leq \|u\|^2 + \|v\|^2.$$

Solution:

$$\begin{aligned}\|u+v\|^2\|u-v\|^2 &= \langle u+v, u+v\rangle\langle u-v, u-v\rangle \\ &= \left(\|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2\right)\left(\|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2\right) \\ &= \|u\|^4 - \langle u,v\rangle\|u\|^2 - \langle v,u\rangle\|u\|^2 + \|u\|^2\|v\|^2 \\ &+ \langle u,v\rangle\|u\|^2 - \langle u,v\rangle^2 - \langle u,v\rangle\langle v,u\rangle + \langle u,v\rangle\|v\|^2 \\ &+ \langle v,u\rangle\|u\|^2 - \langle v,u\rangle\langle u,v\rangle - \langle v,u\rangle^2 + \langle v,u\rangle\|v\|^2 \\ &+ \|u\|^2\|v\|^2 - \langle u,v\rangle\|v\|^2 - \langle v,u\rangle\|v\|^2 + \|v\|^4 \\ &= \|u\|^4 + 2\|u\|^2\|v\|^2 + \|v\|^4 - \langle u,v\rangle^2 - 2\langle u,v\rangle\langle v,u\rangle - \langle v,u\rangle^2 \\ &= \|u\|^4 + 2\|u\|^2\|v\|^2 + \|v\|^4 - (\langle u,v\rangle + \langle v,u\rangle)^2\end{aligned}$$

$$= \|u\|^4 + 2\|u\|^2\|v\|^2 + \|v\|^4 - (\Re\langle u, v\rangle)^2$$
$$\geq \|u\|^4 + 2\|u\|^2\|v\|^2 + \|v\|^4 = \left(\|u\|^2 + \|v\|^2\right)^2$$

Thus

$$\|u+v\|\|u-v\| \leq \|u\|^2 + \|v\|^2 \quad \square$$

23 Suppose $v_1, \dots, v_m \in V$ are such that $\|v_k\| \leq 1$ for each $k = 1, \dots, m$. Show that there exist $a_1, \dots, a_m \in \{1, -1\}$ such that

$$\|a_1 v_1 + \cdots + a_m v_m\| \leq \sqrt{m}.$$

Solution:

The solution to this problem is by induction:

Base case: If m = 1, then we take $a_1 = 1$ and have $v_1 \leq \sqrt{1} = 1 \checkmark$.

Hypothesis: If $m = k$ then there exist $a_1, \dots, a_m \in \{1, -1\}$ such that $\|a_1 v_1 + \cdots + a_k v_k\| \leq \sqrt{k}$.

Inductive step: Let $m = k + 1$, then let $w = a_1 v_1 + \cdots + a_k v_k$.

Choose $a_{k+1} = 1$ if

$$\|w + v_{k+1}\| \leq \|w - v_{k+1}\|$$

otherwise choose $a_{k+1} = -1$. Then we use the parallelogram equality:

$$\|w + a_{k+1}v_{k+1}\|^2 + \|w - a_{k+1}v_{k+1}\|^2 = 2\|w\|^2 + 2\|a_{k+1}v_{k+1}\|^2 \leq 2k + 2 \quad (6.1)$$

But also:

$$\|w + a_{k+1}v_{k+1}\|^2 + \|w - a_{k+1}v_{k+1}\|^2 \geq 2\|w + a_{k+1}v_{k+1}\|^2 \quad (6.2)$$

Combining (6.1) and (6.2), we arrive at:

$$\|w + a_{k+1}v_{k+1}\| \leq \sqrt{k+1} \quad \square$$

24 Prove or give a counterexample: If $\|\cdot\|$ is the norm associated with an inner product on $\mathbb{R}^2$, then there exists $(x, y) \in \mathbb{R}^2$ such that $\|(x, y)\| \neq \max\{x, y\}$.

Solution:

In order to prove this we will try to disprove the negation of "then"-part of the statement above. The negation is that: for all pairs $(x, y) \in \mathbb{R}^2$ it is true that $\|(x, y)\| = \max\{x, y\}$.

Let $x = -1 = y$, then

$$\|(x, x)\| = \|(-1, -1)\| = \max\{-1, -1\} = -1$$

But $\|(x, x)\| = \sqrt{\langle (x, x), (x, x)\rangle} = -1$. So $\langle \vec{x}, \vec{x}\rangle = \pm i$ which contradicts positivity of the inner-product. Thus $\|(x, y)\| \neq \max\{x, y\}$. $\square$

25 Suppose $p > 0$. Prove that there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in \mathbb{R}^2$ if and only if $p = 2$.

Solution:

$\longleftarrow$ If $p = 2$ then the inner product clearly exists, it's the Euclidean inner product.

$\longrightarrow$ If such inner product exists for some p. Note, that $p > 0$, otherwise norm is not defined. Later we discuss $p > 0$.

$$\langle (x, y), (x, y) \rangle = \|(x, y)\|^2 = (x^p + y^p)^{2/p}$$

For $(x, y) = (1, -1)$:

$$\|(1, -1)\| = (1^p + (-1)^p)^{1/p} = (1 + (-1)^p)^{1/p}$$

Note that this norm cannot be equal to zero and thus is defined only for even p. So $p = 2k, k \in \mathbb{N}$

$$\|(x, y)\| = \left(x^{2k} + y^{2k}\right)^{1/2k}.$$

Now we use parallelogram equality on vectors $(0, 1)$ and $(1, 0)$:

$$\begin{aligned}
&\|(1, 0)\|^2 = \left(\left(1^{2k} + 0\right)^{1/2k}\right)^2 = 1 = \|(0, 1)\|^2 \\
&\|(1, 1)\|^2 = \left(1^{2k} + 1^{2k}\right)^{1/k} = 2^{1/k} \\
&\|(1, -1)\|^2 = 2^{1/k}
\end{aligned}$$

$$\begin{aligned}
&\|(1, 1)\|^2 + \|(1, -1)\|^2 = 2\left(\|(1, 0)\|^2 + \|(0, 1)\|^2\right) \\
&2^{1/k} + 2^{1/k} = 2 \cdot (1 + 1) \\
&2^{1/k} = 2 \Rightarrow k = 1 \Rightarrow p = 2 \quad \square
\end{aligned}$$

26 Suppose V is a real inner product space. Prove that

$$\langle u, v \rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}$$

for all $u, v \in V$.

Solution:

$$\begin{aligned}
&\|u+v\|^2 - \|u-v\|^2 = \langle u+v, u+v\rangle - \langle u-v, u-v\rangle = \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle - (\langle u,u\rangle - \langle u,v\rangle \\
&- \langle v,u\rangle + \langle v,v\rangle) = 2\langle u,v\rangle + 2\langle v,u\rangle = 4\langle u,v\rangle
\end{aligned}$$

Thus

$$\langle u,v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4} \quad \text{for all } u,v \in V \quad \square$$

27 Suppose V is a complex inner product space. Prove that

$$\langle u,v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$$

for all $u,v \in V$.

Solution:

$$\begin{aligned}
\|u+v\|^2 &= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle \\
\|u-v\|^2 &= \|u\|^2 + \|v\|^2 - \langle u,v\rangle - \langle v,u\rangle \\
i \cdot \|u+iv\|^2 &= i\left(\|u\|^2 - i\langle u,v\rangle + i\langle v,u\rangle + \|v\|^2\right) \\
i \cdot \|u-iv\|^2 &= i\left(\|u\|^2 + \|v\|^2 + i\langle u,v\rangle - i\langle v,u\rangle\right)
\end{aligned}$$

Combining these we get:

$$\begin{aligned}
&\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2 = 2\langle u,v\rangle + 2\langle v,u\rangle + \\
&+ \langle u,v\rangle - \langle v,u\rangle - (-\langle u,v\rangle) - \langle v,u\rangle = 4\langle u,v\rangle
\end{aligned}$$

Thus

$$\langle u,v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4} \quad \text{for all } u,v \in V \quad \square$$

28 A norm on a vector space U is a function

$$\|\cdot\| : U \to [0,\infty)$$

such that $\|u\| = 0$ if and only if $u = 0$, $\|\alpha u\| = |\alpha|\|u\|$ for all $\alpha \in \mathbb{F}$ and all $u \in U$, and $\|u+v\| \leq \|u\| + \|v\|$ for all $u,v \in U$. Prove that a norm satisfying the parallelogram equality comes from an inner product (in other words, show that if $\|\cdot\|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle\cdot,\cdot\rangle$ on U such that $\|u\| = \langle u,u\rangle^{1/2}$ for all $u \in U$).

29 Suppose $V_1, \ldots, V_m$ are inner product spaces. Show that the equation

$$\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

defines an inner product on $V_1 \times \cdots \times V_m$.

Solution:

Positivity: $\langle (u_1, \ldots, u_m), (u_1, \ldots, u_m) \rangle \geq 0$

Each $\langle u_i, a_i \rangle \geq 0$ for all $u_i \in V_i$, hence their sum is also non-negative. ✓

Definiteness: $\langle (u_1, \ldots, u_m), (u_1, \ldots, u_m) \rangle = 0 \Leftrightarrow u_1, \ldots, u_m = 0$

$\langle u_i, u_i \rangle$ are all nonegative, hence their sum can be zero only if each $\langle u_i, u_i \rangle = 0$ which in turn means every $u_i = 0$. ✓

Additivity in first shot:

$$\begin{aligned}
&\langle (u_1, \ldots, u_m) + (v_1, \ldots, v_m), (w_1, \ldots, w_m) \rangle \\
&= \langle (u_1 + v_1, \ldots, u_m + v_m), (w_1, \ldots, w_m) \rangle \\
&= \langle u_1 + v_1, w_1 \rangle + \cdots + \langle u_m + v_m, w_m \rangle \\
&= \langle u_1, w_1 \rangle + \ldots + \langle u_m, w_m \rangle + \langle v_1, w_1 \rangle + \ldots + \langle v_m, w_m \rangle \\
&= \langle (u_1, \ldots, u_m), (w_1, \ldots, w_m) \rangle + \langle (v_1, \ldots, v_m), (\omega_1, \ldots, \omega_m) \rangle \quad \checkmark
\end{aligned}$$

Homogeneity in first slot:

$$\begin{aligned}
&\langle \lambda (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle (\lambda u_1, \ldots, \lambda u_m), (v_1, \ldots, v_m) \rangle \\
&= \langle \lambda u_1, v_1 \rangle + \ldots + \langle \lambda u_n, v_n \rangle = \lambda \langle u_1, v_1 \rangle + \ldots + \lambda \langle u_m, v_m \rangle \\
&= \lambda (\langle u_1, v_1 \rangle + \ldots + \langle u_m, v_m \rangle) = \lambda \langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle \quad \checkmark
\end{aligned}$$

Conjugate symmetry:

$$\begin{aligned}
&\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle u_1, v_1 \rangle + \ldots + \langle u_m, v_m \rangle = \overline{\langle v_1, u_1 \rangle} + \ldots \\
&+ \overline{\langle v_m, u_m \rangle} = \overline{\langle v_1, u_1 \rangle + \ldots + \langle u_m, u_m \rangle} = \langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle \quad \checkmark
\end{aligned}$$

Thus the inner product is indeed well-defined. □

30 Suppose V is a real inner product space. For $u, v, w, x \in V$, define

$$\langle u + iv, w + ix \rangle_{\mathbb{C}} = \langle u, w \rangle + \langle v, x \rangle + (\langle v, w \rangle - \langle u, x \rangle)i.$$

(a) Show that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ makes $V_{\mathbb{C}}$ into a complex inner product space.

