Chapter 7

Operators on Inner Product Spaces

Contents

7A Self-Adjoint and Normal Operators

1 Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$

Find a formula for $T^*(z_1, \ldots, z_n)$.

Solution:

Suppose $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{F}^n$. Then

$$\begin{aligned}\langle T(z_1, \ldots, z_n), (w_1, \ldots, w_n)\rangle &= \langle (0, z_1, \ldots, z_{n-1}), (w_1, w_2, \ldots, w_n)\rangle \\ &= z_1 w_2 + z_2 w_3 + \cdots + z_{n-1} w_n \\ &= \langle (z_1, \ldots, z_n), (w_2, \ldots, w_n, 0)\rangle\end{aligned}$$

By the definition of the adjoint, we must have

$$T * (w_1, \ldots, w_n) = (w_2, \ldots, w_{n-1}, 0),$$

which is the sought formula for the adjoint. □

2 Suppose $T \in \mathcal{L}(V, W)$. Prove that

$$T = 0 \iff T^* = 0 \iff T^*T = 0 \iff TT^* = 0.$$

Solution:

First equivalence is just the property of a zero map. Indeed, for any $v \in V$, $u \in W$:

$$\langle 0v, u \rangle = 0 = \langle v, 0u \rangle.$$

That is, zero map is "self-adjoint" (although these maps are from different vector spaces).

Third equation follows directly from the second:

$$T^* = 0 \Rightarrow T^*(Tv) = 0 \text{ for every } v \in V \Rightarrow T^*T = 0.$$

Similarly, $T = 0 \Rightarrow TT^* = 0$.

Now suppose $T^*T = 0$. That means for every $v \in V$:

$$\langle T^*Tv, v \rangle = \langle 0, v \rangle = 0$$

and also

$$\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \|Tv\|^2 = 0$$

By the definiteness property of the inner product, $Tv = 0$ for every $v \in V$. Hence $T = 0$.

Similarly, $TT^* = 0$ implies that $T^* = 0$.

Established relations are sufficient to get from any of the stated equations to any other, as desired. $\square$

3 Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that

$$\lambda \text{ is an eigenvalue of } T \iff \overline{\lambda} \text{ is an eigenvalue of } T^*.$$

Solution:

Suppose λ is an eigenvalue of T with a corresponding eigenvector v. That means $(T - \lambda I)$ is not injective, i.e. dimension of its null space is greater than zero. Using properties of adjoint (7.6) and corollary 6.51, we see that

$$\begin{aligned}
\dim \operatorname{range} (T^* - \overline{\lambda} I) &= \dim V - \dim (\operatorname{range} (T^* - \overline{\lambda} I))^\perp \\
&= \dim V - \dim (\operatorname{range} (T - \lambda I)^*)^\perp \\
&= \dim V - \dim \operatorname{null} (T - \lambda I) \\
&< \dim V
\end{aligned}$$

The last inequality implies that $(T^* - \overline{\lambda})$ is not injective (Theorem 3.22), which implies that $\overline{\lambda}$ is an eigenvalue of T^* (Theorem 5.7). $\square$

4 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that

$$U \text{ is invariant under } T \iff U^\perp \text{ is invariant under } T^*.$$

Solution:
Suppose $u \in U$ and $v \in U^\perp$, U is invariant under T. We have

$$\langle Tu, v\rangle = 0$$
$$\langle Tu, v\rangle = \langle u, T^*v\rangle$$

This means $\langle u, T^*v\rangle = 0$ for every choice of u and v. Therefore, $T^*v \in U^\perp$ for every $v \in U^\perp$, hence, $U^\perp$ is invariant under T^*. Changing T to T^* and U to $U^\perp$ gives proof in other direction. □

5 Suppose $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Prove that

$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \|T^* f_1\|^2 + \cdots + \|T^* f_m\|^2.$$

Solution:
Denote a matrix of T with respect to the bases $e_1, \dots, e_n$ and $f_1, \dots, f_m$ by A. Matrix of T^* is then A^* (Theorem 7.9).
Note that $\|Te_j\|^2$ equals sum of elements of the first row of A squared. Similarly, for other vectors e_j. Thus:

$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \sum_{k=1}^{n} |A_{k,1}|^2 + \cdots + \sum_{k=1}^{n} |A_{k,n}|^2 = \sum_{j=1}^{m}\sum_{k=1}^{n} |A_{k,j}|^2.$$

For T^* we have:

$$\|T^* f_1\|^2 + \cdots + \|T^* f_m\|^2 = \sum_{j=1}^{m} |A^*_{j,1}|^2 + \cdots + \sum_{j=1}^{n} |A^*_{j,n}|^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |A^*_{j,k}|^2.$$

By definition of conjugate transpose:

$$\sum_{k=1}^{n}\sum_{j=1}^{m} |A^*_{j,k}|^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |\overline{A_{k,j}}|^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |A_{k,j}|^2,$$

thus leading to the desired equality. □

6 Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective $\iff T^*$ is surjective;

(b) T is surjective $\iff T^*$ is injective.

