Modularity Examples

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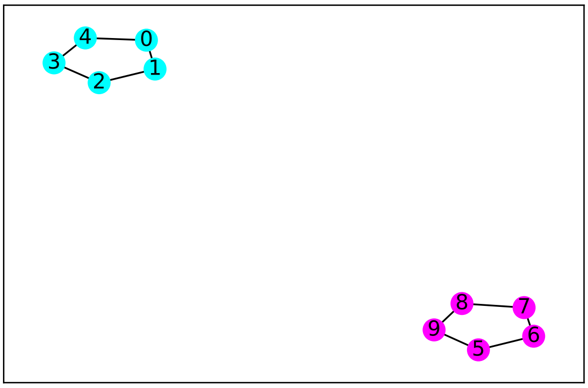
1 The Case of two completely disconnected communities

In this first example, we explore how modularity is affected by exploring the modularity for two completely disconnected sub-networks and comparing when the sub-networks are merged into a single community.

Figure 1: Case 1 graph



Modularity for two communities of 5 nodes

Let **C** be the the community assignment matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Let $\mathbf{A}$ be the graph adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Further let $\mathbf{B}$ be the modularity matrix defined as

$$\mathbf{B} = \mathbf{A} - \frac{1}{2m}\mathbf{r} \otimes \mathbf{r}$$

where

$$\mathbf{r} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

therefore

$$\mathbf{r} \otimes \mathbf{r} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}$$

we also have that

$$m = \frac{1}{2} \sum_i \sum_j \mathbf{A}_{i,j} = 10$$

therefore the modularity matrix can be expressed as

$$\mathbf{B} = \begin{bmatrix} -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \end{bmatrix}$$

We compute the modularity using the following equation

$$M = \frac{1}{4m} Tr(\mathbf{C^T B C})$$

where

$$\mathbf{C^T B C} =$$

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot
\begin{bmatrix}
-\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\
\frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\
-\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\
-\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\
\frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\
-\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} \\
-\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} \\
-\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \\
-\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} \\
-\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20}
\end{bmatrix} \cdot
\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot
\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}
$$

Therefore the modularity equals

$$
M = \frac{1}{40} \times Tr\left(\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \right)
$$

$$
= \frac{1}{40} \times 10 = 0.25
$$

Modularity for a single community of 10 nodes

In our second example we compute the modularity when all of the nodes are grouped into a single cluster. The only computation which we much change is $\mathbf{C^T B C}$. In this scenario, the community assignment matrix is as follows

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{C^TBC} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & \frac{16}{20} \\ -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} & -\frac{4}{20} & \frac{16}{20} & -\frac{4}{20} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

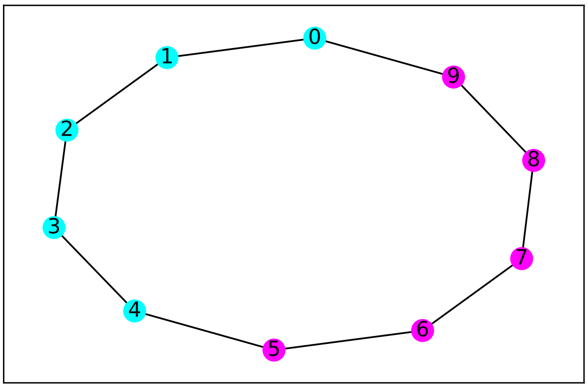
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \frac{1}{40} \times 0 = 0$$

2 The Case of a network divided into 2 communities

Figure 2: Case 1 graph



3 The Case of a singleton community

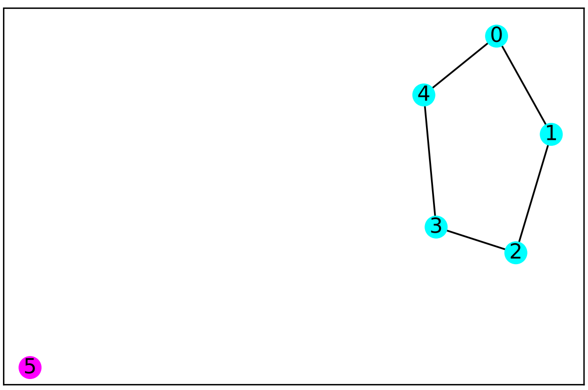
In this second example, we explore how modularity is affected by the presence of a singleton community in a two community graph. Specifically, we explore how modularity changes when from two communities where one community consists of a single isolated node (singleton) and the other community consists of a community of 10 nodes. We two cases for the non-singleton community (i) when all 10-nodes are fully connected vs when all 10 nodes come from a Wattz Strogratz graph with degree 2.

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\mathbf{A}$ be the graph adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3: Case 2 graph



References