(b) Show that if $u, v \in V$, then

$$\langle u, v \rangle_{\mathbb{C}} = \langle u, v \rangle \quad \text{and} \quad \|u + iv\|_{\mathbb{C}}^2 = \|u\|^2 + \|v\|^2.$$

Solution:
We check the properties of an inner product:
Positivity:

$$\langle u+iv, u+iv\rangle_{\mathbb{C}} = \langle u,u\rangle + \langle v,v\rangle + (\langle v,u\rangle - \langle u,v\rangle)i = \langle u,u\rangle + \langle v,v\rangle \geq 0 \quad \checkmark$$

the last equality sign is because for real inner-product spaces $\langle u,v\rangle = \langle v,u\rangle$, and the inequality is because of positivity of the inner product.
Definiteness:
Here we can use the expression above. The sum $\langle u,u\langle + \langle v,v\rangle$ can equal zero only if both terms equal zero (because of positivity). They, in turn, can be zero if and only if $u=0$ and $v=0$. Thus we have:

$$\langle u+iv, u+iv\rangle_{\mathbb{C}} = 0 \Leftrightarrow u+iv = 0 \quad \checkmark$$

Additivity in first slot:
Let $u,v,w,x,y,z \in V$.

$$\begin{aligned}\langle (u+iv)+(w+ix), y+iz\rangle_{\mathbb{C}} &= \langle (u+w)+i(v+x), y+iz\rangle_{\mathbb{C}} \\ &= \langle u+w,y\rangle + \langle v+x,z\rangle + (\langle v+x,y\rangle - \langle u+w,z\rangle)\, i \\ &= \langle u,y\rangle + \langle v,z\rangle + (\langle v,y\rangle - \langle u,z\rangle)i + \langle w,y\rangle + \langle x,z\rangle + (\langle x,y\rangle - \langle w,z\rangle)i \\ &= \langle u+iv, y+iz\rangle_{\mathbb{C}} + \langle w+ix, y+iz\rangle_{\mathbb{C}} \quad \checkmark\end{aligned}$$

Homogeneity in first slot:
Let $\lambda = \alpha + i\beta \in \mathbb{C}$.

$$\begin{aligned}\langle \lambda(u+iv), w+ix\rangle_{\mathbb{C}} &= \langle (\alpha u - \beta v) + i(\alpha v + \beta u), w+ix\rangle_{\mathbb{C}} \\ &= \langle \alpha u + i\alpha v, w+ix\rangle_{\mathbb{C}} + \langle -\beta v + i\beta u, w+ix\rangle_{\mathbb{C}}\end{aligned}$$

$$\begin{aligned}\langle \alpha u + i\alpha v, w+ix\rangle_{\mathbb{C}} &= \langle \alpha u, w\rangle + \langle \alpha v, x\rangle + (\langle \alpha v, w\rangle - \langle \alpha u, x\rangle)i \\ &= \alpha\left(\langle v,x\rangle + (\langle v,w\rangle - \langle u,x\rangle)i\right) \\ &= \alpha\langle u+iv, w+ix\rangle_{\mathbb{C}}\end{aligned}$$

$$\begin{aligned}\langle -\beta v + i\beta u, w+ix\rangle_{\mathbb{C}} &= \langle -\beta v, w\rangle + \langle \beta u, x\rangle + (\langle \beta u, w\rangle - \langle -\beta v, x\rangle)i \\ &= i\beta\left[\langle u,w\rangle + \langle v,x\rangle + (\langle v,w\rangle - \langle u,x\rangle)i\right] \\ &= i\beta\langle u+iv, w+ix\rangle_{\mathbb{C}}\end{aligned}$$

Hence:

$$\begin{aligned}\langle \lambda(u+iv), w+ix\rangle_{\mathbb{C}} &= \alpha\langle u+iv, w+ix\rangle_{\mathbb{C}} + i\beta\langle u+iv, w+ix\rangle_{\mathbb{C}} \\ &= \lambda\langle u+iv, w+ix\rangle_{\mathbb{C}} \quad \checkmark\end{aligned}$$

Conjugate symmetry:

$$\begin{aligned}\langle u+iv, w+ix\rangle_{\mathbb{C}} &= \langle u,w\rangle + \langle v,x\rangle + (\langle v,w\rangle - \langle u,x\rangle)i \\ &= \overline{\langle u,w\rangle + \langle v,x\rangle - (\langle v,w\rangle - \langle u,x\rangle)i} \\ &= \overline{\langle w,u\rangle + \langle x,v\rangle + (\langle x,u\rangle - \langle w,v\rangle)i} \\ &= \overline{\langle w+ix, u+iv\rangle_{\mathbb{C}}} \quad \checkmark\end{aligned}$$

Here we exchanged the order in real-valued inner products.

Thus, $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ makes $V_{\mathbb{C}}$ into a complex inner product space. $\square$

(b) First equation:

$$\langle u,v\rangle_{\mathbb{C}} = \langle u,v\rangle + \langle 0,0\rangle + (\langle 0,v\rangle - \langle u,0\rangle)i = \langle u,v\rangle \quad \checkmark$$

Second equation:

$$\|u+iv\|_{\mathbb{C}}^2 = \langle u+iv, u+iv\rangle_{\mathbb{C}} = \langle u,u\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2 \quad \checkmark$$

31 Suppose $u, v, w \in V$. Prove that

$$\left\|w - \frac{1}{2}(u+v)\right\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$

Solution:

$$\begin{aligned}\left\|\omega - \frac{1}{2}(u+v)\right\|^2 &= \left\langle w - \frac{u+v}{2}, \omega - \frac{u+v}{2}\right\rangle \\ &= \langle w,w\rangle - \left\langle w, \frac{u+v}{2}\right\rangle - \left\langle \frac{u+v}{2}, w\right\rangle + \frac{1}{4}\langle u+v, u+v\rangle \\ &= \langle w,w\rangle - \frac{1}{2}\langle w,u\rangle - \frac{1}{2}\langle w,v\rangle - \frac{1}{2}\langle u,w\rangle - \frac{1}{2}\langle v,w\rangle \\ &\quad + \frac{1}{2}\langle u,u\rangle + \frac{1}{2}\langle v,v\rangle - \frac{1}{4}\|u-v\|^2 \\ &= \frac{1}{2}\left(\langle w,w\rangle - \langle w,u\rangle - \langle u,w\rangle + \langle u,u\rangle\right) \\ &\quad + \frac{1}{2}\left(\langle w,w\rangle - \langle w,v\rangle - \langle v,w\rangle + \langle v,v\rangle\right) - \frac{1}{4}\|u-v\|^2 \\ &= \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4} \quad \square\end{aligned}$$

32 Suppose E is a subset of V with the property that $u, v \in E$ implies $\frac{1}{2}(u+v) \in E$. Let $w \in V$. Show that there is at most one point in E that is closest to w. In other words, show that there is at most one $u \in C$ such that

$$\|w - u\| \leq \|w - x\|$$

for all $x \in E$.

Solution:

From the previous problem we know that:

$$\left\|\omega - \frac{1}{2}(u+v)^2\right\|^2 = \frac{\|\omega - u\|^2 + \|\omega - v\|^2}{2} - \frac{\|u-v\|^2}{4}$$

$$\|\omega - u\|^2 = \frac{1}{2}\|u - v\|^2 + 2\left\|w - \frac{1}{2}(u+v)\right\|^2 - \|\omega - v\|^2$$

Suppose u is one of the closests points in E. Then suppose v is another closest point in E, so that $\|w - u\| = \|w - v\|$ and $u \neq v$. Then $\frac{1}{2}\left(\|w-u\|^2 + \|w-v\|^2\right) = \|w-u\|^2$. Now notice:

$$\|w - u\|^2 = \left\|w - \frac{1}{2}(u+v)\right\|^2 + \frac{1}{4}\|u - v\|^2 > \left\|w - \frac{1}{2}(u+v)\right\|^2$$

where the last inequality is true because $\|u - v\| \geq 0$ for $u \neq v$.

But $\frac{1}{2}(u+v) \in E$ and $\left\|w - \frac{1}{2}(u+v)\right\| < \|w-u\| = \|w-v\|$. That means u and v cannot be even in the set of closest points to w. Thus it can be at most one closest point. $\square$

33 Suppose f, g are differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$.

(a) Show that

$$\langle f(t), g(t)\rangle' = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle.$$

(b) Suppose c is a positive number and $\|f(t)\| = c$ for every $t \in \mathbb{R}$. Show that $\langle f'(t), f(t)\rangle = 0$ for every $t \in \mathbb{R}$.

(c) Interpret the result in (b) geometrically in terms of the tangent vector to a curve lying on a sphere in $\mathbb{R}^n$ centered at the origin.

Solution:
(a)

$$\langle f(t), g(t)\rangle' = \lim_{\Delta t\to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle + \langle f(t), g(t+\Delta t)\rangle - \langle f(t), g(t+\Delta t)\rangle}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{\langle f(t+\Delta t) - f(t), g(t+\Delta t)\rangle}{\Delta t} + \lim_{\Delta t\to 0} \frac{\langle f(t), g(t+\Delta t) - g(t)\rangle}{\Delta t}$$

$$= \left\langle \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, g(t) \right\rangle + \left\langle f(t), \lim_{\Delta t\to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle \quad \square$$

(b)

$$\|f(t)\| = c \quad \Rightarrow \quad \langle f(t), f(t)\rangle = c^2$$

$$\langle f'(t), f(t)\rangle = \langle f(t), f(t)\rangle' - \underbrace{\langle f(t), f'(t)\rangle}_{=\langle f'(t), f(t)\rangle}$$

Hence $\langle f'(t), f(t)\rangle = \frac{1}{2}\langle f(t), f(t)\rangle' = \frac{1}{2} \cdot \frac{d}{dt} c^2 \equiv 0$.

(c) Assume for some $f(t)$: $\langle f(t), f(t)\rangle = f_1^2(t) + \ldots + f_n^2(t) = c^2$
Thus, one can see that $f(t)$ describes a family of parametric curves on an n-dimensional sphere.

Tangent to a curve is given by $f'(t)$. $\langle f'(t), f(t)\rangle = 0$ for all $t \in \mathbb{R}$ means tangent vector is always orthogonal to a curve on the sphere.

31 Use inner products to prove Apollonius's identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

Solution:
Note that $\vec{c} = \vec{a} - \vec{b}$, $\frac{1}{2}\vec{c} = \vec{d} - \vec{b}$ and $\frac{1}{2}\vec{c} = \vec{a} - \vec{d}$. Hence

$$0 = \vec{d} - \vec{b} - \vec{a} + \vec{d} \Rightarrow \vec{d} = \frac{1}{2}(\vec{a} + \vec{b})$$

$$a^2 + b^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 = \frac{1}{2}\left(\|\vec{a} - \vec{b}\|^2 + \|\vec{a} + \vec{b}\|^2\right)$$

$$= \frac{1}{2}\|c\|^2 + \frac{1}{2}\|\vec{a} + \vec{b}\|^2 = \frac{1}{2}\|\vec{c}\|^2 + \frac{1}{2}\|2\vec{d}\|^2 = \frac{1}{2}c^2 + 2d^2 \quad \square$$

32 Fix a positive integer n. The *Laplacian* Δp of a twice differentiable real-valued function p on $\mathbb{R}^n$ is the function on $\mathbb{R}^n$ defined by

$$\Delta p = \frac{\partial^2 p}{\partial x_1^2} + \cdots + \frac{\partial^2 p}{\partial x_n^2}$$

The function p is called *harmonic* if $\Delta p = 0$.

A *polynomial* on $\mathbb{R}^n$ is a linear combination of functions of the form $x_1^{m_1} \cdots x_n^{m_n}$, where $m_1, \ldots, m_n$ are nonnegative integers.

Suppose q is a polynomial on $\mathbb{R}^n$. Prove that there exists a harmonic polynomial p on $\mathbb{R}^n$ such that $p(x) = q(x)$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$.

Solution:

Suppose q is a polynomial of degree $m = \max\{m_1, \ldots, m_n\}$. If $m < 2$, then q is harmonic automatically. Otherwise, let us define an operator T on the vector space of polynomials on $\mathbb{R}^n$ of degree m:

$$Tr = \Delta\left((1 - \|x\|^2)r\right)$$

for every polynomial r in this vector space.

Suppose $\xi \in \operatorname{null} T$.

$$T\xi = \Delta\left((1 - \|x\|^2)\xi\right) = 0$$

That means, $(1 - \|x\|^2)\xi$ is harmonic. Also, $(1 - \|x\|^2)\xi = 0$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$. Hence, $(1 - \|x\|^2)\xi = 0$ for all $x \in \mathbb{R}^n$. $1 - \|x\|^2 \neq 0$ for all possible x, thus ξ must equal zero. That means, T is injective and, because vector space is finite dimensional, is invertible.

Now define

$$p = q + (1 - \|x\|^2)r$$

Obviously, $p = q$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$. We wish for p to be harmonic:

$$\Delta p = 0 = \Delta q + \Delta\left((1 - \|x\|^2)r\right) = \Delta q + Tr \quad \Rightarrow \quad Tr = -\Delta q$$

As T is invertible, we can always choose r such that:

$$r = T^{-1}(\Delta q)$$

Thus, the desired polynomial p is:

$$p = q + (1 - \|x\|^2) \cdot T^{-1}(\Delta q) \qquad \square$$

6B Orthonormal Bases

1 Suppose $e_1, \ldots, e_m$ is a list of vectors in V such that

$$\|a_1e_1 + \cdots + a_me_m\|^2 = |a_1|^2 + \cdots |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$. Show that $e_1, \ldots, e_m$ is an orthonormal list.

Solution:

Take some $k \in \{1, \ldots, m\}$ and choose $a_k = 1$ and $a_j = 0$ for all $j \neq k$. Then:

$$\|a_ke_k\|^2 = |a_k|^2\|e_k\|^2 \quad \text{and} \quad \|a_ke_k\|^2 = |a_k|^2$$

Hence, $\|e_k\| = 1$. Repeating this process for every possibile value of k, we conclude that $\|e_j\| = 1$ for every vector in $e_1, \ldots, e_m$.

Now we want to show that these vectors are all orthogonal to each other. Take any $j, k \in \{1, \ldots, m\}$ such that $j \neq k$ and choose $a_j = a_k = 1$, with all other coefficients being zero.

$$\|e_j + e_k\|^2 = \|e_j\|^2 + \|e_k\|^2 + 2\Re\langle e_j, e_k\rangle$$

But also $\|e_j + e_k\|^2 = 2$. Therefore, $\Re\langle e_j, e_k\rangle = 0$. If $\mathbb{F} = \mathbb{R}$, then we can stop here.

If $\mathfrak{F} = \mathbb{C}$, then for the same pair j, k we choose $a_j = 1$ and $a_k = i$.

$$\|e_j + i \cdot e_k\|^2 = \|e_j\|^2 + \|e_k\|^2 + 2\Im\langle e_j, e_k\rangle$$

And also $\|e_j + i \cdot e_k\|^2 = 2$. Therefore, $\Im\langle e_j, e_k\rangle = 0$.

