hw3 optimization

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1 Problem 4.4

1) Because $\{Q_1, \dots, Q_k\}$ is a finite group, the application $f_i : Q_j \in \mathcal{G} \to Q_i Q_j \in \mathcal{G}$ is bijective. Thus, for all $j = 1, \dots, k$,

$$Q_j \bar{x} = \frac{1}{k} \sum_{i=1}^{k} Q_j Q_i x = \frac{1}{k} \sum_{i=1}^{k} Q_i x = \bar{x}$$

Which proves that $\bar{x} \in \mathcal{F}$

2) First let us establish that for all $i = 1, \dots, k$, $\|Q_i\| = 1$. If it wasn't the case, the sequence $(\|Q_i^n\|)_{n\mathbb{N}}$ would diverge or go to zero. If it diverged, the group would not be finite, which is a contradiction. If it went to zero, it would either do so in a finite number of iteration or in an infinite. If it was infinite, same argument as for the divergence, if it was finite, zero would be part of the group, which is impossible because zero has no inverse.

Thus, by convexity, then because of the invariance of $\mathcal{F}$ by $\mathcal{G}$

$$f(\bar{x}) \leq \frac{1}{k} \sum_{i=1}^{k} f(Q_i x) = \frac{1}{k} \sum_{i=1}^{k} f(x) = f(x)$$

c) Let's suppose there exist a optimal point x_0. Then according to question a) and the $\mathcal{G}$-invariance of the physical set, $\bar{x_0}$ is also in the feasible set. Then because the problem is convex and according to question b), we have $f(\bar{x_0}) \leq f(x_0)$, which is in fact an equality because x_0 is a minimzer, and then because of the convexity of the problem (the solution is unique, local maxima are global maxima), then $x_O = \bar{x_0} \in \mathcal{F}$.

d) Applying the results of the previous question, the minimizer of this problem must belong to the set of vector invariant by permutations. Which is exactly $vect(1, \dots, 1)$, and thus we must look for a vector of shape 1.

2 Problem 4.24

For $p = 2$, simply by rewriting the expression, we obtain the equivalent problem, (where a_i^T are the line of A):

$$\text{minimize } \sum_{i=1}^{n} |a_i^T x - b|^2$$

Because the squared module is equal to the product of the complex with its conjugate, the problem can be rewritten as a least square problem:

$$\text{minimize } \|A'x' - b\|_2^2$$

where, using the notation of problem 4.42,

$$A' = \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \text{ and } x' = \begin{bmatrix} \Re x \\ \Im x \end{bmatrix} \text{ and } b' = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix}$$

And thus, transformaing the least square problem into a quadratic form, we obtain the equivalent problem:

$$\text{minimize } {x'}^T({A'}^T A')x' - 2 \times ({A'}^T b)^T x' - \|b'\|$$

3 Problem 4.42

First, let us call the matrix:

$$X' = \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix}$$

Let us remark that this matrix is symetrical because du to the nature of conjugation on a complex number, the real part of a hermitian matrix is symetrical while the imaginary part is antisymetrical. Thus here the minus sign in front of the imaginary part turns the matrix into a symetrical matrix. Then let us remark that for all $v \in \mathbb{C}$, performing the operation:

$$v^H X v = {v'}^T X' v'$$

Where v' is the vector $(\Re X, \Im X) \in \mathbb{R}^{2n}$. The application $v \in \mathbb{C} \to v' \in \mathbb{R}^{2n}$ is bijective. Indeed:

$${v'}^T X' v' = \langle (Re(v)Re(X) + Im(v)Im(X), -Im(X)Re(v) + Re(X)Im(v)) | v \rangle$$

$$= Re(v)^2 Re(X) + Im(v)Re(v)Im(X) + Re(X)Im(v)^2 - Re(v)Im(X)Im(v)$$

= (Re(v) + i Im(v)) (Re(X) + i Im(X)) (Re(v) + i Im(v))

$$= v^H X v$$

Thus if X is hermitian positive matrix, the solution for both those operation is going to be positive, thus X' is positive semidefinite. And it becomes really easy to formulate the problem with real semidefinite matrices. The constraint simply becomes:

$$x_1 F_1' + \cdots + x_n F_n' + G' \preceq 0$$

4 Problem 4.43

Because $A(x)$ is a weigthed sum of symetrical matrices, it is still symetrical, and thus according to the spectral theorem we obtain the well known property that:

$$\forall u \in \mathbb{R}^n, \lambda_n \leq \frac{u^T A(x) u}{\|u\|^2} \leq \lambda_1$$

With a case of equality on the left when u is the n-th one-hot vector in the base of the the eigenvector of A and a case of equality of the right when u s the first one-hot vector of this base.

a) Thus, the minimization of the maximum eigenvalue is going to be

$$\underset{t}{minimize} t$$
$$\text{subject to } A(x) - tI \preceq 0$$

Where $x\mathbb{R}^n$