Conformal twistor KIDs

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Abstract

Write-up of Edgar's twistor KID calculations from the new notebook.

1 Definitions

We are interested here in the twistor equation:

$$\nabla_{A'(A}\kappa_{B)} = 0,$$

encoded in the vanishing of the *zero quantity* $H_{A'AB} := 2\nabla_{A'(A}\kappa_{B)}$. It will prove useful to define the auxiliary quantity

$$\xi_{A'} := \nabla^{A}{}_{A'}\kappa_A.$$

We also define the zero quantity $S_{A'B'A} := \nabla_{QA'}H_{B'A}{}^{Q}$. This may be expressed alternatively in terms of the auxiliary field$^{\bullet 1}$ $\xi_{A'}$, by an easy computation, as follows:

$$S_{A'B'A} = -\nabla_{AB'}\xi_{A'} + 2\Lambda\epsilon_{A'B'}\kappa_A - 2\Phi_{AQA'B'}\kappa^{Q}. \tag{1}$$

•1: I think we should change the sign in either the definition of S or ξ, to be more consistent.

The other (symmetrised) contraction yields the *Buchdahl constraint*:

$$0 = \nabla_{(A}{}^{A'}H_{|A'|BC)} = \Psi_{ABCD}\kappa^{D},$$

though this won't feature in the calculations here.

2 Wave equations

2.1 For the twistor fields

A short computation shows that

$$\Box\kappa_B = -2\Lambda\kappa_B + \tfrac{2}{3}\nabla^{AA'}H_{A'AB}.$$

Hence, if κ_B solves the twistor equation, then it necessarily satisfies the wave equation

$$\Box\kappa_B + 2\Lambda\kappa_B = 0. \tag{2}$$

We will choose to propagate a twistor candidate according to this equation. It is maybe worth noticing that $S_{A'}{}^{A'}{}_{B} = \nabla^{AA'}H_{A'AB}$. Hence, if κ_A satisfies (2), then necessarily $S_{A'B'A} =$

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$S_{(A'B')A}$, since it is assured that $\nabla^{AA'}H_{A'AB} = 0$.

We will also need a wave equation for the auxiliary field $\xi_{A'}$. To get this, take the contracted derivative of (2) and commute derivatives, finally resulting in

$$\Box\xi_{A'} + 2\Lambda\xi_{A'} - 8(\nabla_{AA'}\Lambda)\kappa^A = 0. \tag{3}$$

Again, if κ_A is a twistor, then the resulting $\xi_{A'}$ solves (3). Given a twistor candidate and its corresponding $\xi_{A'}$, we choose to propagate the latter according to this wave equation.

2.2 For the zero quantities

In order to derive wave equations for the zero quantities, we assume that twistor candidate, κ_A and its auxiliary spinor $\xi_{A'}$ satisfy the wave equations (2)–(3); at the end of the day, these will be satisfied by construction, since the candidate quantities will be propagated off the initial hypersurface using these wave equations.

To get a wave equation for $H_{A'AB}$, simply take the definition of $\boldsymbol{S}$ in terms of $\boldsymbol{H}$ and take a contracted derivative —i.e. consider $\nabla_A{}^{D'}S_{D'A'B}$. Ultimately, we get

$$\Box H_{A'AB} = 8\Lambda H_{A'AB} - 2\Psi_{ABCD}H_{A'}{}^{CD} - 2\Phi_{ADA'D'}H^{D'D}{}_B - 2\nabla_{AD'}S^{D'}{}_{A'B}.$$

It is important to note that this is expressible (in a regular way) in terms of the rescaled Weyl spinor $\phi_{ABCD} = \Theta^{-1}\Psi_{ABCD}$:

$$\Box H_{A'AB} = 8\Lambda H_{A'AB} - 2\Theta\phi_{ABCD}H_{A'}{}^{CD} - 2\Phi_{ADA'D'}H^{D'D}{}_B - 2\nabla_{AD'}S^{D'}{}_{A'B}.$$

To close the system, we need a wave equation for $S_{A'B'A}$. To get this, we will apply the D'Alembertian to (1), commute derivatives, and substitute the wave equation for $\xi_{A'}$, equation (3). Finally, we arrive at•2

•2: Is it possible to simplify this a bit more?

$$\begin{aligned}\Box S_{A'B'A} = 6\Lambda S_{A'B'A} - 4\Phi_{ABC'(A'}S_{B')}{}^{C'B} - 2\Theta\bar{\phi}_{A'B'C'D'}S^{C'D'}{}_A \\ - \tfrac{2}{3}\Phi_{BCA'B'}(\nabla_{AC'}H^{C'BC} + 2\nabla^C{}_{C'}H^{C'}{}_A{}^B) \\ + 4H_{(A'|AB|}\nabla^B{}_{B')}\Lambda - 2(\nabla_{CC'}\Phi_{ABA'B'})H^{C'BC}.\end{aligned} \tag{4}$$

Note that the terms on the right-hand-side are homogeneous in $\boldsymbol{S}$, $\boldsymbol{H}$ and $\nabla\boldsymbol{H}$.

