Chapter 5

Eigenvalues and Eigenvectors

Contents

5A Invariant Subspaces

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

Solution:

(a) Suppose $u \in U$, and because U is a subset of null-space of T, $u \in \operatorname{null} T$. $Tu = 0$ and $0 \in U$. Thus, U is invariant under T. $\square$

(b) Suppose $u \in U$. $Tu \in \operatorname{range} T$, and as $\operatorname{range} T$ is a subset of U, Tu must be an element of U, too. Hence, U is invariant under T. $\square$

2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

Solution:

Suppose $v_k \in V_k$ for every $k \in \{1, \ldots\}$. Each V_k is invariant under T, therefore $Tv_k \in V_k$. Then, for every $v \in V_1 + \cdots + V_m$, which can be written as a linear combination of vectors $v_1, \ldots, v_m$, we can write:

$$Tv = T(a_1 v_1 + \cdots a_m v_m) = a_1 T v_1 + \cdots a_m T v_m$$

So, Tv can be written as a linear combination of vectors from $V_1, \ldots, V_m$. Hence, $Tv \in V_1 + \cdots + V_m$, which means $V_1, \ldots, V_m$ is invariant under T. $\square$

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Solution:

Let us denote subspaces of V invariant under T as U_i. Suppose u is a vector that belongs to the intersection of some collection of such subspaces, $u \in \bigcap_{i=1}^{m} U_i$. It means that $u \in U_i$ for every $i \in \{1, \ldots, m\}$.

Then, $Tu \in U_i$ for every $i \in \{1, \ldots, m\}$, or in other words $Tu \in \bigcap_{i=1}^{m} U_i$. That means, this intersection is invariant under T. This argument works for any collection of U_i, hence the intersection of every collection of subspaces of V invariant under T is invariant under T. $\square$

4 Prove of give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Solution:

Suppose U is neither V, nor $\{0\}$. Let $u_1, \ldots, u_m$ be a basis of U, and $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V. Take some operator T, with its range being V, such that for every u_k:

$$Tu_k = A_{1,k} u_1 + \cdots A_{m,k} u_m + B_{1,k} v_1 + \cdots B_{n,k} v_n$$

with non-zero coefficients $B_{j,k}$. But if these coefficients are not zero, $Tu_k \notin U$, so U is not invariant under such T, which contradicts our initial assumption that U is invariant under every operator on V. Hence we conclude that U must be either $\{0\}$ or V. $\square$

5 Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution:

Let λ be an eigenvalue of T with the eigenvector (x, y). Then:

$$T(x, y) = \lambda(x, y) = (-3y, x)$$

This is equivalent to a system of equations:

$$\begin{aligned} \lambda x &= -3y \\ \lambda y &= x \end{aligned}$$

We can express x from the second equation and insert it into the first.

$$\lambda \cdot \lambda y = -3y$$

Hence the eigenvalue must satisfy the equation $\lambda^2 = -3$. This equation has no real roots, hence the operator T has no eigenvalues.

6 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

Solution:

As in previous problem, we write a system of equations:

$$\begin{aligned} z &= \lambda w \\ w &= \lambda z \end{aligned}$$

Expressing w from the second equation and inserting it into the first gives:

$$z = \lambda^2 z \quad \Rightarrow \quad \lambda^2 = 1$$

Thus we have two eigenvalues:

1. $\lambda_1 = 1$ with eigenvectors of form $v_1 = t(1, 1)$, where $t \in \mathbb{R}$;
2. $\lambda_2 = -1$ with eigenvectors of form $v_1 = t(1, -1)$, where $t \in \mathbb{R}$.

7 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

Solution:

Once again we write a system of equation that is equivalent to a condition of (z_1, z_2, z_3) being an eigenvector:

$$\begin{aligned} 2z_2 &= \lambda z_1 \\ 0 &= \lambda z_2 \\ 5z_3 &= \lambda z_3 \end{aligned}$$

Let us examine the second equation: it tell that either $\lambda = 0$ or $z_2 = 0$.

Assume $\lambda = 0$. Then the third equation tells that $z_3 = 0$, and the first equation tells that $z_2 = 0$ and z_1 is arbitrary.

Now assume $z_2 = 0$ and $\lambda \neq 0$. Then the first equation tells that $z_1 = 0$ and the third equation tells that $\lambda = 5$ and z_3 is arbitrary.

Thus, there are two eigenvalues:

1. $\lambda_1 = 0$ with an eigenvectors of form $v_1 = t(1, 0, 0)$, where $t \in \mathbb{F}$;

2. $\lambda_2 = 5$ with an eigenvectors of form $v_2 = t(0, 0, 1)$, where $t \in \mathbb{F}$.

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

Solution:

Suppose λ is an eigenvalue of P with the corresponding eigenvector u. Then we can write:

$$Pv = \lambda v \quad \text{and} \quad Pv = P^2 v = P(\lambda v) = \lambda^2 v$$

So we have $(\lambda^2 v - \lambda v) = 0$ or $(\lambda^2 - \lambda)v = 0$. This equality can hold if either $v = 0$, or $(\lambda^2 - \lambda) = 0$. The first option is not the case as we supposed that v is an eigenvector. The second option gives the result that $\lambda = 0$ or $\lambda = 1$. $\square$

9 Define $T : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution:

Suppose λ is an eigenvalue of T with corresponding eigenvector p. Then:

$$Tp = \lambda p = p'$$

Write the polynomial p as:

$$p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Its derivative is:

$$p' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots n a_n x^{n-1}$$

Note that from inspection of x^n terms in $p' = \lambda p$ we can get a condition that $\lambda a_n = 0$. Then we do the same for x^{n-1} terms to get $\lambda a_{n-1} = n a_n$. And so on until $\lambda a_0 = a_1$.

Assume $\lambda \neq 0$, so from $\lambda a_n = 0$ we conclude that $a_n = 0$. Then from $\lambda a_{n-1} = n a_n$ we conclude that $a_{n-1} = 0$. And we thus continue until $a_0 = 0$.

Thus, $\lambda \neq 0$ means that $p = 0$, but we assumed that p is eigenvector so it cannot be the case.

Assume $\lambda = 0$. Then from $\lambda a_{n-1} = n a_n$ we see that $a_n = 0$. And thus we continue for every equation $\lambda a_{k-1} = k a_k$ until $\lambda a_0 = a_1$. The coefficient a_0 is here arbitrary, and $p = a_0$.

Hence, the eigenvalue of T is $\lambda = 0$ with eigenvectors of form $p = a_0$, where $a_0 \in \mathbb{R}$.

10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Solution:

Let λ be an eigenvalue of T with the corresponding eigenvector p. Let $p(x)$ has a form: $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$. Then:

$$(Tp)(x) = (\lambda p)(x) = \lambda(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)$$
$$(Tp)(x) = xp'(x) = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4$$

Thus the following equations must be satisfied:

$$\begin{aligned}
\lambda a_0 &= 0, \\
\lambda a_1 &= a_1, \\
\lambda a_2 &= 2a_2, \\
\lambda a_3 &= 3a_3, \\
\lambda a_4 &= 4a_4.
\end{aligned}$$

Suppose in the first equation $a_0 \neq 0$, then $\lambda = 0$ and all other coefficients of $p(x)$ are zero.

If $a_0 = 0$, then other coefficients can be non-zero. Suppose $a_1 \neq 0$, then from the second equation we conclude that $\lambda = 1$. Other equations can thus be satisfied only if $a_2 = a_3 = a_4 = 0$.

Similar reasoning can be applied to all subsequent equations. In the end we have five eigenvalues:

1. $\lambda = 0$ with eigenvectors $p(x) = a$, where $a_0 \in \mathbb{R}$;
2. $\lambda = 1$ with eigenvectors $p(x) = ax$, where $a \in \mathbb{R}$;
3. $\lambda = 2$ with eigenvectors $p(x) = ax^2$, where $a \in \mathbb{R}$;
4. $\lambda = 3$ with eigenvectors $p(x) = ax^3$, where $a \in \mathbb{R}$;
5. $\lambda = 4$ with eigenvectors $p(x) = ax^4$, where $a \in \mathbb{R}$.

11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbb{F}$. Prove that there exists $\delta \geq 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{F}$ such that $0 < |\alpha - \lambda| < \delta$.

Solution:

V is finite-dimensional, so by 5.12, there is a finite number of eigenvalues of T.

For a given α, pick the closest to it eigenvalue of T, μ. Then, choose δ such that $\delta = |\alpha - \mu|$. By construction, there is no other eigenvalue between α and μ, hence any λ such that $0 < |\alpha - \lambda| < \delta$ is not an eigenvalue of T, so $T - \lambda I$ is invertible. $\square$

12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P.

Solution:

Every $v \in V$ can be written uniquely as $v = u + w$ where $u \in U$ and $w \in W$. Suppose some v is an eigenvector with eigenvalue λ. Then

$$Tv = \lambda v = \lambda u + \lambda w = T(u + w) = u$$

This equation can be satisfied if either $\lambda = 1$ and $w = 0$, or $\lambda = 0$ and $u = 0$.

Thus eigenvalues of P are:

1. $\lambda_1 = 1$ with eigenvectors $v_1 = u$, where $u \in U$;
2. $\lambda_2 = 0$ with eigenvectors $v_2 = w$, where $w \in W$.

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(\mathcal{V})$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Solution:

(a) Assume λ is an eigenvalue of T. That means operator $(T - \lambda I)$ is not invertible. Then note that:

$$\begin{aligned} T - \lambda I &= SS^{-1}T - \lambda SS^{-1} = S(S^{-1}T - \lambda S^{-1}) \\ &= S(S^{-1}TSS^{-1} - \lambda S^{-1}) \\ &= S(S^{-1}TS - \lambda I)S^{-1} \end{aligned}$$

As S is invertible, we conclude that $(S^{-1}TS - \lambda I)$ is not invertible. Hence, λ is also an eigenvalue of $S^{-1}TS$.

Now suppose μ is an eigenvalue of $S^{-1}TS$. Applying the same logic to non-invertible operator $(S^{-1}TS - \mu I)$, we get:

$$S^{-1}TS - \mu I = S^{-1}TS - \mu S^{-1}S = S^{-1}(TS - \mu S) = S^{-1}(T - \mu I)S$$

So $T - \mu I$ is not invertible, so μ is also an eigenvalue of T.

Thus we have shown that T and $S^{-1}TS$ have the same eigenvalues. □

(b) If u is an eigenvector of $S^{-1}TS$, then the eigenvector of T with the same eigenvalue is Su.

14 Give and example of an operator on $\mathbb{R}^4$ that has no (real) eigenvalues.

Solution:

Let us define an operator $T \in \mathcal{L}(\mathbb{R}^4)$ as:

$$T(x_1, x_2, x_3, x_4) = (x_2, -2x_1, 3x_4, -4x_3).$$

Indeed, if λ were an eigenvalue of T, then the following system would have solution for at leat one non-zero x_i:

$$\begin{aligned} x_2 &= \lambda x_1 \\ -2x_1 &= \lambda x_2 \\ 3x_4 &= \lambda x_3 \\ -4x_3 &= \lambda x_4 \end{aligned}$$

It follows from the first two equations that $\lambda^2 = -2$ (if x_1 and x_2 are not zero). From the last two equations, it follows that $\lambda^2 = -12$ (if x_3 and x_4 are not zero). Thus $\lambda \notin \mathbb{R}$ and T is the desired operator. □

15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Solution:

We conclude from propositions 3.129 and 3.131 that $S \in \mathcal{L}(V)$ is injective if and only if $S' \in \mathcal{L}(V')$ is injective. This property can be reformulated as: S is not injective if and only if S' is not injective.

Suppose λ is an eigenvalue of T. By 5.7, it is equivalent to $T - \lambda I$ being not injective. As stated above, $T - \lambda I$ is not injective if and only if $(T - \lambda I)'$

is not injective. Using properties of dual maps, we get:

$$(T - \lambda I)' = T' - \lambda I'$$

where I' is an identity operator on dual space. Hence, $T' - \lambda I'$ is not injective and λ is an eigenvalue of T'.

Thus, λ is an eigenvalue of T is and only if it is an eigenvalue of T'. □

16 Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

Solution:

Let v be an eigenvector of T with eigenvalue λ. v can be written in the given basis as:

$$v = a_1 v_1 + \cdots + a_n v_n = \sum_{k=1}^{n} a_k v_k$$

Then we will act on it by the operator T:

$$Tv = T(\sum_{k=1}^{n} a_k v_k) = \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \mathcal{M}(T)_{j,k} v_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_k \mathcal{M}(T)_{j,k} \right) v_j$$

and also:

$$Tv = \lambda v = \sum_{j=1}^{n} \lambda a_j v_j$$

From these two equations we conclude that:

$$\lambda a_j = \sum_{k=1}^{n} a_k \mathcal{M}(T)_{j,k}$$

Take the largest coefficient a_j. Then:

$$\lambda = \sum_{k=1}^{n} \frac{a_k}{a_j} \mathcal{M}(T)_{j,k}$$

Then we examine the absolute value of λ:

$$|\lambda| = \left|\sum_{k=1}^{n} \frac{a_k}{a_j}\mathcal{M}(T)_{j,k}\right| \leq \sum_{k=1}^{n} \left|\frac{a_k}{a_j}\mathcal{M}(T)_{j,k}\right| \leq \sum_{k=1}^{n} |\mathcal{M}(T)_{j,k}| \leq n \max\{|\mathcal{M}(T)_{j,k}|\}$$

where the first inequality comes from properties of absolute value, second inequality from the fact that a_j is largest coefficient, so that $a_k/a_j \leq 1$, and in the third inequlity we replaced matrix elements with the largest matrix element.

Thus we have arrived at the desired inequality. $\square$

17 Suppose $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification $T_{\mathbb{C}}$.

Solution:

Let λ be an eigenvalue of T. That means $T - \lambda I$ is not injective. From *Problem 3B.33* we know that $(T - \lambda I)_{\mathbb{C}}$ is not injective if and only if $T - \lambda I$ is not injective. Notice that for any $u, v \in V$:

$$\begin{aligned}(T - \lambda I)_{\mathbb{C}}(u + iv) &= (T - \lambda I)u + i(T - \lambda I)v = (Tu + iTv) - \lambda(Iu + iIv) \\ &= T_{\mathbb{C}}(u + iv) - \lambda I_{\mathbb{C}}(u + iv) = (T_{\mathbb{C}} - \lambda I_{\mathbb{C}})(u + iv)\end{aligned}$$

So, $(T - \lambda I)_{\mathbb{C}} = T_{\mathbb{C}} - \lambda I_{\mathbb{C}}$ and thus $T_{\mathbb{C}} - \lambda I_{\mathbb{C}}$ is not injective, which means λ is an eigenvalue of $T_{\mathbb{C}}$. $\square$

18 Suppose $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.

Solution:

Suppose $\lambda = a + ib$ is an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $v + iu$. Then:

$$T_{\mathbb{C}}(v + iu) = \lambda(v + iu) = (av + bu) + i(bv + au) = T(v) + iT(u)$$

Thus, $T(v) = av + bu$ and $T(u) = bv + au$. Now examine the combination $\overline{\lambda}(v - iu)$:

$$\overline{\lambda}(v - iu) = (a - ib)(v - iu) = (av + bu) - i(bv + au) = Tu - iTv = T_{\mathbb{C}}(u - iv)$$

Thus, if λ is an eigenvalue of $T_{\mathbb{C}}$ with eigenvector $u + iv$, then $\overline{\lambda}$ is also an eigenvalue of $T_{\mathbb{C}}$ but with eigenvector $u - iv$. Reverse statement is obtained if we change the roles of λ and $\overline{\lambda}$. $\square$

19 Show that the forward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

Solution:

Suppose λ is an eigenvalue of T. Then:

$$T(z_1, z_2, z_3, \dots) = \lambda(z_1, z_2, z_3, \dots) = (0, z_1, z_2, \dots)$$

So, $\lambda z_1 = 0$, $\lambda z_2 = z_1$, etc. If $z_1 \neq 0$, then from the first equation $\lambda = 0$. But it contradicts the second equation as $0 \cdot z_2$ cannot be equal to nonzero number like z_1. Thus we conclude that $z_1 = 0$, and then the second equation turns to $\lambda z_2 = 0$. Repeating the same argument, we arrive at $z_2 = 0$ and $\lambda z_3 = 0$. Continuing this leads to $\lambda z_k = 0$ for every $k \in \mathbb{N}$, which means that the supposed eigenvector is a zero-vector. By definition, 0 is not an eigenvector, hence T has no eigenvectors and no eigenvalues. $\square$

20 Define the backward shift operator $S \in \mathcal{L}(\mathbb{F}^\infty)$ defined by:

$$S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$$

(a) Show that every element of $\mathbb{F}$ is an eigenvalue of S.

