The conformal Killing spinor initial data equations

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Abstract

We obtain necessary and sufficient conditions for an initial data set for the *vacuum conformal Einstein field equations* to give rise to a spacetime development in possession of a Killing spinor. The fact that the conformal Einstein field equations are used in our derivation allows for the possibility that the initial hypersurface be (part of) the conformal boundary $\mathscr{I}$. For conciseness, these conditions are derived assuming that the initial hypersurface is spacelike. Consequently, these equations encode necessary and sufficient conditions for the existence of a Killing spinor in the development of asymptotic initial data on spacelike components of $\mathscr{I}$.

1 Introduction

The discussion of symmetries in General Relativity is ubiquitous. From the question of integrability of the geodesic equations to the existence of explicit solutions to the Einstein field equations and the black hole uniqueness problem, symmetries always play an important role. Symmetry assumptions are usually incorporated into the Einstein field equations —which in vacuum read

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \tag{1}$$

through the use of Killing vectors. From the spacetime point of view, the existence of Killing vectors allows one to perform *symmetry reductions* of the Einstein field equations —see for instance [**?**]. This approach has been exploited in classical uniqueness results such as [**?**]. Closely related to the black hole uniqueness problem, characterisations and classifications of solutions to the Einstein field equations usually exploit the symmetries of the spacetime in one way or another, e.g., in the characterisations of the Kerr spacetime via the *Mars-Simon tensor* —see [**?**, **?**, **?**]. On the other hand, from the point of view of the Cauchy problem, symmetry assumptions should be imposed only at the level of initial data. In this regard, symmetry assumptions can be phrased in terms of the *Killing vector initial data.* The Killing vector initial data equations constitute a set of conditions that an initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ for the Einstein field equations has to satisfy to ensure that the development will contain a Killing vector —see [**?**]. Nevertheless, despite the fact that the existence of Killing vector plays a central role in the discussion of the symmetries, the existence of Killing vectors is sometimes not enough to encode all the symmetries and conserved quantities that a spacetime can posses, e.g., the Carter constant in the Kerr spacetime. To unravel some of these *hidden symmetries* one has analyse the existence of a more fundamental type of objects; *Killing spinors* $\tilde{\kappa}_{AB}$ —in vacuum spacetimes, the existence of a Killing spinor directly implies the existence of a Killing vector. The *Killing spinor initial data equations* have

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been derived in the *physical framework* —governed by the Einstein field equations— in [?]. These equations have been successfully employed in the construction of a geometric invariant which detects whether or not an initial data set corresponds to initial data for the Kerr spacetime —see [?, ?, ?]. This analysis has also been extended to include suitable classes of matter —see [?] for an analogous characterisation of initial data for the Kerr-Newman spacetime. In these characterisations, some asymptotic conditions on the initial data are required. These conditions usually take the form of decay assumptions on $\tilde{\boldsymbol{h}}$, $\tilde{\boldsymbol{K}}$ and $\tilde{\boldsymbol{\kappa}}$ on $\tilde{\mathcal{S}}$, given in terms of asymptotically Cartesian coordinates. Nonetheless, in other approaches, the asymptotic behaviour of the spacetime can be studied in a geometric way through conformal compactifications. The latter is sometimes referred as the Penrose proposal. In this approach one starts with a *physical spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ where $\tilde{\mathcal{M}}$ is a 4-dimensional manifold and $\tilde{\boldsymbol{g}}$ is a Lorentzian metric which is a solution to the Einstein field equations. Then, one introduces a *unphysical spacetime* $(\mathcal{M}, \boldsymbol{g})$ into which $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is conformally embedded. Accordingly, there exists an embedding $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ such that

$$\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}. \tag{2}$$

By suitably choosing the *conformal factor* Ξ the metric $\boldsymbol{g}$ may be well defined at the points where $\Xi = 0$. In such cases, the set of points for where the conformal factor vanishes is at infinity from the physical spacetime perspective. The set

$$\mathscr{I} \equiv \{ p \in \mathcal{M} \;\mid\; \Xi(p) = 0, \;\; \mathbf{d}\Xi(p) \neq 0 \}$$

is called the conformal boundary. However, it can be readily verified that the Einstein field equations are not conformally invariant. Moreover, a direct computation using the conformal transformation formula for the Ricci tensor shows that the vacuum Einstein field equations (1), lead to an equation which is formally singular at the conformal boundary. An approach to deal with this problem was given in [?] where a regular set of equations for the unphysical metric was derived. These equations are known as the *conformal Einstein field equations.* The crucial property of these equations is that they are regular at the points where $\Xi = 0$ and a solution thereof implies whenever $\Xi \neq 0$ a solution to the Einstein field equations —see [?, ?] and [?] for an comprehensive discussion. There are three ways in which these equations can be presented, the metric, the frame and spinorial formulations. These equations have been mainly used in the stability analysis of spacetimes —see for instance [?, ?] for the proof of the global and semiglobal non-linear stability of the de Sitter and Minkowski spacetimes, respectively.

A conformal version of the Killing vector initial data equations using the metric formulation of the conformal Einstein field equations has been obtained in [?]. In the latter reference, intrinsic conditions on an initial hypersurface $\mathcal{S} \subset \mathcal{M}$ of the unphysical spacetime are found such that the development of the data —in the unphysical setting the evolution is governed by the conformal Einstein field equations— gives rise to a conformal Killing vector of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ which, in turn, corresponds to a Killing vector of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Notice that this approach, in particular, allows $\mathcal{S}$ to be determined by $\Xi = 0$ so that it to corresponds to the conformal boundary $\mathscr{I}$. The unphysical Killing vector initial data equations have been derived for the characteristic initial value problem on a cone in [?] and on a spacelike conformal boundary in [?].

For applications involving the the conformal Einstein field equations —say in its spinorial formulation, one frequently has to fix the gauge and write the equations in components. Despite the fact that, at first glance, the conformal Einstein field equations expressed in components with respect to an arbitrary spin frame seem to be overwhelmingly complicated, as shown in [?], symmetry assumptions (spherical symmetry in the latter case) greatly reduce the number of equations to be analysed. In the case of Petrov type D spacetimes, e.g.the Kerr-de Sitter spacetime, the symmetries of the spacetime are closely related to the existence of Killing spinors. Therefore, a natural question in this setting is whether a conformal version of the Killing spinor initial data equations introduced in [?] can be found. In other words, what are the extra conditions that one has to impose on an initial data set for the conformal Einstein field equations so that the arising development contains a Killing spinor? This question is answered in this article by deriving such conditions which we call the *conformal Killing spinor initial data equations*

Despite the fact that the Killing spinor equation is conformally invariant, it is not a priori clear whether the conditions of [?, ?] may be translated directly into the unphysical setting. Indeed, one expects this not to be the case, since the Einstein field equations are not conformally invariant. Moreover, one consideration that is exploited in the discussion of [?] is based on the fact that, on an Einstein spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, a Killing spinor $\tilde{\kappa}_{AB}$ gives rise to a Killing vector $\tilde{\xi}_a$ whose spinorial counterpart is given by $\tilde{\xi}_{AA'} = \tilde{\nabla}_{A'}{}^{Q}\tilde{\kappa}_{QA}$. Nevertheless, this property does not hold in general.

The analysis carried out in this article can be considered as the conformal analogue of the Killing spinor initial data equations derived in [?]. Although the results of [?] may be recovered from the analysis presented here by setting $\Xi = 1$, an important difference is that the set of variables that allow to obtain a closed system of homogeneous wave equations in the present case are different. The need for a different set of *Killing spinor zero-quantities* in the conformal case, can be traced back to the previous observation that in $(\mathcal{M}, \boldsymbol{g})$ the vector $\xi_{AA'} = \nabla_{A'}{}^{Q}\kappa_{QA}$ does not correspond to a (conformal) Killing vector. However, as by product of the present analysis it is shwon that $\xi_{AA'}$ is a Weyl collineation —see [?] for definitions of curvature collineations. Additionally, it is shown that using the conformal factor Ξ, the Killing spinor κ_{AB} and $\xi_{AA'}$, one can construct a conformal Killing vector X_a associated to a Killing vector $\tilde{X}_a$ of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. In the analysis of [?] the fact that $\tilde{\xi}_{AA'} = \tilde{\nabla}_{A'}{}^{Q}\tilde{\kappa}_{QA}$ is a Killing vector is crucial since one propagates off the initial hypersurface, simultaneously, $\tilde{\kappa}_{AA'}$ and the Killing vector $\tilde{\xi}^a$ by introducing $\tilde{S}_{ab} \equiv \tilde{\nabla}_{(a}\tilde{\xi}_{b)}$ as a zero-quantity. Similarly, in the work of [?] where the results of [?] are generalised to the case where $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfies the Einstein-Maxwell equations, the condition $\tilde{S}_{ab} = 0$ is also verified by virtue of the so-called *matter aligment condition*. In the conformal setting analysed in this article, the analogous quantity S_{ab} is not as geometrically motivated as in the previous cases and its usage as a variable in the system does not lead to a closed system of explicitly regular homogeneous wave equations. Here the adjective regular refers to the absence of formally singular terms, such as Ξ^{-1}, in the equations. Instead, the variable that is central for the present analysis turns out to be the so-called *Buchdahl constraint* (and derivatives thereof), which links directly the existence of Killing spinors with the Petrov type of $(\mathcal{M}, \boldsymbol{g})$.

Although the main objective of the present paper is deriving the valence-2 Killing spinor initial data in the conformal setting $(\mathcal{M}, \boldsymbol{g})$, the analogous conditions encoding the existence of a valence-1 Killing spinor are also derived. The latter serves as a warm up exercise for the valence-2 case where one can already observe the above discussed features and understand differences between the derivation of the conditions on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and those on $(\mathcal{M}, \boldsymbol{g})$ in a simpler arena.

For conciseness, the conformal Killing initial data equations are obtained on a spacelike hypersurface $\mathcal{S}$. Nonetheless, a similar computation can be performed on an hypersurface $\mathcal{S}$ with a different causal character. The conditions found in this article have potential applications for the black hole uniqueness problem. In particular, they can be used for an asymptotic characterisation of the Kerr-de Sitter spacetime analogous to [?] in terms of the existence of Killing spinors at the conformal boundary $\mathscr{I}$.

The main results of this article are summarised informally in the following:

Theorem. *If the conformal Killing spinor initial data equations* (??)-(??) *are satisfied on an open set* $\mathcal{U} \subset \mathcal{S}$, *where* $\mathcal{S}$ *is a spacelike hypersurface on which initial data for the conformal Einstein field equations has been prescribed, then, the domain of dependence of* $\mathcal{U}$ *possesses a Killing spinor.*

A precise formulation is the content of Theorem **??** and Proposition **??**.

Involved computations throughout this article were facilitated through the suite `xAct` in `Mathematica`. Note that since the existence of a spinor structure is guaranteed for globally-hyperbolic spacetimes —see Proposition 4 in [?]— the use of spinors is not overly restrictive.