Thus, we have shown that $\langle e_j, e_k\rangle = 0$. As this conclusion is true for every pair of distinct e_j and e_k, the list $e_1, \ldots, e_m$ is orthonormal. $\square$

2 (a) Suppose $\theta \in \mathbb{R}$. Show that both

$$(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta) \quad \text{and} \quad (\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$$

are orthonormal bases of $\mathbb{R}^2$.

(b) Show that each orthonormal basis of $\mathbb{R}^2$ is of the form given by one of the two possibilities of part (a).

Solution:

(a) Firstly, we check the norm of the first list of vectors:

$$\|(\cos\theta, \sin\theta)\| = \left(\cos^2\theta + \sin^2\theta\right)^{1/2} = \sqrt{1} = 1$$

$$\|(-\sin\theta, \cos\theta)\| = \left((-\sin\theta)^2 + \cos^2\theta\right)^{1/2} = \sqrt{1} = 1$$

Then, we check orthogonality:

$$\langle(\cos\theta,\sin\theta),(-\sin\theta,\cos\theta)\rangle = \cos\theta\cdot(-\sin\theta)+\sin\theta\cdot\cos\theta = 0$$

Thus, $(\cos\theta,\sin\theta)$, $(-\sin\theta,\cos\theta)$ is an orthonormal basis of $\mathbb{R}^2$.
In the same way we examine the second list of vectors:

$$\begin{aligned}\|(\sin\theta,-\cos\theta)\| &= \left(\sin^2\theta+(-\cos\theta)^2\right)^{1/2} = \sqrt{1} = 1\\ \langle(\cos\theta,\sin\theta),(\sin\theta,-\cos\theta)\rangle &= \cos\theta\cdot\sin\theta+\sin\theta\cdot(-\cos\theta) = 0\end{aligned}$$

Thus, $(\cos\theta,\sin\theta)$, $(\sin\theta,-\cos\theta)$ is also an orthonormal basis of $\mathbb{R}^2$.
(b) Suppose (x_1,y_1), (x_2,y_2) is some orthonormal basis of $\mathbb{R}^2$. Then the following equations must hold:

$$\begin{cases} x_1^2+y_1^2=1\\ x_2^2+y_2^2=1\\ x_1x_2+y_1y_2=0\end{cases}$$

Because of the first two equations in the system, we can parameterize the variables in the following way.

$$\begin{aligned} x_1 &= \cos\theta, \quad & y_1 &= \sin\theta\\ x_2 &= \sin\theta, \quad & y_2 &= \cos\theta\end{aligned}$$

Then checking the orthogonality:

$$\cos\theta\sin\phi+\sin\theta\cos\phi=0$$

The sum in the left-hand side folds into a sine of a sum:

$$\sin(\theta+\phi)=0 \quad\Rightarrow\quad \theta+\phi=\pi n$$

If n is even, then $\phi = 2k\pi-\theta$ and:

$$\begin{aligned}(x_1,y_1) &= (\cos\theta,\sin\theta)\\ (x_2,y_2) &= (\sin 2k\pi-\theta,\cos 2k\pi-\theta) = (-\sin\theta,\cos\theta)\end{aligned}$$

that is the first possibility from (a).
If n is odd, then $\phi=(2k+1)\pi-\theta$ and:

$$\begin{aligned}(x_1,y_1) &= (\cos\theta,\sin\theta)\\ (x_2,y_2) &= (\sin(2k+1)\pi-\theta,\cos(2k+1)\pi-\theta) = (\sin\theta,-\cos\theta)\end{aligned}$$

that is the second possibility from (a). $\square$

3 Suppose $e_1, \dots, e_m$ is an orthonormal list of vectors in V and $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2 \iff v \in \operatorname{span}(e_1, \dots, e_m)$$

Solution:

$\longleftarrow$ If $v \in \operatorname{span}(e_1, \dots, e_m)$, then we can regard span of a given list of orthonormal vectors as a subspace U of V with the given list as basis and $v \in U$, then the desired equation on the left is true.

$\longrightarrow$ If $v \in V$ and $\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$.

Suppose $v \notin \operatorname{span}(e_1, \dots, e_m)$. We can construct new subspace by extending the given span to $\operatorname{span}(e_1, \dots, e_m, v)$. Applying the Gram-Schmidt procedure to it, we get vector e_{m+1} orthogonal to all other e_i:

$$e_{m+1} = \frac{v - \langle v, e_1\rangle e_1 - \ldots - \langle v, e_m\rangle e_m}{\|v - \langle v, e_1\rangle e_1 - \ldots - \langle v, e_m\rangle e_m\|}$$

Then we take inner product with v:

$$\langle v, e_{m+1}\rangle = \frac{\langle v, v\rangle - |\langle v, e_1\rangle|^2 - \ldots - |\langle v, e_m\rangle|^2}{\|v - \langle v, e_1\rangle e_1 - \ldots - \langle v, e_m\rangle e_m\|} = 0$$

Hence the list $e_1, \dots, e_m$ is sufficient to span v. $\square$

4 Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $\mathcal{C}[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} fg$$

Solution:

First, we check norms:

$$\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}\rangle = \int_{-\pi}^{\pi} \frac{1}{2\pi} dx = \frac{1}{2\pi}(\pi - (-\pi)) = 1$$

$$\langle \frac{\cos kx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}}\rangle = \int_{-\pi}^{\pi} \frac{\cos kx \cdot \cos kx}{\pi} dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(kx) dx = \frac{1}{\pi}\pi = 1$$

$$\langle \frac{\sin kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}\rangle = \int_{-\pi}^{\pi} \frac{\sin kx \cdot \sin kx}{\pi} dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(kx) dx = \frac{1}{\pi}\pi = 1$$

Then we check orthogonality:

$$\langle\frac{1}{\sqrt{2\pi}},\frac{\cos kx}{\sqrt{\pi}}\rangle=\int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}\cdot\frac{\cos kx}{\sqrt{\pi}}dx=\frac{1}{k\pi\sqrt{2}}\cdot\sin kx\Big|_{-\pi}^{\pi}=\frac{\sin\pi k+\sin\pi k}{k\pi\sqrt{2}}=0$$

$$\langle\frac{1}{\sqrt{2\pi}},\frac{\sin kx}{\sqrt{\pi}}\rangle=\int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}\cdot\frac{\sin kx}{\sqrt{\pi}}dx=-\frac{1}{k\pi\sqrt{2}}\cdot\cos kx\Big|_{-\pi}^{\pi}=-\frac{\cos\pi k-\cos\pi k}{k\pi\sqrt{2}}=0$$

$$\langle\frac{\cos kx}{\sqrt{2\pi}},\frac{\cos mx}{\sqrt{2\pi}}\rangle=\int_{-\pi}^{\pi}\frac{\cos(kx)\cos(mx)}{\pi}dx=$$
$$\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\cos((k+m)x)dx+\int_{-\pi}^{\pi}\cos((k-m)x)dx\right)=0$$

$$\langle\frac{\sin kx}{\sqrt{2\pi}},\frac{\sin mx}{\sqrt{2\pi}}\rangle=\int_{-\pi}^{\pi}\frac{\sin(kx)\sin(mx)}{\pi}dx=$$
$$\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\cos((k+m)x)dx-\int_{-\pi}^{\pi}\cos((k-m)x)dx\right)=0$$

$$\langle\frac{\cos kx}{\sqrt{2\pi}},\frac{\sin mx}{\sqrt{2\pi}}\rangle=\int_{-\pi}^{\pi}\frac{\cos(kx)\sin(mx)}{\pi}dx=$$
$$\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\sin((k+m)x)dx-\int_{-\pi}^{\pi}\sin((k-m)x)dx\right)=0$$

Thus, the list in question is indeed orthonormal. □

5 Suppose $f:[-\pi,\pi]\to\mathbb{R}$ is continuous. For each nonnegative integer k, define

$$a_k=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}f(x)\cos(kx)dx\quad\text{and}\quad b_k=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}f(x)\sin(kx)dx$$

Prove that

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)\le\int_{-\pi}^{\pi}f^2.$$

Solution:

As the vector space of continuous functions is infinite dimensional, we can extend the list from Problem 4 to any arbitrarily large n. Using the Bessel's inequality, we get:

$$\left|\langle f, \frac{1}{\sqrt{2\pi}}\rangle\right|^2 + \sum_{k=1}^{\infty}\left(\left|\langle f, \frac{\cos kx}{\sqrt{\pi}}\rangle\right|^2 + \left|\langle f, \frac{\sin kx}{\sqrt{\pi}}\rangle\right|^2\right) \leq \langle f, f\rangle \tag{6.1}$$

Note, that

$$\begin{aligned}
\langle f, \frac{1}{\sqrt{2\pi}}\rangle &= \int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2\pi}}dx = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos(0 \cdot x)dx = a_0 \\
\langle f, \frac{\cos kx}{\sqrt{\pi}}\rangle &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = a_k \\
\langle f, \frac{\sin kx}{\sqrt{\pi}}\rangle &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = b_k \\
\langle f, f\rangle &= \int_{-\pi}^{\pi} f(x)^2 dx
\end{aligned}$$

Inserting it into (6.1), we get:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \int_{-\pi}^{\pi} f^2$$

as desired. □

6 Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V.

(a) Prove that if $v_1, \ldots, v_n$ are vectors in V such that

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}}$$

for each k, then $v_1, \ldots, v_n$ is a basis of V.

(b) Show that there exist $v_1, \ldots, v_n \in V$ such that

$$\|e_k - v_k\| \leq \frac{1}{\sqrt{n}}$$

for each k, but $v_1, \ldots, v_n$ is not linearly independent.

Solution:

(a) V is finite-dimension, so we are left to show that $v_1, \dots, v_n$ is linearly independent. Suppose it is not, so that there exist such $a_1, \dots, a_n$ such that $a_1 v_1 + \cdots + a_n v_n = 0$. Let us examine the following sum:

$$\sum_i^n \|a_i(e_i - v_i)\| = \sum_i^n |a_i| \|e_i - v_i\| < \frac{\sum_i^n |a_i|}{\sqrt{n}}$$

On the other hand, the triangle inequality gives:

$$\sum_i^n \|a_i(e_i - v_i)\| \geq \|\sum_i^n a_i(e_i - v_i)\| = \|\sum_i^n a_i e_i - \sum_i^n a_i v_i\| = \|\sum_i^n a_i e_i\|$$

Thus, we have:

$$\|\sum_i^n a_i e_i\| < \frac{\sum_i^n |a_i|}{\sqrt{n}}$$

Squaring both sides of the inequality, we get:

$$\|\sum_i^n a_i e_i\|^2 = \sum_i^n |a_i|^2 < \frac{\left(\sum_i^n |a_i|\right)^2}{n}$$

We know from *Problem 6A.13* that:

$$n \sum_i^n |a_i|^2 \geq \left(\sum_i^n |a_i|\right)^2$$

Thus we have contradiction and our assumption that $v_1, \dots, v_n$ could be not linearly independent is wrong. Hence, $v_1, \dots, v_n$ is a basis of V indeed. $\square$

(b) Let $V = \mathbb{R}^2$ with Euclidean norm and $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then choose $v_1 = (\frac{1}{2}, -\frac{1}{2})$ and $v_2 = (-\frac{1}{2}, \frac{1}{2})$. Clearly, $v_1 = -v_2$, so the list v_1, v_2 is not linearly independent. Meanwhile:

$$\|e_1 - v_1\| = \sqrt{(1 - \frac{1}{2})^2 + (0 + \frac{1}{2})^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$\|e_2 - v_2\| = \sqrt{(0 + \frac{1}{2})^2 + (1 - \frac{1}{2})^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

7 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ has an upper-triangular matrix with respect to the basis $(1,0,0),(1,1,1),(1,1,2)$. Find an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper-triangular matrix.

Solution:

To find such basis, we have to apply the Gram-Schmidt procedure to the given basis, as it guarantees that T will have an upper-triangular matrix with respect to the produced orthonormal basis.

Let $v_1 = (1,0,0)$, $v_2 = (1,1,1)$, $v_3 = (1,1,2)$. v_1 is already normalized, so we can choose $e_1 = v_1 = (1,0,0)$.

$$\begin{aligned}
&\langle v_2, e_1\rangle = 1\\
&f_2 = v_2 - \langle v_2, e_1\rangle e_1 = (1,1,1) - (1,0,0) = (0,1,1)\\
&\|f_2\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}
\end{aligned}$$

Hence $e_2 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$\begin{aligned}
&\langle v_3, e_1\rangle = 1\\
&\langle v_3, e_2\rangle = 0 + \frac{1}{\sqrt{2}} + 2\cdot\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}\\
&f_3 = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\\
&\quad = (1,1,2) - (1,0,0) - \frac{3}{\sqrt{2}}(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (0, -\frac{1}{2}, \frac{1}{2})\\
&\|f_3\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}
\end{aligned}$$

Hence $e_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and the required orthonormal basis is $(1,0,0)$, $(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

8 Make $\mathcal{P}_2(\mathbb{R})$ into an inner product space by defining $\langle p, q\rangle = \int_0^1 pq$ for all $p, q \in \mathcal{P}_2(\mathbb{R})$.

(a) Apply the Gram-Schmidt procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

(b) The differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbb{R})$ has an upper-triangular matrix with respect to the basis $1, x, x^2$, which is not an orthonormal basis. Find the matrix of the differentiation operator on $\mathcal{P}_2(\mathbb{R})$ with respect to the orthonormal basis produced in (a) and verify that this matrix is upper triangular, as expected from the proof of 6.37.

