Chapter 7

Operators on Inner Product Spaces

Contents

7A Self-Adjoint and Normal Operators

1 Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$

Find a formula for $T^*(z_1, \ldots, z_n)$.

Solution:

Suppose $(z_1, \dots, z_n), (w_1, \dots, w_n) \in \mathbb{F}^n$. Then

$$\begin{aligned}\langle T(z_1, \dots, z_n), (w_1, \dots, w_n) \rangle &= \langle (0, z_1, \dots, z_{n-1}), (w_1, w_2, \dots, w_n) \rangle \\ &= z_1 w_2 + z_2 w_3 + \dots + z_{n-1} w_n \\ &= \langle (z_1, \dots, z_n), (w_2, \dots, w_n, 0) \rangle\end{aligned}$$

By the definition of the adjoint, we must have

$$T * (w_1, \dots, w_n) = (w_2, \dots, w_{n-1}, 0),$$

which is the sought formula for the adjoint. □

2 Suppose $T \in \mathcal{L}(V, W)$. Prove that

$$T = 0 \iff T^* = 0 \iff T^*T = 0 \iff TT^* = 0.$$

Solution:

First equivalence is just the property of a zero map. Indeed, for any $v \in V$, $u \in W$:

$$\langle 0v, u \rangle = 0 = \langle v, 0u \rangle.$$

That is, zero map is "self-adjoint" (although these maps are from different vector spaces).

Third equation follows directly from the second:

$$T^* = 0 \Rightarrow T^*(Tv) = 0 \text{ for every } v \in V \Rightarrow T^*T = 0.$$

Similarly, $T = 0 \Rightarrow TT^* = 0$.

Now suppose $T^*T = 0$. That means for every $v \in V$:

$$\langle T^*Tv, v \rangle = \langle 0, v \rangle = 0$$

and also

$$\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \|Tv\|^2 = 0$$

By the definiteness property of the inner product, $Tv = 0$ for every $v \in V$. Hence $T = 0$.

Similarly, $TT^* = 0$ implies that $T^* = 0$.

Established relations are sufficient to get from any of the stated equations to any other, as desired. □

3 Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that

$$\lambda \text{ is an eigenvalue of } T \iff \overline{\lambda} \text{ is an eigenvalue of } T^*.$$

Solution:
Suppose λ is an eigenvalue of T with a corresponding eigenvector v. That means $(T - \lambda I)$ is not injective, i.e. dimension of its null space is greater than zero. Using properties of adjoint (7.6) and corollary 6.51, we see that

$$\begin{aligned}
\dim \operatorname{range}(T^* - \overline{\lambda} I) &= \dim V - \dim (\operatorname{range}(T^* - \overline{\lambda} I))^{\perp} \\
&= \dim V - \dim (\operatorname{range}(T - \lambda I)^*)^{\perp} \\
&= \dim V - \dim \operatorname{null}(T - \lambda I) \\
&< \dim V
\end{aligned}$$

The last inequality implies that $(T^* - \overline{\lambda})$ is not injective (Theorem 3.22), which implies that $\overline{\lambda}$ is an eigenvalue of T^* (Theorem 5.7). $\square$

4 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that

$$U \text{ is invariant under } T \iff U^{\perp} \text{ is invariant under } T^*.$$

Solution:
Suppose $u \in U$ and $v \in U^{\perp}$, U is invariant under T. We have

$$\begin{aligned}
\langle Tu, v \rangle &= 0 \\
\langle Tu, v \rangle &= \langle u, T^* v \rangle
\end{aligned}$$

This means $\langle u, T^* v \rangle = 0$ for every choice of u and v. Therefore, $T^* v \in U^{\perp}$ for every $v \in U^{\perp}$, hence, $U^{\perp}$ is invariant under T^*. Changing T to T^* and U to $U^{\perp}$ gives proof in other direction. $\square$

5 Suppose $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Prove that

$$\|Te_1\|^2 + \dots + \|Te_n\|^2 = \|T^* f_1\|^2 + \dots + \|T^* f_m\|^2.$$

Solution:
Denote a matrix of T with respect to the bases $e_1, \dots, e_n$ and $f_1, \dots, f_m$ by A. Matrix of T^* is then A^* (Theorem 7.9).

Note that $\|Te_j\|^2$ equals sum of elements of the first row of A squared. Similarly, for other vectors e_j. Thus:

$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \sum_{k=1}^{n} |A_{k,1}|^2 + \cdots + \sum_{k=1}^{n} |A_{k,n}|^2 = \sum_{j=1}^{m}\sum_{k=1}^{n} |A_{k,j}|^2.$$

For T^* we have:

$$\|T^* f_1\|^2 + \cdots + \|T^* f_m\|^2 = \sum_{j=1}^{m} |A^*_{j,1}|^2 + \cdots + \sum_{j=1}^{n} |A^*_{j,n}|^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |A^*_{j,k}|^2.$$

By definition of conjugate transpose:

$$\sum_{k=1}^{n}\sum_{j=1}^{m} |A^*_{j,k}|^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |\overline{A_{k,j}}|^2 = \sum_{k=1}^{n}\sum_{j=1}^{m} |A_{k,j}|^2,$$

thus leading to the desired equality. $\square$

6 Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective $\iff$ T^* is surjective;

(b) T is surjective $\iff$ T^* is injective.

Solution:
(a) We have:

$$\begin{aligned} T \text{ is injective} &\iff \dim \operatorname{null} T = 0 \\ &\iff \dim\,(\operatorname{range} T^*)^\perp = 0 \\ &\iff \operatorname{range} T^* = W \\ &\iff T^* \text{ is surjective.} \end{aligned}$$

Here we used Theorem 3.15 for the first equivalence, Property 7.6 for the second equivalence, Theorem 6.54 for the third equivalence and the last follows from the definition of *surjective*.

(b) We have:

$$\begin{aligned} T \text{ is surjective} &\iff \dim \operatorname{range} T = \dim V \\ &\iff \dim\,(\operatorname{range} T)^\perp = 0 \\ &\iff \dim \operatorname{null} T^* = 0 \\ &\iff T^* \text{ is injective.} \end{aligned}$$

Here we used the same properties of range and null space as in (a), and a different identity from 7.6, relating null space of T^* with range of T. $\square$

7 Prove that if $T \in \mathcal{L}(V, W)$, then

(a) $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$;

(b) $\dim \operatorname{range} T^* = \dim \operatorname{range} T$

Solution:
(a) Using 7.6, 6.51 and Fundamental Theorem of Linear Maps, we get:

$$\begin{aligned} \dim \operatorname{null} T^* &= \dim (\operatorname{range} T)^{\perp} \\ &= \dim W - \dim \operatorname{range} T = \dim W - (\dim V - \dim \operatorname{null} T) \\ &= \dim \operatorname{null} T + \dim W - \dim V. \qquad \square \end{aligned}$$

(b) Here we can use result of part (a) to get:

$$\begin{aligned} \dim \operatorname{range} T^* &= \dim W - \dim \operatorname{null} T^* \\ &= \dim W - (\dim \operatorname{null} T + \dim W - \dim V) \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim \operatorname{range} T. \qquad \square \end{aligned}$$

8 Suppose A is an m-by-n matrix with entries in $\mathbb{F}$. Use (b) in Exercise 7 to prove that the row rank of A equals the column rank of A.
Solution:
Suppose V, W are vector spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$ and A is a matrix of T with respect to some bases.

By Theorem 3.78, column rank of A equals $\dim \operatorname{range} T$. According to the result of *Problem 7A.7b*, $\dim \operatorname{range} T = \dim \operatorname{range} T^*$, which in turn equals column rank of A^*, matrix of T^*, conjugate transpose of A.

Row rank of A equals column rank of A^t. Complex conjugation does not affect column (or row) rank, as span of columns does not change. If $\mathbb{F} = \mathbb{R}$, this is trivially true.

In case $\mathbb{F} = \mathbb{C}$, let k'th entry of j'th column of A^t equal $a_{k,j} = x_{k,j} + iy_{k,j}$. Suppose that rank of A^t equals m; without loss of generality let the first m columns of A^t be linearly independent. Thus, for every $c_1, \dots, c_m$

$$\begin{aligned} c_1 a_{k,1} + \cdots + c_m a_{k,m} &= c_1(x_{k,1} + iy_{k,1}) + \cdots + c_m(x_{k,1} + iy_{k,1}) \\ &= c_1 x_{k,1} + \cdots + c_m x_{k,m} + i(c_1 y_{k,1} + \cdots + c_m y_{k,m}) \\ &\neq 0. \end{aligned}$$

This implies that the sum of real or imaginary parts does not equal zero.

For columns of A^* we have entries $a_{k,j} = x_{k,j} - iy_{k,j}$. The sums are:

$$\begin{aligned}
c_1 a^*_{k,1} + \cdots + c_m a^*_{k,m} &= c_1(x_{k,1} - iy_{k,1}) + \cdots + c_m(x_{k,1} - iy_{k,1}) \\
&= c_1 x_{k,1} + \cdots + c_m x_{k,m} - i(c_1 y_{k,1} + \cdots + c_m y_{k,m}) \\
&\neq 0.
\end{aligned}$$

The last inequality follows from the fact that either the real or imaginary sum does not equal zero. Thus, span of columns of A^* is not less than span of columns of A^t.

Similarly, take a linearly dependent list of columns of A^t and take coefficients c_j that make a linear combination of the columns equal zero. Then, per-row sums of real and imaginary parts of entries of A^t equal zero. Under complex conjugation only the sign of imaginary part changes, therefore, per-row sum of entries of A^* also equals zero. This shows that column rank of A^* is not greater than the column rank of A^t

Thus, we have: column rank of A equals column rank of A^*, which equals to column rank of A^t, which equals to row rank of A, proving the desired equality. □

9 Prove that the product of two self-adjoint operators on V is self-adjoint if and only if the two operators commute.

Solution:

Suppose $T, S \in \mathcal{L}(V)$ are self-adjoint operators. Then we have:

$$TS = ST \quad \iff \quad TS = S^*T^* \quad \iff \quad TS = (TS)^*,$$

where we used definition of self-adjoint and property of the adjoint (7.5 d). □

10 Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if

$$\langle Tv, v\rangle = \langle T^*v, v\rangle$$

for all $v \in V$.

Solution:

First suppose T is self-adjoint. That means $T = T^*$, which trivially leads to equality $\langle Tv, v\rangle = \langle T^*v, v\rangle$.

Now suppose $\langle Tv, v\rangle = \langle T^*v, v\rangle$. Using definition of adjoint, property (7.5 c) $(T^*)^* = T$ and conjugate symmetry of inner products, we get

$$\langle T^*v, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}.$$

Thus, we have $\langle Tv, v\rangle = \overline{\langle Tv, v\rangle}$, which implies that $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$. That by Theorem 7.14 implies that T is self-adjoint, completing the proof. □

11 Define an operator $S : \mathbb{F}^2 \to \mathbb{F}^2$ by $S(w, z) = (-z, w)$.

(a) Find a formula for S^*.

(b) Show that S is normal but not self-adjoint.

(c) Find all eigenvalues of S.

Solution:
(a) To find formula for S^*, suppose $(w, z), (x, y) \in \mathbb{F}^2$. Then:

$$\langle S(w, z), (x, y)\rangle = \langle(-z, w), (x, y)\rangle = -zx + wy = \langle(w, z), (y, -x)\rangle.$$

This implies $S^*(w, z) = (z, -w)$.
(b) Formula for S^* clearly shows that S is not self-adjoint.

$$\begin{aligned} SS^*(w, z) &= S(z, -w) = (w, z) \\ S^*S(w, z) &= S^*(-z, w) = (w, z). \end{aligned}$$

Last two equations show that $SS^* = S^*S$, meaning S is a normal operator.
(c) Suppose λ is an eigenvalue of S. Then:

$$\begin{cases} -z = \lambda w, \\ w = \lambda z. \end{cases}$$

Eliminating z in the second equation via expression in the first we get:

$$w = -\lambda^2 w \quad \Rightarrow \quad (\lambda^2 + 1)w = 0.$$

As we need a non-zero eigenvector, $w \neq 0$. Hence we have equation on eigenvalues:

$$\lambda^2 + 1 = 0.$$

If $\mathbb{F} = \mathbb{R}$, there are no eigenvalues. If $\mathbb{F} = \mathbb{C}$, $\lambda = \pm i$. □

12 An operator $B \in \mathcal{L}(V)$ is called *skew* if

$$B^* = -B.$$

Suppose that $T \in \mathcal{L}(V)$. Prove that T is normal if and only if there exist commuting operators A and B such that A is self-adjoint, B is a skew operator, and $T = A + B$.
Solution:

First suppose T is normal. Let

$$A = \frac{T + T^*}{2} \quad \text{and} \quad B = \frac{T - T^*}{2}. \tag{7.1}$$

Then A is self-adjoint, B is a skew operator, and $T = A + B$. Commutator of A and B equals:

$$\begin{aligned} AB - BA &= \frac{(T + T^*)(T - T^*)}{2} - \frac{(T - T^*)(T + T^*)}{2} \\ &= \frac{T^2 - (T^*)^2 - TT^* + T^*T - T^2 + (T^*)^2 - TT^* + T^*T}{2} \\ &= T^*T - TT^*. \end{aligned} \tag{7.2}$$

Because T is normal, the right side of the equation above equals 0. Thus the operators A and B commute, as desired.