(b) Find all eigenvectors of S.

Solution:

Take some $\lambda \in \mathbb{F}$ and suppose it is an eigenvalue of S.

$$S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots) = \lambda(z_1, z_2, z_3, \dots)$$

Hence, $\lambda z_k = z_{k+1}$ for every $k \in \mathbb{N}$.

If $\lambda = 0$, then we can take $z_1 = 0$ and arbitrary z_2, z_3, etc. So, for $\lambda = 0$, eigenvectors are $(0, z_1, z_2, \dots)$, where $z_k \in \mathbb{F}$.

If $\lambda \neq 0$, then we choose nonzero z_k such that $z_{k+1} = \lambda z_k$. So, for $\lambda \neq 0$, eigenvectors are $(1, \lambda, \lambda^2, \dots)$.

Thus, every $\lambda \in \mathbb{F}$ is an eigenvalue. $\square$

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Solution:

Suppose λ is an eigenvalue of T with eigenvector v: $Tv = \lambda v$. As T is an invertible operator, we write:

$$T^{-1}(\lambda v) = T^{-1}Tv = v = \lambda T^{-1}v$$

Thus, we have $T^{-1}v = (1/\lambda)v$. This shows both required points: λ and $1/\lambda$ are eigenvalues of T and T^{-1} with the same eigenvector v. As $(T^{-1})^{-1} = T$, the argument works in the opposite direction too. $\square$

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u$$

Prove that 3 or -3 is an eigenvalue of T.

Solution:

Take a linear combination $u + w$. If $u + w \neq 0$, then

$$T(u + w) = Tu + Tw = 3w + 3u = 3(u + w)$$

Thus, 3 is an eigenvalue of T.

If $u + w = 0$, then take $u - w$, which in that case is nonzero. Then:

$$T(u - w) = Tu - Tw = 3w - 3u = -3(u - w)$$

Thus, -3 is an eigenvalue of T.

So we have shown that indeed 3 or -3 is an eigenvalue of T. $\square$

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Solution:

Assume λ is an eigenvalue of ST with eigenvector v: $STv = \lambda v$. It can be thought as $S(Tv) = \lambda v$. Now examine the following:

$$TS(Tv) = T(STv) = T(\lambda v) = \lambda Tv$$

Hence, Tv is an eigenvector of TS that has eigenvalue λ. Tv is nonzero, otherwise $S(Tv)$ must be zero, but it is not.

Similar argument (changing roles of S and T) gives that every eigenvalue of TS is also an eigenvalue of ST.

Thus, ST and TS has the same eigenvalues. $\square$

24 Suppose A is an n-by-n matrix with entries in $\mathbb{F}$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tx = Ax$, where elements of $\mathbb{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries of each column of A equals 1. Prove that 1 is an eigenvalue of T.

Solution:

(a) Take $x = (1, 1, \ldots, 1)^t$, i.e. column vector with all entries equal to 1. Then:

$$Ax = \begin{pmatrix} \sum_i^n A_{1,i}x_i \\ \sum_i^n A_{2,i}x_i \\ \vdots \\ \sum_i^n A_{n,i}x_i \end{pmatrix} = \begin{pmatrix} \sum_i^n A_{1,i} \\ \sum_i^n A_{2,i} \\ \vdots \\ \sum_i^n A_{n,i} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

where in the second equals sign we used that every $x_i = 1$ and in the third equals sign we used that the sum of entries in each row equals 1. Thus, x is an eigenvector of T with an eigenvalue 1. □

(b) Let T' be a dual map of T. Then, matrix of T' is a transpose of matrix of T (proposition 3.132), so $\mathcal{M}(T') = A^t$.

As sum of all entries in each *column* of A equals 1, the sum of all entries in each *row* of A^t therefore equals 1. We know from the part (a) of this problem that the operator corresponding to A^t (that is, T') has eigenvalue 1. And by *Problem 5A.15*, the operator T must also have this eigenvalue. □

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Solution:

Assume that u and w are eigenvectors with distinct eigenvalues λ and μ. Let κ be an eigenvalue of T corresponding to $u + w$. κ may be distinct from λ or μ or equal to one of them. Examine the expression $T(u+w) - T(u+w) = 0$:

$$\begin{aligned} T(u + w) - Tu - Tw &= 0 \\ \kappa(u + w) - \lambda u - \mu w &= 0 \\ (\kappa - \lambda)u + (\kappa - \mu)w &= 0 \end{aligned}$$

Thus, we have a linear combination of u and w that is equal to 0. Note, that $\kappa - \lambda$ and $\kappa - \mu$ cannot be equal to zero simultaneously, as $\lambda \neq \mu$.

Hence, u and w are linearly dependent. But we have assumed that these vectors correspond to different eigenvalues, so by Theorem 5.11, they must be linearly independent. That is a contradiction.

Thus, u and w are eigenvectors corresponding to the same eigenvalue. $\square$

26 Suppose $T \in \mathcal{L}$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Solution:

Take any nonzero $v, w \in V$. These two vectors are eigenvectors of T, and so is their linear combination $u + w$. By the result of the previous problem, v and w correspond to the same eigenvalue.

This argument applies to all vectors in V, hence we have $Tv = \lambda v$ for all $v \in V$. At the same time $\lambda I v = \lambda v$ for all $v \in V$. Thus $T = \lambda I$. $\square$

27 Suppose that V is finite-dimensional and $k \in \{1, \ldots, \dim V - 1\}$. Suppose $T \in \mathcal{L}$ is such that every subspace of V of dimension k is invariant under T. Prove that T is a scalar multiple of the identity operator.

Solution:

If $k = 1$, then every vector in V is an eigenvector. By the result of the previous problem, it means that T is a scalar multiple of the identity operator.

Suppose $k \geq 1$. Then take k distinct subspaces of V and construct their intersection. This intersection is either $\{0\}$ or a one-dimensional vector (sub)space. From *Problem 5A.3* we know that such intersection is also invariant under T. Taking arbitrary k-dimensional subspaces we can construct every one-dimensional subspace of V, thus returning to the $k = 1$ case. Hence T is a scalar multiple of the identity operator. $\square$

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

Solution:

$\operatorname{range} T$ is a subspace of V invariant under T. A maximum number of eigenvectors, that are elements of $\operatorname{range} T$, is $\dim \operatorname{range} T$ (5.12).

If $u \in V$ is an eigenvector of T, such that $u \notin \operatorname{range} T$, then the equality:

$$Tu = \lambda u$$

can be satisfied only if $\lambda = 0$. This value of λ is the corresponding eigenvalue.

Thus, there are at most $1 + \dim \operatorname{range} T$ distinct eigenvalues of T. $\square$

29 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and -4, 5 and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Solution:

We know three eigenvalues of T and the dimension of the vector space ($\mathbb{R}^3$) is 3, hence there is no other eigenvalue.

An operator $(T - 9I)$ is invertible, otherwise 9 would have been an eigenvalue of T, which it cannot be. Hence, there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (T - 9I)x = (-4, 5, \sqrt{7})$. □

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution:

Take nonzero $v \in V$. If $(T - 4I)v = 0$, then $Tv = 4v$, so v is an eigenvector and the eigenvalue (λ) is 4.

If $(T - 4I)v \neq 0$, then denote $w = (T - 4I)v$. If $(T - 3I)w = 0$, then $Tw = 3w$, so w is an eigenvector of $T$and $\lambda = 3$.

If $(T - 3I)2 \neq 0$, then denote $u = (T - 3I)w$. Then necessarily $(T - 2I)u = 0$, hence $Tu = 2u$, so u is an eigenvector of $T$and $\lambda = 2$.

Thus we have shown that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$. □

31 Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -I$.

Solution:

Take $(1, 0), (0, 1)$ as a basis of $\mathbb{R}^2$. The desired operator T is "rotation by $\pi/4$" and it is represented by the matrix:

$$\mathcal{M}(T) = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}$$

Indeed:

$$\mathcal{M}(T^4) = (\mathcal{M}(T))^4 = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}^4$$

$$\begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}^2 = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right)^2 & -2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) \\ 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathcal{M}(-I)$$

Thus, $T^4 = -I$.

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

Solution:

Comment: Here we assume that the vector space is over real numbers. Otherwise, every operator would have an eigenvalue, as is proven later in Theorem 5.19.

Rewrite $T^4 = I$ as: $T^4 - I = 0$. We factorize this polynomial applied to an operator to get:

$$(T^2 + I)(T - I)(T + I) = 0$$

1 and -1 are not eigenvalues of T, so $(T - I)$ and $(T + I)$ are injective operators. That means $(T - I)v \neq 0$ and $(T + I)v \neq 0$ for every nonzero $v \in V$. Hence we conclude that $T^2 + I = 0$, or if rewrite, $T^2 = -I$. $\square$

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective is and only if T^m is surjective.

Solution:

(a) If T is injective, then T^m is injective as a composition of operators.

If T^m is injective, then we prove by contradiction. Suppose T is not injective and $v \neq 0$, $v \in T$. Then:

$$T^m v = T^{m-1}(Tv) = T^{m-1}(0) = 0$$

so T^m is also not injective, contrary to our initial assumption.

Hence, T is injective if and only if T^m is injective. $\square$

(b) If T is surjective, then T^m is surjective as a composition of operators.

If T^m is surjective, then we prove by contradiction. Suppose T is not surjective. Take $w \in V$ such that $w \notin \operatorname{range} T$. As T^m is surjective, there exists such $v \in V$ that $T^m v = w$. Then:

$$T^m v = T(T^{m-1}v) = w$$

so $w \in \operatorname{range} T$, contrary to our initial assumption.

Hence, T is surjective if and only if T^m is surjective. $\square$

34 Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Prove that the list $v_1, \dots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

Solution:

Implication from the 'necessary condition' is just Theorem 5.11. So will show only implication from the 'sufficient condition'.

Assume $v_1, \dots, v_m$ is linearly independent list. Extend this list to the basis of V: $v_1, \dots, v_m, u_1, \dots, u_n$. Take an operator $T \in \mathcal{L}(V)$ such that

$$\begin{aligned} Tv_i &= \lambda_i v_i \\ Tu_j &= 0 \end{aligned}$$

for every $i \in \{1, \dots, m\}$ and every $j \in \{1, \dots, n\}$ with λ_i being distinct numbers in $\mathbb{F}$.

These values of Tv_i and Tu_j uniquely define T (by lemma 3.4). Note, that by construction, $v_1, \dots, v_m$ are eigenvectors of T with distinct eigenvalues, hence the desired operator exists. □

35 Suppose $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Solution:

Take differentiation operator $D(f) = f'$. Note that for every $k \in \{1, \dots, n\}$:

$$D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}$$

We see that $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is a list of eigenvectors of D with distinct eigenvalues, hence it is linearly independent. □

36 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Solution:

Take operator $D^2(f) = f''$. Note that for every $k \in \{1, \dots, n\}$:

$$D^2(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x)$$

We see that $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is a list of eigenvectors of D^2 with distinct eigenvalues, hence it is linearly independent. □

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

Solution:

A number λ is an eigenvalue of $\mathcal{A}$ if and only if $(\mathcal{A} - \lambda\mathcal{I})$ is not invertible (here $\mathcal{I}$ is identity operator in $\mathcal{L}(\mathcal{L}(V))$).

Let $S \in \operatorname{null}(\mathcal{A} - \lambda\mathcal{I})$. It means:

$$\begin{aligned}(\mathcal{A} - \lambda\mathcal{I})S &= 0\\ \mathcal{A}(S) - \lambda\mathcal{I}(S) &= 0\\ TS - \lambda S &= 0\\ (T - \lambda I)S &= 0\end{aligned}$$

$S \neq 0$, hence for the last equality to hold, it must be that $\operatorname{null}(T - \lambda I) = \operatorname{range} S \neq \{0\}$. Hence, $(T - \lambda I)$ is not injective. Thus we see that λ is an eigenvalue of $\mathcal{A}$ is and only if λ is an eigenvalue of T. $\square$

38 Suppose V is finite-dimensional , $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U$$

for each $v \in V$.

(a) Show that the definition of T/U makes sense (which requires using the condition that U is invariant under T) and show that T/U is an operator on V/U.

(b) Show that each eigenvalue of T/U is an eigenvalue of T.

Solution:

(a) By definition $v + U = \{v + u : u \in U\}$. So if we act on a linear combination $v + u$ by T we get:

$$T(v + u) = Tv + Tu$$

U is invariant under T: $Tu \in U$. So $(Tv + Tu) \in \{Tv + u : u \in U\}$ and the definition makes sense.

Let us check that T/U is a linear map.

Additivity: Suppose $v, w \in V$. Then:

$$\begin{aligned}(T/U)\left((v + U) + (w + U)\right) &= (T/U)(v + w + U) = T(v + w) + U\\ &= Tv + Tw + U = (Tv + U) + (Tw + U)\\ &= (T/U)(v + U) + (T/U)(w + U) \quad \checkmark\end{aligned}$$

Homogeneity: Suppose $v \in V$ and $\lambda \in \mathbb{F}$.

$$\begin{aligned}(T/U)\,(\lambda(v+U)) &= (T/U)(\lambda v + U) \\ &= T(\lambda v) + U = \lambda T v + U = \lambda(Tv + U) \\ &= \lambda(T/U)(v+U) \quad \checkmark\end{aligned}$$

(b) Suppose λ is an eigenvalue of (T/U) with eigenvector $v + U$.

$$\begin{aligned}(T/U)(v+U) &= Tv + U \\ &= \lambda v + U\end{aligned}$$

Hence $(Tv - \lambda v) \in U$ by lemma 3.101. Denote $u = Tv - \lambda v$, so $Tv = \lambda v + u$.
Take $w \in V$, then:

$$T(v+w) = Tv + Tw = \lambda v + u + Tw$$

We would like to find w such that $v+w$ is an eigenvector of T with eigenvalue λ. For that we need $u + Tw = \lambda w$. Rewriting it, we get:

$$(\lambda I - T)w = u$$

If $(\lambda I - T)$ is not invertible, then $(T - \lambda I)$ is not invertible and hence λ is an eigenvalue of T.
If $(\lambda I - T)$ is invertible, then:

$$w = (\lambda I - T)^{-1} u$$

Which is the sought vector and thus λ is an eigenvalue of T. $\square$

39 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.
Solution:
$\longrightarrow$ Assume T has an eigenvalue. We need the following identity (from Fundamental Theorem of linear maps:

$$\dim \operatorname{range}(T - \lambda I) = \dim V - \dim \operatorname{null}(T - \lambda I)$$

Note, that $T - \lambda I$ is a polynomial $p(z) = z - \lambda$ applied to T. By proposition 5.18, $\operatorname{range}(T - \lambda I)$ is invariant under T.
There is at least eigenvector of T, hence $\dim \operatorname{null}(T - \lambda I) \geq 1$ and therefore $\dim \operatorname{range} T - \lambda I \leq \dim V - 1$.

