Conformal twistor KIDs

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Abstract

Write-up of Edgar's twistor KID calculations from the new notebook.

1 Definitions

We are interested here in the twistor equation:

$$\nabla_{A'(A}\kappa_{B)} = 0,$$

encoded in the vanishing of the *zero quantity* $H_{A'AB} := 2\nabla_{A'(A}\kappa_{B)}$. It will prove useful to define the auxiliary quantity

$$\xi_{A'} := \nabla^{A}{}_{A'}\kappa_A.$$

We also define the zero quantity $S_{A'B'A} := \nabla_{QA'}H_{B'A}{}^{Q}$. This may be expressed alternatively in terms of the auxiliary field$^{\bullet 1}$ $\xi_{A'}$, by an easy computation, as follows:

$$S_{A'B'A} = -\nabla_{AB'}\xi_{A'} + 2\Lambda\epsilon_{A'B'}\kappa_A - 2\Phi_{AQA'B'}\kappa^Q. \tag{1}$$

•1: I think we should change the sign in either the definition of S or ξ, to be more consistent.

The other (symmetrised) contraction yields the *Buchdahl constraint*:

$$0 = \nabla_{(A}{}^{A'}H_{|A'|BC)} = \Psi_{ABCD}\kappa^D,$$

though this won't feature in the calculations here.

2 Wave equations

2.1 For the twistor fields

A short computation shows that

$$\Box\kappa_B = -2\Lambda\kappa_B + \tfrac{2}{3}\nabla^{AA'}H_{A'AB}.$$

Hence, if κ_B solves the twistor equation, then it necessarily satisfies the wave equation

$$\Box\kappa_B + 2\Lambda\kappa_B = 0. \tag{2}$$

We will choose to propagate a twistor candidate according to this equation. It is maybe worth noticing that $S_{A'}{}^{A'}{}_{B} = \nabla^{AA'}H_{A'AB}$. Hence, if κ_A satisfies (2), then necessarily $S_{A'B'A} =$

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$S_{(A'B')A}$, since it is assured that $\nabla^{AA'}H_{A'AB} = 0$.

We will also need a wave equation for the auxiliary field $\xi_{A'}$. To get this, take the contracted derivative of (2) and commute derivatives, finally resulting in

$$\square\xi_{A'} + 2\Lambda\xi_{A'} - 8(\nabla_{AA'}\Lambda)\kappa^{A} = 0. \tag{3}$$

Again, if κ_A is a twistor, then the resulting $\xi_{A'}$ solves (3). Given a twistor candidate and its corresponding $\xi_{A'}$, we choose to propagate the latter according to this wave equation.

2.2 For the zero quantities

In order to derive wave equations for the zero quantities, we assume that twistor candidate, κ_A and its auxiliary spinor $\xi_{A'}$ satisfy the wave equations (2)–(3); at the end of the day, these will be satisfied by construction, since the candidate quantities will be propagated off the initial hypersurface using these wave equations.

To get a wave equation for $H_{A'AB}$, simply take the definition of $\boldsymbol{S}$ in terms of $\boldsymbol{H}$ and take a contracted derivative —i.e. consider $\nabla_{A}{}^{D'}S_{D'A'B}$. Ultimately, we get

$$\square H_{A'AB} = 8\Lambda H_{A'AB} - 2\Psi_{ABCD}H_{A'}{}^{CD} - 2\Phi_{ADA'D'}H^{D'D}{}_{B} - 2\nabla_{AD'}S^{D'}{}_{A'B}.$$

It is important to note that this is expressible (in a regular way) in terms of the rescaled Weyl spinor $\phi_{ABCD} = \Theta^{-1}\Psi_{ABCD}$:

$$\square H_{A'AB} = 8\Lambda H_{A'AB} - 2\Theta\phi_{ABCD}H_{A'}{}^{CD} - 2\Phi_{ADA'D'}H^{D'D}{}_{B} - 2\nabla_{AD'}S^{D'}{}_{A'B}.$$

To close the system, we need a wave equation for $S_{A'B'A}$. To get this, we will apply the D'Alembertian to (1), commute derivatives, and substitute the wave equation for $\xi_{A'}$, equation (3). Finally, we arrive at[•2]