To prove the implication in the other direction, now suppose there exist commuting operators A and B such that A is self-adjoint, B is skew operator, and $T = A + B$. Then $T = AB$. Adding the last two equations and then dividing by 2 produces the equation for A in 7.1. Subtracting the last two equations and then dividing by 2 produces the equation for B in 7.1. Now 7.1 implies 7.2. Because A and B commute, 7.2 implies that T is normal, as desired. □

13 Suppose $\mathbb{F} = \mathbb{R}$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}T = T^*$ for all $T \in \mathcal{L}(V)$.

(a) Find all eigenvalues of $\mathcal{A}$.

(b) Find the minimal polynomial of $\mathcal{A}$.

Solution:
Using property 7.5 c of adjoint, we have:

$$\mathcal{A}^2 T = T \quad \Rightarrow (\mathcal{A}^2 - \mathcal{I})T = 0.$$

Hence the minimal polynomial of $\mathcal{A}$ is $p(z) = z^2 - 1$.

Eigenvalues of $\mathcal{A}$ are roots of the minimal polynomial: ± 1. □

14 Define an inner product on $\mathcal{P}_2(\mathbb{R})$ by $\langle p, q \rangle = \int_0^1 pq$. Define an operator $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by

$$T(ax^2 + bx + c) = bx.$$

(a) Show that with this inner product, the operator T is not self-adjoint.

(b) The matrix of T with respect to the basis $1, x, x^2$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution:

If T were self-adjoint, we would have an equality $\langle Tp, q \rangle = \langle p, Tq \rangle$ for any $p, q \in \mathcal{P}_2(\mathbb{R})$.

Let $p = a_1x^2 + b_1x + c_1$ and $q = a_2x^2 + b_2x + c_2$ for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Then we have:

$$\begin{aligned} \langle Tp, q \rangle &= \langle b_1x, a_2x^2 + b_2x + c_2 \rangle \\ &= b_1 \int_0^1 (a_2x^3 + b_2x^2 + c_2x)dx \\ &= b_1 \left(a_2\frac{x^4}{4} + b_2\frac{x^3}{3} + c_2\frac{x^2}{2} \right) \Big|_0^1 \\ &= \frac{b_1a_2}{4} + \frac{b_1b_2}{3} + \frac{b_1c_2}{2} \end{aligned}$$

$$\begin{aligned} \langle p, Tq \rangle &= \langle a_1x^2 + b_1x + c_1, b_2x \rangle \\ &= b_2 \int_0^1 (a_1x^3 + b_1x^2 + c_1x)dx \\ &= b_2 \left(a_1\frac{x^4}{4} + b_1\frac{x^3}{3} + c_1\frac{x^2}{2} \right) \Big|_0^1 \\ &= \frac{b_2a_1}{4} + \frac{b_1b_2}{3} + \frac{b_2c_1}{2} \end{aligned}$$

Therefore, $\langle Tp, q \rangle \neq \langle p, Tq \rangle$ for all p, q, thus T is not self-adjoint. □

(b) Basis $1, x, x^2$ is *not orthonormal*, while Theorem 7.9 states that matrix of T^* equals complex conjugate transpose of the matrix of T when evaluated in an *orthonormal basis*. Thus, there is no contradiction.

15 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

(a) T is self-adjoint $\iff T^{-1}$ is self-adjoint;

(b) T is normal $\iff T^{-1}$ is normal.

Solution:

(a) Suppose T is self-adjoint. Then by property 7.5 f, we have

$$(T^{-1})^* = (T^*)^{-1} = T^{-1},$$

thus, T^{-1} is self-adjoint.

Changing T to T^{-1} and using property $(T^{-1})^{-1} = T$ (see *Problem 3D.1*), we get the proof in other direction.

(b) Suppose T is normal. Then we have

$$\begin{aligned} T^{-1}(T^{-1})^* &= T^{-1}(T^*)^{-1} = (T^*T)^{-1} \\ &= (TT^*)^{-1} \\ &= (T^*)^{-1}T^{-1} \\ &= (T^{-1})^*T^{-1}, \end{aligned}$$

where we use property of inverse ($(TS)^{-1} = S^{-1}T^{-1}$, see *Problem 3D.2*), property of adjoint 7.5 f and normality of T. This shows that T^{-1} is also normal.

Changing T to T^{-1} and using $(T^{-1})^{-1} = T$, we get the proof in other direction. $\square$

16 Suppose $\mathbb{F} = \mathbb{R}$.

(a) Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

(b) What is the dimension of the subspace of $\mathcal{L}(V)$ in (a) [in terms of $\dim V$]?

Solution:

(a) We need to check three conditions of Theorem 1.34.

- 0 is a self-adjoint operator. ✓
- Suppose S and T are self-adjoint. Then $(S+T)^* = S^* + T^* = S + T$, hence self-adjoint operators are closed under addition. ✓
- Suppose T is self-adjoint and $\alpha \in \mathbb{R}$. Then $(\alpha T)^* = \overline{\alpha} T^* = \alpha T$, hence self-adjoint operators are closed under scalar multiplication. ✓

Thus, the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$. $\square$

(b) Let $e_1, \ldots, e_n$ be an orthonormal basis of V. In this basis, symmetric matrices represent self-adjoint operators on V. Every symmetric matrix can be constructed from a matrix with either only one non-zero entry on the diagonal or two equal non-zero entries ($A_{j,k}$ and $A_{k,j}$). Therefore, dimension of the subspace of self-adjoint operators equals:

$$n + \frac{n^2 - n}{2} = \frac{n^2 + n}{2} = \dim V(\dim V + 1)/2$$

17 Suppose $\mathbb{F} = \mathbb{C}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Solution:

On the complex vector spaces, the set of self-adjoint operators is not closed under scalar multiplication:

$$(\alpha T)^* = \overline{\alpha}\, T^* = \overline{\alpha}\, T.$$

If α has an imaginary part, $\overline{\alpha} \neq \alpha$, hence $(\alpha T)^* \neq \alpha T$. $\square$

18 Suppose $\dim V \geq 2$. Show that the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

Solution:

Suppose $S, T \in \mathcal{L}(V)$ are normal operators that do not commute and $ST^* - T^*S$ has a real component. Then their sum is not a normal operator:

$$\begin{aligned}(S+T)(S+T)^* - (S+T)^*(S+T) &= SS^* + TT^* + ST^* + TS^* \\ &\quad - S^*S - T^*T - S^*T - T^*S \\ &= (ST^* - T^*S) + (TS^* - S^*T) \\ &= (ST^* - T^*S) + (ST^* - T^*S)^* \\ &= 2\Re(ST^* - T^*S) \\ &\neq 0\end{aligned}$$

19 Suppose $T \in \mathcal{L}(V)$ and $\|T^*v\| \leq \|Tv\|$ for every $v \in V$. Prove that T is normal.

Solution:

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V. Then (by *Problem 7A.5*) we have:

$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \|T^*e_1\|^2 + \cdots + \|T^*e_n\|^2. \tag{7.3}$$

This, together with $\|T^*v\| \leq \|Tv\|$, implies that $\|Te_j\|^2 = \|T^*e_j\|^2$ for every e_j in the basis. Indeed, we can rearrange terms in 7.3 as:

$$\|Te_1\|^2 - \|T^*e_1\|^2 = (\|T^*e_2\|^2 - \|Te_2\|^2) + \cdots + (\|T^*e_n\|^2 - \|Te_n\|^2). \tag{7.4}$$

The left-hand side of 7.4 is greater than or equal to zero, meanwhile the right-hand side is less than or equal to zero (as every term on the right side is less than or equal to zero). Therefore, $\|Te_1\| = \|T^*e_1\|$. Similarly, this equality can be shown for any other e_j.

Let $v = \alpha e_1$, then

$$\|Tv\| = \|T(\alpha e_1)\| = |\alpha|\|Te_1\| = |\alpha|\|T^*e_1\| = \|T^*(\alpha e_1)\| = \|T^*v\|.$$

Since the orthonormal basis is arbitrary, we can construct one starting from any arbitrary $v \in V$ using the Gram-Schmidt procedure (6.32). This implies $\|Tv\| = \|T^*v\|$ for every $v \in V$, hence T is normal (Theorem 7.20). $\square$

20 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the following are equivalent.

(a) P is self-adjoint.

(b) P is normal.

(c) There is a subspace U of V such that $P = P_U$.

Solution:

First suppose P is self-adjoint. Then it automatically means P is normal.

Now suppose P is normal. Then, $\operatorname{range} P = \operatorname{range} P^*$ (Theorem 7.21) and $\operatorname{null} P = (\operatorname{range} P^*)^\perp$ (Theorem 7.6), which implies $\operatorname{null} P = (\operatorname{range} P)^\perp$. Thus, we have that every vector in the null space of P is orthogonal to every vector in range of P and $P = P^2$, which implies (by *Problem 6C.9*) that there exists a subspace U of V such that $P = P_U$.

Finally, suppose P is an orthogonal projection on some subspace U of V. Let $u_1, u_2 \in U$ and $w_1, w_2 \in U^\perp$. We have

$$\langle P(u_1 + w_1), u_2 + w_2\rangle = \langle u_1, u_2 + w_2\rangle = \langle u_1, u_2\rangle$$
$$\langle P(u_1 + w_1), u_2 + w_2\rangle = \langle u_1 + w_1, P^*(u_2 + w_2)\rangle.$$

Two equations above imply that $P^*(u_2 + w_2) = u_2$ for any $u_2 \in U$ and $w_2 \in U^\perp$. This coincides with the definition of orthogonal projection on U, therefore $P = P^*$, hence P is self-adjoint.

Thus, we have shown that (a) implies (b), (b) implies (c) and (c) implies (a), hence these three statements are equivalent. $\square$

21 Suppose $D : \mathcal{P}_8(\mathbb{R}) \to \mathcal{P}_8(\mathbb{R})$ is the differentiation operator defined by $Dp = p'$. Prove that there does not exist an inner product on $\mathcal{P}_8(\mathbb{R})$ that makes D a normal operator.

Solution:

Suppose there is an inner product such that D is a normal operator.

By Theorem 7.21, $\operatorname{range} D^* = \operatorname{range} D$.

Now note that for any $p \in \mathcal{P}_8(\mathbb{R})$:

$$\langle Da_0, p\rangle = \langle 0, p\rangle = 0$$
$$\langle Da_0, p\rangle = \langle a_0, D^*p\rangle$$

where a_0 is a constant polynomial. This equation implies that $D^*p \in (\operatorname{span}(1))^{\perp}$. Thus, $\operatorname{range} D \subset (\operatorname{span}(1))^{\perp}$.

At the same time, $Dx = 1 \in \operatorname{range} D$ and $1 \notin (\operatorname{span}(1))^{\perp}$. Hence, our assumption leads to a contradiction, and there is no inner product such that D is a normal operator. $\square$

Comment: This problem can be extended to any finite-dimensional polynomial vector space with dimension greater than 1, as only this fact is used in the proof.

22 Give an example of an operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that T is normal but not self-adjoint.

Solution:

Let T be an operator on $\mathbb{R}^3$, with matrix in standard basis:

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then, the matrix of adjoint operator T^* is:

$$\mathcal{M}(T^*) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

T is clearly not self-adjoint. Yet it is normal, as can be checked by matrix multiplication:

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

23 Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad Tv = 3v, \quad Tw = 4w.$$

Show that $\|T(v+w)\| = 10$.

