Chapter 5

Eigenvalues and Eigenvectors

Contents

5A Invariant Subspaces

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

Solution:

(a) Suppose $u \in U$, and because U is a subset of null-space of T, $u \in \operatorname{null} T$. $Tu = 0$ and $0 \in U$. Thus, U is invariant under T. $\square$

(b) Suppose $u \in U$. $Tu \in \operatorname{range} T$, and as $\operatorname{range} T$ is a subset of U, Tu must be an element of U, too. Hence, U is invariant under T. $\square$

2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

Solution:

Suppose $v_k \in V_k$ for every $k \in \{1, \ldots\}$. Each V_k is invariant under T, therefore $Tv_k \in V_k$. Then, for every $v \in V_1 + \cdots + V_m$, which can be written as a linear combination of vectors $v_1, \ldots, v_m$, we can write:

$$Tv = T(a_1 v_1 + \cdots a_m v_m) = a_1 T v_1 + \cdots a_m T v_m$$

So, Tv can be written as a linear combination of vectors from $V_1, \ldots, V_m$. Hence, $Tv \in V_1 + \cdots + V_m$, which means $V_1, \ldots, V_m$ is invariant under T. □

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Solution:

Let us denote subspaces of V invariant under T as U_i. Suppose u is a vector that belongs to the intersection of some collection of such subspaces, $u \in \bigcap_{i=1}^{m} U_i$. It means that $u \in U_i$ for every $i \in \{1, \ldots, m\}$.

Then, $Tu \in U_i$ for every $i \in \{1, \ldots, m\}$, or in other words $Tu \in \bigcap_{i=1}^{m} U_i$. That means, this intersection is invariant under T. This argument works for any collection of U_i, hence the intersection of every collection of subspaces of V invariant under T is invariant under T. □

4 Prove of give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Solution:

Suppose U is neither V, nor $\{0\}$. Let $u_1, \ldots, u_m$ be a basis of U, and $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V. Take some operator T, with its range being V, such that for every u_k:

$$Tu_k = A_{1,k} u_1 + \cdots A_{m,k} u_m + B_{1,k} v_1 + \cdots B_{n,k} v_n$$

with non-zero coefficients $B_{j,k}$. But if these coefficients are not zero, $Tu_k \notin U$, so U is not invariant under such T, which contradicts our initial assumption that U is invariant under every operator on V. Hence we conclude that U must be either $\{0\}$ or V. □

5 Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution:

Let λ be an eigenvalue of T with the eigenvector (x, y). Then:

$$T(x, y) = \lambda(x, y) = (-3y, x)$$

This is equivalent to a system of equations:

$$\begin{aligned} \lambda x &= -3y \\ \lambda y &= x \end{aligned}$$

We can express x from the second equation and insert it into the first.

$$\lambda \cdot \lambda y = -3y$$

Hence the eigenvalue must satisfy the equation $\lambda^2 = -3$. This equation has no real roots, hence the operator T has no eigenvalues.

6 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

Solution:

As in previous problem, we write a system of equations:

$$\begin{aligned} z &= \lambda w \\ w &= \lambda z \end{aligned}$$

Expressing w from the second equation and inserting it into the first gives:

$$z = \lambda^2 z \quad \Rightarrow \quad \lambda^2 = 1$$

Thus we have two eigenvalues:

1. $\lambda_1 = 1$ with eigenvectors of form $v_1 = t(1, 1)$, where $t \in \mathbb{R}$;
2. $\lambda_2 = -1$ with eigenvectors of form $v_1 = t(1, -1)$, where $t \in \mathbb{R}$.

7 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

Solution:

Once again we write a system of equation that is equivalent to a condition of (z_1, z_2, z_3) being an eigenvector:

$$\begin{aligned} 2z_2 &= \lambda z_1 \\ 0 &= \lambda z_2 \\ 5z_3 &= \lambda z_3 \end{aligned}$$

Let us examine the second equation: it tell that either $\lambda = 0$ or $z_2 = 0$.

Assume $\lambda = 0$. Then the third equation tells that $z_3 = 0$, and the first equation tells that $z_2 = 0$ and z_1 is arbitrary.

Now assume $z_2 = 0$ and $\lambda \neq 0$. Then the first equation tells that $z_1 = 0$ and the third equation tells that $\lambda = 5$ and z_3 is arbitrary.

Thus, there are two eigenvalues:

1. $\lambda_1 = 0$ with an eigenvectors of form $v_1 = t(1, 0, 0)$, where $t \in \mathbb{F}$;

2. $\lambda_2 = 5$ with an eigenvectors of form $v_2 = t(0, 0, 1)$, where $t \in \mathbb{F}$.

