# NOTES ON A PRIMAL DPG FORMULATION FOR CONVECTION-DIFFUSION

Our original variational formulation for the convection-diffusion problem is

$$b(u,v) = \langle \beta_n u, v\rangle_\Gamma + (u, -\beta\nabla v) - \epsilon\left\langle \frac{\partial u}{\partial n}, v\right\rangle_{\Gamma_{in}} + \epsilon\left(\nabla u, \nabla v\right)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}\,, \quad \forall v\in H^1_{\text{out}}$$

where

$$\begin{aligned}\Gamma_{\text{in}} &\coloneqq \{x\in\partial\Omega : \beta\cdot\vec{n}(x)\le 0\}\\ \Gamma_{\text{out}} &\coloneqq \{x\in\partial\Omega : \beta\cdot\vec{n}(x) > 0\}\,.\end{aligned}$$

Boundary conditions can be applied in either a strong or weak fashion in this formulation. The test norm is taken to be

$$\|v\|_V^2 \coloneqq \|\beta\nabla v\|^2 + \epsilon\|\nabla v\|^2 + \|v\|^2$$

This formulation is derived under the assumption that $v$ is $C_0$ continuous throughout the domain. However, if we break that continuity and require only that $v\in H^1(K)$ (i.e. in $H^1$ locally), we can derive a different variational formulation, similar to one introduced in [1]. Integrating by parts, we pick up a boundary term over each element, such that our variational form becomes

$$\begin{aligned} b\left(\left(u,\widehat{f}_n\right), v\right) &= \langle\beta_n u, v\rangle_\Gamma + (u,-\beta\nabla v) - \left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h^0} - \epsilon\left\langle\frac{\partial u}{\partial n}, v\right\rangle_\Gamma + \epsilon\left(\nabla u,\nabla v\right)_{L^2(\Omega)}\\ &= (f,v)_{L^2(\Omega)}\,,\quad \forall v\in H^1_{\text{out}}.\end{aligned}$$

where we have identified the viscous fluxes $\epsilon\frac{\partial u}{\partial n}$ on the interior skeleton $\Gamma_h^0$ as additional unknowns $\widehat{f}_n$.

**1. Interpretation as a nonconforming method for $e$.** The abstract mixed form of the DPG method with $(e,v)\in V$ and $(u,du)\in U$ is given as

$$\begin{aligned}(e,v)_V + b(u,v) &= l(v)\\ b(\delta u, e) &= 0.\end{aligned}$$

This is equivalent to a constrained projection/Riesz inversion

$$(e,v)_V = l(v),\quad (e,v)\in V\cap \text{null}(B^T)$$

where $B : U\to V'$ is defined through $\langle Bu, v\rangle = b(u,v)$.

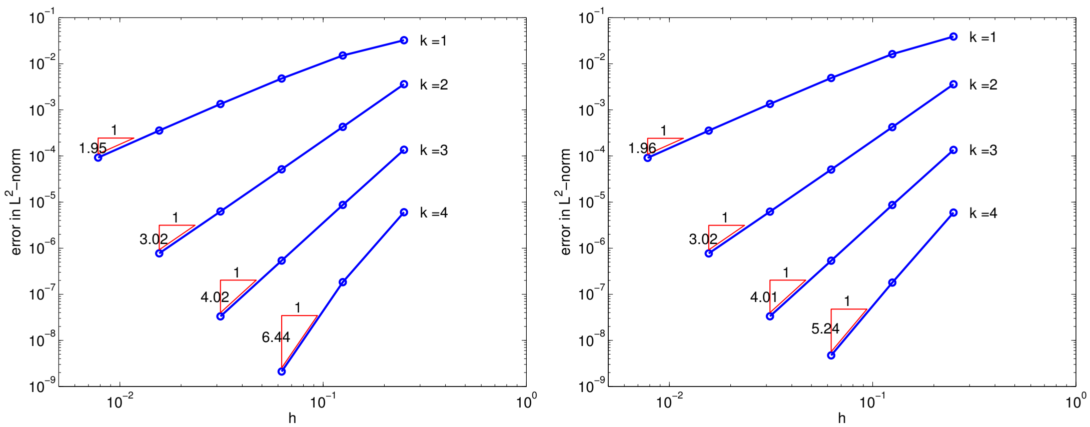
If we choose a broken test space $\left\{v\in L^2(\Omega), v|_K \in V(K)\right\}$, we can enforce continuity between elements by penalizing the jumps of $e$ using a Lagrange multiplier method

