MATH 3431 Spring 2017 Maple Assignment 2
Due: March 28, 2017

Additional Requirements

In addition to the Maple Ground Rules (on the course website),

1. Make sure all graphs have titles, and when you plot more than one expression on a graph, make sure the plot is in black-and-white, that you use different line styles for each expression, and that you include a legend.
2. For each part, make sure you need to type in the boundary conditions and initial function only once. No re-typing and no copying-and-pasting!

Assignment

We use Fourier analysis to solve the heat equation, $u_t = 4u_{xx}$ on $[0, 1]$ subject to different boundary conditions. Note that for questions 1–3, you already know the eigenfunctions, so you do not need to find them.

Do the following for each set of boundary conditions:

a. We know that after separation of variables, $u(t, x) = X(x)T(t)$. Use Maple to find the coefficients of the appropriate Fourier series for X for the given $f(x)$.

b. Assemble a finite Fourier approximation of the given $f(x)$ with N terms (N arbitrary). Make the approximation a function of x and N. An example of how to do so, for Taylor polynomials, is on the course website.

c. Plot $f(x)$ and your series on the same graph for $N = 3$, 5, and 21 (one approximation per graph), and plot the difference (the error in the approximation) on a separate graph. If you can, have the two graphs for each N side-by-side.

d. Assemble the finite Fourier series solution to $u_t = 4u_{xx}$ with the stated initial and boundary conditions with N terms. For parts 1–3 you know what each $T(t)$ looks like, so assembling the solution should be easy.

e. Plot the solution surface (in 3D) for $N = 21$. Comment on the surface. How does the surface look for small t?

f. Plot the finite Fourier series solution with $N = 21$ for $u(t, 0.5)$ for $0 \leq t \leq 1$, that is, make a time plot corresponding to $x = 0.5$ for your approximation.

g. Create an animation of the solution for $N = 21$ as t varies. What happens to skewness as time progresses?

1. $u(t,0) = u(t,1) = 0$, $u(0,x) = x - x^3$. For part (c), comment on the error in the approximation, particularly the relative error, that is, the percent error. You should notice a skewness in the error. Is this surprising?

2. $u_x(t,0) = u(t,1) = 0$, $u(0,x) = x - x^3$. Comment on differences between these approximations and those of the previous question.

3. $u(t,0) = u(t,1) = 0$, $u(0,x) = 1$. For part (c), plot Fourier approximations for $N = 51$ and $N = 101$ as well. Do you think the Fourier approximations will ever be good approximations for all x? For part (g), comment on the quality of the approximation when $t = 0$ and for larger t.

4. We solve $u_t = 4u_{xx}$ with $f(x) = 1 + \frac{1}{2}x - \frac{2}{3}x^2$, and the Robin boundary conditions $X(0) - 2X'(0) = X(1) + X'(1) = 0$.

   (a) Find the eigenfunctions and eigenvalues for X. The eigenfunctions are of the form $A\sin(\sqrt{\lambda}x) + \cos(\sqrt{\lambda}x)$. λ must be determined numerically. Use Maple to find the system of equations λ and A satisfy. Use Maple to find at least 4 positive values for λ to 10 decimal places, and their associated A values. Do this using Maple only. Note that Maple's canned commands are unlikely to work on the system of equations, so you will have to tell Maple more explicitly how to solve the system of equations.

   (b) Assemble the Fourier approximations given by your 4 λ's. Note that the norms of the eigenfunctions now depend on λ.

   (c) Plot $f(x)$ and your approximation on the same graph, and plot the difference on a separate graph. Because f satisfies the boundary conditions, your approximation should be quite good.

   (d) Assemble the first 4 terms of the Fourier solution to $u_u = 4u_{xx}$. Plot the solution in 3D, and create an animation as t varies between 0 and 1.

Maple Assignment # 2
Jarren Ralf

```
> restart; with(plots): assume(n, integer); assume(N, integer); # N
  will be the number of terms in the sum
```

Questions 1 - 3

Use Fourier analysis to solve the heat equation, $u_t = 4\,u_{xx}$ *on* $[0,\ 1]$ *subject* to different boundary and initial conditions.

```
> theHeatEquation := diff(u(t,x), t) = diff(u(t,x), x$2);
  interval := 0..1;
```

$$theHeatEquation := \frac{\partial}{\partial t}\, u(t,x) = \frac{\partial^2}{\partial x^2}\, u(t,x)$$

$$interval := 0\,..1 \qquad \textbf{(1.1)}$$

Question 1

Boundary and Initial Conditions:

```
> initialCondition_Q1  := x-x^3;
  eigenfunctionOfX_Q1  := sin(n*Pi*x);
  eigenfunctionOfT_Q1  := exp(-4*n^2*Pi^2*t);
```

$$initialCondition\_Q1 := -x^3 + x$$

$$eigenfunctionOfX\_Q1 := \sin(n{\sim}\,\pi\, x)$$

$$eigenfunctionOfT\_Q1 := \mathrm{e}^{-4\, n{\sim}^2\, \pi^2\, t} \qquad \textbf{(1.1.1)}$$

We know that the given dirchlett boundary conditions, will produce the eigen functions $X = \sin(n\,\pi\,x)$ for $n \in \mathbb{N}$ and $T = e^{-4\,n^2\,\pi^2\,t}$ for $n \in \mathbb{Z}_{\geq 0}$.

(a)

After the separation of variables, $u(t,x) = X(x)\,T(t)$. Find the coefficients of the Fourier series for X for $f(x) = x - x^3$.

```
> fourierCoefficientsOfX_Q1 := 2*int(initialCondition_Q1 *
  eigenfunctionOfX_Q1, x = interval);
```

$$fourierCoefficientsOfX\_Q1 := \frac{12\ (-1)^{1+n{\sim}}}{n{\sim}^3\ \pi^3} \qquad \textbf{(1.1.1.1)}$$

(b)

Assemble a finite Fourier approximation of the given $f(x)$ with N terms (N arbitrary).

```
> finiteFourierApprox_Q1 := (x, N) -> sum
  (fourierCoefficientsOfX_Q1 * eigenfunctionOfX_Q1, n = 1..
  N);
```

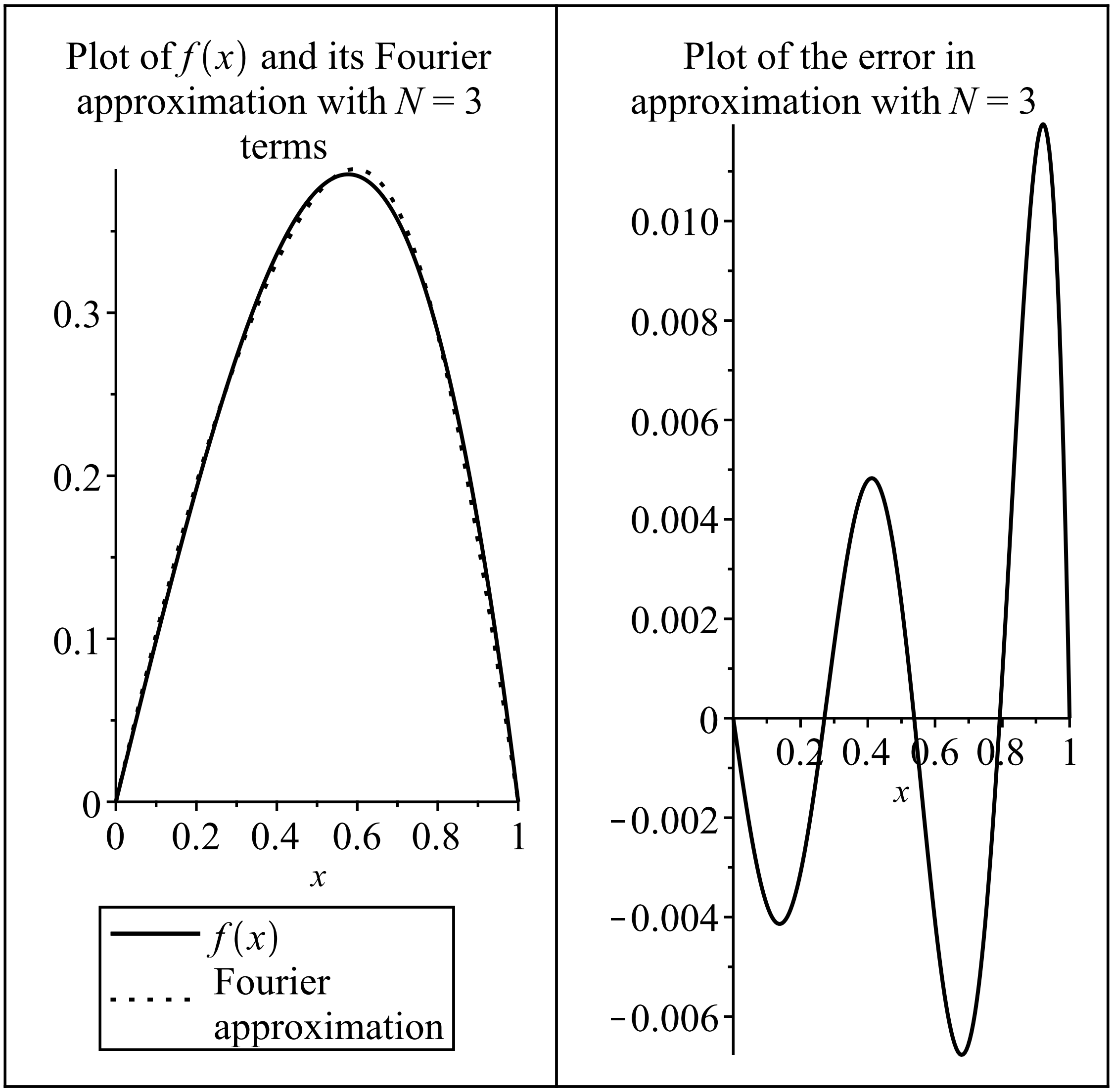
$$finiteFourierApprox\_Q1 := (x, N) \rightarrow \sum_{n=1}^{N} fourierCoefficientsOfX\_Q1\ eigenfunctionOfX\_Q1 \qquad \textbf{(1.1.2.1)}$$

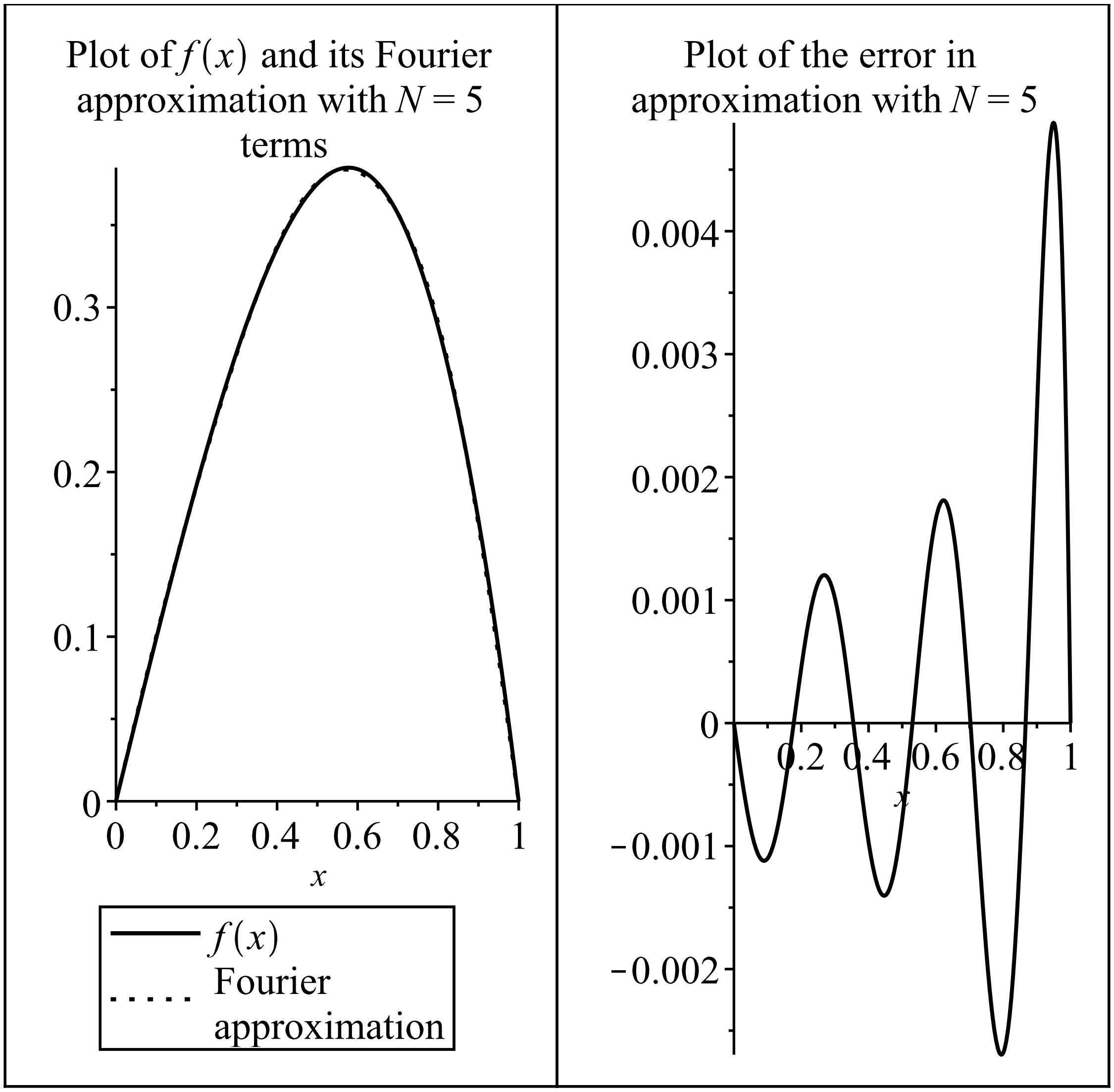
(c)

Plot $f(x) = x - x^3$ and your series on the same graph for $N = 3$, 5, and 21, and plot the difference.