Overview of the article

Section 2 establishes the conventions and notation to be used in the rest of the paper. It also gives an abbridged discussion of the main spinorial identities to be used and the space spinor formalism. Section 3 gives an overview of Killing spinors and their conformal properties. In Section 4 the conformal Einstein field equations are given for later use. In Section 5 the conformal (valence-1 Killing spinor) twistor initial data equations are obtained. In Section **??** the conformal (valence-2) Killing initial data equations are derived and discussed.

2 Notation and spinorial formalism in a nutshell

Spacetime spinor formalism

Upper case Latin indices ${}_{ABC\cdots A'B'C'}$ will be used as abstract indices of the *spacetime spinor* algebra, and the bold numerals ${}_{\mathbf{012}\cdots}$ denote components with respect to a fixed spin dyad $o^A \equiv \epsilon_{\mathbf{0}}{}^A, \iota^A \equiv \epsilon_{\mathbf{1}}{}^A$ —see Penrose & Rindler [**?**] for further details. Although spinor notation will be preferred, for certain computations tensors will be employed. Lower case Latin indices ${}_{a,b,c\ldots}$ will be used as abstract tensor indices. For tensors, our curvature conventions are fixed by

$$\nabla_a \nabla_b \kappa^c - \nabla_b \nabla_a \kappa^c = R_{ab}{}^c{}_d \kappa^d.$$

For spinors, the curvature conventions are fixed via the spinorial Ricci identities which will be written in accordance with the above convention for tensors. To see this, recall that the commutator of covariant derivatives $[\nabla_{AA'}, \nabla_{BB'}]$ can be expressed in terms of the symmetric operator $\square_{AB}$ as

$$[\nabla_{AA'}, \nabla_{BB'}] = \epsilon_{AB}\square_{A'B'} + \epsilon_{A'B'}\square_{AB}$$

where

$$\square_{AB} \equiv \nabla_{Q'(A}\nabla_{B)}{}^{Q'}.$$

The action of the symmetric operator $\square_{AB}$ on valence-1 spinors is encoded in the spinorial Ricci identities

$$\square_{AB}\xi_C = -\Psi_{ABCD}\xi^D + 2\Lambda\xi_{(A}\epsilon_{B)C}, \tag{3a}$$

$$\square_{A'B'}\xi_C = -\xi^A\Phi_{CAA'B'}, \tag{3b}$$

where Ψ_{ABCD} and $\Phi_{AA'BB'}$ and Λ are curvature spinors. The above identities can be extended to higher valence spinors in an analogous way —see [**?**] for further discussion on these identities using different conventions to the ones used in this article. A related identity which will be systematically used in the following discussion is

$$\nabla_{AQ'}\nabla_B{}^{Q'} = \square_{AB} + \frac{1}{2}\epsilon_{AB}\square, \tag{4}$$

where $\square_{AB}$ is the symmetric operator defined above and $\square \equiv \nabla_{AA'}\nabla^{AA'}$.

Space spinor formalism

To have a self-contained discussion in this section the space spinor formalism, originally introduced in [**?**], is briefly recalled —see also [**?**, **?**, **?**]. Let $\tau^{AA'}$ denote the spinorial counterpart of a timelike vector τ^a, normal to a spacelike hypersurface $\mathcal{S}$ and normalised so that $\tau_a\tau^a = 2$. Then, it follows that $\tau_{AA'}\tau^{AA'} = 2$ and, consequently,

$$\tau_{AA'}\tau_B{}^{A'} = \epsilon_{AB}.$$

The covariant derivative $\nabla_{AA'}$ is then decomposed into the *normal* and *Sen* derivatives:

$$\nabla_\tau \equiv \tau^{AA'}\nabla_{AA'},$$

$$\mathcal{D}_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'}.$$

The *Weingarten* spinor and the *acceleration* of the congruence are then defined by

$$K_{ABCD} \equiv \tau_D{}^{C'}\mathcal{D}_{AB}\tau_{CC'},$$
$$K_{AB} \equiv \tau_B{}^{C'}\mathcal{P}\tau_{AC'}.$$

The above can be inverted to obtain the following formulae which will prove useful in the sequel

$$\nabla_\tau \tau_{CC'} = -K_{CD}\tau^D{}_{C'},$$
$$\mathcal{D}_{AB}\tau_{CA'} = -K_{ABCD}\tau^D{}_{A'}.$$

The distribution induced by $\tau_{AA'}$ is integrable if and only $K^D{}_{(AB)D} = 0$, in which case K_{ABCD} describes the extrinsic curvature of the resulting foliation. Nevertheless, this is not required for our subsequent discussion. In other words, we will allow the possibility that the distribution is non-integrable —i.e. the spinor $K^D{}_{(AB)D}$ will not be assumed to vanish.

Defining the spinors $\chi_{AB} \equiv K^D{}_{(AB)D}$, $\chi_{ABCD} \equiv K_{(ABCD)}$ and $\chi \equiv K_{AB}{}^{AB}$, the Weingarten spinor decomposes as follows

$$K_{ABCD} = \chi_{ABCD} - \tfrac{1}{2}\epsilon_{A(C}\chi_{D)B} - \tfrac{1}{2}\epsilon_{B(C}\chi_{D)A} - \tfrac{1}{3}\chi\epsilon_{A(C}\epsilon_{D)B}. \tag{5}$$

For the following discussion we will also need the commutators form with ∇_τ, $\mathcal{D}_{AB}$. To write these commutators in a succinct way, first define

$$\widehat{\square}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\square_{A'B'}$$

from which, proceeding analogously as in [?], one obtains

$$[\nabla_\tau, \mathcal{D}_{AB}] = -\tfrac{1}{2}\chi_{AB} - \square_{AB} + \widehat{\square}_{AB} + K_{(A}{}^D\mathcal{D}_{B)D} - K_{AB}{}^{FG}\mathcal{D}_{FG}, \tag{6}$$

$$\begin{aligned}[\mathcal{D}_{AB}, \mathcal{D}_{CD}] = {}& \tfrac{1}{2}\left(\epsilon_{A(C}\square_{D)B} + \epsilon_{B(C}\square_{D)A}\right) + \tfrac{1}{2}\left(\epsilon_{A(C}\widehat{\square}_{D)B} + \epsilon_{B(C}\widehat{\square}_{D)A}\right)\\ &+ \tfrac{1}{2}\left(K_{CDAB} - K_{ABCD}\right)\nabla_\tau + K_{CDF(A}\mathcal{D}_{B)}{}^F - K_{ABF(C}\mathcal{D}_{D)}{}^F\end{aligned} \tag{7}$$

3 Killing spinors

To start the discussion it is convenient to introduce some notation and definitions. Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ be a 4-dimensional manifold equipped with a Lorentzian metric $\tilde{\boldsymbol{g}}$ and denote by $\tilde{\nabla}$ its associated Levi-Civita connection. For the time being $\tilde{\boldsymbol{g}}$ is not assumed to be a solution to the Einstein field equations (1).

A totally symmetric $\tilde{\kappa}_{A_1\ldots A_p} = \tilde{\kappa}_{(A_1\ldots A_p)}$ valence$-p$ spinor is said to be a (valence$-p$) *Killing spinor* if the following equation is satisfied

$$\tilde{\nabla}_{Q'(Q}\tilde{\kappa}_{A_1\ldots A_p)} = 0. \tag{8}$$

An important property of the Killing spinor equation is that it is conformally-invariant, in other words if $\boldsymbol{g}$ is conformally related to $\tilde{\boldsymbol{g}}$, namely $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ then $\kappa_{A_1\ldots A_q} = \Xi^2\tilde{\kappa}_{A_1\ldots A_q}$ satisfies

$$\nabla_{Q'(Q}\kappa_{A_1\ldots A_p)} = 0,$$

where ∇ is the Levi–Civita connection of $\boldsymbol{g}$.

In this paper we will only focus only the case $p = 1$ and $p = 2$. If $p = 1$, the equation

$$\tilde{\nabla}_{Q'(Q}\tilde{\kappa}_{A)} = 0. \tag{9}$$

is usually referred as the *twistor equation*. We will follow this naming convention and refer to a valence-1 spinor satisfying equation (9) as twistor. Since only the cases $p = 1$ and $p = 2$ will be discussed in this paper, we will refer to the case $p = 1$ as the twistor case and the $p = 2$ as the Killing spinor case. Namely, we will say that a symmetric valence−2 spinor, $\tilde{\kappa}_{AB} = \tilde{\kappa}_{(AB)}$, is a *Killing spinor* if it satisfies the equation

$$\tilde{\nabla}_{A'(A}\tilde{\kappa}_{BC)} = 0. \tag{10}$$

The Killing spinor equation and twistor equations are, in general, overdetermined; in particular, they imply the so-called *Buchdahl constraint*. In the twistor case ($p = 1$) this has the form

$$\tilde{\kappa}^{D}\Psi_{ABCD} = 0,$$

while in the Killing spinor case ($p = 2$) the Buchdahl constraint adquires the form

$$\tilde{\kappa}^{Q}{}_{(A}\Psi_{BCD)Q} = 0,$$

where Ψ_{ABCD} denotes the conformally invariant Weyl spinor. The latter condition restricts Ψ_{ABCD} to be algebraically special. In the twistor case the spacetime is necessarity of Petrov type N or O, hence restricting its applicability for characterisation of black holes. In the Killing spinor case the spacetime is only restricted to be of Petrov type D, N or O.

At first glance, the conformal invariance property of the Killing spinor equation would seem to indicate that the approach leading to the Killing spinor initial data conditions derived in [?] would identically apply for $(\mathcal{M}, \boldsymbol{g})$ with $\tilde{\boldsymbol{g}} = \Xi^2\tilde{\boldsymbol{g}}$. This is not the case simply because the Einstein field equations are not conformally invariant. In other words, in the analysis of [?] the vacuum Einstein field equations (1) were used, and, despite that one can relate R_{ab} with $\tilde{R}_{ab}$ this leads to formally singular terms (terms containing Ξ^{-1}). Moreover, even if one is willing to work with formally singular equations it is not apriori clear that the choice of variables made in [?] will form a closed homogeneous system in the conformal seeting. To understand this second point further, notice that for general manifold with metric $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ —namely $\tilde{\boldsymbol{g}}$ not satifying any field equation— the existence of a Killing spinor $\tilde{\kappa}_{AB}$ is not related directly to the existence of a Killing vector. Nevertheless, if one assumes that $\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein field equations (1) then the concomitant

$$\tilde{\xi}_{AA'} \equiv \tilde{\nabla}^{B}{}_{A'}\tilde{\kappa}_{AB},$$

represents the spinorial counterpart of a complex Killing vector of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ —see [?] for further discussion. This point is subtle and even in the physical (non-conformal) set up if one is to include matter such as the Maxwell field and the analysis of [?] does not straighfowardly apply since further conditions (the matter alignment conditions) —see [?]— need to be propagated.

Remark 1. The notion of Killing spinors is related to that of Killing–Yano tensors. If a Killing spinor $\tilde{\xi}_{AA'}$ is Hermitian, i.e., $\bar{\tilde{\xi}}_{AA'} = \tilde{\xi}_{AA'}$, then one can construct the spinorial counterpart of a *Killing–Yano tensor* $\tilde{\Upsilon}_{ab}$ —i.e. an antisymmetric 2−tensor satisfying $\tilde{\nabla}_{(a}\tilde{\Upsilon}_{b)c} = 0$— as follows

$$\tilde{\Upsilon}_{AA'BB'} = i(\tilde{\kappa}_{AB}\bar{\tilde{\epsilon}}_{A'B'} - \bar{\tilde{\kappa}}_{A'B'}\tilde{\epsilon}_{AB}).$$

Conversely, given a Killing–Yano tensor, one can construct a Killing spinor —see [?, ?, ?].