Solution:

(a) Let $p_1 = 1$, $p_2 = x$, $p_3 = x^2$. $\|p_1\| = 1$, so we can take $e_1 = 1$.

$$\begin{aligned}
\langle p_2, e_1\rangle &= \int_0^1 x \cdot 1 dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}\\
q_2 &= p_2 - \langle p_2, e_1\rangle e_1 = x - \frac{1}{2}\\
\|q_2\|^2 &= \int_0^1 (x - \frac{1}{2})^2 dx = \frac{(x-1/2)^3}{3}\Big|_0^1 = \frac{1}{12}\\
e_2 &= q_2/\|q_2\| = \sqrt{3}(2x-1)
\end{aligned}$$

$$\begin{aligned}
\langle p_3, e_1\rangle &= \int_0^1 x^2 \cdot 1 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}\\
\langle p_3, e_2\rangle &= \int_0^1 x^2\sqrt{3}(2x-1)dx = \sqrt{3}\int_0^1 (2x^3 - x^2)dx\\
&= \sqrt{3}\left(2\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2\sqrt{3}}\\
q_3 &= p_3 - \langle p_3, e_1\rangle e_1 - \langle p_3, e_2\rangle e_2 = x^2 - \frac{1}{3} - \frac{1}{2\sqrt{3}}\sqrt{3}(2x-1) = x^2 - x - \frac{1}{6}\\
\|q_3\|^2 &= \int_0^1 (x^2 - x - \frac{1}{6})^2 dx = \int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right)dx\\
&= \left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{4x^3}{9} - \frac{x^2}{6} + \frac{x}{36}\right)\Big|_0^1 = \frac{1}{180}\\
e_3 &= q_3/\|q_3\| = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)
\end{aligned}$$

Thus, the required orthonormal basis is: $1, \sqrt{3}(2x-1), 6\sqrt{5}(x^2 - x + \frac{1}{6})$.

(b) We examine how the differentiation operator acts on the basis in order to construct the matrix of this operator.

$$\begin{aligned}
D(e_1) &= 1' = 0\\
D(e_2) &= \left(\sqrt{3}(2x-1)\right)' = \sqrt{3}\cdot 2 = 2\sqrt{3}e_1\\
D(e_3) &= \left(6\sqrt{5}(x^2 - x + \frac{1}{6})\right)' = 6\sqrt{5}(2x-1) = \frac{6\sqrt{5}}{\sqrt{3}}e_2
\end{aligned}$$

So the matrix is:

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & \frac{6\sqrt{5}}{\sqrt{3}} \\ 0 & 0 & 0 \end{pmatrix}$$

upper-triangular indeed.

9 Suppose $e_1, \dots, e_m$ is the result of applying the Gram-Schmidt procedure to a linearly independent list $v_1, \dots, v_m$ in V. Prove that $\langle v_k, e_k \rangle > 0$ for each $k = 1, \dots, m$.

Solution:

In the Gram-Schmidt procedure every vector e_k is written as $f_k/\|f_k\|$ where f_k is:

$$f_k = v_k - \langle v_k, e_1 \rangle e_1 - \cdots - \langle v_k, e_{k-1} \rangle e_{k-1}$$

Take the inner product of f_k with v_k:

$$\langle f_k, v_k \rangle = \langle v_k, v_k \rangle - |\langle v_k, e_1 \rangle|^2 - \cdots - |\langle v_k, e_{k-1} \rangle|^2 > 0$$

where we used Bessel's inequality in the end. Here the sign is *greater*, because equality can be only if $v_k \in \operatorname{span}(e_1, \dots, e_{k-1})$, which is not the case.

Thus, $\langle f_k, v_k \rangle > 0$, so $\langle v_k, e_k \rangle > 0$ as well. □

10 Suppose $v_1, \dots, v_m$ is a linearly independent list in V. Explain why the orthonormal list produced by the formulas of the Gram-Schmidt procedure is the only orthonormal list $e_1, \dots, e_m$ in V such the $\langle v_k, e_k \rangle > 0$ and $\operatorname{span}(v_1, \dots, v_k) = \operatorname{span}(e_1, \dots, e_k)$ for each $k = 1, \dots, m$.

Solution:

If we choose $e'_k = -e_k$ for some $k \in \{1, \dots, m\}$, then

$$\langle v_k, e'_k \rangle = -\langle v_k, e_k \rangle < 0$$

Any other option will fail the condition on spans.

The fist vector must always be either $v_1/\|v_1\|$, otherwise spans will be different. Then if we take some e'_2 not generated by the Gram-Schmidt procedure, then it must be a linear combination of "Gram-Schmidt"-vectors, so $\operatorname{span}(e_1, e'_2) \neq \operatorname{span}(e_1, e_2) = \operatorname{span}(v_1, v_2)$. The same logic can be applied at any step of choosing some e_k.

Thus, the only option for both conditions to hold is the list generated by the Gram-Schmidt procedure. □

11 Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that $p(\frac{1}{2}) = \int_0^1 pq$ for every $p \in \mathcal{P}_2(\mathbb{R})$.

Solution:

Here we have a linear functional: $\varphi(p) = p(\frac{1}{2})$; and we need to find a polynomial such that the inner product $\langle p, q\rangle$ represents this linear functional. We already know the orthonormal basis from the *Problem 6B.8*, so we only need to compute the coefficients in the representation.

$$\begin{aligned}
\varphi(e_1) &= e_1(\frac{1}{2}) = 1 \\
\varphi(e_2) &= e_2(\frac{1}{2}) = \sqrt{3}(2 \cdot \frac{1}{2} - 1) = 0 \\
\varphi(e_3) &= e_3(\frac{1}{2}) = 6\sqrt{5}(\frac{1}{4} - \frac{1}{2} + \frac{1}{6}) = -\frac{\sqrt{5}}{2}
\end{aligned}$$

$$q = \overline{\varphi(e_1)e_1} + \overline{\varphi(e_2)e_2} + \overline{\varphi(e_3)e_3} = 1 + \left(-\frac{\sqrt{5}}{2}\right) \cdot 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$$

Thus, $q = -15x^2 + 15x - \frac{3}{2}$.

12 Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$\int_0^1 p(x)\cos(\pi x)dx = \int_0^1 pq$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Solution:

Once again we will use the basis from Problem 8.

$$\begin{aligned}
\varphi(e_1) &= \int_0^1 \cos(\pi x)dx = 0 \\
\varphi(e_2) &= \int_0^1 \sqrt{3}(2x-1)\cos(\pi x)dx = \frac{-4\sqrt{3}}{\pi^2} \\
\varphi(e_3) &= \int_0^1 6\sqrt{5}(x^2 - x + \frac{1}{6})\cos(\pi x)dx = 0
\end{aligned}$$

Hence:

$$q(x) = -\frac{12}{\pi^2}(2x-1)$$

13 Show that a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if and only if the Gram-Schmidt formula in 6.32 produces $f_k = 0$ for some $k \in \{1, \ldots, m\}$.

Solution:

$\longleftarrow$ Suppose $f_k = 0$ for some k. Then:

$$0 = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}$$

or

$$v_k = \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 + \cdots + \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}$$

every f_j in this sum is a linear combination of vectors $v_1, \ldots, v_{k-1}$, hence v_k is a linear combination of vectors $v_1, \ldots, v_{k-1}$ too. It means the list $v_1, \ldots, v_k$ is linearly dependent and hence $v_1, \ldots, v_m$ is also linearly dependent.

$\longrightarrow$ Suppose $v_1, \ldots, v_m$ is linearly dependent. It means, there is $k \in \{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$. Because $\operatorname{span}(v_1, \ldots, v_{k-1}) = \operatorname{span}(e_1, \ldots, e_{k-1})$, $v_k \in \operatorname{span}(e_1, \ldots, e_{k-1})$. The span of orthonormal list $e_1, \ldots, e_{k-1}$ is a subspace and we can regard this list as a basis of this subspace. Then, the vector v_k can be expressed as follows:

$$v_k = \langle v_k, e_1 \rangle + \cdots + \langle v_k, e_{k-1} \rangle$$

Now, let us examine the vector f_k of the Gram-Schmidt procedure:

$$\begin{aligned} f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1} \\ = v_k - \langle v_k, e_1 \rangle e_1 - \cdots - \langle v_k, e_{k-1} \rangle e_{k-1} = 0 \quad \square \end{aligned}$$

14 Suppose V is a real inner product space and $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $e_1, \ldots, e_m$ of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$$

for all $k \in \{1, \ldots, m\}$.

Solution:

Gram-Schmidt procedure gives a list $e_1, \ldots, e_m$ such that $\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$for all $k \in \{1, \ldots, m\}$. We can get other orthonormal lists by modifying the list of Gram-Schmidt procedure. Take some e_k and multiply it by -1. This operation does not change the span of the list, but changes the list itself.

Other options, as we have seen in *Problem 6B.10*, change span, so they cannot be taken.

Therefore we have two options for every vector e_k, making the total number of possible lists equal 2^m. $\square$

15 Suppose $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on V such that $\langle u,v\rangle_1 = 0$ if and only if $\langle u,v\rangle_2 = 0$. Prove that there is a positive number c such that $\langle u,v\rangle_1 = c\langle u,v\rangle_2$ for every $u,v \in V$.

Solution: Suppose v and w are arbitrary non-orthogonal vectors in V, so none of them is zero. Then write as follows:

$$\begin{aligned}
0 &= \langle v,w\rangle_1 - \langle v,w\rangle_1 \cdot \frac{\langle v,v\rangle_1}{\langle v,v\rangle_1} = \langle v,w\rangle_1 - \langle v, \overline{\frac{\langle v,w\rangle_1}{\langle v,v\rangle_1}} v\rangle_1 \\
&= \langle v, w - \frac{\langle w,v\rangle_1}{\langle v,v\rangle_1} v\rangle_1 = 0 = \langle v, w - \frac{\langle w,v\rangle_1}{\langle v,v\rangle_1} v\rangle_2 \\
&= \langle v,w\rangle_2 - \langle v,v\rangle_2 \cdot \frac{\langle v,w\rangle_1}{\langle v,v\rangle_1} \quad \Longrightarrow \quad \langle v,w\rangle_2 = \frac{\|v\|_2^2}{\|v\|_1^2}\langle v,w\rangle_1
\end{aligned}$$

In a similar way we can write:

$$\begin{aligned}
0 &= \langle v,w\rangle_1 - \langle v,w\rangle_1 \cdot \frac{\langle w,w\rangle_1}{\langle w,w\rangle_1} = \langle v,w\rangle_1 - \langle \frac{\langle v,w\rangle_1}{\langle w,w\rangle_1} w, w\rangle_1 \\
&= \langle v - \frac{\langle v,w\rangle_1}{\langle w,w\rangle_1} w, w\rangle_1 = 0 = \langle v - \frac{\langle v,w\rangle_1}{\langle w,w\rangle_1} w, w\rangle_2 \\
&= \langle v,w\rangle_2 - \langle w,w\rangle_2 \cdot \frac{\langle v,w\rangle_1}{\langle w,w\rangle_1} \quad \Longrightarrow \quad \langle v,w\rangle_2 = \frac{\|w\|_2^2}{\|w\|_1^2}\langle v,w\rangle_1
\end{aligned}$$

We see that $\langle v,v\rangle_2/\langle v,v\rangle_1 = \langle w,w\rangle_2/\langle w,w\rangle_1$. But as we took arbitrary non-zero vectors, it follows that the ratio $\langle u,u\rangle_2/\langle u,u\rangle_1$ is the same for any $u \in V$, $u \neq 0$. This ratio is a positive number, as both numenator and denumenator are positive. Thus, there is a positive number c such that $\langle u,v\rangle_1 = c\langle u,v\rangle_2$ for every $u,v \in V$. $\square$

16 Suppose V is finite-dimensional. Suppose $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists a positive number c such that $\|v\|_1 \le c\|v\|_2$ for every $v \in V$.

Solution:

We know from previous problem that there is a positive number k such that $\langle u,v\rangle_1 = k\langle u,v\rangle_2$ for every $u,v \in V$. Let $u = v$. Then we get $\langle v,v\rangle_1 = c\langle v,v\rangle_2$

for every $v \in V$. As k is positive, we can take square root on both sides and get:

$$\|v\|_1 = \sqrt{k}\|v\|_2$$

Choose any $c > \sqrt{k}$ to get the desired inequality $\|v\|_1 \leq c\|v\|_2$. $\square$

17 Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if T is an operator on V such that 1 is the only eigenvalue of T and $\|Tv\| \leq \|v\|$ for all $v \in V$, then T is the identity operator.

Solution:

Let $e_1, \ldots, e_n$ be an orthonormal basis of V such that matrix of T with respect to this basis is upper-triangular. Using 5.41, we conclude that all diagonal entries in the matrix equal 1.

T acts on some basis vector e_k like:

$$Te_k = e_k + \sum_{j=1}^{k-1} A_{j,k} e_j$$

Using the property of T we get:

$$\|Te_k\| \leq \|e_k\| = 1$$

At the same time:

$$\|Te_k\| = \sqrt{\|e_k + \sum_{j=1}^{k-1} A_{j,k} e_j\|^2} = \sqrt{1 + \sum_{j=1}^{k-1} |A_{j,k}|^2}$$

Thus, $\sqrt{1 + \sum_{j=1}^{k-1} |A_{j,k}|^2} \leq 1$, which is possible only if $\sum_{j=1}^{k-1} |A_{j,k}|^2 = 0$.