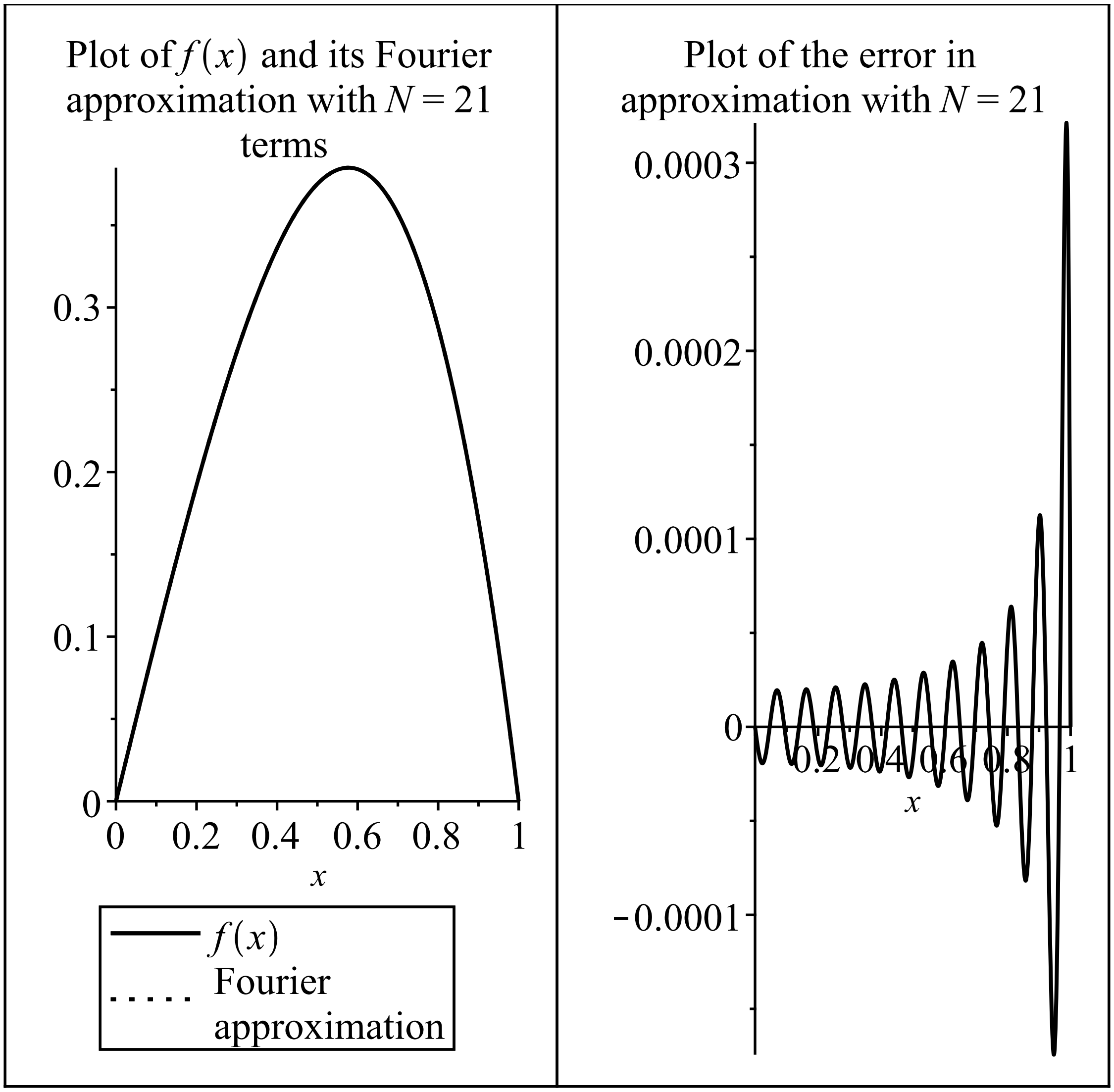
```
> N := 'N': #take away the tilda displayed after the'N' so
  the graph title is in proper format

  A := plot([initialCondition_Q1, finiteFourierApprox_Q1(x,
  3)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=3, " terms")):

  B := plot(initialCondition_Q1 - finiteFourierApprox_Q1(x,
  3), x = interval, colour = black, linestyle = [1,2], title
  = typeset("Plot of the error in approximation with ", N=3)
  ):


  C := plot([initialCondition_Q1, finiteFourierApprox_Q1(x,
  5)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=5, " terms")):

  E := plot(initialCondition_Q1 - finiteFourierApprox_Q1(x,
  5), x = interval, colour = black, linestyle = [1,2], title
  = typeset("Plot of the error in approximation with ", N=5)
  ):


  F := plot([initialCondition_Q1, finiteFourierApprox_Q1(x,
  21)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=21, " terms")):

  G := plot(initialCondition_Q1 - finiteFourierApprox_Q1(x,
  21), x = interval, colour = black, linestyle = [1,2],
  title = typeset("Plot of the error in approximation with
  ", N=21)):

  display(Array([[A, B], [C, E], [F, G]]));
```

Graph of $f(x) = x - x^3$, with its Finite Fourier Approximation with varying Number of Terms, including the Error in Approximation


Plot of $f(x)$ and its Fourier approximation with $N = 3$ terms
0.3
0.2
0.1
0
0 0.2 0.4 0.6 0.8 1
x
$f(x)$
Fourier approximation
Plot of the error in approximation with $N = 3$
0.010
0.008
0.006
0.004
0.002
0
0.2 0.4 0.6 0.8 1
x
−0.002
−0.004
−0.006



Plot of $f(x)$ and its Fourier approximation with $N = 5$ terms
0.3
0.2
0.1
0
0 0.2 0.4 0.6 0.8 1
x
$f(x)$
Fourier approximation
Plot of the error in approximation with $N = 5$
0.004
0.003
0.002
0.001
0
0.2 0.4 0.6 0.8 1
x
−0.001
−0.002



The error in the Fourier approximation is extremely reasonable. Take note that the values on the vertical axis are very small. Hence if you were to view this graph with an integer partitioned graph, you might conclude that the graph is a straight line along the horizontal axis. Additionally, the error seems to be skewed to the left. In otherwords, for smaller x-values, the approximation is better. Finally, The relative error decrease moderately as more terms are used in the approximation.

(d)

Assemble the finite Fourier series solution to $u_t = 4\,u_{xx}$ with N terms.

```
> assume(N, integer); #reinitialize our assumption
  finiteFourierSoln_Q1 := (x, t, N) -> sum
  (eigenfunctionOfT_Q1 * fourierCoefficientsOfX_Q1 *
  eigenfunctionOfX_Q1, n = 1..N);
```

$$finiteFourierSoln\_Q1 := (x, t, N) \rightarrow \sum_{n=1}^{N} eigenfunctionOfT\_Q1\ fourierCoefficientsOfX\_Q1\ eigenfunctionOfX\_Q1 \tag{1.1.4.1}$$

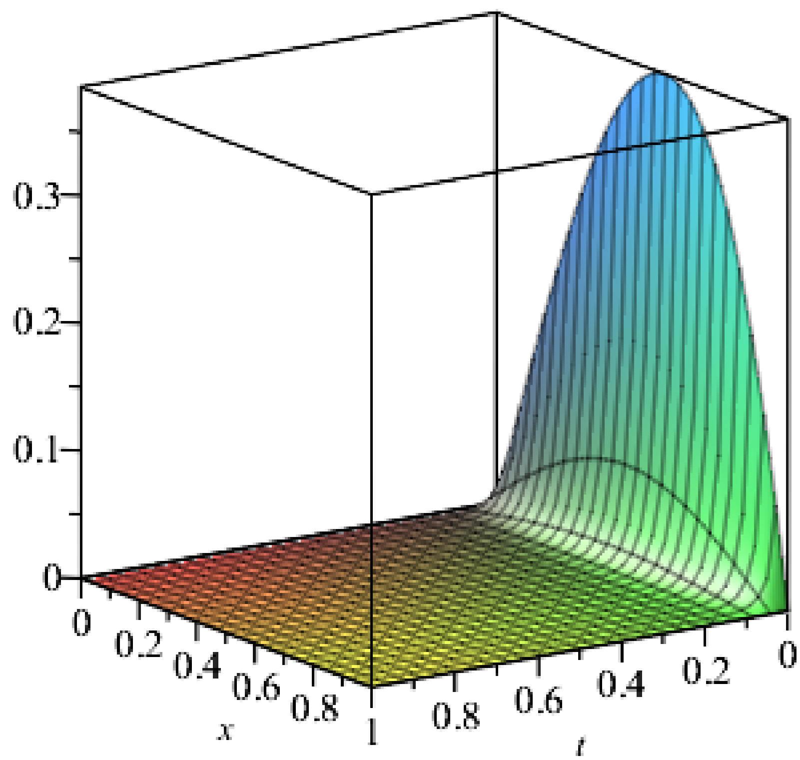
(e)

Plot the solution surface for $N = 21$.

```
> plot3d(finiteFourierSoln_Q1(x, t, 21), t =interval, x =
  interval, title = typeset("Plot of the finite Fourier
  series solution to ", theHeatEquation, " with 21 terms"));
```

Plot of the finite Fourier series solution to

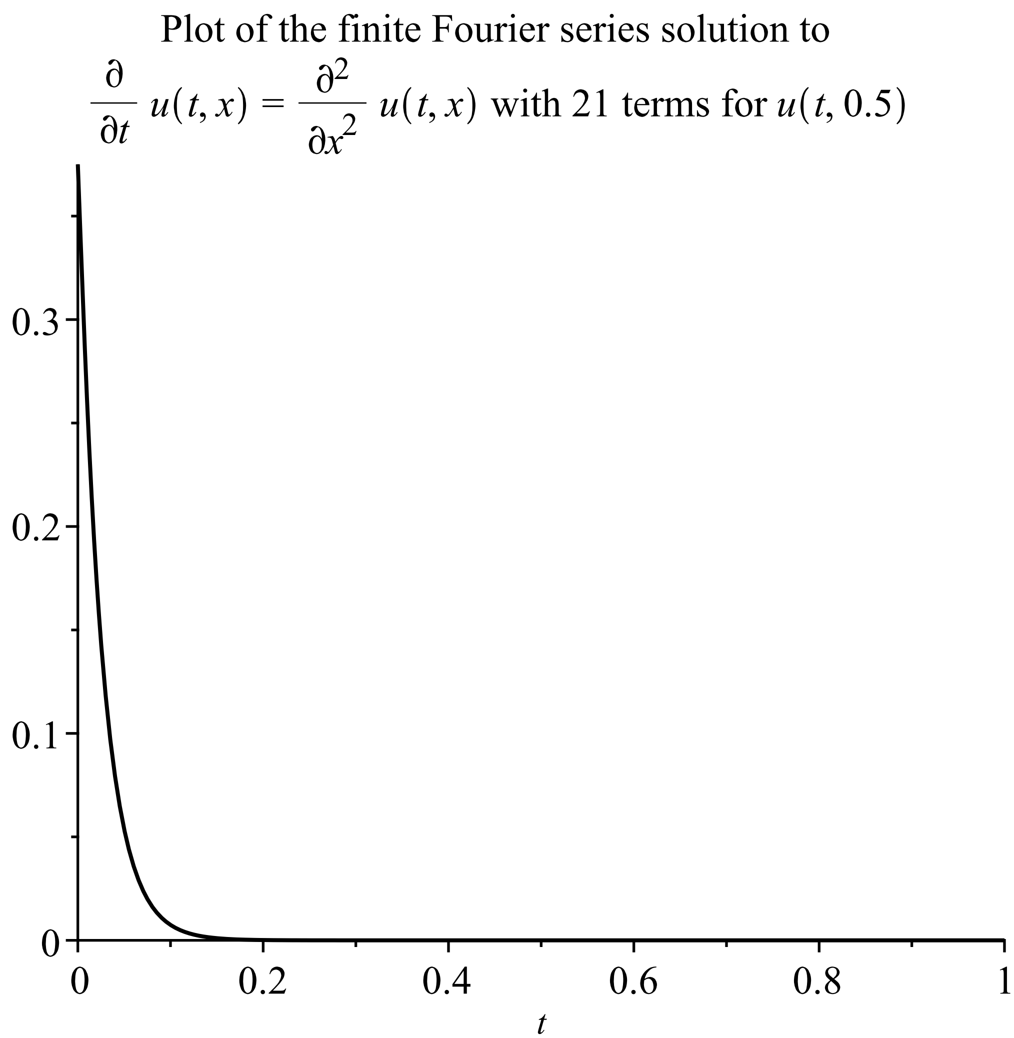
$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) \text{ with 21 terms}$$



(f)

Plot the finite Fourier series with $N = 21$ for $u(t, 0.5)$ for $0 \leq t \leq 1$.

```
> finiteFourierSolnXatOneHalf_Q1 := subs(x = 0.5,
  finiteFourierSoln_Q1(x, t, 21)):
  plot(finiteFourierSolnXatOneHalf_Q1, t = interval, colour
  = black, title = typeset("Plot of the finite Fourier
  series solution to ", theHeatEquation, " with 21 terms for
  ", u(t,0.5)));
```



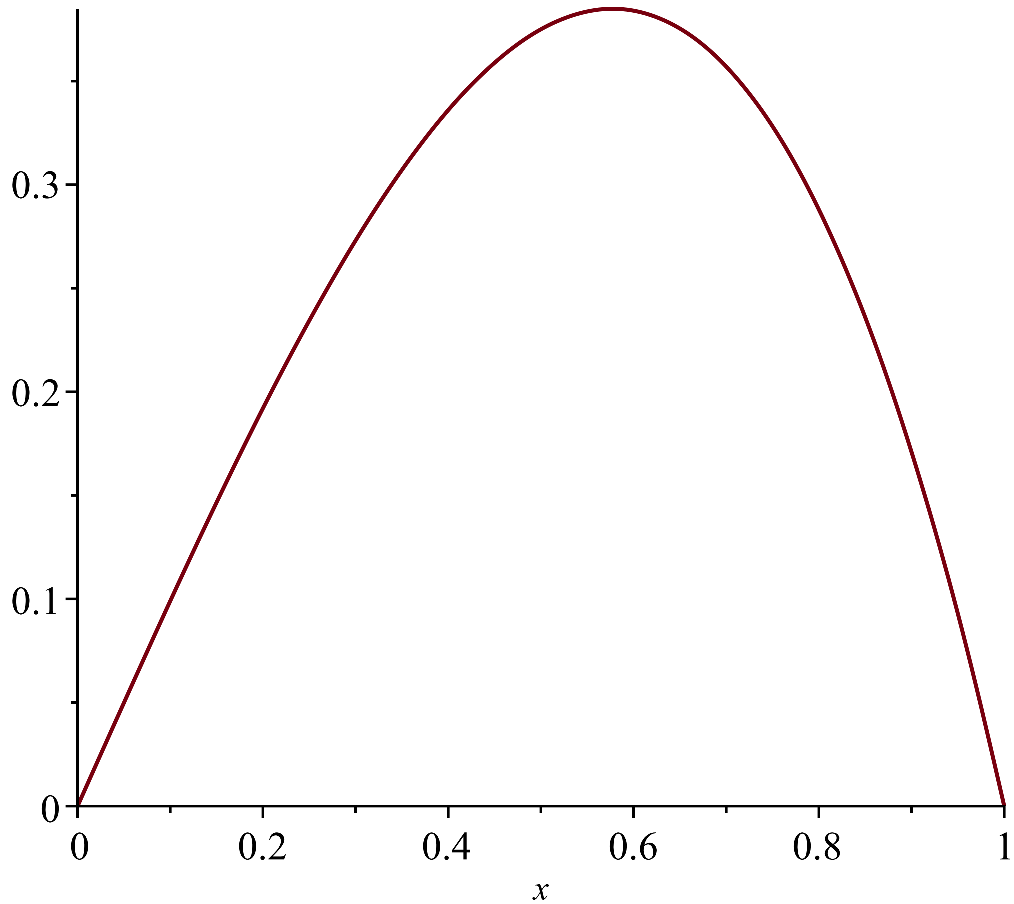
(g)

Create an animation of the solution for $N = 21$ as t varies.

```
> animate(plot, [finiteFourierSoln_Q1(x, t, 21), x =
  interval, title = typeset("Animated plot of the finite
  Fourier series solution to the Heat Equation with 21
  terms")], t = 0..0.15, frames = 100);
```

Animated plot of the finite Fourier series solution to the Heat Equation with 21 terms

$t = 0.$



Question 2

Boundary and Initial Conditions: $u_x(t,0) = u(t,1) = 0, \quad u(0,x) = x - x^3$

```
> initialCondition_Q2  := x-x^3;
  eigenfunctionOfX_Q2  := cos((n+1/2)*Pi*x); #eigenfunction
  eigenfunctionOfT_Q2  := exp(-4*(n+.5)^2*Pi^2*t);
```

$$initialCondition\_Q2 := -x^3 + x$$

$$eigenfunctionOfX\_Q2 := \cos\left(\left(n\sim + \frac{1}{2}\right)\pi x\right)$$

$$eigenfunctionOfT\_Q2 := e^{-4(n\sim + 0.5)^2 \pi^2 t} \tag{1.2.1}$$

We know that the given mixed boundary conditions, will produce the eigen functions $X = \cos\left(n + \frac{1}{2}\right)\pi x$ for $n \in \mathbb{Z}_{\geq 0}$ and $T = e^{-4\left(n + \frac{1}{2}\right)^2 \pi^2 t}$ for $n \in \mathbb{Z}_{\geq 0}$.

