Matter alignment condition notes

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Abstract

Note on the matter alignment condition

1 Matter alignment condition

The physical Einstein-Maxwell equations read:

$$\tilde{\Phi}_{ABA'B'} = 2\tilde{\phi}_{AB}\bar{\tilde{\phi}}_{A'B'}$$

where $\tilde{\Phi}_{ABA'B'}$ and $\tilde{\phi}_{AB}$ denote respectively the physical trace-free Ricci spinor and the physical Maxwell spinor. The physical matter alignment zero-quantity defined as

$$\tilde{\Theta}_{AB} := 2\tilde{\kappa}_{(A}{}^{Q}\tilde{\phi}_{B)Q}$$

The relation between the physical Killing spinor, the physical Maxwell spinor and their unphysical counterparts is given by

$$\kappa_{AB} = \Xi^2\tilde{\kappa}_{AB}, \qquad \phi_{AB} = \Xi^{-1}\tilde{\phi}_{AB}$$

Hence, using that the relation between the physical and unphysical ϵ spinors is

$$\epsilon_{AB} = \Xi\tilde{\epsilon}_{AB}, \qquad \epsilon^{AB} = \Xi^{-1}\tilde{\epsilon}^{AB}$$

and spinor indices of physical quantities, should be moved using the $\tilde{\boldsymbol{\epsilon}}$ spinor one has that

$$\tilde{\kappa}_{A}{}^{Q} = \Xi^{-1}\kappa_{A}{}^{Q}$$

Hence, the matter alignment condition is conformally invariant, in the sense that

$$\tilde{\kappa}_{(A}{}^{Q}\tilde{\phi}_{B)Q} = \kappa_{(A}{}^{Q}\phi_{B)Q}$$

This only means that if the physical matter alignment condition is satisfied then $\boldsymbol{\phi} \propto \boldsymbol{\kappa}$. For notational purposes define the unphysical matter alignment condition zero-quantity as

$$\Theta_{AB} := 2\kappa_{(A}{}^{Q}\phi_{B)Q}$$

Observe that $\Theta_{AB} = 0$ does not imply that the auxiliary vector $\xi_{AA'}$ is a Killing or conformal Killing vector of $(\mathcal{M}, \boldsymbol{g})$. To see this recall that the conformal transfomation formula for the trace-free Ricci spinor is given by

$$\tilde{\Phi}_{ABA'B'} = \Phi_{ABA'B'} + \Xi^{-1}\nabla_{A(A'}\nabla_{B')B}\Xi$$

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Defining the shorthand

$$S_{AA'BB'} := \nabla_{QA'} H_{B'}{}^{Q}{}_{AB}$$

where the auxiliary spinors $\xi_{AA'}$, Q_{AB} and the zero-quantity $H_{A'ABC}$ are as defined as in GasWil22, then a calculation gives

$$S_{AA'BB'} = \frac{1}{2} Q_{AB}\epsilon_{A'B'} - \nabla_{AA'}\xi_{BB'} - \nabla_{BB'}\xi_{AA'} - 6\kappa_{(A}{}^{Q}\Phi_{B)QA'B'}$$

Using the conformal transformation law for the trace-free Ricci spinor, and the relation between the pysical Killing spinors gives

$$\kappa_{A}{}^{Q}\Phi_{BQA'B'} = \Xi\tilde{\kappa}_{A}{}^{Q}\tilde{\Phi}_{BQA'B'} - \Xi^{-1}\kappa_{A}{}^{Q}\nabla_{B(A'}\nabla_{B')Q}\Xi$$

Thus

$$\kappa_{(A}{}^{Q}\Phi_{B)QA'B'} = \Theta_{AB}\bar{\phi}_{A'B'} - \Xi^{-1}\kappa_{(A}{}^{Q}\nabla_{B)(A'}\nabla_{B')Q}\Xi$$

Altogether

$$S_{AA'BB'} = \frac{1}{2} Q_{AB}\epsilon_{A'B'} - 6\Theta_{AB}\bar{\phi}_{A'B'} - \nabla_{AA'}\xi_{BB'} - \nabla_{BB'}\xi_{AA'} + 6\Xi^{-1}\kappa_{(A}{}^{Q}\nabla_{B)(A'}\nabla_{B')Q}\Xi$$

Thus even when the unphysical matter alignment condition $\Theta_{AB} = 0$ and the Killing spinor equation $H_{A'ABC} = 0$ is satisfied (and hence their concomitants $Q_{AB} = 0$ and $S_{AA'BB'} = 0$) the auxiliary vector $\xi_{AA'}$ is not a Killing nor a conformal Killing vector of $(\mathcal{M}, \boldsymbol{g})$.