•2: *Is it possible to simplify this a bit more?*

$$\begin{aligned}\square S_{A'B'A} = 6\Lambda S_{A'B'A} - 4\Phi_{ABC'(A'}S_{B')}{}^{C'B} - 2\Theta\bar{\phi}_{A'B'C'D'}S^{C'D'}{}_{A} \\ - \tfrac{2}{3}\Phi_{BCA'B'}(\nabla_{AC'}H^{C'BC} + 2\nabla^{C}{}_{C'}H^{C'}{}_{A}{}^{B}) \\ + 4H_{(A'|AB|}\nabla^{B}{}_{B')}\Lambda - 2(\nabla_{CC'}\Phi_{ABA'B'})H^{C'BC}.\end{aligned} \tag{4}$$

Note that the terms on the right-hand-side are homogeneous in $\boldsymbol{S}$, $\boldsymbol{H}$ and $\nabla\boldsymbol{H}$.

3 What goes wrong in the higher-valence case?

Define also the "Buchdahl zero quantity":

$$B_{ABCD} = \phi_{F(ABC}\kappa_{D)}{}^{F}.$$

Note that

$$\nabla_{(A}{}^{A'}H_{|A'|BCD)} = 6\Theta B_{ABCD}$$

Can derive equations of the form

$$\square H_{A'ABC} = (\boldsymbol{H}, \nabla\boldsymbol{S}), \tag{5}$$
$$\square H_{A'ABC} = (\boldsymbol{H}, \boldsymbol{B}, \nabla\boldsymbol{B}), \tag{6}$$
$$\square S_{AA'BB'} = (\boldsymbol{H}, \boldsymbol{S}, \Theta\boldsymbol{B}, \nabla\Theta\cdot\nabla\boldsymbol{B}) = (\boldsymbol{H}, \boldsymbol{S}, \nabla\boldsymbol{H}, \nabla\Theta\cdot\nabla\boldsymbol{B}), \tag{7}$$
$$\square B_{ABCD} = (\boldsymbol{H}, \boldsymbol{B}) + \tfrac{2}{3}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} = (\boldsymbol{H}, \boldsymbol{B}) + \tfrac{2}{3}\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD}. \tag{8}$$

The final equality follows from the fact that $\nabla_{(A}{}^{A'}\xi_{B)A'} = 0$, as a consequence of the assumed wave equation for κ_{AB}.[•3]

•3: *Generally speaking, do we need to worry about the fact that $\kappa_{AB}, \xi_{AA'}$ are being propagated independently (though consistently)?*

Note: In the equation for $\boldsymbol{S}$, we couldn't replace the $\nabla \boldsymbol{B}$ with $\nabla\nabla \boldsymbol{H}$ terms even if we wanted to, because then we would have Θ^{-1} factors appearing.

Proposal: Define $F_{A'BCD} := \nabla^{A}{}_{A'} B_{ABCD}$. Then we get a wave equation for B trivially:

$$\Box B_{ABCD} \propto \nabla_{(A}{}^{A'} F_{|A'|BCD)} + \text{curv.} \times \boldsymbol{B}.$$

To get the remaining equation for $F_{A'BCD}$ take a contracted derivative of (8), commute derivatives on the $\nabla_{\boldsymbol{\xi}}\boldsymbol{\phi}$ term and use the fact that $\nabla^{A}{}_{A'}\phi_{ABCD} = 0$.

4 A closed system for the Killing spinor case

Zero quantities:

$$H_{A'ABC}, \qquad B_{ABCD}, \qquad F_{A'BCD} := \nabla^{A}{}_{A'} B_{ABCD}$$

From the previous section, $\nabla_{(A}{}^{A'} H_{|A'|BCD)} = 6\Theta B_{ABCD}$. Additionally, the wave equation for κ_{AB} is equivalent to $\nabla^{AA'} H_{A'ABC} = 0$,

$$Q_{BC} = \nabla^{AA'} H_{A'ABC}$$

$$\Box \kappa_{BC} = -4\Lambda \kappa_{BC} + \Theta \kappa^{AD} \phi_{BCAD} + \tfrac{1}{2} Q_{BC}$$

We have the irreducible decomposition

$$\nabla_{D}{}^{A'} H_{A'ABC} = 6\Theta B_{ABCD} + \tfrac{3}{4}\, Q_{(AB}\epsilon_{C)D} \tag{9}$$