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

Solution:

Suppose λ is an eigenvalue of P with the corresponding eigenvector u. Then we can write:

$$Pv = \lambda v \quad \text{and} \quad Pv = P^2 v = P(\lambda v) = \lambda^2 v$$

So we have $(\lambda^2 v - \lambda v) = 0$ or $(\lambda^2 - \lambda)v = 0$. This equality can hold if either $v = 0$, or $(\lambda^2 - \lambda) = 0$. The first option is not the case as we supposed that v is an eigenvector. The second option gives the result that $\lambda = 0$ or $\lambda = 1$. $\square$

9 Define $T : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution:

Suppose λ is an eigenvalue of T with corresponding eigenvector p. Then:

$$Tp = \lambda p = p'$$

Write the polynomial p as:

$$p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Its derivative is:

$$p' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots n a_n x^{n-1}$$

Note that from inspection of x^n terms in $p' = \lambda p$ we can get a condition that $\lambda a_n = 0$. Then we do the same for x^{n-1} terms to get $\lambda a_{n-1} = n a_n$. And so on until $\lambda a_0 = a_1$.

Assume $\lambda \neq 0$, so from $\lambda a_n = 0$ we conclude that $a_n = 0$. Then from $\lambda a_{n-1} = n a_n$ we conclude that $a_{n-1} = 0$. And we thus continue until $a_0 = 0$.

Thus, $\lambda \neq 0$ means that $p = 0$, but we assumed that p is eigenvector so it cannot be the case.

Assume $\lambda = 0$. Then from $\lambda a_{n-1} = n a_n$ we see that $a_n = 0$. And thus we continue for every equation $\lambda a_{k-1} = k a_k$ until $\lambda a_0 = a_1$. The coefficient a_0 is here arbitrary, and $p = a_0$.

Hence, the eigenvalue of T is $\lambda = 0$ with eigenvectors of form $p = a_0$, where $a_0 \in \mathbb{R}$.

10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Solution:

Let λ be an eigenvalue of T with the corresponding eigenvector p. Let $p(x)$ has a form: $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$. Then:

$$(Tp)(x) = (\lambda p)(x) = \lambda(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)$$
$$(Tp)(x) = xp'(x) = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4$$

Thus the following equations must be satisfied:

$$\begin{aligned} \lambda a_0 &= 0, \\ \lambda a_1 &= a_1, \\ \lambda a_2 &= 2a_2, \\ \lambda a_3 &= 3a_3, \\ \lambda a_4 &= 4a_4. \end{aligned}$$

Suppose in the first equation $a_0 \neq 0$, then $\lambda = 0$ and all other coefficients of $p(x)$ are zero.

If $a_0 = 0$, then other coefficients can be non-zero. Suppose $a_1 \neq 0$, then from the second equation we conclude that $\lambda = 1$. Other equations can thus be satisfied only if $a_2 = a_3 = a_4 = 0$.

Similar reasoning can be applied to all subsequent equations. In the end we have five eigenvalues:

1. λ= 0 with eigenvectors $p(x) = a$, where $a_0 \in \mathbb{R}$;
2. λ= 1 with eigenvectors $p(x) = ax$, where $a \in \mathbb{R}$;
3. λ= 2 with eigenvectors $p(x) = ax^2$, where $a \in \mathbb{R}$;
4. λ= 3 with eigenvectors $p(x) = ax^3$, where $a \in \mathbb{R}$;
5. λ= 4 with eigenvectors $p(x) = ax^4$, where $a \in \mathbb{R}$.

11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbb{F}$. Prove that there exists $\delta \geq 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{F}$ such that $0 < |\alpha - \lambda| < \delta$.

Solution:

V is finite-dimensional, so by 5.12, there is a finite number of eigenvalues of T.

For a given α, pick the closest to it eigenvalue of T, μ. Then, choose δ such that $\delta = |\alpha - \mu|$. By construction, there is no other eigenvalue between α and μ, hence any λ such that $0 < |\alpha - \lambda| < \delta$ is not an eigenvalue of T, so $T - \lambda I$ is invertible. □

12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P.

Solution:

Every $v \in V$ can be written uniquely as $v = u + w$ where $u \in U$ and $w \in W$. Suppose some v is an eigenvector with eigenvalue λ. Then

$$Tv = \lambda v = \lambda u + \lambda w = T(u + w) = u$$

This equation can be satisfied if either $\lambda = 1$ and $w = 0$, or $\lambda = 0$ and $u = 0$.

Thus eigenvalues of P are:

1. $\lambda_1 = 1$ with eigenvectors $v_1 = u$, where $u \in U$;
2. $\lambda_2 = 0$ with eigenvectors $v_2 = w$, where $w \in W$.