Solution:
(a) We have:

$$\begin{aligned} T \text{ is injective} &\iff \dim \operatorname{null} T = 0 \\ &\iff \dim (\operatorname{range} T^*)^\perp = 0 \\ &\iff \operatorname{range} T^* = W \\ &\iff T^* \text{ is surjective.} \end{aligned}$$

Here we used Theorem 3.15 for the first equivalence, Property 7.6 for the second equivalence, Theorem 6.54 for the third equivalence and the last follows from the definition of *surjective*.
(b) We have:

$$\begin{aligned} T \text{ is surjective} &\iff \dim \operatorname{range} T = \dim V \\ &\iff \dim (\operatorname{range} T)^\perp = 0 \\ &\iff \dim \operatorname{null} T^* = 0 \\ &\iff T^* \text{ is injective.} \end{aligned}$$

Here we used the same properties of range and null space as in (a), and a different identity from 7.6, relating null space of T^* with range of T. $\square$

7 Prove that if $T \in \mathcal{L}(V, W)$, then

(a) $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$;

(b) $\dim \operatorname{range} T^* = \dim \operatorname{range} T$

Solution:
(a) Using 7.6, 6.51 and Fundamental Theorem of Linear Maps, we get:

$$\begin{aligned} \dim \operatorname{null} T^* &= \dim (\operatorname{range} T)^\perp \\ &= \dim W - \dim \operatorname{range} T = \dim W - (\dim V - \dim \operatorname{null} T) \\ &= \dim \operatorname{null} T + \dim W - \dim V. \qquad \square \end{aligned}$$

(b) Here we can use result of part (a) to get:

$$\begin{aligned}\dim \operatorname{range} T^* &= \dim W - \dim \operatorname{null} T^* \\ &= \dim W - (\dim \operatorname{null} T + \dim W - \dim V) \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim \operatorname{range} T. \qquad \square\end{aligned}$$

8 Suppose A is an m-by-n matrix with entries in $\mathbb{F}$. Use (b) in Exercise 7 to prove that the row rank of A equals the column rank of A.

Solution:

Suppose V, W are vector spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$ and A is a matrix of T with respect to some bases.

By Theorem 3.78, column rank of A equals $\dim \operatorname{range} T$. According to the result of *Problem 7A.7b*, $\dim \operatorname{range} T = \dim \operatorname{range} T^*$, which in turn equals column rank of A^*, matrix of T^*, conjugate transpose of A.

Row rank of A equals column rank of A^t. Complex conjugation does not affect column (or row) rank, as span of columns does not change. If $\mathbb{F} = \mathbb{R}$, this is trivially true.

In case $\mathbb{F} = \mathbb{C}$, let k'th entry of j'th column of A^t equal $a_{k,j} = x_{k,j} + iy_{k,j}$. Suppose that rank of A^t equals m; without loss of generality let the first m columns of A^t be linearly independent. Thus, for every $c_1, \dots, c_m$

$$\begin{aligned}c_1 a_{k,1} + \dots + c_m a_{k,m} &= c_1(x_{k,1} + iy_{k,1}) + \dots + c_m(x_{k,1} + iy_{k,1}) \\ &= c_1 x_{k,1} + \dots + c_m x_{k,m} + i(c_1 y_{k,1} + \dots + c_m y_{k,m}) \\ &\neq 0.\end{aligned}$$

This implies that the sum of real or imaginary parts does not equal zero.

For columns of A^* we have entries $a_{k,j} = x_{k,j} - iy_{k,j}$. The sums are:

$$\begin{aligned}c_1 a^*_{k,1} + \dots + c_m a^*_{k,m} &= c_1(x_{k,1} - iy_{k,1}) + \dots + c_m(x_{k,1} - iy_{k,1}) \\ &= c_1 x_{k,1} + \dots + c_m x_{k,m} - i(c_1 y_{k,1} + \dots + c_m y_{k,m}) \\ &\neq 0.\end{aligned}$$

The last inequality follows from the fact that either the real or imaginary sum does not equal zero. Thus, span of columns of A^* is not less than span of columns of A^t.

Similarly, take a linearly dependent list of columns of A^t and take coefficients c_j that make a linear combination of the columns equal zero. Then, per-row sums of real and imaginary parts of entries of A^t equal zero. Under complex conjugation only the sign of imaginary part changes, therefore, per-row sum of entries of A^* also equals zero. This shows that column rank of A^* is not greater than the column rank of A^t

Thus, we have: column rank of A equals column rank of A^*, which equals to column rank of A^t, which equals to row rank of A, proving the desired equality. □

9 Prove that the product of two self-adjoint operators on V is self-adjoint if and only if the two operators commute.

Solution:

Suppose $T, S \in \mathcal{L}(V)$ are self-adjoint operators. Then we have:

$$TS = ST \iff TS = S^*T^* \iff TS = (TS)^*,$$

where we used definition of self-adjoint and property of the adjoint (7.5 d). □

10 Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if

$$\langle Tv, v \rangle = \langle T^*v, v \rangle$$

for all $v \in V$.

