The conformal Killing spinor initial data equations

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Abstract

We obtain necessary and sufficient conditions for an initial data set for the conformal Einstein field equations to give rise to a spacetime development in possession of a Killing spinor. This constitutes the conformal analogue of the Killing spinor initial data equations derived in [16]. The fact that the conformal Einstein field equations are used in our derivation allows for the possibility that the initial hypersurface be (part of) the conformal boundary $\mathscr{I}$. For conciseness, these conditions are derived assuming that the initial hypersurface is spacelike. Consequently, these equations encode necessary and sufficient conditions for the existence of a Killing spinor in the development of asymptotic initial data on spacelike components of $\mathscr{I}$.

1 Introduction

The discussion of symmetries in General Relativity is ubiquitous. From the question of integrability of the geodesic equations to the existence of explicit solutions to the Einstein field equations and the black hole uniqueness problem, symmetries always play an important role. Symmetry assumptions are usually incorporated into the Einstein field equations —which in vacuum read

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \tag{1}$$

through the use of Killing vectors. From the spacetime point of view, the existence of Killing vectors allows one to perform *symmetry reductions* of the Einstein field equations —see for instance [33]. This approach has been exploited in classical uniqueness results such as [27]. Closely related to the black hole uniqueness problem, characterisations and classifications of solutions to the Einstein field equations usually exploit the symmetries of the spacetime in one way or another, e.g., in the characterisations of the Kerr spacetime via the *Mars-Simon tensor* —see [18, 19, 28]. On the other hand, from the point of view of the Cauchy problem, symmetry assumptions should be imposed only at the level of initial data. In this regard, symmetry assumptions can be phrased in terms of the *Killing vector initial data.* The Killing vector initial data equations constitute a set of conditions that an initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ for the Einstein field equations has to satisfy to ensure that the development will contain a Killing vector —see [6]. Nevertheless, despite the fact that the existence of Killing vector plays a central role in the discussion of the symmetries, the existence of Killing vectors is sometimes not enough to encode all the symmetries and conserved quantities that a spacetime can posses, e.g., the Carter constant in the Kerr spacetime. To unravel some of these *hidden symmetries* one has analyse the existence of a more fundamental type

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of objects; *Killing spinors* $\tilde{\kappa}_{AB}$ —in vacuum spacetimes, the existence of a Killing spinor directly implies the existence of a Killing vector. The *Killing spinor initial data equations* have been derived in the *physical framework* —governed by the Einstein field equations— in [16]. These equations have been successfully employed in the construction of a geometric invariant which detects whether or not an initial data set corresponds to initial data for the Kerr spacetime —see [2, 3, 4]. This analysis has also been extended to include suitable classes of matter —see [**?**] for an analogous characterisation of initial data for the Kerr-Newman spacetime. In these characterisations, some asymptotic conditions on the initial data are required. These conditions usually take the form of decay assumptions on $\tilde{\boldsymbol{h}}$, $\tilde{\boldsymbol{K}}$ and $\tilde{\boldsymbol{\kappa}}$ on $\tilde{\mathcal{S}}$, given in terms of asymptotically Cartesian coordinates. Nonetheless, in other approaches, the asymptotic behaviour of the spacetime can be studied in a geometric way through conformal compactifications. The latter is sometimes referred as the Penrose proposal. In this approach one starts with a *physical spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ where $\tilde{\mathcal{M}}$ is a 4-dimensional manifold and $\tilde{\boldsymbol{g}}$ is a Lorentzian metric which is a solution to the Einstein field equations. Then, one introduces a *unphysical spacetime* $(\mathcal{M}, \boldsymbol{g})$ into which $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is conformally embedded. Accordingly, there exists an embedding $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ such that

$$\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}. \tag{2}$$

By suitably choosing the *conformal factor* Ξ the metric $\boldsymbol{g}$ may be well defined at the points where $\Xi = 0$. In such cases, the set of points for where the conformal factor vanishes is at infinity from the physical spacetime perspective. The set

$$\mathscr{I} \equiv \big\{ p \in \mathcal{M} \ \mid \ \Xi(p) = 0, \ \ \mathbf{d}\Xi(p) \neq 0 \big\}$$

is called the conformal boundary. However, it can be readily verified that the Einstein field equations are not conformally invariant. Moreover, a direct computation using the conformal transformation formula for the Ricci tensor shows that the vacuum Einstein field equations (1), lead to an equation which is formally singular at the conformal boundary. An approach to deal with this problem was given in [9] where a regular set of equations for the unphysical metric was derived. These equations are known as the *conformal Einstein field equations*. The crucial property of these equations is that they are regular at the points where $\Xi = 0$ and a solution thereof implies whenever $\Xi \neq 0$ a solution to the Einstein field equations —see [9, 11] and [32] for an comprehensive discussion. There are three ways in which these equations can be presented, the metric, the frame and spinorial formulations. These equations have been mainly used in the stability analysis of spacetimes —see for instance [13, 12] for the proof of the global and semiglobal non-linear stability of the de Sitter and Minkowski spacetimes, respectively.

A conformal version of the Killing vector initial data equations using the metric formulation of the conformal Einstein field equations has been obtained in [23]. In the latter reference, intrinsic conditions on an initial hypersurface $\mathcal{S} \subset \mathcal{M}$ of the unphysical spacetime are found such that the development of the data —in the unphysical setting the evolution is governed by the conformal Einstein field equations— gives rise to a conformal Killing vector of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ which, in turn, corresponds to a Killing vector of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Notice that this approach, in particular, allows $\mathcal{S}$ to be determined by $\Xi = 0$ so that it to corresponds to the conformal boundary $\mathscr{I}$. The unphysical Killing vector initial data equations have been derived for the characteristic initial value problem on a cone in [23] and on a spacelike conformal boundary in [24].

For applications involving the the conformal Einstein field equations —say in its spinorial formulation, one frequently has to fix the gauge and write the equations in components. Despite the fact that, at first glance, the conformal Einstein field equations expressed in components with respect to an arbitrary spin frame seem to be overwhelmingly complicated, as shown in [15], symmetry assumptions (spherical symmetry in the latter case) greatly reduce the number of equations to be analysed. In the case of Petrov type D spacetimes, e.g.the Kerr-de Sitter spacetime, the symmetries of the spacetime are closely related to the existence of Killing spinors. Therefore, a natural question in this setting is whether a conformal version of the Killing spinor initial data equations introduced in [16] can be found. In other words, what are the extra conditions that one has to impose on an initial data set for the conformal Einstein field equations so that the arising development contains a Killing spinor? This question is answered in this article by deriving such conditions which we call the *conformal Killing spinor initial data equations*

Despite the fact that the Killing spinor equation is conformally invariant, it is not a priori clear whether the conditions of [16, 3] may be translated directly into the unphysical setting. Indeed, one expects this not to be the case, since the Einstein field equations are not conformally invariant. Moreover, one consideration that is exploited in the discussion of [16] is based on the fact that, on an Einstein spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, a Killing spinor $\tilde{\kappa}_{AB}$ gives rise to a Killing vector $\tilde{\xi}_a$ whose spinorial counterpart is given by $\tilde{\xi}_{AA'} = \tilde{\nabla}_{A'}{}^{Q}\tilde{\kappa}_{QA}$. Nevertheless, this property does not hold in general. In other words, if $(\mathcal{M}, \boldsymbol{g})$, where $\boldsymbol{g}$ is not assumed to satisfy the Einstein field equations, possess a Killing spinor κ_{AB}, then the analogous concomitant $\xi_{AA'} = \nabla_{A'}{}^{Q}\kappa_{QA}$ does not correspond to a Killing vector —not even a conformal Killing vector. This situation is not ameliorated if one assumes that $(\mathcal{M}, \boldsymbol{g})$ satisfies the conformal Einstein field equations. Nevertheless, as discussed in this article, in the latter case one can show that using the conformal factor Ξ, the Killing spinor κ_{AB} and the *auxiliary vector* ξ_a, one can construct a conformal Killing vector X_a associated to a Killing vector $\tilde{X}_a$ of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$.

Although the conditions of [16] may be recovered from the results presented here by setting $\Xi = 1$ an important difference is that the set of variables that allow to obtain a closed system of homogeneous wave equations in the present case are different. The need for a different set of *zero-quantities* to be propagated in the conformal case, can be traced back to the previous observation that in $(\mathcal{M}, \boldsymbol{g})$ the vector $\xi_{AA'} = \nabla_{A'}{}^{Q}\kappa_{QA}$ does not correspond to a (conformal) Killing vector. However, a by product of the present analysis is that $\xi_{AA'}$ is a Weyl collineation —see [17] for definitions of curvature collineations. In the analysis of [16] the fact that $\tilde{\xi}_{AA'} = \tilde{\nabla}_{A'}{}^{Q}\tilde{\kappa}_{QA}$ is a Killing vector is crucial since one propagates off the initial hypersurface, simultaneously, $\tilde{\kappa}_{AA'}$ and the Killing vector $\tilde{\xi}^a$ and introduces $\tilde{S}_{ab} \equiv \tilde{\nabla}_{(a}\tilde{\xi}_{b)}$ as a zero-quantity. Similarly, in the work of [7] where the results of [16] are generalised to the case where $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfies the Einstein-Maxwell equations, the condition $\tilde{S}_{ab} = 0$ is also verified by virtue of the so-called *matter aligment condition*. In the conformal setting analysed in this article, the analogous quantity S_{ab} is not as geometrically motivated as in the previous cases and its usage as a variable in the system does not seem to lead to a closed system of explicitly regular homogeneous wave equations. Here by regular we refer to the absence of formally singular terms, such as Ξ^{-1}, in the equations. Instead, the variable that is central for the present analysis turns out to be the so-called *Buchdahl constraint* (and derivatives thereof), which links directly the existence of Killing spinors with the Petrov type of $(\mathcal{M}, \boldsymbol{g})$.

Although the main objective of the present paper is deriving the valence-2 Killing spinor initial data in the conformal setting $(\mathcal{M}, \boldsymbol{g})$, we also derive the analogous conditions encoding the existence of a valence-1 Killing spinor. The latter serves as a warm up exercise for the valence-2 case where one can already observe and understand the above discussed features and differences between the derivation of the conditions on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and those on $(\mathcal{M}, \boldsymbol{g})$ in a simpler arena.

For conciseness, the conformal Killing initial data equations are obtained on a spacelike hypersurface $\mathcal{S}$. Nonetheless, a similar computation can be performed on an hypersurface $\mathcal{S}$ with a different causal character. The conditions found in this article have potential applications for the black hole uniqueness problem. In particular, they can be used for an asymptotic characterisation of the Kerr-de Sitter spacetime analogous to [20] in terms of the existence of Killing spinors at the conformal boundary $\mathscr{I}$.

The main results of this article are summarised informally in the following:

Theorem. *If the conformal Killing spinor initial data equations* (C1)-(C3) *are satisfied on an open set $\mathcal{U} \subset \mathcal{S}$, where $\mathcal{S}$ is a spacelike hypersurface on which initial data for the conformal Einstein field equations has been prescribed, then, the domain of dependence of $\mathcal{U}$ possesses a Killing spinor.*

A precise formulation is the content of Theorem 2 and Proposition 3.

Involved computations throughout this article were facilitated through the suite `xAct` in `Mathematica`. Note that since the existence of a spinor structure is guaranteed for globally-hyperbolic spacetimes —see Proposition 4 in [32]— the use of spinors is not overly restrictive.

Overview of the article

Section 2 gives an overview of Killing spinors along with their conformal properties. In Section 3 we describe the conformal Einstein field equations, for later use; Section 5 introduces the main objects of interest in the propagation of Killing spinor data, namely the *Killing spinor zero-quantities*. In Section 6 we construct conformally-regular wave equations for the zero-quantities, leading to necessary and sufficient conditions for the existence of a Killing spinor —see Proposition 2. In Section 7 the latter conditions and the space spinor formalism are used to obtain the conformal Killing spinor initial data equations on spacelike hypersurfaces—see Theorem 2.

Notation and conventions

Upper case Latin indices $_{ABC\cdots A'B'C'}$ will be used as abstract indices of the *spacetime spinor* algebra, and the bold numerals $_{\mathbf{012}\cdots}$ denote components with respect to a fixed spin dyad $o^A \equiv \epsilon_{\mathbf{0}}{}^A, \iota^A \equiv \epsilon_{\mathbf{1}}{}^A$ —see Penrose & Rindler [25] for further details. Although spinor notation will be preferred, for certain computations tensors will be employed. Lower case Latin indices $_{a,b,c\ldots}$ will be used as abstract tensor indices. For tensors, our curvature conventions are fixed by

$$\nabla_a \nabla_b \kappa^c - \nabla_b \nabla_a \kappa^c = R_{ab}{}^c{}_d \kappa^d.$$

For spinors, the curvature conventions are fixed via the spinorial Ricci identities which will be written in accordance with the above convention for tensors. To see this, recall that the commutator of covariant derivatives $[\nabla_{AA'}, \nabla_{BB'}]$ can be expressed in terms of the symmetric operator $\square_{AB}$ as

$$[\nabla_{AA'}, \nabla_{BB'}] = \epsilon_{AB}\square_{A'B'} + \epsilon_{A'B'}\square_{AB}$$

where

$$\square_{AB} \equiv \nabla_{Q'(A}\nabla_{B)}{}^{Q'}.$$

The action of the symmetric operator $\square_{AB}$ on valence-1 spinors is encoded in the spinorial Ricci identities

$$\square_{AB}\xi_C = -\Psi_{ABCD}\xi^D + 2\Lambda\xi_{(A}\epsilon_{B)C}, \tag{3a}$$

$$\square_{A'B'}\xi_C = -\xi^A\Phi_{CAA'B'}, \tag{3b}$$

where Ψ_{ABCD} and $\Phi_{AA'BB'}$ and Λ are curvature spinors. The above identities can be extended to higher valence spinors in an analogous way —see [30] for further discussion on these identities using different conventions to the ones used in this article. A related identity which will be systematically used in the following discussion is

$$\nabla_{AQ'}\nabla_B{}^{Q'} = \square_{AB} + \frac{1}{2}\epsilon_{AB}\square, \tag{4}$$

where $\square_{AB}$ is the symmetric operator defined above and $\square \equiv \nabla_{AA'}\nabla^{AA'}$.

Space spinor formalism

To have a self-contained discussion in this section the space spinor formalism, originally introduced in [29], is briefly recalled —see also [16, 3, 32]. Let $\tau^{AA'}$ denote the spinorial counterpart of a timelike vector τ^a, normal to a spacelike hypersurface $\mathcal{S}$ and normalised so that $\tau_a\tau^a = 2$. Then, it follows that $\tau_{AA'}\tau^{AA'} = 2$ and, consequently,

$$\tau_{AA'}\tau_B{}^{A'} = \epsilon_{AB}.$$

The covariant derivative $\nabla_{AA'}$ is then decomposed into the *normal* and *Sen* derivatives:

$$\mathcal{P} \equiv \tau^{AA'}\nabla_{AA'},$$

$$\mathcal{D}_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'}.$$

The *Weingarten* spinor and the *acceleration* of the congruence are then defined by

$$K_{ABCD} \equiv \tau_D{}^{C'} \mathcal{D}_{AB} \tau_{CC'},$$
$$K_{AB} \equiv \tau_B{}^{C'} \mathcal{P} \tau_{AC'}.$$

The above can be inverted to obtain the following formulae which will prove useful in the sequel

$$\mathcal{P}\tau_{CC'} = -K_{CD}\tau^D{}_{C'},$$
$$\mathcal{D}_{AB}\tau_{CA'} = -K_{ABCD}\tau^D{}_{A'}.$$

The distribution induced by $\tau_{AA'}$ is integrable if and only $K^D{}_{(AB)D} = 0$, in which case K_{ABCD} describes the extrinsic curvature of the resulting foliation. Nevertheless, this is not required for our subsequent discussion. In other words, we will allow the possibility that the distribution is non-integrable —i.e. the spinor $K^D{}_{(AB)D}$ will not be assumed to vanish.

Defining the spinors $\chi_{AB} \equiv K^D{}_{(AB)D}$, $\chi_{ABCD} \equiv K_{(ABCD)}$ and $\chi \equiv K_{AB}{}^{AB}$, the Weingarten spinor decomposes as follows

$$K_{ABCD} = \chi_{ABCD} - \tfrac{1}{2}\epsilon_{A(C}\chi_{D)B} - \tfrac{1}{2}\epsilon_{B(C}\chi_{D)A} - \tfrac{1}{3}\chi\epsilon_{A(C}\epsilon_{D)B}. \tag{5}$$

For the following discussion we will also need the commutators form with $\mathcal{P}$, $\mathcal{D}_{AB}$. To write these commutators in a succinct way, first define

$$\widehat{\square}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\square_{A'B'}$$

from which, proceeding analogously as in [3], one obtains

$$[\mathcal{P}, \mathcal{D}_{AB}] = -\tfrac{1}{2}\chi_{AB} - \square_{AB} + \widehat{\square}_{AB} + K_{(A}{}^{D}\mathcal{D}_{B)D} - K_{AB}{}^{FG}\mathcal{D}_{FG}, \tag{6}$$

$$\begin{aligned}[\mathcal{D}_{AB}, \mathcal{D}_{CD}] = {} & \tfrac{1}{2}\left(\epsilon_{A(C}\square_{D)B} + \epsilon_{B(C}\square_{D)A}\right) + \tfrac{1}{2}\left(\epsilon_{A(C}\widehat{\square}_{D)B} + \epsilon_{B(C}\widehat{\square}_{D)A}\right) \\ & + \tfrac{1}{2}\left(K_{CDAB}\mathcal{P} - K_{ABCD}\mathcal{P}\right) + K_{CDF(A}\mathcal{D}_{B)}{}^{F} - K_{ABF(C}\mathcal{D}_{D)}{}^{F} \end{aligned} \tag{7}$$

2 Killing spinors

To start the discussion it is convenient to introduce some notation and definitions. Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ be a 4-dimensional manifold equipped with a Lorentzian metric $\tilde{\boldsymbol{g}}$ and denote by $\tilde{\nabla}$ its associated Levi-Civita connection. For the time being $\tilde{\boldsymbol{g}}$ is not assumed to be a solution to the Einstein field equations (1).

A totally symmetric $\tilde{\kappa}_{A_1\ldots A_q} = \tilde{\kappa}_{(A_1\ldots A_q)}$ valence$-q$ spinor is said to be a Killing spinor if the following equation is satisfied

$$\tilde{\nabla}_{Q'(Q}\tilde{\kappa}_{A_1\ldots A_q)} = 0. \tag{8}$$

An important property of the Killing spinor equation is that it is conformally-invariant, in other words if $\boldsymbol{g}$ is conformally related to $\tilde{\boldsymbol{g}}$, namely $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ then $\kappa_{A_1\ldots A_q} = \Xi^2\tilde{\kappa}_{A_1\ldots A_q}$ satisfies

$$\nabla_{Q'(Q}\kappa_{A_1\ldots A_q)} = 0,$$

where ∇ is the Levi–Civita connection of $\boldsymbol{g}$.