. Contracting with $\nabla^{A}{}_{B'}$ we then derive the following wave equation:

$$\begin{aligned}\Box H_{B'ABC} = 6\Lambda H_{B'ABC} + 12\Theta \nabla_{DB'} B_{ABC}{}^{D} - 12 B_{ABCD} \nabla^{D}{}_{B'}\Theta \\ + \tfrac{3}{2}\nabla_{(A|B'|} Q_{BC)} - 6\Phi_{(A}{}^{D}{}_{|B'}{}^{A'} H_{A'|BC)D}\end{aligned} \tag{10}$$

Similarly, substituting the definition of $F_{A'ABC}$ in terms of B_{ABCD}, it is straightforward to verify the following wave equation for B_{ABCD}:

$$\Box B_{ABCD} = 12\Lambda B_{ABCD} - 6\Theta \phi_{(AB}{}^{FG} B_{CD)FG} + 2\nabla_{AA'} F^{A'}{}_{BCD}. \tag{11}$$

The task remaining is to derive a wave equation for $F_{A'ABC}$. Let us first consider some useful identities:

$$\begin{aligned}\kappa^{DG}\phi_{(ABC}{}^{H}\phi_{F)HDG} &= \kappa_{A}{}^{D}\phi_{(BC}{}^{GH}\phi_{FD)GH} + \kappa_{B}{}^{D}\phi_{(AC}{}^{GH}\phi_{FD)GH} \\ &\quad + \kappa_{C}{}^{D}\phi_{(AB}{}^{GH}\phi_{FD)GH} + \kappa_{F}{}^{D}\phi_{(BC}{}^{GH}\phi_{AD)GH} \\ &= 2\phi_{(AB}{}^{GH} B_{CF)GH}\end{aligned} \tag{12}$$

Using this identity, we can derive the following alternative (non-homogeneous) wave equation for B_{ABCD}:

$$\Box B_{ABCD} = \tfrac{2}{3}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} + 8\Lambda B_{ABCD} - 14\Theta\phi_{(AB}{}^{FG} B_{CD)FG} + \tfrac{2}{3}(\nabla_{FA'}\phi_{G(ABC}) H^{A'}{}_{D)}{}^{FG} \tag{13}$$

where, for convenience, we have defined $\xi_{AA'} := \nabla^{B}{}_{A'}\kappa_{AB}$, as usual. Contrary to previous approaches, however, we won't propagate $\xi_{AA'}$ independently of κ_{AB}; it is simply a convenient shorthand.

Remark 1. *As an aside, note that* (13) *expresses the fact that,* ***given a Killing spinor*** κ_{AB}*, the quantity*

$$\mathcal{L}_{\boldsymbol{\xi}}\phi_{ABCD} \equiv \nabla_{\boldsymbol{\xi}}\phi_{ABCD} + \phi_{F(ABC}\nabla_{D)A'}\xi^{FA'} = \nabla_{\boldsymbol{\xi}}\phi_{ABCD}$$

—the equality being by virtue of the wave equation for κ_{AB} *— vanishes, and in particular that in the physical spacetime the Lie derivative of the Weyl curvature along the Killing vector* $\tilde{\boldsymbol{\xi}}$ *vanishes, as we know should be the case.*

At this point we note that there are no derivatives of zero quantities appearing on the right-hand-side of (13). This, combined with the fact that $\nabla^{A}{}_{A'}\phi_{ABCD}=0$, makes it seems plausible that by applying $\nabla^{A}{}_{A'}$ to (13) we will be able to derive a wave equation for $F_{A'BCD}$. We will see that this does indeed work; the only essential difficulty is in showing that $\nabla^{A}{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD}$ is expressible in terms of the zero quantities $\boldsymbol{H}, \boldsymbol{B}, \boldsymbol{F}$ and their first derivatives only. First, we note the following identities:

$$\kappa^{CD}\nabla_{DG'}\phi_{ABFC} = 2\xi^{C}{}_{G'}\phi_{ABFC} + \phi_{CD(AB}H_{|G'|F)}{}^{CD} - 4F_{G'ABF}$$