Solution:

First suppose T is self-adjoint. That means $T = T^*$, which trivially leads to equality $\langle Tv, v \rangle = \langle T^*v, v \rangle$.

Now suppose $\langle Tv, v \rangle = \langle T^*v, v \rangle$. Using definition of adjoint, property (7.5 c) $(T^*)^* = T$ and conjugate symmetry of inner products, we get

$$\langle T^*v, v \rangle = \langle v, Tv \rangle = \overline{\langle Tv, v \rangle}.$$

Thus, we have $\langle Tv, v \rangle = \overline{\langle Tv, v \rangle}$, which implies that $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$. That by Theorem 7.14 implies that T is self-adjoint, completing the proof. □

11 Define an operator $S : \mathbb{F}^2 \to \mathbb{F}^2$ by $S(w, z) = (-z, w)$.

(a) Find a formula for S^*.

(b) Show that S is normal but not self-adjoint.

(c) Find all eigenvalues of S.

Solution:
(a) To find formula for S^*, suppose $(w,z),(x,y)\in\mathbb{F}^2$. Then:

$$\langle S(w,z),(x,y)\rangle = \langle(-z,w),(x,y)\rangle = -zx+wy = \langle(w,z),(y,-x)\rangle.$$

This implies $S^*(w,z)=(z,-w)$.
(b) Formula for S^* clearly shows that S is not self-adjoint.

$$\begin{aligned} SS^*(w,z) &= S(z,-w) = (w,z) \\ S^*S(w,z) &= S^*(-z,w) = (w,z). \end{aligned}$$

Last two equations show that $SS^*=S^*S$, meaning S is a normal operator.
(c) Suppose λ is an eigenvalue of S. Then:

$$\begin{cases} -z = \lambda w, \\ w = \lambda z. \end{cases}$$

Eliminating z in the second equation via expression in the first we get:

$$w = -\lambda^2 w \quad\Rightarrow\quad (\lambda^2+1)w = 0.$$

As we need a non-zero eigenvector, $w\neq 0$. Hence we have equation on eigenvalues:

$$\lambda^2+1=0.$$

If $\mathbb{F}=\mathbb{R}$, there are no eigenvalues. If $\mathbb{F}=\mathbb{C}$, $\lambda=\pm i$. $\square$

12 An operator $B\in\mathcal{L}(V)$ is called *skew* if

$$B^* = -B.$$

Suppose that $T\in\mathcal{L}(V)$. Prove that T is normal if and only if there exist commuting operators A and B such that A is self-adjoint, B is a skew operator, and $T=A+B$.
Solution:
First suppose T is normal. Let

$$A = \frac{T+T^*}{2} \quad\text{and}\quad B = \frac{T-T^*}{2}. \tag{7.1}$$

Then A is self-adjoint, B is a skew operator, and $T=A+B$. Commutator of A and B equals:

$$\begin{aligned} AB-BA &= \frac{(T+T^*)(T-T^*)}{2} - \frac{(T-T^*)(T+T^*)}{2} \\ &= \frac{T^2-(T^*)^2-TT^*+T^*T-T^2+(T^*)^2-TT^*+T^*T}{2} \qquad (7.2) \\ &= T^*T-TT^*. \end{aligned}$$

Because T is normal, the right side of the equation above equals 0. Thus the operators A and B commute, as desired.

To prove the implication in the other direction, now suppose there exist commuting operators A and B such that A is self-adjoint, B is skew operator, and $T = A + B$. Then $T = AB$. Adding the last two equations and then dividing by 2 produces the equation for A in 7.1. Subtracting the last two equations and then dividing by 2 produces the equation for B in 7.1. Now 7.1 implies 7.2. Because A and B commute, 7.2 implies that T is normal, as desired. $\square$

13 Suppose $\mathbb{F} = \mathbb{R}$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}T = T^*$ for all $T \in \mathcal{L}(V)$.

(a) Find all eigenvalues of $\mathcal{A}$.

(b) Find the minimal polynomial of $\mathcal{A}$.

Solution:
Using property 7.5 c of adjoint, we have:

$$\mathcal{A}^2 T = T \quad \Rightarrow (\mathcal{A}^2 - \mathcal{I})T = 0.$$

Hence the minimal polynomial of $\mathcal{A}$ is $p(z) = z^2 - 1$.

Eigenvalues of $\mathcal{A}$ are roots of the minimal polynomial: ± 1. $\square$

14 Define an inner product on $\mathcal{P}_2(\mathbb{R})$ by $\langle p, q \rangle = \int_0^1 pq$. Define an operator $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by

$$T(ax^2 + bx + c) = bx.$$

(a) Show that with this inner product, the operator T is not self-adjoint.

