BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH97095

MATH97095 (Solutions)

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1. (a) Label the triangle vertices v_1, v_2, v_3, with edges Π_1 joining v_1 and v_2, Π_2 joining v_2 and v_3, and Π_3 joining v_3 and v_1. Let L_i be the non-degenerate affine function that vanishes on Π_i, for $i = 1, 2, 3$. Now let v be the quadratic polynomial that vanishes on all of the vertices and edge centres. If we can show that $v \equiv 0$, then we can conclude that $\mathcal{N}$ determines P. Consider v restricted to Π_1. Since v vanishes at 3 points on Π_1 (two vertices and an edge centre), then by the fundamental theorem of algebra, v vanishes on the entire of Π_1. Hence, $v(x) = L_1(x)Q_1(x)$ where Q_1 is a polynomial of degree at most 1. Similarly, v vanishes everywhere on Π_2, so Q_1 must vanish everywhere on Π_2, except possibly where Π_2 intersects Π_1, since $L_1(x)$ is zero there. However, Q_1 is continuous so must vanish everywhere on Π_2. Hence, $Q_1 = L_2 c$, where c is a constant, and so $v = L_1 L_2 c$. Neither L_1 nor L_2 vanish on the edge centre of Π_3, but v vanishes there, so c must be zero, and so $v \equiv 0$ as required. seen ⇓ 7, A

(b) To each vertex we associate the nodal variable corresponding to point evaluation at that vertex. To each edge we associate the nodal variable corresponding to point evaluation at the centre of that edge. sim. seen ⇓

To check it is C^0, we need to check that we can (a) recover the value of the function at each vertex using nodal variables from that vertex, and (b) recover the value of the function at each edge using nodal variables from the closure of that edge. Showing (a) is immediate. For (b), on the closure of each edge we have two vertex values and an edge value. That is enough to recover the value of the quadratic function restricted to the edge. 7, A

(c) To give a counter example, consider the unit square subdivided into two triangles by the diagonal line $x = y$. We consider the finite element function equal to $x - y$ in the top-left triangle, and equal to zero in the bottom right. This is a piecewise polynomial of degree at most 2 (even though it is only degree 1). The derivative is equal to $(1, -1)$ in the top-left and $(0, 0)$ in the bottom right, and hence it is discontinuous. Hence, the function is only in C^0 and not in C^1, as required. unseen ⇓ 6, A

2. (a) Multiplication by test function, and integration by parts gives

sim. seen ⇓

$$\int_{\Omega} \epsilon uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x - \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}\,S = \int_{\Omega} v\exp(xy)\,\mathrm{d}\,x. \tag{1}$$

Then, application of the boundary condition leads to the variational problem: find $u \in V$ such that

$$a(u, v) := \int_{\Omega} \epsilon uv + \nabla u \cdot \nabla v \,\mathrm{d}\,x = F(v) := \int_{\Omega} v\exp(xy)\,\mathrm{d}\,x, \quad \forall v \in V, \tag{2}$$

where $V \subset H^1$ is a continuous finite element space.

6, A

(b) The Schwarz inequality gives

sim. seen ⇓

$$a(u, v) \leq \epsilon\|u\|_{L^2}\|v\|_{L^2} + |u|_{H^1}|v|_{H^1}, \tag{3}$$

$$\leq \|u\|_{L^2}\|v\|_{L^2} + |u|_{H^1}|v|_{H^1}, \tag{4}$$

$$\leq 2\|u\|_{H^1}\|v\|_{H^1}, \tag{5}$$

so the continuity constant is bounded above by 2 (it is actually 1 from Young's inequality), but we don't require sharper estimates here.

For coercivity,

$$u(v, v) = \epsilon\|v\|_{L^2}^2 + |v|_{H^1}^2 \geq \epsilon\|v\|_{H^1}^2, \tag{6}$$

so the continuity constant is ϵ.