Solution:

Here we use Theorem 7.22. Vectors v and w are eigenvectors of T, corresponding to distinct eigenvalues, hence they are orthogonal. Then we use Pythagorean theorem (6.12) to compute the norm directly:

$$\begin{aligned}\|T(v+w)\| = \|3v + 4w\| &= \sqrt{\|3v\|^2 + \|4w\|^2} \\ &= \sqrt{9\|v\|^2 + 16\|w\|^2} \\ &= \sqrt{9 \cdot 4 + 16 \cdot 4} \\ &= 10. \quad \square\end{aligned}$$

24 Suppose $T \in \mathcal{L}(V)$ and

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + z^m$$

is the minimal polynomial of T. Prove that the minimal polynomial of T^* is

$$\overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + \cdots + \overline{a_{m-1}} z^{m-1} + z^m$$

Solution:

First, note that every $v, u \in V$:

$$\langle p(T)v, u \rangle = 0 = \langle v, (p(T))^* u \rangle \tag{7.5}$$

hence $(p(T))^* = 0$. Expanding adjoint of $p(T)$ we get:

$$\begin{aligned}(p(T))^* &= (a_0 I)^* + (a_1 T)^* + (a_2 T^2)^* + \cdots + (a_m T^{m-1})^* + (T^m)^* \\ &= \overline{a_0} I + \overline{a_1} T^* + \overline{a_2} (T^*)^2 + \cdots + \overline{a_m} (T^*)^{m-1} + (T^*)^m.\end{aligned}$$

Now suppose that there is a polynomial $q(z) \neq (\overline{a_0} + \overline{a_1} z + \cdots + z^m)$ such that $q(T^*) = 0$ and $\deg q \leq \deg p$. Reversing 7.5 with $q(z)$ in place of $p(z)$ we conclude that $\overline{q(T)} = 0$ (that is, $q(T)$ with all coefficients turned into their complex conjugate). That would imply that $p(z)$ is not a minimal polynomial of T, being either of not the least degree or not unique. Hence, we must conclude that

$$\overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + \cdots + \overline{a_{m-1}} z^{m-1} + z^m$$

is a minimal polynomial of T^*. $\square$

25 Suppose $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if T^* is diagonalizable.

Solution:

By Theorem 5.62, T is diagonalizable if and only if the minimal polynomial of T equals $(z-\lambda_1)\dots(z-\lambda_m)$ for some list of distinct $\lambda_1,\dots,\lambda_m \in \mathbb{F}$. Following argument of the previous problem, we see that the minimal polynomial of T^* is $q(z)$ such that $q(T^*) = (p(T))^*$. Thus, we have:

$$\begin{aligned} q(T^*) &= [(T-\lambda_1 I)\dots(T-\lambda_m I)]^* = \\ &= (T-\lambda_m)^* \dots (T-\lambda_1 I)^* \\ &= (T^* - \overline{\lambda_m} I)\dots(T^* - \overline{\lambda_1}). \end{aligned}$$

As $\lambda_1,\dots,\lambda_m$ are distinct, so are $\overline{\lambda_m},\dots,\overline{\lambda_1}$. Thus, the minimal polynomial of T^* has the desired form of a product of distinct $(z-\alpha_i)$ terms, which implies that T^* is diagonalizable.

Reversing proof with T^* in place of T, gives implication in other direction. □

26 Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u\rangle x$ for every $v \in V$.

(a) Prove that if V is a real vector space, then T is self-adjoint if and only if the list u, x is linearly dependent.

(b) Prove that T is normal if and only if the list u, x is linearly dependent.

Solution:

(a) First, suppose that T is self-adjoint. Let v, w be arbitrary vectors in V. Then the inner product of Tv and w is:

$$\langle Tv, w\rangle = \langle \langle v, u\rangle x, w\rangle = \langle v, u\rangle\langle x, w\rangle,$$

and also:

$$\langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, \langle w, u\rangle x\rangle = \langle v, x\rangle\langle w, u\rangle.$$

Thus we have:

$$\begin{aligned} \langle v, u\rangle\langle x, w\rangle - \langle v, x\rangle\langle w, u\rangle = 0 \\ \langle v, \langle x, w\rangle u - \langle w, u\rangle x\rangle = 0 \end{aligned}$$

for every $v, w \in V$. This implies that $\langle x, w\rangle u - \langle w, u\rangle x = 0$, hence u, x is a linearly dependent list.

Now to proof the other direction, suppose u, x is a linearly dependent list. Then $u = \lambda x$, where $\lambda \in \mathbb{R}$. Let $v, w \in V$, then we have:

$$\begin{aligned}
\langle Tv, w\rangle &= \langle\langle v, u\rangle x, w\rangle = \langle\langle v, \lambda x\rangle x, w\rangle \\
&= \lambda\langle v, x\rangle\langle x, w\rangle = \lambda\langle v, x\rangle\langle w, x\rangle \\
&= \langle v, \langle w, \lambda x\rangle x\rangle \\
&= \langle v, \langle w, u\rangle x\rangle \\
&= \langle v, T^*w\rangle.
\end{aligned}$$

This implies that $T^* = T$, i.e. T is self-adjoint. □

(b) Before the proof itself, we must explicitly find T^* for this case. Following the previous part, we have for $v, w \in V$:

$$\langle Tv, w\rangle = \langle v, u\rangle\langle x, w\rangle = \langle v, \langle w, x\rangle u\rangle.$$

Thus, $T^*v = \langle v, x\rangle u$.

Now, we suppose that T is normal. Then

$$TT^*v = T(\langle v, x\rangle u) = \langle\langle v, x\rangle u, u\rangle x = \langle v, x\rangle\|u\|^2 x,$$

and

$$T^*Tv = T^*(\langle v, u\rangle x) = \langle\langle v, u\rangle x, x\rangle u = \langle v, u\rangle\|x\|^2 u.$$

For a normal operator we have $TT^* - T^*T = 0$, hence

$$\langle v, x\rangle\|u\|^2 x = \langle v, u\rangle\|x\|^2 u$$

for every $v \in V$. Thus, the list u, x is linearly dependent.

For a proof in other direction, suppose that $u = \lambda x$, where $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}
\|Tv\| &= \|\langle v, u\rangle x\| = \|\langle v, \lambda x\rangle x\| \\
&= \|\overline{\lambda}\langle v, x\rangle x\| = |\lambda| \cdot \|\langle v, x\rangle x\| \\
&= \|\lambda\langle v, x\rangle x\| = \|\langle v, x\rangle u\| \\
&= \|T^*v\|.
\end{aligned}$$

Thus, by Theorem 7.20, T is normal, completing the proof. □

27 Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

Solution:

Firstly, for $k = 1$, the theorem is obviously true, so we will assume $k \geq 2$ in the rest of the proof.

That $\operatorname{null} T \subseteq \operatorname{null} T^k$ and $\operatorname{range} T \subseteq \operatorname{range} T^k$ (for any operator), is true, as can be easily seen. We will prove the other direction of inclusion.

First, for a self-adjoint operator S (here it will be T^*T), suppose that $v \in \operatorname{null} S^k$. Then we have:

$$0 = \langle S^k v, S^{k-2} v\rangle = \langle S^{k-1} v, S^{k-1} v\rangle.$$

Thus, $\|S^{k-1}v\| = 0$, which implies $S^{k-1}v = 0$, therefore $\operatorname{null} S^k \subseteq \operatorname{null} S^{k-1}$. Repeating the induction on k until $k - 1 = 1$, we have that for every positive integer k, $\operatorname{null} S^k \subseteq \operatorname{null} S$. Hence, $\operatorname{null} S^k = \operatorname{null} S$.

Now we examine a normal operator T. Suppose $v \in \operatorname{null} T^k$ for some positive integer k. Then.

$$T^k v = 0 \Rightarrow (T^*)^k T^k v = 0 \Rightarrow (T^*T)^k v = 0,$$

where the second implication is valid because T and T^* commute. Thus, $v \in \operatorname{null} (T^*T)^k$, which implies $v \in \operatorname{null} T^*T$. Hence

$$0 = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \Longleftrightarrow Tv = 0 \Longleftrightarrow v \in \operatorname{null} T.$$

Thus, we have shown $\operatorname{null} T^k = \operatorname{null} T$ for every positive integer k.

Finally, using that T^k is also a normal operator, we see that

$$\operatorname{range} T^k = (\operatorname{null} (T^k)^*)^\perp = (\operatorname{null} T^k)^\perp = (\operatorname{null} T)^\perp = \operatorname{range} T^* = \operatorname{range} T,$$

completing the proof. $\square$

28 Suppose $T \in \mathcal{L}(V)$ is normal. Prove that if $\lambda \in \mathbb{F}$, then the minimal polynomial of T is not a polynomial multiple of $(x - \lambda)^2$.

Solution:

Let $p(z)$ be a minimal polynomial of T and suppose that it is a polynomial multiple of $(z - \lambda)^2$:

$$p(z) = (z - \lambda)^2 q(z)$$

for some polynomial $q(z)$.

Then we have for every $v \in V$:

$$(T - \lambda I)^2 q(T)v = 0 \Rightarrow q(T)v \in \operatorname{null} (T - \lambda I)^2.$$

By property of normal operator 7.21 (d), $(T - \lambda I)$ is a normal operator. Result of the previous problem thus implies that $q(T)v \in \operatorname{null}(T - \lambda I)$. Thus for every $v \in V$:

$$(T - \lambda I)q(T)v = 0.$$

But this polynomial has a degree less than $p(z)$, contradicting the fact that $p(z)$ is a minimal polynomial of T. Hence, $p(z)$ cannot be a polynomial multiple of $(z - \lambda)^2$ for any $\lambda \in \mathbb{F}$. $\square$

29 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_k\| = \|T^*e_k\|$ for each $k = 1, \ldots, n$, then T is normal.

Solution: Let $\mathbb{F} = \mathbb{R}$ and take the operator T and its adjoint, defined by matrices, with respect to the standard basis:

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad \mathcal{M}(T^*) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

As can be checked with matrix multiplication, these operators do not commute:

$$\mathcal{M}(T) \cdot \mathcal{M}(T^*) = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 2 \\ 3 & 2 & 10 \end{pmatrix}$$
$$\mathcal{M}(T^*) \cdot \mathcal{M}(T) = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 10 \end{pmatrix}.$$

Hence, T is not a normal operator. Meanwhile, for vectors of the basis we have:

$$Te_1 = e_1 + e_2, \quad T^*e_1 = e_1 + e_3,$$
$$Te_2 = 2e_2 + e_3, \quad T^*e_2 = e_1 + 2e_2,$$
$$Te_3 = e_1 + 3e_3, \quad T^*e_3 = e_2 + 3e_3.$$

So we have $\|Te_k\| = \|T^*e_k\|$ for every $k = 1, 2, 3$, but T is not normal, counterproving the statement of the problem. $\square$

30 Suppose that $T \in \mathcal{L}(\mathbb{F}^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$.

Solution:

Vector $(1,1,1)$ is an eigenvector of T with eigenvalue 2; (z_1, z_2, z_3) is an eigenvector of T with eigenvalue 0. By Theorem 7.22, these two vectors are orthogonal. Hence

$$0 = \langle (z_1, z_2, z_3), (1,1,1) \rangle = z_1 + z_2 + z_3,$$

as desired. □

31 Fix a positive integer n. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} fg$, let

$$V = \operatorname{span}(1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx).$$

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that D is normal but not self-adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that T is self-adjoint.

Solution:

(a) Earlier (in *Problem 6B.4*) we have shown that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list. Hence, this list is an orthonormal basis of V. Operator D acts on the basis vectors as follows

$$\begin{aligned} D\left(\frac{1}{\sqrt{2\pi}}\right) &= 0 \\ D\left(\frac{\cos kx}{\sqrt{\pi}}\right) &= -k\frac{\sin kx}{\sqrt{\pi}} \\ D\left(\frac{\sin kx}{\sqrt{\pi}}\right) &= k\frac{\cos kx}{\sqrt{\pi}}. \end{aligned}$$

Thus, in this basis, the matrix of D is

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & n \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Matrix of D^* is a transpose of this matrix. We see that $(\mathcal{M}(D))^t = -\mathcal{M}(D)$, hence $D^* = -D$.

Clearly, D is not self-adjoint, but it is indeed normal:

$$DD^* = D(-D) = -D^2 = (-D)D = D^*D. \quad \square$$

(b) Working in the same basis, we have:

$$\begin{aligned} T(\frac{1}{\sqrt{2\pi}}) &= 0 \\ T(\frac{\cos kx}{\sqrt{\pi}}) &= -k^2 \frac{\cos kx}{\sqrt{\pi}} \\ T(\frac{\sin kx}{\sqrt{\pi}}) &= -k^2 \frac{\sin kx}{\sqrt{\pi}}. \end{aligned}$$

Thus, the matrix of T is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -n^2 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -4 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n^2 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This matrix is symmetric and hence T is a self-adjoint operator. $\square$

32 Suppose $T : V \to W$ is a linear map. Show that under the standard identification of V with V' and the corresponding identification of W with W', the adjoint map $T^* : W \to V$ corresponds to the dual map $T' : W' \to V'$. More precisely, show that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all $w \in W$, where φ_w and φ_{T^*w} are defined as in 6.58.

Solution:

Following Riesz representation theorem, we define $\varphi_v(u)$ as

$$\varphi_v(u) = \langle u, v \rangle,$$

where v, u are either in V, or in W, and we use the inner product defined on the corresponding vector space.

Let $v \in V$, $w \in W$. Then, using definition of dual map and adjoint, we have:

$$\begin{aligned}
(T'(\varphi_w))(v) &= (\varphi_w \circ T)v = \varphi_w(Tv) \\
&= \langle Tv, w \rangle = \langle v, T^*w \rangle \\
&= \varphi_{T^*w}(v),
\end{aligned}$$

as desired. □

7B Spectral Theorem

1 Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

Solution:

Let $T \in \mathcal{L}(V)$ be a normal operator. This means T is diagonalizable in some orthonormal basis. Denote entries on the diagonal of T as $a_j + ib_j$; then entries on the diagonal of T^* are $a_j - ib_j$ (see Theorem 7.9).