(b) The matrix of T with respect to the basis $1, x, x^2$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution:
If T were self-adjoint, we would have an equality $\langle Tp, q \rangle = \langle p, Tq \rangle$ for any $p, q \in \mathcal{P}_2(\mathbb{R})$.

Let $p = a_1x^2+b_1x+c_1$ and $q = a_2x^2+b_2x+c_2$ for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Then we have:

$$\begin{aligned}
\langle Tp, q\rangle &= \langle b_1x, a_2x^2 + b_2x + c_2\rangle \\
&= b_1 \int_0^1 (a_2x^3 + b_2x^2 + c_2x)dx \\
&= b_1 \left(a_2\frac{x^4}{4} + b_2\frac{x^3}{3} + c_2\frac{x^2}{2} \right) \Big|_0^1 \\
&= \frac{b_1a_2}{4} + \frac{b_1b_2}{3} + \frac{b_1c_2}{2}
\end{aligned}$$

$$\begin{aligned}
\langle p, Tq\rangle &= \langle a_1x^2 + b_1x + c_1, b_2x\rangle \\
&= b_2 \int_0^1 (a_1x^3 + b_1x^2 + c_1x)dx \\
&= b_2 \left(a_1\frac{x^4}{4} + b_1\frac{x^3}{3} + c_1\frac{x^2}{2} \right) \Big|_0^1 \\
&= \frac{b_2a_1}{4} + \frac{b_1b_2}{3} + \frac{b_2c_1}{2}
\end{aligned}$$

Therefore, $\langle Tp, q\rangle \neq \langle p, Tq\rangle$ for all p, q, thus T is not self-adjoint. $\square$

(b) Basis $1, x, x^2$ is *not orthonormal*, while Theorem 7.9 states that matrix of T^* equals complex conjugate transpose of the matrix of T when evaluated in an *orthonormal basis*. Thus, there is no contradiction.

15 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

(a) T is self-adjoint $\iff T^{-1}$ is self-adjoint;

(b) T is normal $\iff T^{-1}$ is normal.

Solution:

(a) Suppose T is self-adjoint. Then by property 7.5 f, we have

$$(T^{-1})^* = (T^*)^{-1} = T^{-1},$$

thus, T^{-1} is self-adjoint.

Changing T to T^{-1} and using property $(T^{-1})^{-1} = T$ (see *Problem 3D.1*), we get the proof in other direction.

(b) Suppose T is normal. Then we have

$$\begin{aligned} T^{-1}(T^{-1})^* &= T^{-1}(T^*)^{-1} = (T^*T)^{-1} \\ &= (TT^*)^{-1} \\ &= (T^*)^{-1}T^{-1} \\ &= (T^{-1})^*T^{-1}, \end{aligned}$$

where we use property of inverse ($(TS)^{-1} = S^{-1}T^{-1}$, see *Problem 3D.2*), property of adjoint 7.5 f and normality of T. This shows that T^{-1} is also normal.

Changing T to T^{-1} and using $(T^{-1})^{-1} = T$, we get the proof in other direction. $\square$

16 Suppose $\mathbb{F} = \mathbb{R}$.

(a) Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

(b) What is the dimension of the subspace of $\mathcal{L}(V)$ in (a) [in terms of $\dim V$]?

Solution:
(a) We need to check three conditions of Theorem 1.34.

- 0 is a self-adjoint operator. ✓
- Suppose S and T are self-adjoint. Then $(S+T)^* = S^* + T^* = S + T$, hence self-adjoint operators are closed under addition. ✓
- Suppose T is self-adjoint and $\alpha \in \mathbb{R}$. Then $(\alpha T)^* = \overline{\alpha} T^* = \alpha T$, hence self-adjoint operators are closed under scalar multiplication. ✓

Thus, the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$. $\square$

(b) Let $e_1, \dots, e_n$ be an orthonormal basis of V. In this basis, symmetric matrices represent self-adjoint operators on V. Every symmetric matrix can be constructed from a matrix with either only one non-zero entry on the diagonal or two equal non-zero entries ($A_{j,k}$ and $A_{k,j}$). Therefore, dimension of the subspace of self-adjoint operators equals:

$$n + \frac{n^2 - n}{2} = \frac{n^2 + n}{2} = \dim V(\dim V + 1)/2$$

17 Suppose $\mathbb{F} = \mathbb{C}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Solution:

On the complex vector spaces, the set of self-adjoint operators is not closed under scalar multiplication:

$$(\alpha T)^* = \overline{\alpha}\, T^* = \overline{\alpha}\, T.$$

If α has an imaginary part, $\overline{\alpha} \neq \alpha$, hence $(\alpha T)^* \neq \alpha T$. $\square$

18 Suppose $\dim V \geq 2$. Show that the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

Solution:

Suppose $S, T \in \mathcal{L}(V)$ are normal operators that do not commute and $ST^* - T^*S$ has a real component. Then their sum is not a normal operator:

$$\begin{aligned}(S+T)(S+T)^* - (S+T)^*(S+T) &= SS^* + TT^* + ST^* + TS^* \\ &\quad - S^*S - T^*T - S^*T - T^*S \\ &= (ST^* - T^*S) + (TS^* - S^*T) \\ &= (ST^* - T^*S) + (ST^* - T^*S)^* \\ &= 2\Re(ST^* - T^*S) \\ &\neq 0\end{aligned}$$

19 Suppose $T \in \mathcal{L}(V)$ and $\|T^*v\| \leq \|Tv\|$ for every $v \in V$. Prove that T is normal.

Solution:

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V. Then (by *Problem 7A.5*) we have:

$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \|T^*e_1\|^2 + \cdots + \|T^*e_n\|^2. \tag{7.3}$$

This, together with $\|T^*v\| \leq \|Tv\|$, implies that $\|Te_j\|^2 = \|T^*e_j\|^2$ for every e_j in the basis. Indeed, we can rearrange terms in 7.3 as:

$$\|Te_1\|^2 - \|T^*e_1\|^2 = (\|T^*e_2\|^2 - \|Te_2\|^2) + \cdots + (\|T^*e_n\|^2 - \|Te_n\|^2). \tag{7.4}$$

The left-hand side of 7.4 is greater than or equal to zero, meanwhile the right-hand side is less than or equal to zero (as every term on the right side is less than or equal to zero). Therefore, $\|Te_1\| = \|T^*e_1\|$. Similarly, this equality can be shown for any other e_j.

Let $v = \alpha e_1$, then

$$\|Tv\| = \|T(\alpha e_1)\| = |\alpha|\|Te_1\| = |\alpha|\|T^* e_1\| = \|T^*(\alpha e_1)\| = \|T^* v\|.$$

Since the orthonormal basis is arbitrary, we can construct one starting from any arbitrary $v \in V$ using the Gram-Schmidt procedure (6.32). This implies $\|Tv\| = \|T^*v\|$ for every $v \in V$, hence T is normal (Theorem 7.20). □

20 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the following are equivalent.

(a) P is self-adjoint.

(b) P is normal.

(c) There is a subspace U of V such that $P = P_U$.

Solution:

First suppose P is self-adjoint. Then it automatically means P is normal.

Now suppose P is normal. Then, $\operatorname{range} P = \operatorname{range} P^*$ (Theorem 7.21) and $\operatorname{null} P = (\operatorname{range} P^*)^\perp$ (Theorem 7.6), which implies $\operatorname{null} P = (\operatorname{range} P)^\perp$. Thus, we have that every vector in the null space of P is orthogonal to every vector in range of P and $P = P^2$, which implies (by *Problem 6C.9*) that there exists a subspace U of V such that $P = P_U$.

Finally, suppose P is an orthogonal projection on some subspace U of V. Let $u_1, u_2 \in U$ and $w_1, w_2 \in U^\perp$. We have

$$\begin{aligned}
\langle P(u_1 + w_1), u_2 + w_2\rangle &= \langle u_1, u_2 + w_2\rangle = \langle u_1, u_2\rangle \\
\langle P(u_1 + w_1), u_2 + w_2\rangle &= \langle u_1 + w_1, P^*(u_2 + w_2)\rangle.
\end{aligned}$$

Two equations above imply that $P^*(u_2 + w_2) = u_2$ for any $u_2 \in U$ and $w_2 \in U^\perp$. This coincides with the definition of orthogonal projection on U, therefore $P = P^*$, hence P is self-adjoint.

Thus, we have shown that (a) implies (b), (b) implies (c) and (c) implies (a), hence these three statements are equivalent. □

21 Suppose $D : \mathcal{P}_8(\mathbb{R}) \to \mathcal{P}_8(\mathbb{R})$ is the differentiation operator defined by $Dp = p'$. Prove that there does not exist an inner product on $\mathcal{P}_8(\mathbb{R})$ that makes D a normal operator.

Solution:

Suppose there is an inner product such that D is a normal operator.

By Theorem 7.21, $\operatorname{range} D^* = \operatorname{range} D$.

Now note that for any $p \in \mathcal{P}_8(\mathbb{R})$:

$$\langle Da_0, p\rangle = \langle 0, p\rangle = 0$$
$$\langle Da_0, p\rangle = \langle a_0, D^* p\rangle$$

where a_0 is a constant polynomial. This equation implies that $D^*p \in (\operatorname{span}(1))^\perp$. Thus, $\operatorname{range} D \subset (\operatorname{span}(1))^\perp$.

At the same time, $Dx = 1 \in \operatorname{range} D$ and $1 \notin (\operatorname{span}(1))^\perp$. Hence, our assumption leads to a contradiction, and there is no inner product such that D is a normal operator. $\square$

Comment: This problem can be extended to any finite-dimensional polynomial vector space with dimension greater than 1, as only this fact is used in the proof.