In the sequel $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ will be reserved to denote the *physical spacetime*, in other words, the symbol ˜ will be added to those fields associated with a solution $\tilde{\boldsymbol{g}}$ to the vacuum Einstein field equations (1). Similarly $(\mathcal{M}, \boldsymbol{g})$ will be used to represent the *unphysical spacetime* related to $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ via $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$. —in a slight abuse of notation $\varphi(\tilde{\mathcal{M}})$ and $\mathcal{M}$ will be identified so that the mapping $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ can be omitted.

4 The conformal Einstein field equations

This section contains an abriged discussion of the CFEs in first and second order form. At the end of this section the main technical tool from the theory of partial differential equations to be used for deriving the Killing spinor intial data equations is given.

The conformal Einstein field equations are a conformal formulation of the Einstein field equations. In other words, given a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ satisfying the Einstein field equations, the conformal Einstein field equations encode a system of differential conditions for the curvature and concomitants of the conformal factor associated with $(\mathcal{M}, \boldsymbol{g})$ where $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$. The key property of these equations is that they are regular even at the conformal boundary $\mathscr{I}$, where $\Xi = 0$. This formulation of the conformal Einstein field equations was first given in [?] —see also [?] for a comprehensive discussion.

The metric version of the standard vacuum conformal Einstein field equations are encoded in the following zero-quantities —see [?, ?, ?, ?]:

$$Z_{ab} \equiv \nabla_a\nabla_b\Xi + \Xi L_{ab} - sg_{ab} = 0, \tag{11a}$$
$$Z_a \equiv \nabla_a s + L_{ac}\nabla^c\Xi = 0, \tag{11b}$$
$$\delta_{bac} \equiv \nabla_b L_{ac} - \nabla_a L_{bc} - d_{abcd}\nabla^d\Xi = 0, \tag{11c}$$
$$\lambda_{abc} \equiv \nabla_e d_{abc}{}^e = 0, \tag{11d}$$
$$Z \equiv \lambda - 6\Xi s + 3\nabla_a\Xi\nabla^a\Xi \tag{11e}$$

where Ξ is the conformal factor, L_{ab} is the Schouten tensor, defined in terms of the Ricci tensor R_{ab} and the Ricci scalar R via

$$L_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}, \tag{12}$$

s is the so-called *Friedrich scalar* defined as

$$s \equiv \tfrac{1}{4}\nabla_a\nabla^a\Xi + \tfrac{1}{24}R\Xi \tag{13}$$

and $d^a{}_{bcd}$ denotes the *rescaled Weyl tensor*, defined as

$$d^a{}_{bcd} = \Xi^{-1}C^a{}_{bcd},$$

where $C^a{}_{bcd}$ denotes the Weyl tensor. The geometric meaning of these zero-quantities is the following: The equation $Z_{ab} = 0$ encodes the conformal transformation law between R_{ab} and $\tilde{R}_{ab}$. The equation $Z_a = 0$ is obtained considering $\nabla^a Z_{ab}$ and commuting covariant derivatives. Equations $\delta_{abc} = 0$ and $\lambda_{abc} = 0$ encode the contracted second Bianchi identity. Finally, $Z = 0$ is a constraint in the sense that if it is verified at one point $p \in \mathcal{M}$ then $Z = 0$ holds in $\mathcal{M}$ by virtue of the previous equations. A solution to the metric conformal Einstein field equations consist of a collection of fields

$$\{g_{ab},\ \Xi,\ \nabla_a\Xi, s\ , L_{ab},\ d_{abcd}\}$$

satisfying

$$Z_{ab} = 0, \quad Z_a = 0, \quad \delta_{abc} = 0, \quad \lambda_{abc} = 0, \quad Z = 0. \tag{14}$$

Remark 2. If one opts to use the Ricci tensor R_{ab} instead of the Schouten tensor L_{ab} then the Ricci scalar R appears in the right-hand side of equations but no equation for it has been provided. In the CFEs the Ricci scalar encodes the *conformal gauge source function*, hence there is no equation to fix that variable as it represents a gauge quantity of the formulation.

Since the structural properties of the CFEs are better expressed in spinorial formalism and due to the nature of the applications in this article, the spinorial version of the CFEs will be used. The spinorial translation of the above CFEs zero-quantities render —see [?] for further details.

$$Z_{AA'BB'} = -\Xi\Phi_{ABA'B'} - s\epsilon_{AB}\epsilon_{A'B'} + \Xi\Lambda\epsilon_{AB}\epsilon_{A'B'} + \nabla_{BB'}\nabla_{AA'}\Xi \tag{15a}$$
$$Z_{AA'} = \Lambda\nabla_{AA'}\Xi + \nabla_{AA'}s - \Phi_{ABA'B'}\nabla^{BB'}\Xi \tag{15b}$$
$$\delta_{ABCC'} = \nabla_{A'(A}\Phi_{B)CC'}{}^{A'} - \epsilon_{C(A}\nabla_{B)C'}\Lambda + \phi_{ABCD}\nabla^D{}_{C'}\Xi \tag{15c}$$
$$\Lambda_{CC'AB} = \nabla_{DC'}\phi_{ABC}{}^D \tag{15d}$$

$$Z = \lambda - 6\Xi s + 3\nabla_{AA'}\Xi\nabla^{AA'}\Xi \tag{15e}$$

•1 Similar to the tensorial case, one can choose the Schouten (tensor) spinor or the Ricci (tensor) spinor as a variable. Here the equations have been expressed using the standard curvature spinors of the NP formalism, namely, the trace-free Ricci spinor $\Phi_{ABA'B'}$, the Ricci scalar Λ —in fact $R = 24\Lambda$— and the Weyl spinor Ψ_{ABCD} —see [?, ?]. The rescaled Weyl spinor ϕ_{ABCD} is defined as

•1: *I think the original $z_{AA'BB'CC'}$ does not contain more information than what $\delta_{ABCC'}$ encodes.*

$$\phi_{ABCD} \equiv \Xi^{-1}\Psi_{ABCD}. \tag{16}$$

The CFEs as previously presented can be regarded as a set of covariant conditions for geometric fields on $(\mathcal{M}, \boldsymbol{g})$ and, hence, they do not have a particular PDE character. However, there are, depending on the gauge fixing procedure, different hyperbolic reduction strategies to extract a set of evolution and constraint equations. For the subsequent discussion only the evolution and constraint equations implied by the $\Lambda_{C'AB} = 0$ equation will play a role. A direct calculation using the space spinor formalism shows that $\Lambda_{CC'AB} = 0$ can be recasted as the following system of evolution equation and constraint equations

$$\nabla_\tau \phi_{ABCD} = 2\mathcal{D}_{(A}{}^{F}\phi_{BCD)F}, \qquad \mathcal{D}_{CD}\phi_{AB}{}^{CD} = 0. \tag{17}$$

The evolution and constraint equations associated to the other zero-quantities depend on the particular gauge fixing strategy and will not play a relevant role for the discussion in the next sections.

The CFEs are usually presented as the first order system (14) with the definitions (11), however, for several applications it is convenient to use a second order formulation of the equations. In [?] the tensorial version of the CFEs was recasted as a set of (tensorial) wave equations. Similarly, in [?] a second order form of the spinorial formulation of the CFEs was obtained. This version of the CFEs is particularly suited for the applications of this article, and, in fact, only one of those equations —that for the rescaled Weyl spinor— will be needed •2 . The wave equation for the rescaled Weyl spinor can be succintly obtained from considering $\nabla^{QC'}\Lambda_{C'AB}$. A direct calculation using the identity (4) shows that if $\Lambda_{C'AB} = 0$ then,

•2: *Double check this is true*

$$\square\phi_{ABCF} = 12\Lambda\phi_{ABCF} - 6\Xi\phi_{(AB}{}^{DG}\phi_{CF)DG} \tag{18}$$

A similar calculation can be carried out for the other equations in comprising the CFEs. A full discussion of the *spinorial CFE wave equations* and their equivalence with the standard first order formulation CFEs can be found in [?]. One of the tools used in [?] to show the equivalence between these two set of equations is the uniqueness property to a certain class of wave equations. This same result from the theory of partial differential equations will be used to obtain the main theorem of this article and is presented in the following

•3

•3: *Technical pde theorem moved here*

Theorem 1. *Let $\mathcal{M}$ be a smooth manifold equipped with a Lorentzian metric $\boldsymbol{g}$ and consider the wave equation*

$$\square\underline{u} = h\left(\underline{u},\ \partial\underline{u}\right)$$

where $\underline{u} \in \mathbb{C}^m$ is a complex vector-valued function on $\mathcal{M}$, $h : \mathbb{C}^{2m} \to \mathbb{C}^m$ is a smooth homogeneous function of its arguments and $\square = g^{ab}\nabla_a\nabla_b$. Let $\mathcal{U} \subset \mathcal{S}$ be an open set and $\mathcal{S} \subset \mathcal{M}$ be a spacelike hypersurface with normal τ^a respect to $\boldsymbol{g}$. Then the Cauchy problem

$$\square\underline{u} = h\left(\underline{u},\ \partial\underline{u}\right),$$
$$\underline{u}\,|_{\mathcal{U}} = \underline{u}_0, \quad \nabla_{\boldsymbol{\tau}}\underline{u}\,|_{\mathcal{U}} = \underline{u}_1,$$

where $\underline{u}_0$ and $\underline{u}_1$ are smooth on $\mathcal{U}$ and $\nabla_{\boldsymbol{\tau}} \equiv \tau^\mu\nabla_\mu$, has a unique solution $\underline{u}$ in the domain of dependence of $\mathcal{U}$.

We refer the reader to [?, ?] for a proof —see also Theorem 1 in [?].

Remark 3. Recall that an equation of the above form are said to be *homogeneous in $\underline{u}$ and its first derivatives* if $h\left(\lambda\underline{u},\ \lambda\partial\underline{u}\right) = \lambda h\left(\underline{u},\ \partial\underline{u}\right)$ for all $\lambda \in \mathbb{C}$.

5 Conformal twistor initial data

In this section, the conformal twistor initial data equations are derived. Although the main result of this article is on the conformal valence-2 Killing spinor initial data equations, the twistor case discussed in this section serves as a test case where the main features of the calculation of the next section can be understood in simpler setting.

