Assignment 6

Due date: 1 week after the end of Lesson 8 in class.

Submission Details: You must submit a paper version of your assignment at the beginning of class on the due date and email your code to `louis.saumier@kpu.ca` prior to submitting your paper assignment.

1 [**The Umbrella Problem**] An individual possesses 3 umbrellas that he employs in going from his home to office and vice versa. If he is at home at the beginning of the day and it is raining, then he will take an umbrella with him to the office, provided there is one to be taken. If he is at the office at the end of the day and if it is raining, then he will take an umbrella home, provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (or end) of a day with probability $p = 30\%$. The individual wants to create a model to determine the fraction of time he will get wet.

(a) [**13 pts**] Use the five-step method to answer this question. Use a Markov chain (discrete-time) to model the situation.

(b) [**4 pts**] Compute the sensitivity of your answer to the $p = 30\%$ assumption. Comment on your result.

(c) [**2 pts**] What value of p maximizes the fraction of time he gets wet?

(d) [**4 pts**] Would buying another umbrella reduce the amount of time he gets wet? If so, by how much? Use $p = 30\%$ to answer.

2 [**Drug-Delivery Model**] A pharmaceutical company wants to determine the efficiency of the delivery mechanism of an antibiotic for a specific type of bacteria. This antibiotic is delivered through molecules which have to attach to a site on the surface of the bacteria in order to release the antibiotic into the bacteria. For simplicity, let us call the molecules delivering the antibiotic the acceptable molecules. The surface of a typical bacteria consists of several sites at which foreign molecules—some acceptable and some not—become attached. At a typical site, molecules arrive to the site at a rate of about 30 per second. For a usual dose of this antibiotic given to a patient, acceptable molecules will constitute about 2% of all molecules arriving at the site. Unacceptable molecules stay at the site for about 0.02 seconds and then are ejected, whereas acceptable molecules were designed to stay at the site about four times longer. An arriving molecule will become attached only if the site is free of other molecules.

The pharmaceutical company usually employs two metrics to determine the efficiency of such a delivery system: the percentage of the time a typical site is occupied with an acceptable molecule and the fraction of arriving acceptable molecules that become attached. The company has hired you to estimate these quantities.

(a) [**13 pts**] Use the five-steps method to estimate these quantities. Use a Markov process (continuous-time) for your model.

(b) [**4 pts**] Compute the sensitivity of both your answers in (a) to the rate of 30 arriving molecules per second. Comment on your results.

Assignment # 6

Table of Contents

1. (a) 1
1. (b) 4
1. (c) 5
1. (d) 6
2. (a) 7
2. (b) 9

Jarren Ralf

1. An individual possesses 3 umbrellas that he employs in going from his home to office and vice versa. If he is at home at the beginning of the day and it is raining, then he will take an umbrella with him to the office, provided there is one to be taken. If he is at the office at the end of the day and if it is raining, then he will take an umbrella home, provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (or end) of a day with probability p = 30%. The individual wants to create a model to determine the fraction of time he will get wet.

1. (a)

Use the five-step method to answer this question. Use a Markov chain (discrete-time) to model the situation.

Step 1

Let p_{ij} represent the transition probabilities for going from i umbrellas in one location to j umbrellas in the other. We will also denote this in matrix form by $A = p_{ij}$. We will let π_j be the steady state probability for j umbrellas. Note that the steady state probability is defined as the following:

$\pi_j = \lim_{n \to \infty} p_{ij}^n$ for $j \geq 0$.

- Assumptions -

Assume that the individual in question is a bit of a loser and only goes between work and home. Assume that he only owns 3 umbrellas. Assume that the individual only grabs 1 umbrella (if available) on days that it is raining. Assume that none of his umbrella's are broken so that on days it rains, if he takes an umbrella, he does not get wet. Assume that if the individual leaves work (or home) when it is not raining, that it will not starting raining during his commute, hence catching him without an umbrella and getting him wet. Assume that independent of the past, it rains at the beginning (or end) of a day with probability 30%, i.e. assume the individual lives in Vancouver, BC. Assume that the probability of it raining is independent of the probability of having an umbrella at home or work. We will assume that being wet is binary, i.e. you are either wet or not. We will assume that we are dealing with proper probabilities hence they only take on real number values between 0 and 1 inclusively. We also have the following domain restrictions $X_n = \{0, 1, 2, 3\}$, similiarly, $i, j = \{0, 1, 2, 3\}$ and $n \in N$.

- Equations -

$p_{ij} = P(X_{n+1} = j | X_n = i)$

$p_{12} = p_{21} = p_{30} = 1 - p$

$p_{13} = p_{22} = p_{31} = p$

$p_{03} = 1$

The rest of the entries in the matrix are 0. From this, we can set up our equations for the steady state probabilities.

$\pi_j = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3] \cdot p_{ij}$

$\sum_{i=0}^{3} \pi_i = 1$

$P(isWet) = p \cap \pi_0$

- Objective -

We want to determine the fraction of time that the individual will get wet. In order to find this probability, we need the long term proportion of leaving either the house or work with no umbrella.

Step 2

We will use a discrete time Markov chain to model the situation. We should note that our Markov chain is ergodic, so this in turn ensures that the steady states do indeed exist for this system.

Step 3

We will find the steady state probabilitites in two ways. Firstly we will find them by constructing our state transition matrix and raise it to the power n and find the solution iteratively for a large value of n. Next we will compile our steady state equations from above, by expanding out the matrix, then plug in our probabilities. This leaves us with having to solve the equations below. This should yield the same solution as the iterative process.

