Chapter 5

Eigenvalues and Eigenvectors

Contents

5A Invariant Subspaces

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

Solution:

(a) Suppose $u \in U$, and because U is a subset of null-space of T, $u \in \operatorname{null} T$. $Tu = 0$ and $0 \in U$. Thus, U is invariant under T. $\square$

(b) Suppose $u \in U$. $Tu \in \operatorname{range} T$, and as $\operatorname{range} T$ is a subset of U, Tu must be an element of U, too. Hence, U is invariant under T. $\square$

2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

Solution:

Suppose $v_k \in V_k$ for every $k \in \{1, \ldots\}$. Each V_k is invariant under T, therefore $Tv_k \in V_k$. Then, for every $v \in V_1 + \cdots + V_m$, which can be written as a linear combination of vectors $v_1, \ldots, v_m$, we can write:

$$Tv = T(a_1v_1 + \cdots a_mv_m) = a_1Tv_1 + \cdots a_mTv_m$$

So, Tv can be written as a linear combination of vectors from $V_1, \ldots, V_m$. Hence, $Tv \in V_1 + \cdots + V_m$, which means $V_1, \ldots, V_m$ is invariant under T. $\square$

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Solution:

Let us denote subspaces of V invariant under T as U_i. Suppose u is a vector that belongs to the intersection of some collection of such subspaces, $u \in \bigcap_{i=1}^{m} U_i$. It means that $u \in U_i$ for every $i \in \{1, \ldots, m\}$.

Then, $Tu \in U_i$ for every $i \in \{1, \ldots, m\}$, or in other words $Tu \in \bigcap_{i=1}^{m} U_i$. That means, this intersection is invariant under T. This argument works for any collection of U_i, hence the intersection of every collection of subspaces of V invariant under T is invariant under T. $\square$

4 Prove of give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Solution:

Suppose U is neither V, nor $\{0\}$. Let $u_1, \ldots, u_m$ be a basis of U, and $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V. Take some operator T, with its range being V, such that for every u_k:

$$Tu_k = A_{1,k}u_1 + \cdots A_{m,k}u_m + B_{1,k}v_1 + \cdots B_{n,k}v_n$$

with non-zero coefficients $B_{j,k}$. But if these coefficients are not zero, $Tu_k \notin U$, so U is not invariant under such T, which contradicts our initial assumption that U is invariant under every operator on V. Hence we conclude that U must be either $\{0\}$ or V. $\square$

5 Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution:

Let λ be an eigenvalue of T with the eigenvector (x, y). Then:

$$T(x, y) = \lambda(x, y) = (-3y, x)$$

This is equivalent to a system of equations:

$$\lambda x = -3y$$
$$\lambda y = x$$

We can express x from the second equation and insert it into the first.

$$\lambda \cdot \lambda y = -3y$$

Hence the eigenvalue must satisfy the equation $\lambda^2 = -3$. This equation has no real roots, hence the operator T has no eigenvalues.

6 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

Solution:

As in previous problem, we write a system of equations:

$$z = \lambda w$$
$$w = \lambda z$$

Expressing w from the second equation and inserting it into the first gives:

$$z = \lambda^2 z \quad \Rightarrow \quad \lambda^2 = 1$$

Thus we have two eigenvalues:

1. $\lambda_1 = 1$ with eigenvectors of form $v_1 = t(1, 1)$, where $t \in \mathbb{R}$;
2. $\lambda_2 = -1$ with eigenvectors of form $v_1 = t(1, -1)$, where $t \in \mathbb{R}$.

7 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

Solution:

Once again we write a system of equation that is equivalent to a condition of (z_1, z_2, z_3) being an eigenvector:

$$2z_2 = \lambda z_1$$
$$0 = \lambda z_2$$
$$5z_3 = \lambda z_3$$

Let us examine the second equation: it tell that either $\lambda = 0$ or $z_2 = 0$.

Assume $\lambda = 0$. Then the third equation tells that $z_3 = 0$, and the first equation tells that $z_2 = 0$ and z_1 is arbitrary.

Now assume $z_2 = 0$ and $\lambda \neq 0$. Then the first equation tells that $z_1 = 0$ and the third equation tells that $\lambda = 5$ and z_3 is arbitrary.

Thus, there are two eigenvalues:

1. $\lambda_1 = 0$ with an eigenvectors of form $v_1 = t(1, 0, 0)$, where $t \in \mathbb{F}$;

2. $\lambda_2 = 5$ with an eigenvectors of form $v_2 = t(0, 0, 1)$, where $t \in \mathbb{F}$.

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

Solution:

Suppose λ is an eigenvalue of P with the corresponding eigenvector u. Then we can write:

$$Pv = \lambda v \quad \text{and} \quad Pv = P^2 v = P(\lambda v) = \lambda^2 v$$

So we have $(\lambda^2 v - \lambda v) = 0$ or $(\lambda^2 - \lambda)v = 0$. This equality can hold if either $v = 0$, or $(\lambda^2 - \lambda) = 0$. The first option is not the case as we supposed that v is an eigenvector. The second option gives the result that $\lambda = 0$ or $\lambda = 1$. $\square$

9 Define $T : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution:

Suppose λ is an eigenvalue of T with corresponding eigenvector p. Then:

$$Tp = \lambda p = p'$$

Write the polynomial p as:

$$p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Its derivative is:

$$p' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots n a_n x^{n-1}$$

Note that from inspection of x^n terms in $p' = \lambda p$ we can get a condition that $\lambda a_n = 0$. Then we do the same for x^{n-1} terms to get $\lambda a_{n-1} = n a_n$. And so on until $\lambda a_0 = a_1$.

Assume $\lambda \neq 0$, so from $\lambda a_n = 0$ we conclude that $a_n = 0$. Then from $\lambda a_{n-1} = n a_n$ we conclude that $a_{n-1} = 0$. And we thus continue until $a_0 = 0$.

Thus, $\lambda \neq 0$ means that $p = 0$, but we assumed that p is eigenvector so it cannot be the case.

Assume $\lambda = 0$. Then from $\lambda a_{n-1} = na_n$ we see that $a_n = 0$. And thus we continue for every equation $\lambda a_{k-1} = ka_k$ until $\lambda a_0 = a_1$. The coefficient a_0 is here arbitrary, and $p = a_0$.

Hence, the eigenvalue of T is $\lambda = 0$ with eigenvectors of form $p = a_0$, where $a_0 \in \mathbb{R}$.

10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Solution:

Let λ be an eigenvalue of T with the corresponding eigenvector p. Let $p(x)$ has a form: $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. Then:

$$(Tp)(x) = (\lambda p)(x) = \lambda(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)$$
$$(Tp)(x) = xp'(x) = a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4$$

Thus the following equations must be satisfied:

$$\begin{aligned}
\lambda a_0 &= 0, \\
\lambda a_1 &= a_1, \\
\lambda a_2 &= 2a_2, \\
\lambda a_3 &= 3a_3, \\
\lambda a_4 &= 4a_4.
\end{aligned}$$

Suppose in the first equation $a_0 \neq 0$, then $\lambda = 0$ and all other coefficients of $p(x)$ are zero.

If $a_0 = 0$, then other coefficients can be non-zero. Suppose $a_1 \neq 0$, then from the second equation we conclude that $\lambda = 1$. Other equations can thus be satisfied only if $a_2 = a_3 = a_4 = 0$.

Similar reasoning can be applied to all subsequent equations. In the end we have five eigenvalues:

1. $\lambda = 0$ with eigenvectors $p(x) = a$, where $a_0 \in \mathbb{R}$;
2. $\lambda = 1$ with eigenvectors $p(x) = ax$, where $a \in \mathbb{R}$;
3. $\lambda = 2$ with eigenvectors $p(x) = ax^2$, where $a \in \mathbb{R}$;
4. $\lambda = 3$ with eigenvectors $p(x) = ax^3$, where $a \in \mathbb{R}$;
5. $\lambda = 4$ with eigenvectors $p(x) = ax^4$, where $a \in \mathbb{R}$.

11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbb{F}$. Prove that there exists $\delta \geq 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{F}$ such that $0 < |\alpha - \lambda| < \delta$.

Solution:

V is finite-dimensional, so by 5.12, there is a finite number of eigenvalues of T.

For a given α, pick the closest to it eigenvalue of T, μ. Then, choose δ such that $\delta = |\alpha - \mu|$. By construction, there is no other eigenvalue between α and μ, hence any λ such that $0 < |\alpha - \lambda| < \delta$ is not an eigenvalue of T, so $T - \lambda I$ is invertible. $\square$

12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P.