In this paper we will only focus only the case $q = 1$ and $q = 2$. If $q = 1$, the equation

$$\tilde{\nabla}_{Q'(Q}\tilde{\kappa}_{A)} = 0. \tag{9}$$

is usually referred as the *twistor equation*. We will follow this naming convention and refer to a valence-1 spinor satisfying equation (9) as twistor. Since only the cases $q = 1$ and $q = 2$ will be discussed in this paper, we will refer to the case $q = 1$ as the twistor case and the $q = 2$ as the Killing spinor case. Namely, we will say that a symmetric valence-2 spinor, $\tilde{\kappa}_{AB} = \tilde{\kappa}_{(AB)}$, is a *Killing spinor* if it satisfies the equation

$$\tilde{\nabla}_{A'(A}\tilde{\kappa}_{BC)} = 0. \tag{10}$$

The Killing spinor equation and twistor equations are, in general, overdetermined; in particular, they imply the so-called *Buchdahl constraint*. In the twistor case ($q=1$) this has the form

$$\kappa^{D}\Psi_{ABCD}=0,$$

while in the Killing spinor case ($q=2$) the Buchdahl constraint adquires the form

$$\tilde{\kappa}^{Q}{}_{(A}\Psi_{BCD)Q}=0,$$

where Ψ_{ABCD} denotes the conformally invariant Weyl spinor. The latter condition restricts Ψ_{ABCD} to be algebraically special. In the twistor case the spacetime is necessarity of Petrov type N or O, hence restricting its applicability for characterisation of black holes. In the Killing spinor case the spacetime is only restricted to be of Petrov type D, N or O.

At first glance, the conformal invariance property of the Killing spinor equation would seem to indicate that the approach leading to the Killing spinor initial data conditions derived in [16] would identically apply for $(\mathcal{M},\boldsymbol{g})$ with $\tilde{\boldsymbol{g}}=\Xi^{2}\tilde{\boldsymbol{g}}$. This is not the case simply because the Einstein field equations are not conformally invariant. In other words, in the analysis of [16] the vacuum Einstein field equations $\tilde{R}_{ab}=0$ were used, and, despite that one can relate R_{ab} with $\tilde{R}_{ab}$ this leads to formally singular terms (terms containing Ξ^{-1}). Moreover, even if one is willing to work with formally singular equations it is not apriori clear that the choice of variables made in [16] will form a closed homogeneous system in the conformal seeting. To understand this second point further, notice that for general manifold with metric $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ —namely $\tilde{\boldsymbol{g}}$ not satifying any field equation— the existence of a Killing spinor $\tilde{\kappa}_{AB}$ is not related directly to the existence of a Killing vector. Nevertheless, if one assumes that $\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein field equations (1) then the concomitant

$$\tilde{\xi}_{AA'}\equiv\tilde{\nabla}^{B}{}_{A'}\tilde{\kappa}_{AB},$$

represents the spinorial counterpart of a complex Killing vector of the spacetime $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ —see [16] for further discussion. This point is subtle and even in the physical (non-conformal) set up if one is to include matter such as the Maxwell field and the analysis of [16] does not straighfowardly apply since further conditions (the matter alignment conditions) —see [7]— need to be propagated.

Remark 1. The notion of Killing spinors is related to that of Killing–Yano tensors. If a Killing spinor $\tilde{\xi}_{AA'}$ is Hermitian, i.e., $\bar{\tilde{\xi}}_{AA'}=\tilde{\xi}_{AA'}$, then one can construct the spinorial counterpart of a *Killing–Yano tensor* $\tilde{\Upsilon}_{ab}$ —i.e. an antisymmetric 2−tensor satisfying $\tilde{\nabla}_{(a}\tilde{\Upsilon}_{b)c}=0$— as follows

$$\tilde{\Upsilon}_{AA'BB'}=i(\tilde{\kappa}_{AB}\bar{\tilde{\epsilon}}_{A'B'}-\bar{\tilde{\kappa}}_{A'B'}\tilde{\epsilon}_{AB}).$$

Conversely, given a Killing–Yano tensor, one can construct a Killing spinor —see [**?**, **?**, 26].

In the sequel $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ will be reserved to denote the *physical spacetime*, in other words, the symbol $\tilde{}$ will be added to those fields associated with a solution $\tilde{\boldsymbol{g}}$ to the vacuum Einstein field equations (1). Similarly $(\mathcal{M},\boldsymbol{g})$ will be used to represent the *unphysical spacetime* related to $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ via $\boldsymbol{g}=\Xi^{2}\tilde{\boldsymbol{g}}$. —in a slight abuse of notation $\varphi(\tilde{\mathcal{M}})$ and $\mathcal{M}$ will be identified so that the mapping $\varphi:\tilde{\mathcal{M}}\rightarrow\mathcal{M}$ can be omitted.

3 The conformal Einstein field equations

This section contains an abriged discussion of the CFEs in first and second order form. At the end of this section the main technical tool from the theory of partial differential equations to be used for deriving the Killing spinor intial data equations is given.

The conformal Einstein field equations are a conformal formulation of the Einstein field equations. In other words, given a spacetime $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ satisfying the Einstein field equations, the conformal Einstein field equations encode a system of differential conditions for the curvature and concomitants of the conformal factor associated with $(\mathcal{M},\boldsymbol{g})$ where $\boldsymbol{g}=\Xi^{2}\tilde{\boldsymbol{g}}$. The key property of these equations is that they are regular even at the conformal boundary $\mathscr{I}$, where $\Xi=0$.

This formulation of the conformal Einstein field equations was first given in [9] —see also [32] for a comprehensive discussion.

The metric version of the standard vacuum conformal Einstein field equations are encoded in the following zero-quantities —see [9, 8, 10, 11]:

$$Z_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab} = 0, \tag{11a}$$
$$Z_a \equiv \nabla_a s + L_{ac} \nabla^c \Xi = 0, \tag{11b}$$
$$\delta_{bac} \equiv \nabla_b L_{ac} - \nabla_a L_{bc} - d_{abcd} \nabla^d \Xi = 0, \tag{11c}$$
$$\lambda_{abc} \equiv \nabla_e d_{abc}{}^e = 0, \tag{11d}$$
$$Z \equiv \lambda - 6\Xi s + 3\nabla_a \Xi \nabla^a \Xi \tag{11e}$$

where Ξ is the conformal factor, L_{ab} is the Schouten tensor, defined in terms of the Ricci tensor R_{ab} and the Ricci scalar R via

$$L_{ab} = \frac{1}{2} R_{ab} - \frac{1}{12} R g_{ab}, \tag{12}$$

s is the so-called *Friedrich scalar* defined as

$$s \equiv \tfrac{1}{4} \nabla_a \nabla^a \Xi + \tfrac{1}{24} R \Xi \tag{13}$$

and $d^a{}_{bcd}$ denotes the *rescaled Weyl tensor*, defined as

$$d^a{}_{bcd} = \Xi^{-1} C^a{}_{bcd},$$

where $C^a{}_{bcd}$ denotes the Weyl tensor. The geometric meaning of these zero-quantities is the following: The equation $Z_{ab} = 0$ encodes the conformal transformation law between R_{ab} and $\tilde{R}_{ab}$. The equation $Z_a = 0$ is obtained considering $\nabla^a Z_{ab}$ and commuting covariant derivatives. Equations $\delta_{abc} = 0$ and $\lambda_{abc} = 0$ encode the contracted second Bianchi identity. Finally, $Z = 0$ is a constraint in the sense that if it is verified at one point $p \in \mathcal{M}$ then $Z = 0$ holds in $\mathcal{M}$ by virtue of the previous equations. A solution to the metric conformal Einstein field equations consist of a collection of fields

$$\{g_{ab},\ \Xi,\ \nabla_a \Xi, s\ , L_{ab},\ d_{abcd}\}$$

satisfying

$$Z_{ab} = 0, \quad Z_a = 0, \quad \delta_{abc} = 0, \quad \lambda_{abc} = 0, \quad Z = 0. \tag{14}$$

Remark 2. If one opts to use the Ricci tensor R_{ab} instead of the Schouten tensor L_{ab} then the Ricci scalar R appears in the right-hand side of equations but no equation for it has been provided. In the CFEs the Ricci scalar encodes the *conformal gauge source function*, hence there is no equation to fix that variable since it is a gauge quantity in the formulation.

Since the structural properties of the CFEs are better expressed in spinorial formalism and due to the nature of the applications in this article, the spinorial version of the CFEs will be used. The spinorial translation of the above CFEs zero-quantities render —see [32] for further details.

$$Z_{AA'BB'} = -\,\Xi \Phi_{ABA'B'} - s\epsilon_{AB}\epsilon_{A'B'} + \Xi\Lambda\epsilon_{AB}\epsilon_{A'B'} + \nabla_{BB'}\nabla_{AA'}\Xi \tag{15a}$$
$$Z_{AA'} = \Lambda \nabla_{AA'}\Xi + \nabla_{AA'} s - \Phi_{ABA'B'} \nabla^{BB'} \Xi \tag{15b}$$
$$\delta_{ABCC'} = \nabla_{A'(A}\Phi_{B)CC'}{}^{A'} - \epsilon_{C(A}\nabla_{B)C'}\Lambda + \phi_{ABCD}\nabla^D{}_{C'}\Xi \tag{15c}$$
$$\Lambda_{CC'AB} = \nabla_{DC'}\phi_{ABC}{}^D \tag{15d}$$
$$Z = \lambda - 6\Xi s + 3\nabla_{AA'}\Xi\nabla^{AA'}\Xi \tag{15e}$$

•[1] Similar to the tensorial case, one can choose to use the Schouten (tensor) spinor or the Ricci (tensor) spinor as a variable. Here the equations have been expressed using the standard curvature spinors and notation of the NP formalism, namely, the trace-free Ricci spinor $\Phi_{ABA'B'}$, the Ricci

•*1: I think the original $Z_{AA'BB'CC'}$ does not contain more information than what $\delta_{ABCC'}$ encodes.*

scalar Λ —in fact $R = 24\Lambda$— and the Weyl spinor Ψ_{ABCD}. The rescaled Weyl spinor ϕ_{ABCD} is defined as

$$\phi_{ABCD} \equiv \Xi^{-1}\Psi_{ABCD}. \tag{16}$$

The CFEs as previously presented can be regarded as a set of covariant conditions for geometric fields on $(\mathcal{M}, \boldsymbol{g})$ and they do not have a particular PDE character. However, there are, depending on the gauge fixing procedure, different hyperbolic reduction strategies to extract a set of evolution and constraint equations. For the subsequent discussion only the evolution and constraint equations implied by the $\Lambda_{CC'AB} = 0$ equation will play a role. A direct calculation using the space spinor formalism shows that $\Lambda_{CC'AB} = 0$ can be recasted as the following system of evolution equation and constraint equations

$$\nabla_{\tau}\phi_{ABCD} = 2\mathcal{D}_{(A}{}^{F}\phi_{BCD)F}, \qquad \mathcal{D}_{CD}\phi_{AB}{}^{CD} = 0. \tag{17}$$

The evolution and constraint equations associated to the other zero-quantities depend on the particular gauge fixing strategy and will not play a relevant role for the discussion in the next sections.

The CFEs are usually presented as the first order system (14) with the definitions (11), however, for several applications it is convenient to use a second order formulation of the equations. In [22] the tensorial version of the CFEs was recasted as a set of (tensorial) wave equations. Similarly, in [14] a second order form of the spinorial formulation of the CFEs was obtained. This version of the CFEs is particularly suited for the applications of this article, and, in fact, only one of those equations —that for the rescaled Weyl spinor— will be needed •2 . The wave equation for the rescaled Weyl spinor can be succintly obtained from considering $\nabla^{QC'}\Lambda_{CC'AB}$. A direct calculation using the identity (4) shows that if $\Lambda_{CC'AB} = 0$ then,

•2: Double check this is true

$$\square\phi_{ABCF} = 12\Lambda\phi_{ABCF} - 6\Xi\phi_{(AB}{}^{DG}\phi_{CF)DG} \tag{18}$$

A similar calculation can be carried out for the other equations in comprising the CFEs. A full discussion of the *spinorial CFE wave equations* and their equivalence with the standard first order formulation CFEs can be found in [14]. One of the main tools used in [14] to show the equivalence between these two set of equations is the uniqueness property to a certain class of wave equations. This same result from the theory of partial differential equations will be used to obtain the main result of this article and is presented in the following

•3

•3: Technical pde theorem moved here

Theorem 1. *Let $\mathcal{M}$ be a smooth manifold equipped with a Lorentzian metric $\boldsymbol{g}$ and consider the wave equation*

$$\square\underline{u} = h(\underline{u},\ \partial\underline{u})$$

where $\underline{u} \in \mathbb{C}^m$ is a complex vector-valued function on $\mathcal{M}$, $h : \mathbb{C}^{2m} \to \mathbb{C}^m$ is a smooth homogeneous function of its arguments and $\square = g^{ab}\nabla_a\nabla_b$. Let $\mathcal{U} \subset \mathcal{S}$ be an open set and $\mathcal{S} \subset \mathcal{M}$ be a spacelike hypersurface with normal τ^a respect to $\boldsymbol{g}$. Then the Cauchy problem

$$\square\underline{u} = h(\underline{u},\ \partial\underline{u}),$$
$$\underline{u}|_{\mathcal{U}} = \underline{u}_0, \quad \mathcal{P}\underline{u}|_{\mathcal{U}} = \underline{u}_1,$$

where $\underline{u}_0$ and $\underline{u}_1$ are smooth on $\mathcal{U}$ and $\mathcal{P} \equiv \tau^{\mu}\nabla_{\mu}$, has a unique solution $\underline{u}$ in the domain of dependence of $\mathcal{U}$.

We refer the reader to [32, 31] for a proof —see also Theorem 1 in [16].

Remark 3. Recall that an equation of the above form are said to be *homogeneous in $\underline{u}$ and its first derivatives* if $h(\lambda\underline{u},\ \lambda\partial\underline{u}) = \lambda h(\underline{u},\ \partial\underline{u})$ for all $\lambda \in \mathbb{C}$.

4 Conformal twistor initial data

4.1 Twistor zero-quantities

For the following discussion is convenient to make the following *zero-quantities*

$$H_{A'AB} \equiv 2\nabla_{A'(A}\kappa_{B)}, \tag{19a}$$
$$B_{ABC} \equiv \phi_{ABCD}\kappa^{D}. \tag{19b}$$

The spinors $H_{A'AB}$ and B_{ABC} will be denoted in index free notation as $\boldsymbol{H}$ and $\boldsymbol{B}$ and will be called the twistor zero-quantity and the Buchdahl zero-quantity respectively. The Buchdahl zero-quantity arises as an integrability condition of the twistor equation. To see this, notice that, taking the following derivative of $\boldsymbol{H}$ and substituting the definition (19a) one obtains

$$\nabla_{AA'}H^{A'}{}_{BC} = 2\nabla_{AA'}\nabla_{(B}{}^{A'}\kappa_{C)} = \tfrac{1}{2}\epsilon_{AB}\Box\kappa_{C} + \tfrac{1}{2}\epsilon_{AC}\Box\kappa_{B} + \Box_{BA}\kappa_{C} + \Box_{CA}\kappa_{B}. \tag{20}$$

Symmetrsing and using equation (3a) renders

$$\nabla_{(A|A'|}H^{A'}{}_{BC)} = -2\Psi_{ABCD}\kappa^{D}.$$

The vanishing of the right-hand side of latter equation encodes the Buchdahl constraint, namely the fact that if $(\mathcal{M},\boldsymbol{g})$ admits a twistor then it is necesarilly of Petrov type N or O. To write this in the variables appearing in the conformal Einstein field equations, using the definition of the rescaled Weyl spinor yields

$$\nabla_{(A}{}^{A'}H_{|A'|BC)} = 2\Xi B_{ABC}, \tag{21}$$

which motivates the name for the zero-quantity $\boldsymbol{B}$. Thus, with this notation, it is clear that if the unphysical spacetime $(\mathcal{M},\boldsymbol{g})$ admits a twistor (valence-1 Killing spinor) then following zero-quantities vanish

$$H_{A'AB} = 0, \qquad B_{ABC} = 0. \tag{22}$$

4.2 Twistor auxiliary quantities and the twistor candidate equation

Other useful definition to keep track of the subsequent calculations is the following *auxiliary quantities*

$$Q_{A} \equiv \nabla^{QA'}H_{A'QA} \tag{23a}$$
$$\xi_{A'} \equiv \nabla^{B}{}_{A'}\kappa_{B} \tag{23b}$$

The *auxiliary spinor* $\xi_{A'}$ is merely a convenient placeholder for making irreducible decompositions of derivatives of κ_A such as

$$\nabla_{AA'}\kappa_{B} = \tfrac{1}{2}\epsilon_{AB}\nabla_{CA'}\kappa^{C} + \nabla_{(A|A'|}\kappa_{B)} \tag{24}$$
$$= \tfrac{1}{2}H_{A'AB} - \tfrac{1}{2}\xi_{A'}\epsilon_{AB}. \tag{25}$$

and in principle one can carry out all the calculations without this definition. It is nevertheless illustrative to introduce this shorthand since the analogous quantity in the Killing spinor case $(q = 2)$ will have some geometrical significance.

On the other hand, the *auxiliary quantity* Q_A will be central for the following discussion since it encodes a wave equation for κ_A. To see this, observe that tracing the identity (20) and substituting the definition (23a) gives,

$$Q_{A} = 3\Lambda\kappa_{A} + \tfrac{3}{2}\Box\kappa_{A}. \tag{26}$$

Solving for $\Box\kappa_A$ one has

$$\Box\kappa_{A} = \tfrac{2}{3}Q_{A} - 2\Lambda\kappa_{A}.$$

If the equation $Q_A = 0$ is imposed, then the latter expression can be read as a wave equation for κ_A. This motivates the following definition: a valence-1 spinor η_A satisfying

$$\square \eta_A = -2\Lambda \eta_A \tag{27}$$

will be called a *twistor candidate*. To understand the motivation for this definition and its name, notice that in general, any twistor κ_A trivially satisfies the twistor candiadate equation but not every twistor candidate η_A will solve the twistor equation. In other words,

$$\boldsymbol{H} = 0 \implies \boldsymbol{Q} = 0, \qquad \text{but in general} \qquad \boldsymbol{Q} = 0 \not\Longrightarrow \boldsymbol{H} = 0.$$

However, the initial data $(\nabla_\tau \eta_A, \eta_A)|_{\mathcal{S}}$ for the wave equation (27) has not been fixed yet. The aim of the following calculations is to determine the conditions on the initial data for the twistor candidate such that if propagated off $\mathcal{S}$, using equation (27), then the corresponding twistor candidate η_A is, in fact, a twistor. Namely,

$$\boldsymbol{Q} = 0 \ \ \& \ \ \text{twistor initial data} \quad \Longrightarrow \boldsymbol{H} = 0. \tag{28}$$

The strategy to obtain such conditions on the intial data $(\nabla_\tau \eta_A, \eta_A)|_{\mathcal{S}}$ is to derive a closed system of homogeneous wave equations for the zero-quantities $\boldsymbol{H}$ and $\boldsymbol{B}$ to show that, if trivial initial data for such equations is given, then, using Theorem 1, $\boldsymbol{H} = 0$ and $\boldsymbol{B} = 0$ in the domain of dependence of the data.

