Our original variational formulation for the convection-diffusion problem is

$$b(u,v) = \langle \beta_n u, v\rangle_\Gamma + (u, -\beta\nabla v) - \epsilon \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_{in}} + \epsilon \left(\nabla u, \nabla v\right)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}, \quad \forall v \in H^1_{\text{out}}$$

where

$$\Gamma_{\text{in}} \coloneqq \{x \in \partial\Omega : \beta \cdot \vec{n}(x) < 0\}$$
$$\Gamma_{\text{out}} \coloneqq \{x \in \partial\Omega : \beta \cdot \vec{n}(x) > 0\}.$$

Boundary conditions can be applied in either a strong or weak fashion in this formulation.

This formulation is derived under the assumption that v is C_0 continuous throughout the domain. However, if we break that continuity and require only that $v \in H^1(K)$ (i.e. in H^1 locally), we can derive a different variational formulation. Integrating by parts, we pick up a boundary term over each element, such that our variational form becomes

$$\begin{aligned} b\left(\left(u, \widehat{f}_n\right), v\right) &= \langle \beta_n u, v\rangle_\Gamma + (u, -\beta\nabla v) - \left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_h^0} - \epsilon \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma} + \epsilon \left(\nabla u, \nabla v\right)_{L^2(\Omega)} \\ &= (f,v)_{L^2(\Omega)}, \quad \forall v \in H^1_{\text{out}}. \end{aligned}$$

where we have identified the viscous fluxes $\epsilon\frac{\partial u}{\partial n}$ on the interior skeleton Γ_h^0 as additional unknowns $\widehat{f}_n$.

1. Interpretation as a nonconforming method for e. The abstract mixed form of the DPG method with $(e,v) \in V$ and $(u, du) \in U$ is given as

$$(e,v)_V + b(u,v) = l(v)$$
$$b(\delta u, e) = 0.$$

This is equivalent to a constrained projection/Riesz inversion

$$(e,v)_V = l(v), \quad (e,v) \in V \cap \text{null}(B^T)$$

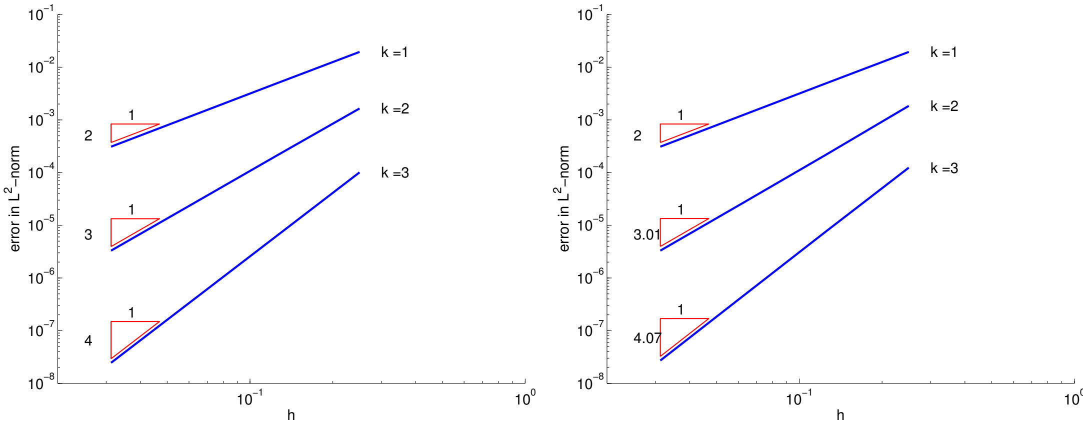
where $B : U \to V'$ is defined through $\langle Bu, v\rangle = b(u,v)$.

If we choose a broken test space $\left\{v \in L^2(\Omega), v|_K \in V(K)\right\}$, we can enforce continuity between elements by penalizing the jumps of e using a Lagrange multiplier method

$$(e,v)_V + b(u,v) + \langle \lambda, [\![v]\!]\rangle_{\Gamma_h} = l(v)$$
$$b(\delta u, e) + \langle \delta\lambda, [\![e]\!]\rangle_{\Gamma_h} = 0.$$

where λ is a function with support only on Γ_h, the mesh skeleton. This is equivalent to the mixed DPG formulation under the bilinear form $b_h((u, \widehat{f}_n), v) \coloneqq b(u,v) + \left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_h}$, with additional trace unknowns $\widehat{f}_n$. The advantage of this formulation is that the degrees of freedom for e can be condensed out and eliminated, leaving a positive definite system for u and $\widehat{f}_n$, and is equivalent to locally computing optimal test functions.

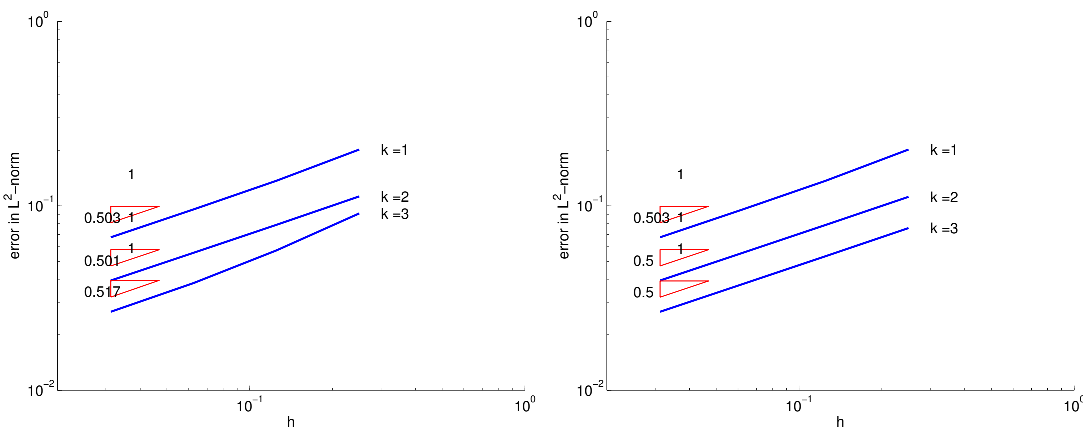
2. Initial numerical results. Both the primal and original C_0 mixed DPG methods for convection-diffusion display optimal rates, and display nearly identical L^2 errors on the same mesh. Below are comparisons of L^2-errors under uniform refinement. The flux variable $\widehat{f}_n$ is taken to be the trace of Raviart-Thomas elements of equal order to the field variables u (this can be relaxed to be $p-1$, where p is the order of u).



(a) Primal DPG

(b) Mixed C_0 DPG

FIG. 2.1. *Rates for primal and C_0 mixed DPG method for $\epsilon = 1.0$.*



(a) Primal DPG

(b) Mixed C_0 DPG

FIG. 2.2. *Rates for primal and C_0 mixed DPG methods for $\epsilon = 1e - 4$*