Solution:

Every $v \in V$ can be written uniquely as $v = u + w$ where $u \in U$ and $w \in W$. Suppose some v is an eigenvector with eigenvalue λ. Then

$$Tv = \lambda v = \lambda u + \lambda w = T(u + w) = u$$

This equation can be satisfied if either $\lambda = 1$ and $w = 0$, or $\lambda = 0$ and $u = 0$.

Thus eigenvalues of P are:

1. $\lambda_1 = 1$ with eigenvectors $v_1 = u$, where $u \in U$;
2. $\lambda_2 = 0$ with eigenvectors $v_2 = w$, where $w \in W$.

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(\mathcal{V})$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Solution:

(a) Assume λ is an eigenvalue of T. That means operator $(T - \lambda I)$ is not invertible. Then note that:

$$\begin{aligned} T - \lambda I &= SS^{-1}T - \lambda SS^{-1} = S(S^{-1}T - \lambda S^{-1}) \\ &= S(S^{-1}TSS^{-1} - \lambda S^{-1}) \\ &= S(S^{-1}TS - \lambda I)S^{-1} \end{aligned}$$

As S is invertible, we conclude that $(S^{-1}TS - \lambda I)$ is not invertible. Hence, λ is also an eigenvalue of $S^{-1}TS$.

Now suppose μ is an eigenvalue of $S^{-1}TS$. Applying the same logic to non-invertible operator $(S^{-1}TS - \mu I)$, we get:

$$S^{-1}TS - \mu I = S^{-1}TS - \mu S^{-1}S = S^{-1}(TS - \mu S) = S^{-1}(T - \mu I)S$$

So $T - \mu I$ is not invertible, so μ is also an eigenvalue of T.

Thus we have shown that T and $S^{-1}TS$ have the same eigenvalues. $\square$

(b) If u is an eigenvector of $S^{-1}TS$, then the eigenvector of T with the same eigenvalue is Su.

14 Give and example of an operator on $\mathbb{R}^4$ that has no (real) eigenvalues.

Solution:

Let us define an operator $T \in \mathcal{L}(\mathbb{R}^4)$ as:

$$T(x_1, x_2, x_3, x_4) = (x_2, -2x_1, 3x_4, -4x_3).$$

Indeed, if λ were an eigenvalue of T, then the following system would have solution for at leat one non-zero x_i:

$$\begin{aligned} x_2 &= \lambda x_1 \\ -2x_1 &= \lambda x_2 \\ 3x_4 &= \lambda x_3 \\ -4x_3 &= \lambda x_4 \end{aligned}$$

It follows from the first two equations that $\lambda^2 = -2$ (if x_1 and x_2 are not zero). From the last two equations, it follows that $\lambda^2 = -12$ (if x_3 and x_4 are not zero). Thus $\lambda \notin \mathbb{R}$ and T is the desired operator. $\square$

15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Solution:

We conclude from propositions 3.129 and 3.131 that $S \in \mathcal{L}(V)$ is injective if and only if $S' \in \mathcal{L}(V')$ is injective. This property can be reformulated as: S is not injective if and only if S' is not injective.

Suppose λ is an eigenvalue of T. By 5.7, it is equivalent to $T - \lambda I$ being not injective. As stated above, $T - \lambda I$ is not injective if and only if $(T - \lambda I)'$

is not injective. Using properties of dual maps, we get:

$$(T - \lambda I)' = T' - \lambda I'$$

where I' is an identity operator on dual space. Hence, $T' - \lambda I'$ is not injective and λ is an eigenvalue of T'.

Thus, λ is an eigenvalue of T is and only if it is an eigenvalue of T'. □

16 Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

Solution:

Let v be an eigenvector of T with eigenvalue λ. v can be written in the given basis as:

$$v = a_1 v_1 + \cdots + a_n v_n = \sum_{k=1}^{n} a_k v_k$$

Then we will act on it by the operator T:

$$Tv = T(\sum_{k=1}^{n} a_k v_k) = \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \mathcal{M}(T)_{j,k} v_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_k \mathcal{M}(T)_{j,k} \right) v_j$$

and also:

$$Tv = \lambda v = \sum_{j=1}^{n} \lambda a_j v_j$$

From these two equations we conclude that:

$$\lambda a_j = \sum_{k=1}^{n} a_k \mathcal{M}(T)_{j,k}$$

Take the largest coefficient a_j. Then:

$$\lambda = \sum_{k=1}^{n} \frac{a_k}{a_j} \mathcal{M}(T)_{j,k}$$

Then we examine the absolute value of λ:

$$|\lambda| = \left|\sum_{k=1}^{n} \frac{a_k}{a_j}\mathcal{M}(T)_{j,k}\right| \leq \sum_{k=1}^{n}\left|\frac{a_k}{a_j}\mathcal{M}(T)_{j,k}\right| \leq \sum_{k=1}^{n}|\mathcal{M}(T)_{j,k}| \leq n\max\{|\mathcal{M}(T)_{j,k}|\}$$

where the first inequality comes from properties of absolute value, second inequality from the fact that a_j is largest coefficient, so that $a_k/a_j \leq 1$, and in the third inequlity we replaced matrix elements with the largest matrix element.

Thus we have arrived at the desired inequality. $\square$

17 Suppose $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification $T_{\mathbb{C}}$.

Solution:

Let λ be an eigenvalue of T. That means $T - \lambda I$ is not injective. From *Problem 3B.33* we know that $(T - \lambda I)_{\mathbb{C}}$ is not injective if and only if $T - \lambda I$ is not injective. Notice that for any $u, v \in V$:

$$\begin{aligned}(T - \lambda I)_{\mathbb{C}}(u + iv) &= (T - \lambda I)u + i(T - \lambda I)v = (Tu + iTv) - \lambda(Iu + iIv) \\ &= T_{\mathbb{C}}(u + iv) - \lambda I_{\mathbb{C}}(u + iv) = (T_{\mathbb{C}} - \lambda I_{\mathbb{C}})(u + iv)\end{aligned}$$

So, $(T - \lambda I)_{\mathbb{C}} = T_{\mathbb{C}} - \lambda I_{\mathbb{C}}$ and thus $T_{\mathbb{C}} - \lambda I_{\mathbb{C}}$ is not injective, which means λ is an eigenvalue of $T_{\mathbb{C}}$. $\square$

18 Suppose $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.

Solution:

Suppose $\lambda = a + ib$ is an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $v + iu$. Then:

$$T_{\mathbb{C}}(v + iu) = \lambda(v + iu) = (av + bu) + i(bv + au) = T(v) + iT(u)$$

Thus, $T(v) = av + bu$ and $T(u) = bv + au$. Now examine the combination $\overline{\lambda}(v - iu)$:

$$\overline{\lambda}(v - iu) = (a - ib)(v - iu) = (av + bu) - i(bv + au) = Tu - iTv = T_{\mathbb{C}}(u - iv)$$

Thus, if λ is an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $u + iv$, then $\overline{\lambda}$ is also an eigenvalue of $T_{\mathbb{C}}$ but with eigenvector $u - iv$. Reverse statement is obtained if we change the roles of λ and $\overline{\lambda}$. $\square$

19 Show that the forward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

Solution:

Suppose λ is an eigenvalue of T. Then:

$$T(z_1, z_2, z_3, \ldots) = \lambda(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots)$$

So, $\lambda z_1 = 0$, $\lambda z_2 = z_1$, etc. If $z_1 \neq 0$, then from the first equation $\lambda = 0$. But it contradicts the second equation as $0 \cdot z_2$ cannot be equal to nonzero number like z_1. Thus we conclude that $z_1 = 0$, and then the second equation turns to $\lambda z_2 = 0$. Repeating the same argument, we arrive at $z_2 = 0$ and $\lambda z_3 = 0$. Continuing this leads to $\lambda z_k = 0$ for every $k \in \mathbb{N}$, which means that the supposed eigenvector is a zero-vector. By definition, 0 is not an eigenvector, hence T has no eigenvectors and no eigenvalues. $\square$

20 Define the backward shift operator $S \in \mathcal{L}(\mathbb{F}^\infty)$ defined by:

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$$

(a) Show that every element of $\mathbb{F}$ is an eigenvalue of S.

(b) Find all eigenvectors of S.

Solution:

Take some $\lambda \in \mathbb{F}$ and suppose it is an eigenvalue of S.

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots) = \lambda(z_1, z_2, z_3, \ldots)$$

Hence, $\lambda z_k = z_{k+1}$ for every $k \in \mathbb{N}$.

If $\lambda = 0$, then we can take $z_1 = 0$ and arbitrary z_2, z_3, etc. So, for $\lambda = 0$, eigenvectors are $(0, z_1, z_2, \ldots)$, where $z_k \in \mathbb{F}$.

