1 Weak boundary conditions

We can derive weak boundary conditions for the convection-diffusion equation. The motivation is that, as $\epsilon \to 0$, the problem should converge to the pure convection problem, which has boundary conditions $u|_{\Gamma_{\text{in}}} = u_{\text{in}}$ and $e|_{\Gamma_{\text{out}}} = 0$. We would like our variational formulation for convection-diffusion problem to converge to these spaces as $\epsilon \to 0$.

For standard convection-diffusion, we have

$$b(u,v) = (-u, \beta \cdot \nabla v)_{L^2(\Omega)} + \epsilon (\nabla u, \nabla v)_{L^2(\Omega)} - \epsilon \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_{\text{in}}}.$$

Imposition of weak boundary conditions is similar to Nitsche's method for weak boundary condition imposition, though it is lacking the penalty term and mesh-dependent parameters found in his method.

$$b(u,v) = (-u, \beta \cdot \nabla v)_{L^2(\Omega)} + \epsilon (\nabla u, \nabla v)_{L^2(\Omega)} - \epsilon \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_{\text{in}}} - \epsilon \left\langle u, \frac{\partial v}{\partial n} \right\rangle_{\Gamma_{\text{out}}}.$$

The term $\epsilon \left\langle u, \frac{\partial v}{\partial n} \right\rangle_{\Gamma_{\text{out}}}$ weakly enforces the outflow condition on u, similarly to the way in which the term $\epsilon \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_{\text{in}}}$ enforces a weak boundary condition on the test space.

2 Deriving the trial norm

Strong convection-diffusion:

$$\nabla \cdot \beta u + \epsilon \triangle u = f$$

Weak form: $v \in H^1_{\text{out}}$

$$b(u,v) = (-u, \beta \cdot \nabla v)_{L^2(\Omega)} + \epsilon (\nabla u, \nabla v)_{L^2(\Omega)} - \epsilon \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_{\text{in}}}$$

Test norm:

$$\|v\|_V^2 = \|\beta \cdot \nabla v\|^2_{L^2(\Omega)} + \epsilon \|\nabla v\|^2_{L^2(\Omega)} + \epsilon \|v\|^2_{\Gamma_{\text{in}}}$$

Cauchy-Schwarz gives

$$|b(u,v)| \leq \|u\| \|\beta \cdot \nabla v\| + \epsilon \|\nabla u\| \|\nabla v\| + \epsilon \left\| \frac{\partial u}{\partial n} \right\|_{\Gamma_{\text{in}}} \|v\|_{\Gamma_{\text{in}}}.$$

Equality holds if each of the paired normed terms are equal. This may not be possible given the trace terms — the supremum of the trace term may not be representable using our trace spaces.

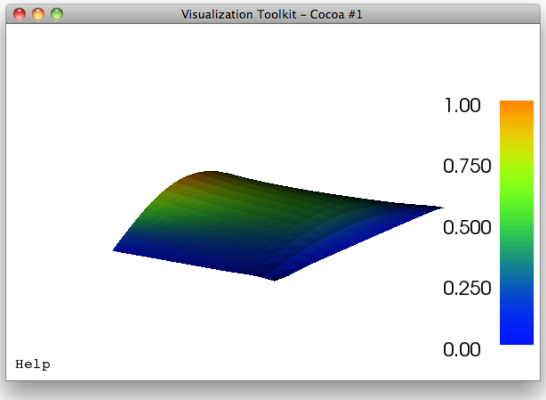
Discrete Cauchy-Schwarz gives

$$|b(u,v)| \leq \left(\|u\|^2 + \epsilon \|\nabla u\|^2 + \epsilon \left\| \frac{\partial u}{\partial n} \right\|_{\Gamma_{\text{in}}} \right)^{1/2} \left(\|\beta \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \epsilon \|v\|_{\Gamma_{\text{in}}} \right)^{1/2}$$

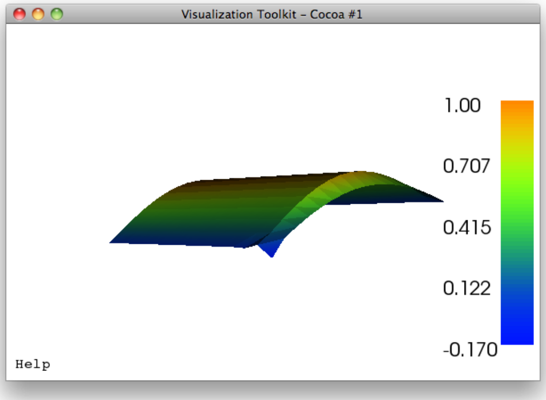
Equality here is easier to show.

If equality of the first Cauchy-Schwarz inequality holds, would it be correct then to claim

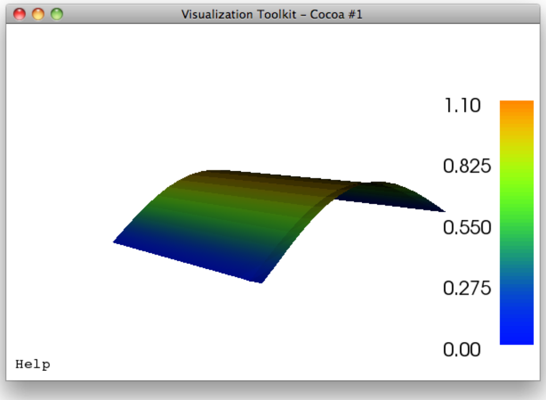
$$\sup_{v \in H^1_{\text{out}} \setminus \{0\}} \frac{b(u,v)}{\|v\|_V} = \left(\|u\|^2 + \epsilon \|\nabla u\|^2 + \epsilon \left\| \frac{\partial u}{\partial n} \right\|_{\Gamma_{\text{in}}} \right)^{1/2} ?$$



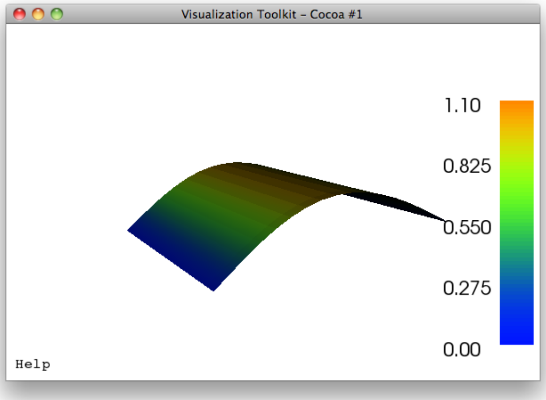
(a) $\epsilon = .1$



(b) $\epsilon = .01$



(c) $\epsilon = .001$



(d) $\epsilon = .0001$