(a)

After the separation of variables, $u(t,x) = X(x)T(t)$. Find the coefficients of the Fourier series for X for $f(x) = x - x^3$.

```
> fourierCoefficientsOfX_Q2 := 2*int(initialCondition_Q2 *
```

```
eigenfunctionOfX_Q2, x = interval);
```

$$fourierCoefficientsOfX\_Q2 := \left(8\left(-4\pi^2 n{\sim}^2 - 4\pi^2 n{\sim} - \pi^2 + 24(-1)^{n\sim} n{\sim}\,\pi + 12\pi(-1)^{n\sim} - 24\right)\right) \Big/ \left(\left(8n{\sim}^3 + 12n{\sim}^2 + 6n{\sim} + 1\right)(2n{\sim}+1)\pi^4\right) \tag{1.2.1.1}$$

(b)

Assemble a finite Fourier approximation of $f(x) = x - x^3$ with N terms (N arbitrary).

```
> finiteFourierApprox_Q2 := (x, N) -> sum
  (fourierCoefficientsOfX_Q2 * eigenfunctionOfX_Q2, n = 1..
  N);
```

$$finiteFourierApprox\_Q2 := (x, N) \rightarrow \sum_{n=1}^{N} fourierCoefficientsOfX\_Q2\ eigenfunctionOfX\_Q2 \tag{1.2.2.1}$$

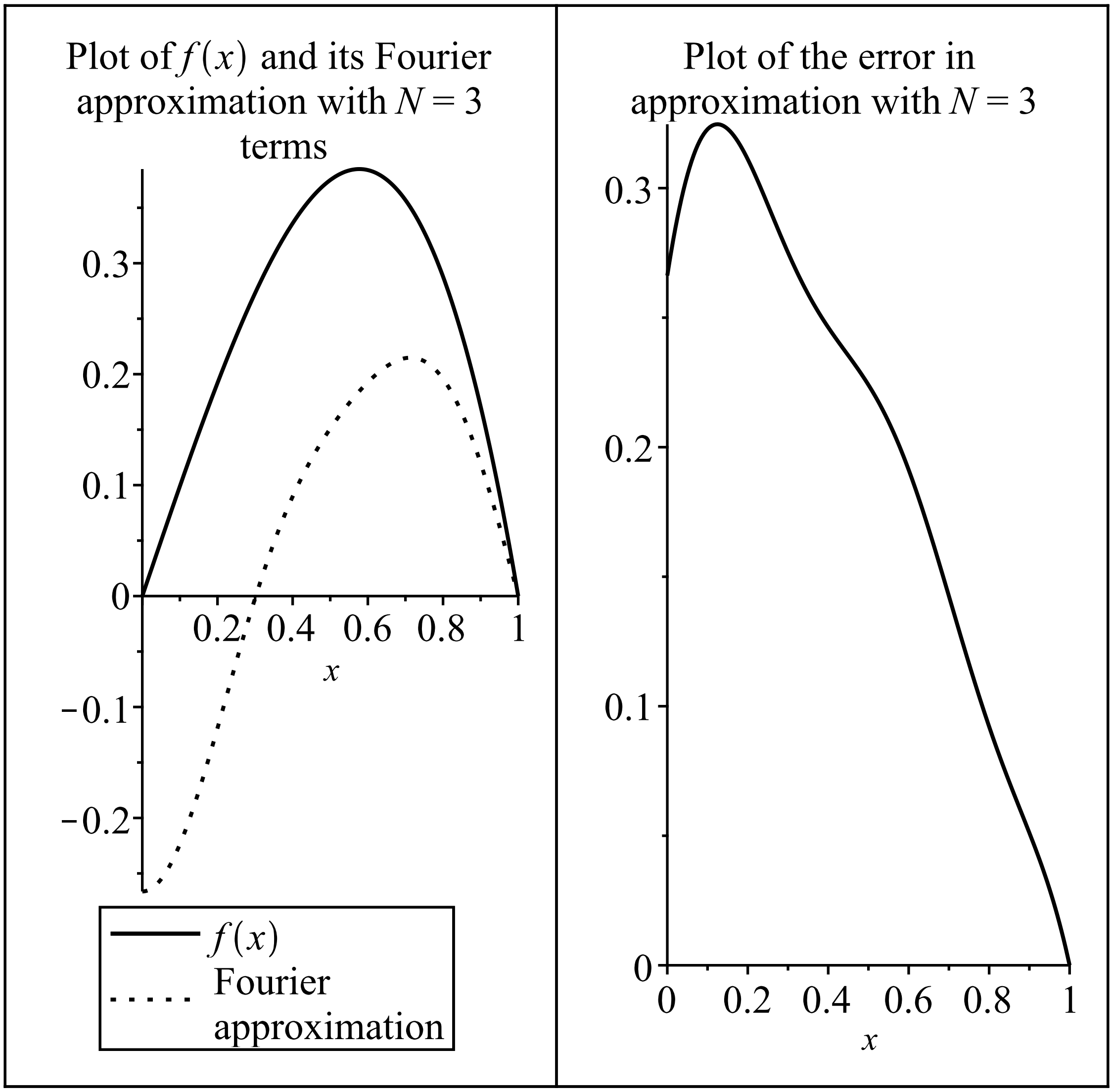
(c)

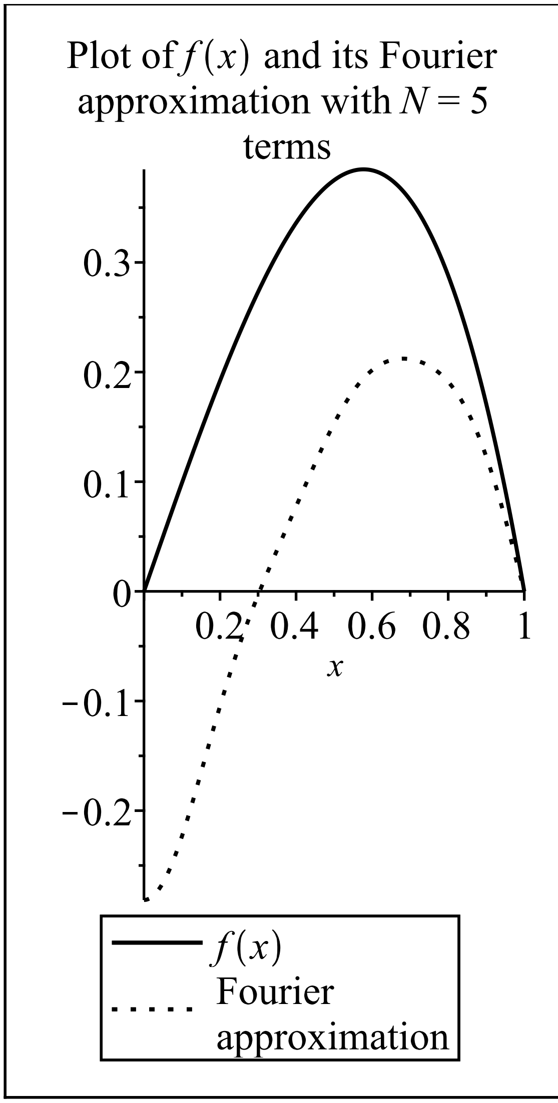
Plot $f(x) = x - x^3$ and your series on the same graph for $N = 3$, 5, and 21, and plot the difference.

```
> N := 'N': #take away the tilda displayed after the'N' so
  the graph title is in proper format

  H := plot([initialCondition_Q2, finiteFourierApprox_Q2(x,
  3)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=3, " terms")):

  J := plot(initialCondition_Q2 - finiteFourierApprox_Q2(x,
  3), x = interval, colour = black, linestyle = [1,2], title
  = typeset("Plot of the error in approximation with ", N=3)
  ):


  K := plot([initialCondition_Q2, finiteFourierApprox_Q2(x,
  5)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=5, " terms")):

  L := plot(initialCondition_Q2 - finiteFourierApprox_Q2(x,
  5), x = interval, colour = black, linestyle = [1,2], title
  = typeset("Plot of the error in approximation with ", N=5)
  ):


  M := plot([initialCondition_Q2, finiteFourierApprox_Q2(x,
  21)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=21, " terms")):

  P := plot(initialCondition_Q2 - finiteFourierApprox_Q2(x,
  21), x = interval, colour = black, linestyle = [1,2],
  title = typeset("Plot of the error in approximation with
```

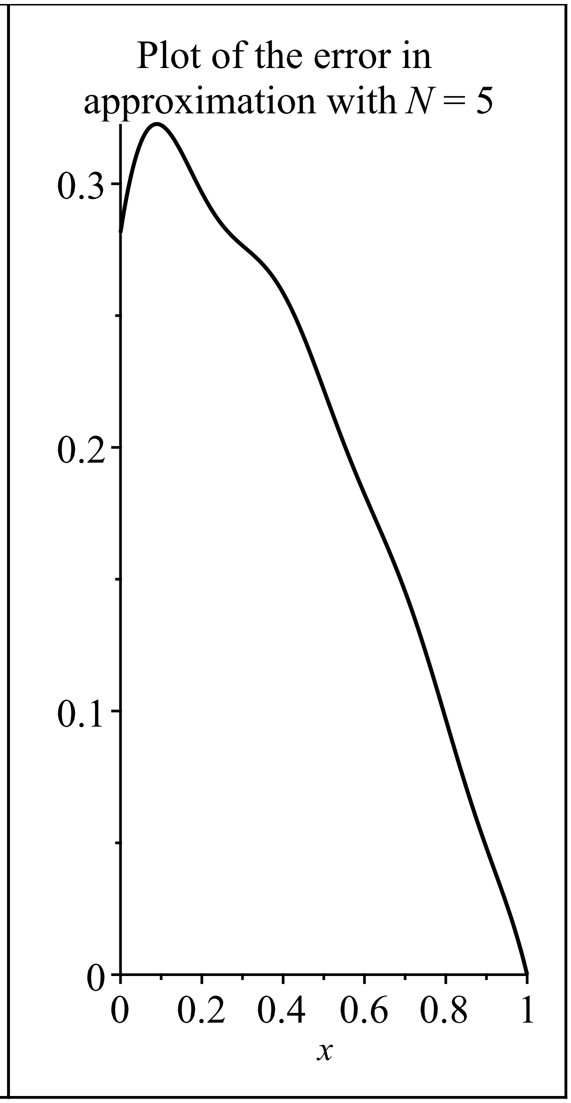
```
", N=21)):
display(Array([[H, J], [K, L], [M, P]]));
```

Graph of $f(x) = x - x^3$, with its Finite Fourier Approximation with varying Number of Terms, including the Error in Approximation

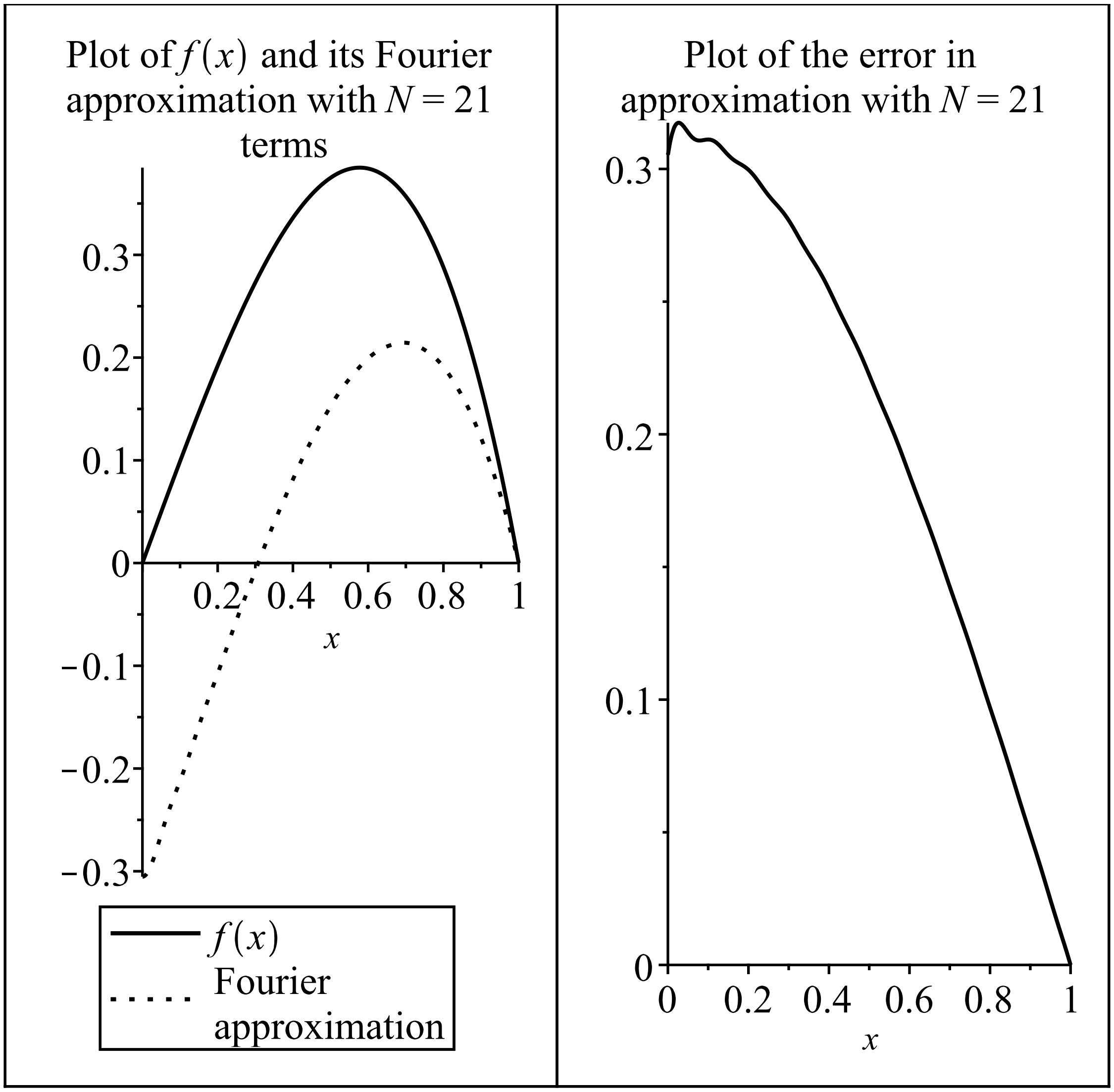

Plot of f(x) and its Fourier approximation with N = 3 terms
0.3
0.2
0.1
0
0.2 0.4 0.6 0.8 1
x
−0.1
−0.2
f(x)
Fourier approximation
Plot of the error in approximation with N = 3
0.3
0.2
0.1
0
0 0.2 0.4 0.6 0.8 1
x



Plot of $f(x)$ and its Fourier approximation with $N = 5$ terms
0.3
0.2
0.1
0
−0.1
−0.2
0.2
0.4
0.6
0.8
1
x
$f(x)$
Fourier approximation



Plot of the error in approximation with $N = 5$
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
x



The Fourier approximation for mixed boundary conditions is terrible. Especially when compared with the plots in the previous question. As the number of terms increases, this approximation doesn't seem to improve very much. It would not be plausable in practice to compute the amount of terms necessary for a reasonable approximation.