5.1 Twistor zero-quantities

For the following discussion is convenient to make the following *zero-quantities*

$$H_{A'AB} \equiv 2\nabla_{A'(A}\kappa_{B)}, \tag{19a}$$
$$B_{ABC} \equiv \phi_{ABCD}\kappa^{D}. \tag{19b}$$

The spinors $H_{A'AB}$ and B_{ABC} will be denoted in index free notation as $\boldsymbol{H}$ and $\boldsymbol{B}$ and will be called the twistor zero-quantity and the Buchdahl zero-quantity respectively. The Buchdahl zero-quantity arises as an integrability condition of the twistor equation. To see this, notice that, taking the following derivative of $\boldsymbol{H}$ and substituting the definition (19a) one obtains

$$\nabla_{AA'}H^{A'}{}_{BC} = 2\nabla_{AA'}\nabla_{(B}{}^{A'}\kappa_{C)} = \tfrac{1}{2}\epsilon_{AB}\Box\kappa_{C} + \tfrac{1}{2}\epsilon_{AC}\Box\kappa_{B} + \Box_{BA}\kappa_{C} + \Box_{CA}\kappa_{B}. \tag{20}$$

Symmetrsing and using equation (3a) renders

$$\nabla_{(A|A'|}H^{A'}{}_{BC)} = -2\Psi_{ABCD}\kappa^{D}.$$

The vanishing of the right-hand side of latter equation encodes the Buchdahl constraint, namely the fact that if $(\mathcal{M}, \boldsymbol{g})$ admits a twistor then it is necesarilly of Petrov type N or O. To write this in the variables appearing in the conformal Einstein field equations, using the definition of the rescaled Weyl spinor yields

$$\nabla_{(A}{}^{A'}H_{|A'|BC)} = 2\Xi B_{ABC}, \tag{21}$$

which motivates the name for the zero-quantity $\boldsymbol{B}$. Thus, with this notation, it is clear that if the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ admits a twistor (valence-1 Killing spinor) then following zero-quantities vanish

$$H_{A'AB} = 0, \qquad B_{ABC} = 0. \tag{22}$$

5.2 Twistor auxiliary quantities and the twistor candidate equation

A useful bookkeeping device for the subsequent calculations are the definitions of following *auxiliary quantities*:

$$Q_{A} \equiv \nabla^{QA'}H_{A'QA} \tag{23a}$$
$$\xi_{A'} \equiv \nabla^{B}{}_{A'}\kappa_{B} \tag{23b}$$

The *auxiliary spinor* $\xi_{A'}$ is merely a convenient placeholder for making irreducible decompositions of derivatives of κ_A such as

$$\nabla_{AA'}\kappa_{B} = \tfrac{1}{2}\epsilon_{AB}\nabla_{CA'}\kappa^{C} + \nabla_{(A|A'|}\kappa_{B)} \tag{24}$$
$$= \tfrac{1}{2}H_{A'AB} - \tfrac{1}{2}\xi_{A'}\epsilon_{AB}. \tag{25}$$

and in principle one can carry out all the calculations without this definition. It is nevertheless illustrative to introduce this shorthand since the analogous quantity in the Killing spinor case $(p = 2)$ will have some geometrical significance.

On the other hand, the *auxiliary quantity* Q_A will be central for the following discussion since it encodes a wave equation for κ_A. To see this, observe that tracing the identity (20) and substituting the definition (23a) gives,

$$Q_{A} = 3\Lambda\kappa_{A} + \tfrac{3}{2}\Box\kappa_{A}. \tag{26}$$

Solving for $\Box\kappa_A$ one has

$$\Box\kappa_A = \tfrac{2}{3}Q_A - 2\Lambda\kappa_A.$$

If the equation $Q_A = 0$ is imposed, then the latter expression can be read as a wave equation for κ_A. This motivates the following definition: a valence-1 spinor η_A satisfying

$$\Box\eta_A = -2\Lambda\eta_A \tag{27}$$

will be called a *twistor candidate*. To understand the motivation for this definition and its name, notice that in general, any twistor κ_A trivially satisfies the twistor candiadate equation but not every twistor candidate η_A will solve the twistor equation. In other words,

$$\boldsymbol{H} = 0 \implies \boldsymbol{Q} = 0, \qquad \text{but in general} \qquad \boldsymbol{Q} = 0 \not\Longrightarrow \boldsymbol{H} = 0.$$

However, the initial data $(\nabla_\tau\eta_A, \eta_A)|_{\mathcal{S}}$ for the wave equation (27) has not been fixed yet. The aim of the following calculations is to determine the conditions on the initial data for the twistor candidate such that if propagated off $\mathcal{S}$, using equation (27), then the corresponding twistor candidate η_A is, in fact, a twistor. Namely,

$$\boldsymbol{Q} = 0 \ \ \& \ \ \text{twistor initial data} \quad \Longrightarrow \boldsymbol{H} = 0. \tag{28}$$

The strategy to obtain such conditions on the intial data $(\nabla_\tau\eta_A, \eta_A)|_{\mathcal{S}}$ is to derive a closed system of homogeneous wave equations for the zero-quantities $\boldsymbol{H}$ and $\boldsymbol{B}$ to show that, if trivial initial data for such equations is given, then, using Theorem 1, $\boldsymbol{H} = 0$ and $\boldsymbol{B} = 0$ in the domain of dependence of the data.

5.3 Wave equations for the zero-quantities

A wave equation for the zero-quantity $\boldsymbol{H}$ can be constructed as follows. From the irreducible decomposition of $\nabla_D{}^{A'}H_{A'AB}$,

$$\nabla_D{}^{A'}H_{A'AB} = \tfrac{1}{3}\epsilon_{BD}\nabla_{CA'}H^{A'}{}_A{}^C + \tfrac{1}{3}\epsilon_{AD}\nabla_{CA'}H^{A'}{}_B{}^C + \nabla_{(A}{}^{A'}H_{|A'|BD)},$$

and the definitions (23a) and equation (21) one has that

$$\nabla_D{}^{A'}H_{A'AB} = 2B_{ABD}\Xi + \tfrac{1}{3}Q_B\epsilon_{AD} + \tfrac{1}{3}Q_A\epsilon_{BD} \tag{29}$$

Applying $\nabla_D{}^{B'}$ to the last expression, and using the identity (4) along with the spinorial Ricci identities (3a)-(3b), renders

$$\Box H_{B'AB} = 6\Lambda H_{B'AB} + 4\Xi\nabla_{DB'}B_{AB}{}^D - 4B_{ABD}\nabla^D{}_{B'}\Xi - 4\Phi_{(A}{}^D{}_{|B'}{}^{A'}H_{A'|B)D} + \tfrac{4}{3}\nabla_{(A|B'|}Q_{B)} \tag{30}$$

To derive a wave equation for $\boldsymbol{B}$, one applies the D'Alembertian operator $\Box$ to the definition in equation (19b) to obtain

$$\Box B_{ABC} = \kappa^D\Box\phi_{ABCD} + \phi_{ABCD}\Box\kappa^D + 2\nabla_{FA'}\phi_{ABCD}\nabla^{FA'}\kappa^D \tag{31}$$

Substituting the definition (19b), the identity (26), and the wave equation satisfied by the rescaled Weyl spinor (18) into the last expression gives

$$\Box B_{ABC} = 10B_{ABC}\Lambda + H^{A'DF}\nabla_{FA'}\phi_{ABCD} - 6\Xi B_{(A}{}^{DF}\phi_{BC)DF} + \tfrac{2}{3}\phi_{ABCD}Q^D \tag{32}$$

Observe that if $Q_A = 0$, namely if the twistor candidate wave equation is imposed then, $\boldsymbol{H}$ and $\boldsymbol{B}$ satisfy the following set of wave equations

$$\Box H_{B'AB} = 6\Lambda H_{B'AB} + 4\Xi\nabla_{DB'}B_{AB}{}^D - 4B_{ABD}\nabla^D{}_{B'}\Xi - 4\Phi_{(A}{}^D{}_{|B'}{}^{A'}H_{A'|B)D} \tag{33a}$$

$$\Box B_{ABC} = 10B_{ABC}\Lambda + H^{A'DF}\nabla_{FA'}\phi_{ABCD} - 6\Xi B_{(A}{}^{DF}\phi_{BC)DF} \tag{33b}$$

Notice that the only place where the CFEs —in their wave equation form— have been used was in equation (31) to substitute the term $\Box\phi_{ABCD}$.

The relevant observation about equations (33a)-(33b) is that they constitute a closed system of *regular and homogeneous* wave equations for $\boldsymbol{H}$ and $\boldsymbol{B}$. Hence prescribing trivial intial data

$$H_{A'AB} = 0, \qquad \nabla_\tau H_{A'AB} = 0, \qquad B_{ABC} = 0, \qquad \nabla_\tau B_{ABC} = 0 \qquad \text{on} \qquad \mathcal{S}$$

and using Theorem 1, which stablishes the uniquess of solutions to wave equations of the type of (33a)-(33b), on has that

$$H_{A'AB} = 0, \qquad B_{ABC} = 0 \qquad \text{on} \qquad \mathcal{D}^+(\mathcal{S}). \tag{34}$$

In turn, substituting the definitions for the zero-quantities $\boldsymbol{H}$ and $\boldsymbol{B}$ into the conditions (34) render a prescription to fix the intial data $(\nabla_\tau\eta_A, \eta_A)|_\mathcal{S}$ for twistor candidate wave equation (27) that ensures that the twistor candidate η_A will corresponds to an actual twistor κ_A in $\mathcal{D}^+(\mathcal{S})$.

This discussion is summarised in the following

Proposition 1. *Given initial data for the conformal field equations on $\mathcal{U} \subseteq \mathcal{S}$ where $\mathcal{S}$ is a spacelike hypersurface $\mathcal{S}$ with normal vector $\tau^{AA'}$, and associated normal derivative $\nabla_\tau \equiv \tau^{AA'}\nabla_{AA'}$, the corresponding spacetime development admits a twistor (valence-1 Killing spinor) in $\mathcal{D}^+(\mathcal{U})$ —the future domain of dependence of $\mathcal{U}$— if and only if*

$$H_{A'AB} = 0, \tag{35a}$$
$$\nabla_\tau H_{A'AB} = 0, \tag{35b}$$
$$B_{ABC} = 0, \tag{35c}$$
$$\nabla_\tau B_{ABC} = 0, \tag{35d}$$

hold on $\mathcal{U}$.

Proof. The *only if* direction is immediate. Suppose, on the other hand, that (35a)-(35d) hold on some $\mathcal{U} \subset \mathcal{S}$ —that is to say, there exist a spinor field κ_A for which (35a)-(35d) are satisfied on $\mathcal{U}$. The latter is then used as initial data for the twistor candidate wave equation

$$\square\kappa_A = -2\Lambda\kappa_A. \tag{36}$$

As the zero-quantities $H_{A'AB}$, B_{ABC} satisfy the homogeneous wave equations (33a)-(33b) then the uniqueness result for homogeneous wave equations, given in Theorem 1, ensures that

$$H_{A'ABC} = 0, \qquad B_{ABC} = 0,$$

in $\mathcal{D}^+(\mathcal{U})$. In other words, κ_A solves the twistor equation on $\mathcal{D}^+(\mathcal{U})$. □

5.4 Comparison between the Killing initial data conditions in the physical and unphysical pictures

The main advantage of the conformal (unphysical) approach to the Einstein field equations is that the conformal boundary $\mathscr{I}$ determined by $\Xi = 0$ is a submanifold of $(\mathcal{M}, \boldsymbol{g})$. This allows, in particular, to consider the $\Xi = 0$ hypersurface as a legitimate hypersurface to prescribe data which can be evolved using regular —without Ξ^{-1}-terms— evolution equations. This set up is particularly atractive to study spacetimes with $\lambda > 0$ in which —given the appropriate conditions— the conformal boundary $\mathscr{I}$ is a spacelike hypersurface and hence one can pose *an asymptotic initial value problem*: an initial value problem where the initial hypersurface is $\mathscr{I}$. On the other hand, the conformal (valence-1 Killing spinor) twistor conditions of proposition (1) allows to identify asymptotic initial data whose development will contain a twistor. Although the twistor case is too restrictive to characterise black hole spacetimes it is still illustrative to compare the derivation of the physical twistor initial data conditions on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and that leading to proposition (1).