4.3 Wave equations for the zero-quantities

A wave equation for the zero-quantity $\boldsymbol{H}$ can be constructed as follows. From the irreducible decomposition of $\nabla_D{}^{A'} H_{A'AB}$,

$$\nabla_D{}^{A'} H_{A'AB} = \tfrac{1}{3}\epsilon_{BD}\nabla_{CA'}H^{A'}{}_A{}^C + \tfrac{1}{3}\epsilon_{AD}\nabla_{CA'}H^{A'}{}_B{}^C + \nabla_{(A}{}^{A'}H_{|A'|BD)},$$

and the definitions (23a) and equation (21) one has that

$$\nabla_D{}^{A'} H_{A'AB} = 2B_{ABD}\Xi + \tfrac{1}{3}Q_B\epsilon_{AD} + \tfrac{1}{3}Q_A\epsilon_{BD} \tag{29}$$

Applying $\nabla_D{}^{B'}$ to the last expression, and using the identity (4) along with the spinorial Ricci identities (3a)-(3b), renders

$$\square H_{B'AB} = 6\Lambda H_{B'AB} + 4\Xi\nabla_{DB'}B_{AB}{}^D - 4B_{ABD}\nabla^D{}_{B'}\Xi - 4\Phi_{(A}{}^D{}_{|B'}{}^{A'}H_{A'|B)D} + \tfrac{4}{3}\nabla_{(A|B'|}Q_{B)} \tag{30}$$

To derive a wave equation for $\boldsymbol{B}$, one applies the D'Alembertian operator $\square$ to the definition in equation (19b) to obtain

$$\square B_{ABC} = \kappa^D \square\phi_{ABCD} + \phi_{ABCD}\square\kappa^D + 2\nabla_{FA'}\phi_{ABCD}\nabla^{FA'}\kappa^D \tag{31}$$

Substituting the definition (19b), the identity (26), and the wave equation satisfied by the rescaled Weyl spinor (18) into the last expression gives

$$\square B_{ABC} = 10B_{ABC}\Lambda + H^{A'DF}\nabla_{FA'}\phi_{ABCD} - 6\Xi B_{(A}{}^{DF}\phi_{BC)DF} + \tfrac{2}{3}\phi_{ABCD}Q^D \tag{32}$$

Observe that if $Q_A = 0$, namely if the twistor candidate wave equation is imposed then, $\boldsymbol{H}$ and $\boldsymbol{B}$ satisfy the following set of wave equations

$$\square H_{B'AB} = 6\Lambda H_{B'AB} + 4\Xi\nabla_{DB'}B_{AB}{}^D - 4B_{ABD}\nabla^D{}_{B'}\Xi - 4\Phi_{(A}{}^D{}_{|B'}{}^{A'}H_{A'|B)D} \tag{33a}$$

$$\square B_{ABC} = 10B_{ABC}\Lambda + H^{A'DF}\nabla_{FA'}\phi_{ABCD} - 6\Xi B_{(A}{}^{DF}\phi_{BC)DF} \tag{33b}$$

Notice that the only place where the CFEs —in their wave equation form— have been used was in equation (31) to substitute the term $\square\phi_{ABCD}$.

The relevant observation about equations (33a)-(33b) is that they constitute a closed system of *regular and homogeneous* wave equations for $\boldsymbol{H}$ and $\boldsymbol{B}$. Hence prescribing trivial intial data

$$H_{A'AB} = 0, \qquad \nabla_\tau H_{A'AB} = 0, \qquad B_{ABC} = 0, \qquad \nabla_\tau B_{ABC} = 0 \qquad \text{on} \qquad \mathcal{S}$$

and using Theorem 1, which stablishes the uniquess of solutions to wave equations of the type of (33a)-(33b), on has that

$$H_{A'AB} = 0, \qquad B_{ABC} = 0, \qquad \text{on} \qquad \mathcal{D}^+(\mathcal{S}). \tag{34}$$

In turn, substituting the definitions for the zero-quantities $\boldsymbol{H}$ and $\boldsymbol{B}$ into the conditions (34) render a prescription to fix the intial data $(\nabla_\tau \eta_A, \eta_A)|_{\mathcal{S}}$ for twistor candidate wave equation (27) that ensures that the twistor candidate η_A corresponds to an actual twistor κ_A.

This discussion is summarised in the following

Proposition 1. *Given initial data for the conformal field equations on $\mathcal{U} \subseteq \mathcal{S}$ where $\mathcal{S}$ is a spacelike hypersurface $\mathcal{S}$ with normal vector $\tau^{AA'}$, and associated normal derivative $\nabla_\tau \equiv \tau^{AA'}\nabla_{AA'}$, the corresponding spacetime development admits a twistor (valence-1 Killing spinor) in $\mathcal{D}^+(\mathcal{U})$ —the future domain of dependence of $\mathcal{U}$— if and only if*

$$H_{A'AB} = 0, \tag{35a}$$
$$\nabla_\tau H_{A'AB} = 0, \tag{35b}$$
$$B_{ABC} = 0, \tag{35c}$$
$$\nabla_\tau B_{ABC} = 0 \tag{35d}$$

hold on $\mathcal{U}$.

Proof. The *only if* direction is immediate. Suppose, on the other hand, that (35a)-(35d) hold on some $\mathcal{U} \subset \mathcal{S}$ —that is to say, there exist a spinor field κ_A for which (35a)-(35d) are satisfied on $\mathcal{U}$. The latter is then used as initial data for the twistor candidate wave equation

$$\square \kappa_A = -2\Lambda \kappa_A \tag{36}$$

As the zero-quantities $H_{A'AB}$, B_{ABC} satisfy the homogeneous wave equations (33a)-(33b) then the uniqueness result for homogeneous wave equations, given in Theorem 1, ensures that

$$H_{A'ABC} = 0, \qquad B_{ABC} = 0,$$

in $\mathcal{D}^+(\mathcal{U})$. In other words, κ_A solves the twistor equation on $\mathcal{D}^+(\mathcal{U})$. $\square$

4.4 Comparisson between the Killing initial data conditions in the physical and unphysical pictures

The main advantage of the conformal (unphysical) approach to the Einstein field equations is that the conformal boundary $\mathscr{I}$ determined by $\Xi = 0$ is a submanifold of $(\mathcal{M}, \boldsymbol{g})$. This allows, in particular, to consider the $\Xi = 0$ hypersurface as a legitimate hypersurface to prescribe data which can be evolved using regular —without Ξ^{-1}-terms— evolution equations. This set up is particularly atractive to study spacetimes with $\lambda > 0$ in which —given the appropriate conditions— the conformal boundary $\mathscr{I}$ is a spacelike hypersurface and hence one can pose *an asymptotic initial value problem*: an initial value problem where the initial hypersurface is $\mathscr{I}$. On the other hand, the conformal (valence-1 Killing spinor) twistor conditions of proposition (1) allows to identify asymptotic initial data whose development will contain a twistor. Although the twistor case is too restrictive to characterise black hole spacetimes it is still illustrative to compare the derivation of the physical twistor initial data conditions on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and that leading to proposition (1).

For the twistor case, one important difference between discussion in [14] using the vacuum Einstein field equations in $(\mathcal{M}, \boldsymbol{g})$ is that the system closes with $\tilde{H}_{A'AB}$ alone and there is no need to introduce the analogous physical Buchdahl zero-quantity $\tilde{B}_{ABC}$. Therefore it is interesting to check if in the conformal case discussed in the previous sections one can also close the system with $H_{A'AB}$ alone.

Applying the D'Alembertian $\square$ to equation (19a), using the definition of the auxiliary quantity Q_A in equation (23a), a direct calculation exploiting the identities (3a)-(3b) and (4), gives

$$\begin{aligned}\square H_{A'AB} =& -2\Psi_{ABCD}H_{A'}{}^{CD} + 6\Lambda H_{A'AB} - 4\Phi_{(A}{}^{C}{}_{|A'}{}^{B'}H_{B'|B)C} \\ & - 2\kappa_{(A}\nabla_{B)A'}\Lambda - 2\kappa^{C}\nabla_{(A}{}^{B'}\Phi_{B)CA'B'} + 2\kappa^{C}\nabla_{DA'}\Psi_{ABC}{}^{D} + \tfrac{4}{3}\nabla_{(A|A'|}Q_{B)}.\end{aligned} \tag{37}$$

If one were discussing the physical case —adding a tilde to every term in (39)— in which the fields are defined on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ which satisfies the vacuum Einstein field equations

$$\tilde{\Lambda} = 0, \qquad \tilde{\Phi}_{AA'BB'} = 0, \tag{38}$$

then, using the Bianchi identity $\tilde{\nabla}^{A}{}_{B'}\Psi_{ABCD} = \tilde{\nabla}^{A'}{}_{(B}\tilde{\Phi}_{CD)A'B'}$ and equations (38), the physical version of equation (39) reduces to

$$\tilde{\square}\tilde{H}_{A'AB} = -2\Psi_{ABCD}\tilde{H}_{A'}{}^{CD} + \tfrac{4}{3}\tilde{\nabla}_{(A|A'|}\tilde{Q}_{B)}. \tag{39}$$

Hence, imposing $\tilde{Q}_A = 0$, the system closes with $\tilde{H}_{A'AB}$ alone.

In the other hand, if one tries to follow the same strategy in the unphysical set up —with $(\mathcal{M}, \boldsymbol{g})$ satisfying the CFEs— one ends up with a formally singular equation. To see this, observe that starting from the identity (39) and using equation (16) along the CFEs zero-quantities

$$\delta_{ABCC'} = 0, \qquad \Lambda_{CC'AB} = 0,$$

as defined in equations (15c)-(15d), a calculation gives

$$\begin{aligned}\square H_{A'AB} =& -\frac{2\nabla^{C}{}_{A'}\Xi\nabla_{(A}{}^{B'}H_{|B'|BC)}}{\Xi} + 6\Lambda H_{A'AB} - 2\Xi\phi_{ABCD}H_{A'}{}^{CD} - 4\Phi_{(A}{}^{C}{}_{|A'}{}^{B'}H_{B'|B)C} \\ & + \tfrac{4}{3}\nabla_{(A|A'|}Q_{B)}.\end{aligned} \tag{40}$$

Hence, setting $Q_A = 0$, renders an homogeneous but *singular equation* —due to the Ξ^{-1} coefficient— for $H_{A'AB}$, for which the theory of behind Theorem 1 does not apply. Arguably, one could try to use the theory of Fuchsian systems to see if the analogous of Theorem 1 applies for the singular equation (40). However, one of the advantages of the conformal approach of the CFEs respect to other approaches to include $\mathscr{I}$ is that one deals with formally regular equations. Therefore, from this perspective, it is preferable to work with explicitly regular equations and hence, it is necessary to introduce B_{ABC} as a further zero-quantity to be propagated. A analogous observation holds for the conformal valence-2 Killing spinor initial data discussion of the following sections, where to close the system in a regular way, one needs to introduce not only the Buchdahl zero-quantity but also its derivative.

4.5 Intrinsic conformal twistor initial data conditions

In this section, the conformal twistor initial data conditions of proposition 1 are written in terms of intrinsic quantities at $\mathcal{S}$. To understand the need of the calculation to be carried out in this section observe that although the conditions of proposition 1 are given on $\mathcal{S}$ it contains not only derivatives tangential to $\mathcal{S}$ but also normal to it. Hence, to obtain genuine intrinsic conditions on $\mathcal{S}$ one needs to remove these normal derivatives.

Using the following definitions,

$$\mathcal{H}_{ABC} \equiv \tau_{(A}{}^{A'}H_{|A'|BC)}, \qquad \mathcal{H}_{A} \equiv \tau^{QA'}H_{A'AQ}, \tag{41}$$

the space spinor split of $H_{A'AB}$ reads,

$$H_{A'AB} = -\tfrac{1}{2}\tau^{C}{}_{A'}\mathcal{H}_{ABC} - \tfrac{1}{6}\tau^{C}{}_{A'}\mathcal{H}_{B}\epsilon_{AC} - \tfrac{1}{6}\tau^{C}{}_{A'}\mathcal{H}_{A}\epsilon_{BC} \tag{42}$$

Hence, the space spinors $\mathcal{H}_{ABC}$ and $\mathcal{H}_A$ contain all the information of $H_{A'AB}$. In other words,

$$H_{A'ABC} = 0 \quad \Longleftrightarrow \quad \mathcal{H}_A = 0 \quad \& \quad \mathcal{H}_{ABC} = 0$$

Substituting the definition (19a) one obtains

$$\mathcal{H}_A = \tfrac{3}{2}\nabla_\tau \kappa_A - \mathcal{D}_{AB}\kappa^B, \qquad \mathcal{H}_{ABC} = 2\mathcal{D}_{(AB}\kappa_{C)}, \tag{43}$$

Then $H_{A'AB}|_{\mathcal{S}} = 0$ imposes then following conditions on the the initial data $(\kappa_A, \nabla_\tau\kappa_A)|_{\mathcal{S}}$ for the twistor candidate wave equation (36):

$$\nabla_\tau\kappa_A = \tfrac{2}{3}\mathcal{D}_{AB}\kappa^B, \qquad \mathcal{D}_{(AB}\kappa_{C)} = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{44}$$

Another set of constraints arise from the conditions $\nabla_\tau H_{A'BC}|_{\mathcal{S}} = 0$ and $B_{ABC}|_{\mathcal{S}} = 0$. These two conditions can be analysed in tandem since $\boldsymbol{B}$ is related to the derivative of $\boldsymbol{H}$. Using the space spinor split of ∇ it follows from the identity (29) that

$$\tau_D{}^{A'}\nabla_\tau H_{A'AB} - 2\tau^{CA'}\mathcal{D}_{DC}H_{A'AB} = 4B_{ABD}\Xi + \tfrac{4}{3}Q_{(A}\epsilon_{B)D} \tag{45}$$

Hence, transvecting with $\tau^D{}_{B'}$ and rearranging gives

$$\nabla_\tau H_{B'AB} = -4B_{ABD}\Xi\tau^D{}_{B'} - 2\tau^{CA'}\tau^D{}_{B'}\mathcal{D}_{DC}H_{A'AB} - \tfrac{4}{3}\tau^D{}_{B'}Q_{(A}\epsilon_{B)D} \tag{46}$$

Hence, if the the twistor candidate wave equation is imposed, namely $Q_A = 0$ then

$$H_{A'AB}|_{\mathcal{S}} = 0 \quad \& \quad B_{ABC}|_{\mathcal{S}} = 0 \implies \nabla_\tau H_{A'AB}|_{\mathcal{S}} = 0. \tag{47}$$

In other words, imposing $\nabla_\tau H_{A'AB}|_{\mathcal{S}} = 0$ is redundant if $H_{A'AB}|_{\mathcal{S}} = 0$ and $B_{ABC}|_{\mathcal{S}} = 0$. Using the definition (19b), the condition $B_{ABC}|_{\mathcal{S}} = 0$ simply reads,

$$\phi_{ABCD}\kappa^D = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{48}$$

Finally, for the condition $\nabla_\tau B_{ABC}|_{\mathcal{S}} = 0$ one has, applying ∇_τ to equation (19b)

$$\nabla_\tau B_{ABC} = \phi_{ABCD}\nabla_\tau\kappa^D + \kappa^D\nabla_\tau\phi_{ABCD} \tag{49}$$

At this point one can exploit the evolution equation for the rescaled Weyl spinor (17) to subsitute for $\nabla_\tau\phi_{ABCD}$ and condition (44) to substitute $\nabla_\tau\kappa_A$ when evaluating at $\mathcal{S}$. Thus

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -2\kappa^D\mathcal{D}_{DF}\phi_{ABC}{}^F + \tfrac{2}{3}\phi_{ABCD}\mathcal{D}^D{}_F\kappa^F = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{50}$$

In fact, the latter expression can be rewritten in terms of $\mathcal{H}_{ABC}|_{\mathcal{S}}$ and $B_{ABC}|_{\mathcal{S}}$ as follows. Swapping indices D and A in equation (50), and exploiting the constraint equation for the rescaled Weyl spinor in expression (17) renders

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -2\kappa^D\mathcal{D}_{AF}\phi_{DBC}{}^F + \tfrac{2}{3}\phi_{ABCD}\mathcal{D}^D{}_F\kappa^F = 0 \qquad \text{on} \qquad \mathcal{S} \tag{51}$$

Applying a $\mathcal{D}_{FQ}$ to the definition (19b) and using the Leibnitz rule, one can replace the first term in the last equation to obtain

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -2\mathcal{D}_{AD}B_{BC}{}^D - 2\phi_{BCDF}\mathcal{D}_A{}^F\kappa^D + \tfrac{2}{3}\phi_{ABCF}\mathcal{D}_D{}^F\kappa^D = 0 \quad \text{on} \quad \mathcal{S} \tag{52}$$

From the irreducible decomposition of $\mathcal{D}_{AB}\kappa_C$ and using the expression for $\mathcal{H}_{ABC}$ equation (43) one has

$$\mathcal{D}_{AB}\kappa_C = \tfrac{1}{2}\mathcal{H}_{ABC} + \tfrac{1}{3}\epsilon_{BC}\mathcal{D}_{AD}\kappa^D + \tfrac{1}{3}\epsilon_{AC}\mathcal{D}_{BD}\kappa^D. \tag{53}$$

Substituting equation (53) into equation (52) one concludes one gets

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -\phi_{BCDF}\mathcal{H}_A{}^{DF} - 2\mathcal{D}_{AD}B_{BC}{}^D = 0 \quad \text{on} \quad \mathcal{S} \tag{54}$$

Hence, overall, the only independent conditions to be imposed are $H_{A'AB}|_{\mathcal{S}} = 0$ and $B_{ABC}|_{\mathcal{S}} = 0$.

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5 Killing spinor zero-quantities

For the subsequent discussion it is convenient to introduce the following spinors

$$H_{A'ABC} \equiv 3\nabla_{A'(A}\kappa_{BC)}, \tag{55a}$$

$$S_{AA'BB'} \equiv \nabla_{QA'}H_{B'}{}^{Q}{}_{AB}, \tag{55b}$$

$$B_{ABCD} \equiv -\frac{1}{6}\nabla_{Q'(A}H^{Q'}{}_{BCD)}. \tag{55c}$$

Observe that if $(\mathcal{M}, \boldsymbol{g})$ admits a Killing spinor κ_{AB} then, by definition, one has

$$H_{A'ABC} = 0, \qquad S_{AA'BB'} = 0, \qquad B_{ABCD} = 0.$$

To see the geometric significance of the above defined zero-quantities it is convenient to introduce the Hermitian spinor

$$\xi_{AA'} \equiv \nabla^{B}{}_{A'}\kappa_{AB}. \tag{56}$$

Observe that the zero-quantity $S_{AA'BB'}$ can be written in terms of $\xi_{AA'}$ as

$$S_{CC'DD'} = -6\kappa_{(D}{}^{A}\Phi_{C)AC'D'} - \nabla_{CC'}\xi_{DD'} - \nabla_{DD'}\xi_{CC'}. \tag{57}$$

Notice that, if $\Phi_{ABA'B'}$ vanishes then $S_{AA'BB'}$ reduces to the Killing vector equation and $\xi_{AA'}$ corresponds to the spinorial counterpart of a Killing vector. To clarify this point further observe that, as a consequence of the conformal properties of the Killing spinor equation, if κ_{AB} is a Killing spinor in the unphysical spacetime then

$$\tilde{\kappa}_{AB} = \frac{1}{\Xi^2}\kappa_{AB}$$

is a Killing spinor of the physical spacetime whenever Ξ is not vanishing. As discussed before, the physical Killing spinor $\tilde{\kappa}_{AB}$ gives rise to a Killing vector $\tilde{\xi}_a$ whose spinorial counterpart is

$$\tilde{\xi}_{AA'} \equiv \tilde{\nabla}^{B}{}_{A'}\tilde{\kappa}_{AB}. \tag{58}$$

Furthermore, if $\tilde{\xi}_a$ is a Killing vector in the physical spacetime then,

$$X_a \equiv \Xi^2\tilde{\xi}_a \tag{59}$$

corresponds to a conformal Killing vector for the unphysical spacetime, namely, it can be verified that

$$\nabla_a X_b + \nabla_b X_a = \frac{1}{2}\nabla^a X_a g_{ab}. \tag{60}$$

This conformal Killing vector additionally satisfy

$$X^a\nabla_a\Xi = \frac{1}{4}\nabla_a X^a. \tag{61}$$

Equations (60)-(61) are the so-called *unphysical Killing equations* —see [24] for a discussion on the unphysical Killing equations. A direct computation shows that given a Killing spinor in the unphysical spacetime κ_{AB}, the concomitant

$$X_{AA'} \equiv \Xi\xi_{AA'} - 3\kappa_{AQ}\nabla_{A'}{}^{Q}\Xi \tag{62}$$

corresponds to the spinorial counterpart of a conformal Killing vector satisfying equations (60) and (61). However, in general, the spinor $\xi_{AA'}$ has no straightforward interpretation and will be regarded as an auxiliary variable for the subsequent discussion. Additionally, observe that the introduction of $\xi_{AA'}$ allows one to write the irreducible decomposition of the gradient of the Killing spinor as

$$\nabla_{AA'}\kappa_{BC} = \frac{1}{3}H_{A'ABC} - \frac{1}{3}\xi_{CA'}\epsilon_{AB} - \frac{1}{3}\xi_{BA'}\epsilon_{AC}. \tag{63}$$