As it is true for any e_k, we conclude that the only non-zero matrix elements are diagonal elements, which are equal to 1. It means, that T is the identity operator. $\square$

18 Suppose $u_1, \ldots, u_m$ is a linearly independent list in V. Show that there exists $v \in V$ such that $\langle u_k, v \rangle = 1$ for all $k \in \{1, \ldots, m\}$.

Solution:

Let $U = \operatorname{span}(u_1, \ldots, u_m)$. For a $w \in U$: $w = a_1 u_1 + \cdots + a_m u_m$, define a linear functional $\varphi(w) = a_1 + \cdots a_m$. By the Riesz representation theorem there is a unique $v \in V$ such that $\varphi(w) = \langle w, v \rangle$.

Now note that for every $k \in \{1, \ldots, m\}$ the value of the linear functional is $\varphi(u_k) = 1$. Thus $\langle u_k, v \rangle = 1$, as desired. $\square$

19 Suppose $v_1, \dots, v_n$ is a basis of V. Prove that there exists a basis $u_1, \dots, u_n$ of V such that

$$\langle v_j, u_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Solution:

Take the dual basis $\varphi_1, \dots, \varphi_n$ of the given basis $v_1, \dots, v_n$. Every linear functional in this dual basis is defined as:

$$\varphi_k(v_j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

By the Riesz representation theorem there exist such vectors $u_1, \dots, u_n$ such that $\langle v_j, u_k \rangle = \varphi_k(v_j)$. That gives the desired values of the inner products. Now we need to show that $u_1, \dots, u_n$ is also a basis of V. Let $a_1, \dots, a_n \in \mathbb{F}$ be such that:

$$a_1 u_1 \cdots a_n u_n = 0$$

Take the inner product of this linear combination with some v_k from the old basis:

$$\langle a_1 u_1 + \cdots + a_n u_n, v_k \rangle = a_k \langle u_k, v_k \rangle = a_k$$

At the same time:

$$\langle a_1 u_1 + \cdots + a_n u_n, v_k \rangle = \langle 0, v_k \rangle = 0$$

We conclude from it that $a_k = 0$ for any $k \in \{1, \dots, n\}$. So $u_1, \dots, u_n$ is linearly independent and hence is a basis of V. $\square$

20 Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional, and $\mathcal{E} \subset \mathcal{L}(V)$ is such that

$$ST = TS$$

for all $S, T \in \mathcal{E}$. Prove that there is an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular matrix.

21 Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional, $T \in \mathcal{L}(V)$, and all eigenvalues of T have absolute value less than 1. Let $\epsilon > 0$. Prove that there exists a positive integer m such that $\|T^m v\| \leq \epsilon \|v\|$ for every $v \in V$.

Solution:

By the Schur's theorem, there is some orthonormal basis $e_1, \ldots, e_n$ such that the matrix of T with respect to this basis is upper-triangular.

Note, that is we change all entries in the matrix of the operator to their absolute values, the norm $\|Tv\|$ is greater or equal to the norm of the original Tv. Also, we can change all diagonal entries (eigenvalues of T) to the maximum eigenvalue. Thus we have constructed an operator A such that:

$$\begin{aligned}
\mathcal{M}(A)_{i,j} &= |\mathcal{M}(T)| \quad \text{if } i \neq j; \\
\mathcal{M}(A)_{i,i} &= \max\{|\lambda_k|\}
\end{aligned}$$

For this operator, $\|Tv\| \leq \|Av\|$ and hence $\|T^m v\| \leq \|A^m v\|$.

Let us denote the value of diagonal elements of $\mathcal{M}(A)$ as λ. Then we can write:

$$A = \lambda I + N$$

where the matrix of N with respect to the basis $e_1, \ldots, e_n$ has all diagonal elements equal to zero.

By 5.27, the minimal polynomial of N is $(z-0)^n = 0$ hence $N^n = 0$. Take any $m \geq n$, then

$$A^m = (\lambda I + N)^m = \lambda^m I + m\lambda^{m-1} N + \cdots + \frac{m!}{(m-n+1)!(n-1)!} \lambda^{m-n+1} N^{n-1}$$

The coefficients in the expansion of A in the limit of $m \to \infty$ are:

$$\lim_{m \to \infty} \frac{m!}{(m-k)!k!} \lambda^{m-k} = \lim_{m \to \infty} \frac{\lambda^m}{k!} = 0 \quad (\text{as } \lambda < 0)$$

Thus, taking m sufficiently large we can make every coefficient in this sum as small as we wish. This sum is finite, so we can make every entry of A^m as small as we wish. Take m such that $\mathcal{M}(A)_{j,k} \leq \epsilon/\sqrt{n}$ for every j and k. Then:

$$\|A^m e_k\|^2 = \left\| \sum_{j=1}^{n} \mathcal{M}(A)_{j,k} e_j \right\|^2 = \sum_{j=1}^{n} |\mathcal{M}(A)_{j,k}|^2 \leq \sum_{j=1}^{n} \frac{\epsilon^2}{n} = \epsilon^2$$

$$\begin{aligned}
\|A^m v\| = \|A^m(a_1 e_1 \cdots a_n e_n)\|^2 &= |a_1|^2 \|A^m e_1\|^2 + \cdots + |a_n|^2 \|A^m e_n\|^2 \\
&\leq |a_1|^2 \epsilon^2 + \cdots + |a_n|^2 \epsilon^2 = \epsilon^2(|a_1|^2 + \cdots + |a_n|^2) = \epsilon^2 \|v\|^2
\end{aligned}$$

Thus, $\|T^m v\| \leq \|A^m v\| \leq \epsilon \|v\|$ as desired. $\square$

22 Suppose $\mathcal{C}[-1,1]$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} fg$$

for all $f, g \in \mathcal{C}[-1,1]$. Let φ be the linear functional on $\mathcal{C}[-1,1]$ defined by $\varphi(f) = f(0)$. Show that there does not exist $g \in \mathcal{C}[-1,1]$ such that

$$\varphi(f) = \langle f, g\rangle$$

for every $f \in \mathcal{C}[-1,1]$.

Solution:

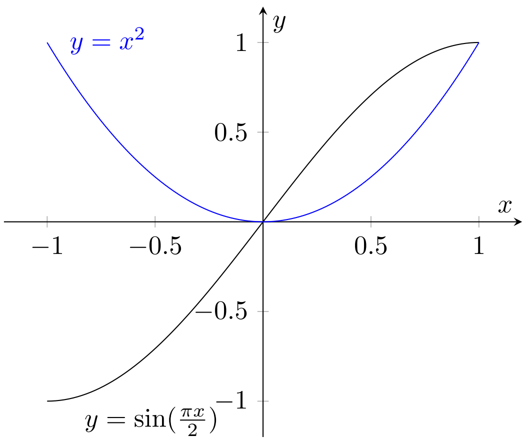

Figure 6.2: Illustration for *Problem 6B.22*

Take two continous functions defined on the interval $[-1,1]$: $f(x) = x^2$ and $h(x) = \sin(\pi x/2)$. The linear functional defined in the problem should give in both cases: $\varphi(f) = f(0) = 0$ and $\varphi(h) = h(0) = 0$. Suppose there exists $g \in \mathcal{C}[-1,1]$ such that it represents the given linear functional.Thus we need that both inner products equal zero:

$$\int_{-1}^{1} x^2 g(x)dx = 0$$
$$\int_{-1}^{1} \sin(\pi x/2)g(x)dx = 0$$

The only function that can make both of these integrals equal zero is $g(x) = 0$. But then such $g(x)$ won't give the correct value of functional of functions such as $p(x) = x^2 + 1$. Hence, there does not exist such $g(x)$. $\square$

6C Orthogonal Complements and Minimization Problems

1 Suppose $v_1, \dots, v_m \in V$. Prove that

$$\{v_1, \dots, v_m\}^\perp = (\operatorname{span}(v_1, \dots, v_m))^\perp$$

Solution:

Suppose $u \in \{v_1, \dots, v_m\}^\perp$. Examine the inner product of u with a vector from $\operatorname{span}(v_1, \dots, v_m)$:

$$\langle a_1 v_1 + \cdots + a_m v_m, u\rangle = a_1 \langle v_1, u\rangle + \cdots + a_m \langle v_m, u\rangle = 0$$

where the last equal sign is there because $u \in \{v_1, \dots, v_m\}^\perp$.

Hence $\{v_1, \dots, v_m\}^\perp \subseteq (\operatorname{span}(v_1, \dots, v_m))^\perp$.

Then, note that $\{v_1, \dots, v_m\} \subset \operatorname{span}(v_1, \dots, v_m)$. Therefore, by 6.48e, $(\operatorname{span}(v_1, \dots, v_m))^\perp \subseteq \{v_1, \dots, v_m\}^\perp$.

Thus we have shown inclusion on both sides, hence $\{v_1, \dots, v_m\}^\perp = (\operatorname{span}(v_1, \dots, v_m))^\perp$. $\square$

2 Suppose U is a subspace of V with basis $u_1, \dots, u_m$ and

$$u_1, \dots, u_m, v_1, \dots, v_n$$

is a basis of V. Prove that if the Gram-Schmidt procedure is applied to the basis of V above, producing a list $e_1, \dots, e_m, f_1, \dots, f_n$, then $e_1, \dots, e_m$ is an orthonormal basis of U and $f_1, \dots, f_n$ is an orthonormal basis of $U^\perp$.

Solution:

The Gram-Schmidt procedure does not change span of the successive lists of vectors. Thus, $\operatorname{span}(e_1, \dots e_m) = \operatorname{span}(u_1, \dots, u_m) = U$, so $e_1, \dots, e_m$ is an orthonormal basis of U.

As the basis produced by the Gram-Schmidt procedure is orthonormal, we can see that any $w = b_1 f_1 + \cdots b_n f_n$ is orthogonal to any vector in U:

$$\langle w, u\rangle = \langle b_1 f_1 + \cdots b_n f_n, a_1 e_1 + \cdots a_m e_m\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i \overline{a_j} \langle f_i, e_j\rangle = 0$$

So, any such vector w is in $U^\perp$.

V is finite-dimensional, as is U, with $\dim V = n + m$ and $\dim U = m$. Hence, by 6.51, $\dim U^\perp = n$. Note, that $f_1, \dots, f_n$ is linearly independent list with length of n and all its vector are in $U^\perp$, hence it must be a basis of $U^\perp$. This list is orthonormal, thus we have shown that $f_1, \dots, f_n$ is an orthonormal basis of $U^\perp$. $\square$

3 Suppose U is the subspace of $\mathbb{R}^4$ defined by

$$U = \operatorname{span}\left((1,2,3,-4),(-5,4,3,2)\right).$$

Find an orthonormal basis of U and an orthonormal basis of $U^{\perp}$.

Solution:

Firstly, we find an orthonormal basis of U. To do that, apply the Gram-Schmidt procedure to the given (linearly independent) list:

$$\|(1,2,3,-4)\| = \sqrt{1+2^2+3^3+(-4)^2} = \sqrt{30}$$

$$e_1 = (1,2,3,-4)/\|(1,2,3,-4)\| = (\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{3}{\sqrt{30}}, -\frac{4}{\sqrt{30}})$$

$$\begin{aligned}
u_2 &= (-5,4,3,2) - \langle(-5,4,3,2),e_1\rangle e_1 \\
\langle(-5,4,3,2),e_1\rangle &= -5\cdot\frac{1}{\sqrt{30}} + 4\cdot\frac{2}{\sqrt{30}} + 3\cdot\frac{3}{\sqrt{30}} - 2\cdot\frac{4}{\sqrt{30}} = \frac{4}{\sqrt{30}} \\
u_2 &= (-5,4,3,2) - \frac{4}{\sqrt{30}}(\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{3}{\sqrt{30}}, -\frac{4}{\sqrt{30}}) \\
&= (-5-\frac{4}{30}, 4-\frac{8}{30}, 3-\frac{12}{30}, 2+\frac{16}{30}) = \left(-\frac{77}{15}, \frac{56}{15}, \frac{13}{5}, \frac{38}{15}\right) \\
\|u_2\| &= \left[\left(-\frac{77}{15}\right)^2 + \left(\frac{56}{15}\right)^2 + \left(\frac{13}{5}\right)^2 + \left(\frac{38}{15}\right)^2\right]^{1/2} = \sqrt{\frac{802}{15}} = \frac{\sqrt{12030}}{15}
\end{aligned}$$

$$e_2 = u_2/\|u_2\| = \left(-\frac{77}{\sqrt{12030}}, \frac{56}{\sqrt{12030}}, \frac{39}{\sqrt{12030}}, \frac{38}{\sqrt{12030}}\right)$$

Secondly, we find an orthonormal basis of $U^{\perp}$. To do that, we extend the given list to a basis: $(1,2,3,-4),(-5,4,3,2),(1,0,0,0),(0,1,0,0)$. Then we continue to apply the Gram-Schmidt procedure in order to get orthonormal vectors f_1, f_2, which will be a basis of $U^{\perp}$.