For the twistor case, one important difference between discussion in [**?**] using the vacuum Einstein field equations in $(\mathcal{M}, \boldsymbol{g})$ is that the system closes with $\tilde{H}_{A'AB}$ alone and there is no need to introduce the analogous physical Buchdahl zero-quantity $\tilde{B}_{ABC}$. Therefore it is interesting to check if in the conformal case discussed in the previous sections one can also close the system with $H_{A'AB}$ alone.

Applying the D'Alembertian $\square$ to equation (19a), using the definition of the auxiliary quantity Q_A in equation (23a), a direct calculation exploiting the identities (3a)-(3b) and (4), gives

$$\begin{aligned}\square H_{A'AB} =& -2\Psi_{ABCD}H_{A'}{}^{CD} + 6\Lambda H_{A'AB} - 4\Phi_{(A}{}^{C}{}_{|A'}{}^{B'}H_{B'|B)C} \\ & -2\kappa_{(A}\nabla_{B)A'}\Lambda - 2\kappa^{C}\nabla_{(A}{}^{B'}\Phi_{B)CA'B'} + 2\kappa^{C}\nabla_{DA'}\Psi_{ABC}{}^{D} + \tfrac{4}{3}\nabla_{(A|A'|}Q_{B)}. \end{aligned} \tag{37}$$

If one were discussing the physical case —adding a tilde to every term in (37)— in which the fields are defined on $(\tilde{\mathcal{M}},\tilde{\boldsymbol{g}})$ which satisfies the vacuum Einstein field equations

$$\tilde{\Lambda} = 0, \qquad \tilde{\Phi}_{AA'BB'} = 0, \tag{38}$$

then, using the Bianchi identity $\tilde{\nabla}^{A}{}_{B'}\Psi_{ABCD} = \tilde{\nabla}^{A'}{}_{(B}\tilde{\Phi}_{CD)A'B'}$ and equations (38), the physical version of equation (37) —formally adding a tilde— reduces to

$$\tilde{\square}\tilde{H}_{A'AB} = -2\Psi_{ABCD}\tilde{H}_{A'}{}^{CD} + \tfrac{4}{3}\tilde{\nabla}_{(A|A'|}\tilde{Q}_{B)}. \tag{39}$$

Hence, imposing $\tilde{Q}_A = 0$, the system closes with $\tilde{H}_{A'AB}$ alone.

In the other hand, if one tries to follow the same strategy in the unphysical set up —with $(\mathcal{M},\boldsymbol{g})$ satisfying the CFEs— one ends up with a formally singular equation. To see this, observe that starting from the identity (37) and using equation (16) along the CFEs zero-quantities

$$\delta_{ABCC'} = 0, \qquad \Lambda_{CC'AB} = 0,$$

as defined in equations (15c)-(15d), a calculation gives

$$\begin{aligned}\square H_{A'AB} =& -\frac{2\nabla^{C}{}_{A'}\Xi\nabla_{(A}{}^{B'}H_{|B'|BC)}}{\Xi} + 6\Lambda H_{A'AB} - 2\Xi\phi_{ABCD}H_{A'}{}^{CD} - 4\Phi_{(A}{}^{C}{}_{|A'}{}^{B'}H_{B'|B)C} \\ & + \tfrac{4}{3}\nabla_{(A|A'|}Q_{B)}. \end{aligned} \tag{40}$$

Hence, setting $Q_A = 0$, renders an homogeneous but *singular equation* —due to the Ξ^{-1} coefficient— for $H_{A'AB}$, for which the theory of behind Theorem 1 does not apply. Arguably, one could try to use the theory of Fuchsian systems to see if the analogous of Theorem 1 applies for the singular equation (40). However, one of the advantages of the conformal approach of the CFEs respect to other approaches to include $\mathscr{I}$ is that one deals with formally regular equations. Therefore, from this perspective, it is preferable to work with explicitly regular equations and hence, it is necessary to introduce B_{ABC} as a further zero-quantity to be propagated. A analogous observation holds for the conformal valence-2 Killing spinor initial data discussion of the following sections, where, to close the system in a regular way, one needs to introduce not only the Buchdahl zero-quantity but also its derivative.

5.5 Intrinsic conformal twistor initial data conditions

In this section, the conformal twistor initial data conditions of proposition 1 are written in terms of intrinsic quantities at $\mathcal{S}$. To understand the need of the calculation to be carried out in this section observe that, although the conditions of proposition 1 are given on $\mathcal{S}$, they contains not only derivatives tangential to $\mathcal{S}$ but also normal to it. Hence, to obtain genuine intrinsic conditions on $\mathcal{S}$ one needs to remove these normal derivatives.

Introducing the following definitions:

$$\mathcal{H}_{ABC} \equiv \tau_{(A}{}^{A'}H_{|A'|BC)}, \qquad \mathcal{H}_{A} \equiv \tau^{QA'}H_{A'AQ}, \tag{41}$$

the space spinor split of $H_{A'AB}$ reads

$$H_{A'AB} = -\tfrac{1}{2}\tau^{C}{}_{A'}\mathcal{H}_{ABC} - \tfrac{1}{6}\tau^{C}{}_{A'}\mathcal{H}_{B}\epsilon_{AC} - \tfrac{1}{6}\tau^{C}{}_{A'}\mathcal{H}_{A}\epsilon_{BC}. \tag{42}$$

Hence, the space spinors $\mathcal{H}_{ABC}$ and $\mathcal{H}_A$ contain all the information of $H_{A'AB}$. In other words,

$$H_{A'ABC} = 0 \quad \Longleftrightarrow \quad \mathcal{H}_A = 0 \quad \& \quad \mathcal{H}_{ABC} = 0$$

Substituting the definition (19a) one obtains

$$\mathcal{H}_A = \tfrac{3}{2}\nabla_\tau \kappa_A - \mathcal{D}_{AB}\kappa^B, \qquad \mathcal{H}_{ABC} = 2\mathcal{D}_{(AB}\kappa_{C)}, \tag{43}$$

Then, $H_{A'AB}|_{\mathcal{S}} = 0$ imposes following conditions on the the initial data $(\kappa_A, \nabla_\tau\kappa_A)|_{\mathcal{S}}$ for the twistor candidate wave equation (36):

$$\nabla_\tau\kappa_A = \tfrac{2}{3}\mathcal{D}_{AB}\kappa^B, \qquad \mathcal{D}_{(AB}\kappa_{C)} = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{44}$$

Another set of constraints arise from the conditions $\nabla_\tau H_{A'BC}|_{\mathcal{S}} = 0$ and $B_{ABC}|_{\mathcal{S}} = 0$. These two conditions can be analysed in tandem since $\boldsymbol{B}$ is related to the derivative of $\boldsymbol{H}$. Using the space spinor split of ∇ it follows, exploting the identity (29), that

$$\tau_D{}^{A'}\nabla_\tau H_{A'AB} - 2\tau^{CA'}\mathcal{D}_{DC}H_{A'AB} = 4B_{ABD}\Xi + \tfrac{4}{3}Q_{(A}\epsilon_{B)D} \tag{45}$$

Transvecting with $\tau^D{}_{B'}$ and rearranging gives

$$\nabla_\tau H_{B'AB} = -4B_{ABD}\Xi\tau^D{}_{B'} - 2\tau^{CA'}\tau^D{}_{B'}\mathcal{D}_{DC}H_{A'AB} - \tfrac{4}{3}\tau^D{}_{B'}Q_{(A}\epsilon_{B)D} \tag{46}$$

Hence, if the the twistor candidate wave equation is imposed, namely $Q_A = 0$, then

$$H_{A'AB}|_{\mathcal{S}} = 0 \quad \& \quad B_{ABC}|_{\mathcal{S}} = 0 \implies \nabla_\tau H_{A'AB}|_{\mathcal{S}} = 0. \tag{47}$$

In other words, imposing $\nabla_\tau H_{A'AB}|_{\mathcal{S}} = 0$ is redundant if $H_{A'AB}|_{\mathcal{S}} = 0$ and $B_{ABC}|_{\mathcal{S}} = 0$ are satisfied. Using the definition (19b), the condition $B_{ABC}|_{\mathcal{S}} = 0$ simply reads,

$$\phi_{ABCD}\kappa^D = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{48}$$

Finally, for the condition $\nabla_\tau B_{ABC}|_{\mathcal{S}} = 0$ one has, applying ∇_τ to equation (19b) that

$$\nabla_\tau B_{ABC} = \phi_{ABCD}\nabla_\tau\kappa^D + \kappa^D\nabla_\tau\phi_{ABCD}. \tag{49}$$

At this point one can exploit the evolution equation for the rescaled Weyl spinor (17) to subsitute for $\nabla_\tau\phi_{ABCD}$ and using equation (43) to substitute $\nabla_\tau\kappa_A$ when evaluating at $\mathcal{S}$. This substitution renders,

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -2\kappa^D\mathcal{D}_{DF}\phi_{ABC}{}^F + \tfrac{2}{3}\phi_{ABCD}\mathcal{D}^D{}_F\kappa^F + \tfrac{2}{3}\phi_{ABCD}\mathcal{H}^D = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{50}$$

In fact, the latter expression can be completly rewritten in terms of $\mathcal{H}_A|_{\mathcal{S}}$, $\mathcal{H}_{ABC}|_{\mathcal{S}}$ and $B_{ABC}|_{\mathcal{S}}$. This can be done as follows: swapping indices D and A in equation (50), and exploiting the constraint equation for the rescaled Weyl spinor in expression (17) gives

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -2\kappa^D\mathcal{D}_{AF}\phi_{DBC}{}^F + \tfrac{2}{3}\phi_{ABCD}\mathcal{D}^D{}_F\kappa^F + \tfrac{2}{3}\phi_{ABCD}\mathcal{H}^D = 0 \qquad \text{on} \qquad \mathcal{S}. \tag{51}$$

Applying a $\mathcal{D}_{FQ}$ to the definition (19b) and using the Leibnitz rule, one can replace the first term in the last equation to obtain

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -2\mathcal{D}_{AD}B_{BC}{}^D - 2\phi_{BCDF}\mathcal{D}_A{}^F\kappa^D + \tfrac{2}{3}\phi_{ABCF}\mathcal{D}_D{}^F\kappa^D + \tfrac{2}{3}\phi_{ABCD}\mathcal{H}^D = 0 \quad \text{on} \quad \mathcal{S}. \tag{52}$$

From the irreducible decomposition of $\mathcal{D}_{AB}\kappa_C$ and using the expression for $\mathcal{H}_{ABC}$ equation (43) one has

$$\mathcal{D}_{AB}\kappa_C = \tfrac{1}{2}\mathcal{H}_{ABC} + \tfrac{1}{3}\epsilon_{BC}\mathcal{D}_{AD}\kappa^D + \tfrac{1}{3}\epsilon_{AC}\mathcal{D}_{BD}\kappa^D. \tag{53}$$

Substituting decomposition (53) into equation (52) renders

$$\nabla_\tau B_{ABC}|_{\mathcal{S}} = -\phi_{BCDF}\mathcal{H}_A{}^{DF} - 2\mathcal{D}_{AD}B_{BC}{}^D + \tfrac{2}{3}\phi_{ABCD}\mathcal{H}^D = 0 \quad \text{on} \quad \mathcal{S} \tag{54}$$

Hence, overall, the only independent conditions to be imposed are $H_{A'AB}|_{\mathcal{S}} = 0$ and $B_{ABC}|_{\mathcal{S}} = 0$.