On the other hand, a direct computation using the definition of $H_{A'ABC}$ shows that the zero-quantity B_{ABCD} encodes the Buchdahl constraint, namely

$$B_{ABCD} = \kappa_{(D}{}^{F}\Psi_{ABC)F}. \tag{64}$$

To complete the discussion observe that if the Killing spinor equation $H_{A'ABC} = 0$ is satisfied, the Killing spinor κ_{AB} and the auxiliary spinor $\xi_{AA'}$ satisfy the following wave equations:

$$\square\kappa_{BC} = -4\Lambda\kappa_{BC} + \kappa^{AD}\Psi_{BCAD}, \tag{65}$$

$$\begin{aligned}\square\xi_{AA'} = &-\tfrac{4}{3}\Delta^{BC}{}_{BA'}\kappa_{AC} - 2\Lambda_{A'A}{}^{BC}\kappa_{BC} - 6\xi_{AA'}\Lambda - 2\xi^{BB'}\Phi_{ABA'B'} \\ &-\tfrac{3}{2}\kappa^{BC}Y_{ABCA'} + \Psi_{ABCD}H_{A'}{}^{BCD} - 12\kappa_{AB}\nabla^{B}{}_{A'}\Lambda - \tfrac{1}{2}\kappa^{BC}\Delta_{(ABC)A'}\end{aligned} \tag{66}$$

The wave equation (65) is derived considering the integrability condition $\nabla^{AA'}H_{A'ABC} = 0$, substituting equation (55a) and exploiting the spinorial Ricci identities (3a)-(3b). To obtain equation (66) observe that from equation (56) one has

$$\square\xi_{AA'} = \nabla_{CC'}\nabla^{CC'}\nabla^{B}{}_{A'}\kappa_{AB}. \tag{67}$$

Commuting covariant derivatives in the last expression renders

$$\square\xi_{AA'} = \square_{A'B'}\nabla^{BB'}\kappa_{AB} + \square^{C}{}_{B}\nabla^{B}{}_{A'}\kappa_{AC} - \nabla_{BA'}\square^{CD}\kappa_{AC} + \nabla^{B}{}_{A'}\square\kappa_{AB} - \nabla^{B}{}_{B'}\square_{A'}{}^{B'}\kappa_{AB}.$$

Then, a lengthy computation using the decomposition of the auxiliary vector as given in equation (63), the Bianchi identities (**??**)-(**??**) and the spinorial Ricci identities (3a)-(3b) render equation (66). The above discussion is summarised in the following

Lemma 1. *Let $(\mathcal{M}, \boldsymbol{g})$ represent a solution to the conformal Einstein field equations admitting a Killing spinor κ_{AB}; namely suppose that*

$$H_{A'ABC} = 0, \quad Z_{AA'BB'} = 0, \quad Z_{AA'} = 0, \quad \Delta_{ABCC'} = 0, \quad \Pi_{AA'BB'} = 0, \quad \Lambda_{AB'BC} = 0.$$

Let $\xi_{AA'}$ denote the auxiliary vector defined as in equation (56)*, then*

$$\nabla_{CC'}\xi_{DD'} + \nabla_{DD'}\xi_{CC'} + 6\kappa_{(D}{}^{A}\Phi_{C)AC'D'} = 0, \qquad \kappa_{(D}{}^{F}\Psi_{ABC)F} = 0.$$

Moreover

$$X_{AA'} = \Xi\xi_{AA'} - 3\kappa_{AQ}\nabla_{A'}{}^{Q}\Xi$$

is the spinorial counterpart of a conformal Killing vector X_a satisfying the unphysical Killing vector equations (60)–(61)*. In addition, the Killing spinor κ_{AB} and the auxiliary vector $\xi_{AA'}$ satisfy the following wave equations*

$$\square\kappa_{BC} = -4\Lambda\kappa_{BC} + \kappa^{AD}\Psi_{BCAD}, \tag{68}$$

$$\square\xi_{AA'} = -6\xi_{AA'}\Lambda - 2\xi^{BB'}\Phi_{ABA'B'} - \tfrac{3}{2}\kappa^{BC}Y_{ABCA'} - 12\kappa_{AB}\nabla^{B}{}_{A'}\Lambda. \tag{69}$$

In the following, we aim to identify the initial data for κ_{AB} which, when propagated according to (68) gives rise to a Killing spinor on the spacetime development. It is important to note that, since κ_{AB} solves equation (68) by construction, equation (68) can be assumed to hold throughout $\mathcal{M}$.

6 Propagation equations

In this section we construct, given a solution to equations (68)-(69) on $\mathcal{M}$, a set of wave equations for the zero-quantities $H_{A'ABC}$ and $S_{AA'BB'}$ which are homogeneous in these zero-quantities and their first derivatives.

6.1 The general strategy

As discussed in detail in Sections 6.3-6.5, deriving the required wave equation for $S_{A'ABC}$ is more involved than the one for $H_{A'ABC}$. In order to obtain an homogeneous wave equation for $S_{AA'BB'}$, we first derive separately two inhomogeneous equations, which when combined yield the desired homogeneous wave equation. The first inhomogeneous equation, derived in Section 6.3, makes use of the definition of $S_{AA'BB'}$ in terms of the auxiliary vector (57). The second inhomogeneous equation, derived in Section 6.5, is obtained through the use of equations (55b)-(55c) and (64). The procedure is summarised in the schematic of Figure 1.

$$
\left.\begin{array}{l}
\mathbf{S} = \mathbf{\Phi} \times \boldsymbol{\kappa} + \boldsymbol{\nabla} \times \boldsymbol{\xi} \longrightarrow \square \mathbf{S} = \mathbf{I}_{sym} + \mathbf{h}_1 \\[2em]
\left.\begin{array}{l}
\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{B} + \mathbf{S} \longrightarrow (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B})_{sym} = \square \mathbf{S}_{sym} + \mathbf{h}_{2\ sym} \\[2em]
\mathbf{B} = \mathbf{\Psi} \times \boldsymbol{\kappa} \longrightarrow (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B})_{sym} = \mathbf{E}_{sym} + \mathbf{h}_{3\ sym}
\end{array}\right\} \square \mathbf{S}_{sym} = \mathbf{E}_{sym} + \mathbf{h}_{sym}
\end{array}\right\} \square \mathbf{S} = \mathbf{h}
$$

Figure 1: Schematic description of the derivation of the wave equation for $S_{AA'BB'}$, given in Sections 6.3-6.5. In this diagram spinors are represented simply by their kernel letters, e.g., $T_{AB\ldots F,C'D'\ldots H'}$ is denoted by $\mathbf{T}$. In addition, the symbol $\times$ has been used to denote, in a schematic way, contractions between spinors. The subscript *sym* has been added to indicate that a given expression is symmetric. The quantities $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ denote homogeneous expressions in $\mathbf{H}$, $\boldsymbol{\nabla} \times \mathbf{H}$, $\mathbf{S}$ and $\boldsymbol{\nabla} \times \mathbf{S}$. The quantities $\mathbf{I}_{sym}$ and $\mathbf{E}_{sym}$ encode the inhomogeneous terms appearing in the corresponding equations.

Once the desired wave equations are obtained the following result for homogeneous wave equations will be used:

Observe that it follows from the uniqueness property of Theorem 1 that the zero-quantities are *propagated*; that is to say that if the initial conditions

$$
\begin{aligned}
H_{A'ABC} &= 0, \\
\mathcal{P}H_{A'ABC} &= 0, \\
S_{AA'BB'} &= 0, \\
\mathcal{P}S_{AA'BB'} &= 0,
\end{aligned}
$$

hold on $\mathcal{U} \subset \mathcal{S}$, then $H_{A'ABC}$ and $S_{AA'BB'}$ vanish identically on the domain of dependence of $\mathcal{U}$. The above initial conditions may then be translated into necessary and sufficient conditions for the Killing spinor candidate, κ_{AB}, restricted to a Cauchy hypersurface $\mathcal{S}$; this is done in Section 7.

6.2 Wave equation for $H_{A'ABC}$

To construct the wave equation for $H_{A'ABC}$ one starts from the identity

$$\square H_{A'ABC} = \nabla_{DD'}\nabla_A{}^{D'} H_{A'BC}{}^D - \nabla_{AD'}\nabla_D{}^{D'} H_{A'}{}^D{}_{BC}.$$

Substituting the definition of $S_{AA'BB'}$, as given in equation (55b), in the second term of the last expression and using equations (3a)-(3b) one obtains

$$\square H_{A'ABC} = 10\Lambda H_{A'ABC} - 4\Psi_{ADF(B} H_{|A'|C)}{}^{DF} - 2\Phi_A{}^D{}_{A'}{}^{D'} H_{D'BCD} - 2\nabla_{AD'} S_B{}^{D'}{}_{CA'}. \quad (70)$$

Exploiting the symmetries of $H_{A'ABC}$ one obtains

$$\square H_{A'ABC} = 10\Lambda H_{A'ABC} + 2\nabla_{(A}{}^{D'} S_{B|D'|C)A'} - 2\Phi_{(A}{}^D{}_{|A'|}{}^{D'} H_{D'|BC)D} - 4\Psi_{(AB}{}^{DF} H_{|A'|C)DF}. \quad (71)$$

Observe that equations (71) and (70) contain the same information as the traces of equation (70) represent identities which follow from the definition of $S_{AA'BB'}$ in terms of $H_{A'ABC}$ as given in equation (55b). These identities will be useful for the subsequent discussion.

$$
\begin{aligned}
\nabla_{AD'}S^{AD'}{}_{CA'} &= -\Psi_{CADB}H_{A'}{}^{ADB} + \Phi^{AD}{}_{A'}{}^{D'}H_{D'CAD}, && (72a)\\
\nabla_{AD'}S_{B}{}^{D'A}{}_{A'} &= -\Psi_{BADC}H_{A'}{}^{ADC} + \Phi^{AD}{}_{A'}{}^{D'}H_{D'BAD}, && (72b)\\
\nabla_{AD'}S^{BD'}{}_{BA'} &= 0. && (72c)
\end{aligned}
$$

Notice that equation (71) is a wave equation homogeneous in $H_{A'ABCD}$, $S_{AA'BB'}$ and their first derivatives.

6.3 Inhomogeneous wave equation for $S_{AA'BB'}$

The purpose of this section is to derive the first of two inhomogeneous wave equations for $S_{AA'BB'}$, which, when combined, will yield the desired homogeneous equation. To proceed, some ancillary spinorial decompositions will be required.

6.3.1 Ancillary decompositions

This section collects some ancillary decompositions which will prove useful for the derivation of the wave equation for $S_{AA'BB'}$. To start the discussion, observe that from expression (55b) it follows that

$$
\nabla_{CC'}\xi^{CC'} = -\frac{1}{2}S_{CC'}{}^{CC'}, \qquad \nabla_{(A|(A'}\xi_{B')|B)} = -\frac{1}{2}S_{(AB)(A'B')} - 3\kappa_{(A}{}^{C}\Phi_{B)CA'B'}. \tag{73}
$$

Using the above expressions the gradient of the auxiliary vector $\xi_{AA'}$ can be decomposed as

$$
\begin{aligned}
\nabla_{AA'}\xi_{BB'} = -3\kappa_{(B}{}^{C}\Phi_{A)CA'C'} - \frac{1}{8}S^{CC'}{}_{CC'}\epsilon_{AB}\epsilon_{A'B'} - \frac{1}{2}S_{(AB)(A'B')}\\
- \frac{1}{2}\epsilon_{A'B'}\nabla_{(A}{}^{C'}\xi_{B)C'} - \frac{1}{2}\epsilon_{AB}\nabla^{C}{}_{(A'}\xi_{B')C}.
\end{aligned} \tag{74}
$$

Using the definitions of $H_{A'ABC}$ and $\xi_{AA'}$ encoded in equations (55a) and (56) respectively, one can reexpress the gradient of the Killing spinor as

$$
\nabla_{AA'}\kappa_{BC} = \tfrac{1}{3}H_{A'ABC} - \tfrac{1}{3}\xi_{CA'}\epsilon_{AB} - \tfrac{1}{3}\xi_{BA'}\epsilon_{AC}. \tag{75}
$$

The irreducible decomposition of the gradient of the trace-free Ricci spinor the second Bianchi identity as expressed in (**??**) and the equation encoded in the zero-quantity (**??**) implies

$$
\begin{aligned}
\nabla_{AA'}\Phi_{BCB'C'} = \tfrac{1}{6}\bar{Y}_{A'B'C'C}\ \epsilon_{AB} + \tfrac{1}{6}\bar{Y}_{A'B'C'B}\epsilon_{AC} + \tfrac{1}{6}Y_{ABCC'}\bar{\epsilon}_{A'B'} + \tfrac{1}{6}Y_{ABCB'}\bar{\epsilon}_{A'C'}\\
- \tfrac{2}{3}\ \bar{\epsilon}_{A'C'}\epsilon_{A(C}\nabla_{B)B'}\Lambda - \tfrac{2}{3}\ \bar{\epsilon}_{A'B'}\epsilon_{A(C}\nabla_{B)C'}\Lambda + \nabla_{(A'|(A}\Phi_{BC)|B'C')}.
\end{aligned} \tag{76}
$$

Another identity that will be used involving second derivatives of the tracefree Ricci spinor is derived as follows: Applying $\nabla^{A}{}_{B'}$ to the zero-quantity encoded in (**??**), and after a lengthy computation using equations (**??**)-(**??**) and (**??**)-(**??**), one obtains the identity

$$
\begin{aligned}
\square\Phi_{BCC'D'} = 8\Lambda\Phi_{BCB'C'} - 2\ \Phi_{B}{}^{A}{}_{C'}{}^{A'}\Phi_{CAB'A'} - 2\ \Phi_{B}{}^{A}{}_{B'}{}^{A'}\Phi_{CAC'A'} - 2\Phi_{BC}{}^{A'D'}\ \Psi_{B'C'A'D'}\\
+ \tfrac{3}{2}\epsilon_{BC}\bar{\epsilon}_{B'C'}\square\Lambda - 2\ \nabla_{(B|(B'}\nabla_{C')|C)}\Lambda + \nabla_{AB'}Y_{B}{}^{A'A}{}_{A'CC'} + 2\nabla_{AB'}\nabla_{BA'}\Phi_{C}{}^{A}{}_{C'}{}^{A'}.
\end{aligned} \tag{77}
$$

Additionally, the following decompositions will be used in the subsequent discussion

$$
\nabla_{AA'}\nabla_{BB'}\Lambda = \tfrac{1}{4}\epsilon_{AB}\bar{\epsilon}_{A'B'}\square\Lambda + \nabla_{(A'|(A}\nabla_{B)|B')}\Lambda, \tag{78}
$$

$$
\begin{aligned}
\xi_{DD'}\nabla_{CC'}\Lambda = \tfrac{1}{4}\xi^{AA'}\epsilon_{CD}\bar{\epsilon}_{C'D'}\nabla_{AA'}\Lambda + \xi_{(C|(C'}\nabla_{D')|D)}\Lambda\\
+\tfrac{1}{2}\bar{\epsilon}_{C'D'}\xi_{(C}{}^{A'}\nabla_{D)A'}\Lambda + \tfrac{1}{2}\epsilon_{CD}\xi^{A}{}_{(C'}\nabla_{|A|D')}\Lambda.
\end{aligned} \tag{79}
$$

6.3.2 Wave equation for $S_{AA'BB'}$

Applying $\nabla_{PP'}\nabla^{PP'}$ to the expression for $S_{AA'BB'}$ in terms of first derivatives of the auxiliary vector as encoded in equation (57) one obtains

$$\square S_{CC'DD'} = -6\nabla_{PP'}\nabla^{PP'}(\kappa_{(C}{}^{Q}\Phi_{D)QC'D'}) - \nabla_{PP'}\nabla^{PP'}\nabla_{CC'}\xi_{DD'} - \nabla_{PP'}\nabla^{PP'}\nabla_{DD'}\xi_{CC'}. \tag{80}$$

Commuting covariant derivatives renders

$$\begin{aligned}
\square S_{CC'DD'} = {} & -6\Phi_{C'D'A(C}\square\kappa_{D)}{}^{A} - 6\kappa_{(C}{}^{A}\square\Phi_{D)AC'D'} - \nabla_{CC'}\square\xi_{DD'} - \nabla_{DD'}\square\xi_{CC'} \\
& - \square_{CA}\nabla^{A}{}_{C'}\xi_{DD'} - \square_{C'A'}\nabla_{C}{}^{A'}\xi_{DD'} - \square_{DA}\nabla^{A}{}_{D'}\xi_{CC'} - \square_{D'A'}\nabla_{D}{}^{A'}\xi_{CC'} \\
& + \nabla_{AC'}\square_{C}{}^{A}\xi_{DD'} + \nabla_{AD'}\square_{D}{}^{A}\xi_{CC'} + \nabla_{CA'}\square_{C'}{}^{A'}\xi_{DD'} + \nabla_{DA'}\square_{D'}{}^{A'}\xi_{CC'} \\
& - 6\nabla_{BA'}\Phi_{DAC'D'}\nabla^{BA'}\kappa_{C}{}^{A} - 6\nabla_{BA'}\Phi_{CAC'D'}\nabla^{BA'}\kappa_{D}{}^{A}.
\end{aligned}$$

At this point one can substitute the wave equation for the Killing spinor, the auxiliary spinor and the tracefree Ricci spinor as given in equations (68), (69) and (77) respectively. Additionally, observe that the the spinorial Ricci identities (3a)-(3b) can be employed to replace the operator $\square_{AB}$ by curvature terms. Substituting equations (68), (69), (77), (64), the conformal Einstein field equations as encoded in (**??**) and the auxiliary results collected in Section 6.3.1, one obtains the following wave equation for $S_{CC'DD'}$

$$\begin{aligned}
\square S_{CC'DD'} = {} & 12B_{CDAB}\Phi^{AB}{}_{C'D'} - 3\xi^{A}{}_{D'}Y_{CDAC'} - 3\xi^{A}{}_{C'}Y_{CDAD'} \\
& + 3\kappa_{D}{}^{A}\nabla_{BC'}Y_{CA}{}^{B}{}_{D'} + 3\kappa_{C}{}^{A}\nabla_{BC'}Y_{DA}{}^{B}{}_{D'} + \tfrac{3}{2}\kappa^{AB}\nabla_{DD'}Y_{CABC'} + \tfrac{3}{2}\kappa^{AB}\nabla_{CC'}Y_{DABD'} \\
& + \tfrac{1}{6}Y_{D}{}^{AB}{}_{D'}H_{C'CAB} + \tfrac{1}{6}Y_{C}{}^{AB}{}_{C'}H_{D'DAB} - \tfrac{1}{3}Y_{C}{}^{AB}{}_{D'}H_{C'DAB} - \tfrac{1}{3}Y_{D}{}^{AB}{}_{C'}H_{D'CAB} \\
& + \tfrac{8}{3}H_{D'CDA}\nabla^{A}{}_{C'}\Lambda + \tfrac{8}{3}H_{C'CDA}\nabla^{A}{}_{D'}\Lambda - \tfrac{2}{3}\bar{Y}_{C'D'}{}^{A'A}H_{A'CDA} - 3\Lambda S^{AA'}{}_{AA'}\epsilon_{CD}\bar{\epsilon}_{C'D'} \\
& - \nabla_{CC'}\left(H_{D'}{}^{ABF}\Psi_{DABF}\right) - \nabla_{DD'}\left(H_{C'}{}^{ABF}\Psi_{CABF}\right) \\
& - 2\Psi_{CDAB}S^{(A}{}_{(C'}{}^{B)}{}_{D')} - 2\bar{\Psi}_{C'D'A'B'}S_{(C}{}^{(A'}{}_{D)}{}^{B')} + 4\Lambda S_{(C|(C'|D)D')} \\
& - 2\Phi_{D}{}^{A}{}_{D'}{}^{A'}(S_{(C|(C'|A)A')} - 2\Phi_{D}{}^{A}{}_{C'}{}^{A'}S_{(C|(D'|A)A)'} - 2\Phi_{C}{}^{A}{}_{D'}{}^{A'}S_{(D|C'|A)A')} \\
& - 2\Phi_{C}{}^{A}{}_{C'}{}^{A'}S_{(D|D'|A)A')} - 2H^{A'}{}_{D}{}^{AB}\nabla_{C'|(C}\Phi_{AB)|D'A')} \\
& - 2H^{A'}{}_{C}{}^{AB}\nabla_{(C'|(D}\Phi_{AB)|D'A')}.
\end{aligned} \tag{81}$$