$\pi_0 = 0.7\pi_3$

$\pi_1 = 0.7\pi_2 + 0.3\pi_3$

$\pi_2 = 0.7\pi_1 + 0.3\pi_2$

$\pi_3 = \pi_0 + 0.3\pi_1$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Lastly, to answer the question, we will find $P(isWet)$. Recall our assumption that the probability of it raining and the probability of having 0 umbrellas is independent. Thus we will compute

$P(isWet) = 0.3\pi_0$.

Step 4

```
clear; clc; close all;
syms pi_0 pi_1 pi_2 pi_3;
format long

% Set constants
p = .3;

% Construct the transition matrix
A = [  0        0        0        1 ;
       0        0      1 - p      p ;
       0      1 - p      p        0 ;
     1 - p      p        0        0];

% Iterate A for large n
product = A;
for i = 1:1000
    product = product*A;
end
disp(product);

% Build System of Equations
eq1 = pi_0 ==                                        pi_3*(1 - p);
eq2 = pi_1 ==                         pi_2*(1 - p)  + pi_3*p;
eq3 = pi_2 ==        pi_1*(1 - p)  + pi_2*p;
eq4 = pi_3 == pi_0 + pi_1*p;
eq5 =    1 == pi_0 + pi_1          + pi_2           + pi_3;

sol = solve([eq1, eq2, eq3, eq4, eq5], [pi_0 pi_1 pi_2 pi_3]);

disp(eval(sol.pi_0));
disp(eval(sol.pi_1));
disp(eval(sol.pi_2));
disp(eval(sol.pi_3));
```

```
  Columns 1 through 3

   0.189189189189185   0.270270270270265   0.270270270270265
   0.189189189189185   0.270270270270265   0.270270270270265
   0.189189189189186   0.270270270270265   0.270270270270265
   0.189189189189186   0.270270270270265   0.270270270270265

  Column 4

   0.270270270270265
   0.270270270270265
   0.270270270270265
   0.270270270270265

   0.189189189189189
```

```
   0.270270270270270

   0.270270270270270

   0.270270270270270
```

We have found the proper steady state probabilities because both methods produce the same values.

```
isWet = p*sol.pi_0;
disp("The probability of getting wet is:");
disp(isWet);
disp(eval(isWet));
```

```
The probability of getting wet is:
21/370

   0.056756756756757
```

Step 5

We can conclude that the probability of getting wet is $\frac{21}{370}$ or 5.68%. As many people would agree, if you do not have the space to carry around an umbrella, it is quite annoying to carry one when it is not raining. Thus, The strategy of leaving a various amount at work and home is a reasonable one.

1. (b)

Compute the sensitivity of your answer to the p = 30% assumption. Comment on your result.

```
clear; clc; close all;

% Declare p are a symbolic variable instead of a fixed constant in
 order to
% compute the sensitivity
syms p pi_0 pi_1 pi_2 pi_3;

% Build System of Equations
eq1 = pi_0 ==                                        pi_3*(1 - p);
eq2 = pi_1 ==                        pi_2*(1 - p)  + pi_3*p;
eq3 = pi_2 ==        pi_1*(1 - p)  + pi_2*p;
eq4 = pi_3 == pi_0 + pi_1*p;
eq5 =    1 == pi_0 + pi_1          + pi_2          + pi_3;

sol = solve([eq1, eq2, eq3, eq4, eq5], [pi_0 pi_1 pi_2 pi_3]);
```

To compute the sensitivity for this question, we need to take a derivative of our desired function, $isWet$ in terms of the variable p. The sensitivity will be $S(isWet, p) = \frac{p}{isWet}\frac{disWet}{dp}$.

```
isWet = p*sol.pi_0;
% Compute Sensitivity of pi_0
derivative = diff(isWet, p);
```

```
sensitivity_symbolic = (p/isWet)*derivative;
sensitivity = eval(subs(sensitivity_symbolic, p, 0.3));

disp('Sensitivity');
disp(sensitivity);
```

```
Sensitivity
   0.652509652509653
```

This means that for a 1% increase in the probability of it raining, the probability that the individual will get wet increases by 0.6525%. If the probability of it raining increases, I would inform the individual that it is more likely he gets wet, and hence recommend he does not walk if he wants to stay dry. But realistically, a 0.6525% increase is rather insignificant. In conclusion, the probability of getting wet is not very sensitive to our assumption of the probability of rain to be 30%.

1. (c)

What value of p maximizes the fraction of time he gets wet.

```
clear; clc; close all;

% We need to be able to solve for p, thus we make it symbolic again
syms p pi_0 pi_1 pi_2 pi_3;

% Build System of Equations
eq1 = pi_0 ==                                            pi_3*(1 - p);
eq2 = pi_1 ==                            pi_2*(1 - p)  + pi_3*p;
eq3 = pi_2 ==         pi_1*(1 - p)  + pi_2*p;
eq4 = pi_3 == pi_0 + pi_1*p;
eq5 =    1 == pi_0 + pi_1           + pi_2           + pi_3;

sol = solve([eq1, eq2, eq3, eq4, eq5], [pi_0 pi_1 pi_2 pi_3]);
```

To maximize the function, we will use calculus I methods of solving for critical points by setting the derivative equal to 0 and solving.

```
isWet = p*sol.pi_0;
criticalPts = solve(diff(isWet));
disp("The critical values for our wet function are:");
disp(eval(criticalPts));
```

```
The critical values for our wet function are:
   0.535898384862246
   7.464101615137754
```