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(\mathcal{V})$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Solution:

(a) Assume λ is an eigenvalue of T. That means operator $(T - \lambda I)$ is not invertible. Then note that:

$$\begin{aligned} T - \lambda I &= SS^{-1}T - \lambda SS^{-1} = S(S^{-1}T - \lambda S^{-1}) \\ &= S(S^{-1}TSS^{-1} - \lambda S^{-1}) \\ &= S(S^{-1}TS - \lambda I)S^{-1} \end{aligned}$$

As S is invertible, we conclude that $(S^{-1}TS - \lambda I)$ is not invertible. Hence, λ is also an eigenvalue of $S^{-1}TS$.

Now suppose μ is an eigenvalue of $S^{-1}TS$. Applying the same logic to non-invertible operator $(S^{-1}TS - \mu I)$, we get:

$$S^{-1}TS - \mu I = S^{-1}TS - \mu S^{-1}S = S^{-1}(TS - \mu S) = S^{-1}(T - \mu I)S$$

So $T - \mu I$ is not invertible, so μ is also an eigenvalue of T.

Thus we have shown that T and $S^{-1}TS$ have the same eigenvalues. $\square$

(b) If u is an eigenvector of $S^{-1}TS$, then the eigenvector of T with the same eigenvalue is Su.

14 Give and example of an operator on $\mathbb{R}^4$ that has no (real) eigenvalues.

Solution:

Let us define an operator $T \in \mathcal{L}(\mathbb{R}^4)$ as:

$$T(x_1, x_2, x_3, x_4) = (x_2, -2x_1, 3x_4, -4x_3).$$

Indeed, if λ were an eigenvalue of T, then the following system would have solution for at leat one non-zero x_i:

$$\begin{aligned} x_2 &= \lambda x_1 \\ -2x_1 &= \lambda x_2 \\ 3x_4 &= \lambda x_3 \\ -4x_3 &= \lambda x_4 \end{aligned}$$

It follows from the first two equations that $\lambda^2 = -2$ (if x_1 and x_2 are not zero). From the last two equations, it follows that $\lambda^2 = -12$ (if x_3 and x_4 are not zero). Thus $\lambda \notin \mathbb{R}$ and T is the desired operator. $\square$

15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Solution:

We conclude from propositions 3.129 and 3.131 that $S \in \mathcal{L}(V)$ is injective if and only if $S' \in \mathcal{L}(V')$ is injective. This property can be reformulated as: S is not injective if and only if S' is not injective.

Suppose λ is an eigenvalue of T. By 5.7, it is equivalent to $T - \lambda I$ being not injective. As stated above, $T - \lambda I$ is not injective if and only if $(T - \lambda I)'$

is not injective. Using properties of dual maps, we get:

$$(T - \lambda I)' = T' - \lambda I'$$

where I' is an identity operator on dual space. Hence, $T' - \lambda I'$ is not injective and λ is an eigenvalue of T'.

Thus, λ is an eigenvalue of T is and only if it is an eigenvalue of T'. $\square$

16 Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

Solution:

Let v be an eigenvector of T with eigenvalue λ. v can be written in the given basis as:

$$v = a_1 v_1 + \cdots + a_n v_n = \sum_{k=1}^{n} a_k v_k$$

Then we will act on it by the operator T:

$$Tv = T(\sum_{k=1}^{n} a_k v_k) = \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \mathcal{M}(T)_{j,k} v_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_k \mathcal{M}(T)_{j,k} \right) v_j$$

and also:

$$Tv = \lambda v = \sum_{j=1}^{n} \lambda a_j v_j$$

From these two equations we conclude that:

$$\lambda a_j = \sum_{k=1}^{n} a_k \mathcal{M}(T)_{j,k}$$

Take the largest coefficient a_j. Then:

$$\lambda = \sum_{k=1}^{n} \frac{a_k}{a_j} \mathcal{M}(T)_{j,k}$$

Then we examine the absolute value of λ:

$$|\lambda| = \left|\sum_{k=1}^{n} \frac{a_k}{a_j}\mathcal{M}(T)_{j,k}\right| \leq \sum_{k=1}^{n}\left|\frac{a_k}{a_j}\mathcal{M}(T)_{j,k}\right| \leq \sum_{k=1}^{n} |\mathcal{M}(T)_{j,k}| \leq n\max\{|\mathcal{M}(T)_{j,k}|\}$$

where the first inequality comes from properties of absolute value, second inequality from the fact that a_j is largest coefficient, so that $a_k/a_j \leq 1$, and in the third inequlity we replaced matrix elements with the largest matrix element.