If $\lambda \neq 0$, then we choose nonzero z_k such that $z_{k+1} = \lambda z_k$. So, for $\lambda \neq 0$, eigenvectors are $(1, \lambda, \lambda^2, \ldots)$.

Thus, every $\lambda \in \mathbb{F}$ is an eigenvalue. $\square$

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Solution:

Suppose λ is an eigenvalue of T with eigenvector v: $Tv = \lambda v$. As T is an invertible operator, we write:

$$T^{-1}(\lambda v) = T^{-1}Tv = v = \lambda T^{-1}v$$

Thus, we have $T^{-1}v = (1/\lambda)v$. This shows both required points: λ and $1/\lambda$ are eigenvalues of T and T^{-1} with the same eigenvector v. As $(T^{-1})^{-1} = T$, the argument works in the opposite direction too. $\square$

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u$$

Prove that 3 or -3 is an eigenvalue of T.

Solution:

Take a linear combination $u + w$. If $u + w \neq 0$, then

$$T(u + w) = Tu + Tw = 3w + 3u = 3(u + w)$$

Thus, 3 is an eigenvalue of T.

If $u + w = 0$, then take $u - w$, which in that case is nonzero. Then:

$$T(u - w) = Tu - Tw = 3w - 3u = -3(u - w)$$

Thus, -3 is an eigenvalue of T.

So we have shown that indeed 3 or -3 is an eigenvalue of T. $\square$

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Solution:

Assume λ is an eigenvalue of ST with eigenvector v: $STv = \lambda v$. It can be thought as $S(Tv) = \lambda v$. Now examine the following:

$$TS(Tv) = T(STv) = T(\lambda v) = \lambda Tv$$

Hence, Tv is an eigenvector of TS that has eigenvalue λ. Tv is nonzero, otherwise $S(Tv)$ must be zero, but it is not.

Similar argument (changing roles of S and T) gives that every eigenvalue of TS is also an eigenvalue of ST.

Thus, ST and TS has the same eigenvalues. $\square$

24 Suppose A is an n-by-n matrix with entries in $\mathbb{F}$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tx = Ax$, where elements of $\mathbb{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries of each column of A equals 1. Prove that 1 is an eigenvalue of T.

Solution:

(a) Take $x = (1, 1, \ldots, 1)^t$, i.e. column vector with all entries equal to 1. Then:

$$Ax = \begin{pmatrix} \sum_i^n A_{1,i}x_i \\ \sum_i^n A_{2,i}x_i \\ \vdots \\ \sum_i^n A_{n,i}x_i \end{pmatrix} = \begin{pmatrix} \sum_i^n A_{1,i} \\ \sum_i^n A_{2,i} \\ \vdots \\ \sum_i^n A_{n,i} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

where in the second equals sign we used that every $x_i = 1$ and in the third equals sign we used that the sum of entries in each row equals 1. Thus, x is an eigenvector of T with an eigenvalue 1. □

(b) Let T' be a dual map of T. Then, matrix of T' is a transpose of matrix of T (proposition 3.132), so $\mathcal{M}(T') = A^t$.

As sum of all entries in each *column* of A equals 1, the sum of all entries in each *row* of A^t therefore equals 1. We know from the part (a) of this problem that the operator corresponding to A^t (that is, T') has eigenvalue 1. And by *Problem 5A.15*, the operator T must also have this eigenvalue. □

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Solution:

Assume that u and w are eigenvectors with distinct eigenvalues λ and μ. Let κ be an eigenvalue of T corresponding to $u+w$. κ may be distinct from λ or μ or equal to one of them. Examine the expression $T(u+w) - T(u+w) = 0$:

$$\begin{aligned} T(u+w) - Tu - Tw &= 0 \\ \kappa(u+w) - \lambda u - \mu w &= 0 \\ (\kappa - \lambda)u + (\kappa - \mu)w &= 0 \end{aligned}$$

Thus, we have a linear combination of u and w that is equal to 0. Note, that $\kappa - \lambda$ and $\kappa - \mu$ cannot be equal to zero simultaneously, as $\lambda \neq \mu$.

Hence, u and w are linearly dependent. But we have assumed that these vectors correspond to different eigenvalues, so by Theorem 5.11, they must be linearly independent. That is a contradiction.

Thus, u and w are eigenvectors corresponding to the same eigenvalue. $\square$

26 Suppose $T \in \mathcal{L}$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Solution:

Take any nonzero $v, w \in V$. These two vectors are eigenvectors of T, and so is their linear combination $u + w$. By the result of the previous problem, v and w correspond to the same eigenvalue.

This argument applies to all vectors in V, hence we have $Tv = \lambda v$ for all $v \in V$. At the same time $\lambda I v = \lambda v$ for all $v \in V$. Thus $T = \lambda I$. $\square$

27 Suppose that V is finite-dimensional and $k \in \{1, \ldots, \dim V - 1\}$. Suppose $T \in \mathcal{L}$ is such that every subspace of V of dimension k is invariant under T. Prove that T is a scalar multiple of the identity operator.

Solution:

If $k = 1$, then every vector in V is an eigenvector. By the result of the previous problem, it means that T is a scalar multiple of the identity operator.

Suppose $k \geq 1$. Then take k distinct subspaces of V and construct their intersection. This intersection is either $\{0\}$ or a one-dimensional vector (sub)space. From *Problem 5A.3* we know that such intersection is also invariant under T. Taking arbitrary k-dimensional subspaces we can construct every one-dimensional subspace of V, thus returning to the $k = 1$ case. Hence T is a scalar multiple of the identity operator. $\square$

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

Solution:

$\operatorname{range} T$ is a subspace of V invariant under T. A maximum number of eigenvectors, that are elements of $\operatorname{range} T$, is $\dim \operatorname{range} T$ (5.12).

If $u \in V$ is an eigenvector of T, such that $u \notin \operatorname{range} T$, then the equality:

$$Tu = \lambda u$$

can be satisfied only if $\lambda = 0$. This value of λ is the corresponding eigenvalue.

Thus, there are at most $1 + \dim \operatorname{range} T$ distinct eigenvalues of T. $\square$

29 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and -4, 5 and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Solution:

We know three eigenvalues of T and the dimension of the vector space ($\mathbb{R}^3$) is 3, hence there is no other eigenvalue.

An operator $(T - 9I)$ is invertible, otherwise 9 would have been an eigenvalue of T, which it cannot be. Hence, there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (T - 9I)x = (-4, 5, \sqrt{7})$. $\square$

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution:

Take nonzero $v \in V$. If $(T-4I)v = 0$, then $Tv = 4v$, so v is an eigenvector and the eigenvalue (λ) is 4.

If $(T - 4I)v \neq 0$, then denote $w = (T - 4I)v$. If $(T - 3I)w = 0$, then $Tw = 3w$, so w is an eigenvector of $T$and $\lambda = 3$.

If $(T-3I)2 \neq 0$, then denote $u = (T-3I)w$. Then necessarily $(T-2I)u = 0$, hence $Tu = 2u$, so u is an eigenvector of $T$and $\lambda = 2$.

Thus we have shown that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. $\square$

31 Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -I$.

Solution:

Take $(1, 0), (0, 1)$ as a basis of $\mathbb{R}^2$. The desired operator T is "rotation by $\pi/4$" and it is represented by the matrix:

$$\mathcal{M}(T) = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}$$

Indeed:

$$\mathcal{M}(T^4) = (\mathcal{M}(T))^4 = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}^4$$

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}^2 = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right)^2 & -2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) \\ 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathcal{M}(-I)$$

Thus, $T^4 = -I$.

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

Solution:

Comment: Here we assume that the vector space is over real numbers. Otherwise, every operator would have an eigenvalue, as is proven later in Theorem 5.19.

Rewrite $T^4 = I$ as: $T^4 - I = 0$. We factorize this polynomial applied to an operator to get:

$$(T^2 + I)(T - I)(T + I) = 0$$

1 and -1 are not eigenvalues of T, so $(T - I)$ and $(T + I)$ are injective operators. That means $(T - I)v \neq 0$ and $(T + I)v \neq 0$ for every nonzero $v \in V$. Hence we conclude that $T^2 + I = 0$, or if rewrite, $T^2 = -I$. $\square$

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective is and only if T^m is surjective.

Solution:

(a) If T is injective, then T^m is injective as a composition of operators.

If T^m is injective, then we prove by contradiction. Suppose T is not injective and $v \neq 0$, $v \in T$. Then:

$$T^m v = T^{m-1}(Tv) = T^{m-1}(0) = 0$$

so T^m is also not injective, contrary to our initial assumption.