Then note that T is self-adjoint if and only if $T = T^*$. This, in turn is equivalent to:

$$a_j + ib_j = a_j - ib_j \iff b_j = 0,$$

for all $j = 1, \ldots, \dim V$, thus, eigenvalues of T are purely real. □

2 Suppose $\mathbb{F} = \mathbb{C}$. Suppose $T \in \mathcal{L}(V)$ is normal and has only one eigenvalue. Prove that T is a scalar multiple of the identity operator.

Solution:

T is normal, hence by Spectral Theorem it is diagonalizable. By Theorem 5.41, entries on the diagonal of T are precisely the eigenvalues of T. T has only one eigenvalue, hence all diagonal entries are equal (say, $\alpha \in \mathbb{C}$). Thus, matrix of T is a scalar multiple of the matrix of the identity operator, and hence, by linear map lemma (3.4), T is a scalar multiple of I. $\square$

3 Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal. Prove the set of eigenvalues of T is contained in $\{0, 1\}$ if and only if there is a subspace U of V such that $T = P_U$.

Solution:

First suppose that there is a subspace U of V such that $T = P_U$. Let $u_1, \ldots, u_m$ be an orthonormal basis of U and $w_1, \ldots, w_n$ be an orthonormal basis of $U^\perp$. The combined list $u_1, \ldots, u_m, w_1, \ldots, w_n$ gives an orthonormal basis of V, because V is a direct sum of U and $U^\perp$ (Theorem 6.49) and every vector in U is orthogonal to every vector in $U^\perp$ by definition.

Note that $Tu_i = P_U u_i = u_i$ and $Tw_j = P_U w_j = 0$. Hence, u_i's are eigenvectors of T with eigenvalue 1, and w_j's are eigenvectors of T with eigenvalue 0. These are the maximum number of (linearly independent) eigenvectors, hence the set of all eigenvalues of T is $\{0, 1\}$.

Now suppose that the set of eigenvalues of T is contained in $\{0, 1\}$. T is normal, hence it is diagonalizable in some orthonormal basis, where basis vectors are eigenvectors of T. Let u_i's be vectors of the basis, corresponding to the eigenvalue 1, and w_i's be vectors of the basis, corresponding to the eigenvalue 0. Let $U = \operatorname{span}(u_1, \ldots, u_m)$ and $W = \operatorname{span}(w_1, \ldots, w_m)$. Note that $V = U \oplus W$, so that any $v \in V$ can be represented as $v = u + w$. Thus, we have:

$$Tv = T(u + w) = u.$$

Hence, T is the orthogonal projection on U, by definition, which completes the proof in other direction. $\square$

4 Prove that a normal operator on a complex inner product space is skew (meaning it equals the negative of its adjoint) if and only if all its eigenvalues are purely imaginary (meaning that they have real part equal to 0).

Solution:

Like in *Problem 7B.1*, we have entries on the diagonal of T equal $a_j + ib_j$ and entries on the diagonal of T^* equal $a_j - ib_j$.

T is skew if and only if $T = -T^*$. This is equivalent to:

$$a_j + ib_j = -a_j + ib_j \iff a_j = 0,$$

for all $j = 1, \ldots, \dim V$, thus, eigenvalues of T are purely imaginary. $\square$

5 Prove or give a counterexample: If $T \in \mathcal{L}(\mathbb{C}^3)$ is a diagonalizable operator, then T is normal (with respect to the usual inner product).

Solution:

Take basis $(1,0,0), (1,1,0), (1,1,1)$; denote these vectors v_1, v_2, v_3, accordingly. Let T be a diagonalizable operator on this basis, acting on the basis as follows:

$$Tv_1 = 2v_1, \quad Tv_2 = 3v_2, \quad Tv_3 = v_3.$$

A vector $v = (x, y, z) \in \mathbb{C}^3$ can be represented as a linear combination of basis vectors as follows:

$$v = (x-y)v_1 + (y-z)v_2 + zv_3.$$

Hence, T acting on v is

$$Tv = 2(x-y)v_1 + 3(y-z)v_2 + zv_3 = (2x + y - 2z, 3y - 2z, z).$$

Now we find the adjoint T^*. Let $u = (a, b, c)$:

$$\begin{aligned}
\langle Tv, u\rangle &= \langle (2x + y - 2z, 3y - 2z, z), (a, b, c)\rangle \\
&= 2ax + ay - 2az + 3by - 2bz + cz \\
&= 2ax + (a + 3b)y + (c - 2a - 2b)z \\
&= \langle (x, y, z), (2a, a + 3b, -2a - 2b + c)\rangle \\
&= \langle v, T^*u\rangle
\end{aligned}$$

Therefore,

$$T^*(x, y, z) = (2x, x + 3b, -2x - 2y + z).$$

Now we calculate norms of Tv and T^*v.

$$\begin{aligned}
\|Tv\| &= \sqrt{4x^2 + 4xy - 8xz + 10y^2 - 16yz + 9z^2}, \\
\|T^*v\| &= \sqrt{9x^2 + 14xy - 4xz + 13y^2 - 4yz + z^2},
\end{aligned}$$

hence $\|Tv\|$ does not equal $\|T^*v\|$ for every $v \in V$, thus T is not normal (Theorem 7.20). $\square$

6 Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Solution: Rearranging terms in $T^9 = T^8$ we get:

$$T^8(T - I) = 0.$$

By Theorem 5.29, $p(z) = z^8(z-1)$ is a polynomial multiple of the minimal polynomial of T. This implies that eigenvalues of T are contained in set $\{0, 1\}$. Using *Problem 7B.3*, we have that there exists subspace $U \subseteq V$ such that $T = P_U$.

By properties of orthogonal projection (6.57), $P_U^2 = P_U$, and by *Problem 7A.20*, P_U is self-adjoint, which completes the proof. $\square$

7 Give an example of an operator T on a complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

Solution:

Let $T \in \mathcal{L}(\mathbb{C}^3)$ be an operator with the following matrix with respect to the standard basis:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

As can be checked with matrix multiplication, matrix of T^2 is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

hence, $T \neq T^2$. But we also have $T^8 = T^9 = 0$. $\square$

8 Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if every eigenvector of T is also an eigenvector of T^*.

Solution:

First, suppose T is normal. Then by Spectral Theorem, V has an orthonormal basis consisting of eigenvectors of T. Matrix of T is diagonal with respect to this basis. By Theorem 7.9, matrix of T^* with respect to the same basis is a conjugate transpose of the matrix of T, hence T^* is diagonalizable with respect to the chosen basis. By Theorem 5.55, those basis vectors are eigenvectors of T^*. Hence every eigenvector of T is an eigenvector of T^*; other linearly independent eigenvectors cannot exist, as every list greater than basis is linearly dependent.

Now suppose that every eigenvector of T is an eigenvector of T^*. Let $e_1, \ldots, e_n$ be an orthonormal basis of V with respect to which T has an upper-triangular matrix (see Theorems 5.47 and 6.37). Matrix of T^* with respect to the same basis is lower triangular.

The vector e_1 is an eigenvector of T, hence it is also an eigenvector of T^*, which means that every entry in the first column of the matrix of T^*

except the first equals zero. This implies that every entry in the first row of the matrix of T except first equals zero. This, in turn, means e_2 is now an eigenvector of T. Continuing this for all rows of the matrix of T, we get that T and T^* are actually diagonalizable with respect to the chosen basis, hence they commute (Theorem 5.76), meaning T is normal. $\square$

9 Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if there exists a polynomial $p \in \mathcal{P}(\mathbb{C})$ such that $T^* = p(T)$.

Solution:

First suppose that there exists a polynomial $p \in \mathcal{P}(\mathbb{C})$ such that $T^* = p(T)$. Then T commutes with T^*:

$$TT^* = Tp(T) = p(T)T = T^*T,$$

hence T is normal.

Now suppose that T is normal. With help of Spectral Theorem, we choose an orthonormal basis of V consisting of eigenvectors of T: $v_1, \ldots, v_m$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. The adjoint T^* is defined by its effect on this basis:

$$T^* v_i = \overline{\lambda_i} v_i, \quad \text{see } \textit{Problem 7A.3}.$$

Let a polynomial $p(z) \in \mathcal{P}(\mathbb{C})$ be such that:

$$p(\lambda_i) = \overline{\lambda_i},$$

for every eigenvalue λ_i of T (such polynomial can be constructed, for example, by Lagrange polynomial method). Then for every v_i of the basis we have:

$$\begin{aligned} p(T)v_i &= (a_0 I + a_1 T + \cdots + a_n T^n)v_i \\ &= (a_0 + \lambda_i + \ldots \lambda_i^n)v_i \\ &= p(\lambda_i)v_i \\ &= \overline{\lambda_i} v_i. \end{aligned}$$

This equation implies that $T^* = p(T)$, as desired. $\square$

10 Suppose V is a complex inner product space. Prove that every normal operator on V has a square root.

Solution:

Every normal operator on a complex inner product space is diagonalizable (Spectral Theorem). Hence, if $e_1, \ldots, e_n$ is a basis of V, with respect to which normal $T \in \mathcal{L}(V)$ is diagonal, we have:

$$Te_i = \alpha_i e_i,$$

where $\alpha_i \in \mathbb{C}$. Define an operator $S \in \mathcal{L}(V)$ as follows:

$$Se_i = \sqrt{\alpha_i} e_i.$$

The square root is well-defined for every complex number, thus S exists indeed. Now it is easy to verify that $S^2 = T$, either with matrix multiplication, or:

$$S^2 e_i = S(Se_i) = S(\sqrt{\alpha_i} e_i) = \sqrt{\alpha_i} \cdot \sqrt{\alpha_i} e_i = \alpha_i e_i.$$

Thus, every normal operator on a complex inner product space has a square root. □

11 Prove that every self-adjoint operator on V has a cube root.

Solution:

Every self-adjoint operator on a real inner product space is diagonalizable (Spectral Theorem). Hence, if $e_1, \ldots, e_n$ is a basis of V, with respect to which normal $T \in \mathcal{L}(V)$ is diagonal, we have:

$$Te_i = \alpha_i e_i,$$

where $\alpha_i \in \mathbb{C}$. Define an operator $S \in \mathcal{L}(V)$ as follows:

$$Se_i = \alpha_i^{1/3} e_i.$$

The cube root is well-defined for every real number, thus S exists indeed. Now it is easy to verify that $S^3 = T$, either with matrix multiplication, or:

$$S^3 e_i = S^2(Se_i) = \alpha_i^{1/3} S(Se_i) = \alpha_i^{2/3} Se_i = \alpha_i e_i.$$

Thus, every self-adjoint operator on a real inner product space has a cube root. □

12 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is normal. Prove that if S is an operator on V that commutes with T, then S commutes with T^*.

Solution:

Suppose T is normal and S commutes with T. From *Problem 7B.9* we know that there exists a polynomial $p(T)$ such that $T^* = p(T)$. Now we have:

$$ST^* = Sp(T) = p(T)S = T^*S,$$

where the second equation holds because S commutes with T. □

13 Without using the complex spectral theorem, use the version of Schur's theorem that applies to two commuting operators (take $\mathcal{E} = \{T, T^*\}$ in *Problem 6B.20*) to give a different proof that if $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal, then T has a diagonal matrix with respect to some orthonormal basis of V.

Solution:

Let V be a complex inner product space and $T \in \mathcal{L}(V)$ be a normal operator. Taking a subspace $\mathcal{E} = \{T, T^*\}$ of $\mathcal{L}(V)$ of commuting operators, we have that by extension of Schur's theorem (*Problem 6B.20*), there exists an orthonormal basis of V, with respect to which both T and T^* have an upper-triangular matrix.

Matrix of T^* (with respect to the chosen basis) is a conjugate transpose of the matrix of T, hence $\mathcal{M}(T^*)$ is lower-triangular. As $\mathcal{M}(T^*)$ is simultaneously upper-triangular and lower-triangular, it must be diagonal. Hence, matrix of T is also diagonal, as desired. □

14 Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

Solution:

We have:

$$T \text{ is self-adjoint } \iff T \text{ is diagonalizable } \iff V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where the first equivalence is due to Real Spectral Theorem (7.29). The second equivalence is due to Theorem 5.55, where we also use the property that all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal. When coming from T being self-adjoint it follows naturally as a property of normal, here specifically self-adjoint, operators (Theorem 7.22). When going in the other direction, this property ensures that the basis can be chosen orthonormal (in order to use Spectral Theorem), as vectors from different eigenspaces are orthogonal and vectors within an eigenspace can be orthogonalized via Gram-Schmidt procedure. □

15 Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

Solution:

We have:

$$T \text{ is normal } \iff T \text{ is diagonalizable } \iff V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where the first equivalence is due to Complex Spectral Theorem (7.31). The second equivalence is due to Theorem 5.55, where we also use the property that all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal. When coming from T being normal adjoint it follows naturally as a property of normal operators (Theorem 7.22). When going in the other direction, this property ensures that the basis can be chosen orthonormal (in order to use Spectral Theorem), as vectors from different eigenspaces are orthogonal and vectors within an eigenspace can be orthogonalized via Gram-Schmidt procedure. □

16 Suppose $\mathbb{F} = \mathbb{C}$ and $\mathcal{E} \subseteq \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if S and T are commuting normal operators for all $S, T \in \mathcal{E}$.