Observe that the latter is an homogeneous expression in $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives except for the term

$$\begin{aligned}
I_{CC'DD'} \equiv {} & 12B_{CDAB}\Phi^{AB}{}_{C'D'} - 3\xi^{A}{}_{D'}Y_{CDAC'} - 3\xi^{A}{}_{C'}Y_{CDAD'} + 3\kappa_{D}{}^{A}\nabla_{BC'}Y_{CA}{}^{B}{}_{D'} \\
& + 3\kappa_{C}{}^{A}\nabla_{BC'}Y_{DA}{}^{B}{}_{D'} + \tfrac{3}{2}\kappa^{AB}\nabla_{CC'}Y_{DABD'} + \tfrac{3}{2}\kappa^{AB}\nabla_{DD'}Y_{CABC'}.
\end{aligned}$$

As an additional remark observe that taking a trace of equation (81) one obtains

$$\begin{aligned}
\square S_{CC'}{}^{C}{}_{D'} = {} & \tfrac{1}{2}\bar{Y}_{D'}{}^{A'B'A}\bar{H}_{AC'A'B'} - \tfrac{1}{2}\bar{Y}_{C'}{}^{A'B'A}\bar{H}_{AD'A'B'} - 6\Lambda\bar{S}^{A'A}{}_{A'A}\bar{\epsilon}_{C'D'} \\
& - \nabla_{AC'}\left(\bar{H}^{AA'B'F'}\bar{\Psi}_{D'A'B'F'}\right) + \nabla_{AD'}\left(\bar{H}^{AA'B'F'}\bar{\Psi}_{C'A'B'F'}\right).
\end{aligned} \tag{82}$$

Observe that the right-hand side of the last equation is an homogeneous expression in $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives. Consequently, exploiting the irreducible decomposition of $S_{AA'BB'}$ to write

$$\begin{aligned}
\square S_{CC'DD'} = {} & \square S_{(CD)(C'D')} - \tfrac{1}{2}\bar{\epsilon}_{C'D'}\square S_{(C}{}^{A'}{}_{D)A'} \\
& - \tfrac{1}{2}\epsilon_{CD}\square S^{A}{}_{(C'|A|D')} + \tfrac{1}{4}\epsilon_{CD}\bar{\epsilon}_{C'D'}\square S^{AA'}{}_{AA'},
\end{aligned} \tag{83}$$

one can re-express equation (81) as

$$\square S_{CC'DD'} = I_{(CD)(C'D')} + F_{CC'DD'}, \tag{84}$$

where $F_{CC'DD'}$ is an homogeneous expression depending on $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives. Notice that the inhomogeneous term $I_{(CD)(C'D')}$ contains the Buchdahl constraint B_{ABCD}. Consequently, one needs to analyse more closely this quantity. An immediate but important observation for the latter discussion is that the main obstruction to obtaining an homogeneous wave equation is contained in the symmetric part of $S_{CDC'D'}$:

$$\square S_{(CD)(C'D')} = I_{(CD)(C'D')} + F_{(CD)(C'D)'}. \tag{85}$$

6.4 Derivatives of the Buchdahl zero-quantity

The presence of the Buchdahl zero-quantity in the inhomogeneous term of equation (85) suggests that it is necessary to find auxiliary identities associated to the Buchdahl constraint. The aim of this section is to derive such identities by expressing $\nabla^{C}{}_{C'}\nabla^{A}{}_{A'}B_{ABCD}$ in two different ways by exploiting equations (55b)-(55c) and (64).

6.4.1 First approach to express second derivatives of the Buchdahl constraint

The irreducible decomposition of $\nabla_{AA'}H^{A'}{}_{BCD}$ and definitions (55b)-(55c) give

$$\nabla_{AA'}H^{A'}{}_{BCD} = -6B_{ABCD} - \tfrac{3}{4}\epsilon_{A(B}S_{C}{}^{A'}{}_{D)A'}. \tag{86}$$

Applying $\nabla^{A}{}_{Q'}$ to the last expression one obtains

$$\nabla^{A}{}_{Q'}\nabla_{AA'}H^{A'}{}_{BCD} = 6\nabla_{AQ'}B_{BCD}{}^{A} - \tfrac{3}{4}\nabla_{Q'(B}S_{C}{}^{A'}{}_{D)A'}. \tag{87}$$

Exploiting the spinorial Ricci identities (3a)-(3b) in equation (87) and rearranging one derives the following expression

$$\begin{aligned}
\nabla_{AQ'}B_{BCD}{}^{A} = {} & \tfrac{1}{12}\square H_{Q'BCD} - \tfrac{1}{2}\Lambda H_{Q'BCD} \\
& + \tfrac{1}{6}\Phi_{D}{}^{A}{}_{Q'}{}^{A'}H_{A'BCA} + \tfrac{1}{6}\Phi_{C}{}^{A}{}_{Q'}{}^{A'}H_{A'BDA} + \tfrac{1}{6}\Phi_{B}{}^{A}{}_{Q'}{}^{A'}H_{A'CDA} \\
& + \tfrac{1}{24}\nabla_{BQ'}S_{C}{}^{A'}{}_{DA'} + \tfrac{1}{24}\nabla_{CQ'}S_{B}{}^{A'}{}_{DA'} + \tfrac{1}{24}\nabla_{DQ'}S_{B}{}^{A'}{}_{CA'}.
\end{aligned}$$

At this point we can substitute for the D'Alembertian of the zero quantity $H_{A'ABC}$ using the wave equation (71). Having done so, one can apply a further derivative to the above equation to obtain

$$\nabla^{C}{}_{C'}\nabla^{A}{}_{A'}B_{ABCD} = \tfrac{1}{36}\square S_{(B|C'|D)A'} - \tfrac{1}{48}\square S_{B}{}^{B'}{}_{DB'}\bar{\epsilon}_{A'C'} + \Sigma_{C'A'BD} + P_{C'A'BD} \tag{88}$$

where $P_{C'A'BD}$ is an homogeneous expression in on $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives, given in appendix 9, and where $\Sigma_{C'A'BD}$ is given by

$$\begin{aligned}
\Sigma_{C'A'BD} \equiv {} & \tfrac{1}{24}\nabla_{AC'}\nabla_{BA'}S^{AB'}{}_{DB'} - \tfrac{1}{36}\nabla_{AC'}\nabla_{BB'}S^{AB'}{}_{DA'} - \tfrac{1}{36}\nabla_{AC'}\nabla_{BB'}S_{D}{}^{B'A}{}_{A'} \\
& + \tfrac{1}{24}\nabla_{AC'}\nabla_{DA'}S_{B}{}^{B'A}{}_{B'} - \tfrac{1}{36}\nabla_{AC'}\nabla_{DB'}S^{AB'}{}_{BA'} - \tfrac{1}{36}\nabla_{AC'}\nabla_{DB'}S_{B}{}^{B'A}{}_{A'}.
\end{aligned}$$

Commuting covariant derivatives and using the identities (72a)-(72c) one can rewrite the above expression as follows

$$\Sigma_{C'A'BD} = \tfrac{1}{18}\square S_{(B|C'|D)A'} + Q_{C'A'BD} \tag{89}$$

where $Q_{C'A'BD}$ is an homogeneous expression in on $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives, given in Appendix 9. Consequently, one has

$$\nabla^{C}{}_{C'}\nabla^{A}{}_{A'}B_{ABCD} = \tfrac{1}{12}\square S_{(B|C'|D)A'} - \tfrac{1}{48}\square S_{B}{}^{B'}{}_{DB'}\bar{\epsilon}_{A'C'} + P_{C'A'BD} + Q_{C'A'BD} \tag{90}$$

6.4.2 Second approach to express second derivatives of the Buchdahl constraint

An alternative way to obtain an expression for $\nabla^{C}{}_{C'}\nabla^{A}{}_{A'}B_{ABCD}$ is to start from equation equation (64) to obtain

$$\nabla^{A}{}_{A'}B_{ABCD} = \Psi_{F(BCD}\nabla^{A}{}_{|A'|}\kappa_{A)}{}^{F} + \kappa_{(D}{}^{F}\nabla^{A}{}_{|A'|}\Psi_{ABC)F}.$$

Exploiting the decomposition (75) and the conformal Einstein field equations as encoded in (**??**) gives

$$\nabla^{A}{}_{A'}B_{ABCD} = \tfrac{1}{2}\xi^{A}{}_{A'}\Psi_{BCDA} - \tfrac{3}{8}\kappa_{(B}{}^{A}Y_{CD)AA'} + \tfrac{1}{4}\Psi_{AF(BC}H_{D)A'}{}^{AF} - \tfrac{1}{4}\kappa^{AF}\nabla_{FA'}\Psi_{BCDA}. \tag{91}$$

Applying $\nabla^{C}{}_{C'}$ to equation (91), a long calculation exploiting the irreducible decomposition of $\nabla_{AA'}\kappa_{BC}$ and $\nabla_{AA'}\xi_{BB'}$, commuting covariant derivatives and using the conformal Einstein field equations as given in (**??**) renders

$$\begin{aligned}
\nabla^{C}{}_{C'}\nabla^{A}{}_{A'}B_{ABCD} = {} & \tfrac{1}{4}\left(\Phi_{D}{}^{F}{}_{A'C'}\Psi_{BACF} - 4\Phi_{A}{}^{F}{}_{A'C'}\Psi_{BDCF} + \Phi_{B}{}^{F}{}_{A'C'}\Psi_{DACF}\right)\kappa^{AC} \\
& + \tfrac{1}{4}\bar{\epsilon}_{A'C'}\left(\Psi_{ACFG}\Psi_{BD}{}^{FG} + 2\Psi_{BA}{}^{FG}\Psi_{DCFG} - 6\Lambda\Psi_{BDAC}\right)\kappa^{AC} \\
& + \tfrac{1}{8}\kappa^{AC}\nabla_{CA'}Y_{BDAC'} + \tfrac{1}{8}\kappa_{D}{}^{A}\nabla_{CC'}Y_{BA}{}^{C}{}_{A'} - \tfrac{1}{4}\xi^{A}{}_{C'}Y_{BDAA'} \\
& + \tfrac{1}{8}\kappa^{AC}\nabla_{CC'}Y_{BDAA'} + \tfrac{1}{8}\kappa_{B}{}^{A}\nabla_{CC'}Y_{DA}{}^{C}{}_{A'} - \tfrac{1}{4}\xi^{A}{}_{A'}Y_{BDAC'} \\
& + \tfrac{1}{4}\Psi_{BDAC}\bar{\epsilon}_{A'C'}\nabla^{(A|B'|}\xi^{C)}{}_{B'} + U_{A'BC'D},
\end{aligned} \tag{92}$$

where $U_{A'BC'D}$ is an homogeneous expression in on $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives, given in Appendix 9.

6.5 Wave equation for $S_{(AB)(A'B')}$

In Section 6.4 two different expressions for $\nabla^{C}{}_{C'}\nabla^{A}{}_{A'}B_{ABCD}$ were computed. Observe that the right-hand side of equation (90) contains $\square S_{(B|C'|D)A'}$ while (92) does not. Consequently, one can use equations (90) and (92) to obtain a wave equation for $S_{(B|C'|D)A'}$. A direct computation using equations (90), (92) and (64) renders

$$\square S_{(AB)(A'B')} = \mathcal{I}_{ABA'B'} + W_{ABA'B'}, \tag{93}$$

where $W_{ABA'B'} = W_{(AB)(A'B')}$ and $\mathcal{I}_{ABA'B'} = \mathcal{I}_{(AB)(A'B')}$ are homogeneous expressions in $H_{A'ABC}$, $S_{AA'BB'}$ and their first derivatives, given by

$$\begin{aligned}
W_{ABA'B'} \equiv {} & 12\left(U_{(AB)(A'B')} - P_{(AB)(A'B')} - Q_{(AB)(A'B')}\right), \\
\mathcal{I}_{ABA'B'} \equiv {} & -12B_{ABCD}\Phi^{CD}{}_{A'B'} + 3\xi^{C}{}_{B'}Y_{ABCA'} + 3\xi^{C}{}_{A'}Y_{ABCB'} \\
& - \tfrac{3}{4}\kappa_{B}{}^{C}\nabla_{DA'}Y_{AC}{}^{D}{}_{B'} - \tfrac{3}{4}\kappa_{A}{}^{C}\nabla_{DA'}Y_{BC}{}^{D}{}_{B'} - \tfrac{3}{2}\kappa^{CD}\nabla_{DB'}Y_{ABCA'} \\
& - \tfrac{3}{4}\kappa_{B}{}^{C}\nabla_{DB'}Y_{AC}{}^{D}{}_{A'} - \tfrac{3}{4}\kappa_{A}{}^{C}\nabla_{DB'}Y_{BC}{}^{D}{}_{A'} - \tfrac{3}{2}\kappa^{CD}\nabla_{DA'}Y_{ABCB'}.
\end{aligned}$$

Through use of the following identity

$$\kappa^{CD}\nabla_{AA'}Y_{BCDB'} = \kappa^{CD}\nabla_{DA'}Y_{ABCB'} - \kappa_{A}{}^{C}\nabla_{DA'}Y_{BC}{}^{D}{}_{B'}$$

it may be shown that in fact

$$\mathcal{I}_{ABA'B'} = -\tfrac{1}{2}I_{(AB)(A'B')}.$$

6.6 Homogeneous wave equation for $S_{AA'BB'}$

As discussed before, the main obstruction to obtaining an homogeneous wave equation is contained to the symmetric part of $S_{ABA'B'}$. More specifically, the obstruction is contained in the terms denoted by $I_{AA'BB'}$ and $\mathcal{I}_{AA'BB'}$ in equations (85) and (93) respectively. However, one can take linear combinations of equations (85) and (93) to remove the inhomogeneous terms. In particular one has

$$3\square S_{(AB)(A'B')} = I_{(AB)(A'B')} + 2\mathcal{I}_{ABA'B'} + F_{(AB)(A'B')} + 2W_{ABA'B'}.$$

After a direct computation using the explicit form of $I_{ABA'B'}$ and $\mathcal{I}_{ABA'B'}$ one concludes that

$$\square S_{(AB)(A'B')} = \frac{1}{3}F_{(AB)(A'B')} + \frac{2}{3}W_{ABA'B'}. \tag{94}$$

Finally, using equation (94), (83) and (84) one obtains

$$\square S_{AA'BB'} = \frac{1}{3}F_{(AB)(A'B')} - \tfrac{1}{2}\bar{\epsilon}_{A'B'}F_{(A}{}^{Q'}{}_{B)Q'} - \tfrac{1}{2}\epsilon_{AB}F^{Q}{}_{(A'|Q|B')} + \tfrac{1}{4}\epsilon_{AB}\bar{\epsilon}_{A'B'}F^{QQ'}{}_{QQ'} + \frac{2}{3}W_{ABA'B'}. \tag{95}$$

Notice that latter encodes an homogeneous expressions for $S_{AA'BB'}$ as $F_{(AB)(A'B')}$ and $W_{ABA'B'}$ represent homogeneous expressions on $H_{A'ABC}$, $S_{AA'BB'}$ and its first derivatives.

We are now in a position to state the following proposition:

Proposition 2. *Given initial data for the alternative conformal field equations on a spacelike hypersurface $\mathcal{S}$ with normal vector $\tau^{AA'}$, and associated normal derivative $\mathcal{P} \equiv \tau^{AA'}\nabla_{AA'}$, the corresponding spacetime development admits a Killing spinor in the domain of dependence of $\mathcal{U} \subset \mathcal{S}$ if and only if*

$$H_{A'ABC} = 0, \tag{96a}$$
$$\mathcal{P}H_{A'ABC} = 0, \tag{96b}$$
$$S_{AA'BB'} = 0, \tag{96c}$$
$$\mathcal{P}S_{AA'BB'} = 0 \tag{96d}$$

hold on $\mathcal{U}$.

Proof. The *only if* direction is immediate. Suppose, on the other hand, that (96a)–(96d) hold on some $\mathcal{U} \subset \mathcal{S}$ —that is to say, there exist spinor fields κ_{AB}, $\xi_{AA'}$ for which (96a)–(96d) are satisfied on $\mathcal{U}$. The latter is then used as initial data for the wave equations

$$\square\kappa_{BC} = -4\Lambda\kappa_{BC} + \kappa^{AD}\Psi_{BCAD},$$
$$\square\xi_{AA'} = -6\xi_{AA'}\Lambda - 2\xi^{BB'}\Phi_{ABA'B'} - \tfrac{3}{2}\kappa^{BC}Y_{ABCA'} - 12\kappa_{AB}\nabla^{B}{}_{A'}\Lambda.$$

As the zero-quantities $H_{A'ABC}$, $S_{AA'BB'}$ satisfy the homogeneous wave equations (71), (95) then the uniqueness result for homogeneous wave equations discussed in Section 6.1 ensures that

$$H_{A'ABC} = 0, \qquad S_{AA'BB'} = 0,$$

in the domain of dependence of $\mathcal{U}$. In other words, κ_{AB} solves the Killing spinor equation on the domain of dependence of $\mathcal{U}$. □

Remark 4. At first glance one might assume that the standard formulation of the conformal Einstein field equations is the appropriate setting for deriving the conditions obtained in this article. Nevertheless some experimentation reveals that instead of a conformally regular system of wave equations for $H_{A'ABC}$ and $S_{AA'BB'}$ one is confronted with an homogeneous Fuchsian system —formally singular at $\Xi = 0$. Although one could potentially still analyse this system and obtain an analogous result to Proposition 2, one would require a uniqueness result for solutions to Fuchsian systems of wave equations. Moreover, following the original spirit of the derivation of the conformal Einstein field equations in [9] one is interested in finding conformally regular equations instead of analysing Fuchsian systems. Fortunately, as shown in Section 6 the alternative formulation of the conformal Einstein field equations given in Section **??** leads to a regular set of wave equations for $H_{A'ABC}$ and $S_{AA'BB'}$.

7 The intrinsic conditions

In this section the conditions (96b)-(96d) are written in terms of intrinsic quantities on $\mathcal{S}$. To do so, the space spinor formalism will be exploited. The discussion given in this section is similar to that of [3]. Notice that, nevertheless, in the discussion given in [3] the Einstein field equations are used to simplify expressions associated with the curvature spinors. In the present analysis the curvature spinors are subject to the alternative conformal Einstein field equations as encoded in the zero-quantities (**??**)-(**??**).