If it is equality, then range $(T - \lambda I)$ is the desired subspace of V.

If it is less than $\dim V - 1$, then we extend a basis of range $(T - \lambda I)$ until we get $\dim V - 1$ vectors in the basis and thus a subspace (let us denote it W) of the desired dimension. W is invariant under $(T - \lambda I)$ by *Problem 5A.1b*. To show that W is also invariant under T, suppose $w_1, w_2 \in W$ are such that $(T - \lambda I)w_1 = w_2$. Then, rearranging the terms, we get:

$$Tw_1 = w_2 + \lambda w_1$$

$(w_2 + \lambda w_2) \in W$, hence $Tw_1 \in W$ and thus we have shown that W is a subspace of V invariant under T with dimension $\dim V - 1$, as desired.

$\longleftarrow$ Assume U is a subspace of V of dimension $\dim V - 1$ that is invariant under T. Examine the operator (T/U) (as in *Problem 5A.38*). It is an operator on V/U — a vector space with dimension (proposition 3.105):

$$\dim V/U = \dim V - \dim U = 1$$

By *Problem 3A.7*, the operator (T/U) is a scalar multiple of identity:

$$(T/U)(v + U) = \lambda(v + U) = \lambda v + U$$

Thus, by definition, λ is an eigenvalue of (T/U) and from *Problem 5A.38* we know that T has the same eigenvalues as (T/U) does. Thus, T has an eigenvalue. $\square$

40 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that:

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Solution:

$$p = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$$
$$p(STS^{-1}) = a_0 + a_1 STS^{-1} + a_2(STS^{-1})^2 + \cdots + a_n(STS^{-1})^n$$

Notice that:

$$(STS^{-1})^2 = STS^{-1}STS^{-1} = ST^2S^{-1}$$
$$(STS^{-1})^3 = STS^{-1}STS^{-1}STS^{-1} = ST^3S^{-1}$$

And so on. Hence:

$$\begin{aligned} p(STS^{-1}) &= a_0 + a_1 STS^{-1} + a_2 ST^2S^{-1} + \cdots + a_n ST^nS^{-1} \\ &= S(a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n)S^{-1} = Sp(T)S^{-1} \quad \square \end{aligned}$$

41 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$.

Solution:

Consider $p(T)u$ for any $u \in U$.

$$p(T)u = (a_0 + a_1 T + \cdots a_n T^n)u = a_0 u + a_1 Tu + \cdots a_n T^n u$$

As U is invariant under T, any $T^k u$ is in U, so as any scalar multiple of $T^k u$. Thus $p(T)u \in U$, which means U is invariant $p(T)$ for any $p \in \mathcal{P}(\mathbb{F})$. $\square$

42 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$.

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all subspaces of $\mathbb{F}^n$ that are invariant under T.

Solution:

(a) Eigenvalues are: 1, 2, ..., n. Corresponding eigenvectors are: $a_1 e_1$, $a_2 e_2$, ..., $a_n e_n$, where $a_1, \ldots, a_n \in \mathbb{F}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. Indeed:

$$T(\ldots, 0, x_k, 0, \ldots) = (\ldots, 0, kx_k, 0, \ldots) = k(\ldots, 0, x_k, 0, \ldots)$$

The dimension of $\mathbb{F}^n$ is n, so there are no more eigenvalues.

(b) Define $U_k = \operatorname{span}(e_k)$. Then the subspaces of $\mathbb{F}^n$ invariant under T are: $\{0\}$, every U_k and every direct sum of any combination of U_k's.

43 Suppose V is finite-dimensional, $\dim V > 1$ and $T \in \mathcal{L}(V)$. Prove that $\{p(T) : p \in \mathcal{P}(\mathbb{F})\} \neq \mathcal{L}(V)$.

Solution:

Denote a set of all $p(T)$ as W. Suppose $W = \mathcal{L}(V)$.

Note, that $Tp(T) = p(T)T$ for every $p(T) \in W$. Denote invertible polynomials of T as $q(T)$. For every such polynomial it is true that $q(T)T = Tq(T)$. And hence $T = q^{-1}(T)Tq(T)$. Examinig the matrix representation of the last equality, we see that

$$\mathcal{M}(T) = \mathcal{M}(q^{-1}Tq) = \mathcal{M}(q(T))^{-1}\mathcal{M}(T)\mathcal{M}(q(T))$$

for every $q(T)$. We supposed that polynomials of T can represent every linear operator on V, hence every invertible polynomial of T represent every invertible linear operator on V. That means the the obtained equality is equivalent to a proposition that matrix representation of T is the same in every basis of V. Thus T is a scalar multiple of identity, by *Problem 3D.19*.

But in the formulation of a problem we didn't restrict the choice of T and for every V with dim > 1, not every T is a scalar multiple of identity. Thus $W \neq \mathcal{L}(V)$. $\square$

5B The Minimal Polynomial

1 Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Solution:

Suppose 9 is an eigenvalue of T^2. Thus, there is nonzero $v \in V$ such that

$$T^2 v = 9v \quad \text{or} \quad (T - 9I)v = 0$$

Factorization of polynomial $T - 9I$ gives:

$$(T - 3I)(T + 3I)v = 0$$

Hence it is either $(T + 3I)v = 0$, so that -3 is an eigenvalue of T, or $(T - 3I)\left((T + 3I)v\right) = 0$, so that 3 is an eigenvalue of T.

To prove in the other direction, suppose that 3 or -3 is an eigenvalue of T with an eigenvector v, then:

$$T^2 v = T(Tv) = T(\lambda v) = \lambda T v = \lambda^2 v$$

For $\lambda = 3$ or -3, $\lambda^2 = 9$, which means 9 is an eigenvalue of T^2. $\square$

2 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dimensional.

Solution:

Let U be a nonzero finite-dimensional subspace of V and be invariant under T. As V is a complex vector space, so is its subspace U, hence $T|_U$ has an eigenvalue by Theorem 5.19, $T|_U u = \lambda u$. Thus, $Tu = T|_U u = \lambda u$, meaning T has an eigenvalue, which contradicts our assumption that T has no eigenvalues.

If U is $\{0\}$ then $T|_U$ can't have any eigenvalues by definition. If U is infinite-dimensional, the existence of an eigenvalue is not obligatory. $\square$

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find the minimal polynomial of T.

Solution:

(a) We use notation $e_1, \ldots, e_n$ for the standard basis of $\mathbb{F}^n$. Suppose λ is an eigenvalue of T, then the system of equations holds:

$$\begin{aligned} \lambda x_1 &= x_1 + \cdots + x_n \\ &\vdots \\ \lambda x_n &= x_n + \cdots + x_n. \end{aligned}$$

Note, that this system is solved by combinations: (i) $x_1 = x_2 = \cdots = x_n = 1$ and $\lambda = n$; (ii) $x_k = 1$, $x_{k+1} = -1$, $x_j = 0$ $(j \neq k, k+1)$ and thus $\lambda = 0$ (for every k running from 1 to $n-1$). In other words, 1 and 0 are eigenvalue of T with eigenvectors $(e_1 + \cdots + e_n)$ and $e_1 - e_2, e_3 - e_2, \ldots, e_{n-1} - e_n$. Thus, we have found n eigenvectors; let us show that this list of vectors is linearly independent (and hence there are no other linearly independent eigenvectors).

Suppose the list $e_1 + \cdots + e_n, e_1 - e_2, \ldots, e_{n-1} - e_n$ is linearly dependent. Then there are such nonzero $a_1, \ldots, a_n \in \mathbb{F}$ such that:

$$a_1(e_1 - e_2) + \cdots + a_{n-1}(e_{n-1} - e_n) + a_n(e_1 + \cdots e_n) = 0$$

Rearranging the terms and collecting them by e_i's gives:

$$(a_1 + a_n)e_1 + (a_2 - a_1 + a_n)e_2 + \cdots + (a_{n-1} - a_{n-2} + a_n)e_{n-1} + (a_n - a_{n-1})e_n = 0$$

The list $e_1, \ldots, e_n$ is linearly independent, hence every coefficient of e_i's must equal zero:

$$\begin{aligned} a_1 + a_n &= 0 \\ a_2 - a_1 + a_n &= 0 \\ a_3 - a_2 + a_n &= 0 \\ &\vdots \\ a_{n-1} - a_{n-2} + a_n &= 0 \\ a_n - a_{n-1} &= 0 \end{aligned}$$

Successively solving equations from first to $(n-1)$'th gives: $a_1 = -a_n$, $a_2 = -2a_n$, $a_3 = -3a_n, \ldots, a_{n-1} = -(n-1)a_n$. Meanwhile, the last equation gives $a_{n-1} = a_n$. $a_n = -(n-1)a_n$ (if $n \neq 0$ as in our case) only if $a_n = 0$, hence all other $a_i = 0$. Thus, the assumption of linear dependence is not correct, and the list $e_1 + \cdots + e_n, e_1 - e_2, \ldots, e_{n-1} - e_n$ is linearly independent. This shows that we indeed found all eigenvalues and all (linearly independent) eigenvectors. $\square$

(b) Let us examine the action of T on any vector in the standard basis:

$$\begin{aligned} Te_i &= e_1 + \cdots + e_n \\ T^2 e_i &= T(Te_i) = T\left(\sum_{j=1}^{n} e_j\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} e_j = n\sum_{j=1}^{n} e_j \end{aligned}$$

Thus we see that $T^2 e_i = nTe_i$. It is true for all basis vectors and because of linearity, for all vectors in $\mathbb{F}^n$. Thus, the minimal polynomial is:

$$p(T) = T^2 - nT; \quad p(z) = z^2 - nz$$

Indeed, zeros of $p(z)$ are the eigenvalues found in (a).

4 Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$, and $\alpha \in \mathbb{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Solution:

$\longrightarrow$ Suppose v is an eigenvector of $p(T)$ with eigenvalue α. By the Fundamental Theorem of Algebra, $p(z) - \alpha$ can be factorized and hence $p(T) - \alpha I = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)$, where λ_k are zeros of $p(z) - \alpha$ (possibly repeated). Then:

$$\left(\sum_{k=0}^{n} a_k T^k - \alpha I\right) v = c(T - \lambda_1 I)(T - \lambda_2 I)\ldots(T - \lambda_n I)v = 0$$

The last equation means that at least one of $(T - \lambda_j I)$ is not invertible, hence λ_j is an eigenvalue of T. Thus, there is some eigenvalue λ of T such that $p(\lambda) = \alpha$.

$\longleftarrow$ Suppose $\alpha = p(\lambda)$ for some eigenvalue of T. Let v be and eigenvector associated with λ. Apply $p(T)$ to v:

$$p(T)v = p(\lambda)v = \alpha v$$

where the first equation sign comes from the fact, shown in the proof of Theorem 5.27. Thus, α is an eigenvalue of $p(T)$. $\square$

5 Give and example of an operator on $\mathbb{R}^2$ that shows the result in Exercise 4 does not hold if $\mathbb{C}$ is replaced with $\mathbb{R}$.

Solution:

If $\mathbb{C}$ is replaced by $\mathbb{R}$ in the previous exercise, the result doesn't hold, because T doesn't have to have an eigenvalue. For example, $T \in \mathcal{L}(\mathbb{R}^2)$: $T(x, y) = (-y, x)$. Here T doesn't have an eigenvalue, but $p(T) = T^2$ does: $T^2 = -I$ and eigenvalue is -1.

6 Suppose $T \in \mathcal{L}(\mathbb{F}^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.

Solution:

Take the standard basis e_1, e_2 of $\mathbb{F}^2$. Then acting by T on it, we get:

$$\begin{aligned} Te_1 &= e_2 \\ Te_2 &= -e_1 \end{aligned}$$

Hence $T^2 e_1 = -e_1$ and the minimal polynomial of T is $T^2 + 1$.

7 (a) Give an example of $S, T \in \mathcal{L}(\mathbb{F}^2)$ such that the minimal polynomial of ST does not equal the minimal polynomial of TS.

(b) Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if at least one of S, T is invertible, then the minimal polynomial of ST equals the minimal polynomial of TS.

Solution:

(a) Take $S, T \in \mathcal{L}(\mathbb{F}^2)$ defined by:

$$T(x, y) = (x + y, 0); \quad S(x, y) = (0, y)$$

Then:

$$\begin{aligned} TS(x, y) &= T(0, y) = (y, 0) \\ ST(x, y) &= S(x + y, 0) = (0, 0) \end{aligned}$$

Here, $ST = 0$ hence the minimal polynomial of ST is $p(z) = 1$. To find minimal polynomial of TS, apply it to the standard basis:

$$\begin{aligned} &TSe_2 = e_1 \\ &(TS)^2 e_2 = TS(e_1) = e_1 \end{aligned}$$

Thus, $(TS)^2 e_2 - TSe_2 = 0$ and the minimal polynomial of TS is $q(z) = z^2 - z$. ST and TS have different zero polynomials, as desired.

(b) Suppose without loss of generality that S is invertible. Then $TS = S^{-1}(ST)S$.

Let $p(z)$ be a minimal polynomial of TS. Then, by *Problem 5A.40*:

$$p(TS) = p(S^{-1}(ST)S) = S^{-1}p(ST)S \tag{5.1}$$

By definition of minimal polynomial, $p(TS)v = 0$ for all $v \in V$. S is invertible, hence $Su = 0$, as well as $S^{-1}u = 0$ for some $u \in V$ if and only if $u = 0$. Thus we conclude that $p(ST)v = 0$ for all $v \in V$.

To prove that $p(z)$ is a minimal polynomial of ST, suppose there is a monic polynomial $q(ST)$ of lesser degree than $p(z)$ such that $q(ST) = 0$. Following eq. 5.1 in reverse order we conclude that $q(TS) = 0$, as well. This contradicts initial assumption that $p(z)$ is the minimal polynomial of TS, hence $p(z)$ is indeed the minimal polynomial of ST. $\square$

8 Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is the operator of counterclockwise rotation by $1°$. Find the minimal polynomial of T.

Solution:

Denote the angle of $1°$ by α. Examine how T acts on e_1 of the standard basis:

$$\begin{aligned} Te_1 &= \cos(\alpha)e_1 + \sin(\alpha)e_2 \\ T^2e_1 &= \cos(\alpha)Te_1 + \sin(\alpha)Te_2 = (\cos^2(\alpha) - \sin^2(\alpha))e_1 + 2\sin(\alpha)\cos(\alpha)e_2 \end{aligned}$$

Then we need to find coefficients c_0, c_1 that solve the following equation:

$$c_0e_1 + c_1Te_1 = -T^2e_1$$

Inserting expressions for Te_1 and T^2e_1 we get:

$$c_0e_1 + c_1\left(\cos(\alpha)e_1 + \sin(\alpha)e_2\right) = \left(\sin^2(\alpha) - \cos^2(\alpha)\right)e_1 - 2\sin(\alpha)\cos(\alpha)e_2$$

This equation is equivalent to a system of two linear equations:

$$\begin{cases} c_0 + c_1\cos(\alpha) = \sin^2(\alpha) - \cos^2\alpha \\ c_1\sin(\alpha) = -2\sin(\alpha)\cos(\alpha) \end{cases}$$

This system is solved by $c_0 = 1$, $c_1 = -2\cos(\alpha)$. Hence, the minimal polynomial of the operator of counterclockwise rotation by $1°$ is:

$$p(z) = z^2 - 2\cos(1°)z + 1 \approx z^2 - 1.9997z + 1$$

9 Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the minimal polynomial of T are rational numbers.