The given in this section can be summarised in the following

Theorem 2. *Consider an initial data set for the vacuum conformal Einstein field equations, as encoded in the CFE zero-quantities* (15a)-(15e)*, on a spacelike hypersurface $\mathcal{S}$ and let $\mathcal{U} \subset \mathcal{S}$ denote an open set. The development of the initial data set will have a twistor (valence-1 Killing spinor) in the domain of dependence of $\mathcal{U}$ if and only if*

$$\mathcal{D}_{(AB}\kappa_{C)} = 0, \qquad \phi_{ABCD}\kappa^D = 0, \tag{55}$$

are satisfied on $\mathcal{U}$. The twistor is obtained evolving according to the wave equation:

$$\square\kappa_A = -2\Lambda\kappa_A. \tag{56}$$

with initial data satisfying conditions in equation (55) *and*

$$\nabla_\tau\kappa_A = \tfrac{2}{3}\mathcal{D}_{AB}\kappa^B. \tag{57}$$

Proof. The analysis of the last subsection shows that conditions

$$H_{A'AB} = 0, \quad \nabla_\tau H_{A'AB} = 0, \quad B_{ABC} = 0, \quad \nabla_\tau B_{ABC} = 0$$

on $\mathcal{U} \subset \mathcal{S}$ are equivalent to the conditions (55). Hence, using Proposition 1 one concludes that if equations (55) hold on $\mathcal{U}$, then the domain of dependence of $\mathcal{U}$ is endowed with a twistor. □

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6 Conformal Killing spinor initial data

6.1 Deriving a wave system

Analogous to the twistor case, we define the zero quantities:

$$H_{A'ABC} = 3\nabla_{A'(A}\kappa_{BC)}, \qquad B_{ABCD} = \kappa_{(A}{}^Q\phi_{BCD)Q}.$$

A short computation shows that

$$\nabla_{(A}{}^{A'}H_{|A'|BCD)} = 6\Xi B_{ABCD}. \tag{58}$$

However, we will see that, unlike the twistor case, one cannot obtain a closed homogeneous wave system in terms of these variables alone. We also define the zero quantity

$$F_{A'BCD} := \nabla^A{}_{A'}B_{ABCD}.$$

We shall see that one can derive a closed homogeneous wave system for $(\boldsymbol{h}, \boldsymbol{B}, \boldsymbol{F})$. •4

•4: *Add factor of one half into def of Q.*

$$Q_{BC} := \nabla^{AA'}H_{A'ABC} \equiv 2(\square\kappa_{BC} + 4\Lambda\kappa_{BC} - \Xi\kappa^{AD}\phi_{BCAD}) = 0. \tag{59}$$

The wave equation for the Killing spinor candidate will then be imposed by setting $Q_{AB} = 0$. It will also prove convenient to define the auxiliary spinor $\xi_{AA'} := \nabla^B{}_{A'}\kappa_{AB}$ in terms of which

$$\nabla_{AA'}\kappa_{BC} = \tfrac{1}{3}H_{A'ABC} - \tfrac{1}{3}\xi_{CA'}\epsilon_{AB} - \tfrac{1}{3}\xi_{BA'}\epsilon_{AC}. \tag{60}$$

Using this, we can show by a straightforward computation that

$$\nabla^Q{}_{A'}H_{B'ABQ} + \tfrac{1}{2}Q_{AB}\epsilon_{A'B'} = \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} + 6\kappa_{(A}{}^Q\Phi_{B)QA'B'}.$$

As remarked in Section ?, $\xi_{AA'}$ is not a Killing vector, in general, but is in the case that the spacetime is physical. From the above, we can derive the following identity

$$\begin{aligned}\nabla_{AA'}\xi_{BB'} = -\tfrac{1}{2}\epsilon_{AB}\nabla^C{}_{(A'}\xi_{|C|B')} - 3\kappa_{(A}{}^C\Phi_{B)CA'B'} - 3\Lambda\kappa_{AB}\epsilon_{A'B'}\\ + \tfrac{3}{4}\Xi\kappa^{CD}\phi_{ABCD}\epsilon_{A'B'} + \tfrac{1}{8}Q_{AB}\epsilon_{A'B'} + \tfrac{1}{2}\nabla^Q{}_{(A'}H_{B')ABC},\end{aligned} \tag{61}$$

which will prove useful later. On the other hand, it is straightforward to show using (58) that

$$\nabla_D{}^{A'} H_{A'ABC} = 6\Xi B_{ABCD} + \tfrac{3}{4}\, Q_{(AB}\epsilon_{C)D}. \tag{62}$$

Contracting with $\nabla^A{}_{B'}$ we then derive the following wave equation:

$$\begin{aligned}\square H_{B'ABC} = 6\Lambda H_{B'ABC} - 12\Xi F_{B'ABC} - 12 B_{ABCD}\nabla^D{}_{B'}\Xi \\ + \tfrac{3}{2}\nabla_{(A|B'|}Q_{BC)} - 6\Phi_{(A}{}^D{}_{|B'}{}^{A'} H_{A'|BC)D}\end{aligned} \tag{63}$$

Similarly, substituting the definition of $F_{A'ABC}$ in terms of B_{ABCD}, it is straightforward to verify the following wave equation for B_{ABCD}: •5

•5: *I haven't carried the **Q** quantities*

$$\square B_{ABCD} = 12\Lambda B_{ABCD} - 6\Xi \phi_{(AB}{}^{FG} B_{CD)FG} + 2\nabla_{AA'}F^{A'}{}_{BCD}. \tag{64}$$

The task remaining is to derive a wave equation for $F_{A'ABC}$. Let us first consider some useful identities:

$$\begin{aligned}2\phi_{(AB}{}^{GH}B_{CF)GH} = \kappa_A{}^D \phi_{(BC}{}^{GH}\phi_{FD)GH} + \kappa_B{}^D\phi_{(AC}{}^{GH}\phi_{FD)GH} \\ + \kappa_C{}^D\phi_{(AB}{}^{GH}\phi_{FD)GH} + \kappa_F{}^D\phi_{(BC}{}^{GH}\phi_{AD)GH}.\end{aligned} \tag{65}$$

Using this, along with the irreducible decomposition

$$\begin{aligned}\phi_{ABCD}\phi_{FGH}{}^D = {} & \tfrac{1}{24}\phi_{DLMP}\phi^{DLMP}\epsilon_{AH}\epsilon_{BG}\epsilon_{CF} + \tfrac{1}{24}\phi_{DLMP}\phi^{DLMP}\epsilon_{AG}\epsilon_{BH}\epsilon_{CF} \\ & + \tfrac{1}{24}\phi_{DLMP}\phi^{DLMP}\epsilon_{AH}\epsilon_{BF}\epsilon_{CG} + \tfrac{1}{24}\phi_{DLMP}\phi^{DLMP}\epsilon_{AF}\epsilon_{BH}\epsilon_{CG} \\ & + \tfrac{1}{24}\phi_{DLMP}\phi^{DLMP}\epsilon_{AG}\epsilon_{BF}\epsilon_{CH} + \tfrac{1}{24}\phi_{DLMP}\phi^{DLMP}\epsilon_{AF}\epsilon_{BG}\epsilon_{CH} \\ & + \tfrac{1}{6}\epsilon_{CH}\phi_{(AB}{}^{DL}\phi_{FG)DL} + \tfrac{1}{6}\epsilon_{CG}\phi_{(AB}{}^{DL}\phi_{FH)DL} + \tfrac{1}{6}\epsilon_{CF}\phi_{(AB}{}^{DL}\phi_{GH)DL} \\ & + \tfrac{1}{6}\epsilon_{BH}\phi_{(AC}{}^{DL}\phi_{FG)DL} + \tfrac{1}{6}\epsilon_{BG}\phi_{(AC}{}^{DL}\phi_{FH)DL} + \tfrac{1}{6}\epsilon_{BF}\phi_{(AC}{}^{DL}\phi_{GH)DL} \\ & + \tfrac{1}{6}\epsilon_{AH}\phi_{(BC}{}^{DL}\phi_{FG)DL} + \tfrac{1}{6}\epsilon_{AG}\phi_{(BC}{}^{DL}\phi_{FH)DL} + \tfrac{1}{6}\epsilon_{AF}\phi_{(BC}{}^{DL}\phi_{GH)DL},\end{aligned}$$

we derive the identity

$$\kappa^{DG}\phi_{(ABC}{}^H\phi_{F)HDG} = 2\phi_{(AB}{}^{GH}B_{CF)GH}. \tag{66}$$

Using the definition of the Buchdahl zero quantity, using the CFE for ϕ_{ABCD}, $\Lambda_{CC'AB} = 0$, along with its wave equation (18), and using the decomposition (60) we get

$$\begin{aligned}\square B_{ABCD} = \tfrac{2}{3}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} + 8\Lambda B_{ABCD} - \tfrac{1}{2}\phi_{(ABC}{}^F Q_{D)F} + \tfrac{2}{3}H^{A'}{}_{(A}{}^{FG}\nabla_{|FA'|}\phi_{BCD)G} \\ - 6\Xi\kappa_{(A}{}^F\phi_{BC}{}^{GH}\phi_{D)FGH} - \Xi\kappa^{FG}\phi_{(ABC}{}^H\phi_{D)FGH}.\end{aligned} \tag{67}$$

Substituting the above identities (65)–(66), we can derive the following alternative (non-homogeneous) wave equation for B_{ABCD}:

$$\square B_{ABCD} = \tfrac{2}{3}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} + 8\Lambda B_{ABCD} - 14\Xi\phi_{(AB}{}^{FG}B_{CD)FG} + \tfrac{2}{3}(\nabla_{FA'}\phi_{G(ABC})H^{A'}{}_{D)}{}^{FG} \tag{68}$$

Remark 4. •6 *The Lie derivative doesn't extend to spinor fields, but let us define*

•6: *Check that the signs are correct*

$$\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD} = \nabla_{\boldsymbol{\xi}}\phi_{ABCD} + \phi_{F(ABC}\nabla_{D)A'}\xi^{FA'},$$

this is simply the spinorialised counterpart of $\mathcal{L}_{\boldsymbol{\xi}}d_{abcd}$. One can then show the following:

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD} &\equiv \nabla_{\boldsymbol{\xi}}\phi_{ABCD} + \phi_{F(ABC}\nabla_{D)A'}\xi^{FA'} \\ &= \nabla_{\boldsymbol{\xi}}\phi_{ABCD} - 6\Lambda\kappa_{(D}{}^F\phi_{ABC)F} - \tfrac{3}{2}\Xi\,\kappa^{FG}\phi_{(ABC}{}^H\phi_{D)FGH} + \tfrac{1}{4}\phi_{F(ABC}\,Q_{D)}{}^F \\ &= \nabla_{\boldsymbol{\xi}}\phi_{ABCD} - 6\Lambda B_{ABCD} - 3\Xi\phi_{(AB}{}^{FG}B_{CD)FG} + \tfrac{1}{4}\phi_{F(ABC}Q_{D)}{}^F\end{aligned}$$

where we are using (61)*, along with the identity*

$$\kappa^{DG}\phi_{(ABC}{}^{H}\phi_{F)DGH} = 2\ B_{(AB}{}^{DG}\phi_{CF)DG}.$$

Given a Killing spinor κ_{AB}, we have $B_{ABCD} = H_{A'ABC} = Q_{AB} = 0$ and hence it follows from (68) *that*

$$\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD} = \nabla_{\boldsymbol{\xi}}\phi_{ABCD} = 0.$$

In particular, we recover the well-known fact that, in the physical spacetime $\mathcal{L}_{\tilde{\boldsymbol{\xi}}}\tilde{C}_{abcd} = 0$ as $\tilde{\xi}_{AA'}$ is a Killing vector.