Recall that our function represents probability. Hence our domain is only defined for the interval $[0, 1]$. Since 7.464 does not belong in the domain we discard it. We will take 0.5359 and plug it into our function, along with the endpoints.

```
disp(eval(subs(isWet, p, criticalPts(1))));
disp(eval(subs(isWet, p, 0)));
disp(eval(subs(isWet, p, 1)));
```

```
0.071796769724491

   0

   0
```

The value of p that maximizes the fraction of time the individual gets wet is 53.5898%. Furthermore, the amount of time the individual is wet, ends up being 7.1797%. It is obvious that the value of p would not be 0. However, you may have expected the function to reach a max when $p = 1$. But with some careful consideration, even if you start off in a state with 0 umbrellas, every event from $n = 2$ and onwards would mean that you carry around the same umbrella from work to home forever.

1. (d)

Would buying another umbrella reduce the amount of time he gets wet? If so, by how much? Use p = 30% to answer.

```
clear; clc; close all;
syms pi_0 pi_1 pi_2 pi_3 pi_4;

% Set constants
p = .3;
probabilityWetPartA = 21/370;

% Build System of Equations
eq1 = pi_0 ==                                               pi_4*(1-
p);
eq2 = pi_1 ==                                  pi_3*(1-p) + pi_4*p;
eq3 = pi_2 ==                     pi_2*(1-p) + pi_3*p;
eq4 = pi_3 ==        pi_1*(1-p) + pi_2*p;
eq5 = pi_4 == pi_0 + pi_1*p;
eq6 =    1 == pi_0 + pi_1       + pi_2       + pi_3       + pi_4;

sol = solve([eq1, eq2, eq3, eq4, eq5, eq6], [pi_0 pi_1 pi_2 pi_3
 pi_4]);

isWet = p*sol.pi_0;
disp("The probability of getting wet is:");
disp(isWet);
disp(eval(isWet));

disp("The probability of getting wet decreases by:");
disp(probabilityWetPartA - isWet);
disp(eval(probabilityWetPartA - isWet));
```

```
The probability of getting wet is:
21/470

   0.044680851063830

The probability of getting wet decreases by:
21/1739
```

```
0.012075905692927
```

If the individual had an extra umbrella then the probability that he would get wet is $\frac{21}{440}$ or 4.468%. This is a decrease of $\frac{21}{1739}$ or 1.208%. This is not too large of a decrease, however, for someone who dispised the rain, I would recommend buying another cheap umbrella (or 2 or 3 ...).

2. A pharmaceutical company wants to determine the effciency of the delivery mechanism of an antibiotic for a specific type of bacteria. This antibiotic is delivered through molecules which have to attach to a site on the surface of the bacteria in order to release the antibiotic into the bacteria. For simplicity, let us call the molecules delivering the antibiotic the acceptable molecules. The surface of a typical bacteria consists of several sites at which foreign molecules|some acceptable and some not|become attached. At a typical site, molecules arrive to the site at a rate of about 30 per second. For a usual dose of this antibiotic given to a patient, acceptable molecules will constitute about 2% of all molecules arriving at the site. Unacceptable molecules stay at the site for about 0.02 seconds and then are ejected, whereas acceptable molecules were designed to stay at the site about four times longer. An arriving molecule will become attached only if the site is free of other molecules.

The pharmaceutical company usually employs two metrics to determine the efficiency of such a delivery system: the percentage of the time a typical site is occupied with an acceptable molecule and the fraction of arriving acceptable molecules that become attached. The company has hired you to estimate these quantities.

2. (a)

Use the five-steps method to estimate these quantities. Use a Markov process (continuous-time) for your model.

Step 1

Let P_i represent the proportion of time spent in state i.

$\pi_j = \lim_{n\to\infty} p_{ij}^n$ for $j \geq 0$.

- *Assumptions* -

Assume that

- *Equations* -

$$\sum_{i=0}^{2} P_i = 1$$

- *Objective* -

We want to determine the percentage of the time a typical site is occupied with an acceptable molecule and the fraction of arriving acceptable molecules that become attached. In order to calculate these quantites, we will need to find steady states for the proportion of time spend in a particular state.

Step 2

We will use a continuous time Markov process to model the situation. We should note that our Markov process is ergodic, so this in turn ensures that the steady states do indeed exist for this system.

Step 3

Step 4

```
clear; clc; close all;
syms p0 p1 p2;

rateMoleculesArrive = 30;
percentAcceptable = .02;
timeBeforeEjected = .02;

unAcceptableArrive = rateMoleculesArrive*(1 - percentAcceptable);
acceptableArrive = rateMoleculesArrive*percentAcceptable;

unAcceptableEjected = 1/timeBeforeEjected;
acceptableEjected = 1/(4*timeBeforeEjected);

eq0 = unAcceptableEjected*p1 + acceptableEjected*p2 ==
 (acceptableArrive + unAcceptableArrive)*p0;
eq1 =                         unAcceptableArrive*p0 ==
 unAcceptableEjected*p1;
eq2 =                           acceptableArrive*p0 ==
 acceptableEjected*p2;
eq3 =                                  p0 + p1 + p2 == 1;

sol = solve([eq0, eq1, eq2, eq3], [p0 p1 p2]);

disp(eval(sol.p0));
disp(eval(sol.p1));
disp(eval(sol.p2));
```

```
   0.611246943765281

   0.359413202933985

   0.029339853300733
```