Thus we have arrived at the desired inequality. $\square$

17 Suppose $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification $T_{\mathbb{C}}$.

Solution:

Let λ be an eigenvalue of T. That means $T - \lambda I$ is not injective. From *Problem 3B.33* we know that $(T - \lambda I)_{\mathbb{C}}$ is not injective if and only if $T - \lambda I$ is not injective. Notice that for any $u, v \in V$:

$$\begin{aligned}(T - \lambda I)_{\mathbb{C}}(u + iv) &= (T - \lambda I)u + i(T - \lambda I)v = (Tu + iTv) - \lambda(Iu + iIv) \\ &= T_{\mathbb{C}}(u + iv) - \lambda I_{\mathbb{C}}(u + iv) = (T_{\mathbb{C}} - \lambda I_{\mathbb{C}})(u + iv)\end{aligned}$$

So, $(T - \lambda I)_{\mathbb{C}} = T_{\mathbb{C}} - \lambda I_{\mathbb{C}}$ and thus $T_{\mathbb{C}} - \lambda I_{\mathbb{C}}$ is not injective, which means λ is an eigenvalue of $T_{\mathbb{C}}$. $\square$

18 Suppose $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.

Solution:

Suppose $\lambda = a + ib$ is an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $v + iu$. Then:

$$T_{\mathbb{C}}(v + iu) = \lambda(v + iu) = (av + bu) + i(bv + au) = T(v) + iT(u)$$

Thus, $T(v) = av + bu$ and $T(u) = bv + au$. Now examine the combination $\overline{\lambda}(v - iu)$:

$$\overline{\lambda}(v - iu) = (a - ib)(v - iu) = (av + bu) - i(bv + au) = Tu - iTv = T_{\mathbb{C}}(u - iv)$$

Thus, if λ is an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $u + iv$, then $\overline{\lambda}$ is also an eigenvalue of $T_{\mathbb{C}}$ but with eigenvector $u - iv$. Reverse statement is obtained if we change the roles of λ and $\overline{\lambda}$. $\square$

19 Show that the forward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

Solution:

Suppose λ is an eigenvalue of T. Then:

$$T(z_1, z_2, z_3, \ldots) = \lambda(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots)$$

So, $\lambda z_1 = 0$, $\lambda z_2 = z_1$, etc. If $z_1 \neq 0$, then from the first equation $\lambda = 0$. But it contradicts the second equation as $0 \cdot z_2$ cannot be equal to nonzero number like z_1. Thus we conclude that $z_1 = 0$, and then the second equation turns to $\lambda z_2 = 0$. Repeating the same argument, we arrive at $z_2 = 0$ and $\lambda z_3 = 0$. Continuing this leads to $\lambda z_k = 0$ for every $k \in \mathbb{N}$, which means that the supposed eigenvector is a zero-vector. By definition, 0 is not an eigenvector, hence T has no eigenvectors and no eigenvalues. $\square$

20 Define the backward shift operator $S \in \mathcal{L}(\mathbb{F}^\infty)$ defined by:

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$$

(a) Show that every element of $\mathbb{F}$ is an eigenvalue of S.

(b) Find all eigenvectors of S.

Solution:

Take some $\lambda \in \mathbb{F}$ and suppose it is an eigenvalue of S.

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots) = \lambda(z_1, z_2, z_3, \ldots)$$

Hence, $\lambda z_k = z_{k+1}$ for every $k \in \mathbb{N}$.

If $\lambda = 0$, then we can take $z_1 = 0$ and arbitrary z_2, z_3, etc. So, for $\lambda = 0$, eigenvectors are $(0, z_1, z_2, \ldots)$, where $z_k \in \mathbb{F}$.

If $\lambda \neq 0$, then we choose nonzero z_k such that $z_{k+1} = \lambda z_k$. So, for $\lambda \neq 0$, eigenvectors are $(1, \lambda, \lambda^2, \ldots)$.

Thus, every $\lambda \in \mathbb{F}$ is an eigenvalue. $\square$

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Solution:

Suppose λ is an eigenvalue of T with eigenvector v: $Tv = \lambda v$. As T is an invertible operator, we write:

$$T^{-1}(\lambda v) = T^{-1}Tv = v = \lambda T^{-1}v$$

Thus, we have $T^{-1}v = (1/\lambda)v$. This shows both required points: λ and $1/\lambda$ are eigenvalues of T and T^{-1} with the same eigenvector v. As $(T^{-1})^{-1} = T$, the argument works in the opposite direction too. $\square$

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u$$

Prove that 3 or -3 is an eigenvalue of T.

Solution:

Take a linear combination $u + w$. If $u + w \neq 0$, then

$$T(u + w) = Tu + Tw = 3w + 3u = 3(u + w)$$

Thus, 3 is an eigenvalue of T.