7.1 Space spinor decompositions and ancillary identities

To obtain intrinsic conditions on $\mathcal{S}$ from equations (96a)-(96d) we start defining the space spinorial counterpart of the zero quantities $H_{A'ABC}$, $S_{AA'BB'}$:

$$H_{ABCD} \equiv \tau_A{}^{A'} H_{A'BCD}, \tag{97a}$$

$$S_{ABCD} \equiv \tau_A{}^{B'} \tau_C{}^{D'} S_{BB'DD'}. \tag{97b}$$

Next, we define the following spinors

$$\xi_{AB} \equiv \mathcal{D}_{(A}{}^{D}\kappa_{B)D},$$
$$\xi \equiv \mathcal{D}^{AB}\kappa_{AB},$$
$$\xi_{ABCD} \equiv \mathcal{D}_{(AB}\kappa_{CD)},$$

in terms of which we have the following irreducible decomposition

$$H_{CABD} = 3\xi_{ABCD} + \tfrac{1}{2}\left(\mathcal{P}\kappa_{BD} + \xi_{BD}\right)\epsilon_{AC} + \tfrac{1}{2}\left(\mathcal{P}\kappa_{AD} + \xi_{AD}\right)\epsilon_{BC} - \tfrac{1}{2}\left(\mathcal{P}\kappa_{AB} + \xi_{AB}\right)\epsilon_{CD}.$$

Additionally, the following decompositions will prove useful

$$\mathcal{D}_{AB}\kappa_{CD} = \xi_{ABCD} - \tfrac{1}{2}\epsilon_{A(C}\xi_{D)B} - \tfrac{1}{2}\epsilon_{B(C}\xi_{D)A} - \tfrac{1}{3}\xi\epsilon_{A(C}\epsilon_{D)B}, \tag{98}$$

$$\begin{aligned}\mathcal{D}_{AB}\xi_{CD} = {}& \tfrac{1}{6}\epsilon_{AD}\epsilon_{BC}\mathcal{D}_{FG}\xi^{FG} + \tfrac{1}{6}\epsilon_{AC}\epsilon_{BD}\mathcal{D}_{FG}\xi^{FG} - \tfrac{1}{4}\epsilon_{BD}\mathcal{D}_{(A}{}^{F}\xi_{C)F} \\ & - \tfrac{1}{4}\epsilon_{BC}\mathcal{D}_{(A}{}^{F}\xi_{D)F} - \tfrac{1}{4}\epsilon_{AD}\mathcal{D}_{(B}{}^{F}\xi_{C)F} - \tfrac{1}{4}\epsilon_{AC}\mathcal{D}_{(B}{}^{F}\xi_{D)F} + \mathcal{D}_{(AB}\xi_{CD)}.\end{aligned} \tag{99}$$

Using the definition of ξ_{AB}, and by commuting derivatives, one obtains the following identities:

$$\mathcal{D}_{AB}\xi^{AB} = -\tfrac{1}{3}\chi\xi + \tfrac{1}{2}\chi^{AB}\xi_{AB} - \tfrac{1}{2}\chi^{AB}\mathcal{P}\kappa_{AB} - \kappa^{AB}\Phi_{AB} + \tfrac{1}{2}\xi^{ABCD}\chi_{ABCD}, \tag{100a}$$

$$\begin{aligned}\mathcal{D}_{A(B}\xi_{D)}{}^{A} = {}& -\tfrac{2}{3}\mathcal{D}_{BD}\xi + \mathcal{D}_{AC}\xi_{BD}{}^{AC} + \chi_{(B}{}^{A}\mathcal{P}\kappa_{D)A} + \tfrac{1}{3}\xi\chi_{BD} \\ & + \tfrac{2}{3}\chi\xi_{BD} - \tfrac{1}{2}\chi_{(B}{}^{A}\xi_{D)A} + \tfrac{1}{2}\xi^{AC}\chi_{BDAC} - \tfrac{1}{2}\chi^{AC}\xi_{BDAC} + \xi_{(B}{}^{ACF}\chi_{D)ACF} \\ & - 4\kappa_{BD}\Lambda - \tfrac{2}{3}\kappa_{BD}\Phi - \kappa_{(B}{}^{A}\Phi_{D)A} + \kappa^{AC}\Theta_{BDAC} + \kappa^{AC}\Psi_{BDAC},\end{aligned} \tag{100b}$$

$$\begin{aligned}\mathcal{D}_{(AB}\xi_{CD)} = {}& 2\mathcal{D}_{(A}{}^{F}\xi_{BCD)F} + \chi_{(AB}\mathcal{P}\kappa_{CD)} \\ & + \tfrac{2}{3}\chi\xi_{ABCD} - \tfrac{1}{3}\xi\chi_{ABCD} + \tfrac{1}{2}\chi_{(AB}\xi_{CD)} - \chi_{(A}{}^{F}\xi_{BCD)F} - \xi_{(A}{}^{F}\chi_{BCD)F} \\ & - \xi_{(AB}{}^{FG}\chi_{CD)FG} - 2\kappa_{(A}{}^{F}\Theta_{BCD)F} - \kappa_{(AB}\Phi_{CD)} - 2\kappa_{(A}{}^{F}\Psi_{BCD)F},\end{aligned} \tag{100c}$$

and similarly,

$$\begin{aligned}\mathcal{P}\xi = {}& -\tfrac{1}{2}K^{AB}\mathcal{P}\kappa_{AB} + \mathcal{D}^{AB}\mathcal{P}\kappa_{AB} - \tfrac{1}{3}\chi\xi + K^{AB}\xi_{AB} \\ & + \chi^{AB}\xi_{AB} + 2\kappa^{AB}\Phi_{AB} - \xi^{ABCD}\chi_{ABCD},\end{aligned} \tag{101a}$$

$$\begin{aligned}\mathcal{P}\xi_{AB} = {}& 4\kappa_{AB}\Lambda - \tfrac{2}{3}\kappa_{AB}\Phi - \kappa^{CD}\Psi_{ABCD} - \tfrac{1}{3}K_{AB}\xi - \tfrac{1}{3}\xi\chi_{AB} - \tfrac{1}{3}\chi\xi_{AB} + \tfrac{1}{2}K^{CD}\xi_{ABCD} \\ & + \tfrac{1}{2}\chi^{CD}\xi_{ABCD} + \kappa^{CD}\Theta_{ABCD} + \tfrac{1}{2}\xi^{CD}\chi_{ABCD} + \tfrac{1}{2}K_{(A}{}^{C}\xi_{B)C} - \kappa_{(A}{}^{C}\Phi_{B)C} \\ & + \tfrac{1}{2}\chi_{(A}{}^{C}\xi_{B)C} + \xi_{(A}{}^{CDF}\chi_{B)CDF} + \mathcal{D}_{(A}{}^{C}\mathcal{P}\kappa_{B)C} - \tfrac{1}{2}K_{(A}{}^{C}\mathcal{P}\kappa_{(B)C},\end{aligned} \tag{101b}$$

$$\begin{aligned}\mathcal{P}\xi_{ABCD} = {}& \mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)} - \tfrac{1}{2}K_{(AB}\mathcal{P}\kappa_{CD)} - \tfrac{1}{3}\chi\xi_{ABCD} - \tfrac{1}{3}\xi\chi_{ABCD} - \tfrac{1}{2}K_{(AB}\xi_{CD)} \\ & + K_{(A}{}^{F}\xi_{BCD)F} + 2\kappa_{(A}{}^{F}\Psi_{BCD)F} - 2\kappa_{(A}{}^{F}\Theta_{BCD)F} - \kappa_{(AB}\Phi_{CD)} \\ & - \tfrac{1}{2}\chi_{(AB}\xi_{CD)} + \chi_{(A}{}^{F}\xi_{BCD)F} - \xi_{(A}{}^{F}\chi_{BCD)F} - \xi_{(AB}{}^{FG}\chi_{CD)FG}.\end{aligned} \tag{101c}$$

Remark 5. If the tracefree Ricci spinor $\Phi_{AA'BB'}$ is made to vanish, then the above identities reduce to those given in [3]. This corresponds to setting $\Xi = 1$ in the alternative CFEs.

7.2 The conditions $H_{A'ABC} = \mathcal{P}H_{A'ABC} = 0$

Given the irreducible decomposition of the zero quantity $H_{A'ABC}$, provided above, the condition $H_{A'ABC} = 0$ is equivalent to

$$
\begin{aligned}
&\xi_{ABCD} = 0, && (102a)\\
&\mathcal{P}\kappa_{AB} = -\xi_{AB}, && (102b)
\end{aligned}
$$

and the condition $\mathcal{P}H_{A'ABC} = 0$ is equivalent to

$$\mathcal{P}H_{ABCD} = \chi_A{}^F H_{FBCD},$$

which, in turn is equivalent to

$$
\begin{aligned}
&\mathcal{P}\xi_{ABCD} = 0, && (103a)\\
&\mathcal{P}^2\kappa_{AB} = -\mathcal{P}\xi_{AB}. && (103b)
\end{aligned}
$$

The wave equation for κ_{AB}, given in equation (68), can be rewritten in terms of the quantities ξ, ξ_{AB}, ξ_{ABCD} as follows

$$
\begin{aligned}
\mathcal{P}^2\kappa_{BC} = &-2\mathcal{D}_{AD}\xi_{BC}{}^{AD} - \tfrac{2}{3}\mathcal{D}_{BC}\xi + 2\mathcal{D}_{(B}{}^{A}\xi_{C)A} - \chi\mathcal{P}\kappa_{BC} - 8\kappa_{BC}\Lambda + 2\kappa^{AD}\Psi_{BCAD}\\
&+ \tfrac{1}{3}K_{BC}\xi + \tfrac{2}{3}\xi\chi_{BC} + K^{AD}\xi_{BCAD} + 2\chi^{AD}\xi_{BCAD} - K_{(B}{}^{A}\xi_{C)A} - 2\chi_{(B}{}^{A}\xi_{C)A}. \quad (104)
\end{aligned}
$$

It is important to note that the Killing spinor satisfies equation (68) by construction, and therefore (104) can be assumed to hold throughout the domain of dependence of $\mathcal{U}$; in particular, we are free to take further $\mathcal{P}-$derivatives of the equation. Through repeated use of the identities (100a)-(100c), (101a)-(101c), along with (102a)–(102b), the above wave equation can be seen to imply (103b), which is therefore trivially satisfied. For future reference note that, using equation (101b), the wave equation for κ_{AB} can alternatively be expressed as

$$
\begin{aligned}
\mathcal{P}^2\kappa_{AB} + \mathcal{P}\xi_{AB} = &-4\kappa_{AB}\Lambda - \chi\mathcal{P}\kappa_{AB} - \tfrac{2}{3}\kappa_{AB}\Phi + \kappa^{CD}\Psi_{ABCD} + \tfrac{1}{3}\xi\chi_{AB} - \tfrac{1}{3}\chi\xi_{AB} + \tfrac{3}{2}K^{CD}\xi_{ABCD}\\
&+ \tfrac{5}{2}\chi^{CD}\xi_{ABCD} + \kappa^{CD}\Theta_{ABCD} + \tfrac{1}{2}\xi^{CD}\chi_{ABCD} - \tfrac{2}{3}\mathcal{D}_{AB}\xi - 2\mathcal{D}_{CD}\xi_{AB}{}^{CD}\\
&+ 2\mathcal{D}_{(A}{}^{C}\xi_{B)C} - 2\mathcal{D}_{(A}{}^{C}\mathcal{P}\kappa_{B)C} - \tfrac{1}{2}K_A{}^C\mathcal{P}\kappa_{B)C} - \kappa_{(A}{}^{C}\Phi_{B)C} - \tfrac{3}{2}\chi_{(A}{}^{C}\xi_{B)C}\\
&+ \xi_{(A}{}^{CDF}\chi_{B)CDF}. \quad (105)
\end{aligned}
$$

Finally, using equations (101c), and (100c) along with (102a)–(102b), one obtains

$$\mathcal{P}\xi_{ABCD} = \kappa_{(A}{}^{F}\Psi_{BCD)F}.$$

Therefore, equation (103a) is equivalent to imposing the Buchdahl constraint, $\kappa_{(A}{}^{F}\Psi_{BCD)F} = 0$, on $\mathcal{U}$.

7.3 The conditions $S_{AA'BB'} = \mathcal{P}S_{AA'BB'} = 0$

Using the definition of S_{ABCD} given in expression (97b) and the expression for $S_{AA'BB'}$ as given in equation (57), a direct computation using the space spinor formalism introduced in Section 1 renders

$$
\begin{aligned}
S_{ABCD} = &K_{C(D|AF|}\mathcal{P}\kappa_{B)}{}^{F} - 6\kappa_{(D}{}^{F}\Phi_{B)FAC} - \tfrac{1}{2}K_{ABCD}\xi - \tfrac{1}{2}K_{CDAB}\xi - K_{CDAF}\xi_B{}^F\\
&- K_{ABCF}\xi_D{}^F + \tfrac{1}{4}K_C{}^F\mathcal{P}\kappa_{DF}\epsilon_{AB} - \tfrac{1}{4}\mathcal{P}^2\kappa_{CD}\epsilon_{AB} + \tfrac{1}{2}\mathcal{P}\xi_{CD}\epsilon_{AB} + \tfrac{1}{4}K_{CD}\xi\epsilon_{AB}\\
&- \tfrac{1}{2}K_C{}^F\xi_{DF}\epsilon_{AB} + \tfrac{1}{4}K_A{}^F\mathcal{P}\kappa_{BF}\epsilon_{CD} - \tfrac{1}{4}\mathcal{P}^2\kappa_{AB}\epsilon_{CD} + \tfrac{1}{2}\mathcal{P}\xi_{AB}\epsilon_{CD} + \tfrac{1}{4}K_{AB}\xi\epsilon_{CD}\\
&- \tfrac{1}{2}K_A{}^F\xi_{BF}\epsilon_{CD} - \tfrac{1}{2}\mathcal{P}\xi\epsilon_{AB}\epsilon_{CD} + \tfrac{1}{2}\mathcal{D}_{AB}\mathcal{P}\kappa_{CD} + \tfrac{1}{2}\mathcal{D}_{CD}\mathcal{P}\kappa_{AB} + \tfrac{1}{2}\epsilon_{CD}\mathcal{D}_{AB}\xi\\
&+ \tfrac{1}{2}\epsilon_{AB}\mathcal{D}_{CD}\xi - \mathcal{D}_{AB}\xi_{CD} - \mathcal{D}_{CD}\xi_{AB}. \quad (106)
\end{aligned}
$$

Using the decompositions (5), (**??**) for K_{ABCD} and Φ_{ABCD}, equation (106) can decomposed in irreducible components. The non-vanishing components (or combinations thereof) of this decomposition are:

$$S_{(ABCD)} = -\xi\chi_{ABCD} - 2\mathcal{D}_{(AB}\xi_{CD)} + \mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)} - 6\kappa_{(A}{}^{F}\Theta_{BCD)F}$$

$$
\begin{aligned}
&\quad - 3\kappa_{(AB}\Phi_{CD)} - \chi_{(AB}\xi_{CD)} + \tfrac{1}{2}\chi_{(AB}\mathcal{P}\kappa_{CD)} - 2\xi_{(A}{}^{F}\chi_{BCD)F} \\
&\quad - \chi_{(ABC}{}^{F}\mathcal{P}\kappa_{D)F}, \qquad (107a)\\
S_{(AB)}{}^{F}{}_{F} - S_{(A}{}^{F}{}_{|F|B)} &= -\tfrac{1}{2}\mathcal{P}\kappa_{AB} + \mathcal{P}^2\kappa_{AB} - 2\mathcal{P}\xi_{AB} - 4\kappa_{AB}\Phi - K_{AB}\xi + \tfrac{2}{3}\chi\xi_{AB} \\
&\quad + 6\kappa^{FC}\Theta_{ABFC} - \mathcal{P}\kappa^{FC}\chi_{ABFC} + 2\xi^{FC}\chi_{ABFC} - 2\mathcal{D}_{AB}\xi \\
&\quad + 2K_{(A}{}^{F}\xi_{B)F} - K_{(A}{}^{F}\mathcal{D}_{AB}\kappa_{B)F} - 6\kappa_{(A}{}^{F}\Phi_{B)F} - 2\chi_{(A}{}^{F}\xi_{B)F} \\
&\quad + \chi_{(A}{}^{F}\mathcal{P}\kappa_{B)F}, \qquad (107b)\\
S^{FG}{}_{FG} + S^{FG}{}_{GF} &= -2\chi\xi - 2\chi^{FG}\mathcal{P}\kappa_{FG} + 4\chi^{FG}\xi_{FG} - 6\kappa^{FG}\Phi_{FG} + 2\mathcal{D}_{FG}\mathcal{P}\kappa^{FG} \\
&\quad - 4\mathcal{D}_{FG}\xi^{FG}, \qquad (107c)\\
S^{FG}{}_{FG} - S^{FG}{}_{GF} &= -K^{FG}\mathcal{P}\kappa_{FG} - 2\mathcal{P}\xi + 2K^{FG}\xi_{FG} + 6\kappa^{FG}\Phi_{FG}. \qquad (107d)
\end{aligned}
$$

Note that in deriving expressions (107a)-(107d) the $H_{ABCD}|_{\mathcal{S}} = 0$ and $\mathcal{P}H_{ABCD}|_{\mathcal{S}} = 0$ conditions have not been used. Taking into account the $H_{ABCD}|_{\mathcal{S}} = 0$ conditions, encoded in equations (102a)-(102b) and (103a)-(103b), and exploiting equations (100a)-(100c), the conditions encoded in (107a)-(107d) reduce to

$$
\begin{aligned}
&\mathcal{P}\xi = \tfrac{3}{2}K^{FG}\xi_{FG} + 3\kappa^{FG}\Phi_{FG}, \qquad (108a)\\
&\mathcal{P}\xi_{AB} = \tfrac{2}{3}\mathcal{D}_{AB}\xi - \tfrac{4}{3}\kappa_{AB}\Phi - \tfrac{1}{3}K_{AB}\xi + \tfrac{1}{3}\chi\xi_{AB} + K_{(A}{}^{F}\xi_{B)F} \\
&\qquad -\chi_{(A}{}^{F}\xi_{B)F} + 2\kappa^{FC}\Theta_{ABFC} - 2\kappa_{(A}{}^{F}\Phi_{B)F} + \xi^{FC}\chi_{ABFC}, \qquad (108b)\\
&\kappa_{(A}{}^{F}\Psi_{BCD)F} = 0. \qquad (108c)
\end{aligned}
$$

Furthermore, one can verify, using the identities (101a)-(101b), that equations (108a)-(108b) are identically satisfied if the intrinsic conditions (102a)-(102b) and (103a) -(103b) hold. Additionally, observe that, as discussed in Section 7.2, the vanishing of the Buchdahl constraint (108c) is obtained through condition (103a). In other words, the $S_{ABCD}|_{\mathcal{S}} = 0$ requirement does not impose any extra conditions than those already encoded in $H_{ABCD}|_{\mathcal{S}} = 0$ and $\mathcal{P}H_{ABCD}|_{\mathcal{S}} = 0$.