The long term steady state proportions as follows : html<table border=1> tr<td> b P /b /td<td> b Percentage /b /td<td> b Fraction /b /td</tr> tr<td> 0 /td<td> .6112 /td<td> 250/409 /td</tr> tr<td> 1 /td<td> .3594 / td<td> 147/409 /td</tr> tr<td> 2 /td<td> .0293 /td<td> 12/409 /td</tr> /table</html>

```
disp("The percentage of time a particular site is occupied by an
 acceptable molecule:");
disp(sol.p2);
disp(eval(sol.p2));

disp("The fraction of arriving acceptable molecules that become
 attached:");
disp(sol.p2/acceptableArrive);
```

```
disp(eval(sol.p2/acceptableArrive));
```

The percentage of time a particular site is occupied by an acceptable molecule:
12/409

0.029339853300733

The fraction of arriving acceptable molecules that become attached:
20/409

0.048899755501222

Step 5

We can conclude that ...

2. (b)

Compute the sensitivity of both your answers in (a) to the rate of 30 arriving molecules per second. Comment on your results.

```
clear; clc; close all;
syms p0 p1 p2 rateMoleculesArrive;

percentAcceptable = .02;
timeBeforeEjected = .02;

unAcceptableArrive = rateMoleculesArrive*(1 - percentAcceptable);
acceptableArrive = rateMoleculesArrive*percentAcceptable;

unAcceptableEjected = 1/timeBeforeEjected;
acceptableEjected = 1/(4*timeBeforeEjected);

eq0 = unAcceptableEjected*p1 + acceptableEjected*p2 ==
 (acceptableArrive + unAcceptableArrive)*p0;
eq1 =                        unAcceptableArrive*p0 ==
 unAcceptableEjected*p1;
eq2 =                          acceptableArrive*p0 ==
 acceptableEjected*p2;
eq3 =                                 p0 + p1 + p2 == 1;

sol = solve([eq0, eq1, eq2, eq3], [p0 p1 p2]);

disp(sol.p2);

% Sensitivity Analysis for acceptable arrived
N = sol.p2;
derivative = diff(N, rateMoleculesArrive);
sensitivity_symbolic = (rateMoleculesArrive/N)*derivative;
sensitivity = eval(subs(sensitivity_symbolic, rateMoleculesArrive,
 30));
```

```
% Sensitivity Analysis for acceptable arrived
N2 = sol.p2/acceptableArrive;
derivative2 = diff(N2, rateMoleculesArrive);
sensitivity_symbolic2 = (rateMoleculesArrive/N2)*derivative2;
sensitivity2 = eval(subs(sensitivity_symbolic2, rateMoleculesArrive,
 30));

disp('Sensitivity');
disp(sensitivity);
disp(sensitivity2);
```

```
(4*rateMoleculesArrive)/(53*rateMoleculesArrive + 2500)

Sensitivity
   0.611246943765281

  -0.388753056234719
```

The sensitivities mean ...

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Assignment # 6

39/40

Jarren Ralf

12.5/13

1. An individual possesses 3 umbrellas that he employs in going from his home to office and vice versa. If he is at home at the beginning of the day and it is raining, then he will take an umbrella with him to the office, provided there is one to be taken. If he is at the office at the end of the day and if it is raining, then he will take an umbrella home, provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (or end) of a day with probability p = 30%. The individual wants to create a model to determine the fraction of time he will get wet.

Contents

- 1. (a)
- 1. (b)
- 1. (c)
- 1. (d)
- 2. (a)
- 2. (b)

1. (a)

Use the five-step method to answer this question. Use a Markov chain (discrete-time) to model the situation.

Step 1

Variable Names	Descriptions
X_n	The number of umbrellas at time *n*
n	Time (half-days)
isWet	The individual gets wet (true/false)

1

Let p_{ij} represent the transition probabilities for going from i umbrellas in one location to j umbrellas in the other. We will also denote this in matrix form by $A = \{p_{ij}\}$ We will let π_j be the steady state probability for j umbrellas. Note that the steady state probability is defined as the following:

$\pi_j = \lim_{n\to\infty} p_{ij}^n$ for $j \geq 0$. → why does steady-state exist? Define the chain first and determine if it is ergodic.

Constants	Value	Descriptions
p	30%	The probability that it rains

- Assumptions -

Assume that the individual in question is a bit of a loser and only goes between work and home. Assume that he only owns 3 umbrellas. Assume that the individual only grabs 1 umbrella (if available) on days that it is raining. Assume that none of his umbrella's are broken so that on days it rains, if he takes an umbrella, he does not get wet. Assume that if the individual

? can't read

leaves work (or home) when it is not raining, that it will not starting raining during his commute, hence catching him without an umbrella and getting him wet. Assume that independent of the past, it rains at the beginning (or end) of a day with probability 30%, i.e. assume the individual lives in Vancouver, BC. Assume that the probability of it raining is independent of the probability of having an umbrella at home or work. We will assume that being wet is binary, i.e. you are either wet or not. We will assume that we are dealing with proper probabilities hence they only take on real number values between 0 and 1 inclusively. We also have the following domain restrictions $X_n = \{0, 1, 2, 3\}$, similiarly, $i, j = \{0, 1, 2, 3\}$ and $n \in N$.