Hence, T is injective if and only if T^m is injective. $\square$

(b) If T is surjective, then T^m is surjective as a composition of operators.

If T^m is surjective, then we prove by contradiction. Suppose T is not surjective. Take $w \in V$ such that $w \notin \operatorname{range} T$. As T^m is surjective, there exists such $v \in V$ that $T^m v = w$. Then:

$$T^m v = T(T^{m-1} v) = w$$

so $w \in \operatorname{range} T$, contrary to our initial assumption.

Hence, T is surjective if and only if T^m is surjective. $\square$

34 Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Prove that the list $v_1, \ldots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \ldots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

Solution:

Implication from the 'necessary condition' is just Theorem 5.11. So will show only implication from the 'sufficient condition'.

Assume $v_1, \dots, v_m$ is linearly independent list. Extend this list to the basis of V: $v_1, \dots, v_m, u_1, \dots, u_n$. Take an operator $T \in \mathcal{L}(V)$ such that

$$Tv_i = \lambda_i v_i$$
$$Tu_j = 0$$

for every $i \in \{1, \dots, m\}$ and every $j \in \{1, \dots, n\}$ with λ_i being distinct numbers in $\mathbb{F}$.

These values of Tv_i and Tu_j uniquely define T (by lemma 3.4). Note, that by construction, $v_1, \dots, v_m$ are eigenvectors of T with distinct eigenvalues, hence the desired operator exists. $\square$

35 Suppose $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Solution:

Take differentiation operator $D(f) = f'$. Note that for every $k \in \{1, \dots, n\}$:

$$D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$$

We see that $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is a list of eigenvectors of D with distinct eigenvalues, hence it is linearly independent. $\square$

36 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Solution:

Take operator $D^2(f) = f''$. Note that for every $k \in \{1, \dots, n\}$:

$$D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$$

We see that $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is a list of eigenvectors of D^2 with distinct eigenvalues, hence it is linearly independent. $\square$

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

Solution:

A number λ is an eigenvalue of $\mathcal{A}$ if and only if $(\mathcal{A} - \lambda\mathcal{I})$ is not invertible (here $\mathcal{I}$ is identity operator in $\mathcal{L}(\mathcal{L}(V))$).

Let $S \in \operatorname{null}(\mathcal{A} - \lambda\mathcal{I})$. It means:

$$\begin{aligned}(\mathcal{A} - \lambda\mathcal{I})S &= 0\\ \mathcal{A}(S) - \lambda\mathcal{I}(S) &= 0\\ TS - \lambda S &= 0\\ (T - \lambda I)S &= 0\end{aligned}$$

$S \neq 0$, hence for the last equality to hold, it must be that $\operatorname{null}(T - \lambda I) = \operatorname{range} S \neq \{0\}$. Hence, $(T - \lambda I)$ is not injective. Thus we see that λ is an eigenvalue of $\mathcal{A}$ is and only if λ is an eigenvalue of T. $\square$

38 Suppose V is finite-dimensional , $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U$$

for each $v \in V$.

(a) Show that the definition of T/U makes sense (which requires using the condition that U is invariant under T) and show that T/U is an operator on V/U.

(b) Show that each eigenvalue of T/U is an eigenvalue of T.

Solution:

(a) By definition $v + U = \{v + u : u \in U\}$. So if we act on a linear combination $v + u$ by T we get:

$$T(v + u) = Tv + Tu$$

U is invariant under T: $Tu \in U$. So $(Tv + Tu) \in \{Tv + u : u \in U\}$ and the definition makes sense.

Let us check that T/U is a linear map.

Additivity: Suppose $v, w \in V$. Then:

$$\begin{aligned}(T/U)\left((v + U) + (w + U)\right) &= (T/U)(v + w + U) = T(v + w) + U\\ &= Tv + Tw + U = (Tv + U) + (Tw + U)\\ &= (T/U)(v + U) + (T/U)(w + U) \quad \checkmark\end{aligned}$$

Homogeneity: Suppose $v \in V$ and $\lambda \in \mathbb{F}$.

$$\begin{aligned}(T/U)\left(\lambda(v+U)\right) &= (T/U)(\lambda v + U) \\ &= T(\lambda v) + U = \lambda T v + U = \lambda(Tv + U) \\ &= \lambda(T/U)(v+U) \quad \checkmark\end{aligned}$$

(b) Suppose λ is an eigenvalue of (T/U) with eigenvector $v+U$.

$$\begin{aligned}(T/U)(v+U) &= Tv + U \\ &= \lambda v + U\end{aligned}$$

Hence $(Tv - \lambda v) \in U$ by lemma 3.101. Denote $u = Tv - \lambda v$, so $Tv = \lambda v + u$.
Take $w \in V$, then:

$$T(v+w) = Tv + Tw = \lambda v + u + Tw$$

We would like to find w such that $v+w$ is an eigenvector of T with eigenvalue λ. For that we need $u + Tw = \lambda w$. Rewriting it, we get:

$$(\lambda I - T)w = u$$

If $(\lambda I - T)$ is not invertible, then $(T - \lambda I)$ is not invertible and hence λ is an eigenvalue of T.
If $(\lambda I - T)$ is invertible, then:

$$w = (\lambda I - T)^{-1} u$$

Which is the sought vector and thus λ is an eigenvalue of T. $\square$

39 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.

Solution:

$\longrightarrow$ Assume T has an eigenvalue. We need the following identity (from Fundamental Theorem of linear maps:

$$\dim \operatorname{range}(T - \lambda I) = \dim V - \dim \operatorname{null}(T - \lambda I)$$

Note, that $T - \lambda I$ is a polynomial $p(z) = z - \lambda$ applied to T. By proposition 5.18, $\operatorname{range}(T - \lambda I)$ is invariant under T.

There is at least eigenvector of T, hence $\dim \operatorname{null}(T - \lambda I) \geq 1$ and therefore $\dim \operatorname{range} T - \lambda I \leq \dim V - 1$.

If it is equality, then range $(T - \lambda I)$ is the desired subspace of V.

If it is less than $\dim V - 1$, then we extend a basis of range $(T - \lambda I)$ until we get $\dim V - 1$ vectors in the basis and thus a subspace (let us denote it W) of the desired dimension. W is invariant under $(T - \lambda I)$ by *Problem 5A.1b*. To show that W is also invariant under T, suppose $w_1, w_2 \in W$ are such that $(T - \lambda I)w_1 = w_2$. Then, rearranging the terms, we get:

$$Tw_1 = w_2 + \lambda w_1$$

$(w_2 + \lambda w_2) \in W$, hence $Tw_1 \in W$ and thus we have shown that W is a subspace of V invariant under T with dimension $\dim V - 1$, as desired.

$\longleftarrow$ Assume U is a subspace of V of dimension $\dim V - 1$ that is invariant under T. Examine the operator (T/U) (as in *Problem 5A.38*). It is an operator on V/U — a vector space with dimension (proposition 3.105):

$$\dim V/U = \dim V - \dim U = 1$$

By *Problem 3A.7*, the operator (T/U) is a scalar multiple of identity:

$$(T/U)(v + U) = \lambda(v + U) = \lambda v + U$$

Thus, by definition, λ is an eigenvalue of (T/U) and from *Problem 5A.38* we know that T has the same eigenvalues as (T/U) does. Thus, T has an eigenvalue. $\square$

40 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that:

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Solution:

$$p = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$$
$$p(STS^{-1}) = a_0 + a_1 STS^{-1} + a_2 (STS^{-1})^2 + \cdots + a_n (STS^{-1})^n$$

Notice that:

$$(STS^{-1})^2 = STS^{-1}STS^{-1} = ST^2S^{-1}$$
$$(STS^{-1})^3 = STS^{-1}STS^{-1}STS^{-1} = ST^3S^{-1}$$

And so on. Hence:

$$\begin{aligned} p(STS^{-1}) &= a_0 + a_1 STS^{-1} + a_2 ST^2 S^{-1} + \cdots + a_n ST^n S^{-1} \\ &= S(a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n)S^{-1} = Sp(T)S^{-1} \quad \square \end{aligned}$$

41 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$.

Solution:

Consider $p(T)u$ for any $u \in U$.

$$p(T)u = (a_0 + a_1 T + \cdots a_n T^n)u = a_0 u + a_1 T u + \cdots a_n T^n u$$

As U is invariant under T, any $T^k u$ is in U, so as any scalar multiple of $T^k u$. Thus $p(T)u \in U$, which means U is invariant $p(T)$ for any $p \in \mathcal{P}(\mathbb{F})$. $\square$

42 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$.

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all subspaces of $\mathbb{F}^n$ that are invariant under T.