Solution:

If there exists an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix, then the conclusion follows directly, as any diagonalizable operators commute, and because they are diagonalizable with respect to the orthonormal basis, these operators are normal (Spectral Theorem).

Now suppose that for any $S, T \in \mathcal{E}$, S and T are commuting normal operators. *Problem 6B.20* implies that there exists an orthonormal basis of V with respect which every element of V has an upper-triangular matrix.

Suppose $T \in \mathcal{E}$. By the conclusion above, it has an upper-triangular matrix with respect to the chosen orthonormal basis. We will show that this matrix is actually a diagonal matrix.

As in the proof of the complex spectral theorem, we see that

$$\begin{aligned} \|Te_1\|^2 &= |a_{1,1}|^2, \\ \|T^*e_1\|^2 &= |a_{1,1}|^2 + |a_{1,2}|^2 + \cdots + |a_{1,n}|^2. \end{aligned}$$

Because T is normal, $\|Te_1\| = \|T^*e_1\|$. Thus the two equations above imply that all entries in the first row of the matrix of T, except possibly the first entry $a_{1,1}$, equal zero. This, in turn, implies that

$$\begin{aligned} \|Te_2\|^2 &= |a_{2,2}|^2, \quad \text{because we showed that } a_{1,2} = 0, \\ \|T^*e_2\|^2 &= |a_{2,2}|^2 + |a_{2,3}|^2 + \cdots + |a_{2,n}|^2. \end{aligned}$$

Because T is normal, $\|Te_2\| = \|T^* e_2\|$. Thus, the two equations above imply that all entries in the second row of the matrix of T, except possibly the first entry $a_{2,2}$, equal zero.

Continuing in this fashion, we see that all non-diagonal entries in the matrix of T equal 0. As it holds for any operator in $\mathcal{E}$, we actually showed that the chosen basis is indeed the orthonormal basis with respect to which every element of $\mathcal{E}$ has a diagonal matrix. $\square$

17 Suppose $\mathbb{F} = \mathbb{R}$ and $\mathcal{E} \subseteq \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if S and T are commuting self-adjoint operators for all $S, T \in \mathcal{E}$.

Solution:

If there exists an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix, then the conclusion follows directly, as any diagonalizable operators commute, and because they are diagonalizable with respect to the orthonormal basis, these operators are self-adjoint (Spectral Theorem).

Now suppose that for any $S, T \in \mathcal{E}$, S and T are commuting self-adjoint operators. *Problem 6B.20* implies that there exists an orthonormal basis of V with respect which every element of V has an upper-triangular matrix. Let A be a matrix of an operator T with respect to the chosen basis. By Theorem 7.9 (for $\mathbb{F} = \mathbb{C}$), $A^t = A$. We also have $A_{i,j} = A_{j,i}$. This implies that all non-diagonal entries of A equal zero. Hence, all elements of $\mathcal{E}$ have a diagonal matrix with respect to the chosen orthonormal basis. $\square$

18 Give an example of a real inner product space V, an operator $T \in \mathcal{L}(V)$, and real numbers b, c with $b^2 < 4c$ such that

$$T^2 + bT + cI$$

is not invertible.

Solution:

Let $V = \mathbb{R}^2$ and take T such that with respect to the standard basis its matrix is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now take $b = 0$ and $c = 1$ ($b^2 < 4c$). We have

$$\begin{aligned} \mathcal{M}(T^2 + bT + cI) &= \mathcal{M}(T^2 + I) = \mathcal{M}(T^2) + \mathcal{M}(I) \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, $T^2 + I = 0$, which is not invertible, as desired. $\square$

19 Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T.

(a) Prove that $U^\perp$ is invariant under T.

(b) Prove that $T|_U \in \mathcal{L}(U)$ is self-adjoint.

(c) Prove that $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

Solution:

(a) By the result of *Problem 7A.4*, $U^\perp$ is invariant under T^*. Here T is self-adjoint, i.e. $T^* = T$, which implies $U^\perp$ is invariant under T.

(b) Let $u_1, u_2 \in U$. First treating u_1, u_2 as vectors of V with the inner product defined on V, we have

$$\langle Tu_1, u_2 \rangle = \langle u_1, Tu_2 \rangle,$$

as T is self-adjoint.

Because U is invariant under T, we can treat T in the equation above as $T|_U$ and have inner product on U. Thus, we have

$$\langle T|_U u_1, u_2 \rangle = \langle u_1, T|_U u_2 \rangle.$$

Thus, $T|_U$ is self-adjoint.

(c) Substituting $U^\perp$ for U in the proof above (valid because of the part (a)), we have a proof that $T|_{U^\perp}$ is self-adjoint. $\square$

20 Suppose $T \in \mathcal{L}(V)$ is normal and U is a subspace of V that is invariant under T.

(a) Prove that $U^\perp$ is invariant under T.

(b) Prove that U is invariant under T^*.

(c) Prove that $(T|_U)^* = (T^*)|_U$.

(d) Prove that $T|_U \in \mathcal{L}(U)$ and $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ are normal operators.

Solution:

Let $u_1, \ldots, u_n$ be an orthonormal basis of U and $w_1, \ldots, w_m$ be an orthonormal basis of $U^\perp$. Then $u_1, \ldots, u_n, w_1, \ldots, w_m$ is an orthonormal basis of V. The matrix of T with respect to this basis is:

$$\mathcal{M}(T) = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} & a_{1,n+1} & \ldots & a_{1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} & a_{n,n+1} & \ldots & a_{n,n+m} \\ 0 & \ldots & 0 & a_{n+1,n+1} & \ldots & a_{n+1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{n+m,n+1} & \ldots & a_{n+m,n+m} \end{pmatrix},$$

where the lower left block is all zeros, because U is invariant under T.

Now consider the sum:

$$\sum_{k=1}^{n} \|Te_k\|^2 = \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{j,k}|^2. \tag{7.6}$$

Because T is normal, we have

$$\sum_{k=1}^{n} \|Te_k\|^2 = \sum_{k=1}^{n} \|T^*e_k\|^2$$

Using Theorem 7.9, we obtain

$$\sum_{k=1}^{n} \|T^*e_k\|^2 = \sum_{k=1}^{n} \sum_{j=1}^{n+m} |\overline{a}_{k,j}|^2 = \sum_{k=1}^{n} \sum_{j=1}^{n+m} |a_{k,j}|^2 \tag{7.7}$$

Now subtracting 7.6 from 7.7 (with interchanged dummy indices), we get

$$\sum_{k=1}^{n} \sum_{j=n+1}^{n+m} |a_{k,j}|^2 = 0.$$

This equation holds if and only if $a_{k,j} = 0$ for all $k = 1, \ldots, n$ and $j = n+1, \ldots, n+m$.

Therefore, the matrix of T has the form

$$\mathcal{M}(T) = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & a_{n+1,n+1} & \ldots & a_{n+1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{n+m,n+1} & \ldots & a_{n+m,n+m} \end{pmatrix},$$

which proves that $U^\perp$ is invariant under T and (upon taking the complex conjugate transpose of the matrix) that U is invariant under $T^\perp$.

Comparison of the matrices with respect to the chosen basis prooves (c). Matrix of $T|_U$ is the upper left block of the matrix of T; matrix of $(T|_U)^*$ is thus

$$\begin{pmatrix} \overline{a}_{1,1} & \dots & \overline{a}_{n,1} \\ \vdots & \ddots & \vdots \\ \overline{a}_{1,n} & \dots & \overline{a}_{n,n} \end{pmatrix}. \tag{7.8}$$

The matrix of T^* is complex conjugate transpose of T, and matrix of $(T^*)|_U$ is its upper left block, which gives the same matrix as 7.8. Hence, $(T|_U)^* = (T^*)|_U$.

Finally, that $T|_U$ and $T|_{U^\perp}$ are normal follows from the normality of T and that U and $U^\perp$ are invariant under both T and T^*. $\square$

21 Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that

$$T^2 - 5T + 6I = 0.$$

Solution:

T is self-adjoint and hence by Spectral Theorem, is diagonalizable. Then, by Theorems 5.27 and 5.62, the minimal polynomial of T is $(z-2)(z-3) = z^2 - 5z + 6$, as 2 and 3 are the only eigenvalues of T. By definition of a minimal polynomial:

$$p(T) = T^2 - 5T + 6I = 0. \quad \square$$

22 Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^3)$ such that 2 and 3 are the only eigenvalues of T and $T^2 - 5T + 6I \neq 0$.

Solution:

Take $T \in \mathcal{L}(\mathbb{C}^3)$ such that with respect to the standard basis, its matrix is:

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

One can then check with the matrix operations, that $T^2 - 5T + 6I \neq 0$:

$$\mathcal{M}(T^2 - 5T + 6I) = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}^2 - 5\begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

23 Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon.$$

Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

Solution:

T is self-adjoint, hence by the Spectral Theorem, there exists an orthonormal basis $e_1, \ldots, e_n$ of V, with respect to which the matrix of T is diagonal. That means

$$Te_k = \lambda_k e_k,$$

where λ_k is an eigenvalue of T (repetitions are possible).

Now express v via basis vectors:

$$v = a_1 e_1 + \ldots a_n e_n.$$

Here we also have

$$\sum_{k=1}^{n} |a_k|^2 = 1, \tag{7.9}$$

because $\|v\| = 1$. Now recast the inequality $\|Tv - \lambda v\| < \epsilon$ as $\|Tv - \lambda v\|^2 < \epsilon^2$. Note that in the left part of the inequality we have

$$\begin{aligned}
\|Tv - \lambda v\|^2 &= \left\| \sum_{k=1}^{n} a_k \lambda_k e_k - \lambda \sum_{k=1}^{n} a_k e_k \right\|^2 \\
&= \left\| \sum_{k=1}^{n} a_k (\lambda_k - \lambda) e_k \right\|^2 \\
&= \sum_{k=1}^{n} |a_k (\lambda_k - \lambda)|^2 = \sum_{k=1}^{n} |a_k|^2 |\lambda_k - \lambda|^2 \\
&\geq |\lambda' - \lambda|^2 \sum_{k=1}^{n} |a_k|^2 \\
&= |\lambda' - \lambda|^2.
\end{aligned}$$

The third equality is valid because the list $e_1, \ldots, e_n$ is orthonormal (Theorem 6.24). The last equality is valid because of 7.9. For the inequality we took an eigenvalue λ' of T such that $|\lambda' - \lambda|$ is the smallest amongst $|\lambda_k - \lambda|$. Combining this result with the given inequality we have:

$$|\lambda' - \lambda|^2 < \epsilon^2 \quad \Rightarrow \quad |\lambda - \lambda'| < \epsilon. \quad \square$$

24 Suppose U is a finite-dimensional vector space and $T \in \mathcal{L}(U)$.

(a) Suppose $\mathbb{F} = \mathbb{R}$. Prove that T is diagonalizable if and only if there is a basis of U such that the matrix of T with respect to this basis equals its transpose.

(b) Suppose $\mathbb{F} = \mathbb{C}$. Prove that T is diagonalizable if and only if there is a basis of U such that the matrix of T with respect to this basis commutes with its conjugate transpose.

Solution:

(a) First suppose that T is diagonalizable. Then the conclusion follows immediately, as any diagonal matrix equals its transpose.

Now suppose that there exists a basis of U such that the matrix of T with respect to this basis equals its transpose. Denote that matrix as A.

Consider a standard orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ where $n = \dim U$. Let $S \in \mathcal{L}(\mathbb{R}^n)$ be such operator that its matrix with respect to the standard basis is A. The transpose of A is a matrix of S^* with respect to the standard basis. By the Real Spectral Theorem, there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors $x_1, \ldots, x_n$ of S, such that its matrix is diagonal with respect that basis. Now we represent eigenvectors $x_1, \ldots, x_n$ as column-matrices with respect to the standard basis as:

$$x_j = \begin{pmatrix} x_{j,1} \\ \vdots \\ x_{j,n} \end{pmatrix}.$$

As these vectors are the eigenvectors of S, represented by the matrix A, we have:

$$A \cdot x = \begin{pmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,n} \\ A_{1,2} & A_{2,2} & \ldots & A_{2,n} \\ \vdots & & \ddots & \vdots \\ A_{1,n} & A_{2,n} & \ldots & A_{n,n} \end{pmatrix} \begin{pmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,n} \end{pmatrix} = \begin{pmatrix} \sum_i A_{1,i} x_{j,i} \\ \sum_i A_{2,i} x_{j,i} \\ \vdots \\ \sum_i A_{n,i} x_{j,i} \end{pmatrix} = \lambda_j \begin{pmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,n} \end{pmatrix},$$

where λ_j is an eigenvalue of the corresponding eigenvector. From here we get a useful relation:

$$\sum_i A_{k,i} x_{j,i} = \lambda_j x_{j,k}, \tag{7.10}$$

for every $k = 1, \ldots, n$.