- Equations -

$$p_{ij} = P(X_{n+1} = j | X_n = i)$$

$$p_{12} = p_{21} = p_{30} = 1 - p$$

0.5

$$p_{13} = p_{22} = p_{31} = p$$

$$p_{03} = 1$$

The rest of the entries in the matrix are 0. From this, we can set up our equations for the steady state probabilities.

$$\pi_j = [\pi_0 \; \pi_1 \; \pi_2 \; \pi_3] \cdot p_{ij}$$

$$\sum_{i=0}^{3} \pi_i = 1$$

$$P(isWet) = p \cap \pi_0$$

- Objective -

We want to determine the fraction of time that the individual will get wet. In order to find this probability, we need the long term proportion of leaving either the house or work with no umbrella. 1

Step 2

-0.5 why? explain

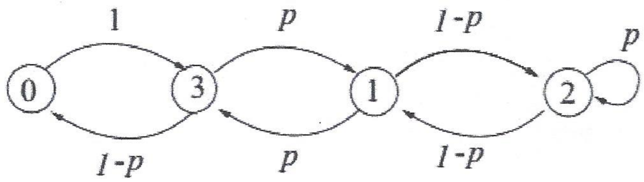
We will use a discrete time Markov chain to model the situation. We should note that our Markov chain is ergodic, so this in turn ensures that the steady states do indeed exist for this system. 1

Step 3

We will find the steady state probabilitites in two ways. Firstly we will find them by constructing our state transition matrix and raise it to the power n and find the solution iteratively for a large value of n. Next we will compile our steady state equations from above, by expanding out the matrix, then plug in our probabilities. This leaves us with having to solve the equations below. This should yield the same solution as the iterative process.

```
figure
imshow(imread('1.jpg'))
title('State Transition Diagram')
```

State Transition Diagram



2

$\pi_0 = 0.7\pi_3$

$\pi_1 = 0.7\pi_2 + 0.3\pi_3$

$\pi_2 = 0.7\pi_1 + 0.3\pi_2$

$\pi_3 = \pi_0 + 0.3\pi_1$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

2

Lastly, to answer the question, we will find $P(isWet)$. Recall our assumption that the probability of it raining and the probability of having 0 umbrellas is independent. Thus we will compute

1

$P(isWet) = 0.3\pi_0.$

Step 4

```
clear; clc; close all;
syms pi_0 pi_1 pi_2 pi_3;

% Set constants
p = .3;

% Construct the transition matrix
A = [  0          0          0          1 ;
       0          0        1 - p        p ;
       0        1 - p        p          0 ;
     1 - p        p          0          0];

% Iterate A for large n
product = A;
for i = 1:1000
    product = product*A;
end
disp(product);

% Build System of Equations
eq1 = pi_0 ==                                        pi_3*(1 - p);
eq2 = pi_1 ==                         pi_2*(1 - p)  + pi_3*p;
eq3 = pi_2 ==          pi_1*(1 - p)  + pi_2*p;
eq4 = pi_3 == pi_0 + pi_1*p;
eq5 =     1 == pi_0 + pi_1            + pi_2            + pi_3;
```

```
sol = solve([eq1, eq2, eq3, eq4, eq5], [pi_0 pi_1 pi_2 pi_3]);

disp(eval(sol.pi_0));
disp(eval(sol.pi_1));
disp(eval(sol.pi_2));
disp(eval(sol.pi_3));
```

```
0.1892    0.2703    0.2703    0.2703
0.1892    0.2703    0.2703    0.2703
0.1892    0.2703    0.2703    0.2703
0.1892    0.2703    0.2703    0.2703

0.1892

0.2703

0.2703

0.2703
```

We have found the proper steady state probabilities because both methods produce the same values.

π	Percentage	Fraction
0	18.9189189	7/37
1	27.0270270	10/37
2	27.0270270	10/37
3	27.0270270	10/37

3

```
isWet = p*sol.pi_0;
disp("The probability of getting wet is:");
disp(isWet);
disp(eval(isWet));
```

```
The probability of getting wet is:
21/370

    0.0568
```

Step 5

We can conclude that the probability of getting wet is $\frac{21}{370}$ or 5.68%. As many people would agree, if you do not have the space to carry around an umbrella, it is quite annoying to carry one when it is not raining. Thus, The strategy of leaving a various amount at work and home is a reasonable one. 1

1. (b)

Compute the sensitivity of your answer to the p = 30% assumption. Comment on your result.

```
clear; clc; close all;

% Declare p are a symbolic variable instead of a fixed constant in order to
% compute the sensitivity
syms p pi_0 pi_1 pi_2 pi_3;

% Build System of Equations
eq1 = pi_0 ==                                         pi_3*(1 - p);
eq2 = pi_1 ==                          pi_2*(1 - p)  + pi_3*p;
eq3 = pi_2 ==           pi_1*(1 - p)  + pi_2*p;
eq4 = pi_3 == pi_0 + pi_1*p;
eq5 =    1 == pi_0 + pi_1            + pi_2          + pi_3;

sol = solve([eq1, eq2, eq3, eq4, eq5], [pi_0 pi_1 pi_2 pi_3]);
```

4/4

To compute the sensitivity for this question, we need to take a derivative of our desired function, $isWet$ in terms of the variable p. The sensitivity will be $S(isWet, p) = \frac{p}{isWet}\frac{disWet}{dp}$.

```
isWet = p*sol.pi_0;
% Compute Sensitivity of pi_0
derivative = diff(isWet, p);
sensitivity_symbolic = (p/isWet)*derivative;
sensitivity = eval(subs(sensitivity_symbolic, p, 0.3));

disp('Sensitivity');
disp(sensitivity);
```

```
Sensitivity
    0.6525
```

This means that for a 1% increase in the probability of it raining, the probability that the individual will get wet increases by 0.6525%. If the probability of it raining increases, I would inform the individual that it is more likely he gets wet, and hence recommend he does not walk if he wants to stay dry. But realistically, a 0.6525% increase is rather insignificant. In conclusion, the probability of getting wet is not very sensitive to our assumption of the probability of rain to be 30%.

