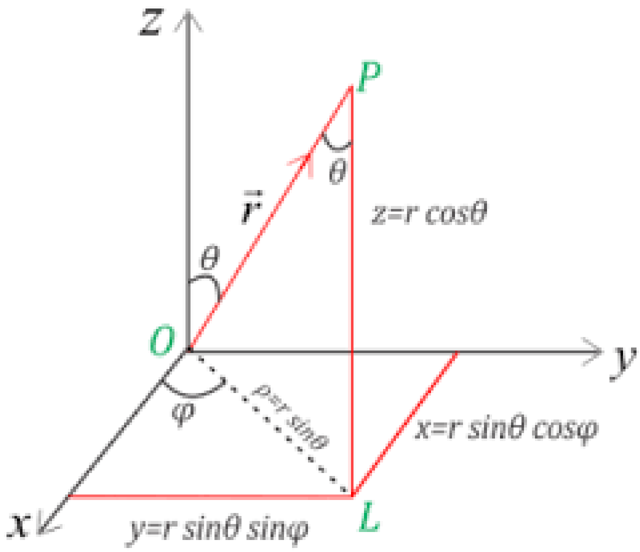
Spherical Coordinates $\iff$ Cartesian Coordinates



Spherical Coordinates:

$r = \sqrt{x^2 + y^2 + z^2}$

$\phi = \arctan(\frac{y}{x})$

$\theta = \arccos(\frac{y}{r}) = \arccos(\frac{y}{\sqrt{x^2 + y^2 + z^2}})$

Cartesian Coordinates:

$x = r\sin(\theta)\cos(\phi)$

$y = r\sin(\theta)\sin(\phi)$

$z = r\cos(\theta)$

Vertical Disparity

Proof. We seek to find the derivative of θ w.r.t the vertical axis z, namely $\frac{\partial\theta}{\partial z}$, which relates the baseline dz to the vertical angular disparity $\gamma = d\theta$.

$$\frac{\partial\theta}{\partial z} = \frac{\partial\theta}{\partial u}\frac{\partial u}{\partial z}, \tag{1}$$

with

$$u(z) = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \tag{2}$$

and the left hand side derivative being:

$$\frac{\partial\theta}{\partial u} = -\frac{1}{\sqrt{1 - u^2}}, \tag{3}$$

while the right hand side one being:

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{v}{w}\right) = \frac{w(\partial v/\partial z) - v(\partial w/\partial z)}{w^2}, \tag{4}$$

with

$$v(z) = z, \quad \text{and}, \quad w(z) = \sqrt{x^2 + y^2 + z^2}. \tag{5}$$

The derivatives of 5 are:

$$\partial v/\partial z = 1, \quad \partial w/\partial z = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{6}$$

Assigning these into 4:

$$\frac{\sqrt{x^2 + y^2 + z^2} - \frac{z^2}{\sqrt{x^2+y^2+z^2}}}{x^2 + y^2 + z^2} = \frac{\frac{x^2+y^2+z^2-z^2}{\sqrt{x^2+y^2+z^2}}}{x^2 + y^2 + z^2} = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} \tag{7}$$

Using the result of 2 and 3, as well as 6 into 1 we get:

$$\begin{align}
\frac{\partial \theta}{\partial z} &= -\frac{1}{\sqrt{1 - \frac{z^2}{x^2+y^2+z^2}}} \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}} \tag{8}\\
&= -\frac{1}{\sqrt{\frac{x^+y^2+z^2-z^2}{x^2+y^2+z^2}}} \frac{x^2 + y^2}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}} \tag{9}\\
&= -\frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2}} \frac{x^2 + y^2}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}} \tag{10}\\
&= -\frac{\sqrt{x^2 + y^2}}{(x^2 + y^2 + z^2)} \tag{11}\\
&= -\frac{\sqrt{x^2 + y^2}}{\sqrt{(x^2 + y^2 + z^2)}} \frac{1}{\sqrt{(x^2 + y^2 + z^2)}} \tag{12}\\
& \tag{13}
\end{align}$$

Yet, $r = \sqrt{x^2 + y^2 + z^2}$, which is the 3D segment $\overrightarrow{\mathbf{OP}}$, while $d = \sqrt{x^2 + y^2}$ is the 3D segment $\overrightarrow{\mathbf{OL}}$, and thus:

$$\frac{\partial \theta}{\partial z} = -\frac{d}{r}\frac{1}{r}. \tag{14}$$

But from the triangle $\triangle\mathbf{OPL}$, we get that $\frac{d}{r} = \cos(\frac{\pi}{2} - \theta) = \sin(\theta)$, and therefore:

$$\frac{\partial\theta}{\partial z} = -\frac{\sin(\theta)}{r}. \tag{15}$$

$\square$