hw13 Optimization

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1 Problem 11.14

First let us remark that, for all $x^0 \in R^n$:

$$x^{k+1} = W^k x^0$$

Therefore, using the hint:

$$x^{k+1} = \sum_{i=1}^{n} \lambda_i^k v_i u_i^T x^0$$

But, because the absolute values of the eigenvalues are either 1 or inferior to 1, when k goes to infinity, we get:

$$x^{k+1} = \sum_{i=1}^{n} \lambda_i^k v_i u_i^T x^0 \underset{k \to \infty}{\to} v_1 u_1^T x^0$$

And thus:

$$x^* = v_1 u_1^T x^0$$

Then we can conclude that $v_1 u_1^T = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$

2 Problem 11.15

Let us prove the equivalence by prooving the chain of implication. First of all, if $x_1 = \cdots = x_n = x^*$, then $x = x^* 1$ and therefore, because $W1 = 1$, by linearity, we get $(I - W)x = x^* = x^*(1 - W1) = 0$ which proves the first implication (i) $\implies$ (ii).

Then, for (ii) $\implies$ (iii), clearly, if $(I - W)x = 0$, then $x^T(I - W)x = 0$ and thus $\|x\|_{I-W} = O$, which proves: (ii) $\implies$ (iii). We can not chain (iii) $\implies$ (iv), so we will prove the reverse: (iii) $\implies$ (ii). Provided: $1 = \lambda_1 > \lambda_2 \cdots \leq_n$, the eigenvalues of $I - W$ will all be strictly positive and strictly inferior to one, except the first which will be zero.

For (ii) $\implies$ (iv), if $(I - W)x = 0$, then $Wx = x$ if W is invertible, when multiplying by the inverse, we get: $x = W^{-1}$, ie $(W^{-1} - I)x = 0$, which is what we had to prove. In fact, we will prove the equvalence outright by noting that we can reverse the calculation to get $W^{-1}x = x \equiv x = Wx$ which gives (ii) $\equiv$ (iv). It follows easily from that (iv) $\equiv$ (v), the proof is the same as (ii) $\equiv$ (iii).

Then, assuming (ii), we get $U^2x = 0$, ie $Ux \in Ker(U)$.

3 Problem 10.3

Because of lemma 3 and the maximal nonotonicity of A, we have :

$$\langle x^\infty, u^\infty \rangle \leq F_A(x^\infty, u^\infty) \leq lim\, min_{k\to\infty} F(x^k, u^k) =$$
$$lim\, min_{k\to\infty} \langle u^k, x^k \rangle = \langle x^\infty, u^\infty \rangle$$

The right equality stems from lemma 3 which gives $u^k \in x^k \equiv F(x^k, u^k) = \langle u^k, x^k \rangle$. The line show the results that need to be shown.

4 Problem 10.6

First, let us remark that there exist a bijection from $Gra\ A \to GraA^{1,-1}$. Therefore, let's suppose we had an element $X = ((x, v), (v, y)) \notin GraA^{1,-1}$ such that $A \cup X$ is maximal monotone, then applying that bijection, with $X' = ((x, y), (u, v))$, $A \cup X'$ would be a monotone operator, which is impossible since A is already maximal. Therefore $A^{1,-1}$ is maximal.

Second, let $((x_1, v_1), (v_1, y_1))$ and $((x_2, u_2), (v_2, y_2))$ be element of $GraA^{1,-1}$. We wnt to prove the property :

$$\left\langle \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ u_2 \end{pmatrix} | \begin{pmatrix} v_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} v_2 \\ y_2 \end{pmatrix} \right\rangle \geq 0$$

But using the maximal monotonicity of A, we already have:

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} | \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle \geq 0$$

And indeed, these two scalar products are equal because they are both equal to:

$$\langle x_1 | x_2 \rangle + \langle y_1 | y_2 \rangle + \langle u_1 | u_2 \rangle + \langle v_1 | v_2 \rangle$$

This way we have shown that $A^{1,-1}$ is maximal monotone.

5 Problem 10.7

Because F is CCP, thanks to theorem 1, its subgradient is a maximal monotone operator. Furthermore, we can compute its subgradient as it is actually:

$$\partial F(x,y) = \begin{bmatrix} \partial_x L(x, v^* \\ v^* \end{bmatrix}$$

where $v^{ast} = agrmax\{L(x,v) + \langle y|v\rangle\}$. Then, we have to notive that because L is convex concave, for $h_{xy} : v \to L(x,v) + \langle y|v\rangle$, because v^* is a maximizer of h :

$$\partial h_{xy} = \partial_y L(x, v^{ast}) + y = 0$$

And therefore:

$$\partial_y L(x, v^{ast}) = y$$

This is important as then:

$$\partial L(x, v^*) = \begin{bmatrix} \partial_x L(x, v^*) \\ \partial_y L(x, v^*) \end{bmatrix} = \begin{bmatrix} \partial_x L(x, v^* \\ y \end{bmatrix}$$

Which clearly shows that ∂L is a partial inverse of ∂F (with y and v^* swapped for the partial inversion).

Then, because ∂L is a partial inverse of ∂F, it is also maximal monotone thanks to the previous exercice, which closes the proof of this exercice.

6 Problem 10.10