1. (c)

What value of p maximizes the fraction of time he gets wet.

```
clear; clc; close all;

% We need to be able to solve for p, thus we make it symbolic again
syms p pi_0 pi_1 pi_2 pi_3;

% Build System of Equations
```

```
eq1 = pi_0 ==                                              pi_3*(1 - p);
eq2 = pi_1 ==                              pi_2*(1 - p)   + pi_3*p;
eq3 = pi_2 ==               pi_1*(1 - p)  + pi_2*p;
eq4 = pi_3 == pi_0 + pi_1*p;
eq5 =     1 == pi_0 + pi_1              + pi_2            + pi_3;

sol = solve([eq1, eq2, eq3, eq4, eq5], [pi_0 pi_1 pi_2 pi_3]);
```

To maximize the function, we will use calculus I methods of solving for critical points by setting the derivative equal to 0 and solving.

```
isWet = p*sol.pi_0;
criticalPts = solve(diff(isWet));
disp("The critical values for our wet function are:");
disp(eval(criticalPts));
```

2/2

```
The critical values for our wet function are:
    0.5359
    7.4641
```

Recall that our function represents probability. Hence our domain is only defined for the interval $[0, 1]$. Since 7.464 does not belong in the domain we discard it. We will take 0.5359 and plug it into our function, along with the endpoints.

```
disp(eval(subs(isWet, p, criticalPts(1))));
disp(eval(subs(isWet, p, 0)));
disp(eval(subs(isWet, p, 1)));
```

```
    0.0718

     0

     0
```

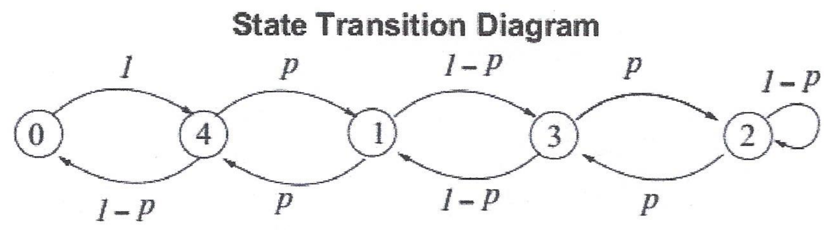
The value of p that maximizes the fraction of time the individual gets wet is 53.5898%. Furthermore, the amount of time the individual is wet, ends up being 7.1797%. It is obvious that the value of p would not be 0. However, you may have expected the function to reach a max when $p = 1$. But with some careful consideration, even if you start off in a state with 0 umbrellas, every event from $n = 2$ and onwards would mean that you carry around the same umbrella from work to home forever.

4/4

1. (d)

Would buying another umbrella reduce the amount of time he gets wet? If so, by how much? Use p = 30% to answer.

```
figure
imshow(imread('2.jpg'))
title('State Transition Diagram')
```




```
clear; clc; close all;

syms pi_0 pi_1 pi_2 pi_3 pi_4;

% Set constants
p = .3;
probabilityWetPartA = 21/370;

% Build System of Equations
eq1 = pi_0 ==                                                   pi_4*(1-p);
eq2 = pi_1 ==                                      pi_3*(1-p) + pi_4*p;
eq3 = pi_2 ==                         pi_2*(1-p) + pi_3*p;
eq4 = pi_3 ==            pi_1*(1-p) + pi_2*p;
eq5 = pi_4 == pi_0 + pi_1*p;
eq6 =     1 == pi_0 + pi_1        + pi_2        + pi_3        + pi_4;

sol = solve([eq1, eq2, eq3, eq4, eq5, eq6], [pi_0 pi_1 pi_2 pi_3 pi_4]);

isWet = p*sol.pi_0;
disp("The probability of getting wet is:");
disp(isWet);
disp(eval(isWet));

disp("The probability of getting wet decreases by:");
disp(probabilityWetPartA - isWet);
disp(eval(probabilityWetPartA - isWet));
```

```
The probability of getting wet is:
21/470

    0.0447

The probability of getting wet decreases by:
21/1739

    0.0121
```

If the individual had an extra umbrella then the probability that he would get wet is $\frac{21}{440}$ or 4.468%. This is a decrease of $\frac{21}{1739}$ or 1.208%. This is not too large of a decrease, however, for someone who despises the rain, I would recommend buying another cheap umbrella (or 2 or 3 ...).

2. A pharmaceutical company wants to determine the effciency of the delivery mechanism of an antibiotic for a specific type of bacteria. This antibiotic is delivered through molecules which have to attach to a site on the surface of the bacteria in order to release the antibiotic into the bacteria. For simplicity, let us call the molecules delivering the antibiotic the acceptable molecules. The surface of a typical bacteria consists of several sites at which foreign molecules--some acceptable and some not--become attached. At a typical site, molecules arrive to the site at a rate of about 30 per second. For a usual dose of this antibiotic given to a patient, acceptable molecules will constitute about 2% of all molecules arriving at the site. Unacceptable molecules stay at the site for about 0.02 seconds and then are ejected, whereas acceptable molecules were designed to stay at the site about four times longer. An arriving molecule will become attached only if the site is free of other molecules.

The pharmaceutical company usually employs two metrics to determine the efficiency of such a delivery system: the percentage of the time a typical site is occupied with an acceptable molecule and the fraction of arriving acceptable molecules that become attached. The company has hired you to estimate these quantities.