22 Give an example of an operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that T is normal but not self-adjoint.

Solution:

Let T be an operator on $\mathbb{R}^3$, with matrix in standard basis:

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then, the matrix of adjoint operator T^* is:

$$\mathcal{M}(T^*) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

T is clearly not self-adjoint. Yet it is normal, as can be checked by matrix multiplication:

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

23 Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad Tv = 3v, \quad Tw = 4w.$$

Show that $\|T(v+w)\| = 10$.

Solution:

Here we use Theorem 7.22. Vectors v and w are eigenvectors of T, corresponding to distinct eigenvalues, hence they are orthogonal. Then we use Pythagorean theorem (6.12) to compute the norm directly:

$$\begin{aligned}\|T(v+w)\| &= \|3v+4w\| = \sqrt{\|3v\|^2 + \|4w\|^2} \\ &= \sqrt{9\|v\|^2 + 16\|w\|^2} \\ &= \sqrt{9 \cdot 4 + 16 \cdot 4} \\ &= 10. \quad \square\end{aligned}$$

24 Suppose $T \in \mathcal{L}(V)$ and

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + z^m$$

is the minimal polynomial of T. Prove that the minimal polynomial of T^* is

$$\overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + \cdots + \overline{a_{m-1}} z^{m-1} + z^m$$

Solution:

First, note that every $v, u \in V$:

$$\langle p(T)v, u\rangle = 0 = \langle v, (p(T))^* u\rangle \tag{7.5}$$

hence $(p(T))^* = 0$. Expanding adjoint of $p(T)$ we get:

$$\begin{aligned}(p(T))^* &= (a_0 I)^* + (a_1 T)^* + (a_2 T^2)^* + \cdots + (a_m T^{m-1})^* + (T^m)^* \\ &= \overline{a_0} I + \overline{a_1} T^* + \overline{a_2} (T^*)^2 + \cdots + \overline{a_m} (T^*)^{m-1} + (T^*)^m.\end{aligned}$$

Now suppose that there is a polynomial $q(z) \neq (\overline{a_0} + \overline{a_1} z + \cdots + z^m)$ such that $q(T^*) = 0$ and $\deg q \leq \deg p$. Reversing 7.5 with $q(z)$ in place of $p(z)$ we conclude that $\overline{q(T)} = 0$ (that is, $q(T)$ with all coefficients turned into their complex conjugate). That would imply that $p(z)$ is not a minimal polynomial of T, being either of not the least degree or not unique. Hence, we must conclude that

$$\overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + \cdots + \overline{a_{m-1}} z^{m-1} + z^m$$

is a minimal polynomial of T^*. $\square$

25 Suppose $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if T^* is diagonalizable.

Solution:

By Theorem 5.62, T is diagonalizable if and only if the minimal polynomial of T equals $(z-\lambda_1)\dots(z-\lambda_m)$ for some list of distinct $\lambda_1, \dots, \lambda_m \in \mathbb{F}$. Following argument of the previous problem, we see that the minimal polynomial of T^* is $q(z)$ such that $q(T^*) = (p(T))^*$. Thus, we have:

$$\begin{aligned} q(T^*) &= [(T-\lambda_1 I)\dots(T-\lambda_m I)]^* = \\ &= (T-\lambda_m)^* \dots (T-\lambda_1 I)^* \\ &= (T^* - \overline{\lambda_m} I)\dots(T^* - \overline{\lambda_1}). \end{aligned}$$

As $\lambda_1, \dots, \lambda_m$ are distinct, so are $\overline{\lambda_m}, \dots, \overline{\lambda_1}$. Thus, the minimal polynomial of T^* has the desired form of a product of distinct $(z-\alpha_i)$ terms, which implies that T^* is diagonalizable.

Reversing proof with T^* in place of T, gives implication in other direction. □

26 Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u\rangle x$ for every $v \in V$.

(a) Prove that if V is a real vector space, then T is self-adjoint if and only if the list u, x is linearly dependent.

(b) Prove that T is normal if and only if the list u, x is linearly dependent.

Solution:

(a) First, suppose that T is self-adjoint. Let v, w be arbitrary vectors in V. Then the inner product of Tv and w is:

$$\langle Tv, w\rangle = \langle \langle v, u\rangle x, w\rangle = \langle v, u\rangle\langle x, w\rangle,$$

and also:

$$\langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, \langle w, u\rangle x\rangle = \langle v, x\rangle\langle w, u\rangle.$$

Thus we have:

$$\begin{aligned} \langle v, u\rangle\langle x, w\rangle - \langle v, x\rangle\langle w, u\rangle = 0 \\ \langle v, \langle x, w\rangle u - \langle w, u\rangle x\rangle = 0 \end{aligned}$$

for every $v, w \in V$. This implies that $\langle x, w\rangle u - \langle w, u\rangle x = 0$, hence u, x is a linearly dependent list.