Solution:

Take any vector w from the basis of V such that $v \notin T$. Then $Tw, T^2w, \ldots,$ $T^{\dim V}w$ are linear combinations of basis vectors with rational coefficients (for Tw it follows from the fact that all entries of the matrix of T in the basis under consideration are rational; for T^kw the coefficients are combinations of sums and products of the matrix entries, hence are rational too). Suppose $c_0, c_1, \ldots, c_{n-1}$ $(n \leq \dim V)$ are coefficients of the minimal polynomial. It means these coefficients are solution of:

$$c_0 + c_1Tw + \cdots + c_{n-1}T^{n-1}w = T^nw.$$

This equation is equivalent to a system of n linear equations. Linear equations with rational coefficients have rational solutions, which means the minimal polynomial of T has rational coefficients.

10 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1}v)$$

for all integers $m \geq \dim V - 1$.

Solution:

If $m = \dim V - 1$, then the proposition is trivially true.

Suppose $m \geq \dim V - 1$.

The list $v, Tv, \ldots, T^{\dim V-1}v$ is of length $\dim V$, so there is no list of larger length that can be linearly independent (otherwise we would have contradiction with Theorem 2.22). Hence, the list $v, Tv, \ldots, T^m v$ is definitely linearly dependent.

Let k be the greatest number such that the list $v, Tv, \ldots, T^k v$ is linearly independent. By the linear dependence lemma (2.19):

$$\begin{aligned}
&\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^k v)\\
&\operatorname{span}(v, Tv, \ldots, T^{\dim V-1} v) = \operatorname{span}(v, Tv, \ldots, T^k v)
\end{aligned}$$

Hence we conclude that $\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V-1}v)$ for all $m \geq \dim V - 1$ indeed. $\square$

11 Suppose V is a two-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $T^2 - (a+d)T + (ad-bc)I = 0$.

(b) Show that the minimal polynomial of T equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad-bc) & \text{otherwise.} \end{cases}$$

Solution:

(a) To show the desired result, it is sufficient to show that $\mathcal{M}(T^2 - (a + d)T + (ad - bc)I) = \mathcal{M}(0)$.

$$\mathcal{M}(T^2) = \mathcal{M}(T) \cdot \mathcal{M}(T) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & ac + cd \\ ab + bd & bc + d^2 \end{pmatrix}$$

$$\begin{aligned} \text{Desired matrix} &= \begin{pmatrix} a^2 + bc - (a+d)a + ad - bc & ac + cd - (a+d)c + 0 \\ ab + bd - (a+d)b + 0 & bc + d^2 - (a+d)d + ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Thus we have the desired equality.

(b) First, suppose $b = c = 0$ and $a = d$. Then, the matrix of T is:

$$\mathcal{M}(T) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a \cdot \mathcal{M}(I)$$

Hence, $T - aI = 0$ and the minimal polynomial in that case is $p(z) = z - a$.

Second, suppose the constraints on a, b, c, d are not satisfied. That means, T is not a multiple of an identity operator, hence its minimal polynomial has degree greater than 1.

We have shown in part (a) that the monic polynomial $p(z) = z^2 - (a + d)z + (ad - bc)$ applied to T gives zero operator. Hence, it is the minimal polynomial of T. $\square$

12 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$. Find the minimal polynomial of T.

Solution:

In *Problem 5A.42* we have shown that T has eigenvalues: $1, 2, \ldots, n$. By Theorem 5.27, proposition (b), the minimal polynomial is:

$$p(z) = (z-1)(z-2)\cdots(z-n)$$

This polynomial has degree $n = \dim \mathbb{F}^n$, hence no factor in braces is repeated.

13 Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Prove that there exists a unique $r \in \mathcal{P}(\mathbb{F})$ such that $p(T) = r(T)$ and $\deg r$ is less than the degree of the minimal polynomial of T.

Solution:

Suppose $p(z)$ has degree less than the degree of the minimal polynomial. Then take $r(z) = p(z)$.

Suppose the degree of $p(z)$ is greater or equal to the degree of the minimal polynomial. Denote minimal polynomial by $q(z)$. Then applying polynomial division algorithm to $p(z)$ gives:

$$p(z) = q(z)s(z) + r(z)$$

with $r \in \mathcal{P}(\mathbb{F})$ having degree less than $\deg q$. Now note that:

$$p(T) = q(T)s(T) + r(T) = 0 \cdot s(T) + r(T) = r(T) \quad \square$$

14 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has minimal polynomial $4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1}.

Solution:

Examine the following:

$$\begin{aligned} p(T)T^{-5} &= 0(T^{-5}) = 0 \\ p(T)T^{-5} &= 4T^{-5} + 5T^{-4} - 6T^{-3} - 7T^{-2} + 2T^{-1} + 1 \end{aligned}$$

Hence, the minimal polynomial of T^{-1} is

$$q(z) = z^5 + \frac{5}{4}z^4 - \frac{3}{2}z^3 - \frac{7}{4}z^2 + \frac{1}{2}z + \frac{1}{4}.$$

0 is not the root of $p(z)$ hence it is not an eigenvalue of T and hence T^{-1} has the same number of eigenvalues as T (*Problem 5A.21*). Combining it with Theorem 5.27, we get that minimal polynomial of T^{-1} should be of the same degree as minimal polynomial of T. The obtained $q(z)$ meets this criterion. $\square$

15 Suppose V is finite-dimensional complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$. Define $f : \mathbb{C} \mapsto \mathbb{R}$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I)$$

Prove that f is not a continuous function.

Solution:

By Theorem 5.19, there is some eigenvalue λ of T. Suppose, its corresponding eigenvectors are $v_1, \dots, v_k$. Thus, $\operatorname{null}(T - \lambda I) = \operatorname{span}(v_1, \dots, v_k)$ and by the Fundamental Theorem of Linear Maps, $\dim \operatorname{range}(T - \lambda I) = n - k$.

We have shown in *Problem 5A.11* that there is arbitrarily small neighborhood of eigenvalue (particularly) in which all the numbers make $T - \alpha I$ invertible. If $T - \alpha I$ is invertible, then $\dim \operatorname{range}(T - \alpha I) = n$. Thus, $f(\lambda)$ have discontinuity at least at every eigenvalue, and thus it is not a continuous function. $\square$

16 Suppose $a_0, \dots, a_{n-1} \in \mathbb{F}$. Let T be the operator on $\mathbb{F}^n$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & -a_2 \\ & & \ddots & & \vdots \\ & & & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}$$

Here all entries of the matrix are 0 except for all 1's on the line under the diagonal and the entries in the last column (some of which might also be 0). Show that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n.$$

Solution:

Firstly, let us examine how T acts on the standard basis:

$$\begin{aligned} Te_1 &= -a_0 e_n \\ Te_k &= e_{k-1} - a_{k-1} e_n \quad \text{for } k \in \{2, \dots, n\} \end{aligned}$$

Then take e_1 as a trial vector as successively apply powers of T to it. Such successive application leads to:

$$T^n e_1 = -a_0 e_1 - a_1 T e_1 + \cdots + a_{n-1} T^{n-1} e_1$$

We will show this by induction. The base case is $k = 2$:

$$T^2 e_1 = -a_0 \left(e_{n-1} - a_{n-1} e_n\right) = -a_0 e_{n-1} + a_{n-1} T e_1$$

Then, for every $k \in \{2, \dots, n\}$ we suppose that:

$$T^k e_1 = -a_0 e_{n-k+1} - \left(a_{n-k+1} T e_1 + \cdots + a_{n-1} T^{k-1} e_1\right) \tag{5.2}$$

If eq. 5.2 is true for k, then examine case of $k+1$.

$$\begin{aligned}
T^{k+1} e_1 = T(T^k e_1) &= -a_0 T e_{n-k+1} - \left(a_{n-k+1} T^2 e_1 + \cdots + a_{n-1} T^{(k+1)-1} e_1\right) \\
&= -a_0 (e_{n-k-1+1} - a_{n-k-1+1} e_n) \\
&\quad - \left(a_{n-k+1} T^2 e_1 + \cdots + a_{n-1} T^{(k+1)-1} e_1\right) \\
&= -a_0 e_{n-(k+1)+1} \\
&\quad - \left(a_{n-(k+1)+1} T e_1 + a_{n-k+1} T^2 e_1 + \cdots + a_{n-1} T^{(k+1)-1} e_1\right)
\end{aligned}$$

Hence, eq. 5.2 is true by induction. Inserting $k = n$ in it, we obtain the desired relation.

The obtained expression on $T^n e_1$ in terms of all other powers of T is unique as $e_1, Te_1, \dots, T^{n-1} e_1$ is a linearly independent list. Indeed, every subsequent $T^k e_1$ (except $T^n e_1$) has one additional basis vector, and thus it is not a liner combination of all previous terms.

Hence,

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

is a minimal polynomial of T. $\square$

17 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbb{F}$. Show that the minimal polynomial of $T - \lambda I$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

Solution:

Note that

$$q(T - \lambda I) = p\left((T - \lambda I) + \lambda I\right) = p(T) = 0$$

Suppose there is a monic polynomial $r(z)$ with degree less than $\deg q$ such that $r(T - \lambda I) = 0$. Then if we rewrite $r(T - \lambda I)$ in terms of T, then we get another polynomial $s(T)$. As we just rearranged expression, $s(T) = 0$. But $\deg s = \deg r < \deg p$, contradicting the fact that p is the minimal polynomial of T. Hence, q is indeed the minimal polynomial of $T - \lambda I$. $\square$

18 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbb{F}\backslash\{0\}$. Show that the minimal polynomial of λT is the polynomial q defined by $q(z) = \lambda^{\deg p} p\left(\frac{z}{\lambda}\right)$.

Solution:

Note that

$$q(\lambda T) = \lambda^{\deg p} p\left(\frac{\lambda T}{\lambda}\right) = \lambda^{\deg p} p(T) = 0$$

Here, the factor before $p(z/\lambda)$ makes $q(z)$ a monic polynomial. The rest is to show that $q(z)$ has minimal degree.

Suppose $r(\lambda T) = 0$ and $\deg r < \deg q = \deg p$. Then viewing expression for $r(\lambda T)$ as a polynomial of T shows that it is some $s(T)$ such that $s(T) = 0$ and $\deg s < \deg p$ contradicting the fact that p is the minimal polynomial of T. Hence, q is the minimal polynomial of λT. $\square$

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbb{F})\}.$$

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

Solution:

Let p be the minimal polynomial of T. Then p being the minimal polynomial means that the list $v, Tv, \ldots, T^{\deg p} v$ is linearly dependent for all $v \in V$, while the list $v, Tv, \ldots, T^{\deg p - 1} v$ is linearly independent for some $v \in V$. Hence, the list $I, T, T, \ldots, T^{\deg p-1}$ is linearly independent list of maximal length with elements from $\mathcal{E}$. Thus, this list is the basis of $\mathcal{E}$ and $\mathcal{E}$ has dimension $\deg p$. $\square$

20 Suppose $T \in \mathcal{L}(\mathbb{F}^4)$ is such that the eigenvalues of T are 3, 5, 8. Prove that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

Solution:

Eigenvalues of T are zeros of the minimal polynomial. Let p be the minimal polynomial of T, so

$$p(z) = (z - 3)(z - 5)(z - 8) \cdot q(z)$$

Degree of $p(z)$ is at most 4, hence $\deg q$ is at most 1. If $p(z)$ had non-real zeros, they would come in pairs and $\deg q$ would be at least 2 (lemmas 4.14 and 4.16). Thus, $q(z)$ is a repeated factor $(z - 3)$, $(z - 5)$, or $(z - 8)$.

It means $(z - 3)^2(z - 5)^2(z - 8)^2$ is a polynomial multiple of the minimal polynomial. Hence, by Theorem 5.29, $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$. $\square$

21 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of T has degree at most $1 + \dim \operatorname{range} T$.

Solution:

Suppose $w \in \operatorname{null} T$, then if $p(z)$ is the minimal polynomial of T, then:

$$p(T)w = 0 = a_0 w + a_1 T w + \cdots + a_n T^n w = a_0 w$$

If $\operatorname{null} T = \{0\}$, then by the Fundamental Theorem of linear maps $\dim \operatorname{range} T = \dim V$, and we get the desired result as the degree of the minimal polynomial is at most $\dim V$ by 5.22. If $\operatorname{null} T \neq \{0\}$, then $a_0 = 0$.

Let $m = \dim \operatorname{range} T$. Range of T is invariant under T, so every $T^k v \in \operatorname{range} T$. A list of at most m vectors in $\operatorname{range} T$ can be linearly independent. Hence, the longest linearly independent list of powers of T applied to a vector is $Tv, T^2 v, \ldots, T^m v$ for all $v \in V$. Thus, necessarily there are such $c_1, \cdots, c_m$ that

$$T^{m+1} v = c_0 T v + \cdots + c_m T^m v$$

for all $v \in V$. Hence, the minimal polynomial of T has degree at most $1 + \dim \operatorname{range} T$. $\square$

22 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $I \in \operatorname{span}(T, T^2, \ldots, T^{\dim V})$.

Solution:

$\longrightarrow$ Suppose T is invertible. Then by lemma 5.32, the constant term of the minimal polynomial of T is nonzero. Hence, for every $v \in V$:

$$c_0 I v + c_1 T v + \cdots + c_m T^m v = 0$$

where m is the degree of the minimal polynomial. As it is true for every $v \in V$, we can rewrite it as:

$$I = -\frac{c_1}{c_0} T + \cdots + \frac{c_m}{c_0} T^m$$

Thus, $I \in \operatorname{span} T, \ldots, T^m$. Moreover, every other power of T is in the same span, hence $\operatorname{span} T, \ldots, T^m = \operatorname{span} T, \ldots, T^{\dim V}$ and thus $I \in \operatorname{span} T, \ldots, T^{\dim V}$.

$\longleftarrow$ Suppose $I \in \operatorname{span} T, \ldots, T^{\dim V}$. Let m be the smallest number (less that $\dim V$), for which it holds that there are nonzero $c_1, \ldots, c_m$ such that $I = c_1 T + \cdots + c_m T^m$. Rearranging the terms on the same side and dividing by c_m gives the minimal polynomial of T. It has nonzero constant term, hence by 5.32, T is invertible. $\square$

23 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that if $v \in V$, then $\operatorname{span}(v, Tv, \ldots, T^{n-1}v)$ is invariant under T.

Solution:

Let $v \in V$, m is degree of the minimal polynomial. Consider list $v, Tv, \ldots, T^{m-1}v$. $T^m v$ can be expressed as

$$T^m v = -c_0 v - c_1 Tv - \cdots - c_{m-1} T^{m-1} v$$

where c_j are coefficients of the minimal polynomial. Hence, $T(T^{m-1})v = T^m v$ is in $\operatorname{span}(v, Tv, \ldots, T^{m-1}v)$. Any other power is trivially in the same span:

$$T(v) = Tv \in \operatorname{span}(v, Tv, \ldots, T^{m-1}v),$$

$$T(T^k v) = T^{k+1} v \in \operatorname{span}(v, Tv, \ldots, T^{m-1}v),$$

where $k < (m-1)$. Thus, $\operatorname{span}(v, Tv, \ldots, T^{m-1}v)$ is invariant under T.