$$\begin{aligned}(e,v)_V + b(u,v) + \langle\lambda, [\![v]\!]\rangle_{\Gamma_h} &= l(v)\\ b(\delta u, e) + \langle\delta\lambda, [\![e]\!]\rangle_{\Gamma_h} &= 0.\end{aligned}$$

where $\lambda$ is a function with support only on $\Gamma_h$, the mesh skeleton. This is equivalent to the mixed DPG formulation under the bilinear form $b_h((u,\widehat{f}_n),v) \coloneqq b(u,v) + \left\langle\widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_h}$, with additional trace unknowns $\widehat{f}_n$. The advantage of this formulation is that the degrees of freedom for $e$ can be condensed out and eliminated, leaving a positive definite system for $u$ and $\widehat{f}_n$, and is equivalent to locally computing optimal test functions. This idea for the discretization of $e$ is very similar to the primal hybrid finite element method introduced in [2].

**2. Initial numerical results.** We refer to the mixed formulation of the DPG method using $C_0$ test spaces as the $C_0$ mixed (DPG) method. The formulation with Lagrange multipliers as unknowns on the mesh skeleton will be referred to as the primal (DPG) method.

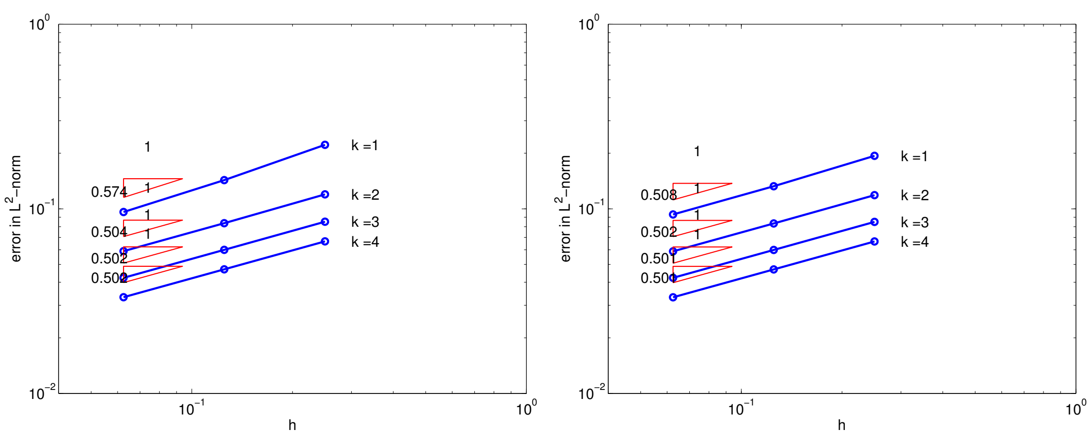
**2.1. Erikkson-Johnson problem.** Both the primal and original $C_0$ mixed DPG methods for convection-diffusion display optimal rates, and display nearly identical $L^2$ errors on the same mesh. Below are comparisons of $L^2$-errors under uniform refinement. The flux variable $\widehat{f}_n$ is taken to be the trace of Raviart-Thomas elements of equal order to the field variables $u$ (this can be relaxed to be $p-1$, where $p$ is the order of $u$).