Solution:

(a) Eigenvalues are: 1, 2, ..., n. Corresponding eigenvectors are: $a_1 e_1$, $a_2 e_2$, ..., $a_n e_n$, where $a_1, \ldots, a_n \in \mathbb{F}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. Indeed:

$$T(\ldots, 0, x_k, 0, \ldots) = (\ldots, 0, kx_k, 0, \ldots) = k(\ldots, 0, x_k, 0, \ldots)$$

The dimension of $\mathbb{F}^n$ is n, so there are no more eigenvalues.

(b) Define $U_k = \operatorname{span}(e_k)$. Then the subspaces of $\mathbb{F}^n$ invariant under T are: $\{0\}$, every U_k and every direct sum of any combination of U_k's.

43 Suppose V is finite-dimensional, $\dim V > 1$ and $T \in \mathcal{L}(V)$. Prove that $\{p(T) : p \in \mathcal{P}(\mathbb{F})\} \neq \mathcal{L}(V)$.

Solution:

Denote a set of all $p(T)$ as W. Suppose $W = \mathcal{L}(V)$.

Note, that $Tp(T) = p(T)T$ for every $p(T) \in W$. Denote invertible polynomials of T as $q(T)$. For every such polynomial it is true that $q(T)T = Tq(T)$. And hence $T = q^{-1}(T)Tq(T)$. Examinig the matrix representation of the last equality, we see that

$$\mathcal{M}(T) = \mathcal{M}(q^{-1}Tq) = \mathcal{M}(q(T))^{-1}\mathcal{M}(T)\mathcal{M}(q(T))$$

for every $q(T)$. We supposed that polynomials of T can represent every linear operator on V, hence every invertible polynomial of T represent every invertible linear operator on V. That means the the obtained equality is equivalent to a proposition that matrix representation of T is the same in every basis of V. Thus T is a scalar multiple of identity, by *Problem 3D.19*.

But in the formulation of a problem we didn't restrict the choice of T and for every V with dim > 1, not every T is a scalar multiple of identity. Thus $W \neq \mathcal{L}(V)$. $\square$

5B The Minimal Polynomial

1 Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Solution:

Suppose 9 is an eigenvalue of T^2. Thus, there is nonzero $v \in V$ such that

$$T^2 v = 9v \quad \text{or} \quad (T - 9I)v = 0$$

Factorization of polynomial $T - 9I$ gives:

$$(T - 3I)(T + 3I)v = 0$$

Hence it is either $(T + 3I)v = 0$, so that -3 is an eigenvalue of T, or $(T - 3I)\left((T + 3I)v\right) = 0$, so that 3 is an eigenvalue of T.

To prove in the other direction, suppose that 3 or -3 is an eigenvalue of T with an eigenvector v, then:

$$T^2 v = T(Tv) = T(\lambda v) = \lambda T v = \lambda^2 v$$

For $\lambda = 3$ or -3, $\lambda^2 = 9$, which means 9 is an eigenvalue of T^2. $\square$

2 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dimensional.

Solution:

Let U be a nonzero finite-dimensional subspace of V and be invariant under T. As V is a complex vector space, so is its subspace U, hence $T|_U$ has an eigenvalue by Theorem 5.19, $T|_U u = \lambda u$. Thus, $Tu = T|_U u = \lambda u$, meaning T has an eigenvalue, which contradicts our assumption that T has no eigenvalues.

If U is $\{0\}$ then $T|_U$ can't have any eigenvalues by definition. If U is infinite-dimensional, the existence of an eigenvalue is not obligatory. $\square$

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n)$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find the minimal polynomial of T.

Solution:

(a) We use notation $e_1, \dots, e_n$ for the standard basis of $\mathbb{F}^n$. Suppose λ is an eigenvalue of T, then the system of equations holds:

$$\begin{aligned} \lambda x_1 &= x_1 + \cdots + x_n \\ &\vdots \\ \lambda x_n &= x_n + \cdots + x_n. \end{aligned}$$

Note, that this system is solved by combinations: (i) $x_1 = x_2 = \cdots = x_n = 1$ and $\lambda = n$; (ii) $x_k = 1$, $x_{k+1} = -1$, $x_j = 0$ ($j \neq k, k+1$) and thus $\lambda = 0$ (for every k running from 1 to $n-1$). In other words, 1 and 0 are eigenvalue of T with eigenvectors $(e_1 + \cdots + e_n)$ and $e_1 - e_2, e_3 - e_2, \dots, e_{n-1} - e_n$. Thus, we have found n eigenvectors; let us show that this list of vectors is linearly independent (and hence there are no other linearly independent eigenvectors).

Suppose the list $e_1 + \cdots + e_n, e_1 - e_2, \dots, e_{n-1} - e_n$ is linearly dependent. Then there are such nonzero $a_1, \dots, a_n \in \mathbb{F}$ such that:

$$a_1(e_1 - e_2) + \cdots + a_{n-1}(e_{n-1} - e_n) + a_n(e_1 + \cdots e_n) = 0$$

Rearranging the terms and collecting them by e_i's gives:

$$(a_1 + a_n)e_1 + (a_2 - a_1 + a_n)e_2 + \cdots + (a_{n-1} - a_{n-2} + a_n)e_{n-1} + (a_n - a_{n-1})e_n = 0$$

The list $e_1, \dots, e_n$ is linearly independent, hence every coefficient of e_i's must equal zero:

$$\begin{aligned} a_1 + a_n &= 0 \\ a_2 - a_1 + a_n &= 0 \\ a_3 - a_2 + a_n &= 0 \\ &\vdots \\ a_{n-1} - a_{n-2} + a_n &= 0 \\ a_n - a_{n-1} &= 0 \end{aligned}$$

Successively solving equations from first to $(n-1)$'th gives: $a_1 = -a_n$, $a_2 = -2a_n$, $a_3 = -3a_n, \dots, a_{n-1} = -(n-1)a_n$. Meanwhile, the last equation gives $a_{n-1} = a_n$. $a_n = -(n-1)a_n$ (if $n \neq 0$ as in our case) only if $a_n = 0$, hence all other $a_i = 0$. Thus, the assumption of linear dependence is not correct, and the list $e_1 + \cdots + e_n, e_1 - e_2, \ldots, e_{n-1} - e_n$ is linearly independent. This shows that we indeed found all eigenvalues and all (linearly independent) eigenvectors. □

(b) Let us examine the action of T on any vector in the standard basis:

$$Te_i = e_1 + \cdots + e_n$$

$$T^2 e_i = T(Te_i) = T\left(\sum_{j=1}^{n} e_j\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} e_j = n\sum_{j=1}^{n} e_j$$

Thus we see that $T^2 e_i = nTe_i$. It is true for all basis vectors and because of linearity, for all vectors in $\mathbb{F}^n$. Thus, the minimal polynomial is:

$$p(T) = T^2 - nT; \quad p(z) = z^2 - nz$$

Indeed, zeros of $p(z)$ are the eigenvalues found in (a).

4 Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$, and $\alpha \in \mathbb{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Solution:

$\longrightarrow$ Suppose v is an eigenvector of $p(T)$ with eigenvalue α. By the Fundamental Theorem of Algebra, $p(z) - \alpha$ can be factorized and hence $p(T) - \alpha I = c(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)$, where λ_k are zeros of $p(z) - \alpha$ (possibly repeated). Then:

$$\left(\sum_{k=0}^{n} a_k T^k - \alpha I\right) v = c(T - \lambda_1 I)(T - \lambda_2 I) \ldots (T - \lambda_n I)v = 0$$

The last equation means that at least one of $(T - \lambda_j I)$ is not invertible, hence λ_j is an eigenvalue of T. Thus, there is some eigenvalue λ of T such that $p(\lambda) = \alpha$.

$\longleftarrow$ Suppose $\alpha = p(\lambda)$ for some eigenvalue of T. Let v be and eigenvector associated with λ. Apply $p(T)$ to v:

$$p(T)v = p(\lambda)v = \alpha v$$

where the first equation sign comes from the fact, shown in the proof of Theorem 5.27. Thus, α is an eigenvalue of $p(T)$. □

5 Give and example of an operator on $\mathbb{R}^2$ that shows the result in Exercise 4 does not hold if $\mathbb{C}$ is replaced with $\mathbb{R}$.

Solution:

If $\mathbb{C}$ is replaced by $\mathbb{R}$ in the previous exercise, the result doesn't hold, because T doesn't have to have an eigenvalue. For example, $T \in \mathcal{L}(\mathbf{R}^2)$: $T(x,y) = (-y,x)$. Here T doesn't have an eigenvalue, but $p(T) = T^2$ does: $T^2 = -I$ and eigenvalue is -1.

