NOTES ON A NONCONFORMING FEM

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1. A hybrid nonconforming method. Consider a coercive variational formulation

$$a(u,v) = l(v), \quad \forall v \in V.$$

We assume further that $a(u,v)$ is coercive when restricted to a single element $u, v \in V(K)$. Standard methods approximate u using a conforming subspace of V. If we choose instead a broken test space

$$V_h \coloneqq \left\{ v \in L^2(\Omega), \left. v \right|_K \in V(K) \right\},$$

we can enforce continuity between elements by penalizing the jumps of e using a Lagrange multiplier method

$$\begin{aligned} a_h(u,v) + \langle \lambda, [\![v]\!] \rangle_{\Gamma_h} &= l(v) \\ \langle \delta\lambda, [\![u]\!] \rangle_{\Gamma_h} &= 0. \end{aligned}$$

where $a_h(u,v) = \sum a_K(u,v)$, and λ is a function with support only on Γ_h, the mesh skeleton. The advantage of this formulation is that the degrees of freedom for e can be condensed out and eliminated, leaving a positive definite system for λ.

2. Stability analysis. We define norms on our variables u and λ:

$$\begin{aligned} \|u\|_{\Omega_h}^2 &\coloneqq \sum_{K \in \Omega_h} a_K(u,u) \\ \|\lambda\|_{\Gamma_h} &\coloneqq \min_{q \in Q, \gamma_Q(q)_{\Gamma_h} = \lambda} \|q\|_Q \,. \end{aligned}$$

where λ is discretized as the trace of some space Q with trace $\gamma_Q(q)$, such that $\gamma_Q(q)$ is dual to $\gamma_V(u)$s, the trace of u, on Γ_h. We also define an auxiliary norm on the jumps of functions in V_h

$$\|[\![v]\!]\|_{\Gamma_h} \coloneqq \sup_{\lambda} \frac{\langle \lambda, [\![v]\!] \rangle_{\Gamma_h}}{\|\lambda\|_{\Gamma_h}}$$

Since the condensed system is equivalent to the mixed system, we can use Brezzi mixed theory to analyze the stability of this method. We require two conditions

- Inf-sup relating the two spaces

$$\inf \sup \frac{\langle \lambda, [\![u]\!] \rangle_{\Gamma_h}}{\|\lambda\|_{\Gamma_h} \|u\|_{\Omega_h}} \geq \gamma_0 > 0,$$

- Inf-sup in the kernel: for $u_0 \in U_0 \coloneqq \left\{ u \in V : \langle \lambda, [\![u]\!] \rangle_{\Gamma_h} = 0 \right\}$

$$\inf \sup \frac{a_h(u_0, v)}{\|u_0\|_{\Omega_h} \|v\|_{\Omega_h}} \geq \gamma_0 > 0,$$

The second condition holds trivially due to coercivity of $a_h(u,v)$ on K, while the first condition reduces to

$$\|[\![u]\!]\|_{\Gamma_h} \geq \gamma_0 \|u\|_{\Omega_h} \,.$$

In other words, the jumps of u should be bounded from below by the nonconforming broken norm.

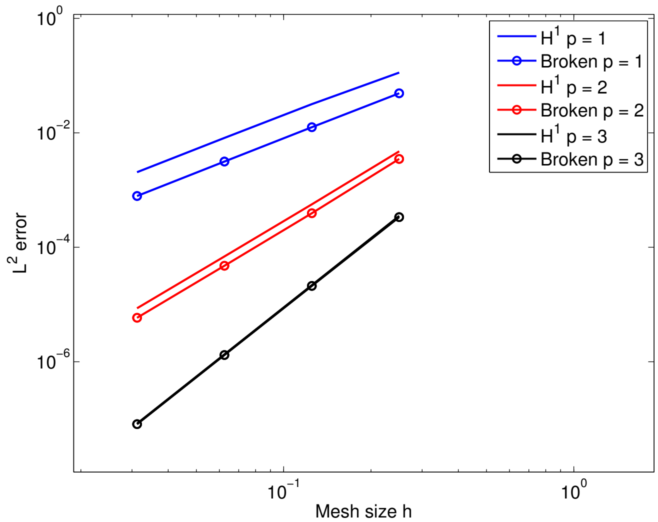

FIG. 2.1. *L^2 errors for conforming vs nonconforming discretizations.*

2.1. Example: Poisson's equation with first order term. We consider the equation with $\alpha > 0$

$$-\Delta u + \alpha u = f$$

which gives the following conforming variational formulation

$$a(u, v) := (\nabla u, \nabla v)_\Omega + \alpha \, (u, v)_\Omega = (f, v)_\Omega \, .$$

The nonconforming formulation is given by

$$a_h(u, \lambda, v) := \sum_{K \in \Omega_h} \left[a_K(u, v) + \langle \lambda, v \rangle_{\partial K} \right] = \sum_{K \in \Omega_h} a_K(u, v) + \langle \lambda, [\![v]\!] \rangle_{\Gamma_h}$$

In this specific case, λ is the trace of Raviart-Thomas elements, with corresponding norm

$$\|\lambda\|_{\Gamma_h} := \min_{q \in H(\mathrm{div}),\, q \cdot n|_{\Gamma_h} = \lambda} \|q\|_{H(\mathrm{div})} \, .$$