2. (a)

12.5/13

Use the five-steps method to estimate these quantities. Use a Markov process (continuous-time) for your model.

Step 1

Variable Names	Descriptions
X_t	The occupancy of a site on a bacteria for time *t*
t	Time (seconds)
isAttached	The molecules attach to a site on the bacteria(true/false)

1

X	Description of State
0	The site on the bacteria is unoccupied
1	The site on the bacteria is occupied by an unacceptable molecule
2	The site on the bacteria is occupied by an acceptable molecule

why does steady-state exist?

Let P_i represent the proportion of time spent in state i. This is defined interms of the steady-state distribution and the associated rates for each time inbetween states.

Constants	Value	Descriptions
rateMoleculesArrive	30	The rate at which the molecules arrive at a site by (1/second)
percentAcceptable	.02	Proportion of acceptable molecules that arrive at a site
timeBeforeEjected	.02	The time an unacceptable molecule stays at a site for before being ejected (s)
acceptableArrive	---	The rate of acceptable molecules arriving at a site (1/s)
unAcceptableArrive	---	? The rate of unacceptable molecules arriving at a site (1/s)
acceptableEjected	---	The rate of acceptable molecules being ejected from a site (1/s)

unAcceptableEjected	---	The rate of unacceptable molecules being ejected from a site (1/s)

- Assumptions -

Assume that there are no acceptable molecules already present within the patient. Lets assume every site is independent of one another, hence the proability of latching onto any site is not affected at all by neighbouring sites. Assume that the propabibility of latching onto any site is fixed and equal to every other site. Assume that a molecule is either acceptable or not acceptable. An addendum to this would be that acceptable molecules are never ineffective but rather always potent. Assume that a molecule will only every occupy one site at a time. Assume that the dosage is consistent with every trial we observe. Assume that acceptable molecules do stay on the bacteria four times longer than unacceptable ones. Assume that a molecule doesn't attach to a site that is already occupied. Assume that the two metrics chosen are indeed a good measure of the effectiveness of the antibiotic. Assume that the probability of an acceptable molecule arriving and the probability of a site being empty are independent. We also have the following domain restrictions: $X_t = \{0, 1, 2\}$, $t \in \{R | t \geq 0\}$. Lastly assume that isAttached is a binary variable such that a molecule is either attached or not attached. 1

- Equations -

$P_i = \frac{(\pi_i/\lambda_i)}{(\pi_0/\lambda_0)+(\pi_1/\lambda_1)+(\pi_2/\lambda_2)}$ for $i = 1, 2, 3$

$$\sum_{i=0}^{2} P_i = 1$$

To simplify our work, let us set up the following equations.

$$acceptableArrive = rateMoleculesArrive(1 - percentAcceptable)$$

$$unAcceptableArrive = rateMoleculesArrive \cdot percentAcceptable$$

$$acceptableEjected = \frac{1}{timeBeforeEjected}$$

$$unAcceptableEjected = \frac{1}{4 \cdot timeBeforeEjected}$$

- Objective -

We want to determine the percentage of the time a typical site is occupied with an acceptable molecule and the fraction of arriving acceptable molecules that become attached. In order to calculate these quantites, we will need to find steady states for the proportion of time spend in a particular state. 1

Step 2

We will use a continuous time Markov process to model the situation. We should note that our Markov process is ergodic, so this in turn ensures that the steady states do indeed exist for this system. 1

-0.5 why? explain

Step 3

Since the probabilities are not available to us, we must use the fluid-flow analogy to help us solve the problem. If we were to imagine a fluid flowing in and out of each state, then in order to remain in equilibrium, the fluid must satidfy the law of conservation of mass. Alas we will model each of our states with the following equation:

Rate flowing into state i = Rate flowing out of state i for every i.

This yields the following equations:

$$50P_1 + \frac{25}{2}P_2 == 30P_0$$

2

$$\frac{147}{5}P_0 == 50P_1$$

$$\frac{3P_0}{5} == \frac{25P_2}{2}$$

$$P_0 + P_1 + P_2 == 1$$

We then solve the above four equations to find the long term proportion of time being in a particular state. Specifically, the pharmaceutical company wants to know what $p2$ is. Additionally, we can calculate the the fraction of arriving acceptable molecules that become attached by the following formula,

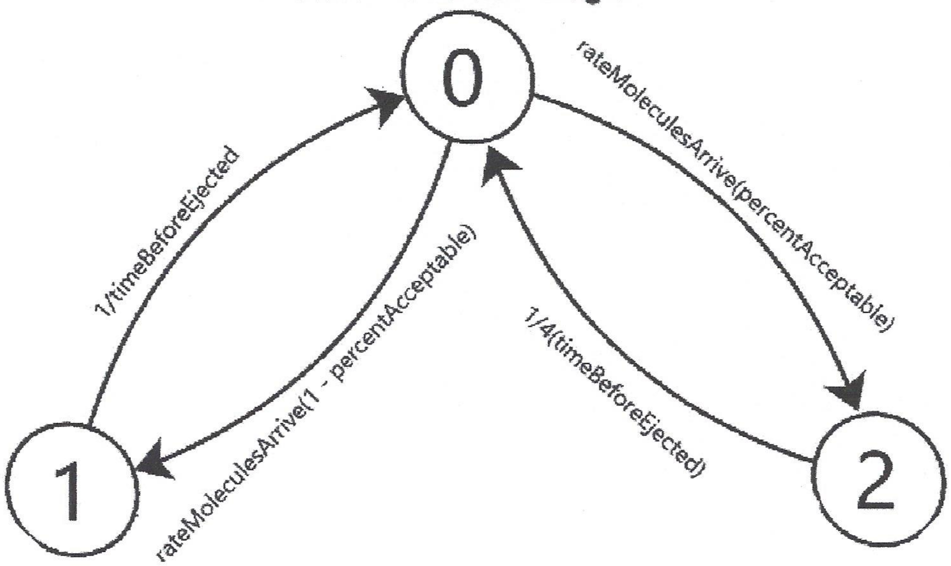
$$isAttached = P_0 \cap acceptableArrive.$$