3 What goes wrong in the higher-valence case?

Define also the "Buchdahl zero quantity":

$$B_{ABCD} = \phi_{F(ABC}\kappa_{D)}{}^F.$$

Note that

$$\nabla_{(A}{}^{A'}H_{'|BCD)} = 6\Theta B_{ABCD}$$

Can derive equations of the form

$$\Box H_{A'ABC} = (\boldsymbol{H}, \nabla\boldsymbol{S}), \tag{5}$$
$$\Box H_{A'ABC} = (\boldsymbol{H}, \boldsymbol{B}, \nabla\boldsymbol{B}), \tag{6}$$
$$\Box S_{AA'BB'} = (\boldsymbol{H}, \boldsymbol{S}, \Theta\boldsymbol{B}, \nabla\Theta\cdot\nabla\boldsymbol{B}) = (\boldsymbol{H}, \boldsymbol{S}, \nabla\boldsymbol{H}, \nabla\Theta\cdot\nabla\boldsymbol{B}), \tag{7}$$
$$\Box B_{ABCD} = (\boldsymbol{H}, \boldsymbol{B}) + \tfrac{2}{3}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} = (\boldsymbol{H}, \boldsymbol{B}) + \tfrac{2}{3}\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD}. \tag{8}$$

The final equality follows from the fact that $\nabla_{(A}{}^{A'}\xi_{B)A'} = 0$, as a consequence of the assumed wave equation for κ_{AB}.•3

•3: Generally speaking, do we need to worry about the fact that $\kappa_{AB}, \xi_{AA'}$ are being propagated independently (though consistently)?

Note: In the equation for $\boldsymbol{S}$, we couldn't replace the $\nabla\boldsymbol{B}$ with $\nabla\nabla\boldsymbol{H}$ terms even if we wanted to, because then we would have Θ^{-1} factors appearing.

Proposal: Define $F_{A'BCD} := \nabla^{A}{}_{A'}B_{ABCD}$. Then we get a wave equation for B trivially:

$$\Box B_{ABCD} \propto \nabla_{(A}{}^{A'}F_{|A'|BCD)} + \text{curv.} \times \boldsymbol{B}.$$

To get the remaining equation for $F_{A'BCD}$ take a contracted derivative of (8), commute derivatives on the $\nabla_{\boldsymbol{\xi}}\boldsymbol{\phi}$ term and use the fact that $\nabla^{A}{}_{A'}\phi_{ABCD} = 0$.

4 A closed system for the Killing spinor case

Zero quantities:

$$H_{A'ABC}, \qquad B_{ABCD}, \qquad F_{A'BCD} := \nabla^{A}{}_{A'}B_{ABCD}$$

From the previous section, $\nabla_{(A}{}^{A'}H_{'|BCD)} = 6\Theta B_{ABCD}$. Additionaly, the wave equation for κ_{AB} is equivalent to $\nabla^{AA'}H_{A'ABC} = 0$, hence we have $\nabla_{A}{}^{A'}H_{A'BCD} = 6\Theta B_{ABCD}$. Contracting with $\nabla^{A}{}_{B'}$ we then derive the following wave equation:

$$\Box H_{A'ABC} = ... \tag{9}$$

Similarly, substituting the definition of $F_{A'ABC}$ in terms of B_{ABCD}, it is straightforward to verify the following wave equation for B_{ABCD}:

$$\Box B_{ABCD} = 12\Lambda B_{ABCD} - 6\Theta\phi_{(AB}{}^{FG}B_{CD)FG} + 2\nabla_{AA'}F^{A'}{}_{BCD}. \tag{10}$$

The task remaining is to derive a wave equation for $F_{A'ABC}$. Let us first consider some useful identities:

$$\begin{aligned}\kappa^{DG}\phi_{(ABC}{}^{H}\phi_{F)HDG} &= \kappa_{A}{}^{D}\phi_{(BC}{}^{GH}\phi_{FD)GH} + \kappa_{B}{}^{D}\phi_{(AC}{}^{GH}\phi_{FD)GH}\\ &\quad + \kappa_{C}{}^{D}\phi_{(AB}{}^{GH}\phi_{FD)GH} + \kappa_{F}{}^{D}\phi_{(BC}{}^{GH}\phi_{AD)GH}\\ &= 2\phi_{(AB}{}^{GH}B_{CF)GH}\end{aligned} \tag{11}$$

Using this identity, we can derive the following alternative (non-homogeneous) wave equation for B_{ABCD}:

$$\Box B_{ABCD} = \tfrac{2}{3}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} + 8\Lambda B_{ABCD} - 14\Theta\phi_{(AB}{}^{FG}B_{CD)FG} + \tfrac{2}{3}(\nabla_{FA'}\phi_{G(ABC})H^{A'}{}_{D)}{}^{FG} \tag{12}$$

where, for convenience, we have defined $\xi_{AA'} := \nabla^{B}{}_{A'}\kappa_{AB}$, as usual. Contrary to previous approaches, however, we won't propagate $\xi_{AA'}$ independently of κ_{AB}; it is simply a convenient shorthand.

Remark 1. *As an aside, note that* (12) *expresses the fact that,* ***given a Killing spinor*** κ_{AB}*, the quantity*

$$\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD} \equiv \nabla_{\boldsymbol{\xi}}\phi_{ABCD} + \phi_{F(ABC}\nabla_{D)A'}\xi^{FA'} = \nabla_{\boldsymbol{\xi}}\phi_{ABCD}$$

—the equality being by virtue of the wave equation for κ_{AB}*— vanishes, and in particular that in the physical spacetime the Lie derivative of the Weyl curvature along the Killing vector* $\tilde{\boldsymbol{\xi}}$ *vanishes, as we know should be the case.*

At this point we note that there are no derivatives of zero quantities appearing on the right-hand-side of (12). This, combined with the fact that $\nabla^{A}{}_{A'}\phi_{ABCD} = 0$, makes it seems plausible that by applying $\nabla^{A}{}_{A'}$ to (12) we will be able to derive a wave equation for $F_{A'BCD}$. We will see that this does indeed work; the only essential difficulty is in showing that $\nabla^{A}{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD}$ is expressible in terms of the zero quantities $\boldsymbol{H}, \boldsymbol{B}, \boldsymbol{F}$ and their first derivatives only. This is done in the following identity:

$$\nabla^{A}{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} = \tag{13}$$