If $u + w = 0$, then take $u - w$, which in that case is nonzero. Then:

$$T(u - w) = Tu - Tw = 3w - 3u = -3(u - w)$$

Thus, -3 is an eigenvalue of T.

So we have shown that indeed 3 or -3 is an eigenvalue of T. $\square$

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Solution:

Assume λ is an eigenvalue of ST with eigenvector v: $STv = \lambda v$. It can be thought as $S(Tv) = \lambda v$. Now examine the following:

$$TS(Tv) = T(STv) = T(\lambda v) = \lambda Tv$$

Hence, Tv is an eigenvector of TS that has eigenvalue λ. Tv is nonzero, otherwise $S(Tv)$ must be zero, but it is not.

Similar argument (changing roles of S and T) gives that every eigenvalue of TS is also an eigenvalue of ST.

Thus, ST and TS has the same eigenvalues. $\square$

24 Suppose A is an n-by-n matrix with entries in $\mathbb{F}$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tx = Ax$, where elements of $\mathbb{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries of each column of A equals 1. Prove that 1 is an eigenvalue of T.

Solution:

(a) Take $x = (1, 1, \ldots, 1)^t$, i.e. column vector with all entries equal to 1. Then:

$$Ax = \begin{pmatrix} \sum_i^n A_{1,i}x_i \\ \sum_i^n A_{2,i}x_i \\ \vdots \\ \sum_i^n A_{n,i}x_i \end{pmatrix} = \begin{pmatrix} \sum_i^n A_{1,i} \\ \sum_i^n A_{2,i} \\ \vdots \\ \sum_i^n A_{n,i} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

where in the second equals sign we used that every $x_i = 1$ and in the third equals sign we used that the sum of entries in each row equals 1. Thus, x is an eigenvector of T with an eigenvalue 1. □

(b) Let T' be a dual map of T. Then, matrix of T' is a transpose of matrix of T (proposition 3.132), so $\mathcal{M}(T') = A^t$.

As sum of all entries in each *column* of A equals 1, the sum of all entries in each *row* of A^t therefore equals 1. We know from the part (a) of this problem that the operator corresponding to A^t (that is, T') has eigenvalue 1. And by *Problem 5A.15*, the operator T must also have this eigenvalue. □

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Solution:

Assume that u and w are eigenvectors with distinct eigenvalues λ and μ. Let κ be an eigenvalue of T corresponding to $u+w$. κ may be distinct from λ or μ or equal to one of them. Examine the expression $T(u+w) - T(u+w) = 0$:

$$\begin{aligned} T(u + w) - Tu - Tw &= 0 \\ \kappa(u + w) - \lambda u - \mu w &= 0 \\ (\kappa - \lambda)u + (\kappa - \mu)w &= 0 \end{aligned}$$

Thus, we have a linear combination of u and w that is equal to 0. Note, that $\kappa - \lambda$ and $\kappa - \mu$ cannot be equal to zero simultaneously, as $\lambda \neq \mu$.

Hence, u and w are linearly dependent. But we have assumed that these vectors correspond to different eigenvalues, so by Theorem 5.11, they must be linearly independent. That is a contradiction.

Thus, u and w are eigenvectors corresponding to the same eigenvalue. $\square$

26 Suppose $T \in \mathcal{L}$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Solution:

Take any nonzero $v, w \in V$. These two vectors are eigenvectors of T, and so is their linear combination $u + w$. By the result of the previous problem, v and w correspond to the same eigenvalue.

This argument applies to all vectors in V, hence we have $Tv = \lambda v$ for all $v \in V$. At the same time $\lambda I v = \lambda v$ for all $v \in V$. Thus $T = \lambda I$. $\square$

27 Suppose that V is finite-dimensional and $k \in \{1, \ldots, \dim V - 1\}$. Suppose $T \in \mathcal{L}$ is such that every subspace of V of dimension k is invariant under T. Prove that T is a scalar multiple of the identity operator.

Solution:

If $k = 1$, then every vector in V is an eigenvector. By the result of the previous problem, it means that T is a scalar multiple of the identity operator.

Suppose $k \geq 1$. Then take k distinct subspaces of V and construct their intersection. This intersection is either $\{0\}$ or a one-dimensional vector (sub)space. From *Problem 5A.3* we know that such intersection is also invariant under T. Taking arbitrary k-dimensional subspaces we can construct every one-dimensional subspace of V, thus returning to the $k = 1$ case. Hence T is a scalar multiple of the identity operator. $\square$

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

Solution:

$\operatorname{range} T$ is a subspace of V invariant under T. A maximum number of eigenvectors, that are elements of $\operatorname{range} T$, is $\dim \operatorname{range} T$ (5.12).