At this point we note that there are no derivatives of zero quantities appearing on the right-hand-side of (68). This, combined with the fact that $\nabla^{A}{}_{A'}\phi_{ABCD} = 0$, suggests that by applying $\nabla^{A}{}_{A'}$ to (68) we will be able to derive a wave equation for $F_{A'BCD}$. We will see that this does indeed work; the only essential difficulty is in showing that $\nabla^{A}{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD}$ is expressible in terms of the zero quantities $\boldsymbol{H}, \boldsymbol{B}, \boldsymbol{F}$ and their first derivatives only. First, we note the following identities:

$$\kappa^{CD}\nabla_{DG'}\phi_{ABFC} = 2\xi^{C}{}_{G'}\phi_{ABFC} + \phi_{CD(AB}H_{|G'|F)}{}^{CD} - 4F_{G'ABF} \tag{69}$$

which can be used to derive the more general identity

$$\begin{aligned}
\kappa_{A}{}^{F}\nabla_{FF'}\phi_{BCDG} = \kappa_{A}{}^{F}\nabla_{DF'}\phi_{BCGF}& \\
= \tfrac{1}{3}\xi_{AF'}\phi_{BCDG} - \tfrac{1}{3}\xi^{F}{}_{F'}\phi_{BCGF}\epsilon_{AD} - \tfrac{1}{3}\xi^{F}{}_{F'}\phi_{(BC|DF}\epsilon_{A|G)}& \\
- \tfrac{2}{3}\xi^{F}{}_{F'}\phi_{(B|D|C|F}\epsilon_{A|G)} + \epsilon_{AD}F_{F'BCG} + \nabla_{(B|F'}B_{A|CG)D} + \tfrac{1}{3}\phi_{(BC|D}{}^{F}H_{F'A|G)F}& \\
+ 2\epsilon_{A(B}F_{|F'|CG)D} - \tfrac{1}{3}\phi_{(BC}{}^{FH}H_{|F'|G)FH}\epsilon_{AD} - \tfrac{1}{6}\phi_{(BC}{}^{FH}H_{|F'DFH}\epsilon_{A|G)}& \\
- \tfrac{1}{3}\phi_{(B|D}{}^{FH}H_{F'|C|FH}\epsilon_{A|G)} - \tfrac{1}{6}\phi_{D(B}{}^{FH}H_{|F'|C|FH}\epsilon_{A|G)}.&
\end{aligned} \tag{70}$$

Using the irreducible decomposition

$$\phi_{ABCD}\phi_{FG}{}^{CD} = \tfrac{1}{6}\phi_{CDHL}\phi^{CDHL}\epsilon_{AG}\epsilon_{BF} + \tfrac{1}{6}\phi_{CDHL}\phi^{CDHL}\epsilon_{AF}\epsilon_{BG} + \phi_{(AB}{}^{CD}\phi_{FG)CD} \tag{71}$$

we can show that

$$\begin{aligned}
\kappa^{AD}\phi_{AD}{}^{GH}\nabla_{HA'}\phi_{BCFG} &= 4\xi^{A}{}_{A'}\phi_{(BC}{}^{DG}\phi_{F)ADG} + \tfrac{1}{2}\phi_{ADGH}\phi^{ADGH}H_{A'BCF} \\
&\quad - 4B_{(B}{}^{ADG}\nabla_{|AA'|}\phi_{CF)DG} - 8\phi_{(BC}{}^{AD}F_{|A'|F)ADG} \\
&\quad - 4\phi_{(B}{}^{ADG}\nabla_{|AA'|}B_{CF)DG} - \tfrac{1}{3}\phi_{(BC}{}^{AD}\phi_{|AD}{}^{GH}H_{A'|F)GH} \\
&\quad - \tfrac{2}{3}\phi_{(B}{}^{ADG}\phi_{C|AD}{}^{H}H_{A'|F)GH}
\end{aligned} \tag{72}$$

We can then derive•7 ...

•7: *Check the last equality —had to typeset manually*

$$\begin{aligned}
\nabla^{A}{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} &= \nabla_{\boldsymbol{\xi}}(\nabla^{A}{}_{A'}\phi_{ABCD}) + (\nabla^{A}{}_{A'}\xi^{FF'})\nabla_{FF'}\phi_{ABCD} + \xi^{FF'}\left[\nabla^{A}{}_{A'}, \nabla_{FF'}\right]\phi_{ABCD} \\
&= (\nabla^{A}{}_{A'}\xi^{FF'})\nabla_{FF'}\phi_{ABCD} + \xi^{FF'}\left[\nabla^{A}{}_{A'}, \nabla_{FF'}\right]\phi_{ABCD} \\
&= 6\Lambda\xi^{D}{}_{A'}\phi_{ABCD} - \xi^{DF'}\Phi_{D}{}^{F}{}_{A'F'}\phi_{ABCF} - 3\xi^{DF'}\Phi_{(A}{}^{F}{}_{|A'F'|}\phi_{BC)DF} \\
&\quad - 3\Xi\xi^{D}{}_{A'}\phi_{DFG(A}\phi_{BC)}{}^{FG} - 3\Lambda\kappa^{DF}\nabla_{FA'}\phi_{ABCD} + \tfrac{1}{8}Q^{DF}\nabla_{FA'}\phi_{ABCD} \\
&\quad - \tfrac{3}{2}\kappa^{DF}\Phi_{D}{}^{G}{}_{A'}{}^{F'}\nabla_{FF'}\phi_{ABCG} - \tfrac{3}{2}\kappa^{DF}\Phi_{D}{}^{G}{}_{A'}{}^{F'}\nabla_{GF'}\phi_{ABCF} \\
&\quad + \tfrac{3}{4}\Xi\kappa^{DF}\phi_{DF}{}^{GH}\nabla_{HA'}\phi_{ABCG} - \tfrac{1}{2}(\nabla_{FF'}\phi_{ABCD})\nabla_{Q(A'}H_{F')F}{}^{DQ} \\
&= -3\Phi_{A}{}^{D}{}_{A'}{}^{F'}F_{F'BCD} - \tfrac{3}{8}\Xi\phi_{DFGH}\phi^{DFGH}H_{A'ABC} + \Lambda\phi_{BCDF}H_{A'A}{}^{DF} \\
&\quad - \tfrac{1}{8}Q^{DF}\nabla_{FA'}\phi_{ABCD} - \tfrac{3}{2}\Phi^{DF}{}_{A'}{}^{F'}\nabla_{FF'}B_{ABCD} + \tfrac{1}{4}(\nabla_{DA'}H^{F'DFG})\nabla_{GF'}\phi_{ABCF} \\
&\quad + \tfrac{1}{4}(\nabla_{D}{}^{F'}H_{A'}{}^{DFG})\nabla_{GF'}\phi_{ABCF} + 12\Lambda F_{A'ABC} + 6\Xi\nabla^{G}{}_{A'}\left(B_{(AB}{}^{DF}\phi_{CG)DF}\right) \\
&\quad - \tfrac{9}{2}\Phi_{(B}{}^{D}{}_{|A'}{}^{F'}F_{F'A|CD)} - \tfrac{3}{2}\Phi^{DF}{}_{A'}{}^{F'}\nabla_{(B|F'}B_{A|CD)F} + \tfrac{1}{4}\Xi\phi_{(BC}{}^{GH}\phi_{DF)GH}H_{A'A}{}^{DF} \\
&\quad + 2\Lambda\phi_{DF(AB}H_{|A'|C)}{}^{DF} + 3\Xi\phi_{(BC}{}^{DF}F_{|A'A|D)F} + 6\Xi\phi_{A(B}{}^{DF}F_{|A'|CD)F} \\
&\quad + \tfrac{3}{8}\Phi_{(B}{}^{D}{}_{|A'|}{}^{F'}\phi_{CD)}{}^{FG}H_{F'AFG} + \tfrac{9}{8}\Phi_{(B}{}^{D}{}_{|A'}{}^{F'}\phi_{A|C}{}^{FG}H_{|F'|D)FG}
\end{aligned}$$

$$
\begin{aligned}
&+ \Phi_A{}^D{}_{A'}{}^{F'}\phi_{(BC}{}^{FG}H_{|F'|D)FG} - \tfrac{1}{2}\Phi^{DF}{}_{A'}{}^{F'}\phi_{A(BC}{}^{G}H_{|F'|D)FG} \\
&- \tfrac{1}{2}\Phi^{DF}{}_{A'}{}^{F'}\phi_{A(B|D}{}^{G}H_{F'|CF)G} + \tfrac{1}{6}\Xi H_{A'A}{}^{DF}\phi_{(BC}{}^{GH}\phi_{DF)GH} \\
&+ \tfrac{1}{6}\Xi H_{A'B}{}^{DF}\phi_{(CA}{}^{GH}\phi_{DF)GH} + \tfrac{1}{6}\Xi H_{A'C}{}^{DF}\phi_{(AB}{}^{GH}\phi_{DF)GH} \qquad (73)
\end{aligned}
$$

where the third equality follows from (61), $\Lambda_{CC'AB} = 0$ and by expanding the commutator; the fourth equality follows from (70)–(72). Note that the final expression is homogeneous in the zero quantities $(\boldsymbol{H}, \boldsymbol{B}, \boldsymbol{F})$ and their first derivatives.•8