Since these are assumed to be independent probabilities, we can calculate the following as such:

$$isAttached = \frac{3P_0}{5}.$$

```
I = imread('3.jpg');
R = imresize(I, 0.6, 'nearest');
figure
imshow(R)
title('State Transition Diagram')
```

State Transition Diagram



2

Step 4

```
clear; clc; close all;
syms p0 p1 p2;

% Set constants
rateMoleculesArrive = 30;
  percentAcceptable = .02;
  timeBeforeEjected = .02;

   acceptableArrive = rateMoleculesArrive*percentAcceptable;
 unAcceptableArrive = rateMoleculesArrive*(1 - percentAcceptable);
  acceptableEjected = 1/(4*timeBeforeEjected);
unAcceptableEjected = 1/timeBeforeEjected;

eq0 = unAcceptableEjected*p1 +   acceptableEjected*p2 == ...
       (acceptableArrive     + unAcceptableArrive)*p0;
eq1 =                        unAcceptableArrive*p0 == unAcceptableEjected*p1;
eq2 =                          acceptableArrive*p0 ==   acceptableEjected*p2;
eq3 =                                 p0 + p1 + p2 == 1;

sol = solve([eq0, eq1, eq2, eq3], [p0 p1 p2]);

disp(eval(sol.p0));
disp(eval(sol.p1));
disp(eval(sol.p2));
```

3

```
0.6112

0.3594

0.0293
```

The long term steady state proportions as follows :

P	Percentage	Fraction
0	.6112	250/409
1	.3594	147/409
2	.0293	12/409

```
disp("The percentage of time a particular site is occupied by an acceptable molecule:");
disp(sol.p2);
disp(eval(sol.p2));

isAttached = sol.p0*acceptableArrive;
disp("The fraction of arriving acceptable molecules that become attached:");
disp(isAttached);
disp(eval(isAttached));
```

```
The percentage of time a particular site is occupied by an acceptable molecule:
12/409

    0.0293

The fraction of arriving acceptable molecules that become attached:
150/409

    0.3667
```

Step 5

1

We can conclude that the percentage of time that a typical site is occupied by an acceptable molecule is $\frac{12}{409}$ or 2.934%. Also, the fraction of arriving acceptable molecules that become attached is $\frac{150}{409}$ or 36.67%. The strength of this antibiotic is the fraction of arriving acceptable molecules that end up attached to sites on the bacteria.

2. (b)

4/4

Compute the sensitivity of both your answers in (a) to the rate of 30 arriving molecules per second. Comment on your results.

```
clear; clc; close all;
syms p0 p1 p2 rateMoleculesArrive;

% Set constants
percentAcceptable = .02;
timeBeforeEjected = .02;
```

```
  acceptableArrive = rateMoleculesArrive*percentAcceptable;
 unAcceptableArrive = rateMoleculesArrive*(1 - percentAcceptable);
  acceptableEjected = 1/(4*timeBeforeEjected);
unAcceptableEjected = 1/timeBeforeEjected;

eq0 = unAcceptableEjected*p1 +   acceptableEjected*p2 == ...
       (acceptableArrive     + unAcceptableArrive)*p0;
eq1 =                      unAcceptableArrive*p0 == unAcceptableEjected*p1;
eq2 =                        acceptableArrive*p0 ==   acceptableEjected*p2;
eq3 =                               p0 + p1 + p2 == 1;

sol = solve([eq0, eq1, eq2, eq3], [p0 p1 p2]);

% Sensitivity Analysis for time occupied by acceptable molecule
N = sol.p2;
derivative = diff(N, rateMoleculesArrive);
sensitivity_symbolic = (rateMoleculesArrive/N)*derivative;
sensitivity = eval(subs(sensitivity_symbolic, rateMoleculesArrive, 30));

% Sensitivity Analysis for fraction of acceptable molecules that attach
isAttached = sol.p0*acceptableArrive;
derivative2 = diff(isAttached, rateMoleculesArrive);
sensitivity_symbolic2 = (rateMoleculesArrive/isAttached)*derivative2;
sensitivity2 = eval(subs(sensitivity_symbolic2, rateMoleculesArrive, 30));

disp('Sensitivities');
disp(sensitivity);
disp(sensitivity2);
```

```
Sensitivities
    0.6112

    0.6112
```

We observe that for every 1% increase in the rate molecules arrive at a typical site, that the percentage of time an acceptable molecule occupies that site increases by about .6112%. If the pharmaceutical company wanted to increase the effectiveness of the treatment, they could perhaps administer a high dosage, ultimately increasing the rate at which molecules arrive at the site. This will not change the proportion of time drastically. The conclusion is that the time spend for a site to be occupied by an acceptable is not very sensitive to this value. Finally, for every 1% increase in the rate molecules arrive at a typical site, the fraction of arriving acceptable molecules that become attached increases by .6112%.