6 Suppose $T \in \mathcal{L}(\mathbb{F}^2)$ is defined by $T(w,z) = (-z,w)$. Find the minimal polynomial of T.

Solution:

Take the standard basis e_1, e_2 of $\mathbb{F}^2$. Then acting by T on it, we get:

$$\begin{aligned} Te_1 &= e_2 \\ Te_2 &= -e_1 \end{aligned}$$

Hence $T^2 e_1 = -e_1$ and the minimal polynomial of T is $T^2 + 1$.

7 (a) Give an example of $S, T \in \mathcal{L}(\mathbb{F}^2)$ such that the minimal polynomial of ST does not equal the minimal polynomial of TS.

(b) Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if at least one of S, T is invertible, then the minimal polynomial of ST equals the minimal polynomial of TS.

Solution:

(a) Take $S, T \in \mathcal{L}(\mathbb{F}^2)$ defined by:

$$T(x,y) = (x+y, 0); \quad S(x,y) = (0,y)$$

Then:

$$\begin{aligned} TS(x,y) &= T(0,y) = (y,0) \\ ST(x,y) &= S(x+y,0) = (0,0) \end{aligned}$$

Here, $ST = 0$ hence the minimal polynomial of ST is $p(z) = 1$. To find minimal polynomial of TS, apply it to the standard basis:

$$\begin{aligned} &TSe_2 = e_1 \\ &(TS)^2 e_2 = TS(e_1) = e_1 \end{aligned}$$

Thus, $(TS)^2 e_2 - TSe_2 = 0$ and the minimal polynomial of TS is $q(z) = z^2 - z$. ST and TS have different zero polynomials, as desired.

(b) Suppose without loss of generality that S is invertible. Then $TS = S^{-1}(ST)S$.

Let $p(z)$ be a minimal polynomial of TS. Then, by *Problem 5A.40*:

$$p(TS) = p(S^{-1}(ST)S) = S^{-1}p(ST)S \tag{5.1}$$

By definition of minimal polynomial, $p(TS)v = 0$ for all $v \in V$. S is invertible, hence $Su = 0$, as well as $S^{-1}u = 0$ for some $u \in V$ if and only if $u = 0$. Thus we conclude that $p(ST)v = 0$ for all $v \in V$.

To prove that $p(z)$ is a minimal polynomial of ST, suppose there is a monic polynomial $q(ST)$ of lesser degree than $p(z)$ such that $q(ST) = 0$. Following eq. 5.1 in reverse order we conclude that $q(TS) = 0$, as well. This contradicts initial assumption that $p(z)$ is the minimal polynomial of TS, hence $p(z)$ is indeed the minimal polynomial of ST. $\square$

8 Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is the operator of counterclockwise rotation by $1°$. Find the minimal polynomial of T.

Solution:

Denote the angle of $1°$ by α. Examine how T acts on e_1 of the standard basis:

$$\begin{aligned}
&Te_1 = \cos(\alpha)e_1 + \sin(\alpha)e_2\\
&T^2e_1 = \cos(\alpha)Te_1 + \sin(\alpha)Te_2 = (\cos^2(\alpha) - \sin^2(\alpha))e_1 + 2\sin(\alpha)\cos(\alpha)e_2
\end{aligned}$$

Then we need to find coefficients c_0, c_1 that solve the following equation:

$$c_0e_1 + c_1Te_1 = -T^2e_1$$

Inserting expressions for Te_1 and T^2e_1 we get:

$$c_0e_1 + c_1(\cos(\alpha)e_1 + \sin(\alpha)e_2) = \left(\sin^2(\alpha) - \cos^2(\alpha)\right)e_1 - 2\sin(\alpha)\cos(\alpha)e_2$$

This equation is equivalent to a system of two linear equations:

$$\begin{cases} c_0 + c_1\cos(\alpha) = \sin^2(\alpha) - \cos^2\alpha \\ c_1\sin(\alpha) = -2\sin(\alpha)\cos(\alpha) \end{cases}$$

This system is solved by $c_0 = 1$, $c_1 = -2\cos(\alpha)$. Hence, the minimal polynomial of the operator of counterclockwise rotation by $1°$ is:

$$p(z) = z^2 - 2\cos(1°)z + 1 \approx z^2 - 1.9997z + 1$$

9 Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the minimal polynomial of T are rational numbers.

Solution:

Take any vector w from the basis of V such that $v \notin T$. Then $Tw, T^2w, \ldots,$ $T^{\dim V}w$ are linear combinations of basis vectors with rational coefficients (for Tw it follows from the fact that all entries of the matrix of T in the basis under consideration are rational; for $T^k w$ the coefficients are combinations of sums and products of the matrix entries, hence are rational too). Suppose $c_0, c_1, \ldots, c_{n-1}$ $(n \leq \dim V)$ are coefficients of the minimal polynomial. It means these coefficients are solution of:

$$c_0 + c_1 Tw + \cdots + c_{n-1}T^{n-1}w = T^n w.$$

This equation is equivalent to a system of n linear equations. Linear equations with rational coefficients have rational solutions, which means the minimal polynomial of T has rational coefficients.

10 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1}v)$$

for all integers $m \geq \dim V - 1$.

Solution:

If $m = \dim V - 1$, then the proposition is trivially true.

Suppose $m \geq \dim V - 1$.

The list $v, Tv, \ldots, T^{\dim V - 1}v$ is of length $\dim V$, so there is no list of larger length that can be linearly independent (otherwise we would have contradiction with Theorem 2.22). Hence, the list $v, Tv, \ldots, T^m v$ is definitely linearly dependent.

Let k be the greatest number such that the list $v, Tv, \ldots, T^k v$ is linearly independent. By the linear dependence lemma (2.19):

$$\begin{aligned} \operatorname{span}(v, Tv, \ldots, T^m v) &= \operatorname{span}(v, Tv, \ldots, T^k v) \\ \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1}v) &= \operatorname{span}(v, Tv, \ldots, T^k v) \end{aligned}$$

Hence we conclude that $\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1}v)$ for all $m \geq \dim V - 1$ indeed. $\square$

11 Suppose V is a two-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $T^2 - (a+d)T + (ad-bc)I = 0$.

(b) Show that the minimal polynomial of T equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

Solution:

(a) To show the desired result, it is sufficient to show that $\mathcal{M}(T^2 - (a+d)T + (ad-bc)I) = \mathcal{M}(0)$.

$$\mathcal{M}(T^2) = \mathcal{M}(T) \cdot \mathcal{M}(T) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & ac + cd \\ ab + bd & bc + d^2 \end{pmatrix}$$

$$\begin{aligned} \text{Desired matrix} &= \begin{pmatrix} a^2 + bc - (a+d)a + ad - bc & ac + cd - (a+d)c + 0 \\ ab + bd - (a+d)b + 0 & bc + d^2 - (a+d)d + ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Thus we have the desired equality.

(b) First, suppose $b = c = 0$ and $a = d$. Then, the matrix of T is:

$$\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a \cdot \mathcal{M}(I)$$

Hence, $T - aI = 0$ and the minimal polynomial in that case is $p(z) = z - a$.

Second, suppose the constraints on a, b, c, d are not satisfied. That means, T is not a multiple of an identity operator, hence its minimal polynomial has degree greater than 1.

We have shown in part (a) that the monic polynomial $p(z) = z^2 - (a+d)z + (ad-bc)$ applied to T gives zero operator. Hence, it is the minimal polynomial of T. $\square$

12 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$. Find the minimal polynomial of T.

Solution:

In *Problem 5A.42* we have shown that T has eigenvalues: $1, 2, \ldots, n$. By Theorem 5.27, proposition (b), the minimal polynomial is:

$$p(z) = (z-1)(z-2)\cdots(z-n)$$

This polynomial has degree $n = \dim \mathbb{F}^n$, hence no factor in braces is repeated.

13 Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Prove that there exists a unique $r \in \mathcal{P}(\mathbb{F})$ such that $p(T) = r(T)$ and $\deg r$ is less than the degree of the minimal polynomial of T.

Solution:

Suppose $p(z)$ has degree less than the degree of the minimal polynomial. Then take $r(z) = p(z)$.

Suppose the degree of $p(z)$ is greater or equal to the degree of the minimal polynomial. Denote minimal polynomial by $q(z)$. Then applying polynomial division algorithm to $p(z)$ gives:

$$p(z) = q(z)s(z) + r(z)$$

with $r \in \mathcal{P}(\mathbb{F})$ having degree less than $\deg q$. Now note that:

$$p(T) = q(T)s(T) + r(T) = 0 \cdot s(T) + r(T) = r(T) \quad \square$$

14 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has minimal polynomial $4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1}.