which can be used to derive the more general identity

$$\begin{aligned}\kappa_{A}{}^{F}\nabla_{DF'}\phi_{BCGF} = {}& \tfrac{1}{3}\xi_{AF'}\phi_{BCDG} - \tfrac{1}{3}\xi^{F}{}_{F'}\phi_{BCGF}\epsilon_{AD} + \epsilon_{AD}F_{F'BCG} + \nabla_{(B|F'}B_{A|CG)D} \\ & + \tfrac{1}{3}\phi_{(BC|D}{}^{F}H_{F'A|G)F} + 2\epsilon_{A(B}\nabla^{F}{}_{|F'|}B_{C|D|G)F} - \tfrac{1}{3}\xi^{F}{}_{F'}\phi_{(BC|DF}\epsilon_{A|G)} \\ & - \tfrac{2}{3}\xi^{F}{}_{F'}\phi_{(B|D|C|F}\epsilon_{A|G)} - \tfrac{1}{3}\phi_{(BC}{}^{FH}H_{|F'|G)FH}\epsilon_{AD} - \tfrac{1}{6}\phi_{(BC}{}^{FH}H_{|F'DFH}\epsilon_{A|G)} \\ & - \tfrac{1}{3}\phi_{(B|D}{}^{FH}H_{F'|C|FH}\epsilon_{A|G)} - \tfrac{1}{6}\phi_{D(B}{}^{FH}H_{|F'|C|FH}\epsilon_{A|G)}\end{aligned} \tag{14}$$

$$\begin{aligned}\kappa^{AD}\phi_{AD}{}^{GH}\nabla_{HA'}\phi_{BCFG} = {}& 4\xi^{A}{}_{A'}\phi_{(BC}{}^{DG}\phi_{F)ADG} + \tfrac{1}{2}\phi_{ADGH}\phi^{ADGH}H_{A'BCF} \\ & + 4B_{(BC}{}^{AD}\nabla^{G}{}_{|A'|}\phi_{F)ADG} - 4B_{(B}{}^{ADG}\nabla_{|AA'|}\phi_{CF)DG} \\ & - 8\phi_{(BC}{}^{AD}\nabla^{G}{}_{|A'|}B_{F)ADG} - 4\phi_{(B}{}^{ADG}\nabla_{|AA'|}B_{CF)DG} \\ & - \tfrac{1}{3}\phi_{(BC}{}^{AD}\phi_{|AD}{}^{GH}H_{A'|F)GH} - \tfrac{2}{3}\phi_{(B}{}^{ADG}\phi_{C|AD}{}^{H}H_{A'|F)GH}\end{aligned} \tag{15}$$

We can then derive...

$$\nabla^{A}{}_{A'}\nabla_{\boldsymbol{\xi}}\phi_{ABCD} = \tag{16}$$

4.1 Intrinsic conditions

We consider κ_{AB} defined over $\mathcal{S}$ satisfying

$$\tau_{(A}{}^{A'}H_{|A'|BCD)} \equiv \mathcal{D}_{(AB}\kappa_{CD)} = 0, \tag{17a}$$

$$B_{ABCD} \equiv \kappa_{(A}{}^{F}\phi_{BCD)F} = 0 \tag{17b}$$

on $\mathcal{S}$. We evolve the Killing spinor candidate according to () and the initial condition

$$\mathcal{P}\kappa_{BD} = -\xi_{BD} \tag{18}$$

on $\mathcal{S}$, with $\xi_{AB} := \mathcal{D}_{(A}{}^{C}\kappa_{B)C}$. Starting from (9), and performing the decomposition of the covariant derivative, we get

$$\nabla_{\boldsymbol{\tau}}H_{B'ABC} + 2\tau^{D}{}_{B'}\tau^{FA'}\mathcal{D}_{DF}H_{A'ABC} = -12\Theta\tau^{D}{}_{B'}B_{ABCD} + \tfrac{3}{2}\tau_{(A|B'|}Q_{BC)}$$

At this point we see that if ()–() hold then $\nabla_{\boldsymbol{\tau}}H_{B'ABC}=0$ on $\mathcal{S}$. Now we move onto B_{ABCD}. A similar computation to the twistor case yields

$$\nabla_{\boldsymbol{\tau}}B_{ABCD} = 2\kappa_{(A}{}^{F}\mathcal{D}_{B}{}^{G}\phi_{CD)FG} + \phi_{(ABC}{}^{F}\xi_{D)F} \tag{19}$$