Now we go back to U with its basis $u_1, \dots, u_n$, with respect to which the matrix of T equals A. Construct vectors in U as follows:

$$v_j = \sum_{k=1}^{n} x_{j,k} u_k,$$

for every $j = 1, \dots, n$. These are the eigenvectors of T. Indeed, we have:

$$\begin{aligned} Tv_j &= \sum_{k=1}^{n} x_{j,k} T u_k = \sum_{k=1}^{n} x_{j,k} \sum_{l=1}^{n} A_{l,k} u_l \\ &= \sum_{k,l=1}^{n} x_{j,k} A_{l,k} u_l = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} A_{l,k} x_{j,k} \right) u_l \\ &= \sum_{l=1}^{n} \lambda_j x_{j,l} u_l \\ &= \lambda_j v_j. \end{aligned}$$

Here in the fifth equality we used (7.10).

Lastly, we need to show that the list $v_1, \dots, v_n$ is linearly independent. Without loss of generality, suppose $v_1, \dots, v_s$ is linearly dependent list (if not, just choose and relabel the vectors that give the linearly dependent list). By definition, it means there exist $a_1, \dots, a_s \in \mathbb{R}$ such that

$$a_1 v_1 + \cdots + a_s v_s = 0.$$

Expanding v_j's as linear combinations of u_i's we obtain:

$$\sum_{l=1}^{s} a_l \sum_{k=1}^{n} x_{l,k} u_k = \sum_{k=1}^{n} \left(\sum_{l=1}^{s} a_l x_{l,k} \right) u_k = 0.$$

As $u_1, \dots, u_n$ is basis and hence linearly independent, we get that

$$\sum_{l=1}^{s} a_l x_{l,k} = 0$$

for every $k = 1, \dots, n$. That, in turn, means the list $x_1, \dots, x_s$ is linearly dependent. That contradict our initial identification of $x_1, \dots, x_n$ as a basis of $\mathbb{R}^n$. Thus, the list $v_1, \dots, v_n$ is linearly independent and hence is a basis of U. Now we combine it with Theorem 5.55 to get the desired result that T is diagonalizable. $\square$

(b) First suppose that T is diagonalizable with respect to some basis $u_1, \ldots, u_n$. Matrix of T with respect to this basis is diagonal; hence the conjugate transpose of this matrix is also diagonal. Anu two diagonal matrices commute, giving the desired result.

Now suppose that there exists a basis $u_1, \ldots, u_n$ of U such that the matrix of T with respect to this basis commutes with its conjugate transpose. Denote this matrix by A.

Here we consider a standard basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ where $n = \dim U$. Let $S \in \mathcal{L}(\mathbb{C}^n)$ be an operator whose matrix with respect to the standard basis is A. The conjugate transpose of A is a matrix of S^* with respect to the standard basis. By the Complex Spectral Theorem, there exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors $x_1, \ldots, x_n$ of S, such that its matrix is diagonal with respect that basis.

The rest of the proof is the same as in the real case of part (a), with appropriate substitution of $\mathbb{R}$ on $\mathbb{C}$. Thus, T is diagonalizable. $\square$

25 Suppose that $T \in \mathcal{L}(V)$ and there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that if $k \in \{1, \ldots, n\}$, then the pseudoinverse $T^\dagger$ satisfies the equation

$$T^\dagger e_k = \begin{cases} \frac{1}{\lambda_k} e_k & \text{if } \lambda_k \neq 0, \\ 0 & \text{if } \lambda_k = 0. \end{cases}$$

Solution:

Relabel vectors of the given orthonormal basis such that vectors $e_1, \ldots, e_m$ have all non-zero eigenvectors and $e_{m+1}, \ldots, e_n$ have zero as an eigenvalue. Then we have:

$$V = U \oplus U^\perp$$

where $U = E(0, T) = \operatorname{null} T$ and $U^\perp = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Also note that such representation implies that $\operatorname{range} T = (\operatorname{null} T)^\perp$ (see also *Problem 5D.3*).

First suppose $w \in U$. Then by definition of the pseudoinverse we have

$$T^\dagger w = (T|_{(\operatorname{null} T)^\perp})^{-1} P_{\operatorname{range} T} w = (T|_{(\operatorname{null} T)^\perp})^{-1} P_{(\operatorname{null} T)^\perp} w = 0.$$

Now examine $U^\perp$. The operator $T|_{U^\perp}$ is invertible and diagonalizable. By the result of *Problem 5D.7*, an eigenspace of $T|_{U^\perp}$ of an eigenvalue λ_j is an eigenspace of $(T|_{U^\perp})^{-1}$ of eigenvalue $1/\lambda_j$.

If $w \in U^\perp$, then $P_{\operatorname{range} T} w = P_{U^\perp} w = w$. That concludes the proof giving the desired

$$T^\dagger e_k = \begin{cases} \frac{1}{\lambda_k} e_k & \text{if } \lambda_k \neq 0, \\ 0 & \text{if } \lambda_k = 0. \end{cases} \qquad \square$$

7C Positive Operators

1 Suppose $T \in \mathcal{L}(V)$. Prove that if both T and $-T$ are positive operators, then $T = 0$.

Solution:

We have for every $v \in V$:

$$0 = \langle 0v, v \rangle = \langle (T - T)v, v \rangle = \langle Tv, v \rangle + \langle (-T)v, v \rangle.$$

Sum of two nonnegative terms equal zero, hence each of the terms also equals zero. Thus, we have that T is self-adjoint (by definition of positive operator) and $\langle Tv, v \rangle = 0$ for every $v \in V$. That implies $T = 0$, by Theorem 7.16, as desired. $\square$

2 Suppose $T \in \mathcal{L}(\mathbb{F}^4)$ is the operator whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that T is an invertible positive operator.

Solution:

Let $v \in \mathbb{F}^4$ and $v = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$. T acting on v gives

$$Tv = (2a_1 - a_2)e_1 + (-a_1 + 2a_2 - a_3)e_2 + (-a_2 + 2a_3 - a_4)e_3 + (-a_3 + 2a_4)e_4$$

Note, that T is surjective, hence by Theorem 3.63, T is invertible.

The inner product $\langle Tv, v \rangle$ equals

$$\begin{aligned} \langle Tv, v \rangle &= \overline{a_1}(2a_1 - a_2) + (-a_1 + 2a_2 - a_3)\overline{a_2} \\ &\quad + (-a_2 + 2a_3 - a_4)\overline{a_3} + (-a_3 + 2a_4)\overline{a_4} \\ &= 2|a_1|^2 - \overline{a_1}a_2 - a_1\overline{a_2} + 2|a_2|^2 - \overline{a_2}a_3 - a_2\overline{a_3} \\ &\quad + 2|a_3|^2 - \overline{a_3}a_4 - a_3\overline{a_4} + 2|a_4|^2 \\ &= |a_1|^2 + |a_4|^2 + |a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_3 - a_4|^2 \\ &\geq 0. \end{aligned}$$

This shows that T is a positive operator, thus completing the proof. $\square$

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Show that T is a positive operator.

Solution:

Let $v = a_1 e_1 + \cdots + a_n e_n$. Then:

$$Tv = (a_1 + \cdots + a_n)e_1 + \ldots (a_1 + \cdots + a_n)e_n.$$

The inner product $\langle Tv, v \rangle$ equals

$$\begin{aligned}
\langle Tv, v \rangle &= \overline{a_1}(a_1 + \cdots + a_n) + \cdots + \overline{a_n}(a_1 + \cdots + a_n) \\
&= |a_1|^2 + \overline{a_1}a_2 + \cdots + \overline{a_1}a_n \\
&\quad + |a_2|^2 + \overline{a_2}a_1 + \cdots + \overline{a_2}a_n + \ldots \\
&\quad + |a_n|^2 + \overline{a_n}a_2 + \cdots + \overline{a_n}a_{n-1} \\
&= \sum_{i=1}^{n} |a_i|^2 + 2 \sum_{i}^{n-1} \sum_{k=i+1}^{n} \Re a_i a_k \\
&= |a_1 + \cdots + a_n|^2 \\
&\geq 0
\end{aligned}$$

hence the operator T is a positive operator. $\square$

4 Suppose n is an integer with $n > 1$. Show that there exists an n-by-n matrix A such that all of the entries of A are positive numbers and $A = A^*$, but the operator on $\mathbb{F}^n$ whose matrix (with respect to the standard basis) equals A is not a positive operator.

Solution:

Referring to Theorem 7.38, we see that we should look for an operator T (with matrix A) which has at least one negative eigenvalue. The following matrix confirms to this criterion:

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix}.$$

This is a symmetric matrix, hence it equals its complex conjugate transpose.

This operator has a negative eigenvalue -4 with eigenvector $v = (1, 0, -1)$. Hence, for this vector we have:

$$\langle Tv, v \rangle = \langle -4(1, 0, -1), (1, 0, -1) \rangle = -8 < 0.$$

So, T is not a positive operator, as desired. $\square$

5 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that T is a positive operator if and only if for every orthonormal basis $e_1, \dots, e_n$ of V, all entries on the diagonal of $\mathcal{M}(T, (e_1, \dots, e_n))$ are nonnegative numbers.

Solution:

For any orthonormal basis, entries on the diagonal of a matrix equal the inner product $\langle Te_j, e_j \rangle$.

If T is a positive operator, than $\langle Te_j, e_j \rangle \geq 0$ irregarless of a chosen basis.

Going in the other direction, if all entries on the diagonal of the matrix of T are nonnegative for every orthonormal basis, then we can choose an orthonormal basis, with respect to which T is diagonal (we can do that by the Spectral Theorem). All entries on the diagonal are thus nonnegative and by Theorem 7.38, it follows that T is a positive operator. $\square$

6 Prove that the sum of two positive operators on V is a positive operator.

Solution:

Suppose $T, S \in \mathcal{L}(V)$ are both positive operators. Then for $T+S$ we have:

$$\langle (T+S)v, v \rangle = \langle Tv, v \rangle + \langle Sv, v \rangle \geq 0,$$

where the first equality is true because an inner product has additivity in first slot, and the inequality holds because each term in the sum is nonnegative. Thus, the sum of tho positive operators is a positive operator. $\square$

7 Suppose $S \in \mathcal{L}(V)$ is an invertible positive operator and $T \in \mathcal{L}(V)$ is a positive operator. Prove that $S+T$ is invertible.

Solution:

Suppose $S+T$ is not invertible. That means (Theorem 3.65) $S+T$ is not injective. Let $v \in V$ is a nonzero vector such that $(S+T)v = 0$. Then:

$$Sv = -Tv,$$

Now we take an inner product of both parts with $v \neq 0$.

$$\langle Sv, v \rangle \geq 0, \quad \text{because } S \text{ is a positive operator.}$$
$$\langle -Tv, v \rangle = -\langle Tv, v \rangle \leq 0, \quad \text{because } T \text{ is a positive operator.}$$

These inequalities can be true simultaneously only if $Tv = Sv = 0$. That cannot be, as S is an invertible, hence an injective operator, and $Sv = 0$ if and only if $v = 0$, which is not the case here.

Thus, our assumption that $S+T$ is not invertible is false, thus proving the proposition. $\square$

8 Suppose $T \in \mathcal{L}(V)$. Prove that T is a positive operator if and only if the pseudoinverse $T^\dagger$ is a positive operator.

Solution:

First suppose that T is positive. Then by Theorem 7.38, there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T with eigenvalues $\lambda_1, \ldots, \lambda_n$. By the result of *Problem 7B.25*, the pseudoinverse satisfies the equation:

$$T^\dagger e_k = \begin{cases} \frac{1}{\lambda_k} e_k & \text{if } \lambda_k \neq 0, \\ 0 & \text{if } \lambda_k = 0. \end{cases}$$

One can see that $T^\dagger$ is diagonalizable with respect to the basis $e_1, \ldots, e_n$ with only nonnegative numbers on the diagonal (i.e. eigenvalues). Thus, $T^\dagger$ is a positive operator, by Theorem 7.38.

Now suppose that $T^\dagger$ is a positive operator. It means it is a self-adjoint and hence a normal operator. This implies that $V = \operatorname{null} T^\dagger \oplus \operatorname{range} T^\dagger$ (Theorem 7.21), and $\operatorname{range} T^\dagger = (\operatorname{null} T^\dagger)^\perp$ and $\operatorname{null} T^\dagger = (\operatorname{range} T^\dagger)^\perp$ (Theorem 7.6).