Solution:

Examine the following:

$$\begin{aligned} p(T)T^{-5} &= 0(T^{-5}) = 0 \\ p(T)T^{-5} &= 4T^{-5} + 5T^{-4} - 6T^{-3} - 7T^{-2} + 2T^{-1} + 1 \end{aligned}$$

Hence, the minimal polynomial of T^{-1} is

$$q(z) = z^5 + \frac{5}{4}z^4 - \frac{3}{2}z^3 - \frac{7}{4}z^2 + \frac{1}{2}z + \frac{1}{4}.$$

0 is not the root of $p(z)$ hence it is not an eigenvalue of T and hence T^{-1} has the same number of eigenvalues as T (*Problem 5A.21*). Combining it with Theorem 5.27, we get that minimal polynomial of T^{-1} should be of the same degree as minimal polynomial of T. The obtained $q(z)$ meets this criterion. $\square$

15 Suppose V is finite-dimensional complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$. Define $f : \mathbb{C} \mapsto \mathbb{R}$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I)$$

Prove that f is not a continuous function.

Solution:

By Theorem 5.19, there is some eigenvalue λ of T. Suppose, its corresponding eigenvectors are $v_1, \ldots, v_k$. Thus, $\operatorname{null}(T - \lambda I) = \operatorname{span}(v_1, \ldots, v_k)$ and by the Fundamental Theorem of Linear Maps, $\dim \operatorname{range}(T - \lambda I) = n - k$.

We have shown in *Problem 5A.11* that there is arbitrarily small neighborhood of eigenvalue (particularly) in which all the numbers make $T - \alpha I$ invertible. If $T - \alpha I$ is invertible, then $\dim \operatorname{range}(T - \alpha I) = n$. Thus, $f(\lambda)$ have discontinuity at least at every eigenvalue, and thus it is not a continuous function. $\square$

16 Suppose $a_0, \ldots, a_{n-1} \in \mathbb{F}$. Let T be the operator on $\mathbb{F}^n$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & \ddots & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{pmatrix}$$

Here all entries of the matrix are 0 except for all 1's on the line under the diagonal and the entries in the last column (some of which might also be 0). Show that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n.$$

Solution:

Firstly, let us examine how T acts on the standard basis:

$$\begin{aligned} Te_1 &= -a_0 e_n \\ Te_k &= e_{k-1} - a_{k-1} e_n \quad \text{for } k \in \{2, \ldots, n\} \end{aligned}$$

Then take e_1 as a trial vector as successively apply powers of T to it. Such successive application leads to:

$$T^n e_1 = -a_0 e_1 - a_1 T e_1 + \cdots + a_{n-1} T^{n-1} e_1$$

We will show this by induction. The base case is $k = 2$:

$$T^2 e_1 = -a_0 \left(e_{n-1} - a_{n-1} e_n\right) = -a_0 e_{n-1} + a_{n-1} T e_1$$

Then, for every $k \in \{2, \ldots, n\}$ we suppose that:

$$T^k e_1 = -a_0 e_{n-k+1} - \left(a_{n-k+1} T e_1 + \cdots + a_{n-1} T^{k-1} e_1\right) \tag{5.2}$$

If eq. 5.2 is true for k, then examine case of $k+1$.

$$\begin{aligned}
T^{k+1} e_1 = T(T^k e_1) &= -a_0 T e_{n-k+1} - \left(a_{n-k+1} T^2 e_1 + \cdots + a_{n-1} T^{(k+1)-1} e_1\right) \\
&= -a_0 (e_{n-k-1+1} - a_{n-k-1+1} e_n) \\
&\quad - \left(a_{n-k+1} T^2 e_1 + \cdots + a_{n-1} T^{(k+1)-1} e_1\right) \\
&= -a_0 e_{n-(k+1)+1} \\
&\quad - \left(a_{n-(k+1)+1} T e_1 + a_{n-k+1} T^2 e_1 + \cdots + a_{n-1} T^{(k+1)-1} e_1\right)
\end{aligned}$$

Hence, eq. 5.2 is true by induction. Inserting $k = n$ in it, we obtain the desired relation.

The obtained expression on $T^n e_1$ in terms of all other powers of T is unique as $e_1, Te_1, \ldots, T^{n-1} e_1$ is a linearly independent list. Indeed, every subsequent $T^k e_1$ (except $T^n e_1$) has one additional basis vector, and thus it is not a liner combination of all previous terms.

Hence,

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

is a minimal polynomial of T. $\square$

17 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbb{F}$. Show that the minimal polynomial of $T - \lambda I$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

Solution:

Note that

$$q(T - \lambda I) = p\left((T - \lambda I) + \lambda I\right) = p(T) = 0$$

Suppose there is a monic polynomial $r(z)$ with degree less than $\deg q$ such that $r(T - \lambda I) = 0$. Then if we rewrite $r(T - \lambda I)$ in terms of T, then we get another polynomial $s(T)$. As we just rearranged expression, $s(T) = 0$. But $\deg s = \deg r < \deg p$, contradicting the fact that p is the minimal polynomial of T. Hence, q is indeed the minimal polynomial of $T - \lambda I$. $\square$

18 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbb{F}\backslash\{0\}$. Show that the minimal polynomial of λT is the polynomial q defined by $q(z) = \lambda^{\deg p} p\left(\frac{z}{\lambda}\right)$.

Solution:

Note that

$$q(\lambda T) = \lambda^{\deg p} p\left(\frac{\lambda T}{\lambda}\right) = \lambda^{\deg p} p(T) = 0$$

Here, the factor before $p(z/\lambda)$ makes $q(z)$ a monic polynomial. The rest is to show that $q(z)$ has minimal degree.

Suppose $r(\lambda T) = 0$ and $\deg r < \deg q = \deg p$. Then viewing expression for $r(\lambda T)$ as a polynomial of T shows that it is some $s(T)$ such that $s(T) = 0$ and $\deg s < \deg p$ contradicting the fact that p is the minimal polynomial of T. Hence, q is the minimal polynomial of λT. $\square$

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbb{F})\}.$$

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

Solution:

Let p be the minimal polynomial of T. Then p being the minimal polynomial means that the list $v, Tv, \dots, T^{\deg p} v$ is linearly dependent for all $v \in V$, while the list $v, Tv, \dots, T^{\deg p - 1} v$ is linearly independent for some $v \in V$. Hence, the list $I, T, T^{,} \dots, T^{\deg p - 1}$ is linearly independent list of maximal length with elements from $\mathcal{E}$. Thus, this list is the basis of $\mathcal{E}$ and $\mathcal{E}$ has dimension $\deg p$. $\square$

20 Suppose $T \in \mathcal{L}(\mathbb{F}^4)$ is such that the eigenvalues of T are 3, 5, 8. Prove that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

Solution:

Eigenvalues of T are zeros of the minimal polynomial. Let p be the minimal polynomial of T, so

$$p(z) = (z - 3)(z - 5)(z - 8) \cdot q(z)$$

Degree of $p(z)$ is at most 4, hence $\deg q$ is at most 1. If $p(z)$ had non-real zeros, they would come in pairs and $\deg q$ would be at least 2 (lemmas 4.14 and 4.16). Thus, $q(z)$ is a repeated factor $(z - 3)$, $(z - 5)$, or $(z - 8)$.

It means $(z - 3)^2(z - 5)^2(z - 8)^2$ is a polynomial multiple of the minimal polynomial. Hence, by Theorem 5.29, $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$. $\square$

21 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of T has degree at most $1 + \dim \operatorname{range} T$.

Solution:

Suppose $w \in \operatorname{null} T$, then if $p(z)$ is the minimal polynomial of T, then:

$$p(T)w = 0 = a_0 w + a_1 Tw + \cdots + a_n T^n w = a_0 w$$

If $\operatorname{null} T = \{0\}$, then by the Fundamental Theorem of linear maps $\dim \operatorname{range} T = \dim V$, and we get the desired result as the degree of the minimal polynomial is at most $\dim V$ by 5.22. If $\operatorname{null} T \neq \{0\}$, then $a_0 = 0$.

Let $m = \dim \operatorname{range} T$. Range of T is invariant under T, so every $T^k v \in \operatorname{range} T$. A list of at most m vectors in $\operatorname{range} T$ can be linearly independent. Hence, the longest linearly independent list of powers of T applied to a vector is $Tv, T^2 v, \ldots, T^m v$ for all $v \in V$. Thus, necessarily there are such $c_1, \cdots, c_m$ that

$$T^{m+1} v = c_0 Tv + \cdots + c_m T^m v$$

for all $v \in V$. Hence, the minimal polynomial of T has degree at most $1 + \dim \operatorname{range} T$. $\square$

22 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $I \in \operatorname{span}(T, T^2, \ldots, T^{\dim V})$.