(d)

Assemble the finite Fourier series solution to $u_t = 4\, u_{xx}$ with N terms.

```
> assume(N, integer); #reinitialize our assumption
  finiteFourierSoln_Q2 := (x, t, N) -> sum
  (eigenfunctionOfT_Q2 * fourierCoefficientsOfX_Q2 *
  eigenfunctionOfX_Q2, n = 1..N);
```

$$finiteFourierSoln\_Q2 := (x, t, N) \rightarrow \sum_{n=1}^{N} eigenfunctionOfT\_Q2\ fourierCoefficientsOfX\_Q2\ eigenfunctionOfX\_Q2 \qquad \textbf{(1.2.4.1)}$$

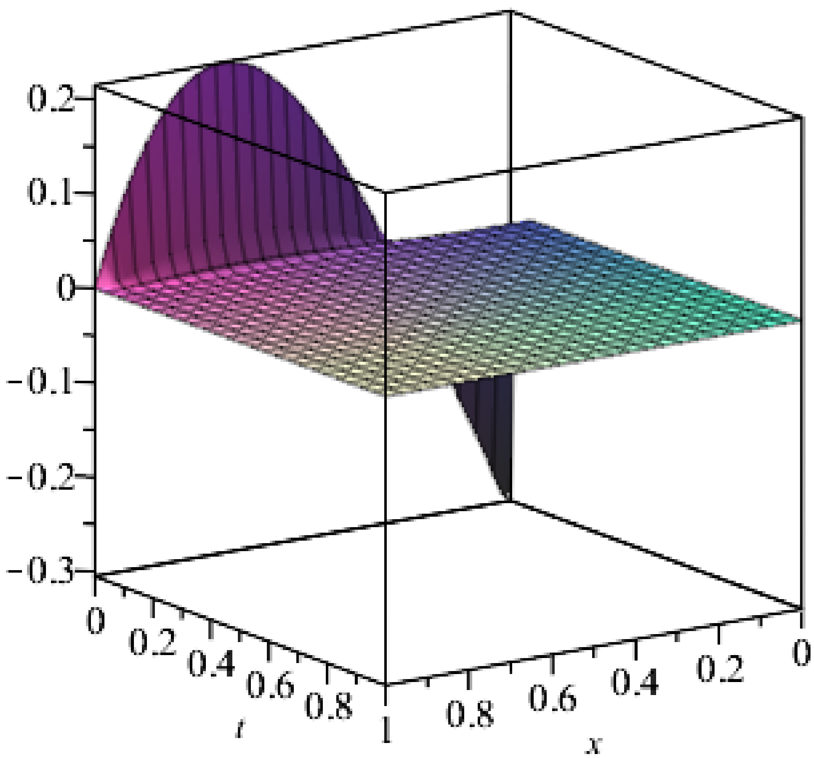
(e)

Plot the solution surface for $N = 21$.

```
> plot3d(finiteFourierSoln_Q2(x, t, 21), x = interval, t =
  interval, title = typeset("Plot of the finite Fourier
```

```
series solution to ", theHeatEquation, " with 21 terms"));
```

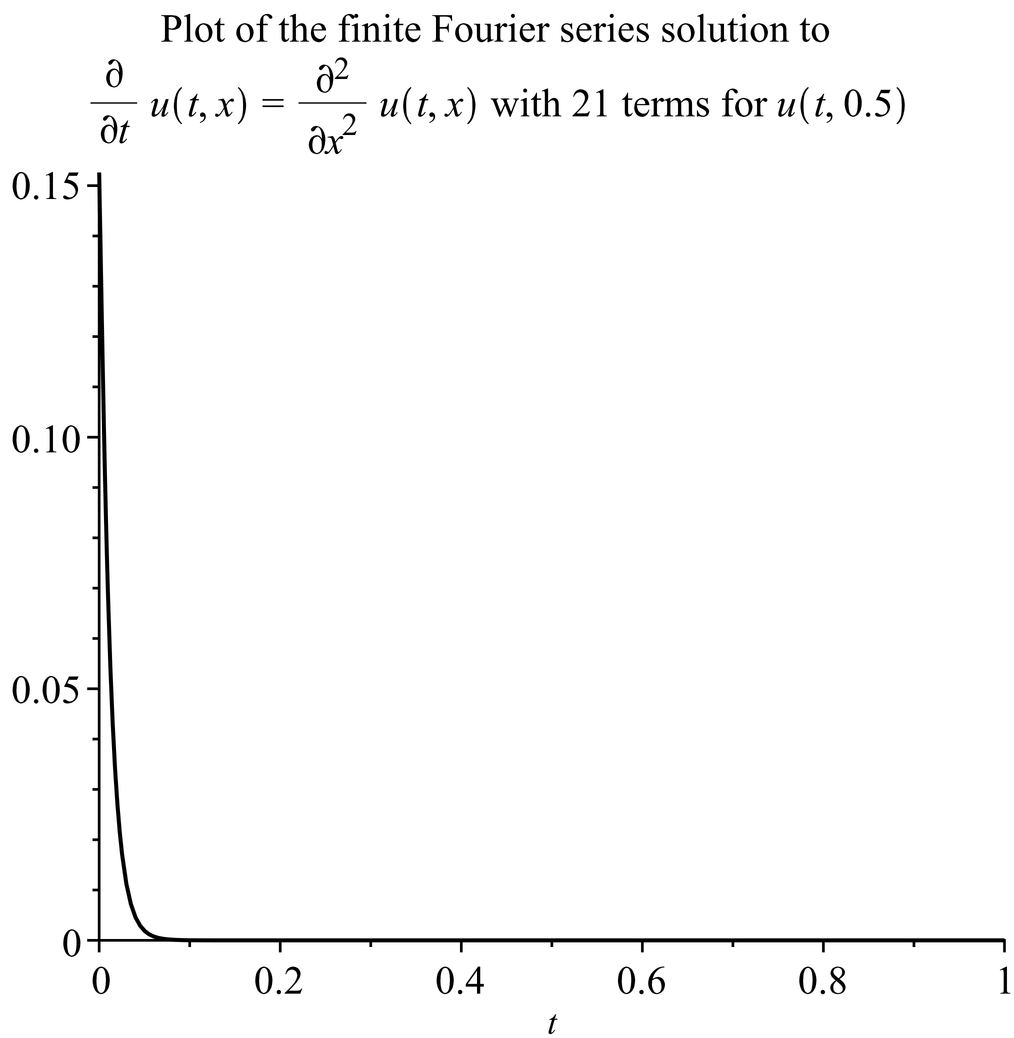
Plot of the finite Fourier series solution to
$$\frac{\partial}{\partial t}\, u(t,x) = \frac{\partial^2}{\partial x^2}\, u(t,x) \text{ with 21 terms}$$



(f)

Plot the finite Fourier series with $N = 21$ for $u(t, 0.5)$ for $0 \le t \le 1$.

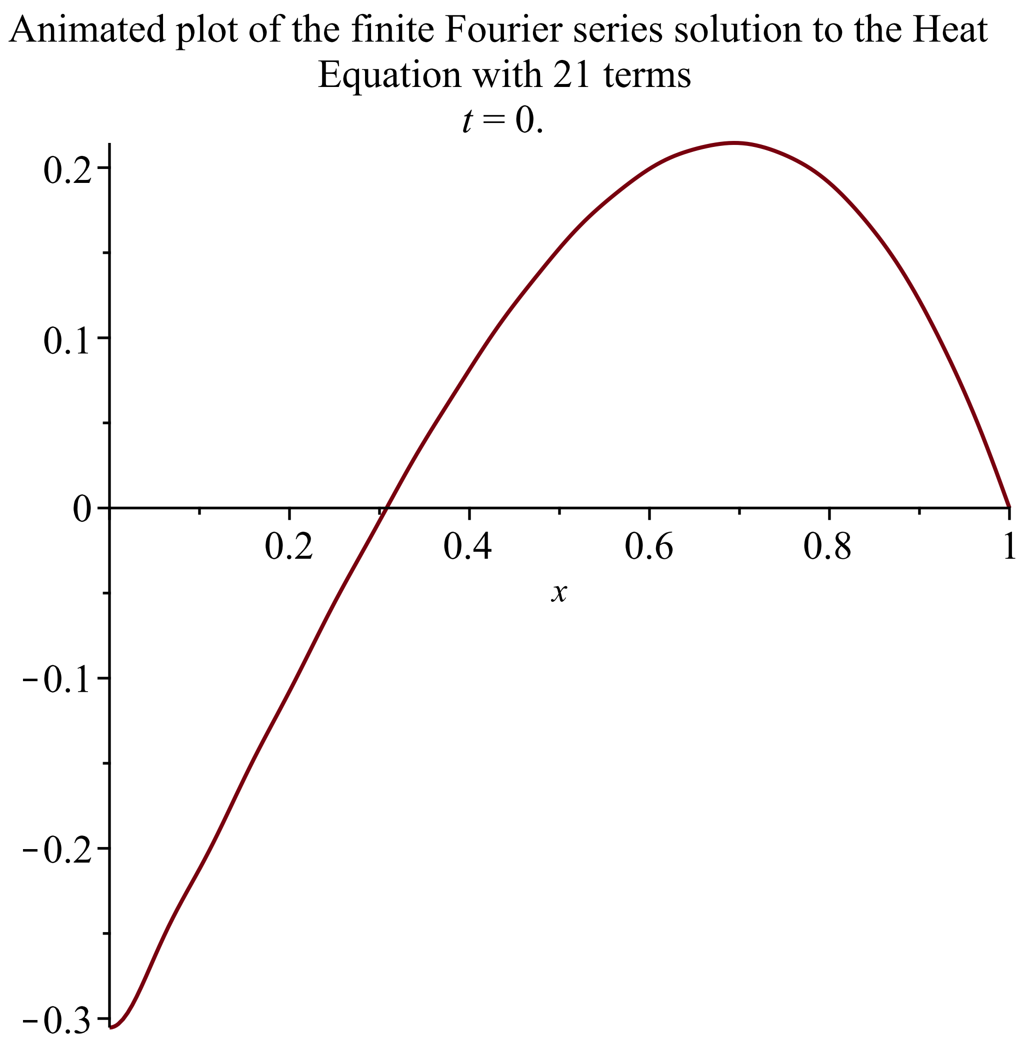
```
> finiteFourierSolnXatOneHalf_Q2 := subs(x = 0.5,
  finiteFourierSoln_Q2(x, t, 21)):
  plot(finiteFourierSolnXatOneHalf_Q2, t = interval, colour
  = black, title = typeset("Plot of the finite Fourier
  series solution to ", theHeatEquation, " with 21 terms for
  ", u(t,0.5)));
```



(g)

Create an animation of the solution for $N = 21$ as t varies.

```
> animate( plot, [finiteFourierSoln_Q2(x, t, 21), x =
  interval, title = typeset("Animated plot of the finite
  Fourier series solution to the Heat Equation with 21
  terms")], t = 0..0.065, frames = 100);
```



Question 3

Boundary and Initial Conditions: $u(t, 0) = u(t, 1) = 0, \quad u(0, x) = 1$

```
> initialCondition_Q3 := 1;
  eigenfunctionOfX_Q3 := sin(n*Pi*x);
  eigenfunctionOfT_Q3 := exp(-4*n^2*Pi^2*t);
```

$$initialCondition\_Q3 := 1$$

$$eigenfunctionOfX\_Q3 := \sin(n{\sim}\,\pi\, x)$$

$$eigenfunctionOfT\_Q3 := e^{-4\, n{\sim}^2\, \pi^2\, t} \tag{1.3.1}$$

We know that the given dirchlett boundary conditions, will produce the eigen functions $X = \sin(n\,\pi\,x)$ for $n \in \mathbb{N}$ and $T = e^{-4\,n^2\,\pi^2\,t}$ for $n \in \mathbb{Z}_{\geq 0}$.

(a)

After the separation of variables, $u(t, x) = X(x)\,T(t)$. Find the coefficients of the Fourier series for X for $f(x) = 1$.

```
> fourierCoefficientsOfX_Q3 := 2*int(eigenfunctionOfX_Q1, x
  = interval);
```

(1.3.1.1)

$$fourierCoefficientsOfX\_Q3 := \frac{2\left(-(-1)^{n\sim}+1\right)}{n\sim\,\pi} \quad \textbf{(1.3.1.1)}$$

(b)

Assemble a finite Fourier approximation of $f(x) = 1$ with N terms (N arbitrary).

```
> finiteFourierApprox_Q3 := (x, N) -> sum
  (fourierCoefficientsOfX_Q3 * eigenfunctionOfX_Q3, n = 1..
  N);
```

$$finiteFourierApprox\_Q3 := (x, N) \rightarrow \sum_{n=1}^{N} fourierCoefficientsOfX\_Q3\ eigenfunctionOfX\_Q3 \quad \textbf{(1.3.2.1)}$$