If $u \in V$ is an eigenvector of T, such that $u \notin \operatorname{range} T$, then the equality:

$$Tu = \lambda u$$

can be satisfied only if $\lambda = 0$. This value of λ is the corresponding eigenvalue.

Thus, there are at most $1 + \dim \operatorname{range} T$ distinct eigenvalues of T. $\square$

29 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and -4, 5 and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Solution:

We know three eigenvalues of T and the dimension of the vector space ($\mathbb{R}^3$) is 3, hence there is no other eigenvalue.

An operator $(T - 9I)$ is invertible, otherwise 9 would have been an eigenvalue of T, which it cannot be. Hence, there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (T - 9I)x = (-4, 5, \sqrt{7})$. $\square$

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution:

Take nonzero $v \in V$. If $(T-4I)v = 0$, then $Tv = 4v$, so v is an eigenvector and the eigenvalue (λ) is 4.

If $(T - 4I)v \neq 0$, then denote $w = (T - 4I)v$. If $(T - 3I)w = 0$, then $Tw = 3w$, so w is an eigenvector of $T$and $\lambda = 3$.

If $(T-3I)2 \neq 0$, then denote $u = (T-3I)w$. Then necessarily $(T-2I)u = 0$, hence $Tu = 2u$, so u is an eigenvector of $T$and $\lambda = 2$.

Thus we have shown that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. $\square$

31 Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -I$.

Solution:

Take $(1, 0), (0, 1)$ as a basis of $\mathbb{R}^2$. The desired operator T is "rotation by $\pi/4$" and it is represented by the matrix:

$$\mathcal{M}(T) = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}$$

Indeed:

$$\mathcal{M}(T^4) = (\mathcal{M}(T))^4 = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}^4$$

$$\begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}^2 = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right)^2 & -2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) \\ 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathcal{M}(-I)$$

Thus, $T^4 = -I$.

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

Solution:

Comment: Here we assume that the vector space is over real numbers. Otherwise, every operator would have an eigenvalue, as is proven later in Theorem 5.19.

Rewrite $T^4 = I$ as: $T^4 - I = 0$. We factorize this polynomial applied to an operator to get:

$$(T^2 + I)(T - I)(T + I) = 0$$

1 and -1 are not eigenvalues of T, so $(T - I)$ and $(T + I)$ are injective operators. That means $(T - I)v \neq 0$ and $(T + I)v \neq 0$ for every nonzero $v \in V$. Hence we conclude that $T^2 + I = 0$, or if rewrite, $T^2 = -I$. $\square$

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective is and only if T^m is surjective.

Solution:

(a) If T is injective, then T^m is injective as a composition of operators.

If T^m is injective, then we prove by contradiction. Suppose T is not injective and $v \neq 0$, $v \in T$. Then:

$$T^m v = T^{m-1}(Tv) = T^{m-1}(0) = 0$$

so T^m is also not injective, contrary to our initial assumption.

Hence, T is injective if and only if T^m is injective. $\square$

(b) If T is surjective, then T^m is surjective as a composition of operators.

If T^m is surjective, then we prove by contradiction. Suppose T is not surjective. Take $w \in V$ such that $w \notin \operatorname{range} T$. As T^m is surjective, there exists such $v \in V$ that $T^m v = w$. Then:

$$T^m v = T(T^{m-1}v) = w$$

so $w \in \operatorname{range} T$, contrary to our initial assumption.

Hence, T is surjective if and only if T^m is surjective. $\square$

34 Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Prove that the list $v_1, \ldots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \ldots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

Solution:

Implication from the 'necessary condition' is just Theorem 5.11. So will show only implication from the 'sufficient condition'.

Assume $v_1, \ldots, v_m$ is linearly independent list. Extend this list to the basis of V: $v_1, \ldots, v_m, u_1, \ldots, u_n$. Take an operator $T \in \mathcal{L}(V)$ such that

$$\begin{aligned} Tv_i &= \lambda_i v_i \\ Tu_j &= 0 \end{aligned}$$

for every $i \in \{1, \ldots, m\}$ and every $j \in \{1, \ldots, n\}$ with λ_i being distinct numbers in $\mathbb{F}$.