If $m = n$, then we are done. If $m < n$, then we have linearly dependent list $v, Tv, \ldots, T^n v$, and hence by linear dependence lemma $\operatorname{span}(v, Tv, \ldots, T^{n-1}v) = \operatorname{span}(v, Tv, \ldots, T^{m-1})$. Thus, $\operatorname{span}(v, Tv, \ldots, T^{n-1}v)$ is invariant under T. $\square$

24 Suppose V is finite-dimensional complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that 5 and 6 are eigenvalues of T and that T has no other eigenvalues. Prove that $(T - 5I)^{\dim V - 1}(T - 6I)^{\dim V - 1} = 0$.

Solution:

Eigenvalues of T are zeros of the minimal polynomial. Let p be the minimal polynomial of T, so

$$p(z) = (z-5)(z-6) \cdot q(z)$$

T has no other eigenvalues, while V is a complex vector space. Hence, $q(z) = (z-5)^x (z-6)^y$ where x, y are some nonnegative integers. Degree of $p(z)$ is at most $\dim V$, hence $\deg q$ is at most $\dim V - 1$. Moreover, the degree of each of the two factors is at most $\dim V - 1$. It means $(z-5)^{\dim V - 1}(z-6)^{\dim V - 1}$ is a polynomial multiple of the minimal polynomial. Hence, by Theorem 5.29, $(T - 5I)^{\dim V - 1}(T - 6I)^{\dim V - 1} = 0$. $\square$

25 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T.

(a) Prove that the minimal polynomial of T is a polynomial multiple of the minimal polynomial of the quotient operator T/U.

(b) Prove that

$$(\text{minimal polynomial of } T|_U) \times (\text{minimal polynomial of } T/U)$$

is a polynomial multiple of the minimal polynomial of T.

Solution:

(a) Let q be a minimal polynomial of T/U and p be a minimal polynomial of T. Referring to lemma 3.105, $\dim V/U \leq \dim V$, hence $\deg q \leq \deg p$. Then note:

$$p(T/U)(v+U) = (p(T)/U)\,(v+U) = p(T)v + U = 0 + U$$

Thus, $p(T/U) = 0$ for all $(v+U) \in V/U$. By proposition 5.29, $p(z)$ is a polynomial multiple of $q(z)$. $\square$

(b) Let q be a minimal polynomial of (T/U), s be a minimal polynomial of $T|_U$ and p be a minimal polynomial of T.

Note that in order q to be a minimal polynomial of T/U we need that $q(T)v \in U$ for all $v \in V$:

$$q(T/U)(v+U) = (q(T)/U)(v+U) = q(T)v + U = 0 + U \quad \Rightarrow q(T)v \in U$$

Then for any $v \in V$:

$$(sq)(T)v = s(T)\,(q(T)v) = 0$$

where the last equality sign comes from the fact that s is the minimal polynomial of $T|_U$.

Thus, $(sq)(T) = 0$ and therefore (by proposition 5.29) it is a polynomial multiple of the minimal polynomial of T. $\square$

26 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Prove that the set of eigenvalues of T equals the union of the set of eigenvalues of $T|_U$ and the set of eigenvalues of T/U.

Solution:

From *Problem 5B.25* we know that the product of minimal polynomials of $T|_U$ and T/U is a polynomial multiple of the minimal polynomial of T:

$$p = sq \cdot r$$

where we used the same notation as in previous problem. Suppose r has factors, not present in p. This means that either $T|_U$ or T/U has eigenvalues that are not eigenvalues of T. This is a contradiction. Hence, the set of eigenvalues of T is a union of the set of eigenvalues of $T|_U$ and the set of eigenvalues of T/U. $\square$

27 Suppose $\mathbb{F} = \mathbb{R}$, V is finite-dimensional, and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T_{\mathbb{C}}$ equals the minimal polynomial of T.

Solution:

Let p be a minimal polynomial of T and q be a minimal polynomial of $T_{\mathbb{C}}$. Note that:

$$p(T_{\mathbb{C}})(v + iu) = p(T)v + ip(T)u = 0 + i \cdot 0 = 0$$

Hence, $p(z)$ is a polynomial multiple of $q(z)$. At the same time:

$$q(T_{\mathbb{C}})(v + iu) = q(T)v + iq(T)u$$

which is true if and only if $q(T) = 0$ for all $v \in V$. Thus, $q(T)$ is a polynomial multiple of $p(z)$. The fact that $p = qr$ and $q = ps$, where r and s are some polynomials means that both r and s must equal 1. Thus, $p = q$, that is, T and $T_{\mathbb{C}}$ have the same minimal polynomial. $\square$

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T' \in \mathcal{L}(V')$ equals the minimal polynomial of T.

Solution:

Note that for any $p \in \mathcal{P}(\mathbb{F})$:

$$\begin{aligned} p(T')(\varphi) &= \left(a_0 I' + a_1 T' + \cdots + a_m (T')^m\right)(\varphi) \\ &= a_0 \varphi \circ I + a_1 \varphi \circ T + \cdots + a_m \varphi T^m \\ &= \varphi \circ p(T) = \left(p(T)\right)'(\varphi) \end{aligned}$$

Using *Problem 3F.16* we arrive at:

$$p(T') = 0 \iff p(T) = 0$$

Hence, the minimal polynomial of $T' \in \mathcal{L}(V')$ equals the minimal polynomial of T. $\square$

29 Show that every operator on a finite-dimensional vector space of dimension at least two has an invariant subspace of dimension two.

Solution:

We will prove this by induction.

Let V be a two-dimensional vector space. Then V is such two-dimensional space, invariant under any operator on it.

Now let V be a finite-dimensional vector space such that $\dim V > 2$ and suppose that every operator on a vector space of dimension less than $\dim V$ and greater or equal than 2 has an invariant subspace of dimension 2.

Take any $T \in \mathcal{L}(V)$. Then by the Fundamental Theorem of Linear Maps:

$$\dim V = \dim \operatorname{range} T + \dim \operatorname{null} T$$

At least one of the terms in the sum on the right is greater or equal than 2. Take the one with the dimension greater than 1 and call it U. Both range and null-space of T are invariant under T, so U is invariant under T. Moreover, closing our attention on $T|_U$, we see that U has a subspace of dimension 2 that is invariant under $T|_U$. This is also a subspace of V. Thus, V has a subspace of dimension 2 invariant under T. $\square$

5C Upper-Triangular Matrices

1 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and T^2 has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some basis of V.

Solution:

Let $V = \mathbb{R}^2$ and T be an operator of rotation by $\pi/2$:

$$Te_1 = e_2, \quad Te_2 = -e_1$$

where e_1, e_2 is the standard basis. From Example 5.9a we know that this operator has no eigenvalues, and hence its minimal polynomial cannot be written in form $(z-\lambda_1)\cdots(z-\lambda_n)$ (Theorem 5.27), which implies (by Theorem 5.44) that there isn't a basis in which T has an upper-triangular matrix.

At the same time, for T^2:

$$T^2e_1 = -e_1, \quad T^2e_2 = -e_2$$

so the matrix of T^2 in the standard basis is:

$$\mathcal{M}(T^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and it is upper-triangular.

Thus, we have shown a counterexample. $\square$

2 Suppose A and B are upper-triangular matrices of the same size, with $\alpha_1, \ldots, \alpha_n$ on the diagonal of A and $\beta_1, \ldots, \beta_n$ on the diagonal of B.

(a) Show that $A+B$ is an upper-triangular matrix with $\alpha_1+\beta_1, \ldots, \alpha_n+\beta_n$ on the diagonal.

(b) Show that AB is an upper-triangular matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diagonal.

Solution:

(a) Using definition of matrix addition (3.34), element on row j, column k of $A+B$ is the sum of elements $A_{j,k}$ and $B_{j,k}$. Thus, diagonal elements are:

$$(A+B)_{j,j} = A_{j,j} + B_{j,j} = \alpha_j + \beta_j$$

The elements under the diagonal of both A and B equal zero, hence their sum is also zero. Thus, $A+B$ is an upper-triangular matrix, as desired. $\square$

(b) Using definition of matrix multiplication (3.41), we get that the diagonal elements are:

$$(AB)_{j,j} = \sum_{r=1}^{n} A_{j,r} B_{r,j}$$

note that as both A and B are upper-triangular, $A_{j,r} = 0$ if $j > r$, and $B_{r,j} = 0$ if $r > j$. Thus, $A_{j,r}B_{r,j} \neq 0$ if and only if $r = j$. Thus we have shown that the diagonal elements of AB are $\alpha_1\beta_1, \dots, \alpha_n\beta_n$.

Now examine elements under the diagonal, $(AB)_{j,k}$ with $j > k$.

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}$$

In this case, again, $B_{r,k} = 0$ if $k > r$. But if we take nonzero $B_{r,k}$, that is, such that $r < k$, then as $r < j$, we conclude that $r < j$ and hence $A_{j,r} = 0$. Thus, elements of AB under the diagonal are all zero and therefore AB is an upper-triangular matrix. $\square$

3 Suppose $T \in \mathcal{L}(V)$ is invertible and $v_1, \dots, v_n$ is a basis of V with respect to which the matrix of T is upper triangular, with $\lambda_1, \dots, \lambda_n$ on the diagonal. Show that the matrix of T^{-1} is also upper triangular with respect to the basis $v_1, \dots, v_n$, with

$$\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$$

on the diagonal.

Solution:

As T is an upper-triangular matrix, $U_k = \operatorname{span}(v_1, \dots, v_k)$ is invariant under T for each $k = 1, \dots, n$ (by Theorem 5.39). That is, for every $u \in U_k$, there is $Tu \in U_k$. T is invertible, so for every $w \in U_k$ there is a unique $u \in U_k$

such that $Tu = w$. In other words, $T^{-1}w = u \in U_k$, for every $w \in U_k$, which means that U_k is invariant under T^{-1}.

Thus we have shown that T^{-1} has an upper-triangular matrix with respect to same basis $v_1, \ldots, v_n$ (by 5.39).

Now we use result of *Problem 5C.2* (b). The product TT^{-1} is identity operator I. The diagonal elements of TT^{-1} are $\lambda_1(T^{-1})_{1,1}, \ldots, \lambda_n(T^{-1})_{n,n}$ and they all are equal to 1. Hence, the diagonal elements of T^{-1} are

$$\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$$

as desired. □

4 Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

Solution:

Such an operator is an operator on $\mathbb{F}$ of counterclockwise rotation by 90°. Indeed, its matrix is:

$$\mathcal{M}(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

But this operator is invertible, inverse being rotation clockwise by 90°, with matrix:

$$\mathcal{M}(T^{-1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

5 Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Solution:

Take $V = \mathbb{R}^2$. Then such operator would be $T(x, y) = T(x - y, y - x)$. Indeed, in the standard basis its matrix is:

$$\mathcal{M} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

In order to convince oneself that this operator is not invertible, note what it does to the vector (1,1):

$$T(1, 1) = (1 - 1, 1 - 1) = (0, 0)$$

Hence, T is not injective and therefore not invertible. □

6 Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional, and $T \in \mathcal{L}(V)$. Prove that if $k \in \{1, \ldots, \dim V\}$, then V has a k-dimensional subspace invariant under T.

Solution:

By the Theorem 5.27, minimal polynomial of V has the form $(z-\lambda_1)\cdots(z-\lambda_m)$, where $\lambda_1, \ldots, \lambda_m$ are eigenvalues of T. That implies (by Theorem 5.44) that T has an upper-triangular matrix with respect to some basis $v_1, \ldots, v_{\dim V}$.

This in turn is equivalent to the fact that span $(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, \dim V$. Note that each span $(v_1, \ldots, v_k)$ is a k-dimensional subspace of V, as lists $v_1, \ldots, v_k$ are linearly independent.

Thus, indeed an operator on a complex finite-dimensional vector space has a subspace of dimension $k \in \{1, \ldots, \dim V\}$ that is invariant under this operator. $\square$

7 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) Prove that there exists a unique monic polynomial p_v of smallest degree such that $p_v(T)v = 0$.

(b) Prove that the minimal polynomial of T is a polynomial multiple of p_v.

Solution:

(a) The list $v, Tv, \ldots, T^{\dim V}v$ has length $1+\dim V$ and thus is linearly dependent. By the linear dependence lemma (2.19), there is the smallest positive integer $m < \dim V$ such that $T^m v$ is a linear combination of $v, Tv, \ldots, T^{m-1}v$. Thus, there exist scalars $c_0, c_1, \ldots, c_{m-1} \in \mathbb{F}$ such that

$$c_0 v + c_1 Tv + \cdots + c_{m-1}T^{m-1}v + T^m v = 0 \tag{5.3}$$

Define a monic polynomial $p_v \in \mathcal{P}_m(\mathbb{F})$ by

$$p_v(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m$$

Then 5.3 implies that $p_v(T)v = 0$. Thus, there is a polynomial of smallest degree with the desired property.

To prove the uniqueness part, suppose there exists $q_v \in \mathcal{P}(\mathbb{F})$ of the same degree as p_v such that $q_v(T)v = 0$. Then $(p_v - q_v)(T)v = 0$ and also $\deg(p_v - q_v) < \deg p$. If $p_v - q_v$ were not equal to 0, then we could divide $p_v - q_v$ by the coefficient of the highest-order term in $p_v - q_v$ to get a monic polynomial of smaller degree that p_v that when applied to T sends v to 0, which cannot be. Thus $p_v - q_v = 0$, as desired. $\square$

(b) Let $p \in \mathcal{P}(\mathbb{F})$ be the minimal polynomial of T. By the division algorithm for polynomials (4.9), there exist polynomials $s, r \in \mathcal{P}(\mathbb{F})$ such that

$$p = p_v s + r$$

and $\deg r < \deg p_v$. We have:

$$0 = p(T) = p_v(T)s(T) + r(T)$$

The equation above implies that $r = 0$, otherwise dividing r by its highest-degree coefficient would produce a monic polynomial that when applied to T sends v to 0, and that cannot be as demonstrated in part (a). Thus, we have $p = p_v s$. Hence, the minimal polynomial of T is a polynomial multiple of p_v, as desired. $\square$

8 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and there exists a nonzero vector $v \in V$ such that $T^2 v + 2Tv = -2v$.

(a) Prove that if $\mathbb{F} = \mathbb{R}$, then there does not exist a basis of V with respect to which T has an upper-triangular matrix.

(b) Prove that if $\mathbb{F} = \mathbb{C}$ and A is an upper-triangular matrix that equals the matrix of T with respect to some basis of V, then $-1 + i$ or $-1 - i$ appears on the diagonal of A.

Solution:

Comment: Here it is obviously assumed that Tv is not a scalar multiple of v.

(a) Rewrite the given equation as:

$$(T^2 + 2T + 2I)v = 0$$

Then we see that the polynomial $p_v = z^2 + 2z + 2$ is the monic polynomial with properties described in the previous problem.

For $\mathbb{F} = \mathbb{R}$ this polynomial has no roots (its value is always positive), hence it cannot be factorized into the form $(z - \lambda_1)(z - \lambda_2)$. *Problem 5C.7*(b) states that the minimal polynomial of T is a polynomial multiple of p_v. This implies that the minimal polynomial of T cannot be factorized into degree 1 factors, thus, by Theorem 5.44, T doesn't have an upper-triangular matrix with respect to any basis. $\square$

(b) If $\mathbb{F} = \mathbb{C}$, then T has an upper-triangular matrix with respect to some basis (5.47) and p_v can be factorized as:

$$p_v(z) = z^2 + 2z + 2 = (z - (-1 + i))(z - (-1 - i))$$

By Theorem 5.27, the minimal polynomial of T has the form $(z - \lambda_1) \cdots (z - \lambda_m)$, where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T.