Now, to analyse the conditions imposed by requiring $\mathcal{P}S_{ABCD} = 0$, observe that

$$
\tau_A{}^{B'}\tau_C{}^{D'}\mathcal{P}S_{BB'DD'} = \mathcal{P}S_{ABCD} - K^{F}{}_{C}S_{ABFD} - K^{F}{}_{A}S_{CDFB}. \qquad (109)
$$

Consequently, if the conditions $S_{ABCD}|_{\mathcal{S}} = 0$ hold, then, it is enough to analyse the restriction imposed by $\mathcal{P}S_{ABCD}|_{\mathcal{S}} = 0$. Taking a $\mathcal{P}$-derivative of equations (107a)-(107d) and exploiting the space spinor formalism one obtains

$$
\begin{aligned}
\mathcal{P}S_{(ABCD)} &= -\xi\mathcal{P}\chi_{ABCD} - \chi_{ABCD}\mathcal{P}\xi - 2\mathcal{P}\mathcal{D}_{(AB}\xi_{CD)} + \mathcal{P}\mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)} \\
&\quad - 6\kappa_{(A}{}^{F}\mathcal{P}\Theta_{BCD)F} - 3\kappa_{(AB}\mathcal{P}\Phi_{CD)} - \chi_{(AB}\mathcal{P}\xi_{CD)} + \tfrac{1}{2}\chi_{(AB}\mathcal{P}^2\kappa_{CD)} \\
&\quad - \xi_{(AB}\mathcal{P}\chi_{CD)} - 2\xi_{(A}{}^{F}\mathcal{P}\chi_{BCD)F} + 6\Theta_{(ABC}{}^{F}\mathcal{P}\kappa_{D)F} - 3\Phi_{(AB}\mathcal{P}\kappa_{CD)} \\
&\quad + 2\chi_{(ABC}{}^{F}\mathcal{P}\xi_{D)F} - \chi_{(ABC}{}^{F}\mathcal{P}^2\kappa_{D)F} + \tfrac{1}{2}\mathcal{P}\kappa_{(AB}\mathcal{P}\chi_{CD)} \\
&\quad + \mathcal{P}\kappa_{A}{}^{F}\mathcal{P}\chi_{BCD)F}, \qquad (110a)\\
\mathcal{P}(S_{(AB)}{}^{F}{}_{F} - S_{(A}{}^{F}{}_{|F|B)}) &= \mathcal{P}^3\kappa_{AB} - 2\mathcal{P}^2\xi_{AB} - \tfrac{1}{3}\mathcal{P}\kappa_{AB}\mathcal{P}\chi - \tfrac{1}{3}\chi\mathcal{P}^2\kappa_{AB} - 4\kappa_{AB}\mathcal{P}\Phi - K_{AB}\mathcal{P}\xi \\
&\quad + \tfrac{2}{3}\chi\mathcal{P}\xi_{AB} + 6\kappa^{FC}\mathcal{P}\Theta_{ABFC} - \mathcal{P}\kappa^{FC}\mathcal{P}\chi_{ABFC} - 2\mathcal{P}\mathcal{D}_{AB}\xi - 4\Phi\mathcal{P}\kappa_{AB} \\
&\quad -\xi\mathcal{P}K_{AB} + \tfrac{2}{3}\xi_{AB}\mathcal{P}\chi + 2\xi^{FC}\mathcal{P}\chi_{ABFC} + 6\Theta_{ABFC}\mathcal{P}\kappa^{FC} - \chi_{ABFC}\mathcal{P}\xi^{FC} \\
&\quad + 2K_{(A}{}^{F}\mathcal{P}\xi_{B)F} - K_{(A}{}^{F}\mathcal{P}^2\kappa_{B)F} - 6\kappa_{(A}{}^{F}\mathcal{P}\Phi_{B)F} - 2\chi_{(A}{}^{F}\mathcal{P}\xi_{B)F} \\
&\quad + \chi_{(A}{}^{F}\mathcal{P}^2\kappa_{B)F} - 2\xi_{(A}{}^{F}\mathcal{P}K_{B)F} + 2\xi_{(A}{}^{F}\mathcal{P}\chi_{B)F} + 6\Phi_{(A}{}^{F}\mathcal{P}\kappa_{B)F} \\
&\quad - \mathcal{P}K_{(A}{}^{F}\mathcal{P}\kappa_{B)F} - \mathcal{P}\kappa_{(A}{}^{F}\mathcal{P}\chi_{B)F}, \qquad (110b)\\
\mathcal{P}(S^{FG}{}_{FG} + S^{FG}{}_{GF}) &= -2\chi\mathcal{P}\xi - 2\mathcal{P}\kappa^{FG}\mathcal{P}\chi_{FG} - 6\kappa^{FG}\mathcal{P}\Phi_{FG} + 2\mathcal{P}\mathcal{D}_{FG}\mathcal{P}\kappa^{FG} - 4\mathcal{P}\mathcal{D}_{FG}\xi^{FG} \\
&\quad - 2\xi\mathcal{P}\chi - 2\chi^{FG}\mathcal{P}^2\kappa_{FG} + 4\chi^{FG}\mathcal{P}\xi_{FG} + 4\xi^{FG}\mathcal{P}\chi_{FG} - 6\Phi^{FG}\mathcal{P}\kappa_{FG}, \\
&\qquad (110c)\\
\mathcal{P}(S^{FG}{}_{FG} - S^{FG}{}_{GF}) &= -\mathcal{P}K^{FG}\mathcal{P}\kappa_{FG} - K^{FG}\mathcal{P}^2\kappa_{FG} - 2\mathcal{P}^2\xi + 2K^{FG}\mathcal{P}\xi_{FG} + 6\kappa^{FG}\mathcal{P}\Phi_{FG}
\end{aligned}
$$

$$+ 2\xi^{FG}\mathcal{P}K_{FG} + 6\Phi^{FG}\mathcal{P}\kappa_{FG}. \tag{110d}$$

Observe that, in deriving equations (110a)-(110d), the conditions $H_{ABCD}|_{\mathcal{S}} = 0$ and $\mathcal{P}H_{ABCD}|_{\mathcal{S}} = 0$ were not used. Similar to the discussion leading to equations (108a)-(108c), exploiting $H_{ABCD}|_{\mathcal{S}} = 0$ and $\mathcal{P}H_{ABCD}|_{\mathcal{S}} = 0$ leads to simpler expressions. To implement this computation it will proof convenient to derive some ancillary results first. This is done in the following.

Applying the commutator (6) to $\mathcal{P}\kappa_{CD}$ and exploiting the intrinsic conditions encoded in (102b) and (103b) one obtains

$$\mathcal{P}\mathcal{D}_{AB}\mathcal{P}\kappa_{CD}+\mathcal{D}_{AB}\mathcal{P}\xi_{CD} = \tfrac{1}{2}K_{AB}\mathcal{P}\xi_{CD}+\square_{AB}\xi_{CD}-\widehat{\square}_{AB}\xi_{CD}-K_{(A}{}^{F}\mathcal{D}_{B)F}\xi_{CD}+K_{ABFG}\mathcal{D}^{FG}\xi_{CD}.$$

Similarly, using again the commutator (6) applied now to ξ_{CD} and exploiting the intrinsic conditions encoded in (102b) and (103b) one obtains

$$\mathcal{P}\mathcal{D}_{AB}\xi_{CD}-\mathcal{D}_{AB}\mathcal{P}\xi_{CD} = -\tfrac{1}{2}K_{AB}\mathcal{P}\xi_{CD}-\square_{AB}\xi_{CD}+\widehat{\square}_{AB}\xi_{CD}+K_{(A}{}^{F}\mathcal{D}_{B)F}\xi_{CD}-K_{ABFG}\mathcal{D}^{FG}\xi_{CD}.$$

Comparing the last two expressions one concludes that

$$\mathcal{P}\mathcal{D}_{AB}\mathcal{P}\kappa_{CD} + \mathcal{P}\mathcal{D}_{AB}\xi_{CD} = 0. \tag{111}$$

Additionally, observe that taking a $\mathcal{P}-$derivative to equations (101a)-(101b) and exploiting equations (100a)-(100b) along with conditions (102a)-(102b) and (103a)-(103b) we obtain

$$\begin{aligned}
\mathcal{P}^2\xi =& \tfrac{3}{2}K^{AB}\mathcal{P}\xi_{AB} + 3\kappa^{AB}\mathcal{P}\Phi_{AB} + \tfrac{3}{2}\xi^{AB}\mathcal{P}K_{AB} - 3\xi^{AB}\Phi_{AB}, \qquad (112)\\
\mathcal{P}^2\xi_{AB} =& -\tfrac{4}{3}\kappa_{AB}\mathcal{P}\Phi - \tfrac{1}{3}K_{AB}\mathcal{P}\xi + \tfrac{1}{3}\chi\mathcal{P}\xi_{AB} + K_{(B}{}^{C}\mathcal{P}\xi_{A)C} + 2\kappa^{CD}\mathcal{P}\Theta_{ABCD}\\
& - 2\kappa_{(B}{}^{C}\mathcal{P}\Phi_{A)C} - \tfrac{2}{3}\mathcal{P}\mathcal{D}_{AB}\xi + \tfrac{1}{3}\xi\mathcal{P}K_{AB} - \chi_{(A}{}^{C}\mathcal{P}\xi_{B)C} + \tfrac{1}{3}\xi_{AB}\mathcal{P}\chi + \tfrac{4}{3}\Phi\xi_{AB}\\
& - \xi_{(A}{}^{C}\mathcal{P}K_{B)C} + \xi_{(A}{}^{C}\mathcal{P}\chi_{B)C} + \xi^{CD}\mathcal{P}\chi_{ABCD} - 2\xi^{CD}\Theta_{ABCD} + \xi_{A}{}^{C}\Phi_{BC}\\
& + \chi_{ABCD}\mathcal{P}\xi^{CD}. \qquad (113)
\end{aligned}$$

Using the above expressions along with equations (111), (99), (100a)-(100c), (101a)-(101c), and the $H_{ABCD}|_{\mathcal{S}}$ and $\mathcal{P}H_{ABCD}|_{\mathcal{S}} = 0$ conditions encoded in equations (102a)-(102b) and (103a)-(103b) we obtain

$$6\kappa_{(A}{}^{F}\mathcal{P}\Psi_{BCD)F} + 6\Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{114a}$$

$$\mathcal{P}^3\kappa_{AB} + \mathcal{P}^2\xi_{AB} = 0. \tag{114b}$$

To simplify equation (114b) we can exploit the wave equation for κ_{AB} as expressed in equation (105). Taking a $\mathcal{P}-$derivative of the latter equations and using the identities (100a)-(100c) (101a)-(101c), and the $H_{ABCD}|_{\mathcal{S}}$ and $\mathcal{P}H_{ABCD}|_{\mathcal{S}} = 0$ conditions encoded in equations (102a)-(102b) and (103a)-(103b) one obtains

$$\mathcal{P}^3\kappa_{AB} + \mathcal{P}^2\xi_{AB} = 0.$$

Consequently, equation (114a) contains the only independent condition encoded by $\mathcal{P}S_{AA'BB'}|_{\mathcal{S}} = 0$. Finally, one can exploit the conformal field equation encoded in the zero-quantity (**??**) to express the $\mathcal{P}$ derivative of the Weyl spinor in terms of intrinsic quantities at $\mathcal{S}$. To do so, let

$$\Lambda_{ABCD} \equiv \tau_{A}{}^{A'}\Lambda_{A'BCD},$$

$$Y_{ABCD} \equiv \tau_{D}{}^{D'}Y_{ABCD'}.$$

Exploiting the space spinor formalism one obtains

$$\Lambda_{ABCD} = \tfrac{1}{2}\mathcal{P}\Psi_{ABCD} - \tfrac{1}{2}Y_{ABCD} + \mathcal{D}_{QD}\Psi_{ABC}{}^{Q},$$

from which one obtains evolution and constraint equations encoded in

$$\Lambda_{(ABCD)} = 0, \qquad \Lambda_{AB}{}^{Q}{}_{Q} = 0,$$

given explicitly by

$$\mathcal{P}\Psi_{ABCD} + 2\mathcal{D}_{Q(D}\Psi_{ABC)}{}^{Q} - Y_{ABCD} = 0, \tag{115a}$$
$$\mathcal{D}^{PQ}\Psi_{PQAB} - \tfrac{1}{2}Y_{AB}{}^{Q}{}_{Q} = 0. \tag{115b}$$

Using the evolution equation encoded in expression (115a), the condition given in equation (114a) reads

$$\mathcal{P}S_{(ABCD)} = 6\kappa_{(A}{}^{F}Y_{BCD)F} + 12\kappa_{(A}{}^{F}\mathcal{D}_{|F|}{}^{G}\Psi_{BCD)G} + 6\Psi_{(ABC}{}^{F}\xi_{D)F} \tag{116}$$

We are now in a position to formulate the main Theorem of this article:

Theorem 2. *Consider an initial data set for the (alternative) conformal Einstein field equations, as encoded in the zero-quantities* (**??**)-(**??**)*, on a spacelike hypersurface $\mathcal{S}$ and let $\mathcal{U} \subset \mathcal{S}$ denote an open set. The development of the initial data set will have a Killing spinor in the domain of dependence of $\mathcal{U}$ if and only if*

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0, \tag{C1}$$
$$\kappa_{(A}{}^{F}\Psi_{BCD)F} = 0, \tag{C2}$$
$$\kappa_{(A}{}^{F}Y_{BCD)F} + 2\kappa_{(A}{}^{F}\mathcal{D}_{|F|}{}^{G}\Psi_{BCD)G} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{C3}$$

are satisfied on $\mathcal{U}$. The Killing spinor is obtained evolving according to the wave equation (68) *with initial data satisfying conditions* (C1)-(C3) *and*

$$\mathcal{P}\kappa_{AB} = -\xi_{AB}. \tag{118}$$

Proof. The prior discussion of this section establishes that the conditions

$$H_{A'ABC} = 0,$$
$$\mathcal{P}H_{A'ABC} = 0,$$
$$S_{AA'BB'} = 0,$$
$$\mathcal{P}S_{AA'BB'} = 0$$

on $\mathcal{U} \subset \mathcal{S}$ are equivalent to (C1)–(C3). Hence, appealing to Proposition 2, we see that if (C1)–(C3) hold on $\mathcal{U}$, then the domain of dependence of $\mathcal{U}$ is endowed with a Killing spinor. □

Definition. The equations (C1)-(C3) will be referred to as the *conformal Killing spinor initial data equations*, and a solution, κ_{AB}, thereof a *Killing spinor candidate.*

Remark 6. The conditions (C1)-(C3) are a highly-overdetermined system of equations. It therefore follows that, while they are to be read as equations for the Killing spinor candidate, κ_{AB}, the existence of a non-trivial solution to the these equations places strong restrictions on the initial data and, consequently, on the resulting spacetime. Observe that (C2) implies that the restriction of the Weyl spinor to $\mathcal{S}$ is algebraically special. It will be seen in Section 8 that, equation, (C3) places further constraints on curvature associated to initial data for the (alternative) CFEs, in the sense of restricting various components of the Cotton spinor, when expressed in terms of a suitably-adapted spin dyad.

Remark 7. While the analysis in this article is carried out via the spinor formalism, we remark here that the main results could alternatively be rewritten in tensorial terms; the above Theorem may be reframed in terms of the existence of a Killing–Yano tensor (rather than of a Killing spinor) on the spacetime development.

The conditions (C1)-(C3) were derived from (96a)-(96d) exploiting the space spinor formalism adapted to a timelike Hermitian spinor $\tau^{AA'}$ corresponding to the normal vector to the initial hypersurface $\mathcal{S}$. Nevertheless, conditions (96a)-(96d) are irrespective of the causal nature of $\mathcal{S}$, consequently, a similar analysis to that given in Section 7 can be used to identify spinorial Killing initial data for the conformal Einstein field equations on a timelike or null hypersurface as well.

The initial hypersurface $\mathcal{S}$ can be chosen to determined by the condition $\Xi = 0$ so that $\mathcal{S}$ corresponds to the conformal boundary $\mathscr{I}$. In this case, conditions (C1)-(C3) provide with conditions on *asymptotic initial data* that ensure the existence of a Killing spinor in the development of this data. This Killing spinor can be used to construct a conformal Killing vector in the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ corresponding to a Killing vector of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ —see Lemma 1. On the other hand, setting $\Xi = 1$, so that we have Cauchy data for the Einstein field equations, the Cotton tensor (spinor) vanishes and conditions (C1)-(C3) reduce to the conditions given in [2, 3, 4]. Note that, while condition (C3) trivialises in this case as it to follow as a consequence of (C1)-(C2) —see [5] for a detailed discussion of this. Nevertheless, in the general case ($\Xi \neq 1$), (C3) encodes non-trivial information about the Cotton spinor and cannot be eliminated by virtue of the conditions (C1)-(C2) alone —see Remark 6.

8 Further analysis of the conformal Killing spinor initial data equations

In this section the conditions (C1)-(C3) are further analysed by expressing them in components with respect to a spin dyad adapted to the Killing spinor κ_{AB}. The ultimate goal of this section is to show that, in contrast to the $\Xi = 1$ case —see [5], the condition (C3) is in general non-trivial; that is to say that it does not follow as a consequence of conditions (C1)-(C2). Rather, we will see that (C3) captures essential information about the Cotton spinor, and may only be eliminated from the conformal Killing spinor initial data equations by additionally constraining certain components of the Cotton spinor.

Recalling that $Y_{ABCD} = Y_{(ABC)D}$ let $Y_{\boldsymbol{i}}$ with $\boldsymbol{i} = \boldsymbol{0}, ..., \boldsymbol{7}$ denote the components of Y_{ABCD} respect to a spin dyad, $\{o^A, \iota^A\}$, normalised as $o_A \iota^A = 1$. In other words, let

$$
\begin{aligned}
Y_{\boldsymbol{0}} &= \iota^A \iota^B \iota^C \iota^D Y_{ABCD}, & Y_{\boldsymbol{4}} &= \iota^A o^B o^C \iota^D Y_{ABCD}, \\
Y_{\boldsymbol{1}} &= \iota^A \iota^B \iota^C o^D Y_{ABCD}, & Y_{\boldsymbol{5}} &= \iota^A o^B o^C o^D Y_{ABCD}, \\
Y_{\boldsymbol{2}} &= \iota^A \iota^B o^C \iota^D Y_{ABCD}, & Y_{\boldsymbol{6}} &= o^A o^B o^C \iota^D Y_{ABCD}, \\
Y_{\boldsymbol{3}} &= \iota^A \iota^B o^C o^D Y_{ABCD}, & Y_{\boldsymbol{7}} &= o^A o^B o^C o^D Y_{ABCD}.
\end{aligned}
$$

Using the latter notation Y_{ABCD} is expressed as follows

$$
\begin{aligned}
Y_{ABCD} = {} & Y_{\boldsymbol{0}} o_A o_B o_C o_D - Y_{\boldsymbol{1}} o_A o_B o_C \iota_D - 3Y_{\boldsymbol{2}} o_D o_{(A} o_B \iota_{C)} + 3Y_{\boldsymbol{3}} \iota_D o_{(A} o_B \iota_{C)} \\
& + 3Y_{\boldsymbol{4}} o_D o_{(A} \iota_B \iota_{C)} - 3Y_{\boldsymbol{5}} \iota_D o_{(A} \iota_B \iota_{C)} - Y_{\boldsymbol{6}} o_D \iota_A \iota_B \iota_C + Y_{\boldsymbol{7}} \iota_A \iota_B \iota_C \iota_D.
\end{aligned}
\tag{119}
$$

The results of this section are summarised in the following Proposition:

Proposition 3. *If $\kappa_{AB}\kappa^{AB} \neq 0$ then there exists a dyad, $\{o, \iota\}$, and some real-valued function $\varkappa$ for which*

$$\kappa_{AB} = e^{\varkappa} o_{(A} \iota_{B)}.$$

In terms of this adapted dyad, and assuming (C1)–(C2), the condition (C3) *is then equivalent to*

$$Y_{\boldsymbol{0}} = Y_{\boldsymbol{1}} = Y_{\boldsymbol{6}} = Y_{\boldsymbol{7}} = 0.$$

On the other hand, if $\kappa_{AB}\kappa^{AB} = 0$ then there exists a dyad, $\{o, \iota\}$, for which $\kappa_{AB} = o_A o_B$, in terms of which condition (C3) *is equivalent to*

$$Y_{\boldsymbol{2}} = Y_{\boldsymbol{3}} = Y_{\boldsymbol{4}} = Y_{\boldsymbol{5}} = Y_{\boldsymbol{6}} = Y_{\boldsymbol{7}} = 0.$$

Cases *i)* and *ii)* are dealt with separately in the remainder of this section.