Note that the quantity on the right-hand-side is intrinsic to $\mathcal{S}$; we add it to our list of conditions (for the meantime). Consider now the quantity $F_{A'ABC}$. By definition, $F_{A'ABC} = \nabla^{D}{}_{A'}B_{ABCD}$ and so, decomposing the covariant derivative

$$F_{A'BCD} = \tfrac{1}{2}\tau^{A}{}_{A'}\nabla_{\boldsymbol{\tau}}B_{ABCD} - \tau^{F}{}_{A'}\mathcal{D}^{A}{}_{F}B_{ABCD}$$

Hence, given the vanishing of B_{ABCD} and the right-hand-side of (19), we see that $F_{A'ABC}=0$ on $\mathcal{S}$, too. Thus, all that remains to show is to find intrinsic conditions under which $\nabla_{\boldsymbol{\tau}}F_{A'ABC}=0$ on $\mathcal{S}$. Now, combining our two wave equations for B_{ABCD}, namely () and (), we see that

$$\begin{aligned}\nabla_{FA'}F^{A'}{}_{BCD} = {}& \tfrac{1}{3}\nabla_{\xi}\phi_{BCDF} - 2\Lambda B_{BCDF} + 3\Theta B_{(BC}{}^{AG}\phi_{D)FAG} - 7\Theta B_{(BC}{}^{AG}\phi_{DF)AG} \\ & - \tfrac{1}{4}\phi_{(BCD}{}^{A}Q_{F)A} + \tfrac{1}{3}H^{A'}{}_{(B}{}^{AG}\nabla_{|AA'|}\phi_{CDF)G}.\end{aligned} \tag{20}$$

Decomposing, we have

$$\begin{aligned}\nabla_{\boldsymbol{\tau}} F_{B'BCD} + 2\tau^{A}{}_{B'}\tau^{FA'}\mathcal{D}_{AF}F_{A'BCD} = {}& \tfrac{2}{3}\nabla_{\xi}\phi_{BCDA}\tau^{A}{}_{B'} - 4\Lambda B_{BCDA}\tau^{A}{}_{B'} + 2\Theta B_{CD}{}^{FG}\tau^{A}{}_{B'}\phi_{BAFG} \\ & + 2\Theta B_{BD}{}^{FG}\tau^{A}{}_{B'}\phi_{CAFG} + 2\Theta B_{BC}{}^{FG}\tau^{A}{}_{B'}\phi_{DAFG} \\ & - 14\Theta\tau^{A}{}_{B'}B_{(BC}{}^{FG}\phi_{DA)FG} - \tfrac{1}{2}\tau^{A}{}_{B'}\phi_{(BCD}{}^{F}Q_{A)F} \\ & \qquad + \tfrac{2}{3}\tau^{A}{}_{B'}H^{A'}{}_{(B}{}^{FG}\nabla_{|FA'|}\phi_{CDA)G}. \end{aligned} \tag{21}$$

Hence, if $F_{A'ABC} = H_{A,ABC} = B_{ABCD} = 0$ on $\mathcal{S}$, then we will also have $\nabla_{\boldsymbol{\tau}} F_{A'ABC} = 0$ on $\mathcal{S}$ provided

$$\nabla_{\boldsymbol{\xi}}\phi_{ABCD} = 0$$

on $\mathcal{S}$. Decomposing the covariant derivative, using the evolution equation for rescaled Weyl () along with (18) we have

$$\nabla_{\xi}\phi_{ABCD} = \xi\mathcal{D}_{D}{}^{F}\phi_{ABCF} + \tfrac{3}{2}\xi^{FG}\mathcal{D}_{FG}\phi_{ABCD}$$

with $\xi := \mathcal{D}^{AB}\kappa_{AB}$ •4 . At this point then, we have reduced our initial conditions for the zero quantities to the following set of intrinsic conditions:

•4: *give decomposition of* $\xi_{AA'}$

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0, \tag{22a}$$

$$B_{ABCD} \equiv \kappa_{(A}{}^{F}\phi_{BCD)F} = 0, \tag{22b}$$

$$2\kappa_{(A}{}^{F}\mathcal{D}_{B}{}^{G}\phi_{CD)FG} + \phi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{22c}$$

$$\xi^{FG}\mathcal{D}_{FG}\phi_{ABCD} + \tfrac{2}{3}\xi\mathcal{D}_{(A}{}^{F}\phi_{BCD)F} = 0. \tag{22d}$$