These values of Tv_i and Tu_j uniquely define T (by lemma 3.4). Note, that by construction, $v_1, \ldots, v_m$ are eigenvectors of T with distinct eigenvalues, hence the desired operator exists. $\square$

35 Suppose $\lambda_1, \ldots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Solution:

Take differentiation operator $D(f) = f'$. Note that for every $k \in \{1, \ldots, n\}$:

$$D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$$

We see that $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is a list of eigenvectors of D with distinct eigenvalues, hence it is linearly independent. $\square$

36 Suppose that $\lambda_1, \ldots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Solution:

Take operator $D^2(f) = f''$. Note that for every $k \in \{1, \ldots, n\}$:

$$D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$$

We see that $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ is a list of eigenvectors of D^2 with distinct eigenvalues, hence it is linearly independent. $\square$

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

Solution:

A number λ is an eigenvalue of $\mathcal{A}$ if and only if $(\mathcal{A}-\lambda\mathcal{I})$ is not invertible (here $\mathcal{I}$ is identity operator in $\mathcal{L}(\mathcal{L}(V))$).

Let $S \in \operatorname{null}(\mathcal{A}-\lambda\mathcal{I})$. It means:

$$\begin{aligned}(\mathcal{A}-\lambda\mathcal{I})S &= 0\\ \mathcal{A}(S)-\lambda\mathcal{I}(S) &= 0\\ TS-\lambda S &= 0\\ (T-\lambda I)S &= 0\end{aligned}$$

$S \neq 0$, hence for the last equality to hold, it must be that $\operatorname{null}(T-\lambda I) = \operatorname{range} S \neq \{0\}$. Hence, $(T-\lambda I)$ is not injective. Thus we see that λ is an eigenvalue of $\mathcal{A}$ is and only if λ is an eigenvalue of T. $\square$

38 Suppose V is finite-dimensional , $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v+U) = Tv+U$$

for each $v \in V$.

(a) Show that the definition of T/U makes sense (which requires using the condition that U is invariant under T) and show that T/U is an operator on V/U.

(b) Show that each eigenvalue of T/U is an eigenvalue of T.

Solution:

(a) By definition $v+U = \{v+u : u \in U\}$. So if we act on a linear combination $v+u$ by T we get:

$$T(v+u) = Tv+Tu$$

U is invariant under T: $Tu \in U$. So $(Tv+Tu) \in \{Tv+u : u \in U\}$ and the definition makes sense.

Let us check that T/U is a linear map.

Additivity: Suppose $v, w \in V$. Then:

$$\begin{aligned}(T/U)\left((v+U)+(w+U)\right) &= (T/U)(v+w+U) = T(v+w)+U\\ &= Tv+Tw+U = (Tv+U)+(Tw+U)\\ &= (T/U)(v+U)+(T/U)(w+U) \quad \checkmark\end{aligned}$$

Homogeneity: Suppose $v \in V$ and $\lambda \in \mathbb{F}$.

$$\begin{aligned}(T/U)\left(\lambda(v+U)\right) &= (T/U)(\lambda v + U) \\ &= T(\lambda v) + U = \lambda T v + U = \lambda(Tv + U) \\ &= \lambda(T/U)(v+U) \quad \checkmark\end{aligned}$$

(b) Suppose λ is an eigenvalue of (T/U) with eigenvector $v + U$.

$$\begin{aligned}(T/U)(v+U) &= Tv + U \\ &= \lambda v + U\end{aligned}$$

Hence $(Tv - \lambda v) \in U$ by lemma 3.101. Denote $u = Tv - \lambda v$, so $Tv = \lambda v + u$.

Take $w \in V$, then:

$$T(v+w) = Tv + Tw = \lambda v + u + Tw$$

We would like to find w such that $v+w$ is an eigenvector of T with eigenvalue λ. For that we need $u + Tw = \lambda w$. Rewriting it, we get:

$$(\lambda I - T)w = u$$

If $(\lambda I - T)$ is not invertible, then $(T - \lambda I)$ is not invertible and hence λ is an eigenvalue of T.

If $(\lambda I - T)$ is invertible, then:

$$w = (\lambda I - T)^{-1} u$$

Which is the sought vector and thus λ is an eigenvalue of T. $\square$

39 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.

Solution:

$\longrightarrow$ Assume T has an eigenvalue. We need the following identity (from Fundamental Theorem of linear maps:

$$\dim \operatorname{range}(T - \lambda I) = \dim V - \dim \operatorname{null}(T - \lambda I)$$

Note, that $T - \lambda I$ is a polynomial $p(z) = z - \lambda$ applied to T. By proposition 5.18, range $(T - \lambda I)$ is invariant under T.

There is at least eigenvector of T, hence $\dim \operatorname{null}(T - \lambda I) \geq 1$ and therefore $\dim \operatorname{range} T - \lambda I \leq \dim V - 1$.

If it is equality, then range $(T - \lambda I)$ is the desired subspace of V.