•8: *This isn't necessarily in its simplest form.*

Then,

$$
\begin{aligned}
\Box F_{A'BCD} &= \Box(\nabla^A{}_{A'}B_{ABCD}) \\
&= \left[\Box, \nabla^A{}_{A'}\right] B_{ABCD} + \nabla^A{}_{A'}\Box B_{ABCD} \\
&= \left[\Box, \nabla^A{}_{A'}\right] B_{ABCD} + \tfrac{2}{3}\nabla^A{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} \\
&\quad + \nabla^A{}_{A'}\left(8\Lambda B_{ABCD} - 14\Xi\phi_{(AB}{}^{FG}B_{CD)FG} + \tfrac{2}{3}(\nabla_{FA'}\phi_{G(ABC})H^{A'}{}_{D)}{}^{FG}\right) \\
&= \tfrac{2}{3}\nabla^A{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} - 6\Lambda F_{A'BCD} - 6\Phi_{(B}{}^{A}{}_{|A'}{}^{B'}F_{B'|CD)A} - 9B_{BCDA}\nabla^A{}_{A'}\Lambda \\
&\quad + 3B_{(BC}{}^{FG}\phi_{D)AFG}\nabla^A{}_{A'}\Xi + 3B_{AF(BC}\nabla^{FB'}\Phi_{D)}{}^{A}{}_{A'B'} - 6\Xi\phi_{(B}{}^{AFG}\nabla_{|GA'|}B_{CD)AF} \\
&\quad + \nabla^A{}_{A'}\left(8\Lambda B_{ABCD} - 14\Xi\phi_{(AB}{}^{FG}B_{CD)FG} + \tfrac{2}{3}(\nabla_{FA'}\phi_{G(ABC})H^{A'}{}_{D)}{}^{FG}\right),
\end{aligned}
$$

where we are using (68) in the third line and in the fourth we are expanding out the commutator and using the Bianchi identites. Substituting (73), we obtain a homogeneous expression in $(\boldsymbol{H}, \boldsymbol{B}, \boldsymbol{F})$ and their first derivatives, as required, and we are done. •9

•9: *Simplifications possible.*

Proposition 2. *Given initial data for the conformal field equations on $\mathcal{U} \subseteq \mathcal{S}$ where $\mathcal{S}$ is a spacelike hypersurface $\mathcal{S}$ with normal vector $\tau^{AA'}$, and associated normal derivative $\nabla_\tau \equiv \tau^{AA'}\nabla_{AA'}$, the corresponding spacetime development admits a (valence-2) Killing spinor in $\mathcal{D}^+(\mathcal{U})$ —the future domain of dependence of $\mathcal{U}$— if and only if*

$$ H_{A'ABC} = B_{ABCD} = F_{A'ABC} = 0 \qquad (74a) $$
$$ \nabla_\tau H_{A'ABC} = \nabla_\tau B_{ABCD} = \nabla_\tau F_{A'ABC} = 0. \qquad (74b) $$

hold on $\mathcal{U}$.

6.2 Intrinsic conditions

In this section, we aim to reduce (74a)–(74b) to a set of intrinsic conditions —that is to say, conditions on κ_{AB} and its spatial derivatives that are computable at the level of initial data. We consider κ_{AB} defined over $\mathcal{S}$ satisfying

$$ \tau_{(A}{}^{A'}H_{|A'|BCD)} \equiv \mathcal{D}_{(AB}\kappa_{CD)} = 0, \qquad (75a) $$
$$ B_{ABCD} \equiv \kappa_{(A}{}^{F}\phi_{BCD)F} = 0 \qquad (75b) $$

on $\mathcal{S}$. We evolve the Killing spinor candidate according to (59) and the initial condition

$$ \nabla_{\boldsymbol{\tau}}\kappa_{BC} = -\xi_{BC} \qquad (76) $$

on $\mathcal{S}$, with $\xi_{AB} := \mathcal{D}_{(A}{}^{C}\kappa_{B)C}$. This can be easily shown to be equivalent to $\tau^{AA'}H_{A'ABC} = 0$, and so (76) and (75a) together encode $H_{A'ABC} = 0$ on $\mathcal{S}$. Decomposing the covariant derivative, we have on $\mathcal{S}$

$$
\begin{aligned}
\xi_{AB'} = -\tau_B{}^{A'}\tau^B{}_{B'}\xi_{AA'} &= \tfrac{1}{2}\tau^B{}_{B'}\nabla_{\boldsymbol{\tau}}\kappa_{AB} + \tau^B{}_{B'}\mathcal{D}_{BC}\kappa_A{}^C. \\
&= \tfrac{1}{2}\tau^B{}_{B'}\nabla_{\boldsymbol{\tau}}\kappa_{AB} - \tau^B{}_{B'}\xi_{AB} + \xi\tau_{AB'} \\
&= -\tfrac{3}{2}\tau^B{}_{B'}\xi_{AB} + \xi\tau_{AB'} \qquad (77)
\end{aligned}
$$

where in the last line we are using (76). Starting from (62), and performing the decomposition of the covariant derivative, we get

$$\nabla_{\boldsymbol{\tau}} H_{B'ABC} + 2\tau^{D}{}_{B'}\tau^{FA'}\mathcal{D}_{DF}H_{A'ABC} = -12\Xi\tau^{D}{}_{B'}B_{ABCD} + \tfrac{3}{2}\tau_{(A|B'|}Q_{BC)}.$$

At this point we see that if ()–() hold then $\nabla_{\boldsymbol{\tau}} H_{B'ABC} = 0$ on $\mathcal{S}$. Now we move onto B_{ABCD}. A similar computation to the twistor case yields

$$\nabla_{\boldsymbol{\tau}} B_{ABCD} = 2\kappa_{(A}{}^{F}\mathcal{D}_{B}{}^{G}\phi_{CD)FG} + \phi_{(ABC}{}^{F}\xi_{D)F} \tag{78}$$

Note that the quantity on the right-hand-side is intrinsic to $\mathcal{S}$; we add it to our list of conditions (for the meantime). Consider now the quantity $F_{A'ABC}$; by definition, $F_{A'ABC} = \nabla^{D}{}_{A'}B_{ABCD}$ and so, decomposing the covariant derivative

$$F_{A'BCD} = \tfrac{1}{2}\tau^{A}{}_{A'}\nabla_{\boldsymbol{\tau}} B_{ABCD} - \tau^{F}{}_{A'}\mathcal{D}^{A}{}_{F}B_{ABCD}.$$

Hence, given the vanishing of B_{ABCD} and the right-hand-side of (78), we see that $F_{A'ABC} = 0$ on $\mathcal{S}$, too. Thus, all that's left is to find intrinsic conditions under which $\nabla_{\boldsymbol{\tau}} F_{A'ABC} = 0$ on $\mathcal{S}$. Now, combining our two wave equations for B_{ABCD}, (64) and (68), we see that

$$\begin{aligned}\nabla_{FA'}F^{A'}{}_{BCD} = \tfrac{1}{3}\nabla_{\xi}\phi_{BCDF} - 2\Lambda B_{BCDF} + 3\Xi B_{(BC}{}^{AG}\phi_{D)FAG} - 7\Xi B_{(BC}{}^{AG}\phi_{DF)AG}\\ - \tfrac{1}{4}\phi_{(BCD}{}^{A}Q_{F)A} + \tfrac{1}{3}H^{A'}{}_{(B}{}^{AG}\nabla_{|AA'|}\phi_{CDF)G}.\end{aligned} \tag{79}$$

Decomposing the covariant derivative, we have

$$\begin{aligned}\nabla_{\boldsymbol{\tau}} F_{B'BCD} &+ 2\tau^{A}{}_{B'}\tau^{FA'}\mathcal{D}_{AF}F_{A'BCD}\\ &= \tau^{A}{}_{B'}\big(\tfrac{2}{3}\nabla_{\xi}\phi_{BCDA} - 4\Lambda B_{BCDA} + 6\Xi B_{(BC}{}^{FG}\phi_{D)AFG}\\ &-14\Xi B_{(BC}{}^{AG}\phi_{DA)FG} - \tfrac{1}{2}\phi_{(BCD}{}^{F}Q_{A)F} + \tfrac{2}{3}H^{A'}{}_{(B}{}^{FG}\nabla_{|FA'|}\phi_{CDA)G}\Big).\end{aligned} \tag{80}$$

Hence, if $F_{A'ABC} = H_{A,ABC} = B_{ABCD} = 0$ on $\mathcal{S}$, then we will also have $\nabla_{\boldsymbol{\tau}} F_{A'ABC}\big|_{\mathcal{S}} = 0$ if and only if

$$\nabla_{\boldsymbol{\xi}}\phi_{ABCD}\big|_{\mathcal{S}} = 0.$$

Decomposing the covariant derivative, using the evolution equation for rescaled Weyl (17), along with (77) we have

$$\nabla_{\boldsymbol{\xi}}\phi_{ABCD} = \xi\mathcal{D}_{(A}{}^{F}\phi_{BCD)F} + \tfrac{3}{2}\xi^{FG}\mathcal{D}_{FG}\phi_{ABCD}$$

with $\xi := \mathcal{D}^{AB}\kappa_{AB}$•10 . At this point then, we have reduced our initial conditions for the zero quantities to the following set of intrinsic conditions: •10: give decomposition of $\xi_{AA'}$

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0, \tag{81a}$$
$$\kappa_{(A}{}^{F}\phi_{BCD)F} = 0, \tag{81b}$$
$$2\kappa_{(A}{}^{F}\mathcal{D}_{B}{}^{G}\phi_{CD)FG} + \phi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{81c}$$
$$\xi^{FG}\mathcal{D}_{FG}\phi_{ABCD} + \tfrac{2}{3}\xi\mathcal{D}_{(A}{}^{F}\phi_{BCD)F} = 0. \tag{81d}$$

We will now explore potential redundancy in the above conditions...

Conclusions

In this article a *conformal* version of the Killing spinor initial data equations given in [**?**] are derived. By conformal it is understood that $(\mathcal{M}, \boldsymbol{g})$ is conformally related to an Einstein spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Consequently, we call these conditions the *conformal Killing spinor initial data equations.* The existence of a non-trivial solution of this system of equations is a necessary and sufficient condition for the existence of a Killing spinor on the development. The conditions are intrinsic to a spacelike hypersurface $\mathcal{S} \subset \mathcal{M}$. In the case where the conformal rescaling is trivial, $\Xi = 1$, the conditions reduce to those given in [?]. These conditions contain one differential condition and two algebraic conditions. The differential condition corresponds to the so-called *spatial Killing spinor equation*. Notice that the conformal approach followed in this article —i.e., use of the conformal Einstein field equations— opens the possibility to allow $\mathcal{S}$ to be determined by $\Xi = 0$ so that it to corresponds to the conformal boundary $\mathscr{I}$. The analysis given in this article already shows that in a potential characterisation of the Kerr-de Sitter spacetime, via the existence of Killing spinors at the (spacelike) conformal boundary.

Nonetheless, future applications are not restricted to the analysis of de-Sitter like spacetimes. To see this, notice that, the most delicate part of the analysis consisted on finding a system of homogeneous wave equations for $H_{A'ABC}$ and B_{ABCD} and $F_{A'ABC}$. This system of wave equations in turn, leads to conditions (**??**)-(**??**) which are irrespective of the causal nature of $\mathcal{S}$. Consequently, one could investigate the analogous conditions to those derived in Section **??** considering a timelike or null hypersurface $\mathcal{S}$ instead. In the latter case one could consider the conformal boundary of an asymptotically flat spacetime. In the case of a timelike hypersurface $\mathcal{S}$, the analogous conditions could be useful for the analysis of anti-de Sitter like spacetimes.

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