(a) Primal DPG

(b) Mixed $C_0$ DPG

FIG. 2.1. *Rates for primal and $C_0$ mixed DPG method for $\epsilon = 1.0$.*




(a) Primal DPG

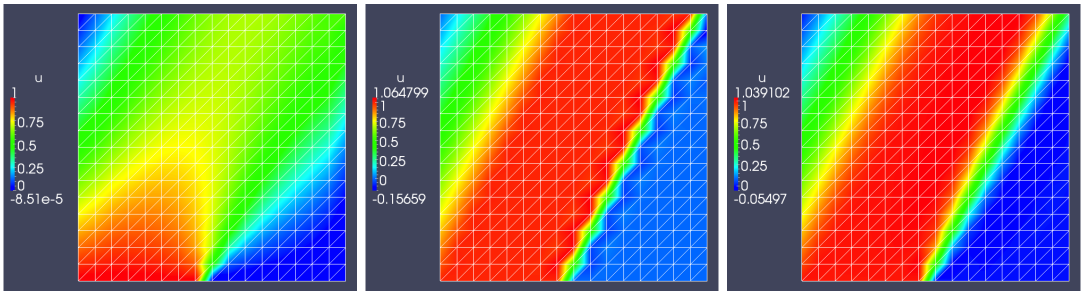
(b) Mixed $C_0$ DPG

FIG. 2.2. *Rates for primal and $C_0$ mixed DPG methods for $\epsilon = 1e-4$*

**2.2. Crosswind diffusion.** The primal DPG method appears to suffer from over-diffusion in the crosswind direction. We illustrate this by using a model problem for pure convection ($\epsilon = 0$), where $\beta = [.5, 1]$ and $f = 0$. Boundary conditions are set such that

$$u = \begin{cases} 1 \text{ if } y = 0, x < .5 \\ 1 - y \text{ if } x = 0 \end{cases}$$

The discontinuous inflow is convected in a non-diffusive manner across the domain. Figure 2.3 shows the result of varying choices of $p_f$, the order of the flux variable, and $\triangle p$, the degree




(a) $p = 1$, $p_f = 1$, $\triangle p = 1$ (b) $p = 1$, $p_f = 2$, $\triangle p = 1$ (c) $p = 1$, $p_f = 2$, $\triangle p = 2$

FIG. 2.3. *Comparisons of solutions for the primal DPG method for pure convection with varying orders for fluxes and test space.*

of enrichment in the space. Note that, for $p_f = p + \triangle p$, we recover exactly a $C_0$-conforming discretization of the test space (simply note that the error orthogonality condition implies that $\left\langle \delta\widehat{f}_n, [\![e]\!] \right\rangle_{\Gamma_h^0} = 0$. Choosing $\delta\widehat{f}_n = [\![e]\!]$ shows that $[\![e]\!] = 0$). Notice also that, if we approximate optimal test functions with an norder $p+2$ polynomial space and the order of the flux variable is only $p+1$, we still end up with an overly diffusive method.

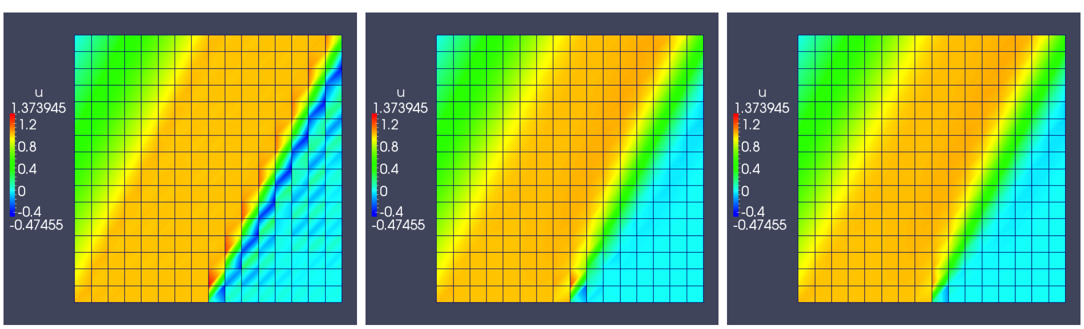
We believe this to be related to the fact that, for $\epsilon = 0$, the test norm becomes