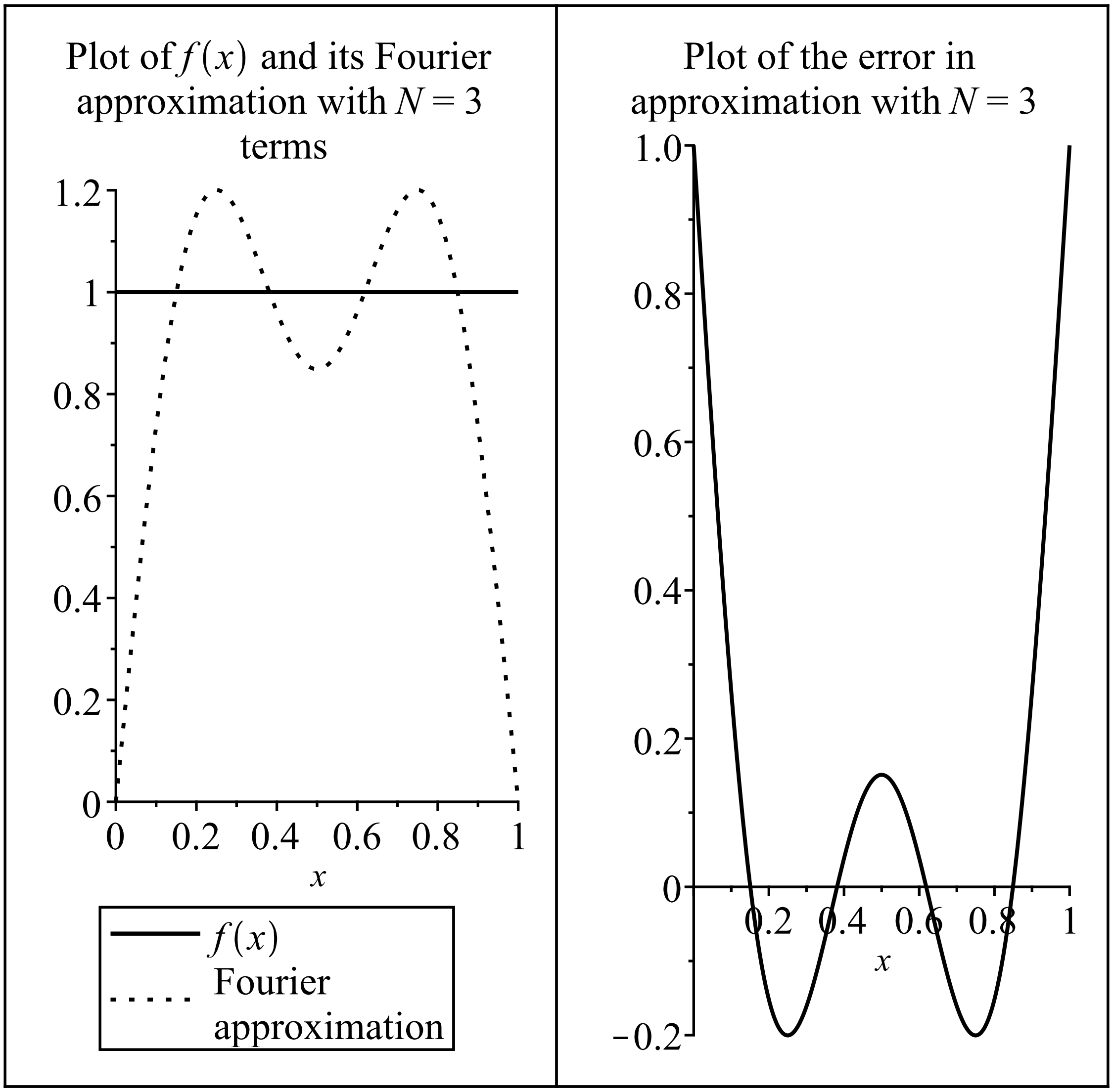
(c)

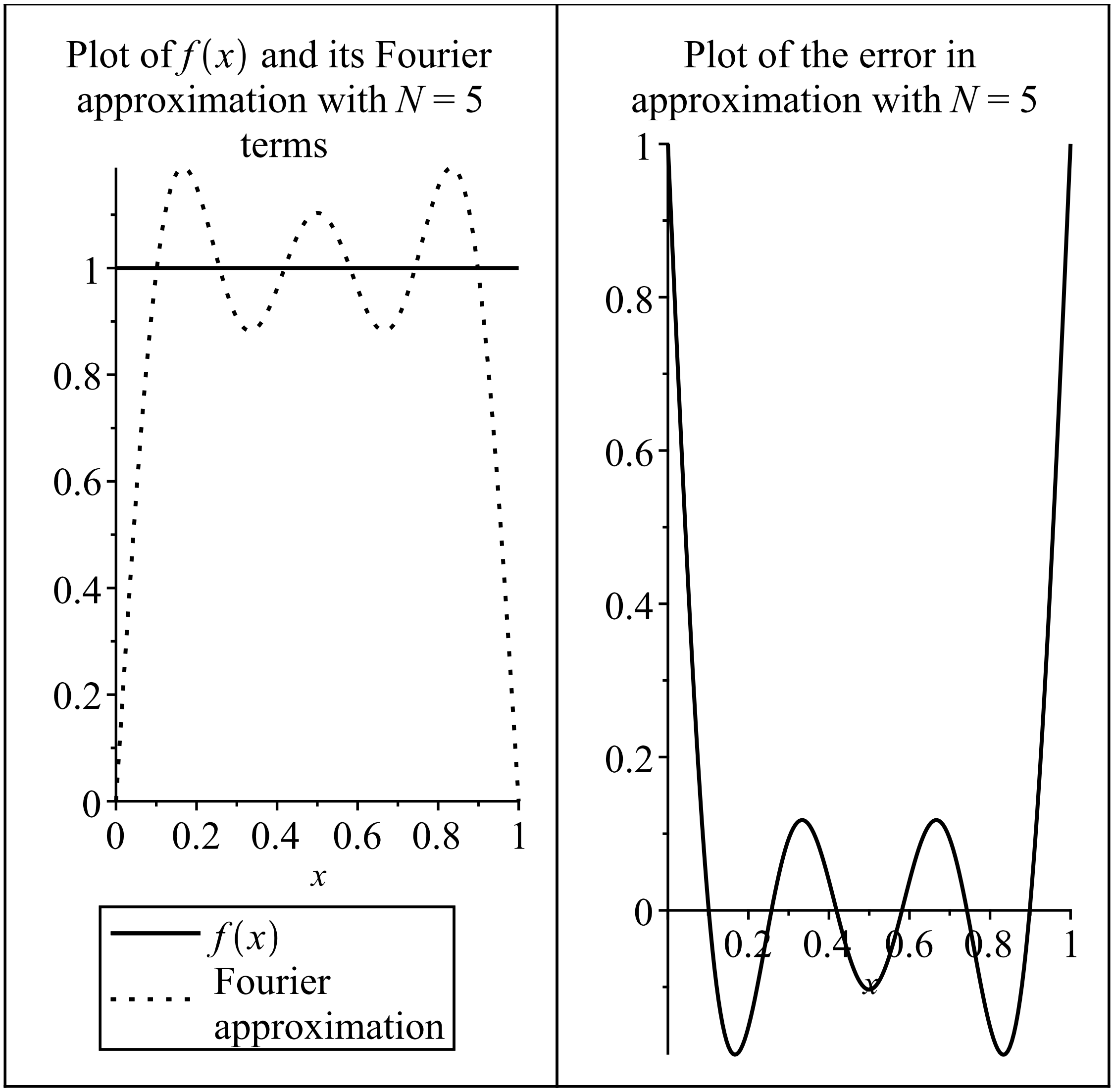
Plot $f(x) = 1$ and your series on the same graph for $N = 3$, 5, 21, 51, and 101, and plot the difference.

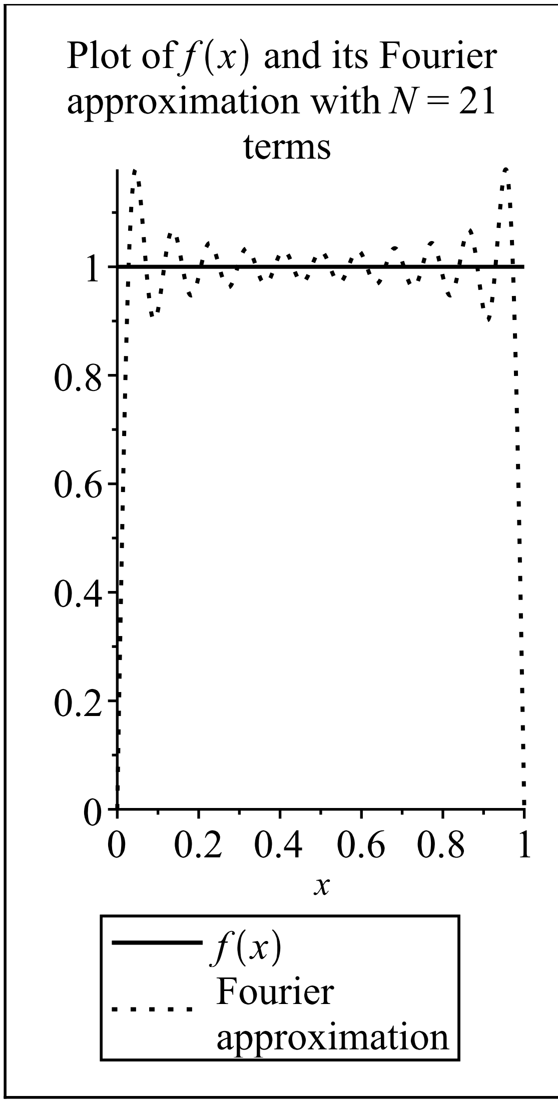
```
> N := 'N': #take away the tilda displayed after the'N' so
  the graph title is in proper format

  Q := plot([initialCondition_Q3, finiteFourierApprox_Q3(x,
  3)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=3, " terms")):

  R := plot(initialCondition_Q3 - finiteFourierApprox_Q3(x,
  3), x = interval, colour = black, linestyle = [1,2], title
  = typeset("Plot of the error in approximation with ", N=3)
  ):


  S := plot([initialCondition_Q3, finiteFourierApprox_Q3(x,
  5)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=5, " terms")):

  T := plot(initialCondition_Q3 - finiteFourierApprox_Q3(x,
  5), x = interval, colour = black, linestyle = [1,2], title
  = typeset("Plot of the error in approximation with ", N=5)
  ):


  U := plot([initialCondition_Q3, finiteFourierApprox_Q3(x,
  21)], x = interval, colour = black, linestyle = [1,2],
  legend = [typeset(f(x)), "Fourier approximation"], title =
  typeset("Plot of ", f(x), " and its Fourier approximation
  with ", N=21, " terms")):

  V := plot(initialCondition_Q3 - finiteFourierApprox_Q3(x,
  21), x = interval, colour = black, linestyle = [1,2],
  title = typeset("Plot of the error in approximation with
  ", N=21)):
```

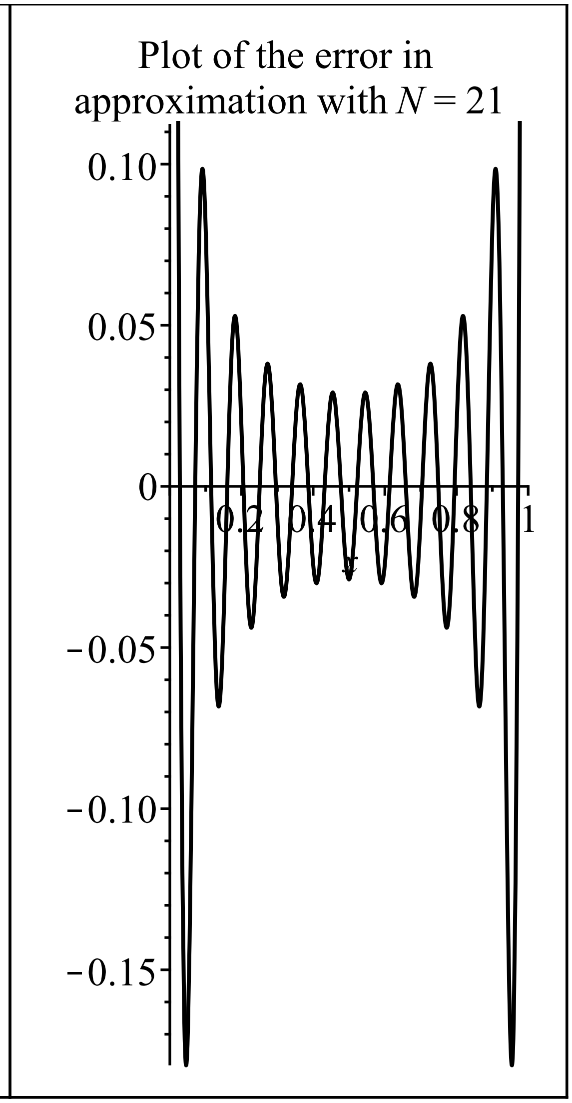
```
W := plot([initialCondition_Q3, finiteFourierApprox_Q3(x,
51)], x = interval, colour = black, linestyle = [1,2],
legend = [typeset(f(x)), "Fourier approximation"], title =
typeset("Plot of ", f(x), " and its Fourier approximation
with ", N=51, " terms")):

X := plot(initialCondition_Q3 - finiteFourierApprox_Q3(x,
51), x = interval, colour = black, linestyle = [1,2],
title = typeset("Plot of the error in approximation with
", N=51)):


Y := plot([initialCondition_Q3, finiteFourierApprox_Q3(x,
101)], x = interval, colour = black, linestyle = [1,2],
legend = [typeset(f(x)), "Fourier approximation"], title =
typeset("Plot of ", f(x), " and its Fourier approximation
with ", N=101, " terms")):

Z := plot(initialCondition_Q3 - finiteFourierApprox_Q3(x,
101), x = interval, colour = black, linestyle = [1,2],
title = typeset("Plot of the error in approximation with
", N=101)):

display(Array([[Q, R], [S, T], [U, V], [W, X], [Y, Z]]));
```

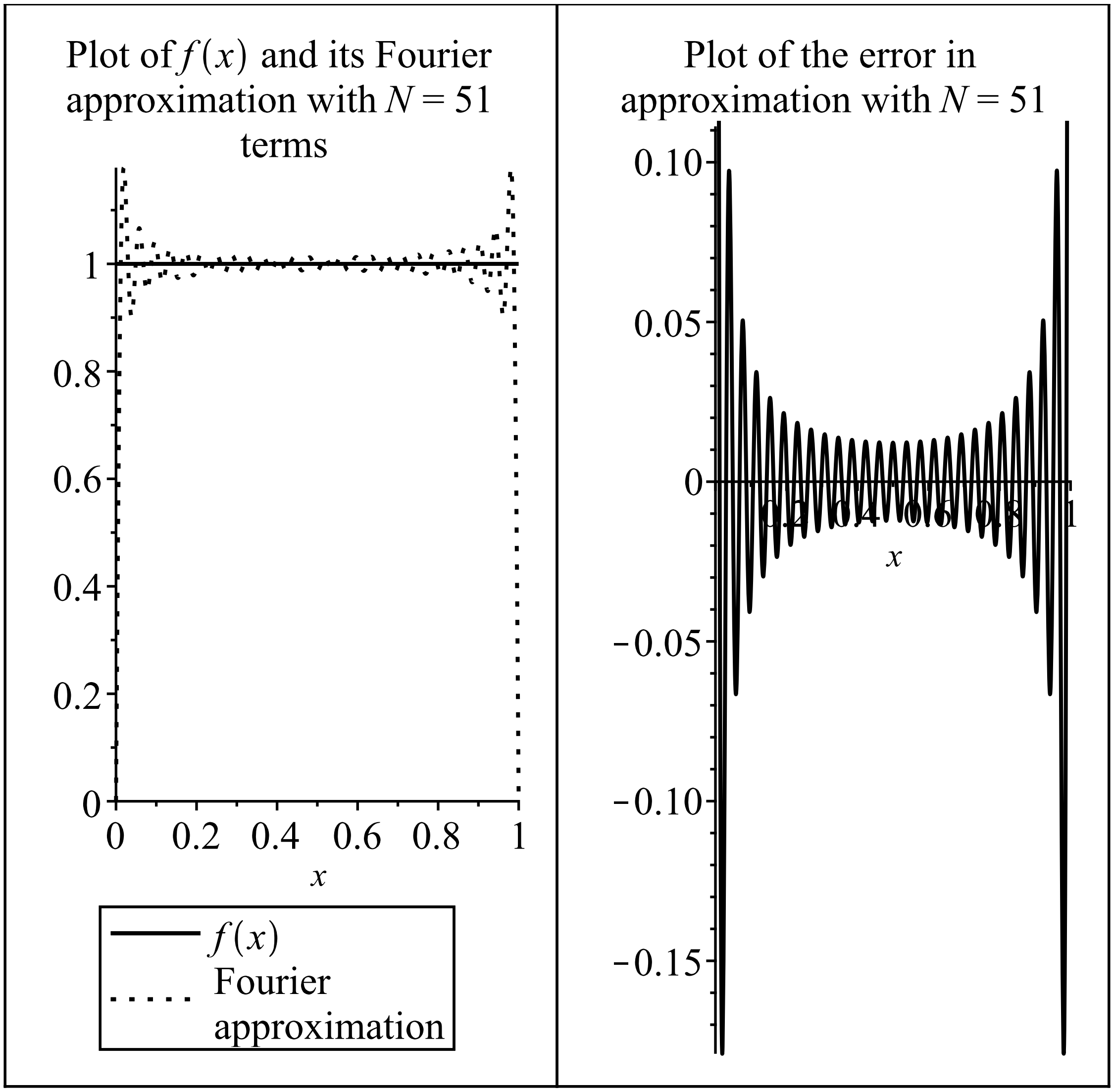
Graph of $f(x) = x - x^3$, with its Finite Fourier Approximation with varying Number of Terms, including the Error in Approximation