If it is less than $\dim V - 1$, then we extend a basis of range $(T - \lambda I)$ until we get $\dim V - 1$ vectors in the basis and thus a subspace (let us denote it W) of the desired dimension. W is invariant under $(T - \lambda I)$ by *Problem 5A.1b*. To show that W is also invariant under T, suppose $w_1, w_2 \in W$ are such that $(T - \lambda I)w_1 = w_2$. Then, rearranging the terms, we get:

$$Tw_1 = w_2 + \lambda w_1$$

$(w_2 + \lambda w_2) \in W$, hence $Tw_1 \in W$ and thus we have shown that W is a subspace of V invariant under T with dimension $\dim V - 1$, as desired.

$\longleftarrow$ Assume U is a subspace of V of dimension $\dim V - 1$ that is invariant under T. Examine the operator (T/U) (as in *Problem 5A.38*). It is an operator on V/U — a vector space with dimension (proposition 3.105):

$$\dim V/U = \dim V - \dim U = 1$$

By *Problem 3A.7*, the operator (T/U) is a scalar multiple of identity:

$$(T/U)(v + U) = \lambda(v + U) = \lambda v + U$$

Thus, by definition, λ is an eigenvalue of (T/U) and from *Problem 5A.38* we know that T has the same eigenvalues as (T/U) does. Thus, T has an eigenvalue. $\square$

40 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that:

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Solution:

$$p = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$$
$$p(STS^{-1}) = a_0 + a_1 STS^{-1} + a_2(STS^{-1})^2 + \cdots + a_n(STS^{-1})^n$$

Notice that:

$$(STS^{-1})^2 = STS^{-1}STS^{-1} = ST^2S^{-1}$$
$$(STS^{-1})^3 = STS^{-1}STS^{-1}STS^{-1} = ST^3S^{-1}$$

And so on. Hence:

$$\begin{aligned} p(STS^{-1}) &= a_0 + a_1 STS^{-1} + a_2 ST^2S^{-1} + \cdots + a_n ST^nS^{-1} \\ &= S(a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n)S^{-1} = Sp(T)S^{-1} \quad \square \end{aligned}$$

41 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$.

Solution:

Consider $p(T)u$ for any $u \in U$.

$$p(T)u = (a_0 + a_1T + \cdots a_nT^n)u = a_0u + a_1Tu + \cdots a_nT^nu$$

As U is invariant under T, any T^ku is in U, so as any scalar multiple of T^ku. Thus $p(T)u \in U$, which means U is invariant $p(T)$ for any $p \in \mathcal{P}(\mathbb{F})$. $\square$

42 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$.

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all subspaces of $\mathbb{F}^n$ that are invariant under T.

Solution:

(a) Eigenvalues are: 1, 2, ..., n. Corresponding eigenvectors are: a_1e_1, a_2e_2, ..., a_ne_n, where $a_1, \ldots, a_n \in \mathbb{F}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. Indeed:

$$T(\ldots, 0, x_k, 0, \ldots) = (\ldots, 0, kx_k, 0, \ldots) = k(\ldots, 0, x_k, 0, \ldots)$$

The dimension of $\mathbb{F}^n$ is n, so there are no more eigenvalues.

(b) Define $U_k = \operatorname{span}(e_k)$. Then the subspaces of $\mathbb{F}^n$ invariant under T are: $\{0\}$, every U_k and every direct sum of any combination of U_k's.

43 Suppose V is finite-dimensional, $\dim V > 1$ and $T \in \mathcal{L}(V)$. Prove that $\{p(T) : p \in \mathcal{P}(\mathbb{F})\} \neq \mathcal{L}(V)$.

Solution:

Denote a set of all $p(T)$ as W. Suppose $W = \mathcal{L}(V)$.

Note, that $Tp(T) = p(T)T$ for every $p(T) \in W$. Denote invertible polynomials of T as $q(T)$. For every such polynomial it is true that $q(T)T = Tq(T)$. And hence $T = q^{-1}(T)Tq(T)$. Examinig the matrix representation of the last equality, we see that

$$\mathcal{M}(T) = \mathcal{M}(q^{-1}Tq) = \mathcal{M}(q(T))^{-1}\mathcal{M}(T)\mathcal{M}(q(T))$$

for every $q(T)$. We supposed that polynomials of T can represent every linear operator on V, hence every invertible polynomial of T represent every invertible linear operator on V. That means the the obtained equality is equivalent to a proposition that matrix representation of T is the same in every basis of V. Thus T is a scalar multiple of identity, by *Problem 3D.19*.

But in the formulation of a problem we didn't restrict the choice of T and for every V with $\dim > 1$, not every T is a scalar multiple of identity. Thus $W \neq \mathcal{L}(V)$. $\square$