Result of *Problem 5C.7*(b) shows that the minimal polynomial of T is a polynomial multiple of p_v, hence it contains factors $(z-(-1+i))$ and $(z-(-1-i))$; furthermore, $-1+i$ and $-1-i$ are thus eigenvalues of T. By Proposition 5.41, these numbers are diagonal entries of the upper-triangular matrix of T. $\square$

9 Suppose B is a square matrix with complex entries. Prove that there exists an invertible square matrix A with complex entries such that $A^{-1}BA$ is an upper-triangular matrix.

Solution:

Let V be a complex vector space and $v_1, \dots, v_n$ is a basis of V. Then we can suppose that B is a matrix of some operator $T \in \mathcal{L}(V)$ with respect to that basis.

According to Theorem 5.47, T has an upper-triangular matrix with respect to some basis of V. Denote this basis as $w_1, \dots, w_n$. So, $C = \mathcal{M}(T, (w_1, \dots, w_n))$ is an upper-triangular matrix. By Theorem 3.84, there exists matrix:

$$A = \mathcal{M}(I, (w_1, \dots, w_n), (v_1, \dots, v_n))$$

such that $C = A^{-1}BA$. A is a square matrix and is invertible (lemma 3.82), as desired. $\square$

10 Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Show that the following are equivalent.

(a) The matrix of T with respect to $v_1, \dots, v_n$ is lower triangular.

(b) $\operatorname{span}(v_k, \dots, v_n)$ is invariant under T for each $k = 1, \dots, n$.

(c) $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$ for each $k = 1, \dots, n$.

Solution:

First suppose (a) holds. To prove that (b) holds, suppose $k \in \{1, \dots, n\}$. If $j \in \{1, \dots, n\}$, then

$$Tv_j \in \operatorname{span}(v_j, \dots, v_n)$$

because the matrix of T with respect to $v_1, \dots, v_n$ is lower-triangular. Because $\operatorname{span}(v_j, \dots, v_n) \subseteq \operatorname{span}(v_k, \dots, v_n)$ if $j \geq k$, we see that:

$$Tv_j \in \operatorname{span}(v_1, \dots, v_k)$$

for each $j \in \{1, \dots, k\}$. Thus $\operatorname{span}(v_k, \dots, v_n)$ is invariant under T, competing the proof that (a) implies (b)

Now suppose (b) holds, so $\operatorname{span}(v_k, \ldots, v_n)$ is invariant under T for each $k = 1, \ldots, n$. In particular, $Tv_k \in \operatorname{span}(v_k, \ldots, v_n)$ for each $k = 1, \ldots, n$. Thus, (b) implies (c).

Now suppose (c) holds, so $Tv_k \in \operatorname{span}(v_k, \ldots, v_n)$ for each $k = 1, \ldots, n$. This means that when writing each Tv_k as a linear combination of the basis vectors $v_1, \ldots, v_n$, we need to use only the vectors $v_k, \ldots, v_n$. Hence all entries above the diagonal of $\mathcal{M}(T)$ are 0. Thus $\mathcal{M}(T)$ is a lower-triangular matrix, completing the proof that (c) implies (a).

We have shown that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a), which shows that (a), (b) and (c) are equivalent. □

11 Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if $T \in \mathcal{L}(V)$, then there exists a basis of V with respect to which T has a lower-triangular matrix.

Solution:

By Proposition 5.47, T has an upper-triangular matrix with respect to some basis $u_1, \ldots, u_n$. That is equivalent to fact that $Tu_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$ (Theorem 5.39).

Take another basis of V $v_1, \ldots, v_n$ such that $v_l = un - l$ for each $k = 1, \ldots, n$. Then the fact above can be rewritten as:

$$Tv_l \in \operatorname{span}(v_n, \ldots, v_l) = \operatorname{span}(v_l, \ldots, v_n) \quad \text{for each } l \in 1, \ldots, n$$

By the result of the previous problem, that is equivalent to T having a lower-triangular matrix with respect to basis $v_1, \ldots, v_n$. □

12 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V, and U is a subspace of V that is invariant under T.

(a) Prove that $T|_U$ has an upper-triangular matrix with respect to some basis of U.

(b) Prove that the quotient operator T/U has an upper-triangular matrix with respect to some basis of V/U.

Solution:

(a) Let p be a minimal polynomial of T. T has an upper-triangular matrix, hence $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ (Theorem 5.44).

Let q be a minimal polynomial of $T|_U$. By Theorem 5.31, q is a polynomial multiple of p. Thus, q has a form $q(z) = (z - \lambda_l) \cdots (z - \lambda_r)$ for some

$\lambda_l, \ldots, \lambda_r \in \mathbb{F}$, which implies that $T|_U$ has an upper-triangular matrix with respect to some basis (Theorem 5.44). $\square$

(b) We know from the result of *Problem 5B.25* that the minimal polynomial of T is a polynomial multiple of the minimal polynomial of T/U. By the same argument as in part (a), T/U has an upper-triangular matrix with respect to some basis. $\square$

13 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose there exists a subspace U of V that is invariant under T such that $T|_U$ has an upper-triangular matrix with respect to some basis of U and also T/U has an upper-triangular matrix with respect to some basis of V/U. Prove that T has an upper-triangular matrix with respect to some basis of V.

Solution:

Let p be the minimal polynomial of $T|_U$ and q be the minimal polynomial of T/U. By Theorem 5.44 both p and q have factorization into the factors of degree 1. From the *Problem 5B.25* we know that pq is a polynomial multiple of the minimal polynomial of T. Hence, it also has form $(z-\lambda_1)\cdots(z-\lambda_m)$, which implies that T has an upper-triangular matrix with respect to some basis. $\square$

14 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an upper-triangular matrix with respect to some basis of V if and only if the dual operator T' has an upper-triangular matrix with respect to some basis of the dual space V'.

Solution:

According to the result of *Problem 5B.28*, T and T' have the same minimal polynomial. Thus if T or T' has an upper-triangular matrix with respect to some basis, then the minimal polynomial of both of them has the form $(z-\lambda_1)\cdots(z-\lambda_m)$, and hence the other ($T'$ or T, respectively) also has an upper-triangular matrix. $\square$

5D Diagonalizable Operators

1 Suppose V is finite-dimensional complex vector space and $T \in \mathcal{L}(V)$.

(a) Prove that if $T^4 = I$, then T is diagonalizable.

(b) Prove that if $T^4 = T$, then T is diagonalizable.

(c) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ such that $T^4 = T^2$ and T is not diagonalizable.

Solution:

(a) We can rewrite equation $T^4 = I$, as $T^4 - I = 0$, which means that for $p(z) = z^4 - 1$ we have $p(T) = 0$ and hence $p(z)$ is a polynomial multiple of the minimal polynomial of T (by 5.29). $p(z)$ can be factorized as:

$$p(z) = (z-1)(z+1)(z-i)(z+i)$$

This implies that the minimal polynomial of T has a form $(z-\lambda_1)\dots(z-\lambda_m)$ for some distinct $\lambda_1, \dots, \lambda_m$. Therefore, by Theorem 5.62, T is diagonalizable. □

(b) In the similar fashion, rewrite $T^4 = T$ as $T^4 - T = 0$, which means $p(z) = z^4 - z$ is a polynomial multiple of the minimal polynomial of T. Then we factorize it:

$$p(z) = (z-0)(z-1)(z + \frac{1+i\sqrt{3}}{2})(z + \frac{1-i\sqrt{3}}{2}).$$

Thus, the minimal polynomial of T has a form $(z-\lambda_1)\dots(z-\lambda_m)$ for some distinct $\lambda_1, \dots, \lambda_m$. Therefore, by Theorem 5.62, T is diagonalizable. □

(c) Take the standard basis e_1, e_2 of $\mathbb{C}^2$ and let the operator T be defined by:

$$Te_1 = 0 \quad \text{and} \quad Te_2 = e_1.$$

Then we have $T^2 e_2 = T(e_1) = 0$, which implies that $T^2 = 0$ and thus $p(z) = z^2$ is the minimal polynomial of T. $p(z)$ has two identical roots ($z = 0$), hence by Theorem 5.62, T is not diagonalizable. But $T^4 = T^2 = 0$, so we have found the desired operator T. □

2 Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V. Prove that if $\lambda \in \mathbb{F}$, then λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

Solution:

When $\mathcal{M}(T)$ is a diagonal matrix with respect to basis $v_1, \dots, v_n$ and λ_i is on a diagonal, $Tv_i = \lambda_i v_i$. Suppose λ_i appears on a diagonal m times. Every appearence of λ_i on a diagonal corresponds to some vector v_j in the basis. These v_j's are basis vectors, so they are linearly-independent. At the same time, these vectors are elements of the eigenspace $E(\lambda_i, T)$. As these vectors are linearly independent and are in the same subspace, we conclude $m \le \dim E(\lambda_i, T)$.

By Theorem 5.55, V is a direct sum of all eigenspaces, thus its dimension is equal to the sum of dimensions of all eigenspaces. If at least one eigenspace has dimension strictly greater than number of appearences of its eigenvector, then $\dim V$ is less than the number of the basis vectors, which it cannot be. Thus, every eigenvalue appears exactly $\dim E(\lambda, T)$ times on the diagonal. $\square$

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that if the operator T is diagonalizable, then $V = \operatorname{null} T \oplus \operatorname{range} T$.

Solution:

Suppose $\operatorname{null} T = \{0\}$. Then is straightforward that $V = \operatorname{null} T \oplus \operatorname{range} T$.
Now suppose $\operatorname{null} T \neq \{0\}$. Then 0 is an eigenvalue of T.

Let $\lambda_1, \ldots, \lambda_{m+1}$ be distinguishable eigenvalues of T; and choose them in such a way that $\lambda_1 = 0$. Then, by 5.55:

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_{m+1}, T) \tag{5.4}$$

Moreover, V has a basis $v_1, \ldots, v_n$ that consists of eigenvectors of T.

Now we must show that $\operatorname{range} T = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Relabel basis vectors such that $v_1, \ldots, v_k$ are basis vectors that are not in $\operatorname{null} T$, i.e. not eigenvectors in $E(0, T)$, then $v_1, \ldots, v_k, v_{k+1}, \ldots v_n$ is a full basis. Note that

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 T v_1 + \cdots + a_k T v_k$$

Vectors in $Tv_1, \ldots, Tv_k$ are scalar multiples of the basis-vectors, as we chose basis vectors to be eigenvectors. Therefore, the list $v1, \ldots, v_k$ spans $\operatorname{range} T$. Hence, $\operatorname{range} T = E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T)$, where we conclude equality from the fact that the right part has the same basis as the left. Rewriting Eq. 5.4 using this fact gives:

$$V = \operatorname{null} T \oplus \operatorname{range} T \quad \square$$

4 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$.

(b) $V = \operatorname{null} T + \operatorname{range} T$.

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

Solution:

(a) $\Rightarrow$ (b) by definition of a direct sum.
(a) $\iff$ (c) by 1.46.

To end the proof we can show that (b) implies (c). Thus, suppose (b) is true. Then by 2.43:

$$\begin{aligned} \dim V &= \dim(\operatorname{range} T + \operatorname{null} T) \\ &= \dim \operatorname{range} T + \dim \operatorname{null} T - \dim(\operatorname{null} T \cap \operatorname{range} T) \end{aligned}$$

On the other hand, by fundamental theorem of linear maps:

$$\dim V = \dim(\operatorname{range} T) + \dim(\operatorname{null} T)$$

These two equations imply $\dim(\operatorname{null} T \cap \operatorname{range} T) = 0$, and therefore $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

Thus, we have shown that (c) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c), meaning these three conditions are equivalent. $\square$

5 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

Solution:

$\longrightarrow$ If T has a diagonal matrix with respect to some basis:

$$\mathcal{M}(T) = \begin{pmatrix} a_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{pmatrix},$$

then for every $\lambda \in \mathbb{C}$ an operator $T - \lambda I$ has a matrix (in the same basis):

$$\mathcal{M}(T - \lambda I) = \begin{pmatrix} a_{1,1} - \lambda & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} - \lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} - \lambda \end{pmatrix}.$$

So it is also a diagonal matrix, and by *Problem 5D.4*: $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$.

$\longleftarrow$ Now suppose that for every $\lambda \in \mathbb{C}$: $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$.

Suppose T has m distinct eigenvalues $\lambda_1, \dots, \lambda_m$ (as V is a complex vector space, $m > 0$). For each these eigenvalues we have $V = \operatorname{null}(T - \lambda_i I) \oplus \operatorname{range}(T - \lambda_i I) = E(\lambda_i, T) \oplus \operatorname{range}(T - \lambda_i I)$. Sum of all the eigenspaces is a direct sum (Theorem 5.54). Suppose this sum does not equal V, then:

$$V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T) \oplus U$$

with $U \neq \{0\}$.

From the decomposition of V into null space and range of $T - \lambda_i I$ and from our construction of U we conclude that U is a subspace of every range $(T - \lambda_i I)$. Thus

$$U = \bigcap_k \operatorname{range}(T - \lambda_k I).$$

Let $u \in U$ such that $u \neq 0$. U being an intersection of subspaces invariant under T, is itself invariant under T. As U is a nonzero complex subspace, $T|_U$ has an eigenvalue on U and a corresponding eigenvector. Returning to V, we see that this eigenvalue is also an eigenvalue of T on V. But we already used all possible eigenvalues and their eigenvectors in constructing $E(\lambda_i, T)$. Thus, U must equal $\{0\}$.

Therefore,

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

which means T is diagonalizable. $\square$

6 Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

Solution:

By Theorem 5.54, $\sum_i \dim E(\lambda_i, T) \leq \dim V$. If both $T - 2I$ and $T - 6I$ are non-invertible, then both 2 and 6 are eigenvalues of T. Their eigenspaces has dimension at least 1. Hence sum of dimensions of all eigenspaces is $E(8, T) + E(2, T) + E(6, T) \geq 4 + 1 + 1 = 6$. Such situation cannot be as $\dim \mathbb{F}^5 = 5$. Hence we conclude that the negation of "$T - 2I$ and $T - 6I$ are not invertible", that is "$T - 2I$ or $T - 6I$ is invertible". $\square$

7 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

$$E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1})$$

for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

Solution:

Suppose λ is an eigenvalue of T. We know from *Problem 5A.21* that $1/\lambda$ is then an eigenvalue of T^{-1}, and the eigenvectors of these operators are the same. As eigenvectors are the same, they span the same subspace of V. Hence $E(\lambda, T) = E(1/\lambda, T)$. If $1/\lambda$ is an eigenvalue of T^{-1}, then λ is an eigenvalue of T and the same reasoning applies.

Suppose λ is not an eigenvalue. It means $E(\lambda, T) = \{0\}$. By the same *Problem 5A.21*, $1/\lambda$ is not an eigenvalue of T^{-1}, thus $E(1/\lambda, T) = E(\lambda, T) = \{0\}$, which ends the proof. $\square$

8 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct nonzero eigenvalues of T. Prove that

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T.$$

Solution:

Consider direct sum $E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$; denote it as W. Take bases of every eigenspace and combine them into a basis of W: $v_1, \ldots, v_k$. Note that these vectors are eigenvectors of T.