$$\|v\|_V^2 \coloneqq \|\beta\nabla v\|^2 + \|v\|^2$$

which does not yield a well-defined trace when $\beta_n = 0$ (in the cross-stream directions). This is further confirmed when comparing the primal hybrid method of [2] to a standard $C_0$ finite element method for a manufactured solution of $(v, \delta v)_V = (f, v)$; as $\epsilon \to 0$, the error in the primal hybrid solution becomes much larger (two orders of magnitude larger for $p = 3$) than the error in the $C_0$ solution.

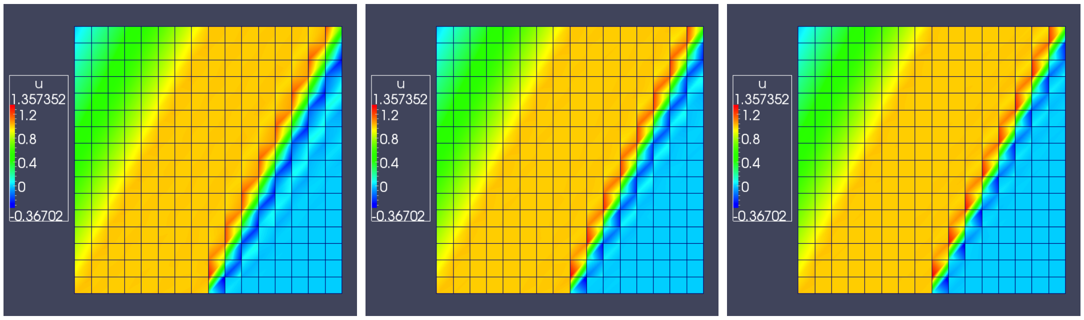
**2.3. DPG with the ultra-weak formulation.** The DPG method with ultra-weak variational formulation [3, 4] behaves very similarly. We take again the pure convection equation and discretize it with $u \in L^2(\Omega)$ instead of $u \in C_0(\Omega)$. The test norm and variational form are identical to the previous example. The problem is implemented in Camellia [5].

Figures 2.4 and 2.5 illustrate the effect of increasing the degree of enrichment of the test space. In Figure 2.4, the field and flux variable are set to a fixed polynomial order while the test space order is increased, which, as observed previously, produces an overly diffusive solution. In Figure 2.5, the flux polynomial order is increased along with the test space order, and produces minimally diffusive solutions that approach the $L^2$ best approximation as $\triangle p$ increases.




(a) $p = 1$, $p_f = 1$, $\triangle p = 1$ (b) $p = 1$, $p_f = 1$, $\triangle p = 2$ (c) $p = 1$, $p_f = 1$, $\triangle p = 3$

FIG. 2.4. *Comparisons of solutions ultra-weak DPG for pure convection with fixed flux order and increasing test space order. Image colors are scaled to the solution for $\triangle p = 1$.*




(a) $p = 1$, $p_f = 2$, $\triangle p = 1$ (b) $p = 1$, $p_f = 3$, $\triangle p = 2$ (c) $p = 1$, $p_f = 4$, $\triangle p = 3$

Fig. 2.5. *Comparisons of solutions for ultra-weak DPG for pure convection with increasing flux and test space order. Image colors are scaled to the solution for $\triangle p = 1$.*

**3. Conclusions.** These three numerical experiments appear to imply that, for the DPG method with optimal test functions, setting $p_f = p$ may control error along the streamline in a very strong manner, but that $p_f = p + \triangle p$ is required for control over error in the cross-stream direction.

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