$$\begin{aligned}
w_1 &= (1,0,0,0) - \langle(1,0,0,0),e_1\rangle e_1 - \langle(1,0,0,0),e_2\rangle e_2 \\
\langle(1,0,0,0),e_1\rangle &= \frac{1}{\sqrt{30}} \\
\langle(1,0,0,0),e_2\rangle &= -\frac{77}{\sqrt{12030}}
\end{aligned}$$

$$w_1 = \left(1 - \frac{1}{30} - \frac{5929}{12030}, -\frac{2}{30} + \frac{4312}{12030}, -\frac{3}{30} + \frac{3003}{12030}, \frac{4}{30} + \frac{2926}{123030}\right)$$
$$= \left(\frac{190}{401}, \frac{117}{401}, \frac{60}{401}, \frac{151}{401}\right)$$
$$\|w_1\| = \sqrt{\frac{190}{401}}$$

$$f_1 = \left(\sqrt{\frac{190}{401}}, \frac{117}{\sqrt{76190}}, \frac{60}{\sqrt{76190}}, \frac{151}{\sqrt{76190}}\right)$$

$$w_2 = (0,1,0,0) - \langle (0,1,0,0), e_1\rangle e_1 - \langle (0,1,0,0), e_2\rangle e_2 - \langle (0,1,0,0), f_1\rangle f_1$$
$$\langle (0,1,0,0), e_1\rangle = \frac{2}{\sqrt{30}}$$
$$\langle (0,1,0,0), e_2\rangle = \frac{56}{\sqrt{12030}}$$
$$\langle (0,1,0,0), f_1\rangle = \frac{117}{\sqrt{76190}}$$
$$w_2 = \left(-\frac{2}{30} + \frac{4312}{12030} - \frac{22230}{76190}, 1 - \frac{4}{30} - \frac{3136}{12030} - \frac{13689}{76190}, \ddots\right.$$
$$\left. -\frac{6}{30} - \frac{2184}{12030} - \frac{7020}{76190}, \frac{8}{30} - \frac{2128}{12030} - \frac{17669}{76190}\right)$$
$$= \left(0, \frac{81}{190}, -\frac{9}{19}, -\frac{27}{190}\right)$$
$$\|w_2\| = \frac{9\sqrt{190}}{190}$$

$$f_2 = \left(0, \frac{9\sqrt{190}}{190}, -\frac{\sqrt{190}}{19}, -\frac{3\sqrt{190}}{190}\right)$$

4 Suppose $e_1, \ldots, e_n$ is a list of vectors in V with $\|e_k\| = 1$ for each $k = 1, \ldots, n$ and

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$$

for all $v \in V$. Prove that $e_1, \ldots, e_n$ is an orthonormal basis of V.

Solution:

At the first step, we will show that $e_1, \ldots, e_n$ is an orthonormal list of vectors. Consider e_1. Its squared norm is: $\|e_1\|^2 = 1$. At the same time:

$$\|e_1\|^2 = |\langle e_1, e_1\rangle|^2 + |\langle e_1, e_2\rangle|^2 + \cdots + |\langle e_1, e_n\rangle|^2 = \|e_1\|^2 + |\langle e_1, e_2\rangle|^2 + \cdots + |\langle e_1, e_n\rangle|^2$$

Hence, $|\langle e_1, e_2\rangle|^2 + \cdots + |\langle e_1, e_n\rangle|^2 = 0$. As it is a sum of non-negative terms, we must conclude that $\langle e_1, e_k\rangle = 0$ for every $k \neq 1$.

The same logic can be applied to any e_j in the given list. Thus we have shown that the vectors in the list $e_1, \ldots, e_n$ are mutually orthogonal. As the norm of every vector in this list equals 1, it is the orthonormal list.

Now we will show that the list $e_1, \ldots, e_n$ spans the whole V. Let $U = \operatorname{span}(e_1, \ldots, e_n) \subseteq V$. $e_1, \ldots, e_n$ is a basis of U, because it is linearly independent list that spans U. Suppose $w \in V$ is such a vector that $w \in U^\perp$, then

$$\|w\| = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 = 0 + \cdots + 0 = 0$$

where we used the definition of orthogonal complement to write $\langle w, e_k\rangle = 0$. By the definiteness property of inner products, $w = 0$. Hence, $U^\perp = \{0\}$ and by 6.54, $U = V$.

Thus, $e_1, \ldots, e_n$ is an orthonormal basis of V. $\square$

5 Suppose that V is finite-dimensional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity operator on V.

Solution:

By 6.49, every $v \in V$ can be represented as:

$$v = u + w$$

where $u \in U$ and $w \in U^\perp$. Write w as:

$$w = v - u = Iv - P_U v = (I - P_U)v$$

So, we see that $P_{U^\perp} = I - P_U$. $\square$

6 Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that

$$T = TP_{(\operatorname{null} T)^\perp} = P_{\operatorname{range} T} T.$$

Solution:

First, note that $Tv \in \operatorname{range} T$, hence $P_{\operatorname{range} T} Tv = Tv$ for every $v \in V$. Thus we have shown that $T = P_{\operatorname{range} T} T$.

Then, note that because $\operatorname{null} T$ is a subspace of V, by 6.49 every $v \in V$ can be represented as:

$$v = u + w$$

where $u \in \operatorname{null} T$ and $w \in (\operatorname{null} T)^\perp$. Then apply T on v:

$$Tv = T(u + w) = Tu + Tw = 0 + Tw = Tw = T(P_{(\operatorname{null} T)^\perp} v) = TP_{(\operatorname{null} T)^\perp} v$$

for every $v \in V$. Thus, we have shown that $T = TP_{(\operatorname{null} T)^\perp}$. $\square$

7 Suppose that X and Y are finite-dimensional subspaces of V. Prove that $P_X P_Y = 0$ if and only if $\langle x, y\rangle = 0$ for all $x \in X$ and all $y \in Y$.

Solution:

$\longrightarrow$ Suppose $P_X P_Y = 0$. Consider a vector $v \in V$:

$$P_X P_Y v = 0$$

If $P_Y v = 0$ for all $v \in V$, then $Y = \{0\}$, and the conclusion $\langle x, y\rangle = 0$ follows immediately. Let $Y \neq \{0\}$. Then, $P_Y v$ can be represented as:

$$P_Y v = u + w$$

where $u \in X$, $w \in X^{\perp}$. Then

$$P_X(P_Y v) = P_X(u + w) = P_X u + P_X w = u + 0 = 0$$

Hence, $u = 0$ for any choice of $v \in V$. Thus, $P_Y v \in X^{\perp}$. At the same time $P_y v \in \operatorname{range} P_Y$ and $\operatorname{range} P_Y = Y$. Therefore, $Y \subseteq X^{\perp}$. The last statement means that $\langle x, y\rangle = 0$ for all $x \in X$ and all $y \in Y$.

$\longleftarrow$ Suppose $\langle x, y\rangle = 0$ for all x and y.

Then we can state that $Y \subseteq X^{\perp}$. As $Y = \operatorname{range} P_Y$, it immediately follows that $P_X P_Y v = P_X(P_Y v) = 0$ for any $v \in V$ (see property 6.57c). $\square$

8 Suppose U is a finite-dimensional subspace of V and $v \in V$. Define a linear functional $\varphi : U \to \mathbb{F}$ by

$$\varphi(u) = \langle u, v\rangle$$

for all $u \in U$. By the Riesz representation theorem as applied to the inner product space U, there exists a unique vector $w \in U$ such that

$$\varphi(u) = \langle u, w\rangle$$

for all $u \in U$. Show that $w = P_U v$.

Solution:

By 6.49, $V = U \oplus U^{\perp}$. Hence, we can uniquely write $v = w + \xi$, where $w \in U$ and $\xi \in U^{\perp}$. Then:

$$\varphi(u) = \langle u, v\rangle = \langle u, w + \xi\rangle = \langle u, w\rangle + \langle u, \xi\rangle = \langle u, w\rangle + 0 = \langle u, w\rangle \quad \square$$

9 Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$. Prove that there exists a subspace U of V such that $P = P_U$.

Solution:

Because every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$, $\operatorname{null} P$ is a subspace of $(\operatorname{range} P)^\perp$. By the Fundamental Theorem of linear maps: $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$. Hence $\operatorname{null} P = (\operatorname{range} P)^\perp$ and:

$$V = \operatorname{null} P \oplus \operatorname{range} P$$

Take $U = \operatorname{range} P$, then $U^\perp = \operatorname{null} P$. Suppose $u \in U$, then:

$$P^2 u = Pu \quad \text{and also} \quad P^2 u = P(Pu)$$

Thus, $P(Pu - u) = 0$, which means $(Pu - u) \in \operatorname{null} P = U^\perp$. At the same time, $u \in U$ and $Pu \in U$, which means $(Pu - u) \in U$. Therefore, $(Pu - u) \in U \cap U^\perp = \{0\}$. Thus we have shown, that

$$Pu = u$$

for all $u \in U$.

Now take arbitrary $v \in V$. It can be uniquely represented as $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then:

$$Pv = P(u + w) = Pu + Pw = Pu + 0 = Pu$$

where we used the fact that $U^\perp = \operatorname{null} P$.

Thus, U is the required subspace of V. $\square$

10 Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \le \|v\|$$

for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.

Solution:

By *Problem 3B.27*, $V = \operatorname{null} P \oplus \operatorname{range} P$.

Let $U = \operatorname{range} P$, $u \in U$ and $v \in V$ such that $u = Pv$. Using $P^2 = P$ we see that:

$$P^2 v = Pv \Rightarrow Pu = u$$

Now we will show that $\operatorname{null} P = U^\perp$. Take $u \in U$ and $w \in \operatorname{null} P$. Then write:

$$\|P(u + \lambda w)\| \le \|u + \lambda w\|$$
$$\|P(u + \lambda w)\| = \|Pu + \lambda Pw\| = \|Pu\| = \|u\|$$

Hence, $\|u\| \leq \|u + \lambda w\|$ for all $\lambda \in \mathbb{F}$. From *Problem 6A.6* , we deduce that $\langle u, w\rangle = 0$. As we took arbitrary u and w, it means $\operatorname{null} P = U^{\perp}$.

Hence, $U = \operatorname{range} P$ is the required subspace. □

11 Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Solution:

Suppose $u \in U$. Then we can write:

$$\begin{aligned} P_U T P_U u &= P_U T u \\ T P_U u &= T u \end{aligned}$$

U is invariant under T if and only if $Tu \in U$, and hence if and only if $P_U T u = Tu$. Thus we have shown that $P_U T P_U = T P_U$ for every $u \in U$.

Now suppose $w \in U^{\perp}$, then $P_U w = 0$ and:

$$\begin{aligned} P_U T P_U w &= P_U T(0) = 0 \\ T P_U w &= T(0) = 0 \end{aligned}$$

Thus U is invariant under T if and only if $P_U T P_U = T P_U$ for every $v \in V$. □

12 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that U and $U^{\perp}$ are both invariant under T if and only if $P_U T = T P_U$.

Solution:

$\longrightarrow$ Suppose U and $U^{\perp}$ are invariant under T. Take $u \in U$. Then, $Tu \in U$ and:

$$P_U T u = P_U(Tu) = Tu \quad \text{and} \quad T P_U u = Tu$$

so $P_U T = T P_U$ for all $u \in U$.

Then take $w \in U^{\perp}$. $U^{\perp}$ is invariant under T means that $Tw \in U^{\perp}$. Hence:

$$P_U T w = P_U(Tw) = 0 \quad \text{and} \quad T P_U w = T(0) = 0$$

so $P_U T = T P_U$ for all $w \in U^{\perp}$. As $V = U \oplus U^{\perp}$, any $v \in V$ can be uniquely written as $v = u + w$ with some $u \in U$ and $w \in U^{\perp}$. So we have:

$$P_U T v = P_U T(u+w) = P_U T u + P_U T w = T P_U u + T P_U w = T P_U(u+w) = T P_U v$$

Thus, $P_U T = T P_U$ for every $v \in V$.

$\longleftarrow$ Suppose $P_U T = T P_U$ for every $v \in V$.

Take $u \in U$. Notice, that $T P_U u = Tu$ and $T P_U u = P_U T u \in U$. So, $Tu \in U$ and hence U is invariant under T.

Take $w \in U^{\perp}$. $T P_U w = T(0) = 0$ and $T P_U w = P_U T w = 0$. Therefore, it must be that $Tw \in U^{\perp}$, so $U^{\perp}$ is also invariant under T. □

13 Suppose $\mathbb{F} = \mathbb{R}$ and V is finite-dimensional. For each $v \in V$, let φ_v denote the linear functional on V defined by

$$\varphi_v(u) = \langle u, v \rangle$$

for all $u \in V$.

(a) Show that $v \mapsto \varphi_v$ is an injective linear map from V to V'.

(b) Use (a) and a dimension-counting argument to show that $v \mapsto \varphi_v$ is an isomorphism from V onto V'.

Solution:

(a) We will show that a linear map $S : v \mapsto \varphi_v$ is injective by showing that its null space contains only 0. Suppose $w \in \operatorname{null} S$. Then $\langle u, w \rangle = 0$ for every $u \in V$. Take $u = \lambda w$, where $\lambda \neq 0$, then

$$\langle \lambda w, w \rangle = \lambda \langle w, w \rangle = 0$$

Therefore, by the definiteness property of inner products, $w = 0$. Hence, S is injective.

(b) By the Fundamental theorem of linear maps:

$$\dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$$

As $\operatorname{null} S = \{0\}$, we conclude that $\dim \operatorname{range} S = \dim V$.

Moreover, by 3.111 $\dim V' = \dim V$. Hence, S is also surjective and thus it is an invertible linear map from V to V', *i.e.* isomorphism. $\square$

14 Suppose that $e_1, \ldots, e_n$ is an orthonormal basis of V. Explain why the dual basis of $e_1, \ldots, e_n$ is $\varphi_{e_1}, \ldots, \varphi_{e_n}$ under the identification of V' with V profided by the Riesz representation theorem.