6, A

(c) Combining Ceá's lemma and the interpolation error estimate, we get

unseen ⇓

$$\|u - u_h\|_{H^1} \leq \frac{2}{\epsilon}h\|u\|_{H^2}. \tag{7}$$

As $\epsilon \to 0$, the error estimate becomes unbounded.

8, B

3. (a) We work with the geometric decomposition, assigning the two tangential components on each edge to that edge. Then, we require that neighbouring cells have the same nodal variable values from that edge (taking care of orientation). Since the tangential component of the function is a linear scalar function restricted to the edge, and the nodal variables give two values of that function on the edge, the tangential component is determined from those values and will be continuous across the edge. unseen ⇓ 6, C

(b) If we have $\phi \in P_2$, then ϕ is continuous across an edge. Therefore the tangential component of the derivative is the same on either side of the edge. Further, the derivative is a linear vector field, so $\nabla\phi \in V$. unseen ⇓ 8, D

(c) For $u \in V$, the weak curl is $Du \in L^2$ such that sim. seen ⇓

$$Du|_K = \nabla^\perp \cdot u|_K, \tag{8}$$

for each triangle K in the mesh. To check that this is indeed the weak curl, take a C_0^∞ function (infinitely differentiable and all derivatives vanish on the boundary) ϕ, and compute

$$-\int_\Omega \phi Du \,\mathrm{d}\,x = -\sum_K \int_K \phi Du \,\mathrm{d}\,x, \tag{9}$$

$$= \sum_K \int_K \phi\nabla^\perp \cdot u \,\mathrm{d}\,x - \sum_K \int_{\partial K} \phi u \cdot n_K^\perp \,\mathrm{d}\,S, \tag{10}$$

$$= \int_\Omega \phi\nabla^\perp \cdot u \,\mathrm{d}\,x - \sum_K \int_{\partial K} \phi(u^+ \cdot (n^+)^\perp + u^- \cdot (n^-)^\perp) \,\mathrm{d}\,S, \tag{11}$$

$$= \int_\Omega \phi\nabla^\perp \cdot u \,\mathrm{d}\,x - \int_{\partial\Omega} \underbrace{\phi = 0}(u^+ \cdot (n^+)^\perp + u^- \cdot (n^-)^\perp) \,\mathrm{d}\,S, \tag{12}$$

$$-\int_\Gamma \phi \underbrace{(u^+ \cdot (n^+)^\perp + u^- \cdot (n^-)^\perp)}_{=0} \,\mathrm{d}\,S, \tag{13}$$

$$= \int_\Omega \phi\nabla^\perp \cdot u \,\mathrm{d}\,x, \tag{14}$$

where n_K is the outward pointing normal to each triangle K, and the facet integrals vanish due to continuity of the tangential component. This matches the definition of the weak curl. 6, C

4. (a) We have

unseen ⇓

$$\int_\Omega u^{k+1}v + \nabla u^{k+1}\cdot\nabla v\,\mathrm{d}\,x = \int_\Omega u^k v + \nabla u^k\cdot\nabla v\,\mathrm{d}\,x + \mu\left(G[v] - \int_\Omega a(x)u^k v + b(x)\nabla u^k\cdot\nabla v\,\mathrm{d}\,x\right), \quad \forall v\in V, \tag{15}$$

Assuming that the limit exists, we take the limit, and get

$$\int_\Omega u^* v + \nabla u^*\cdot\nabla v\,\mathrm{d}\,x = \int_\Omega u^* v + \nabla u^*\cdot\nabla v\,\mathrm{d}\,x + \mu\left(G[v] - \int_\Omega a(x)u^* v + b(x)\nabla u^*\cdot\nabla v\,\mathrm{d}\,x\right), \quad \forall v\in V, \tag{16}$$

which reduces to

$$\mu\left(G[v] - \int_\Omega a(x)u^* v + b(x)\nabla u^*\cdot\nabla v\,\mathrm{d}\,x\right) = 0, \quad \forall v\in V, \tag{17}$$

which implies u^* solves our equation for $\mu > 0$.