Using the result of *Problem 6C.20*, we can deduce the following:

$$\begin{aligned} \operatorname{range} T^\dagger &= (\operatorname{null} T^\dagger)^\perp = ((\operatorname{range} T)^\perp)^\perp = \operatorname{range} T \\ \operatorname{null} T^\dagger &= (\operatorname{range} T^\dagger)^\perp = ((\operatorname{null} T)^\perp)^\perp = \operatorname{null} T. \end{aligned}$$

Now any $v \in V$ can be represented as $v = u + w$, where $u \in \operatorname{null} T = \operatorname{null} T^\dagger$ and $w \in \operatorname{range} T = \operatorname{range} T^\dagger$.

For $u \in \operatorname{null} T$ we have $Tu = 0$, hence $\langle Tu, u \rangle = 0$.

For $w \in \operatorname{range} T^\dagger = (\operatorname{null} T)^\perp$ we have a unique element of $y \in \operatorname{range} T$ such that $T^\dagger y = w$. As $T^\dagger$ is positive, we have:

$$\langle T^\dagger y, y \rangle \geq 0 \Rightarrow \langle w, Tw \rangle \geq 0 \Rightarrow \langle Tw, w \rangle \geq 0.$$

Now we have for any $v \in V$:

$$\begin{aligned} \langle Tv, v \rangle &= \langle Tu + Tw, u + w \rangle = \langle Tw, u + w \rangle \\ &= \langle Tw, u \rangle + \langle Tw, w \rangle \\ &= \langle Tw, w \rangle \\ &\geq 0 \end{aligned}$$

where we used $\langle Tw, u \rangle = 0$ in the last equality, because $Tw \in \operatorname{range} T = (\operatorname{null} T)^\perp$ and $u \in \operatorname{null} T$, so Tw and u are orthogonal.

We have shown that if $T^\dagger$ is a positive operator, then T is also a positive operator, thus completing the proof. $\square$

9 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

Solution:

Let $w \in W$. Then we have:

$$\langle S^*TSw, w\rangle = \langle T(Sw), Sw\rangle \geq 0.$$

The first equality comes from the definition of an adjoint, the second equality comes from T being a positive operator. Thus, S^*TS is a positive operator on W. $\square$

10 Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \quad \text{and} \quad Tw = v.$$

Prove that $v = w$.

Solution:

Examine the following:

$$\begin{aligned}
\langle T(v-w), v-w\rangle &= \langle Tv, v\rangle - \langle Tw, v\rangle - \langle Tv, w\rangle + \langle Tw, w\rangle \\
&= \langle w, v\rangle - \langle v, v\rangle - \langle w, w\rangle + \langle v, w\rangle \\
&= 2\Re\langle v, w\rangle - \|v\|^2 - \|w\|^2 \\
&\leq |\langle v, w\rangle| - \|v\|^2 - \|w\|^2 \\
&\leq 2\|v\|\|w\| - \|v\|^2 - \|w\|^2 \\
&= -(\|v\| - \|w\|)^2 \\
&\leq 0.
\end{aligned}$$

Here the transition from the fourth to the fifth line is due to the Cauchy-Schwarz inequality (6.14).

On the other hand, T is a positive operator, so

$$\langle T(v-w), v-w\rangle \geq 0.$$

Thus we conclude that $\langle T(v-w), v-w\rangle = 0$.

From the Theorem 7.43 it follows that $T(v-w) = 0$. On the other hand:

$$T(v-w) = Tv - Tw = w - v.$$

Hence, $w - v = 0$, or $v = w$. $\square$

11 Suppose T is a positive operator on V and U is a subspace of V invariant under T. Prove that $T|_U \in \mathcal{L}(U)$ is a positive operator on U.

Solution:

As U is invariant under T, the inner product $\langle T|_U u, u\rangle_U$ defined on U makes sense. Thus for every $u \in U \subseteq V$ we have:

$$\langle T|_U u, u\rangle_U = \langle Tu, u\rangle_V \geq 0.$$

So $T|_U$ is a positive operator. □

12 Suppose $T \in \mathcal{L}(V)$ is a positive operator. Prove that T^k is a positive operator for every positive integer k.

Solution:

Let $v \in V$. Then:

$$\langle T^k v, v\rangle = \langle Tv, (T^*)^{k-1} v\rangle \geq 0,$$

so T^k is a positive operator for every positive integer k. □

13 Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $\alpha \in \mathbb{R}$.

(a) Prove that $T - \alpha I$ is a positive operator if and only if α is less than or equal to every eigenvalue of T.

(b) Prove that $\alpha I - T$ is a positive operator if and only if α is greater than or equal to every eigenvalue of T.

Solution:

(a) According to Theorem 7.38, an operator is positive if and only if all its eigenvalues are nonnegative. Operator αI is diagonal in every basis, hence T and αI are simultaneously diagonalizable and thus they commute. By Theorem 5.81, each eigenvalue of $T - \alpha I$ is an eigenvalue of T minus an eigenvalue of αI. The only eigenvalue of αI is α. Thus for each eigenvalue λ_i of T we have

$$T - \alpha I \text{ is a positive operator} \iff \lambda_i - \alpha \geq 0 \iff \lambda_i \geq \alpha. \quad \square$$

(b) Similarly to (a), we have for each eigenvalue λ_i of T:

$$\alpha I - T \text{ is a positive operator} \iff \alpha - \lambda_i \geq 0 \iff \alpha \geq \lambda_i. \quad \square$$

14 Suppose T is a positive operator on V and $v_1, \dots, v_m \in V$. Prove that

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\langle Tv_k, v_j\rangle \geq 0.$$

Solution:
Using the additivity in both slots of an inner product we have:

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{k=1}^{m}\langle Tv_k, v_j\rangle &= \langle Tv_k, \sum_{j=1}^{m} v_j\rangle \\
&= \langle \sum_{k=1}^{m} Tv_k, \sum_{j=1}^{m} v_j\rangle \\
&= \langle \sum_{j=1}^{m} Tv_j, \sum_{j=1}^{m} v_j\rangle \\
&\geq 0.
\end{aligned}$$

Where we relabelled dummy indices in the last equality, thus obtaining an expression of $\langle Tv, v\rangle$ type and producing the desired inequality. $\square$

15 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that there exist positive operators $A, B \in \mathcal{L}(V)$ such that

$$T = A - B \qquad \text{and} \qquad \sqrt{T^*T} = A + B \qquad \text{and} \qquad AB = BA = 0.$$

Solution:
Denote $\sqrt{T^*T}$ as S. Then we can construct operators A and B as follows:

$$A = \frac{T+S}{2}, \quad B = \frac{S-T}{2}.$$

These operators satisfy the first two properties. We need to show that they are positive and their composition is zero.

Take an orthonormal basis $e_1, \dots, e_n$ of V with respect to which T is diagonal. All diagonal entries of T, i.e. eigenvalues, are real (Theorem 7.12). Now note that $S^2 = T^*T$ has the same eigenvectors with eigenvalues $\lambda_1^2, \dots, \lambda_n^2$. Its positive square root S also has eigenvectors $e_1, \dots, e_n$, but with eigenvalues $|\lambda_1|, \dots, |\lambda_n|$.

Using the rule of matrix summation, we have that the matrices of A and B are diagonal with entries being equal:

$$\mathcal{M}(A)_{j,j} = \frac{\lambda_j + |\lambda_j|}{2}, \quad \mathcal{M}(B)_{j,j} = \frac{|\lambda_j| - \lambda_j}{2}.$$

If $\lambda_j \geq 0$, $\mathcal{M}(A)_{j,j} = \lambda_j$ and $\mathcal{M}(B)_{j,j} = 0$.

If $\lambda_j < 0$, $\mathcal{M}(A)_{j,j} = 0$ and $\mathcal{M}(B)_{j,j} = |\lambda_j|$.

Thus, A and B have diagonal matrices with respect to the orthonormal basis $e_1, \ldots, e_n$ with only nonnegative numbers on the diagonal. It means that A and B are positive operators (Theorem 7.38).

At last we have:

$$AB = \frac{T+S}{2}\frac{S-T}{2} = \frac{1}{4}(TS - T^2 + S^2 - ST) = \frac{1}{4}(T^*T - T^2) = 0$$

$$BA = \frac{S-T}{2}\frac{T+S}{2} = \frac{1}{4}(ST + S^2 - T^2 - TS) = \frac{1}{4}(T^*T - T^2) = 0.$$

Here we used the fact that S and T commute (they are simultaneously diagonalizable) and that T is self-adjoint. □

16 Suppose T is a positive operator on V. Prove that

$$\operatorname{null}\sqrt{T} = \operatorname{null} T \qquad \text{and} \qquad \operatorname{range}\sqrt{T} = \operatorname{range} T.$$

Solution:

Let $R = \sqrt{T}$, hence $T = R^2$. R is self-adjoint (Theorem 7.38), hence it is normal. By the result of *Problem 7A.27*

$$\operatorname{null} R = \operatorname{null} R^2 \qquad \text{and} \qquad \operatorname{range} R = \operatorname{range} R^2,$$

proving the desired result. □

17 Suppose that $T \in \mathcal{L}(V)$ is a positive operator. Prove that there exists a polynomial p with real coefficients such that $\sqrt{T} = p(T)$.

Solution:

Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of T. Then $p(T)$ and $\sqrt{T}$ have the same eigenvectors, but with eigenvalues $p(\lambda_1), \ldots, p(\lambda_m)$ and $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}$. We need that for every $j = 1, \ldots, m$

$$p(\lambda_j) = \sqrt{\lambda_j}.$$

$\lambda_j, \sqrt{\lambda_j} \in \mathbb{R}$ for every $j = 1, \ldots, m$. Thus, by the result of *Problem 4.7*, there exists a unique polynomial $p \in \mathcal{P}_{m-1}(\mathbb{R})$ satisfying the equation above. That is the sought polynomial $p(T) = \sqrt{T}$ (take an orthonormal basis that diagonalizes T and use linear map lemma 3.4). □

18 Suppose S and T are positive operators on V. Prove that ST is a positive operator if and only if S and T commute.

Solution:

First suppose that ST is a positive operator. Thus, we have that T, S and ST are self-adjoint operators. Now it is easy to show that S and T commute:

$$ST = (ST)^* = T^*S^* = TS.$$

Now suppose S and T commute. Rearranging the sequence of equalities above we can first show that ST is self-adjoint:

$$ST = TS = T^*S^* = (ST)^*.$$

As S and T are commuting self-adjoint operators, they are simultaneously diagonalizable. Now the ST is also a diagonal matrix with eintries on the diagonal being equal to a product of corresponding diagonal entries of S and T. Thus, the matrix of ST (with respect to the same basis) has all nonnegative diagonal entries, which means ST is a positive operator. $\square$

19 Show that the identity operator on $\mathbb{F}^2$ has infinitely many self-adjoint square roots.

Solution:

Take a standard orthonormal basis of $\mathbb{F}^2$. A self-adjoint square root of the identity operator have the matrix with respect to this basis in the following form:

$$\mathcal{M}(\sqrt{I}) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{pmatrix}.$$

As it is a square root of the identity operator we must have $\mathcal{M}(I) = \mathcal{M}(\sqrt{I})\mathcal{M}(\sqrt{I})$. Thus:

$$\mathcal{M}(\sqrt{I}) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{pmatrix} = \begin{pmatrix} \alpha^2 + |\beta|^2 & (\alpha+\gamma)\beta \\ (\alpha+\gamma)\overline{\beta} & \gamma^2 + |\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we have two options: $\beta = 0$ or $\beta \neq 0$. If $\beta = 0$, then $\alpha = \pm 1$ and $\gamma = \pm 1$. If $\beta \neq 0$, then we must have $\alpha = -\gamma$, and β is determined from $\alpha + |\beta|^2 = 1$. Without loss of generality, assume $\beta \in \mathbb{R}$. Then pairs (α, β) and (γ, β) correspond to points (x, y) and $(-x, y)$ on the unit circle of $\mathbb{R}^2$. These points determine the matrix elements of $\mathcal{M}(\sqrt{I})$ and thus determine the operator $\sqrt{I}$. There are infinitely many such points, therefore there are infinitely many self-adjoint square roots of the identity operator. $\square$

20 Suppose $T \in \mathcal{L}(V)$ and $e_1, \dots, e_n$ is an orthonormal basis of V. Prove that T is a positive operator if and only if there exist $v_1, \dots, v_n \in V$ such that

$$\langle Te_k, e_j \rangle = \langle v_k, v_j \rangle$$

for all $j, k = 1, \dots, n$.

Solution:

First suppose that T is a positive operator. Then $T = R^*R$ for some $R \in \mathcal{L}(V)$ (Theorem 7.38). Then for all $j, k = 1, \dots, n$ we have:

$$\langle Te_k, e_j \rangle = \langle R^*Re_k, e_j \rangle = \langle Re_k, Re_j \rangle.$$

Take $v_j = Rv_j$ for each $j \in \{1, \dots, n\}$. That is the desired list $v_1, \dots, v_n \in V$.