Plot of f(x) and its Fourier approximation with N = 3 terms
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x
f(x)
Fourier approximation
Plot of the error in approximation with N = 3
1.0
0.8
0.6
0.4
0.2
0
-0.2
0.2
0.4
0.6
0.8
1
x

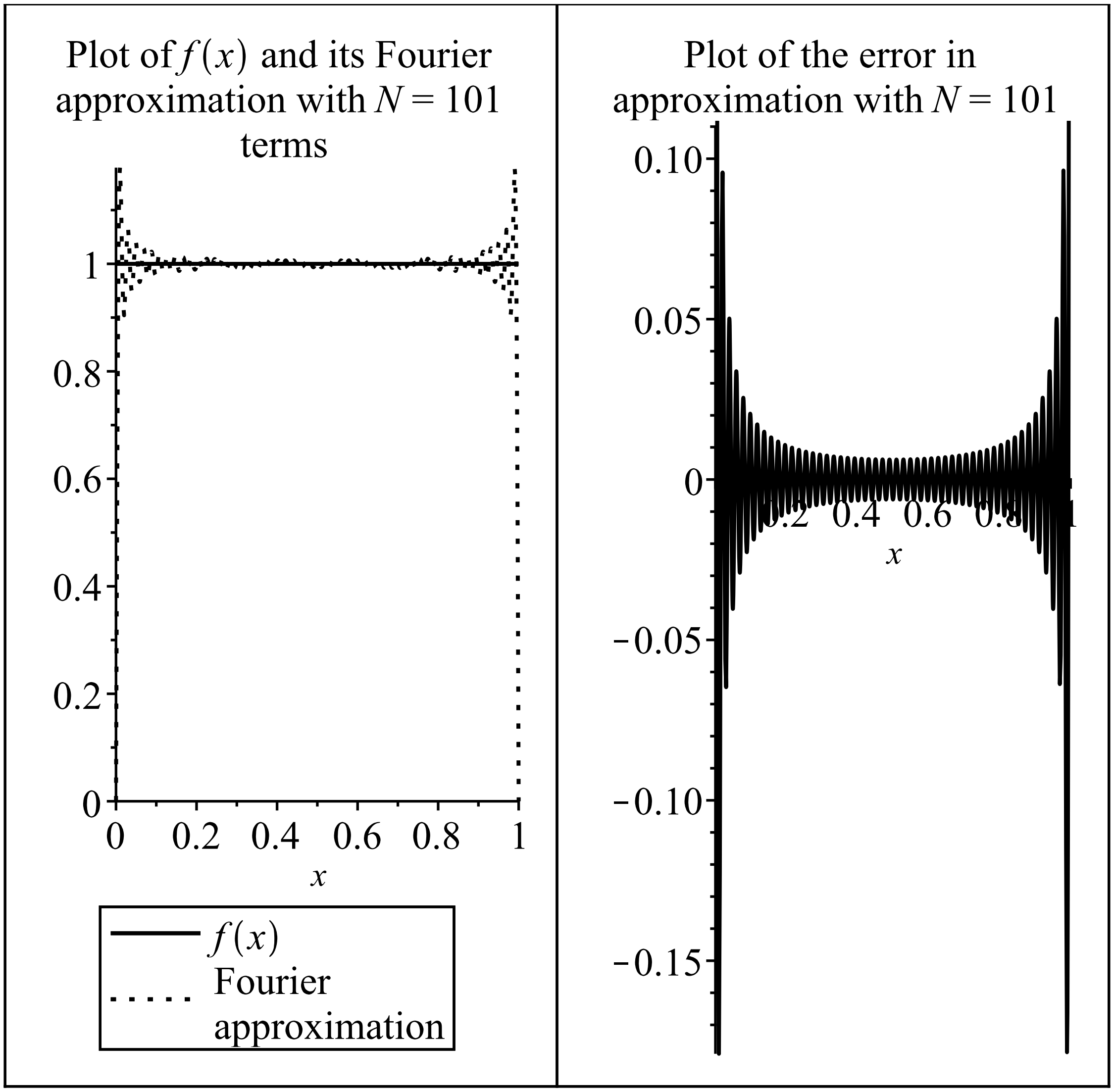

Plot of $f(x)$ and its Fourier approximation with $N = 5$ terms
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1
x
$f(x)$
Fourier approximation
Plot of the error in approximation with $N = 5$
1
0.8
0.6
0.4
0.2
0
0.2 0.4 0.6 0.8 1
x


Plot of f(x) and its Fourier approximation with N = 21 terms
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x
f(x)
Fourier approximation


Plot of the error in approximation with N = 21
0.10
0.05
0
−0.05
−0.10
−0.15
0.2
0.4
0.6
0.8
1
x



Plot of $f(x)$ and its Fourier approximation with $N = 51$ terms
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x
$f(x)$
Fourier approximation
Plot of the error in approximation with $N = 51$
0.10
0.05
0
−0.05
−0.10
−0.15
0.2
0.4
0.6
0.8
1
x



Fourier approximations for $u(0, x) = 1$ will never be a good for all x. No matter how many terms are used in the sum, observe that near the endpoints the approximation is garbage. Even in the centre of the interval after using one hundred terms to approximate, the error is about 0.01.

(d)

Assemble the finite Fourier series solution to $u_t = 4\, u_{xx}$ with N terms.

```
> assume(N, integer); #reinitialize our assumption
  finiteFourierSoln_Q3 := (x, t, N) -> sum
  (eigenfunctionOfT_Q3 * fourierCoefficientsOfX_Q3 *
  eigenfunctionOfX_Q3, n = 1..N);
```

$$finiteFourierSoln\_Q3 := (x, t, N) \rightarrow \sum_{n=1}^{N} eigenfunctionOfT\_Q3\ fourierCoefficientsOfX\_Q3\ eigenfunctionOfX\_Q3 \tag{1.3.4.1}$$

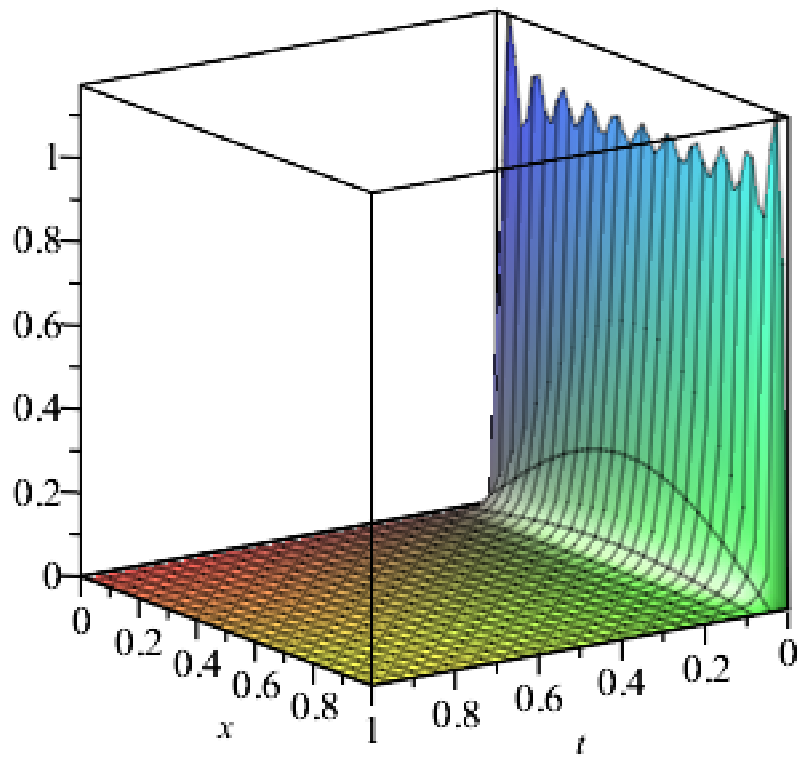
(e)

Plot the solution surface for $N = 21$.

```
> plot3d(finiteFourierSoln_Q3(x, t, 21), t = interval, x =
  interval, title = typeset("Plot of the finite Fourier
```

```
series solution to ", theHeatEquation, " with 21 terms"));
```

Plot of the finite Fourier series solution to
$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x)$ with 21 terms



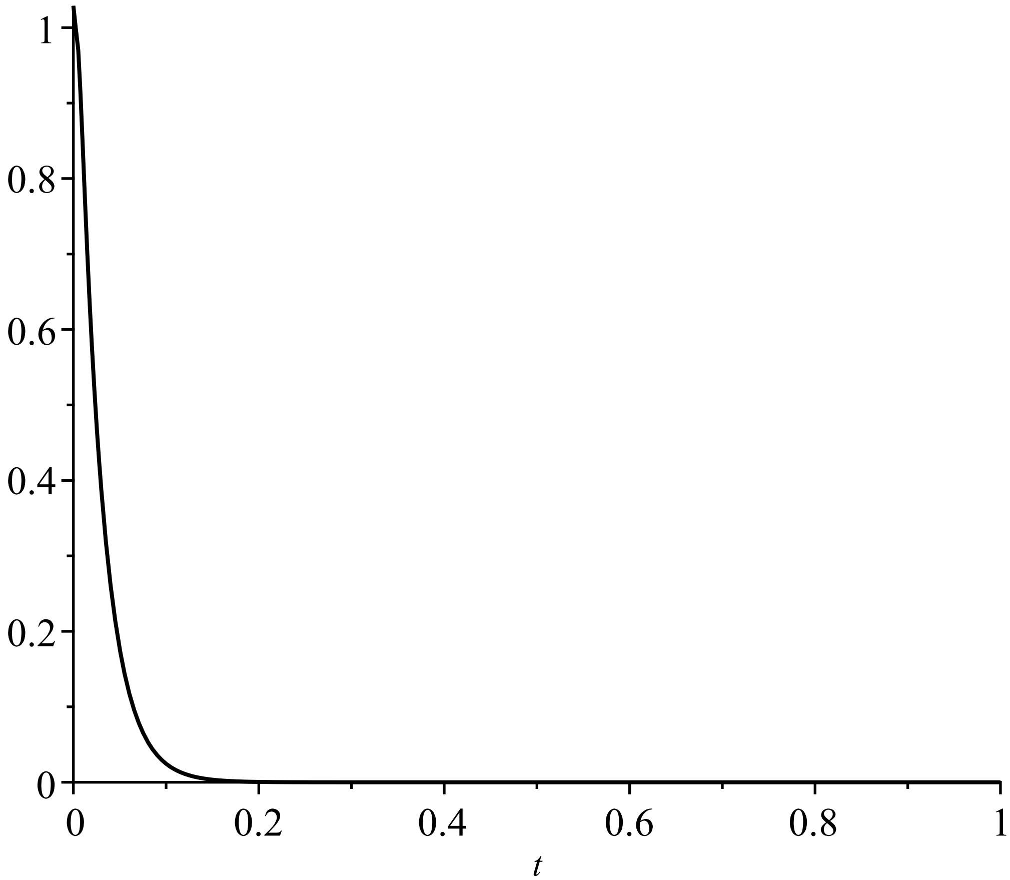
(f)

Plot the finite Fourier series with $N = 21$ for $u(t, 0.5)$ for $0 \le t \le 1$.

```
> finiteFourierSolnXatOneHalf_Q3 := subs(x = 0.5,
  finiteFourierSoln_Q3(x, t, 21)):
  plot(finiteFourierSolnXatOneHalf_Q3, t = interval, colour
  = black, title = typeset("Plot of the finite Fourier
  series solution to ", theHeatEquation, " with 21 terms for
  ", u(t,0.5)));
```

Plot of the finite Fourier series solution to

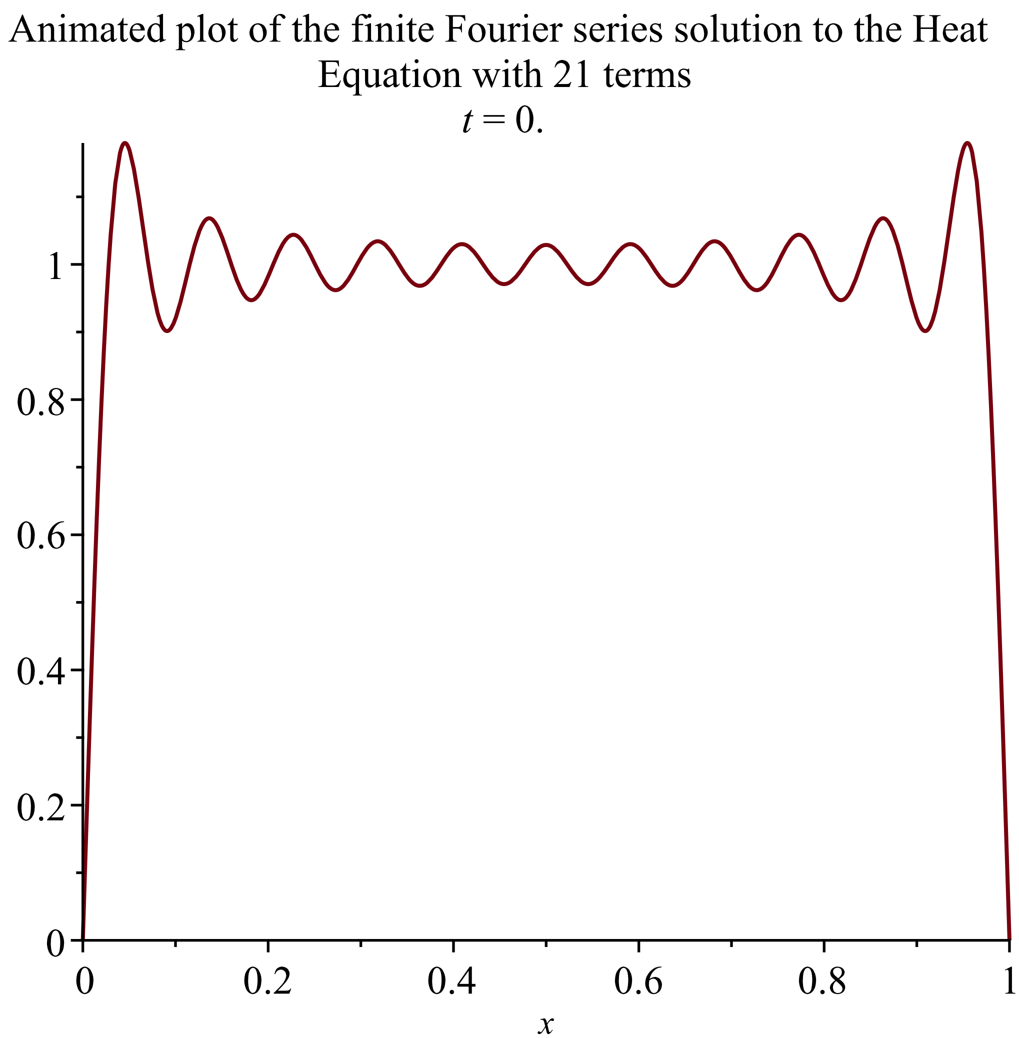
$$\frac{\partial}{\partial t}\, u(t, x) = \frac{\partial^2}{\partial x^2}\, u(t, x) \text{ with 21 terms for } u(t, 0.5)$$



(g)

Create an animation of the solution for $N = 21$ as t varies.

```
> animate( plot, [finiteFourierSoln_Q3(x, t, 21), x =
  interval, title = typeset("Animated plot of the finite
  Fourier series solution to the Heat Equation with 21
  terms")], t = 0..0.15, frames = 100);
```



The approximation when $t = 0$ is nonsense. The constant function $f(x) = 1$ is not represented very well. However, as t increases, the approximation improves tremendously and seems to react as expected.