For any $w \in W$: $Tw \in \operatorname{range} T$. As $w = a_1 v_1 + \cdots + a_k v_k$, where v_i are the eigenvectors of T, $Tw = a_1 \lambda_1 v_1 + \cdots a_k \lambda_m v_k$. As a_i's are arbitrary and all λ_i's are nonzero, that means Tw is a linear combination of k linearly-independent vectors with k equals sum of all the dimensions of the eigenspaces. But there can't be more linearly-independent vectors than basis vectors of a vector space—here basis vectors of range T. Thus

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T,$$

as desired. □

9 Suppose $R, T \in \mathcal{L}(\mathbb{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{F}^3)$ such that $R = S^{-1} T S$.

Comment: This can be proven much shorty using Theorem 3.84, but at the time of writing the proof I've forgotten about it. Anyway, I've decided to leave it like that, for this proof gives some additional insight into why 3.84 works.

Solution:

Let v_1, v_2, v_3 be the eigenvectors of T corresponding to eigenvalues 2, 6 and 7; and let u_1, u_2, u_3 be eigenvectors of R corresponding to eigenvalues 2, 6 and 7. As $\dim \mathbb{F}^3 = 3$, R and T are diagonalizable (by 5.58) and hence v_1, v_2, v_3 and u_1, u_2, u_3 are two bases of $\mathbb{F}^3$ (because eigenvectors of distinct eigenvalues are linearly-independent).

Express vectors of these two bases in terms of each other:

$$v_i = a_{1,i} u_1 + a_{2,i} u_2 + a_{3,1} u_3 \quad \text{and} \quad u_i = b_{1,i} v_1 + b_{2,i} v_2 + b_{3,i} v_3.$$

Then we examine how R acts on the basis v_1, v_2, v_3:

$$\begin{aligned}
Rv_i &= a_{1,i}Ru_1 + a_{2,i}Ru_2 + a_{3,i}Ru_3 = a_{1,i}\lambda_1 u_1 + a_{2,i}\lambda_2 u_2 + a_{3,i}\lambda_3 u_3 \\
&= \lambda_1 a_{1,i}b_{1,1}v_1 + \lambda_1 a_{1,i}b_{2,1}v_2 + \lambda_1 a_{1,i}b_{3,1}v_3 \\
&\quad + \lambda_2 a_{2,i}b_{1,2}v_1 + \lambda_2 a_{2,i}b_{2,2}v_2 + \lambda_2 a_{2,i}b_{3,2}v_3 \\
&\quad + \lambda_3 a_{3,i}b_{1,3}v_1 + \lambda_3 a_{3,i}b_{2,3}v_2 + \lambda_3 a_{3,i}b_{3,3}v_3 \\
&= (\lambda_1 a_{1,i}b_{1,1} + \lambda_2 a_{2,i}b_{1,2} + \lambda_3 a_{3,i}b_{1,3})v_1 \\
&\quad + (\lambda_1 a_{1,i}b_{2,1} + \lambda_2 a_{2,i}b_{2,2} + \lambda_3 a_{3,i}b_{2,3})v_2 \\
&\quad + (\lambda_1 a_{1,i}b_{3,1} + \lambda_2 a_{2,i}b_{3,2} + \lambda_3 a_{3,i}b_{3,3})v_3
\end{aligned}$$

This implies that matrix of R in basis v_1, v_2, v_3 is:

$$\begin{aligned}
\mathcal{M}(R) &= \begin{bmatrix} \lambda_1 a_{1,1}b_{1,1} + \lambda_2 a_{2,1}b_{1,2} + \lambda_3 a_{3,1}b_{1,3} & \cdots & \cdots \\ \lambda_1 a_{1,1}b_{2,1} + \lambda_2 a_{2,1}b_{2,2} + \lambda_3 a_{3,1}b_{2,3} & \cdots & \cdots \\ \lambda_1 a_{1,1}b_{3,1} + \lambda_2 a_{2,1}b_{3,2} + \lambda_3 a_{3,3}b_{3,3} & \cdots & \cdots \end{bmatrix} \\
&= \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix} \cdot \begin{bmatrix} a_{1,1}\lambda_1 & a_{1,2}\lambda_1 & a_{1,3}\lambda_1 \\ a_{2,1}\lambda_2 & a_{2,2}\lambda_2 & a_{3,3}\lambda_2 \\ a_{3,1}\lambda_3 & a_{3,2}\lambda_3 & a_{3,3}\lambda_3 \end{bmatrix} \\
&= \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \cdot \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{3,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \\
&= \mathcal{M}(B) \cdot \mathcal{M}(T) \cdot \mathcal{M}(A)
\end{aligned}$$

Thus $R = BTA$, where linear transformations A and B are determined by the matrices above.

Now we will show that $AB = BA = I$ and thus $S = A$ and $S^{-1} = B$. Examine how AB acts on vectors v_1, v_2, v_3:

$$\begin{aligned}
AB(v_i) &= A(Bv_i) = A(b_{1,i}v_1 + b_{2,i}v_2 + b_{3,i}v_3) = b_{1,i}Av_1 + b_{2,i}Av_2 + b_{3,i}Av_3 \\
&= b_{1,i}(a_{1,1}v_1 + a_{2,1}v_2 + a_{3,1}v_3) + b_{2,i}(a_{1,2}v_1 + a_{2,2}v_2 + a_{3,2}v_3) \\
&\quad + b_{3,i}(a_{1,3}v_1 + a_{2,3}v_2 + a_{3,3}v_3) \\
&= (b_{1,i}a_{1,1} + b_{2,i}a_{1,2} + b_{3,i}a_{1,3})v_1 + (b_{1,i}a_{2,1} + b_{2,i}a_{2,2} + b_{3,i}a_{2,3})v_2 \\
&\quad + (b_{1,i}a_{3,1} + b_{2,i}a_{3,2} + b_{3,i}a_{3,3})v_3
\end{aligned}$$

Note that (from the way $a_{i,j}$ and $b_{i,j}$ are defined):

$$\begin{aligned}u_i &= b_{1,i}v_1 + b_{2,i}v_2 + b_{3,i}v_3 \\ &= b_{1,i}(a_{1,1}u_1 + a_{2,1}u_2 + a_{3,1}u_3) + b_{2,i}(a_{1,2}u_1 + a_{2,2}u_2 + a_{3,2}u_3) \\ &\quad + b_{3,i}(a_{1,3}u_1 + a_{2,3}u_2 + a_{3,3}u_3) \\ &= (b_{1,i}a_{1,1} + b_{2,i}a_{1,2} + b_{3,i}a_{1,3})u_1 + (b_{1,i}a_{2,1} + b_{2,i}a_{2,2} + b_{3,i}a_{2,3})u_2 \\ &\quad + (b_{1,i}a_{3,1} + b_{2,i}a_{3,2} + b_{3,i}a_{3,3})u_3\end{aligned}$$

As here left and right part of the expression is the same, we conclude that the only nonzero terms are $\sum_j^3 b_{j,i}a_{i,j}$ and these terms equal one. This implies that $AB(v_i) = v_i$ for all three v_i's, which means $AB = I$.

By the result of *Problem 3D.24*, we also have $BA = I$. So, indeed A and B are inverses of each other.

Thus we have shown that there exists $S \in \mathcal{L}(\mathbb{F}^3)$ such that $R = S^{-1}TS$. $\square$

10 Find $R, T \in \mathcal{L}(\mathbb{F}^4)$ such that R and T each have 2, 6, 7 as eigenvalues, R and T have no other eigenvalues, and there does not exist an invertible operator $S \in \mathcal{L}(\mathbb{F}^4)$ such that $R = S^{-1}TS$.

Solution:

Suppose that $R, T \in \mathbb{F}^4$ have matrices with respect to the standard basis:

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

$$\mathcal{M}(R) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Clearly R and T have only 2, 6 and 7 as eigenvalues. Denote an eigenvalue of T corresponding to vector e_i from the basis as λ_i^T, and similarly λ_i^R for R.

Suppose there is $S \in \mathcal{L}(\mathbb{F}^4)$ such that $R = S^{-1}TS$. Let $v_i \in \mathbb{F}^4$ be a nonzero vector such that $Sv_i = e_i$; then $S^{-1}e_i = v_i$ (such v_i exists as S is invertible). Hence

$$S^{-1}TSv_i = S^{-1}Te_i = S^{-1}\left(\lambda_i^T e_i\right) = \lambda_i^T S^{-1}e_i = \lambda_i^T v_i$$

But we know that $S^{-1}TS = R$, and $Re_j = \lambda_j^R e_j$ Thus v_i is one of the basis vectors or a linear combination of basis vectors corresponding to the same eigenvalue. Also note that in such case it must be that $\lambda_i^T = \lambda_j^R$

Firstly, consider $\lambda_1^R = 2$. Its eigenvector is e_1. Now we find corresponding basis vector e_j such that $Se_1 = e_j$ and $\lambda_1^R = \lambda_j^T$. The only option is pair $\left(e_1, \lambda_1^T\right)$, as only $\lambda_1^T = 2$.

Now consider $\lambda_2^R = 2$. By the procedure above we conclude that $Se_2 = e_1$. But thus we have $Se_1 = Se_2$, so S isn't injective and hence is not invertible. That contradicts our assumption that S is invertible, therefore, the given $R, T \in \mathcal{L}(\mathbb{F}^4)$ are such that no invertible operator $S \in \mathcal{L}(\mathbb{F}^4)$ can satisfy $R = S^{-1}TS$. $\square$

11 Find $T \in \mathcal{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.

Solution:

Let e_1, e_2, e_3 be a standard basis of $\mathbb{C}^3$. Construct T in the following way:

$$Te_1 = 6e_1, \quad Te_2 = 7e_2, \quad Te_3 = e_1$$

Vectors e_1 and e_2 are eigenvectors of T and are basis vectors at the same time.

Suppose there is another eigenvector of T — v_3, which is linearly independent of e_1 and e_2. Such vector must be some linear combination of three standard basis vectors with nonzero coefficient of e_3. As eigenvectors are determined up to a scalar multiple let $v_3 = e_3 + a_1 e_1 + a_2 e_2$. Examine how T acts on v_3:

$$Tv_3 = T\left(e_3 + a_1 e_1 + a_2 e_2\right) = e_1 + 6a_1 e_1 + 7a_2 e_2$$

On the other hand:

$$Tv_3 = \lambda v_3 = \lambda\left(e_3 + a_1 e_1 + a_2 e_2\right) = \lambda e_3 + a_1 \lambda e_1 + a_2 \lambda e_2$$

From these two expressions we deduce that:

$$\begin{aligned} \lambda e_3 &= \left(6a_1 - \lambda a_1 + 1\right) e_1 + a_2(7 - \lambda) e_2 \\ e_3 &= \frac{6a_1 - \lambda a_1 + 1}{\lambda} e_1 + \frac{7 - \lambda}{\lambda} a_2 e_2 \end{aligned}$$

so e_3 is a linear combination of e_1 and e_2, which it cannot be by definition.

Thus, there is no third eigenvector of T and T has only two eigenvalues with one-dimensional eigenspaces. So we have $\dim E(6, T) + \dim E(7, T) = 2 < \dim \mathbb{C}^3$, thus T is not diagonalizable (by Theorem 5.55). $\square$

12 Suppose $T \in \mathcal{L}\left(\mathbb{C}^3\right)$ is such that 6 and 7 are eigenvalues of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$. Prove that there exists $(x, y, z) \in \mathbb{F}^3$ such that

$$T(z_1, z_2, z_2) = (6 + 8z_1, 6 + 8z_2, 13 + 8z_3).$$

Solution:
Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be eigenvectors of T:

$$\begin{aligned} T\left(x_1, y_1, z_1\right) &= \left(6x_1, 6y_1, 6z_1\right) \\ T\left(x_2, y_2, z_2\right) &= \left(7x_2, 7y_2, 7z_2\right). \end{aligned}$$

For T to be non-diagonalizable, there must be no other eigenvector and no other eigenvalue. 8 is not an eigenvalue of T, so $T - 8I$ is invertible.

Examine $(6, 7, 13) \in \mathbb{C}^3$. There is some $v \in \mathbb{C}^3$ such that $(T - 8I)v = (6, 7, 13)$. That is equivalent to $Tv - 8v = (6, 7, 13)$, which implies

$$Tv = 8v + (6, 7, 13)$$

Rewriting v as (x, y, z), we get:

$$T(x, y, z) = 8\left(x, y, z\right) + (6, 7, 13) = (6 + 8x, 7 + 8y, 13 + 8z)$$

Thus, we have shown that a desired vector exists. $\square$

13 Suppose A is a diagonal matrix with distinct entries on the diagonal and B is a matrix of the same size as A. Show that $AB = BA$ if and only if B is a diagonal matrix.

Solution:
We will consider element on the i^{th} row and j^{th} column of AB and BA.
Firstly, suppose $AB = BA$. Then:

$$\begin{aligned} (AB)_{i,j} &= \sum_k A_{i,k} B_{k,j} = A_{i,i} B_{i,j}, \\ (BA)_{i,j} &= \sum_k B_{i,k} A_{k,j} = B_{i,j} A_{j,j}, \end{aligned}$$

where we used the fact that A is a diagonal matrix. As $AB = BA$, we may write:

$$A_{i,i} B_{i,j} = B_{i,j} A_{j,j}.$$

This equality can be true in two cases: (1) $B_{i,j} = 0$; (2) $B_{i,j}$ is nonzero, hence $A_{i,i} = A_{j,j}$, which can be true if and only if $i = j$, as A has distinct entries on the diagonal. Thus, we have shown that only nonzero elements of B are diagonal entries.

Now suppose that B is a diagonal matrix. Then we have:

$$(AB)_{i,j} = \sum_k A_{i,k}B_{k,j} = A_{i,i}B_{i,j}\delta_{i,j},$$
$$(BA)_{i,j} = \sum_k B_{i,k}A_{k,j} = B_{i,j}A_{j,j}\delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker's delta. For $i \neq j$, we have $(AB)_{i,j} = (BA)_{i,j} = 0$, and for $i = j$, we have $(AB)_{i,j} = (BA)_{i,j} = A_{i,i}B_{i,i}$. This shows that $AB = BA$, completing the proof. $\square$

14 (a) Give and example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbb{F} = \mathbb{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

Solution:

(a) Consider $\mathbb{F}^4$ and an operator $T \in \mathcal{L}(\mathbb{F}^4)$ such that its minimal polynomial is:

$$p(z) = z^2(z^2 - 1) = z^4 - z^2.$$

Using the result of *Problem 5B.16* we can construct matrix of T in the standard basis, which thus defines T:

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Minimal polynomial of T has a repeated factor, so it is not diagonalizable (Theorem 5.62).

Now consider T^2. Denote T^2 as S. Then note that

$$p(T) = T^4 - T^2 = 0 = (T^2)^2 - (T^2) = S^2 - 1 = (S-1)(S+1) = q(S)$$

So, $q(z) = (z-1)(z+1)$ is a polynomial multiple of the minimal polynomial of S. Writing the matrix of S explicitly gives:

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Thus, S is not a scalar multiple of the identity operator and thus its minimal polynomial must be of degree 2. The polynomial $q(z)$ is this minimal polynomial. It has two non-repeated factors, therefore S is diagonalizable.

(b) First, suppose that T is diagonalizable. Then the matrix of T in some basis is diagonal, the matrix of T^2 is $\mathcal{M}(T^2) = \mathcal{M}(T) \cdot \mathcal{M}(T)$ is also diagonal and so on. Thus, if T is diagonalizable, then T^k is diagonalizable for every positive integer k.

