Modularity Examples

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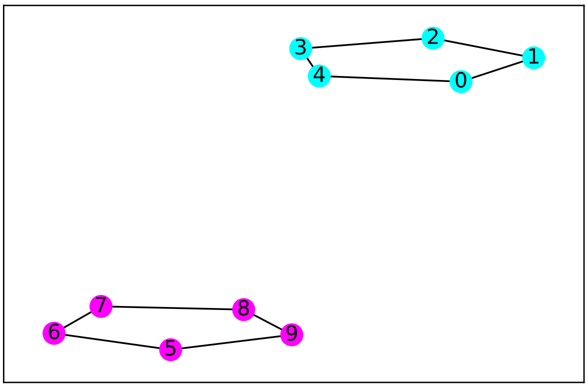
1 Case 1: Two completely disconnected communities

In this first case, we explore how modularity is influenced by community structure exploring an elementary example consisting of two completely disconnected sub-networks of 5 nodes each. These two sub-networks correspond to two communities. We focus on two examples: **Example 1** consists of calculating the modularity when the nodes in sub-networks assigned to their correct communities and **Example 2** consists of calculating the modularity when all the nodes are assigned to a single community **Figure 1**. For all examples, we use the following formulation of the modularity

$$Q = \sum_{k=1}^{K} \frac{\Sigma_{in,k}}{2m} - \left(\frac{\Sigma_{tot,k}}{2m}\right)^2$$

where $\Sigma_{in,k}$ is the total number of edges within the k^{th} community (multiplied by 2), $\sigma_{tot,k}$ is the total number of edges incident to each node in community k, and m is the total number of edges in the network

Figure 1: Case 1 graph



Example 1: Modularity for two communities of 5 nodes

We start by computing the modularity for community one (teal). The structure of each sub-network is identical which greatly simplified the calculations. For community 1, the number of edges within is $\Sigma_{in,1} = 2 \times 5$, the number of incident edges is $\Sigma_{tot,1} = 10$ and $m = 10$. Therefore the modularity of community 1 is

$$Q_1 = \frac{10}{2(10)} - \frac{10^2}{4(100)} = 0.25$$

similarly for community 2 we have

$$Q_2 = \frac{10}{2(10)} - \frac{10^2}{4(100)} = 0.25$$

Therefore the total modularity is the sum of the modularity of each community:

$$Q = \sum_{k=1}^{2} Q_k = 0.5$$

Example 2: Modularity for a single community of 10 nodes

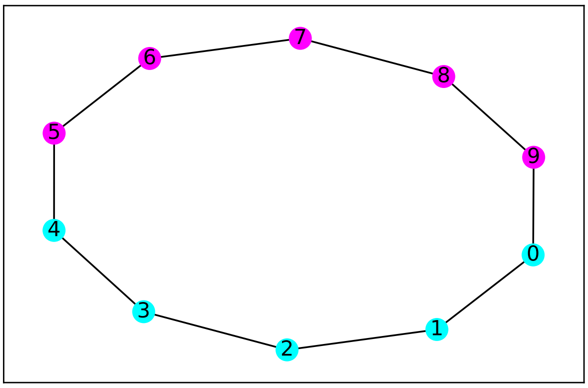
In our second example we compute the modularity when all of the nodes are grouped into a single community. The is gives $\Sigma_{in,1} = 2(10) = 20$, $\Sigma_{tot,1} = 2(10) = 20$, $m = 10$ which gives:

$$Q = \frac{20}{2(10)} - \frac{20^2}{4(100)} = 1 - 1 = 0$$

2 Case 2: A single network divided into 2 communities

In this second case, we explore how modularity is influenced by community structure through another elementary example in which we have a single ring-structured community which is divided evenly into two sub-networks of 5 nodes each **Figure 2**. We again focus on two examples: **Example 1** calculating the modularity when the nodes are divided into two communities and **Example 2** calculating the modularity when all the nodes in the ring are assigned to a single community.

Figure 2: Case 2 graph



Example 1: Modularity for a ring divided into two communities of 5 nodes

We start by computing the individual modularities for each community. Again structure of each sub-network is identical making the calculations simple. For community 1, the number of edges within is $\Sigma_{in,1} = 2 \times 4$, the number of incident edges is $\Sigma_{tot,1} = 10$ and $m = 10$. Therefore the modularity of community 1 is

$$Q_1 = \frac{8}{2(10)} - \frac{10^2}{4(20)} = 0.15$$

similarly for community 2 we have

$$Q_2 = \frac{10}{2(20)} - \frac{10^2}{4(20)} = 0.15$$

Therefore the total modularity is the sum of the modularity of each community:

$$Q = \sum_{k=1}^{2} Q_k = 0.3$$

Example 2: Modularity for all nodes in a single community

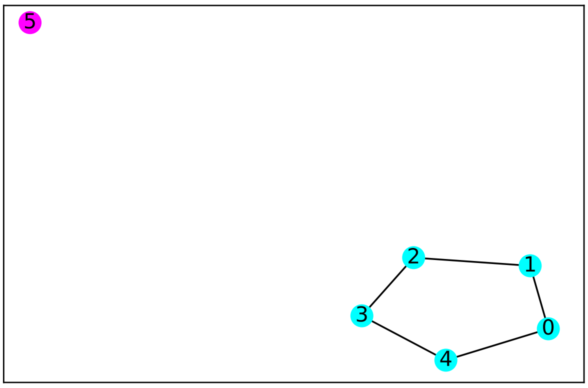
In our second example we compute the modularity when all of the nodes are grouped into a single community. The is gives $\Sigma_{in,1} = 2(10) = 20$, $\Sigma_{tot,1} = 2(10) = 20$, $m = 10$ which gives:

$$Q = \frac{20}{2(10)} - \frac{20^2}{4(100)} = 1 - 1 = 0$$

3 Case 3: The case of a singleton community

In this second example, we explore how modularity is affected by the presence of a singleton community **Figure 3**. In **Example 1** we calculate the modularity when there are two communities where one community consists of a single isolated node (singleton) and the other community consists of a community of 5 nodes. In **Example 2** we expand on **Example 1** by allowing the singleton community to have a single edge connecting it the primary network of 5 nodes **Figure 4**. Lastly, in **Examples 3-4** we calculate the modularity for **Examples 1-2** when grouping all nodes into a single community.

Figure 3: Case 3 graph



Example 1: Modularity for a ring divided into two communities of 5 nodes

We start by computing the individual modularities for each community. For the singleton community, the number of edges within is $\Sigma_{in,1} = 0$, the number of incident edges is $\Sigma_{tot,1} = 5$ and $m = 5$. Therefore the modularity of the singleton community is

$$Q_1 = \frac{0}{2(5)} - \frac{0^2}{4(25)} = 0$$

For the second community of five nodes in a ring structure, we have

$$Q_2 = \frac{10}{2(5)} - \frac{10^2}{4(25)} = 0$$

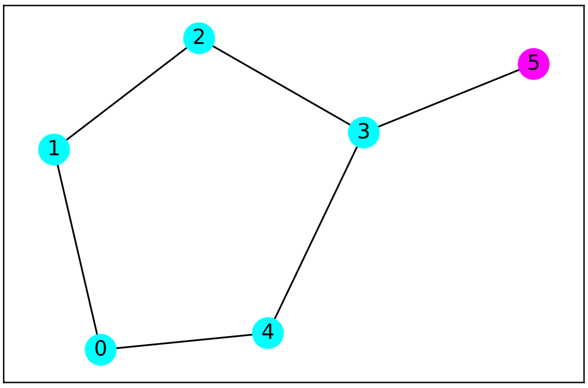
Therefore the total modularity is the sum of the modularity of each community:

$$Q = \sum_{k=1}^{K} Q_k = 0$$

Example 2: Modularity for all nodes in a single community

In our second example we compute the modularity similar to **Example 1** but now the singleton community share a single edge to the other community of give nodes. The choice of which node the edge connects to is arbitrary and does not effect the modularity.

Figure 4: Case 3(b) graph



For the singleton community, the number of edges within is still $\Sigma_{in,1} = 0$, the number of incident edges is now $\Sigma_{tot,1} = 6$ and $m = 6$. Therefore the modularity of the singleton community is

$$Q_1 = \frac{0}{2(6)} - \frac{1^2}{4(36)} = -0.00694$$

The same value as before. For the second community of five nodes in a ring structure, we have

$$Q_2 = \frac{10}{2(6)} - \frac{11^2}{4(100)} = -0.00694$$

Therefore the total modularity is the sum of the modularity of each community:

$$Q = \sum_{k=1}^{K} Q_k = -0.01389$$

Example 3: Example 1 with a single community

In this third example we compute the modularity when all of the nodes from the graph in **Example 1** are grouped into a single community. The is gives $\Sigma_{in} = 2(5) = 10$, $\Sigma_{tot} = 2(5) = 10$, $m = 5$ which gives:

$$Q = \frac{10}{2(5)} - \frac{10^2}{4(25)} = 1 - 1 = 0$$

Example 4: Example 2 with a single community

Lastly, for the fourth example we compute the modularity when all of the nodes from the graph in **Example 2** are grouped into a single community. The is gives $\Sigma_{in} = 2(6) = 12$, $\Sigma_{tot} = 2(6) = 12$, $m = 6$ which gives:

$$Q = \frac{12}{2(6)} - \frac{12^2}{4(36)} = 1 - 1 = 0$$

Case 1 In Linear Form

Please refer to the network in **Figure 1**:

Example 1: Modularity for two communities of 5 nodes

Let $\mathbf{C}$ be the the community assignment matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Let $\mathbf{A}$ be the graph adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Further let $\mathbf{B}$ be the modularity matrix defined as

$$\mathbf{B} = \mathbf{A} - \frac{1}{2m}\mathbf{r} \otimes \mathbf{r}$$

where

$$\mathbf{r} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

therefore

$$\mathbf{r} \otimes \mathbf{r} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}$$

we also have that

$$m = \frac{1}{2} \sum_i \sum_j \mathbf{A}_{i,j} = 10$$

therefore the modularity matrix can be expressed as

$$\mathbf{B} = \begin{bmatrix} -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \end{bmatrix}$$

We compute the modularity using the following equation

$$M = \frac{1}{4m} Tr(\mathbf{C^T B C})$$

where

$$\mathbf{C^T B C} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Therefore the modularity equals

$$M = \frac{1}{40} \times Tr \left(\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \right)$$

$$= \frac{1}{40} \times 10 = 0.25$$

Example 2: Modularity for a single community of 10 nodes

In our second example we compute the modularity when all of the nodes are grouped into a single cluster. The only computation which we much change is $\mathbf{C^T B C}$. In this scenario, the community assignment matrix is as follows

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{C^T B C} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \frac{1}{40} \times 0 = 0$$

A Case 3

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\mathbf{A}$ be the graph adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

References