Question 4

We solve $u_t = 4\,u_{xx}$ on $[0, 1]$ with $f(x) = 1 + \frac{1}{2}x - \frac{2}{3}x^2$, and the Robin boundary conditions $X(0) - 2\,X'(0) = X(1) + X'(1) = 0$.

```
> initialCondition_Q4 := x -> 1 + 1/2*x - 2/3*x^2;
```

$$initialCondition\_Q4 := x \rightarrow 1 + \frac{1}{2}\,x - \frac{2}{3}\,x^2 \tag{2.1}$$

(a)

Find the eigenfunctions and eigenvalues for X.

```
> A := 'A': B := 'B': #clear variables used above in plotting
  the array of graphs

  solution_Q4 := x -> A*sin(sqrt(lambda)*x)+B*cos(sqrt(lambda)*
  x);
```

$$solution\_Q4 := x \rightarrow A \sin\left(\sqrt{\lambda}\, x\right) + B \cos\left(\sqrt{\lambda}\, x\right) \quad \textbf{(2.1.1)}$$

```
> robinBoundaryCondition := [solution_Q4(0)-2*D(solution_Q4)(0)
=0, solution_Q4(1)+D(solution_Q4)(1)=0];
```

$$robinBoundaryCondition := \left[B - 2A\sqrt{\lambda} = 0, A \sin\left(\sqrt{\lambda}\right) + B \cos\left(\sqrt{\lambda}\right) + A\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) - B\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) = 0\right] \quad \textbf{(2.1.2)}$$

```
> solveFirstRobinBoundaryEquationFor_B := solve
(robinBoundaryCondition[1], B);
```

$$solveFirstRobinBoundaryEquationFor\_B := 2A\sqrt{\lambda} \quad \textbf{(2.1.3)}$$

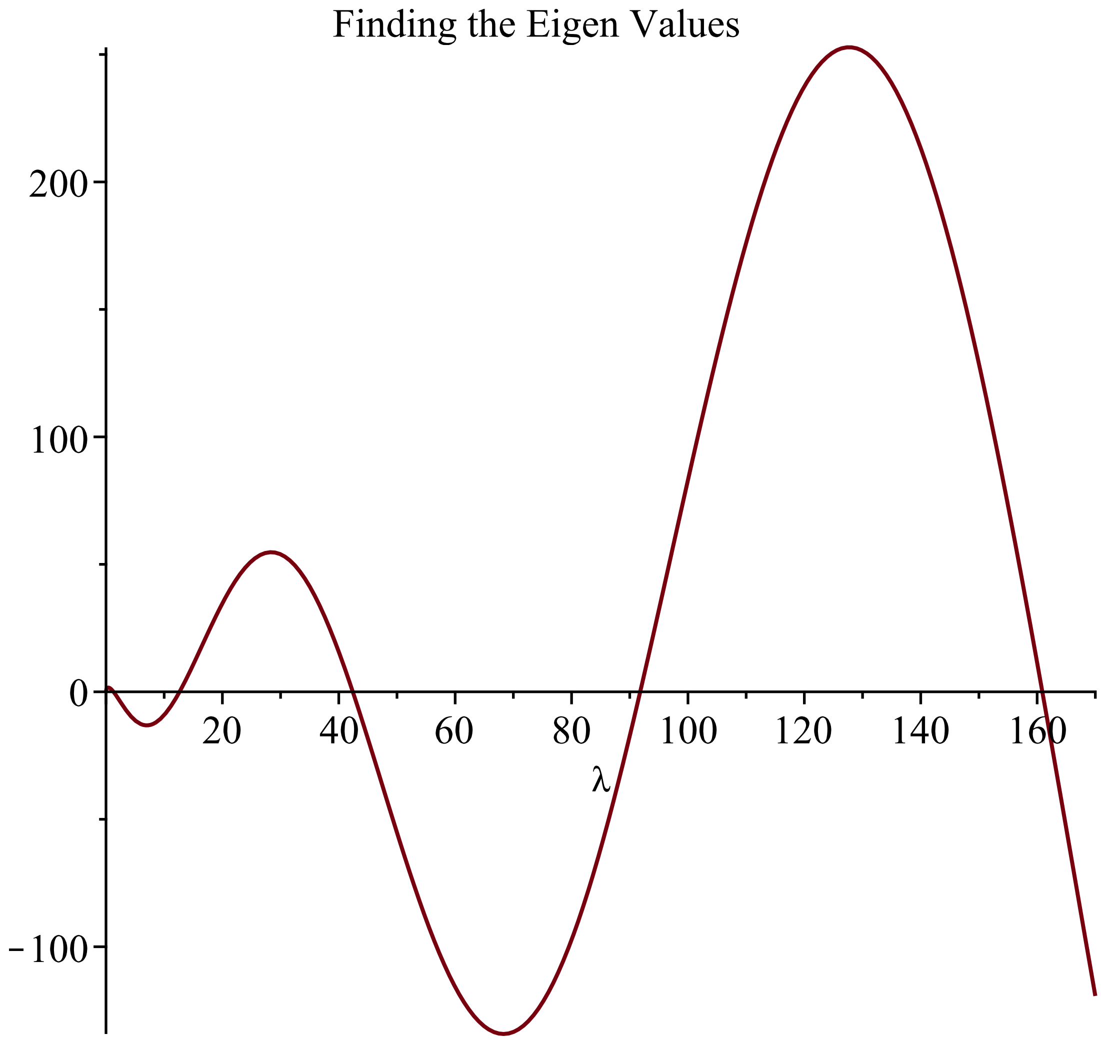
```
> substitute_B_intoSecondRobinBoundaryEquation := subs(B =
solveFirstRobinBoundaryEquationFor_B, robinBoundaryCondition
[2]);
```

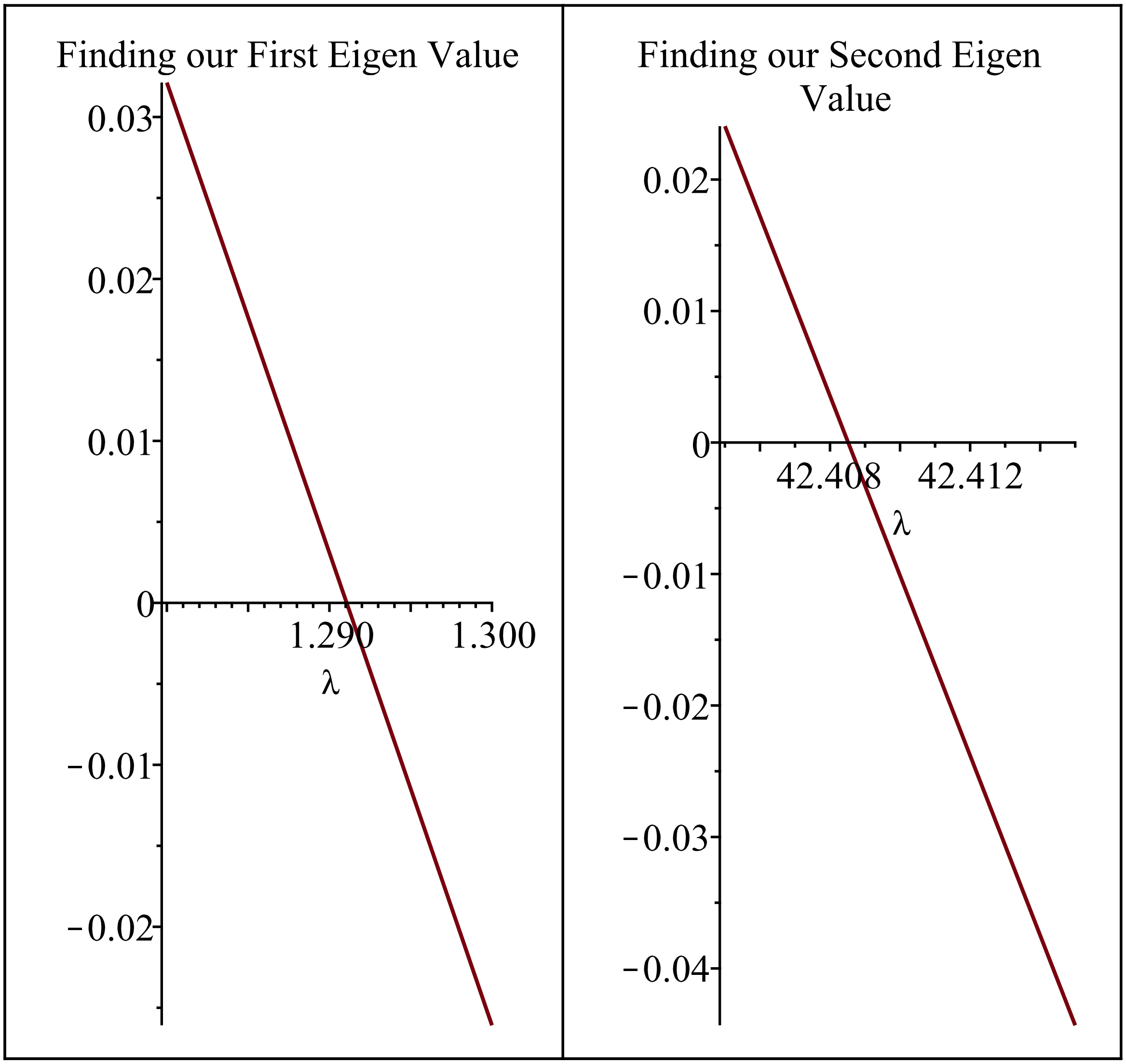
$$substitute\_B\_intoSecondRobinBoundaryEquation := A \sin\left(\sqrt{\lambda}\right) + 3A\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) - 2A\lambda\sin\left(\sqrt{\lambda}\right) = 0 \quad \textbf{(2.1.4)}$$

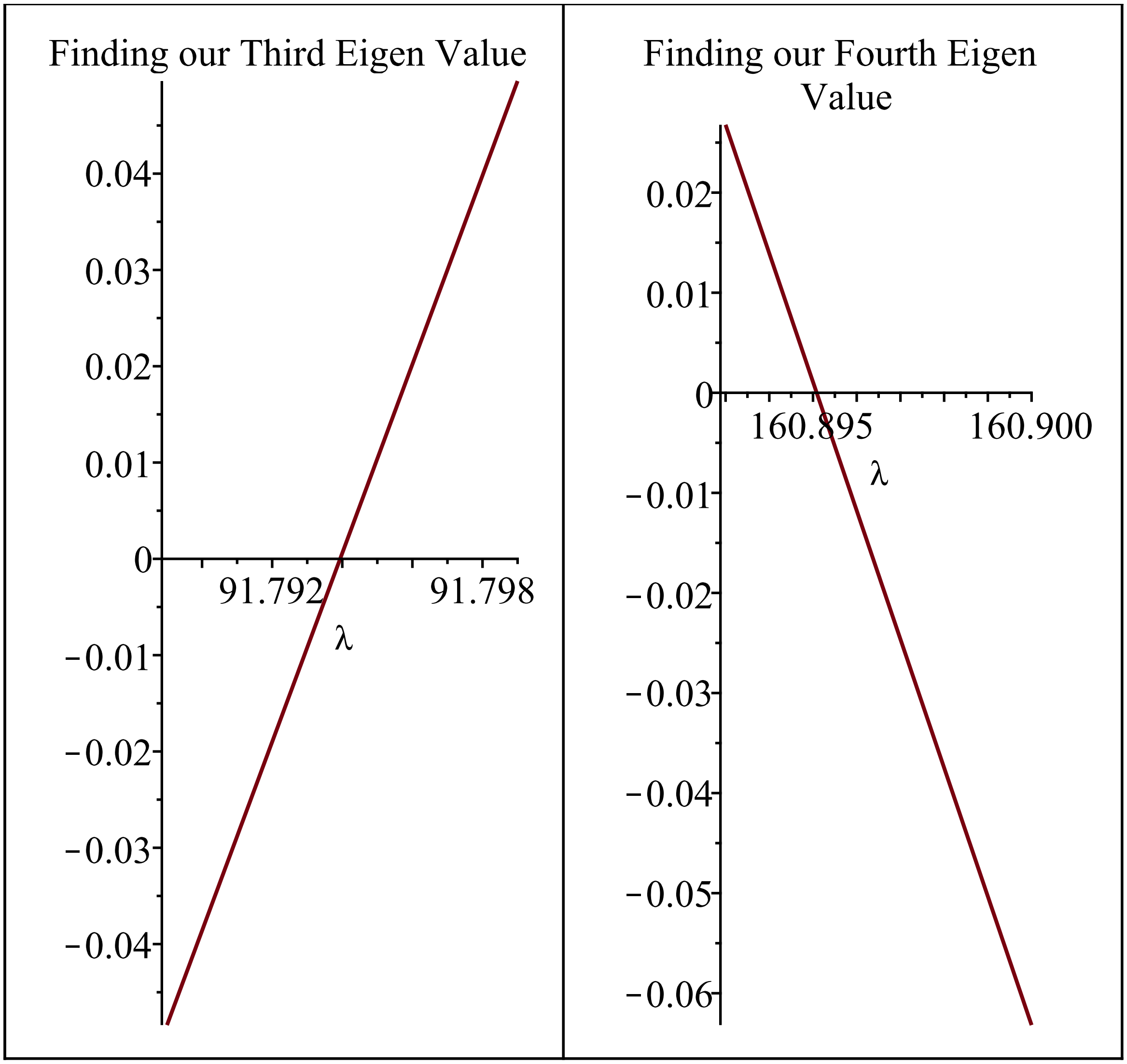
```
> findEigenValues := subs(A=1,
substitute_B_intoSecondRobinBoundaryEquation);
```

$$findEigenValues := \sin\left(\sqrt{\lambda}\right) + 3\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) - 2\lambda\sin\left(\sqrt{\lambda}\right) = 0 \quad \textbf{(2.1.5)}$$

```
> plot(lhs(findEigenValues), lambda = 0..170, title = "Finding
the Eigen Values");

A := plot(3*lambda^(1/2)*cos(lambda^(1/2))-2*sin(lambda^(1/2)
)*lambda+sin(lambda^(1/2)), lambda = 1.28..1.3, title =
"Finding our First Eigen Value"):

B := plot(3*lambda^(1/2)*cos(lambda^(1/2))-2*sin(lambda^(1/2)
)*lambda+sin(lambda^(1/2)), lambda = 42.405..42.415, title =
"Finding our Second Eigen Value"):

C := plot(3*lambda^(1/2)*cos(lambda^(1/2))-2*sin(lambda^(1/2)
)*lambda+sin(lambda^(1/2)), lambda = 91.789..91.799, title =
"Finding our Third Eigen Value"):

E := plot(3*lambda^(1/2)*cos(lambda^(1/2))-2*sin(lambda^(1/2)
)*lambda+sin(lambda^(1/2)), lambda = 160.893..160.90, title =
"Finding our Fourth Eigen Value"):


display(Array([[A, B], [C, E]]));
```