Now suppose that T^k is diagonalizable for some positive integer k. Consider the minimal polynomial of T^k. By Theorem 5.62, it has the form:

$$p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$$

for some distinct $\lambda_1, \ldots, \lambda_m$. As T is invertible, so T^k is also invertible (induction on k and application of the result of *Problem 3D.11*). Hence, by Theorem 5.7, every λ_j is nonzero.

Note that the polynomial

$$q(z) = p(z^k) = (z^k - \lambda_1) \cdots (z^k - \lambda_m)$$

is a polynomial multiple of the minimal polynomial of T, as $q(T) = p(T^4) = 0$. Every factor in brackets can be rewritten in terms of k roots of λ_j:

$$\left(z - |\lambda_j|^{1/k}\right)\left(z - |\lambda_j|^{1/k} e^{i\frac{2\pi}{k}}\right)\left(z - |\lambda_j|^{1/k} e^{i\frac{2\pi \cdot 2}{k}}\right) \cdots \left(z - |\lambda_j|^{1/k} e^{i\frac{2\pi(k-1)}{k}}\right).$$

All these roots are distinct, because no λ_j equals zero. Thus, $q(z)$ can be written as a product of $k \cdot m$ non-repeated factors, hence the minimal polynomial of T also has the form $(z - \mu_1) \cdots (z - \mu_n)$ for some $\mu_1, \ldots, \mu_n \in \mathbb{C}$. That means T is a diagonalizable operator, which completes the second part of the proof. $\square$

15 Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Prove that the following are equivalent.

(a) T is diagonalizable.

(b) There does not exist $\lambda \in \mathbb{C}$ such that p is a polynomial multiple of $(z - \lambda)^2$.

(c) p and its derivative p' have no zeros in common.

(d) The greatest common divisor of p and p' is the constant polynomial 1.

Solution:

By Theorem 5.62, T is diagonalizable if and only if p has the form $(z - \lambda_1)\cdots(z - \lambda_m)$ for some distinct $\lambda_1, \ldots, \lambda_m$. This, in turn, is equivalent to there being no $\lambda \in \mathbb{C}$ such that p is a polynomial multiple of $(z - \lambda)^2$ (otherwise, this factor would appear twice).

Thus, (a) is equivalent to (b).

Also, by *Problem 4.8*, all zeros of p are distinct if and only if p and its derivative p' have no zeros in common. This shows that (a) and (c) are equivalent.

Now we will show that (c) is true if and only if (d) is true. By the Fundamental Theorem of Algebra (second version), p and p' have unique factorizations:

$$p(z) = (z - \lambda_1)\cdots(z - \lambda_m)$$
$$p'(z) = (z - \mu_1)\cdots(z - \mu_{m-1})$$

Suppose that q is the greater common divisor of p and p'. It also has a unique factorization:

$$q(z) = (z - \tau_1)\cdots(z - \tau_k)$$

Because these factorizations are unique, zeros of q are also zeros of p and p'.

And now we see that p and p' have no zeros in common if and only if q is a constant polynomial, that is, $q = 1$.

Thus we have shown that (b) $\Leftrightarrow$ (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). $\square$

16 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Prove that a subspace U of V is invariant under T if and only if there exist subspaces $U_1, \ldots, U_m$ of V such that $U_k \subseteq E(\lambda_k, T)$ for each k and $U = U_1 \oplus \cdots \oplus U_m$.

Solution:

$\longrightarrow$ As T is a diagonalizable operator, its restriction to U, $T|_U$ is a diagonalizable operator on U, eigenvalues of which are a subset of the eigenvalues of T.

Denoting by $E(\lambda_k, T|_U)$ eigenspaces of $T|_U$ corresponding to an eigenvalue λ_k, and letting $E(\lambda_j, T|_U) = \{0\}$ for eigenvalue λ_j of T that is not an eigenvalue of $T|_U$, we can write $U = E(\lambda_1, T|_U)\cdots E(\lambda_m, T|_U)$ using Theorem 5.55. As U is a subspace of V, it is easy to see that $E(\lambda_k, T|_U) \subseteq E(\lambda_k, T)$ (any eigenvector $v_k \in U$ is also an eigenvector $v_k \in V$ with the same eigenvalue). Now let $U_k = E(\lambda_k, T|_U)$ to get the desired result:

$$U = U_1 \oplus \cdots \oplus U_m$$

$\longleftarrow$ Suppose U is a subspace of V and there exist subspaces $U_1, \ldots, U_m$ of V such that $U_k \subseteq E(\lambda_k, T)$ and $U = U_1 \oplus \cdots \oplus U_m$. From the last expression it follows that for any $u \in U$ there is a unique decomposition:

$$u = u_1 + \cdots + u_m,$$

where $u_k \in U_k$. Now apply operator T to it.

$$Tu = T(u_1 + \cdots u_m) = \lambda_1 u_1 + \cdots \lambda_m u_m.$$

As U is a direct sum of $U_1, \ldots, U_m$, the last sum is an element of U. Thus, we have $Tu \in U$ for every $u \in U$, which means that U is invariant under T. This completes the second part of the proof. $\square$

17 Suppose V is finite-dimensional. Prove that $\mathcal{L}(V)$ has a basis consisting of diagonalizable operators.

Solution:

Take some basis $v_1, \ldots, v_n$ of V and a list of distinct scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. Then construct operators T_1, T_2, etc. in such a way that for each of the operators, the vectors in the basis of V are eigenvalues of that operator with an eigenvalue from the list $\lambda_1, \ldots, \lambda_n$ (eigenvalues are not repeated). These operators are clearly distinct from each other as they act differently on the basis vectors (linear map lemma, 3.4).

The total number of operators that can be constructed this way is equal to the number of ordered permutations of $(\lambda_1, \ldots, \lambda_n)$ that is equal to $n!$. The maximum number of linearly independent elements of the vector space is equal to the dimension of that vector space. For our case of $\mathcal{L}(V)$, the dimension is equal to n^2 (see lemma 3.71), and $n! \geq n^2$ for positive integers.

Thus, we can pick any n^2 linearly independent operators from $T_1, T_2, \ldots, T_{n!}$. These operators are diagonalizable by construction, so we get a basis of $\mathcal{L}(V)$ consisting of diagonalizable operators, as desired. $\square$

18 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Prove that the quotient operator T/U is a diagonalizable operator on V/U.

Solution:

The operator T is diagonalizable, hence its minimal polynomial has the form $(z - \lambda_1) \cdots (z - \lambda_m)$ for some distinct $\lambda_1, \ldots, \lambda_m$ (Theorem 5.62).

From the result of part (a) of *Problem 5B.25* we know that the minimal polynomial of T is a polynomial multiple of the minimal polynomial of T/U. This implies that the minimal polynomial of T/U has the form $(z-\mu_1) \cdots (z-$

μ_l), where $\mu_1, \ldots, \mu_l$ are distinct numbers taken from the list $\lambda_1, \ldots, \lambda_m$. This, in turn, means that T/U is a diagonalizable operator on V/U. $\square$

19 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists a subspace U of V that is invariant under T such that $T|_U$ and T/U are both diagonalizable, then T is diagonalizable.

Solution:

This proposition is false. The counterexample is an operator T of on $\mathbb{F}^2$ defined as $T(x, y) = (x, x + y)$, with a subspace $U = \operatorname{span}(e_2)$, where $e_2 = (0, 1)$ is a vector of the standard basis of $\mathbb{F}^2$.

Operators $T|_U$ and T/U are diagonalizable, because they are operators on one-dimensional subspaces. In fact, both these operators are just identity operators on the corresponding vector spaces.

The operator T, on the other hand, is not diagonalizable. Let us find the minimal polynomial of T.

$$
\begin{aligned}
&Te_1 = e_1 + e_2 \\
&Te_2 = e_2 \\
&T^2(e_1) = T(e_1 + e_2) = e_1 + 2e_2 = e_1 + 2(Te_1 - e_1) = 2Te_1 - e_1
\end{aligned}
$$

Thus, $T^2 - 2T + I = 0$ and the minimal polynomial of T is $p(z) = (z - 1)^2$. It has a repeated factor, therefore T is not diagonalizable. $\square$

20 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if the dual operator T' is diagonalizable.

Solution:

From *Problem 5B.28*, we know that the minimal polynomial of T equals the minimal polynomial of T'. So the minimal polynomial of T has the form $(z-\lambda_1)\cdots(z-\lambda_m)$ for some distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ if and only if the minimal polynomial of T' has the same form. This, by Theorem 5.62, means that T is diagonalizable if and only if T' is diagonalizable. $\square$

21 The *Fibonacci sequence* $F_0, F_1, \ldots$ is defined by

$$F_0 = 0,\ F_1 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 2.$$

Define $T \in \mathcal{L}\left(\mathbb{R}^2\right)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each positive integer n.

(b) Find the eigenvalues of T.

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(d) Use the solution to part (c) to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each nonnegative integer n.

(e) Use part (d) to conclude that if n is a nonnegative integer, then the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Solution:

(a) To show this, we will use mathematical induction.

Base case: for $n = 1$ we have $T(0,1) = (1, 0+1) = (1,1) = (F_1, F_2)$

Now assume that for some $n = k$, an expression $T^k(0,1) = (F_k, F_{k+1})$ is true. Then for $n = k+1$ we have:

$$\begin{aligned} T^{k+1}(0,1) &= T\left(T^k(0,1)\right) = T\left(F_k, F_{k+1}\right) = \left(F_{k+1}, F_k + F_{k+1}\right) \\ &= \left(F_{k+1}, F_{k+2}\right) = \left(F_{k+1}, F_{(k+1)+1}\right) \end{aligned}$$

So, $T^n(0,1) = (F_n, F_{n+1})$ for $n = k+1$ and therefore, by induction, it is true for every positive integer n. $\square$

(b) Let (x,y) be an eigenvector of T. Then $T(x,y) = (y, x+y) = \lambda(x,y)$. This is equivalent to the system of equations:

$$\begin{cases} y = \lambda x, \\ x + y = \lambda y. \end{cases}$$

Using the first equation to substitute y in the second equation, we get:

$$x + \lambda x = \lambda^2 x, \quad \text{hence} \quad \lambda^2 x - \lambda x - x = 0, \text{ or } \left(\lambda^2 - \lambda - 1\right) x = 0$$

This means $x = 0$ or $\lambda^2 - \lambda - 1 = 0$. If $x = 0$, then $y = 0$, but such solution is not an eigenvector by definition. Hence, the only option is the second expression, solutions to which are:

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

which are the sought eigenvalues of T

(c) First, consider the eigenvalue $\lambda_1 = (1+\sqrt{5})/2$.

Let $x = 1$, then $y = \lambda x = \frac{1+\sqrt{5}}{2}$ and the eigenvector is:

$$\left(1, \frac{1+\sqrt{5}}{2}\right)$$

Second, consider the eigenvalue $\lambda_2 = (1-\sqrt{5})/2$.

Let $x = 1$ then $y = \lambda x = \frac{1-\sqrt{5}}{2}$ and the eigenvector is:

$$\left(1, \frac{1-\sqrt{5}}{2}\right)$$

Thus, the basis of $\mathbb{R}^2$ consisting of eigenvectors of T is $\left(1, \frac{1+\sqrt{5}}{2}\right), \left(1, \frac{1-\sqrt{5}}{2}\right)$.

(d) Let $v_1 = \left(1, \frac{1+\sqrt{5}}{2}\right), v_2 = \left(1, \frac{1-\sqrt{5}}{2}\right)$. Note that

$$\frac{1}{\sqrt{5}}(v_1 - v_2) = \frac{1}{\sqrt{5}}\left(1-1, \frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) = (0,1).$$

Then, we act on $(0,1)$ by T^n

$$\begin{aligned} T^n(0,1) = (F_n, F_{n+1}) &= \frac{1}{\sqrt{5}} T^n(v_1 - v_2) \\ &= \frac{1}{\sqrt{5}}(T^n v_1 - T^n v_2) = \frac{1}{\sqrt{5}}\left(\lambda_1^n v_1 - \lambda_2^2 v_2\right) \\ &= \left(\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n), \frac{1}{\sqrt{5}}\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right)\right) \end{aligned}$$

Thus $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

(e) Let us estimate the absolute value of λ_2^n term:

$$\left|\frac{1-\sqrt{5}}{2}\right| \approx 0.62 \quad \Rightarrow \quad \left|\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < 1, \text{ hence } \left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < 0.5$$

From result of part (d), we know that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right)$$

Then we estimate the differce:

$$\left| F_n - \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right| = \left| - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \right| < 0.5$$

So F_n differs from $\frac{1}{\sqrt{5}} \left(\frac{1+1\sqrt{5}}{2} \right)^n$ by less than $\frac{1}{2}$. F_n is an interger, so it is in fact the closest integer to $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$. $\square$

22 Suppose $T \in \mathcal{L}(V)$ and A is an n-by-n matrix that is the matrix of T with respect to some basis of V. Prove that if

$$|A_{j,j}| > \sum_{k=1, k \neq j}^{n} |A_{j,k}|$$

for each $j \in \{1, \ldots, n\}$, then T is invertible.

Solution:

The Gershogorin disk theorem (5.67) states that every eigenvalue λ of T is contained in some Gershogorin disk of T:

$$|\lambda - A_{j,j}| \leq \sum_{k=1, k \neq j}^{n} |A_{j,k}|$$

for some $j \in \{1, \ldots, n\}$. This expression can be restated as:

$$|A_{j,j} - \lambda| \leq \sum_{k=1, k \neq j}^{n} |A_{j,k}| \tag{5.5}$$

Using properties of the absolute value we then write:

$$|A_{j,j} - \lambda| \geq |A_{j,j}| - |\lambda| \tag{5.6}$$

Combining Eqs. 5.5 and 5.6 and rearranging the terms, we come to

$$|\lambda| \geq |A_{j,j}| - \sum_{k=1, k \neq j}^{n} |A_{j,k}| > 0.$$

To write the last inequality sign, we used the property given in the problem. Thus, all eigenvalues of T must be nonzero, which by lemma 5.7 guarantees that T is invertible. $\square$

23 Suppose the definition of the Gershogorin disks is changed so that the radius of the k'th disk is the sum of the absolute values of the entries in column (instead of row) k of A, excluding the diagonal entry. Show that the Gershogorin disk theorem (5.67) still holds with this changed definition.

Definition: Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Let A denote the matrix of T with respect to this basis. A *Gershogorin-like disk* of T with respect to the basis $v_1, \ldots, v_n$ is a set of the form

$$\left\{ z \in \mathbb{F} : |z - A_{j,j}| \leq \sum_{k=1, k\neq j}^{n} |A_{k,j}| \right\},$$

where $j \in \{1, \ldots, n\}$.

Solution:

Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of T. The result of *Problem 5A.15* tells that λ is also necessarily an eigenvalue of the dual map T'. Let B denote the matrix of T' with respect to the corresponding dual basis. In this basis the Gershogorin disk theorem (5.67) holds, so

$$|\lambda - B_{j,j}| \leq \sum_{k=1, k\neq j}^{n} |B_{j,k}|.$$

By lemma 3.132, the matrix of T' is transpose of matrix of T, which means $B_{j,k} = A_{k,j}$. Rewriting the expression above using this relation gives:

$$|\lambda - A_{j,j}| \leq \sum_{k=1, k\neq j}^{n} |A_{k,j}|.$$

So the Gershogorin disk theorem also holds for Gershogorin-like disks. □