The first condition then requires only a mesh-independent Poincare inequality for broken H^1 functions, given in Lemma 4.2 of [1].

$$\|v\|_{\Omega_h} \leq C \left(\|\nabla v\|_{\Omega_h} + \|[\![v]\!]\|_{\Gamma_h} \right)$$

Both the primal and original C_0 mixed DPG methods for convection-diffusion display optimal L^2 error convergence rates of h^{p+1}, and display nearly identical L^2 errors on the same mesh, though the nonconforming discretization delivers a noticably smaller L^2 error for $p = 1$. Below are comparisons of L^2-errors under uniform refinement. The flux variable λ is taken to be the trace of Raviart-Thomas elements of equal order to the field variables u (which gives a polynomial degree of $p - 1$ on the edge). If we decrease our polynomial degree to be Raviart-Thomas elements of order $p - 1$ instead of p, the rate of convergence falls to h^p for L^2 errors.[1]

2.2. Example: Anisotropic Poisson's equation with first order term. We consider the equation with $\alpha > 0$

$$-\nabla \cdot (K \nabla u) + \alpha u = f$$

[1]In general, the optimal rate of convergence appears to be limited to $p_f + 1$.

Order	Conforming	Non-conforming
$p = 1$	1.98898	2.00114
$p = 2$	3.00635	3.01681
$p = 3$	4.01242	4.00886

(a) $p_f = p$

Order	Non-conforming
$p = 1$	N/A
$p = 2$	2.00317
$p = 3$	2.98382

(b) $p_f = p - 1$

FIG. 2.2. *Rates of convergence for conforming vs nonconforming discretizations.*

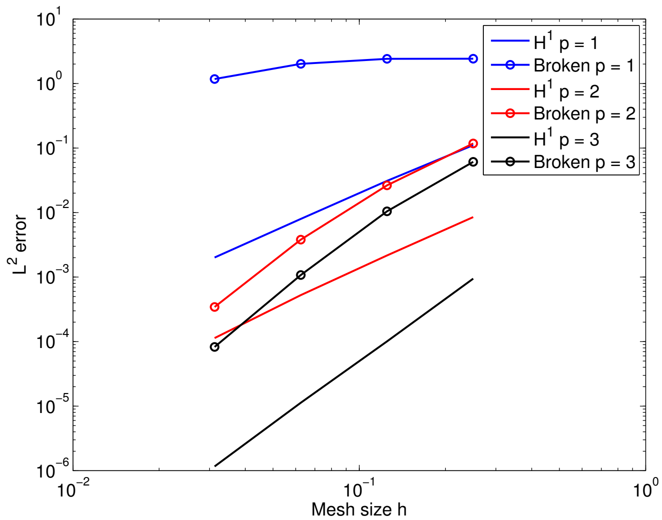

FIG. 2.3. *L^2 errors for conforming vs nonconforming discretizations with $\epsilon = 10^{-4}$.*

for the specific example

$$K = \beta\beta^T + \epsilon I$$

with $\epsilon \geq 0$. The conforming and nonconforming formulations are given similarly to before, with λ as the trace of Raviart-Thomas elements. We note that

$$\|u\|_{\Omega_h}^2 = \|\beta \cdot \nabla u\|^2 + \epsilon \|\nabla u\|^2 + \|u\|^2$$

which implies that, as $\epsilon \to 0$, the trace of u on element boundaries where $\beta_n = 0$ becomes ill-defined.

The first stability condition implied by the Poincare inequality on a nonconforming space is dependent on the constant C. It's possible to derive a similar Poincare inequality for $\|u\|_{\Omega_h}$, though the constant will likely be proportional to ϵ^{-1}, which causes the degradation seen in Figure 2.3 and Table 2.4. We observe non-optimal rates of convergence for both the conforming and nonconforming method, but the L^2 error in the nonconforming method is much higher.

REMARK 1. *To Fred: I believe this poor performance when ϵ is small is responsible for the loss of streamline diffusion stability. I'm not sure if DPG with the ultra-weak formulation delivers better: we tried a second Erikkson-Johnson problem to test the behavior of DPG with respect to a near-internal layer (see [2, 3] for details - we approximate a discontinuity with 25 terms of a Fourier series and used that inflow data to drive the Erikkson-Johnson problem). It appears to behave pretty robustly for that problem over a range of ϵ as well, though the fact that it was only a "near-discontinuity" may have helped.*

REFERENCES

[1] L. Demkowicz and J. Gopalakrishnan. Analysis of the DPG method for the Poisson equation. *SIAM J. Numer. Anal.*, 49(5):1788–1809, September 2011.

Order	Conforming	Non-conforming
$p = 1$	1.98810	0.78235
$p = 2$	2.21456	3.46313
$p = 3$	3.28074	3.70192

FIG. 2.4. *Rates of convergence for conforming vs nonconforming discretizations for* $\epsilon = 10^{-4}$.

[2] J. Chan, N. Heuer, T. Bui-Thanh, and L. Demkowicz. Robust DPG method for convection-dominated diffusion problems II: A natural inflow condition. Technical Report 21, ICES, June 2012. submitted to Comput. Math. Appl.

[3] L. Demkowicz and N. Heuer. Robust DPG method for convection-dominated diffusion problems. Technical Report 11-33, ICES, 2011.