Now to proof the other direction, suppose u, x is a linearly dependent list. Then $u = \lambda x$, where $\lambda \in \mathbb{R}$. Let $v, w \in V$, then we have:

$$\begin{aligned}
\langle Tv, w\rangle &= \langle\langle v, u\rangle x, w\rangle = \langle\langle v, \lambda x\rangle x, w\rangle \\
&= \lambda\langle v, x\rangle\langle x, w\rangle = \lambda\langle v, x\rangle\langle w, x\rangle \\
&= \langle v, \langle w, \lambda x\rangle x\rangle \\
&= \langle v, \langle w, u\rangle x\rangle \\
&= \langle v, T^* w\rangle.
\end{aligned}$$

This implies that $T^* = T$, i.e. T is self-adjoint. □

(b) Before the proof itself, we must explicitly find T^* for this case. Following the previous part, we have for $v, w \in V$:

$$\langle Tv, w\rangle = \langle v, u\rangle\langle x, w\rangle = \langle v, \langle w, x\rangle u\rangle.$$

Thus, $T^* v = \langle v, x\rangle u$.

Now, we suppose that T is normal. Then

$$TT^* v = T(\langle v, x\rangle u) = \langle\langle v, x\rangle u, u\rangle x = \langle v, x\rangle\|u\|^2 x,$$

and

$$T^* Tv = T^*(\langle v, u\rangle x) = \langle\langle v, u\rangle x, x\rangle u = \langle v, u\rangle\|x\|^2 u.$$

For a normal operator we have $TT^* - T^*T = 0$, hence

$$\langle v, x\rangle\|u\|^2 x = \langle v, u\rangle\|x\|^2 u$$

for every $v \in V$. Thus, the list u, x is linearly dependent.

For a proof in other direction, suppose that $u = \lambda x$, where $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}
\|Tv\| &= \|\langle v, u\rangle x\| = \|\langle v, \lambda x\rangle x\| \\
&= \|\overline{\lambda}\langle v, x\rangle x\| = |\lambda| \cdot \|\langle v, x\rangle x\| \\
&= \|\lambda\langle v, x\rangle x\| = \|\langle v, x\rangle u\| \\
&= \|T^* v\|.
\end{aligned}$$

Thus, by Theorem 7.20, T is normal, completing the proof. □

27 Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

Solution:

Firstly, for $k = 1$, the theorem is obviously true, so we will assume $k \geq 2$ in the rest of the proof.

That $\operatorname{null} T \subseteq \operatorname{null} T^k$ and $\operatorname{range} T \subseteq \operatorname{range} T^k$ (for any operator), is true, as can be easily seen. We will prove the other direction of inclusion.

First, for a self-adjoint operator S (here it will be T^*T), suppose that $v \in \operatorname{null} S^k$. Then we have:

$$0 = \langle S^k v, S^{k-2} v\rangle = \langle S^{k-1} v, S^{k-1} v\rangle.$$

Thus, $\|S^{k-1}v\| = 0$, which implies $S^{k-1}v = 0$, therefore $\operatorname{null} S^k \subseteq \operatorname{null} S^{k-1}$. Repeating the induction on k until $k - 1 = 1$, we have that for every positive integer k, $\operatorname{null} S^k \subseteq \operatorname{null} S$. Hence, $\operatorname{null} S^k = \operatorname{null} S$.

Now we examine a normal operator T. Suppose $v \in \operatorname{null} T^k$ for some positive integer k. Then.

$$T^k v = 0 \Rightarrow (T^*)^k T^k v = 0 \Rightarrow (T^*T)^k v = 0,$$

where the second implication is valid because T and T^* commute. Thus, $v \in \operatorname{null} (T^*T)^k$, which implies $v \in \operatorname{null} T^*T$. Hence

$$0 = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \iff Tv = 0 \iff v \in \operatorname{null} T.$$

Thus, we have shown $\operatorname{null} T^k = \operatorname{null} T$ for every positive integer k.

Finally, using that T^k is also a normal operator, we see that

$$\operatorname{range} T^k = (\operatorname{null} (T^k)^*)^\perp = (\operatorname{null} T^k)^\perp = (\operatorname{null} T)^\perp = \operatorname{range} T^* = \operatorname{range} T,$$

completing the proof. $\square$

28 Suppose $T \in \mathcal{L}(V)$ is normal. Prove that if $\lambda \in \mathbb{F}$, then the minimal polynomial of T is not a polynomial multiple of $(x - \lambda)^2$.

Solution:

Let $p(z)$ be a minimal polynomial of T and suppose that it is a polynomial multiple of $(z - \lambda)^2$:

$$p(z) = (z - \lambda)^2 q(z)$$

for some polynomial $q(z)$.