6, B

(b) First we use that

unseen ⇓

$$G[v] = \int_\Omega a(x)uv + b(x)\nabla u\cdot\nabla v\,\mathrm{d}\,x, \quad \forall v\in V, \tag{18}$$

so

$$\begin{aligned} &\mu\left(G[v] - \int_\Omega a(x)u^k v + b(x)\nabla u^k\cdot\nabla v\,\mathrm{d}\,x\right)\\ &= \mu\left(\int_\Omega a(x)uv + b(x)\nabla u\cdot\nabla v\,\mathrm{d}\,x - \int_\Omega a(x)u^k v + b(x)\nabla u^k\cdot\nabla v\,\mathrm{d}\,x\right)\\ &= \mu\left(\int_\Omega a(x)(u-u^k)v + b(x)\nabla(u-u^k)\cdot\nabla v\,\mathrm{d}\,x\right), \quad \forall v\in V. \end{aligned} \tag{19}$$

Then we get

$$\int_\Omega \epsilon^{k+1}v + \nabla\epsilon^{k+1}\cdot\nabla v\,\mathrm{d}\,x = \int_\Omega \epsilon^k v + \nabla\epsilon^k\cdot\nabla v\,\mathrm{d}\,x - \mu\int_\Omega a(x)\epsilon^k v + b(x)\nabla\epsilon^k\cdot\nabla v\,\mathrm{d}\,x, \quad \forall v\in V. \tag{20}$$

6, B

(c) Equation (20) is of the form $a(u,v) = F(v)$, with

unseen ⇓

$$a(u,v) = \int_\Omega uv + \nabla u\cdot\nabla v\,\mathrm{d}\,x, \quad F(v) = \int_\Omega \epsilon^k v + \nabla\epsilon^k\cdot\nabla v\,\mathrm{d}\,x - \mu\int_\Omega a(x)\epsilon^k v + b(x)\nabla\epsilon^k\cdot\nabla v\,\mathrm{d}\,x. \tag{21}$$

In this case, $a(\cdot,\cdot)$ *is* the H^1 inner product, so the continuity and coercivity constants are both 1. We have

$$|F(v)| \le (1-\mu\beta)\|\epsilon^k\|_{L^2}\|v\|_{L^2} + (1-\mu\beta)|\epsilon^k|_{H^1}|v|_{L^2} \le 2(1-\mu\beta)\|\epsilon^k\|_{H^1}\|v\|_{H^1}, \tag{22}$$

(sharper estimates are possible but the question doesn't require them). Hence, from Lax-Milgram, we have

$$\|\epsilon^{k+1}\|_{H^1} < (1-2\mu\beta)x\|\epsilon^k\|_{H^1}, \tag{23}$$

and so the norm of the error is guaranteed to reduce if $2\mu\beta < 1$.

8, D

5. (a) unseen ⇓

$$(B^*p)[v] = (Bv)[p] = b(v,p). \tag{24}$$

6, M

(b) We have seen ⇓

$$\|B^*p\|_{V'} = \sup_{0\neq v\in V} \frac{(B^*p)[v]}{\|v\|_V} = \sup_{0\neq v\in V} \frac{b(v,p)}{\|v\|_V} \geq \beta \|p\|_Q, \tag{25}$$

by the inf-sup condition. Assume that there exists p, q such that $B^*p = B^*q$. Linearity means that $B^*(p-q) = 0$. Then,

$$0 = \|B^*(p-q)\|_{V'} \geq \beta\|p-q\|_Q \implies p = q. \tag{26}$$

Hence B^* maps each element of Q to a different element of V', i.e. it is injective.

7, M

(c) $\tilde{\nabla}$ corresponds to the operator B^*, followed by the Riesz map back into V. Since B^* is injective, and the Riesz map is invertible, therefore $\tilde{\nabla}$ is also injective. 7, M

Review of mark distribution:

Total A marks: 32 of 32 marks
Total B marks: 20 of 20 marks
Total C marks: 12 of 12 marks
Total D marks: 16 of 16 marks
Total marks: 100 of 80 marks
Total Mastery marks: 20 of 20 marks