Finding the Eigen Values
200
100
0
-100
20
40
60
80
100
120
140
160
λ


Finding our First Eigen Value
0.03
0.02
0.01
0
-0.01
-0.02
1.290
1.300
λ
Finding our Second Eigen Value
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
42.408
42.412
λ



Now we will guess at the values of lambda and use fsolve to aquire more decimal places.

```
> B := 'B': #clear this variable used in the arrays above

  eigenValue_1 := evalf[11](fsolve(findEigenValues, lambda =
  1.290..1.292));
  eigenValue_2 := evalf[12](fsolve(findEigenValues, lambda =
  42.408..42.41));
  eigenValue_3 := evalf[12](fsolve(findEigenValues, lambda =
  91.79..91.796));
  eigenValue_4 := evalf[13](fsolve(findEigenValues, lambda =
  160.895..160.896));
  eigenValue := [eigenValue_1, eigenValue_2, eigenValue_3,
  eigenValue_4]:
```

$$eigenValue\_1 := 1.2910617252$$

$$eigenValue\_2 := 42.4085185010$$

$$eigenValue\_3 := 91.7939388388$$

$$eigenValue\_4 := 160.8950838685 \tag{2.1.6}$$

```
> eigenfunctionOfX_Q4 := (x, n) -> sin(sqrt(eigenValue[n])*x)
  +2*sqrt(eigenValue[n])*cos(sqrt(eigenValue[n])*x);
  eigenfunctionOfT_Q4 := (t, n) -> exp(-4*eigenValue[n]*t);
```

$$eigenfunctionOfX\_Q4 := (x, n) \rightarrow \sin\left(\sqrt{eigenValue_n}\, x\right) + 2\sqrt{eigenValue_n}\, \cos\left(\sqrt{eigenValue_n}\, x\right)$$

$$eigenfunctionOfT\_Q4 := (t, n) \rightarrow e^{-4\, eigenValue_n\, t} \quad \textbf{(2.1.7)}$$

(b)

Assemble the Fourier approximations given by your 4 λ's.

```
> fourierCoefficientsOfX_Q4 := n -> int(initialCondition_Q4(x)
  * eigenfunctionOfX_Q4(x,n), x = interval) / (int
  (eigenfunctionOfX_Q4(x,n)^2, x = interval));
```

$$fourierCoefficientsOfX\_Q4 := n \rightarrow \frac{int(initialCondition\_Q4(x)\ eigenfunctionOfX\_Q4(x, n), x = interval)}{int\left(eigenfunctionOfX\_Q4(x, n)^2, x = interval\right)} \quad \textbf{(2.2.1)}$$

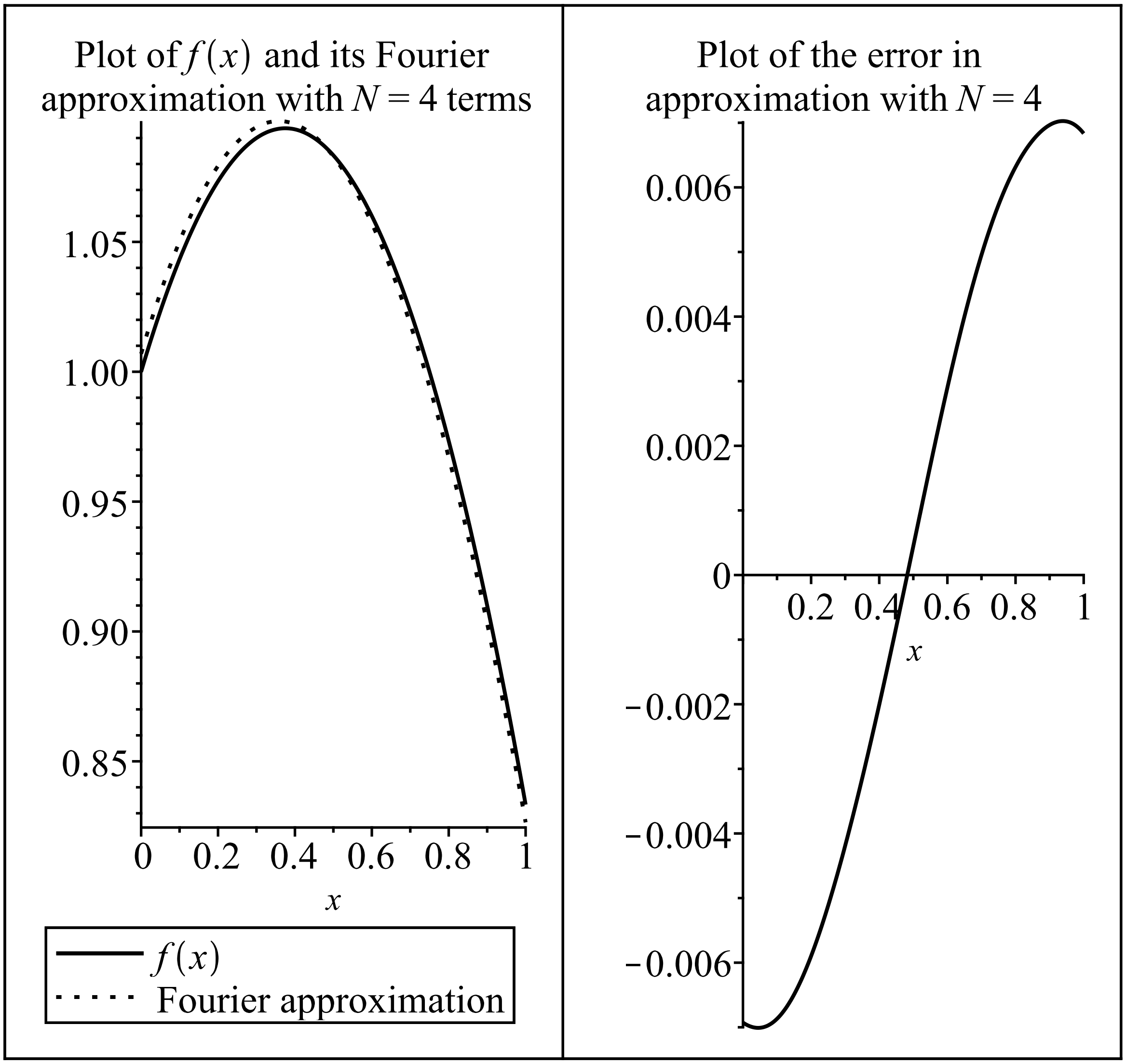
```
> finiteFourierApprox_Q4 := (x, N) -> sum
  (fourierCoefficientsOfX_Q4(n) * eigenfunctionOfX_Q4(x,n), n =
  1..N);
```

$$finiteFourierApprox\_Q4 := (x, N) \rightarrow \sum_{n=1}^{N} fourierCoefficientsOfX\_Q4(n)\ eigenfunctionOfX\_Q4(x, n) \quad \textbf{(2.2.2)}$$

(c)

Plot $f(x)$ and your approximation on the same graph, and plot the difference on a separate graph.

```
> E := 'E': F := 'F': #clear these variables used in the arrays
  above
  N := 'N': #take away the tilda displayed after the'N' so the
  graph title is in proper format

  E := plot([initialCondition_Q4(x), finiteFourierApprox_Q4(x,
  4)], x = interval, colour = black, linestyle = [1,2], legend
  = [typeset(f(x)), "Fourier approximation"], title = typeset
  ("Plot of ", f(x), " and its Fourier approximation with ", N=
  4, " terms")):

  F := plot(initialCondition_Q4(x) - finiteFourierApprox_Q4(x,
  4), x = interval, colour = black, linestyle = [1,2], title =
  typeset("Plot of the error in approximation with ", N=4)):


  display(Array([E, F]));
```

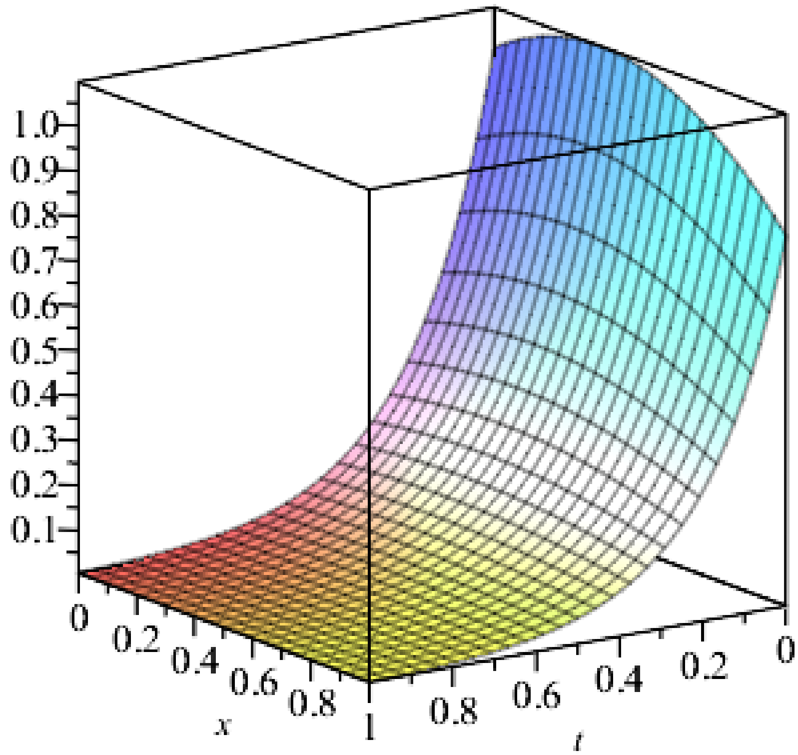


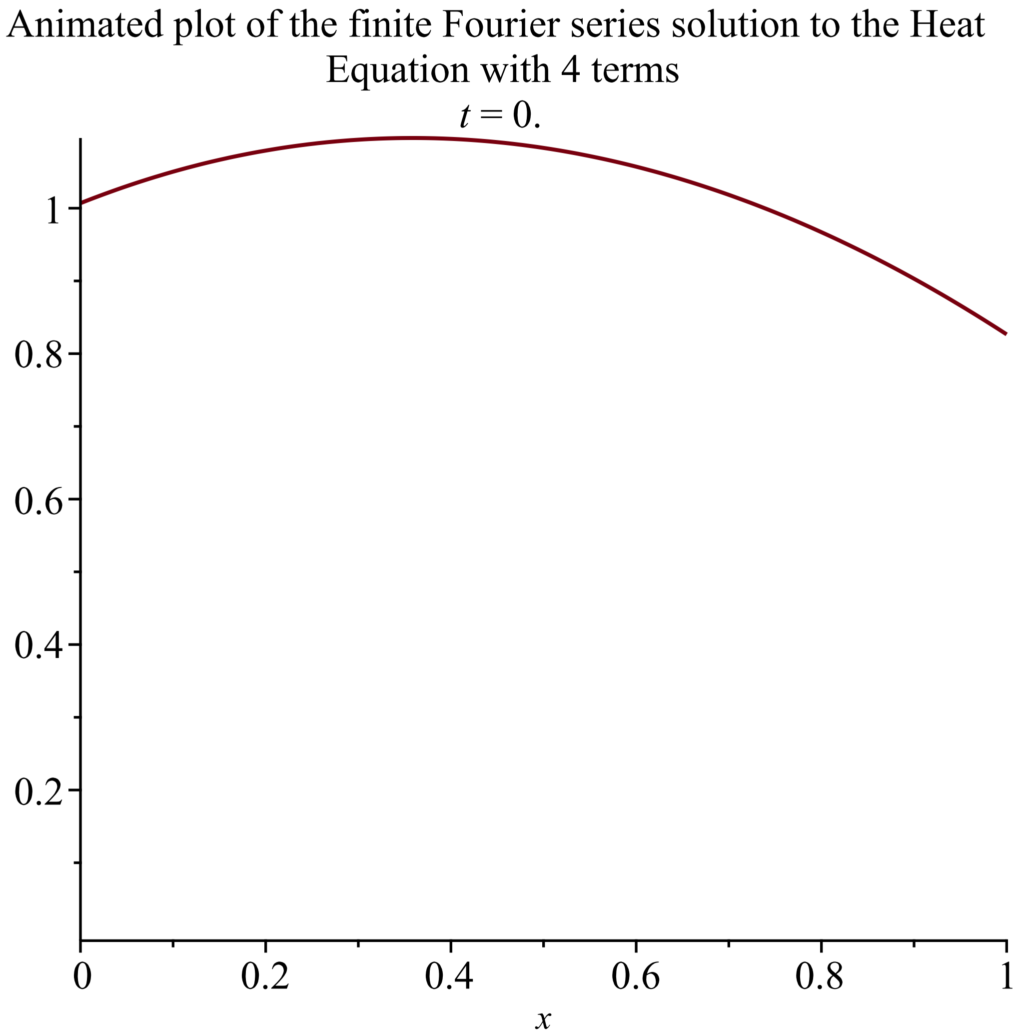
(d)

Assemble the first 4 terms of the Fourier solution to $u_t = 4\,u_{xx}$. Plot the solution and create an animation as t varies between 0 and 1.

```
> assume(N, integer);
> finiteFourierSoln_Q4 := (x, t, N) -> sum(eigenfunctionOfT_Q4
  (t, n) * fourierCoefficientsOfX_Q4(n) * eigenfunctionOfX_Q4
  (x, n), n = 1..N):
  finiteFourierSoln_Q4(x, t, 4):

> plot3d(finiteFourierSoln_Q4(x, t, 4), t = interval, x =
  interval, title = typeset("Plot of the finite Fourier series
  solution to ", theHeatEquation, " with 4 terms"));

  animate( plot, [finiteFourierSoln_Q4(x, t, 4), x = interval,
  title = typeset("Animated plot of the finite Fourier series
  solution to the Heat Equation with 4 terms")], t = 0..1.6,
  frames = 100);
```

Plot of the finite Fourier series solution to
$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x)$ with 4 terms




Animated plot of the finite Fourier series solution to the Heat Equation with 4 terms
t = 0.
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
x