Solution:

From the notion of φ_v of the Riesz representation theorem, we can write:

$$\varphi_{e_j}(e_k) = \langle e_k, e_j \rangle$$

The inner product $\langle e_k, e_j \rangle$ equals 1 only if $j = k$, and otherwise equals 0. Thus, the list $\varphi_{e_1}, \ldots, \varphi_{e_n}$ satisfies the definition of a dual basis (3.112).

15 In $\mathbb{R}^4$, let

$$U = \operatorname{span}\left((1,1,0,0),(1,1,1,2)\right).$$

Find $u \in U$ such that $\|u - (1,2,3,4)\|$ is as small as possible.

Solution:

To find the desired u, we first apply the Gram-Schmidt procedure to the given spanning list of U.

$$e_1 = (1,1,0,0)/\|(1,1,0,0)\| = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$

$$\begin{aligned}
f_2 &= (1,1,1,2) - \langle (1,1,1,2), e_1\rangle e_1 \\
\langle (1,1,1,2), e_1\rangle &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \\
f_2 &= (1,1,1,2) - \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) = (0,0,1,2) \\
\|f_2\| &= \sqrt{1+2^2} = \sqrt{5}
\end{aligned}$$

$$e_2 = f_2/\|f_2\| = \left(0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

Let us use v for $(1,2,3,4)$. The desired u is:

$$u = P_U v = \langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2$$

The inner products are:

$$\begin{aligned}
\langle v, e_1\rangle &= \frac{1}{\sqrt{2}} + 2\cdot\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \\
\langle v, e_2\rangle &= 2\cdot\frac{1}{\sqrt{5}} + 4\cdot\frac{2}{\sqrt{5}} = 2\sqrt{5}
\end{aligned}$$

Hence:

$$u = \frac{3\sqrt{2}}{2}\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) + 2\sqrt{5}\cdot\left(0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \left(\frac{3}{2}, \frac{3}{2}, 2, 4\right)$$

16 Suppose $\mathcal{C}[-1,1]$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} fg$$

for all $f, g \in \mathcal{C}[-1,1]$. Let U be the subspace of $\mathcal{C}[-1,1]$ defined by

$$U = \{f \in \mathcal{C}[-1,1] : f(0) = 0\}.$$

(a) Show that $U^\perp = \{0\}$.

(b) Show that 6.49 and 6.52 do not hold without the finite-dimensional hypothesis.

Solution:

(a) See Fig. 6.2. Both $\sin x$ and x^2 are in U, while the only continuous function that is orthogonal to both of them is $g(x) = 0$. So, $U^\perp = \{0\}$.

(b) 6.49 states that $V = U \oplus U^\perp$. Here clearly $U \neq \mathcal{C}$, so $\mathcal{C}$ cannot be a direct sum of U and $\{0\}$.

6.52 states that $U = (U^\perp)^\perp$. In this case, $(U^\perp)^\perp = \mathcal{C} \neq U$.

17 Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and $\int_0^1 |2 + 3x - p(x)|^2 dx$ is as small as possible.

Solution:

A general polynomial in $\mathcal{P}_3(\mathbb{R})$ can be written as:

$$q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

The conditions $p(0) = 0$ and $p'(0) = 0$ mean that $a_0 = 0$ and $a_1 = 0$ for such polynomials. Thus, the polynomials of interested can be written as:

$$p(x) = ax^2 + bx^3$$

and they form a two-dimensional subspace of $\mathcal{P}_3(\mathbb{R})$. Given the integral in the problem, we will define the inner product on $\mathcal{P}_3(\mathbb{R})$ as:

$$\langle p, q \rangle = \int_0^1 pq$$

Apply the Gram-Schmidt procedure to this subspace.

$$f_1 = x^2$$

$$\|f_1\|^2 = \int_0^1 (x^2)^2 dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$$

$$e_1 = \sqrt{5}x^2$$

$$f_2 = x^3 - \langle x^3, e_1\rangle e_1$$
$$\langle x^3, e_1\rangle = \int_0^1 x^3 \cdot \sqrt{5}x^2 dx = \sqrt{5}\frac{x^6}{6}\Big|_0^1 = \frac{\sqrt{5}}{6}$$
$$f_2 = x^3 - \frac{\sqrt{5}}{6} \cdot \sqrt{5}x^2 = x^3 - \frac{5}{6}x^2$$
$$\|f_2\|^2 = \int_0^1 (x^3 - \frac{5}{6}x^2)^2 dx = \frac{1}{252}$$

$$e_2 = 6\sqrt{7}x^3 - 5\sqrt{7}x^2$$

The desired polynomial $p(x)$ is:

$$p(x) = \langle 2+3x, e_1\rangle e_1 + \langle 2+3x, e_2\rangle e_2$$

The inner products are:

$$\langle 2+3x, e_1\rangle = \int_0^1 (2+3x) \cdot \sqrt{5}x^2 dx = \frac{17\sqrt{5}}{12}$$
$$\langle 2+3x, e_2\rangle = \int_0^1 (2+3x)(6\sqrt{7}x^3 - 5\sqrt{7}x^2) dx = -\frac{29\sqrt{7}}{60}$$

Hence:

$$p(x) = \frac{17\sqrt{5}}{12} \cdot \sqrt{5}x^3 - \frac{29\sqrt{7}}{60}6\sqrt{7}x^3 + \frac{29\sqrt{7}}{60} \cdot 5\sqrt{7}x^2 = -\frac{793}{60}x^3 + \frac{1015}{60}x^2$$

18 Find $p \in \mathcal{P}_5(\mathbb{R})$ that makes $\int_{-\pi}^{\pi} |\sin x - p(x)|^2 dx$ as small as possible.

Solution:

Firstly, we will find an orthonormal basis of $\mathcal{P}_5(\mathbb{R})$ with the inner product given by:

$$\langle p, q\rangle = \int_{-\pi}^{\pi} pq$$

$$f_1 = 1, \qquad \|f_1\| = \sqrt{2\pi}$$
$$e_1 = f_1/\|f_1\| = 1/\sqrt{2\pi}$$

$$f_2 = x - \langle x, e_1 \rangle e_1$$
$$\langle x, e_1 \rangle = \int_{-\pi}^{\pi} \frac{x}{\sqrt{2\pi}} dx = 0$$
$$\|f_2\|^2 = \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3}$$
$$e_2 = f_2 / \|f_2\| = \sqrt{\frac{3}{2\pi^3}} x$$

$$f_3 = x^2 - \langle x^2, e_1 \rangle e_1 - \langle x^2, e_2 \rangle e_2$$
$$\langle x^2, e_1 \rangle = \int_{-\pi}^{\pi} \frac{x^2}{\sqrt{2\pi}} dx = \frac{\sqrt{2\pi^5}}{3}$$
$$\langle x^2, e_2 \rangle = \int_{-\pi}^{\pi} \sqrt{\frac{3}{2\pi^3}} x^3 dx = 0$$
$$f_3 = x^2 - \frac{\sqrt{2\pi^5}}{3} \cdot \frac{1}{\sqrt{2\pi}} = x^2 - \frac{\pi^2}{3}$$
$$\|f_3\|^2 = \int_{-\pi}^{\pi} \left(x^2 - \frac{\pi^2}{3} \right)^2 dx = \frac{8\pi^5}{45}$$
$$e_3 = f_3 / \|f_3\| = \frac{3\sqrt{5}}{\sqrt{8\pi^5}} \left(x^2 - \frac{\pi^2}{3} \right)$$

$$f_4 = x^3 - \langle x^3, e_1 \rangle e_1 - \langle x^3, e_2 \rangle e_2 - \langle x^3, e_3 \rangle e_3$$
$$\langle x^3, e_1 \rangle = \int_{-\pi}^{\pi} \frac{x^3}{\sqrt{2\pi}} dx = 0$$
$$\langle x^3, e_2 \rangle = \int_{-\pi}^{\pi} x^4 \sqrt{\frac{3}{2\pi^3}} dx = \frac{\sqrt{6\pi^7}}{5}$$
$$\langle x^3, e_3 \rangle = \int_{-\pi}^{\pi} x^3 \frac{3\sqrt{5}}{\sqrt{8\pi^5}} \left(x^2 - \frac{\pi^2}{3} \right) dx = 0$$
$$f_4 = x^3 - \frac{\sqrt{6\pi^7}}{5} \cdot \sqrt{\frac{3}{2\pi^3}} x = x^3 - \frac{3\pi^2}{5} x$$
$$\|f_4\|^2 = \int_{-\pi}^{\pi} \left(x^3 - \frac{3\pi^2}{5} x \right)^2 dx = \frac{8\pi^7}{175}$$
$$e_4 = f_4 / \|f_4\| = \frac{5\sqrt{7}}{2\sqrt{2\pi^7}} \left(x^3 - \frac{3\pi^2}{5} x \right)$$

$$f_5 = x^4 - \langle x^4, e_1\rangle e_1 - \langle x^4, e_2\rangle e_2 - \langle x^4, e_3\rangle e_3 - \langle x^4, e_4\rangle e_4$$
$$\langle x^4, e_1\rangle = \int_{-\pi}^{\pi} \frac{x^4}{\sqrt{2\pi}} dx = \frac{\sqrt{2\pi^9}}{5}$$
$$\langle x^4, e_2\rangle = \int_{-\pi}^{\pi} x^4 \sqrt{\frac{3}{2\pi^3}} x dx = 0$$
$$\langle x^4, e_3\rangle = \int_{-\pi}^{\pi} x^4 \frac{3\sqrt{5}}{\sqrt{8\pi^5}} \left(x^2 - \frac{\pi^2}{3}\right) dx = \frac{4\sqrt{2\pi^9}}{7\sqrt{5}}$$
$$\langle x^4, e_4\rangle = \int_{-\pi}^{\pi} x^4 \frac{5\sqrt{7}}{2\sqrt{2\pi^7}} \left(x^3 - \frac{3\pi^2}{5} x\right) dx = 0$$
$$f_5 = x^4 - \frac{\sqrt{2\pi^9}}{5} \cdot \frac{1}{\sqrt{2\pi}} - \frac{4\sqrt{2\pi^9}}{7\sqrt{5}} \cdot \frac{3\sqrt{5}}{\sqrt{8\pi^5}} \left(x^2 - \frac{\pi^2}{3}\right)$$
$$= x^4 - \frac{6\pi^2}{7} x^2 + \frac{3\pi^4}{35}$$
$$\|f_5\|^2 = \int_{-\pi}^{\pi} \left(x^4 - \frac{6\pi^2}{7} x^2 + \frac{3\pi^4}{35}\right)^2 dx = \frac{128\pi^9}{11025}$$

$$e_5 = f_5/\|f_5\| = \frac{105}{8\pi^4\sqrt{2\pi}} \left(x^4 - \frac{6\pi^2}{7} x^2 + \frac{3\pi^4}{35}\right)$$

$$f_6 = x^5 - \langle x^5, e_1\rangle e_1 - \langle x^5, e_2\rangle e_2 - \langle x^5, e_3\rangle e_3 - \langle x^5, e_4\rangle e_4 - \langle x^5, e_5\rangle e_5$$
$$\langle x^5, e_1\rangle = \int_{-\pi}^{\pi} \frac{x^5}{\sqrt{2\pi}} dx = 0$$
$$\langle x^5, e_2\rangle = \int_{-\pi}^{\pi} x^5 \sqrt{\frac{3}{2\pi^3}} x dx = \frac{\pi^5\sqrt{6\pi}}{7}$$
$$\langle x^5, e_3\rangle = \int_{-\pi}^{\pi} x^5 \frac{3\sqrt{5}}{\sqrt{8\pi^5}} \left(x^2 - \frac{\pi^2}{3}\right) dx = 0$$
$$\langle x^5, e_4\rangle = \int_{-\pi}^{\pi} x^5 \frac{5\sqrt{7}}{2\sqrt{2\pi^7}} \left(x^3 - \frac{3\pi^2}{5} x\right) dx = \frac{4\pi^5\sqrt{2\pi}}{9\sqrt{7}}$$
$$\langle x^5, e_5\rangle = \int_{-\pi}^{\pi} x^5 \frac{105}{8\pi^4\sqrt{2\pi}} \left(x^4 - \frac{6\pi^2}{7} x^2 + \frac{3\pi^4}{35}\right) dx = 0$$

$$
\begin{aligned}
f_6 &= x^5 - \frac{\pi^5\sqrt{6\pi}}{7} \cdot \sqrt{\frac{3}{2\pi^3}}x - \frac{4\pi^5\sqrt{2\pi}}{9\sqrt{7}} \cdot \frac{5\sqrt{7}}{2\sqrt{2\pi^7}}\left(x^3 - \frac{3\pi^2}{5}x\right) \\
&= x^5 - \frac{10\pi^2}{9}x^3 + \frac{5\pi^4}{21}x \\
\|f_6\|^2 &= \int_{-\pi}^{\pi}\left(x^5 - \frac{10\pi^2}{9}x^3 + \frac{5\pi^4}{21}x\right)^2 dx = \frac{128\pi^11}{43659}
\end{aligned}
$$

$$
e_6 = f_6/\|f_6\| = \frac{2\sqrt{11}}{16\pi^5\sqrt{2\pi}}\left(63x^5 - 70\pi^2x^3 + 15\pi^4x\right)
$$