Then we have for every $v \in V$:

$$(T - \lambda I)^2 q(T)v = 0 \Rightarrow q(T)v \in \operatorname{null} (T - \lambda I)^2.$$

By property of normal operator 7.21 (d), $(T - \lambda I)$ is a normal operator. Result of the previous problem thus implies that $q(T)v \in \operatorname{null}(T - \lambda I)$. Thus for every $v \in V$:

$$(T - \lambda I)q(T)v = 0.$$

But this polynomial has a degree less than $p(z)$, contradicting the fact that $p(z)$ is a minimal polynomial of T. Hence, $p(z)$ cannot be a polynomial multiple of $(z - \lambda)^2$ for any $\lambda \in \mathbb{F}$. $\square$

29 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_k\| = \|T^* e_k\|$ for each $k = 1, \ldots, n$, then T is normal.

Solution: Let $\mathbb{F} = \mathbb{R}$ and take the operator T and its adjoint, defined by matrices, with respect to the standard basis:

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad \mathcal{M}(T^*) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

As can be checked with matrix multiplication, these operators do not commute:

$$\mathcal{M}(T) \cdot \mathcal{M}(T^*) = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 2 \\ 3 & 2 & 10 \end{pmatrix}$$
$$\mathcal{M}(T^*) \cdot \mathcal{M}(T) = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 10 \end{pmatrix}.$$

Hence, T is not a normal operator. Meanwhile, for vectors of the basis we have:

$$\begin{aligned} Te_1 &= e_1 + e_2, & T^* e_1 &= e_1 + e_3, \\ Te_2 &= 2e_2 + e_3, & T^* e_2 &= e_1 + 2e_2, \\ Te_3 &= e_1 + 3e_3, & T^* e_3 &= e_2 + 3e_3. \end{aligned}$$

So we have $\|Te_k\| = \|T^* e_k\|$ for every $k = 1, 2, 3$, but T is not normal, counterproving the statement of the problem. $\square$

30 Suppose that $T \in \mathcal{L}(\mathbb{F}^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$.

Solution:

Vector $(1,1,1)$ is an eigenvector of T with eigenvalue 2; (z_1, z_2, z_3) is an eigenvector of T with eigenvalue 0. By Theorem 7.22, these two vectors are orthogonal. Hence

$$0 = \langle (z_1, z_2, z_3), (1,1,1) \rangle = z_1 + z_2 + z_3,$$

as desired. □

31 Fix a positive integer n. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} fg$, let

$$V = \operatorname{span}(1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx).$$

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that D is normal but not self-adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that T is self-adjoint.

Solution:

(a) Earlier (in *Problem 6B.4*) we have shown that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list. Hence, this list is an orthonormal basis of V. Operator D acts on the basis vectors as follows

$$\begin{aligned} D(\frac{1}{\sqrt{2\pi}}) &= 0 \\ D(\frac{\cos kx}{\sqrt{\pi}}) &= -k\frac{\sin kx}{\sqrt{\pi}} \\ D(\frac{\sin kx}{\sqrt{\pi}}) &= k\frac{\cos kx}{\sqrt{\pi}}. \end{aligned}$$

Thus, in this basis, the matrix of D is

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & n \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Matrix of D^* is a transpose of this matrix. We see that $(\mathcal{M}(D))^t = -\mathcal{M}(D)$, hence $D^* = -D$.

Clearly, D is not self-adjoint, but it is indeed normal:

$$DD^* = D(-D) = -D^2 = (-D)D = D^*D. \quad \square$$

(b) Working in the same basis, we have:

$$\begin{aligned} T\left(\frac{1}{\sqrt{2\pi}}\right) &= 0 \\ T\left(\frac{\cos kx}{\sqrt{\pi}}\right) &= -k^2 \frac{\cos kx}{\sqrt{\pi}} \\ T\left(\frac{\sin kx}{\sqrt{\pi}}\right) &= -k^2 \frac{\sin kx}{\sqrt{\pi}}. \end{aligned}$$

Thus, the matrix of T is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -n^2 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -4 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n^2 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This matrix is symmetric and hence T is a self-adjoint operator. $\square$

32 Suppose $T : V \to W$ is a linear map. Show that under the standard identification of V with V' and the corresponding identification of W with W', the adjoint map $T^* : W \to V$ corresponds to the dual map $T' : W' \to V'$. More precisely, show that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all $w \in W$, where φ_w and φ_{T^*w} are defined as in 6.58.

Solution:

Following Riesz representation theorem, we define $\varphi_v(u)$ as

$$\varphi_v(u) = \langle u, v \rangle,$$

where v, u are either in V, or in W, and we use the inner product defined on the corresponding vector space.

Let $v \in V$, $w \in W$. Then, using definition of dual map and adjoint, we have:

$$\begin{aligned}
\left(T'(\varphi_w)\right)(v) &= (\varphi_w \circ T)v = \varphi_w(Tv) \\
&= \langle Tv, w \rangle = \langle v, T^*w \rangle \\
&= \varphi_{T^*w}(v),
\end{aligned}$$

as desired. $\square$