Now suppose there exist $v_1, \dots, v_n \in V$ such that

$$\langle Te_k, e_j \rangle = \langle v_k, v_j \rangle$$

for all $j, k = 1, \dots, n$. Let $v = a_1e_1 + \dots + a_ne_n$ for some $a_1, \dots, a_n \in \mathbb{F}$. Then we have:

$$\begin{aligned}
\langle Tv, v \rangle &= \langle T \sum_k^n a_k e_k, \sum_j^n a_j e_j \rangle = \langle \sum_k^n a_k Te_k, \sum_j^n a_j e_j \rangle \\
&= \sum_k^n a_k \langle Te_k, \sum_j^n a_j e_j \rangle = \sum_k^n a_k \sum_j^n \overline{a_j} \langle Te_k, e_j \rangle \\
&= \sum_k^n a_k \sum_j^n \overline{a_j} \langle v_k, v_j \rangle \\
&= \sum_k^n a_k \langle v_k, \sum_j^n a_j v_j \rangle = \langle \sum_k^n a_k v_k, \sum_j^n a_j v_j \rangle \\
&= \left\| \sum_k^n a_k v_k \right\|^2 \geq 0.
\end{aligned}$$

Thus, T is a positive operator, which completes the proof. $\square$

21 Suppose n is a positive integer. The n-by-n *Hilbert matrix* is the n-by-n matrix whose entry in row j, column k is $\frac{1}{j+k-1}$. Suppose $T \in \mathcal{L}(V)$ is an operator whose matrix with respect to some orthonormal basis of V is the n-by-n Hilbert matrix. Prove that T is a positive invertible operator.

Solution:

Here we use the fact that $\langle Te_k, e_j \rangle$ is an entry in the matrix of T at k'th column, j'th row. We need to find $v_1, \dots, v_n \in V$ such that the condition of *Problem 7C.20* holds; that will proove that T is a positive operator.

Take $\mathcal{P}_{n-1}(\mathbb{F})$, which is isomorphic to V. We will make (an arbitrary!) correspondence between the orthonormal basis $e_1, \dots, e_n$ of V and an orthonormal basis of $\mathcal{P}_{n-1}(\mathbb{F})$ (obtained, for example, by Gram-Schmidt procedure from the standard $1, x, x^2, \dots$ basis). That will define a map $S : V \to \mathcal{P}_{n-1}(\mathbb{F})$ (linear map lemma 3.4), that is our 'correspondence' between V and $\mathcal{P}_{n-1}(\mathbb{F})$.

Let us define the inner product on V in such a way that it corresponds to an inner product on $\mathcal{P}_{n-1}(\mathbb{F})$:

$$\langle v, w \rangle = \langle Sv, Sw \rangle = \int_0^1 pq$$

where $v, w \in V$, and $p = Sv$, $q = Sw$.

S is a surjective map, hence it is invertible. Choose $v_1, \dots, v_n \in V$ such that $v_k = S^{-1}(x^k)$. Now we have:

$$\begin{aligned}
\langle v_j, v_k \rangle &= \langle x^j, x^k \rangle \\
&= \int_0^1 x^j x^k dx = \int_0^1 x^{j+k} dx \\
&= \frac{1}{j+k-1} \\
&= \langle Te_j, e_k \rangle.
\end{aligned}$$

Thus, T is a positive operator. We need to show that T is invertible. Suppose it is not. Then there exists nonzero $v \in V$ such that $Tv = 0$. Then it is also true that $\langle Tv, v \rangle = 0$.

Expressing v via basis vectors, $v = a_1 e_1 + \cdots + a_n e_n$, we have:

$$\begin{aligned}
\langle Tv, v \rangle &= \langle T \sum_j^n a_j e_j, \sum_k^n a_k e_k \rangle = \sum_j^n a_j \sum_k^n \overline{a_k} \langle Te_j, e_k \rangle \\
&= \sum_j^n a_j \sum_k^n \overline{a_k} \langle v_j, v_k \rangle = \langle \sum_j^n a_j v_j, \sum_k^n v_k \rangle \\
&= \left\| \sum_k^n a_k v_k \right\|^2 \geq 0.
\end{aligned}$$

Because of our assumption, $\langle Tv, v \rangle = 0$, it follows that $\sum_k^n a_k v_k = 0$, that is the list $v_1, \dots, v_n$ is not linearly dependent. That contradicts the invertibility of S which must send every linearly independent list to another linearly independent list (see *Problem 3B.9*). Thus, T is an invertible positive operator, as desired. $\square$

22 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $u \in V$ is such that $\|u\| = 1$ and $\|Tu\| \geq \|Tv\|$ for all $v \in V$ with $\|v\| = 1$. Show that u is an eigenvector of T corresponding to the largest eigenvalue of T.

Solution:

T is a positive operator, hence it is self-adjoint, hence it is diagonalizable by Spectral Theorem. As T is diagonalizable, V is a direct sum of the eigenspaces of T (*Problem 7B.14* and *Problem 7B.15*), $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$ and all pairs of eigenvectors corresponding to different eigenvalues are orthogonal. Let λ_n be the largest eigenvector of these.

We can write u as a sum $u = v_1 + \cdots + v_n$, where each v_j is a unit vecotor and $v_j \in E(\lambda_j, T)$. We have $\|u\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2 = 1$, and

$$\begin{aligned}\|Tu\|^2 = \|\lambda_1 v_1 + \dots \lambda_n v_n\|^2 &= \lambda_1^2 \|v_1\|^2 + \cdots + \lambda_n^2 \|v_n\|^2 \\ &\leq \lambda_n^2 (\|v_1\|^2 + \cdots + \|v_n\|^2) = \lambda_n^2.\end{aligned}$$

Now let $w_n \in E(\lambda_n, T)$ be a unit vector. We must have $\|Tu\|^2 \geq \|Tw_n\|^2$, so

$$\|Tu\|^2 \geq \|Tw_n\|^2 = \lambda_n^2 \|w_n\|^2 = \lambda_n^2.$$

Thus, we came to the $\lambda_n^2 \leq \|Tu\|^2 \leq \lambda_n^2$. Therefore, $\|Tu\|^2 = \lambda_n^2$. Writing u as a sum of $v_1, \dots, v_n$ one again, we have

$$\begin{aligned}\lambda_1^2 \|v_1\|^2 + \cdots + \lambda_n^2 \|v_n\|^2 &= \lambda_n^2 \\ \lambda_1^2 \|v_1\|^2 + \cdots + \lambda_n^2 (\|v_n\|^2 - 1) &= 0\end{aligned}$$

If the largest eigenvalue is zero, then it is the only eigenvalue of T (eigenvalues of a positive operator must be nonnegative), and the conclusion is obviously true.

If $\lambda_n \neq 0$, then we must have $\|v_j\| = 0$ except for $j = n$, for which $\|v_n\| = 1$. Thus we left with $u = v_n$, so indeed u is an eigenvector of T corresponding to the largest eigenvalue. $\square$

23 For $T \in \mathcal{L}(V)$ and $u, v \in V$, define $\langle u, v \rangle_T$ by $\langle u, v \rangle_T = \langle Tu, v \rangle$.

(a) Suppose $T \in \mathcal{L}(V)$. Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot \rangle$.

(b) Prove that every inner product on V is of the form $\langle \cdot, \cdot \rangle_T$ for some positive invertible operator $T \in \mathcal{L}(V)$.

Solution:

(a) Regardless of what type of operator T is, additivity, homogeneity and conjugate symmetry hold:

$$\langle u + w, v\rangle_T = \langle T(u+w), v\rangle = \langle Tu, v\rangle + \langle Tw, v\rangle = \langle u, v\rangle_T + \langle w, v\rangle_T$$
$$\langle \lambda u, v\rangle_T = \langle T(\lambda u), v\rangle = \langle \lambda Tu, v\rangle = \lambda\langle Tu, v\rangle = \lambda\langle u, v\rangle_T$$
$$\langle u, v\rangle_T = \langle Tu, v\rangle = \overline{\langle v, Tu\rangle} = \overline{\langle Tv, u\rangle} = \overline{\langle v, u\rangle_T}$$

We need to show positivity and definiteness.

Positivity condition reads:

$$\langle v, v\rangle_T = \langle Tv, v\rangle \geq 0,$$

which is true for every $v \in V$ if and only if T is a positive operator.

Definiteness condition reads:

$$\langle v, v\rangle_T = \langle Tv, v\rangle = 0 \text{ if and only if } v = 0.$$

As T is a positive operator $\langle Tv, v\rangle = 0$ implies that $Tv = 0$. From here we have:

$$(Tv = 0 \text{ if and only if } v = 0) \Longleftrightarrow T \text{ is invertible},$$

thus completing the proof. □

(b) Let $\langle \cdot, \cdot \rangle$ be some reference inner product on V and $\langle \cdot, \cdot \rangle_d$ is another inner product on V.

Let $\phi_1, \ldots, \phi_n$ be a (dual) basis of V'. By the Riesz representation theorem, for each $j \in \{1, \ldots, n\}$ there is a unique $v_j \in V$ such that:

$$\phi_j(u) = \langle u, v_j\rangle,$$

for every $u \in V$. Similarly, for each $j \in \{1, \ldots, n\}$ there is a unique $w_j \in V$ such that:

$$\phi_j(u) = \langle u, w_j\rangle_d,$$

for every $u \in V$. Lists $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are linearly independent. To show this, suppose they were not. Let $v_m = a_1 v_1 + \cdots + a_{m-1} v_{m-1}$ for some nonzero $a_1, \ldots, a_{m-1} \in \mathbb{F}$. Then we would have:

$$\begin{aligned}\phi_m(u) = \langle u, v_m\rangle &= \langle u, a_1 v_1 + \cdots + a_{m-1} v_{m-1}\rangle \\ &= \overline{a_1}\langle u, v_1\rangle + \cdots + \overline{a_{m-1}}\langle u, v_{m-1}\rangle \\ &= \overline{a_1}\phi_1(u) + \cdots + \overline{a_{m-1}}\phi_{m-1}(u)\end{aligned}$$

for every $u \in V$. This cannot be, as $\phi_1, \dots, \phi_n$ is a basis, hence is a linearly independent list. Similar reasoning applies to $w_1, \dots, w_n$. This shows that $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are linearly independent indeed. Moreover, length of these two lists equals $\dim V$, so they are two bases of V. Now define operator T such that $T(v_j) = w_j$. Let $v = a_1 v_1 + \dots a_n v_n$. We have

$$\begin{aligned}
\langle u, v \rangle_d &= \phi(u) = \overline{a_1}\phi_1(u) + \cdots + \overline{a_n}\phi_n(u) \\
&= \overline{a_1}\langle u, w_1 \rangle + \cdots + \overline{a_n}\langle u, w_n \rangle \\
&= \overline{a_1}\langle u, Tv_1 \rangle + \cdots + \overline{a_n}\langle u, Tv_n \rangle \\
&= \langle u, Tv \rangle = \langle T^* u, v \rangle.
\end{aligned}$$

Using result of part (a) of this exercise, we have that T^* is an invertible positive operator. That means $T^* = T$, so the T defined above is the sought operator. □

24 Suppose S and T are positive operators on V. Prove that

$$\operatorname{null}(S+T) = \operatorname{null} S \cap \operatorname{null} T$$

Solution:
First suppose that $v \in \operatorname{null} S$ and $v \in \operatorname{null} T$. Then:

$$(T + S)v = Tv + Sv = 0 + 0 = 0,$$

thus $\operatorname{null} S \cap \operatorname{null} T \subseteq \operatorname{null}(S+T)$.

Now suppose $v \in \operatorname{null}(S+T)$. Examine the following inner product:

$$\langle (S+T)v, v \rangle = 0 = \langle Sv, v \rangle + \langle Tv, v \rangle.$$

S and T are positive operators, hence $\langle Sv, v \rangle$ and $\langle Tv, v \rangle$ are nonnegative numbers. This implies that both of these equal zero, $\langle Sv, v \rangle = 0$ and $\langle Tv, v \rangle = 0$. Theorem 7.43 then implies that $Sv = 0$ and $Tv = 0$, that is $v \in \operatorname{null} S \cap \operatorname{null} T$. Hence, $\operatorname{null} S \cap \operatorname{null} T \subseteq \operatorname{null}(S+T)$.

Combining with inclusion in the other direction, we have the desired equality. □

25 Let T be the second derivative operator in Exercise 31(b) in Section 7A. Show that $-T$ is a positive operator.

Solution:
Let D denote the derivative operator, and T denote the second derivative operator, as in *Problem 7A.31.*

Note that:

$$-T = -D^2 = (-D)D = D^*D.$$

Now Theorem 7.38 condition (f) implies that $-T$ is a positive operator.