Remark 8. Note that if the spacetime is of Type O, i.e. $\Psi_{ABCD} = 0$, then it follows from the conformal field equations (namely, the equation $\Lambda_{A'ABC} = 0$) that $Y_{ABCC'} = 0$ and hence that (C2), (C3) trivialise, leaving only (C1).

8.1 Type D Case: $\kappa_{AB}\kappa^{AB} \neq 0$

If $\kappa_{AB}\kappa^{AB} \neq 0$ then one can choose a normalised spin dyad $\{o_A, \iota_B\}$ with $o_A\iota^A = 1$, adapted to κ_{AB}. In other words, such that

$$\kappa_{AB} = e^{\varkappa} o_{(A}\iota_{B)}, \tag{120}$$

where $\varkappa$ is a scalar field. Similarly, condition (C2) implies that

$$\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}, \tag{121}$$

where ψ is a scalar field. Using these expressions condition (C1) implies the following equations

$$o^A o^B o^C \mathcal{D}_{BC} o_A = 0, \tag{122a}$$
$$o^A o^B \mathcal{D}_{AB}\varkappa = -2o^A o^B \iota^C \mathcal{D}_{BC} o_A, \tag{122b}$$
$$o^A \iota^B \mathcal{D}_{AB}\varkappa = \tfrac{1}{2} o^A o^B \iota^C \mathcal{D}_{AB}\iota_C - \tfrac{1}{2} o^A \iota^B \iota^C \mathcal{D}_{BC} o_A, \tag{122c}$$
$$\iota^A \iota^B \mathcal{D}_{AB}\varkappa = 2o^A \iota^B \iota^C \mathcal{D}_{AC}\iota_B, \tag{122d}$$
$$\iota^A \iota^B \iota^C \mathcal{D}_{BC}\iota_A = 0. \tag{122e}$$

Additionally, using equation (120) the spinor ξ_{AB} can be expressed as

$$\begin{aligned} e^{-\varkappa}\xi_{AB} = {}& \tfrac{1}{2} o_{(A}\mathcal{D}_{B)}{}^C \iota_C - \tfrac{1}{2} o^C \mathcal{D}_{(A|C|}\iota_{B)} + \tfrac{1}{2}\iota_{(A}\mathcal{D}_{B)}{}^C o_C - \tfrac{1}{2}\iota^C \mathcal{D}_{(A|C|}o_{B)} \\ & - \tfrac{1}{2} o_{(A}\iota^C \mathcal{D}_{B)C}\varkappa - \tfrac{1}{2} o^C \iota_{(A}\mathcal{D}_{B)C}\varkappa. \end{aligned} \tag{123}$$

Using equations (121) and (119) the constraint equations encoded in $\Lambda_{AB}{}^Q{}_Q = 0$ as given by (115b) imply

$$o^A o^B \mathcal{D}_{AB}\psi = 3Y_5 - 3Y_6 - 2o^A \psi \mathcal{D}_{AB} o^B + 4o^A o^B \psi \iota^C \mathcal{D}_{BC} o_A + 2o^A o^B o^C \psi \mathcal{D}_{BC}\iota_A, \tag{124a}$$
$$\begin{aligned} o^A \iota^B \mathcal{D}_{AB}\psi = {}& -\tfrac{3}{2} Y_3 + \tfrac{3}{2} Y_4 - \psi \iota^A \mathcal{D}_{AB} o^B - o^A \psi \mathcal{D}_{AB}\iota^B - \tfrac{1}{2} o^A o^B \psi \iota^C \mathcal{D}_{AB}\iota_C \\ & - o^A \psi \iota^B \iota^C \mathcal{D}_{AC} o_B + \tfrac{1}{2} o^A \psi \iota^B \iota^C \mathcal{D}_{BC} o_A + o^A o^B \psi \iota^C \mathcal{D}_{BC}\iota_A, \end{aligned} \tag{124b}$$
$$\iota^A \iota^B \mathcal{D}_{AB}\psi = 3Y_1 - 3Y_2 - 2\psi \iota^A \mathcal{D}_{AB}\iota^B - 4o^A \psi \iota^B \iota^C \mathcal{D}_{AC}\iota_B - 2\psi \iota^A \iota^B \iota^C \mathcal{D}_{BC} o_A, \tag{124c}$$

while condition (C3) is equivalent to

$$\psi o^A o^B o^C \mathcal{D}_{BC} o_A - Y_{\mathbf{7}} = 0, \tag{125a}$$
$$\psi o^A o^B \iota^C \mathcal{D}_{AB} o_C - \psi o^A \mathcal{D}_{AB} o^B - \tfrac{1}{6} o^A o^B \mathcal{D}_{AB}\psi + \tfrac{1}{2} Y_{\mathbf{5}} - \tfrac{1}{6} Y_{\mathbf{6}} = 0, \tag{125b}$$
$$\psi o^A o^B \iota^C \mathcal{D}_{AB}\iota_C - \psi o^A \iota^B \iota^C \mathcal{D}_{BC} o_A - 4\psi \mathcal{D}_{AB}(o^A \iota^B) - 2o^A \iota^B \mathcal{D}_{AB}\psi - 3Y_{\mathbf{3}} + 3Y_{\mathbf{4}} = 0, \tag{125c}$$
$$\psi \iota^A \iota^B \iota^C \mathcal{D}_{BC} o_A + \psi \iota^A \mathcal{D}_{AB}\iota^B + \tfrac{1}{6}\iota^A \iota^B \mathcal{D}_{AB}\psi + \tfrac{1}{2} Y_{\mathbf{2}} - \tfrac{1}{6} Y_{\mathbf{1}} = 0, \tag{125d}$$
$$\psi \iota^A \iota^B \iota^C \mathcal{D}_{BC}\iota_A - Y_{\mathbf{0}} = 0. \tag{125e}$$

A computation using equations (123) with (122a)-(122e) and (124a)-(124c), shows that the condition (C3) implies

$$Y_{\mathbf{0}} = Y_{\mathbf{1}} = Y_{\mathbf{6}} = Y_{\mathbf{7}} = 0. \tag{126}$$

The converse also holds. That is to say, if equation (126) along with (C1)-(C2) are satisfied, and assuming one has initial data for the alternative CFEs —so that, in particular, $\Lambda_{AB}{}^Q{}_Q = 0$— then condition (C3) holds.

8.2 Type N Case: $\kappa_{AB}\kappa^{AB} = 0$

If $\kappa_{AB}\kappa^{AB} = 0$ then one can choose a normalised spin dyad $\{o_A, \iota_B\}$ such that

$$\kappa_{AB} = o_A o_B. \tag{127}$$

Condition (C2) in this adapted dyad implies

$$\Psi_{ABCD} = \psi o_{(A}o_B o_C o_{D)}. \tag{128}$$

Using equation (127) one observes that condition (C1) implies

$$o^A o^B o^C \mathcal{D}_{AB} o_C = 0, \tag{129a}$$
$$o^A o^B \iota^C \mathcal{D}_{(AB} o_{C)} = 0, \tag{129b}$$
$$o^A \iota^B \iota^C \mathcal{D}_{(AB} o_{C)} = 0, \tag{129c}$$
$$\iota^A \iota^B \iota^C \mathcal{D}_{AB} o_C = 0. \tag{129d}$$

Additionally, using equation (127) the spinor ξ_{AB} can be expressed as

$$\xi_{AB} = -\tfrac{1}{2} o^C \mathcal{D}_{AC} o_B - \tfrac{1}{2} o_B \mathcal{D}_{AC} o^C - \tfrac{1}{2} o^C \mathcal{D}_{BC} o_A - \tfrac{1}{2} o_A \mathcal{D}_{BC} o^C. \tag{130}$$

Using equations (128) and (119) the constraint equations encoded in $\Lambda_{AB}{}^Q{}_Q = 0$ as given by equation (115b) imply

$$Y_{\mathbf{5}} - Y_{\mathbf{6}} = 0, \tag{131a}$$
$$Y_{\mathbf{3}} - Y_{\mathbf{4}} = 0, \tag{131b}$$
$$o^A o^B \mathcal{D}_{AB} \psi = \tfrac{1}{2} Y_{\mathbf{1}} - \tfrac{1}{2} Y_{\mathbf{2}} - 2 o^A \psi \mathcal{D}_{AB} o^B - 2 o^A o^B \iota^C \psi \mathcal{D}_{AB} o_C, \tag{131c}$$

Observe that in contrast with the case discussed in Section 8.1, constraints (131a)-(131b) immediately imply algebraic dependence of various components of the Cotton spinor. In this case (C3) is equivalent to

$$Y_{\mathbf{5}} = Y_{\mathbf{7}} = 0, \tag{132a}$$
$$o^A o^B o^C \psi \mathcal{D}_{BC} o_A - \tfrac{1}{2} Y_{\mathbf{3}} = 0, \tag{132b}$$
$$o^A o^B \psi \iota^C \mathcal{D}_{(AB} o_{C)} - \tfrac{1}{2} Y_{\mathbf{2}} = 0. \tag{132c}$$

A computation using equations (130), (129a)-(129d) and (131b)-(131c) shows that condition (C3) implies

$$Y_{\mathbf{2}} = Y_{\mathbf{3}} = Y_{\mathbf{4}} = Y_{\mathbf{5}} = Y_{\mathbf{6}} = Y_{\mathbf{7}} = 0. \tag{133}$$

Again, the converse holds so that condition (C3) may be replaced with equation (133). Collecting together both cases, Proposition 3 follows immediately.

Conclusions

In this article a *conformal* version of the Killing spinor initial data equations given in [16] are derived. By conformal it is understood that $(\mathcal{M}, \boldsymbol{g})$ is conformally related to an Einstein spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Consequently, we call these conditions the *conformal Killing spinor initial data equations*. The existence of a non-trivial solution of this system of equations is a necessary and sufficient condition for the existence of a Killing spinor on the development. The conditions are intrinsic to a spacelike hypersurface $\mathcal{S} \subset \mathcal{M}$. In the case where the conformal rescaling is trivial, $\Xi = 1$, the conditions reduce to those given in [3]. These conditions contain one differential condition and two algebraic conditions. The differential condition corresponds to the so-called *spatial Killing spinor equation*. The first algebraic condition corresponds to the restriction of the Buchdahl constraint on the initial hypersurface and the second imposes restrictions on the Cotton spinor of the initial data set. Moreover, it was shown that, in a spin dyad adapted to the Killing spinor, these conditions can be used along with the conformal Einstein field equations to show that certain components (at least half of them) of the Cotton spinor $Y_{ABCA'}$ have to vanish on the initial hypersurface $\mathcal{S}$. Notice that the conformal approach followed in this article —i.e., use of the (alternative) conformal Einstein field equations— opens the possibility to allow $\mathcal{S}$ to be determined by $\Xi = 0$ so that it to corresponds to the conformal boundary $\mathscr{I}$. The analysis given in this article already shows that in a potential characterisation of the Kerr-de Sitter spacetime, via the existence of Killing spinors at the conformal boundary, the Cotton spinor will play a replant role. This is not unexpected since the conformal boundary of the Kerr-de Sitter spacetime is conformally flat —see [1, 21]. Therefore, the Cotton tensor associated with asymptotic initial data corresponding to the Kerr-de Sitter spacetime vanishes. Nonetheless, future applications are not restricted to the analysis of de-Sitter like spacetimes. To see this, notice that, the most delicate part of the analysis consisted on finding a system of homogeneous wave equations for $H_{A'ABC}$ and $S_{AA'BB'}$. This system of wave equations in turn, leads to conditions (96a)-(96d) which are irrespective of the causal nature of $\mathcal{S}$. Consequently, one could investigate the analogous conditions to those derived in Section 7 considering a timelike or null hypersurface $\mathcal{S}$ instead. In the latter case one could consider the conformal boundary of an asymptotically flat spacetime. In the case of a timelike hypersurface $\mathcal{S}$, the analogous conditions could be useful for the analysis of anti-de Sitter like spacetimes.

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9 Appendix

$$
\begin{aligned}
P_{C'A'BD} \equiv\; & -\tfrac{1}{18}Y_{D}{}^{AC}{}_{C'}H_{A'BAC} - \tfrac{1}{18}Y_{B}{}^{AC}{}_{C'}H_{A'DAC} + \tfrac{1}{36}\Phi_{DAC'B'}S^{AB'}{}_{BA'} \\
& + \tfrac{1}{36}\Phi_{BAC'B'}S^{AB'}{}_{DA'} - \tfrac{1}{24}\Phi_{BAA'C'}S^{AB'}{}_{DB'} - \tfrac{1}{36}\Lambda S_{BA'DC'} + \tfrac{1}{36}\Phi_{DAC'B'}S_{B}{}^{B'A}{}_{A'} \\
& - \tfrac{1}{24}\Phi_{DAA'C'}S_{B}{}^{B'A}{}_{B'} + \tfrac{1}{36}\bar{\Psi}_{A'C'B'D'}S_{B}{}^{B'}{}_{D}{}^{D'} - \tfrac{1}{12}\Lambda S_{BC'DA'} - \tfrac{1}{36}\Lambda S_{DA'BC'} \\
& + \tfrac{1}{36}\Phi_{BAC'B'}S_{D}{}^{B'A}{}_{A'} + \tfrac{1}{36}\bar{\Psi}_{A'C'B'D'}S_{D}{}^{B'}{}_{B}{}^{D'} - \tfrac{1}{12}\Lambda S_{DC'BA'} + \tfrac{1}{36}\Lambda S_{B}{}^{B'}{}_{DB'}\bar{\epsilon}_{A'C'} + \\
& \tfrac{1}{36}\Lambda S_{D}{}^{B'}{}_{BB'}\bar{\epsilon}_{A'C'} + \tfrac{1}{9}H_{B'BDC}\nabla_{AC'}\Phi^{AC}{}_{A'}{}^{B'} - \tfrac{1}{9}\Psi_{BDCF}\nabla_{AC'}H_{A'}{}^{ACF} \\
& + \tfrac{1}{3}\Lambda\nabla_{AC'}H_{A'BD}{}^{A} - \tfrac{1}{3}H_{A'BDA}\nabla^{A}{}_{C'}\Lambda + \tfrac{1}{9}\Phi_{D}{}^{A}{}_{A'}{}^{B'}\nabla_{CC'}H_{B'BA}{}^{C} \\
& + \tfrac{1}{9}\Phi^{AC}{}_{A'}{}^{B'}\nabla_{CC'}H_{B'BDA} + \tfrac{1}{9}\Phi_{B}{}^{A}{}_{A'}{}^{B'}\nabla_{CC'}H_{B'DA}{}^{C} - \tfrac{1}{9}H_{B'DAC}\nabla^{C}{}_{C'}\Phi_{B}{}^{A}{}_{A'}{}^{B'} \\
& - \tfrac{1}{9}H_{B'BAC}\nabla^{C}{}_{C'}\Phi_{D}{}^{A}{}_{A'}{}^{B'} - \tfrac{1}{9}H_{A'}{}^{ACF}\nabla_{FC'}\Psi_{BDAC} + \tfrac{1}{9}\Psi_{DACF}\nabla^{F}{}_{C'}H_{A'B}{}^{AC} \\
& + \tfrac{1}{9}\Psi_{BACF}\nabla^{F}{}_{C'}H_{A'D}{}^{AC}
\end{aligned}
$$

$$
\begin{aligned}
Q_{C'A'BD} \equiv\; & \tfrac{1}{18}\Phi_{DAC'B'}S^{AB'}{}_{BA'} + \tfrac{1}{18}\Phi_{BAC'B'}S^{AB'}{}_{DA'} - \tfrac{1}{12}\Phi_{BAA'C'}S^{AB'}{}_{DB'} - \tfrac{1}{18}\Lambda S_{BA'DC'} \\
& + \tfrac{1}{18}\Phi_{DAC'B'}S_{B}{}^{B'A}{}_{A'} - \tfrac{1}{12}\Phi_{DAA'C'}S_{B}{}^{B'A}{}_{B'} + \tfrac{1}{18}\bar{\Psi}_{A'C'B'D'}S_{B}{}^{B'}{}_{D}{}^{D'} - \tfrac{1}{6}\Lambda S_{BC'DA'} \\
& - \tfrac{1}{18}\Lambda S_{DA'BC'} + \tfrac{1}{18}\Phi_{BAC'B'}S_{D}{}^{B'A}{}_{A'} + \tfrac{1}{18}\bar{\Psi}_{A'C'B'D'}S_{D}{}^{B'}{}_{B}{}^{D'} - \tfrac{1}{6}\Lambda S_{DC'BA'} \\
& + \tfrac{1}{12}\Psi_{BDAC}S^{AB'C}{}_{B'}\bar{\epsilon}_{A'C'} - \tfrac{7}{36}\Lambda S_{B}{}^{B'}{}_{DB'}\bar{\epsilon}_{A'C'} - \tfrac{1}{36}\Lambda S_{D}{}^{B'}{}_{BB'}\bar{\epsilon}_{A'C'} \\
& - \tfrac{1}{12}\Lambda S^{AB'}{}_{AB'}\epsilon_{BD}\bar{\epsilon}_{A'C'} - \tfrac{1}{18}H_{B'DAC}\nabla_{BC'}\Phi^{AC}{}_{A'}{}^{B'} + \tfrac{1}{18}H_{A'}{}^{ACF}\nabla_{BC'}\Psi_{DACF} \\
& + \tfrac{1}{18}\Psi_{DACF}\nabla_{BC'}H_{A'}{}^{ACF} - \tfrac{1}{18}\Phi^{AC}{}_{A'}{}^{B'}\nabla_{BC'}H_{B'DAC} - \tfrac{1}{18}H_{B'BAC}\nabla_{DC'}\Phi^{AC}{}_{A'}{}^{B'} \\
& + \tfrac{1}{18}H_{A'}{}^{ACF}\nabla_{DC'}\Psi_{BACF} + \tfrac{1}{18}\Psi_{BACF}\nabla_{DC'}H_{A'}{}^{ACF} - \tfrac{1}{18}\Phi^{AC}{}_{A'}{}^{B'}\nabla_{DC'}H_{B'BAC}
\end{aligned}
$$

$$
\begin{aligned}
U_{A'BC'D} \equiv\; & -\tfrac{1}{24}Y_{D}{}^{AC}{}_{C'}H_{A'BAC} - \tfrac{1}{24}Y_{B}{}^{AC}{}_{C'}H_{A'DAC} - \tfrac{1}{24}Y_{D}{}^{AC}{}_{A'}H_{C'BAC} - \tfrac{1}{24}Y_{B}{}^{AC}{}_{A'}H_{C'DAC} \\
& - \tfrac{1}{12}\Psi_{BDCF}\nabla_{AC'}H_{A'}{}^{ACF} - \tfrac{1}{12}H_{C'}{}^{ACF}\nabla_{FA'}\Psi_{BDAC} - \tfrac{1}{12}H_{A'}{}^{ACF}\nabla_{FC'}\Psi_{BDAC} \\
& + \tfrac{1}{12}\Psi_{DACF}\nabla^{F}{}_{C'}H_{A'B}{}^{AC} + \tfrac{1}{12}\Psi_{BACF}\nabla^{F}{}_{C'}H_{A'D}{}^{AC} - \tfrac{1}{4}\Psi_{BDAC}S^{(A}{}_{(A'}{}^{C)}{}_{C')}
\end{aligned}
$$

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