NOTES ON C_0 DPG METHODS

JESSE CHAN, JOHN A. EVANS

1. Uzawa vs matrix-free methods. We have a saddle point problem

$$\left[\begin{array}{cc} R_V & B \\ B^T & 0 \end{array}\right]\left[\begin{array}{c} e \\ u \end{array}\right] = \left[\begin{array}{c} f \\ 0 \end{array}\right]$$

Assumptions:

- We want to avoid solving the saddle point problem (indefinite, large)
- We cannot form the Schur complement

  $$S = B^T R_V^{-1} B$$

  as it's completely dense due to the globally coupled nature of R_V^{-1}.

The two methods left are matrix-free methods for solving $Su = f$ and inexact Uzawa algorithms for the saddle point problem. Each method relies heavily on the assumption that R_V is easily invertible (the coercive and positive-definite nature of the operator should lend itself well to fast CG and multigrid-based solvers).

The inexact Uzawa method is for the DPG saddle point problem

$$\begin{aligned} e_{i+1} &= e_i + \tilde{R_V}^{-1}\left(f - R_V e_i + B u_i\right) \\ u_{i+1} &= u_i + \tilde{S}^{-1}\left(B^T e\right) \end{aligned}$$

where $\tilde{S}^{-1}$ and $\tilde{R_V}^{-1}$ are meant to approximate the inverses of R_V and S. We assume for now that $\tilde{R_V}^{-1}$ is computable exactly, which gives us

$$\begin{aligned} e_{i+1} &= R_V^{-1}(f - B u_i) \\ u_{i+1} &= u_i + \tilde{S}^{-1}\left(B^T e_i\right). \end{aligned}$$

It can be shown that for the Uzawa iteration error $E_i = u - u_i$,

$$E_{i+1} = \left(I - \tilde{S}^{-1} B^T R_V^{-1} B\right) E_i.$$

If $\tilde{S}^{-1}$ is computed exactly as S^{-1}, the Uzawa method essentially reduces down to a matrix-free method, where the difficulty of the method becomes in preconditioning S effectively. The benefit of the Uzawa algorithm compared to matrix-free methods is the relaxation of the preconditioning required on $\tilde{S}$, in return for more iterations of the Uzawa algorithm loop.