The desired polynomial $p(x)$ is given by the orthogonal projection:

$$
\begin{aligned}
p(x) = \langle \sin x, e_1\rangle e_1 + \langle \sin x, e_2\rangle e_2 + \langle \sin x, e_3\rangle e_3 + \langle \sin x, e_4\rangle e_4 \\
+ \langle \sin x, e_5\rangle e_5 + \langle \sin x, e_6\rangle e_6
\end{aligned}
$$

We calculate the inner products:

$$
\begin{aligned}
\langle \sin x, e_1\rangle &= \int_{-\pi}^{\pi} \sin x \cdot \frac{1}{\sqrt{2\pi}}dx = 0 \\
\langle \sin x, e_2\rangle &= \int_{-\pi}^{\pi} \sin x \cdot \sqrt{\frac{3}{2\pi^3}}xdx = \sqrt{\frac{6}{\pi}} \\
\langle \sin x, e_3\rangle &= \int_{-\pi}^{\pi} \sin x \cdot \frac{3\sqrt{5}}{\sqrt{8\pi^5}}\left(x^2 - \frac{\pi^2}{3}\right)dx = 0 \\
\langle \sin x, e_4\rangle &= \int_{-\pi}^{\pi} \sin x \cdot \frac{5\sqrt{7}}{2\sqrt{2\pi^7}}\left(x^3 - \frac{3\pi^2}{5}x\right)dx = \sqrt{\frac{14}{\pi^5}}(\pi^2 - 15) \\
\langle \sin x, e_5\rangle &= \int_{-\pi}^{\pi} \sin x \cdot \frac{105}{8\pi^4\sqrt{2\pi}}\left(x^4 - \frac{6\pi^2}{7}x^2 + \frac{3\pi^4}{35}\right)dx = 0 \\
\langle \sin x, e_6\rangle &= \int_{-\pi}^{\pi} \sin x \cdot \frac{2\sqrt{11}}{16\pi^5\sqrt{2\pi}}\left(63x^5 - 70\pi^2x^3 + 15\pi^4x\right)dx \\
&= \sqrt{\frac{22}{\pi^9}}\left(\pi^4 - 105\pi^2 + 945\right)
\end{aligned}
$$

Then we get:

$$
\begin{aligned}
p(x) = \frac{693}{8\pi^{10}}(\pi^4 - 105\pi^2 + 945)x^5 - \frac{315}{4\pi^8}\left(\pi^4 - 125\pi^2 + 1155\right)x^3 \\
+ \frac{105}{8\pi^6}\left(\pi^4 - 153\pi^2 + 1485\right)x
\end{aligned}
$$

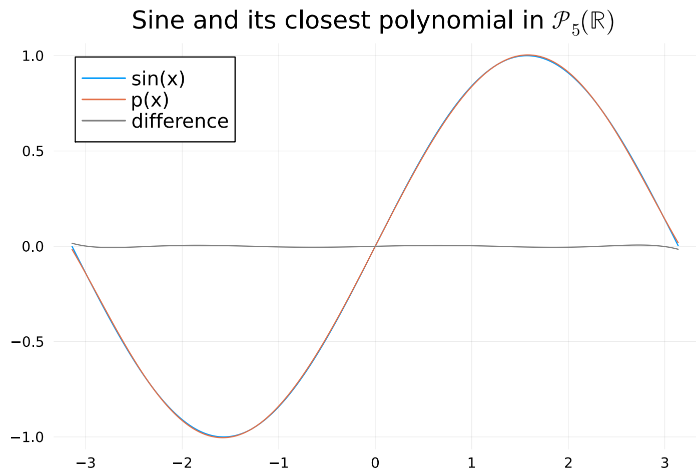

Figure 6.3: Illustration for *Problem 6C.18*. Maximal difference is ≈ 0.016.

19 Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is an orthogonal projection of V onto some subspace of V. Prove that $P^\dagger = P$.

Solution:

Let us denote a subspace, for which P is an orthogonal projection, as U. Suppose $v \in V$. We can write it as: $v = u + w$, where $u \in U$ and $w \in U^\perp$.

By the definition of a pseudoinverse, $P^\dagger v = (P|_{(\operatorname{null} P)^\perp})^{-1} P_{\operatorname{range} P} v$. Note that $\operatorname{range} P = U$ and $(\operatorname{null} P)^\perp = \operatorname{range} P = U$, so $P_{\operatorname{range} P} = P$ and $P|_{(\operatorname{null} P)^\perp} = P|_U$.

$$
\begin{aligned}
&P^\dagger v = P^\dagger(u + w) = P^\dagger u + P^\dagger w \\
&P^\dagger w = (P|_U)^{-1} P w = (P|_U)^{-1}(0) = 0 \\
&P^\dagger u = (P|_U)^{-1} P u = (P|_U)^{-1} u = u
\end{aligned}
$$

The last equality is due to the fact, that P sends vectors of U to themselves. Hence, we get

$$P^\dagger v = u + 0 = u$$

for all $v \in V$. So, $P^\dagger = P$. $\square$

20 Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that

$$\operatorname{null} T^\dagger = (\operatorname{range} T)^\perp \quad \text{and} \quad \operatorname{range} T^\dagger = (\operatorname{null} T)^\perp.$$

Solution:

$T|_{(\operatorname{null} T)^\perp}$ is invertible, so if $v \in \operatorname{null} T^\dagger$ and $v \neq 0$, then $v \in \operatorname{null} P_{\operatorname{range} T}$. So

$$\operatorname{null} T^\dagger = \operatorname{null} P_{\operatorname{range} T} = (\operatorname{range} P_{\operatorname{range} T})^\perp = (\operatorname{range} T)^\perp$$

From the properties of operators and definition of pseudoinverse it is clear, that $\operatorname{range} T^\dagger \subseteq \operatorname{range}(T|_{(\operatorname{null} T)^\perp})^{-1}$. At the same time, $(T|_{(\operatorname{null} T)^\perp})^{-1}$ is a linear map from $\operatorname{range} T$ to $(\operatorname{null} T)^\perp$. The $\operatorname{range} T$ is wholly covered by the $P_{\operatorname{range} T}$, hence we must conclude that $\operatorname{range} T^\dagger = \operatorname{range}(T|_{(\operatorname{null} T)^\perp})^{-1}$. The last of what we need is that $\operatorname{range}(T|_{(\operatorname{null} T)^\perp})^{-1} = (\operatorname{null} T)^\perp$. Indeed, $(T|_{(\operatorname{null} T)^\perp})^{-1}$ is an invertible map, so the desired conclusion follows immediately. Thus,

$$\operatorname{range} T^\dagger = (\operatorname{null} T)^\perp \quad \square$$

21 Suppose $T \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}^2)$ is defined by

$$T(a, b, c) = (a + b + c, 2b + 3c).$$

(a) For $(x, y) \in \mathbb{F}^2$, find a formula for $T^\dagger(x, y)$.

(b) Verify that the equation $TT^\dagger = P_{\operatorname{range} T}$ from 6.69(b) holds with the formula for $T^\dagger$ obtained in (a).

(c) Verify that the equation $T^\dagger T = P_{(\operatorname{null} T)^\perp}$ from 6.69(c) holds with the formula for $T^\dagger$ obtained in (a).

Solution:

(a) Note that $\operatorname{range} T = \mathbb{F}^2$ and $\operatorname{null} T = \{(a, b, c) \in \mathbb{F}^3 : a + b + c = 0 \text{ and } 2b + 3c = 0\}$. The list of one vector $(1, -3, 2)$ spans $\operatorname{null} T$, and we can take it as a basis of $\operatorname{null} T$.

Now suppose $(x, y) \in \mathbb{F}^2$. Then:

$$T^\dagger(x, y) = (T|_{(\operatorname{null} T)^\perp})^{-1} P_{\operatorname{range} T}(x, y) = (T|_{(\operatorname{null} T)^\perp})^{-1}(x, y)$$

The right side of the equation is a vector $(a, b, c) \in \mathbb{F}^3$ such that $T(a, b, c) = (x, y)$ and $(a, b, c) \in (\operatorname{null} T)^\perp$. In other words:

$$\begin{aligned} &a + b + c = x \\ &2b + 3c = y \\ &a - 3b + 2c = 0 \end{aligned}$$

The first two equations are equivalent to $T(a,b,c) = (x,y)$ and the third equation is the condition on orthogonality to $(1,-3,2)$. Solving this system of equations, we get:

$$a = \frac{1}{14}(11x - 5y); \quad b = \frac{1}{14}(3x + y); \quad c = \frac{1}{14}(-2x + 4y)$$

Hence

$$T^\dagger(x,y) = \frac{1}{14}(13x - 5y, 3x + y, -2x + 4y)$$

(b) Indeed:

$$\begin{aligned} TT^\dagger(x,y) &= T(\frac{1}{14}(11x - 5y, 3x + y, -2x + 4y)) \\ &= \frac{1}{14}(13x - 5y + 3x + y - 2x + 4y, 2(3x + y) + 3(-2x + 4y)) \\ &= \frac{1}{14}(14x, 14y) = (x,y) = P_{\mathrm{range}\, T}(x,y) \quad \checkmark \end{aligned}$$

(c) First, we will decompose (a,b,c) into $v \in \mathrm{null}\, T$ and $u \in (\mathrm{null}\, T)^\perp$.

$$\begin{aligned} v &= \frac{\langle (a,b,c), (1,-3,2)\rangle}{\|(1,-3,2)\|^2}(1,-3,2) \\ &\langle (a,b,c), (1,-3,2)\rangle = a - 3b + 2c \\ &\|(1,-3,2)\|^2 = 1 + 9 + 4 = 14 \\ v &= \frac{a - 3b + 2c}{14}(1,-3,2) \\ u &= (a,b,c) - v = \frac{1}{14}(13a + 3v - 2c, 3a + 5b + 6c, -2a + 6b - 10c) \end{aligned}$$

Hence $P_{(\mathrm{null}\, T)^\perp}(a,b,c) = \frac{1}{14}(13a + 3v - 2c, 3a + 5b + 6c, -2a + 6b - 10c)$. Now we calculate $T^\dagger T(a,b,c)$:

$$\begin{aligned} T^\dagger T(a,b,c) &= T^\dagger(a + b + c, 2b + 3c) \\ &= \frac{1}{14}\,(13(a + b + c) - 5(2b + 3c), 3(a + b + c) + 2b + 3c, \\ &\qquad -2(a + b + c) + 4(2b + 3c)) \\ &= \frac{1}{14}\,(13a + 3b - 2c, 3a + 5b + 6c, -2a + 6b + 10c) \\ &= P_{(\mathrm{null}\, T)^\perp}(a,b,c) \quad \checkmark \end{aligned}$$

22 Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$TT^\dagger T = T \quad \text{and} \quad T^\dagger T T^\dagger = T^\dagger.$$

Solution:

Property 6.69b tells that $TT^\dagger = P_{\operatorname{range} T}$. For any $v \in V$, $Tv \in \operatorname{range} T$, hence

$$TT^\dagger Tv = P_{\operatorname{range} T}(Tv) = Tv$$

which shows that $TT^\dagger T = T$.

Note that $\operatorname{range} T^\dagger = (\operatorname{null} T)^\perp$ (*Problem 6C.20*) and by property 6.69c, $T^\dagger T = P_{(\operatorname{null} T)^\perp}$. Then:

$$T^\dagger T T^\dagger w = P_{(\operatorname{null} T)^\perp}(T^\dagger w) = T^\dagger w$$

for all $w \in W$. Hence, $T^\dagger T T^\dagger = T^\dagger$. $\square$

23 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$(T^\dagger)^\dagger = T.$$

Solution:

First, we use definition of pseudoinverse to write:

$$(T^\dagger)^\dagger = (T^\dagger|_{(\operatorname{null} T^\dagger)^\perp})^{-1} P_{\operatorname{range} T^\dagger} = (T^\dagger|_{\operatorname{range} T})^{-1} P_{(\operatorname{null} T)^\perp}$$
$$T^\dagger|_{\operatorname{range} T} = \left((T|_{(\operatorname{null} T)^\perp})^{-1} P_{\operatorname{range} T}\right)|_{\operatorname{range} T} = (T|_{(\operatorname{null} T)^\perp})^{-1}(P_{\operatorname{range} T})|_{\operatorname{range} T}$$
$$(T^\dagger)^\dagger = (P_{\operatorname{range} T})|_{\operatorname{range} T} T|_{(\operatorname{null} T)^\perp} P_{(\operatorname{null} T)^\perp} = T|_{(\operatorname{null} T)^\perp} P_{(\operatorname{null} T)^\perp}$$

Suppose $v \in V$ and $v = u + w$ such that $u \in (\operatorname{null} T)^\perp$ and $w \in \operatorname{null} T$. Then:

$$\begin{aligned}(T^\dagger)^\dagger v = (T^\dagger)^\dagger u + (T^\dagger)^\dagger w &= T|_{(\operatorname{null} T)^\perp} P_{(\operatorname{null} T)^\perp} u + T|_{(\operatorname{null} T)^\perp} P_{(\operatorname{null} T)^\perp} w \\ &= T|_{(\operatorname{null} T)^\perp} P_{(\operatorname{null} T)^\perp} u = Tu = Tu + Tw = T(u + w) = Tv\end{aligned}$$

for all $v \in V$. Thus, $(T^\dagger)^\dagger = T$. $\square$