Solution:

$\longrightarrow$ Suppose T is invertible. Then by lemma 5.32, the constant term of the minimal polynomial of T is nonzero. Hence, for every $v \in V$:

$$c_0 Iv + c_1 Tv + \cdots + c_m T^m v = 0$$

where m is the degree of the minimal polynomial. As it is true for every $v \in V$, we can rewrite it as:

$$I = -\frac{c_1}{c_0} T + \cdots + \frac{c_m}{c_0} T^m$$

Thus, $I \in \operatorname{span} T, \ldots, T^m$. Moreover, every other power of T is in the same span, hence $\operatorname{span} T, \ldots, T^m = \operatorname{span} T, \ldots, T^{\dim V}$ and thus $I \in \operatorname{span} T, \ldots, T^{\dim V}$.

$\longleftarrow$ Suppose $I \in \operatorname{span} T, \ldots, T^{\dim V}$. Let m be the smallest number (less that $\dim V$), for which it holds that there are nonzero $c_1, \ldots, c_m$ such that $I = c_1 T + \cdots + c_m T^m$. Rearranging the terms on the same side and dividing by c_m gives the minimal polynomial of T. It has nonzero constant term, hence by 5.32, T is invertible. $\square$

23 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that if $v \in V$, then $\operatorname{span}(v, Tv, \dots, T^{n-1}v)$ is invariant under T.

Solution:

Let $v \in V$, m is degree of the minimal polynomial. Consider list $v, Tv, \dots, T^{m-1}v$. $T^m v$ can be expressed as

$$T^m v = -c_0 v - c_1 Tv - \cdots - c_{m-1} T^{m-1} v$$

where c_j are coefficients of the minimal polynomial. Hence, $T(T^{m-1})v = T^m v$ is in $\operatorname{span}(v, Tv, \dots, T^{m-1}v)$. Any other power is trivially in the same span:

$$T(v) = Tv \in \operatorname{span}(v, Tv, \dots, T^{m-1}v),$$

$$T(T^k v) = T^{k+1} v \in \operatorname{span}(v, Tv, \dots, T^{m-1}v),$$

where $k < (m-1)$. Thus, $\operatorname{span}(v, Tv, \dots, T^{m-1}v)$ is invariant under T.

If $m = n$, then we are done. If $m < n$, then we have linearly dependent list $v, Tv, \dots, T^n v$, and hence by linear dependence lemma $\operatorname{span}(v, Tv, \dots, T^{n-1}v) = \operatorname{span}(v, Tv, \dots, T^{m-1})$. Thus, $\operatorname{span}(v, Tv, \dots, T^{n-1}v)$ is invariant under T. $\square$

24 Suppose V is finite-dimensional complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that 5 and 6 are eigenvalues of T and that T has no other eigenvalues. Prove that $(T-5I)^{\dim V - 1}(T-6I)^{\dim V - 1} = 0$.

Solution:

Eigenvalues of T are zeros of the minimal polynomial. Let p be the minimal polynomial of T, so

$$p(z) = (z-5)(z-6) \cdot q(z)$$

T has no other eigenvalues, while V is a complex vector space. Hence, $q(z) = (z-5)^x(z-6)^y$ where x, y are some non-negative integers. Degree of $p(z)$ is at most $\dim V$, hence $\deg q$ is at most $\dim V - 1$. Moreover, the degree of each of the two factors is at most $\dim V - 1$. It means $(z-5)^{\dim V - 1}(z-6)^{\dim V - 1}$ is a polynomial multiple of the minimal polynomial. Hence, by Theorem 5.29, $(T-5I)^{\dim V - 1}(T-6I)^{\dim V - 1} = 0$. $\square$

25 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T.

(a) Prove that the minimal polynomial of T is a polynomial multiple of the minimal polynomial of the quotient operator T/U.

(b) Prove that

$$(\text{minimal polynomial of } T|_U) \times (\text{minimal polynomial of } T/U)$$

is a polynomial multiple of the minimal polynomial of T.

Solution:

(a) Let q be a minimal polynomial of T/U and p be a minimal polynomial of T. Referring to lemma 3.105, $\dim V/U \leq \dim V$, hence $\deg q \leq \deg p$. Then note:

$$p(T/U)(v+U) = (p(T)/U)\,(v+U) = p(T)v + U = 0 + U$$

Thus, $p(T/U) = 0$ for all $(v+U) \in V/U$. By proposition 5.29, $p(z)$ is a polynomial multiple of $q(z)$. □

(b) Let q be a minimal polynomial of (T/U), s be a minimal polynomial of $T|_U$ and p be a minimal polynomial of T.

Note that in order q to be a minimal polynomial of T/U we need that $q(T)v \in U$ for all $v \in V$:

$$q(T/U)(v+U) = (q(T)/U)(v+U) = q(T)v + U = 0 + U \qquad \Rightarrow q(T)v \in U$$

Then for any $v \in V$:

$$(sq)(T)v = s(T)\,(q(T)v) = 0$$

where the last equality sign comes from the fact that s is the minimal polynomial of $T|_U$.

Thus, $(sq)(T) = 0$ and therefore (by proposition 5.29) it is a polynomial multiple of the minimal polynomial of T. □

26 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Prove that the set of eigenvalues of T equals the union of the set of eigenvalues of $T|_U$ and the set of eigenvalues of T/U.

Solution:

From *Problem 5B.25* we know that the product of minimal polynomials of $T|_U$ and T/U is a polynomial multiple of the minimal polynomial of T:

$$p = sq \cdot r$$

where we used the same notation as in previous problem. Suppose r has factors, not present in p. This means that either $T|_U$ or T/U has eigenvalues that are not eigenvalues of T. This is a contradiction. Hence, the set of eigenvalues of T is a union of the set of eigenvalues of $T|_U$ and the set of eigenvalues of T/U. □

27 Suppose $\mathbb{F} = \mathbb{R}$, V is finite-dimensional, and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T_{\mathbb{C}}$ equals the minimal polynomial of T.

Solution:

Let p be a minimal polynomial of T and q be a minimal polynomial of $T_{\mathbb{C}}$. Note that:

$$p(T_{\mathbb{C}})(v + iu) = p(T)v + ip(T)u = 0 + i \cdot 0 = 0$$

Hence, $p(z)$ is a polynomial multiple of $q(z)$. At the same time:

$$q(T_{\mathbb{C}})(v + iu) = q(T)v + iq(T)u$$

which is true if and only if $q(T) = 0$ for all $v \in V$. Thus, $q(T)$ is a polynomial multiple of $p(z)$. The fact that $p = qr$ and $q = ps$, where r and s are some polynomials means that both r and s must equal 1. Thus, $p = q$, that is, T and $T_{\mathbb{C}}$ have the same minimal polynomial. $\square$

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T' \in \mathcal{L}(V')$ equals the minimal polynomial of T.

Solution:

Note that for any $p \in \mathcal{P}(\mathbb{F})$:

$$\begin{aligned} p(T')(\varphi) &= \left(a_0 I' + a_1 T' + \cdots + a_m (T')^m\right)(\varphi) \\ &= a_0 \varphi \circ I + a_1 \varphi \circ T + \cdots + a_m \varphi T^m \\ &= \varphi \circ p(T) = (p(T))'(\varphi) \end{aligned}$$

Using *Problem 3F.16* we arrive at:

$$p(T') = 0 \iff p(T) = 0$$

Hence, the minimal polynomial of $T' \in \mathcal{L}(V')$ equals the minimal polynomial of T. $\square$

29 Show that every operator on a finite-dimensional vector space of dimension at least two has an invariant subspace of dimension two.

Solution:

We will prove this by induction.

Let V be a two-dimensional vector space. Then V is such two-dimensional space, invariant under any operator on it.

Now let V be a finite-dimensional vector space such that $\dim V > 2$ and suppose that every operator on a vector space of dimension less than $\dim V$ and greater or equal than 2 has an invariant subspace of dimension 2.

Take any $T \in \mathcal{L}(V)$. Then by the Fundamental Theorem of Linear Maps:

$$\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T$$

At least one of the terms in the sum on the right is greater or equal than 2. Take the one with the dimension greater than 1 and call it U. Both range and null-space of T are invariant under T, so U is invariant under T. Moreover, closing our attention on $T|_U$, we see that U has a subspace of dimension 2 that is invariant under $T|_U$. This is also a subspace of V. Thus, V has a subspace of dimension 2 invariant under T. $\square$