hw6 Optimisation

Guillaume Jarry

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1 Problem 2.14

Let L be the Lipschitz factor of our opertor T, and let $(x_n)_{n\in N}$ be a sequel of element of R^n such that $\forall n \in N x_n = T^n(x_0)$. By the contractive property of $T`weget$:

$$\|x_{n+1} - x_n\| \leq L\|x_n - x_{n-1}\|$$

Thus, by induction, we get the following formula:

$$\|x_{n+1} - x_n\| \leq L^n\|x_1 - x_0\|$$

Then, because $L < 1$, then the sequence $\|x_{n+1} - x_n\|_{n\in N}$ converges towards zero. Thus, the sequence $(x_n)_{n\in N}$ is a Cauchy-sequence and then because R^n is a Banach space the sequence converges.

Then, let us suppose that there exist two fixed point to the operator T, a_1 and a_2. Then in that case:

$$\|T(a_1) - T(a_2)\| = \|a_1 - a_2\|$$

This is impossible because T is contractive.

2 Problem 2.15

First let us understand that a zero of T is a fixed point of the operator $Id - T$. Then, for this operator, we can compute by linearity of the scalar product:

$$\|x - T(x) - y + T(y)\|^2 = \|x - y\|2 + \|T(x) - T(y)\|2 - 2\langle T(x) - T(y)|x - y\rangle$$

Then, by μ-monotonicity of T (thus T is also μ-Lipschitz) we get:

$$\|x - T(x) - y + T(y)\|^2 \leq \|x - y\|^2(1 - 2\mu + \mu^2) = \|x - y\|^2(1 - \mu)^2$$

Thus, $|1-\mu|$ is the Lipschitz-factor of the operator $Id-T$. If $|1-\mu| < 1$, this operator is contractive and according to the above problem, we can say that T has exactly one zero. If $|mu - 1| \geq 1$, then $\mu \geq 2$ whch gives :

3 Problem 2.16

First let use the hint then apply the Jensen inequality to it:

$$\|(I - \alpha\nabla f)(x) - (I - \alpha\nabla f)(y)\|^2 \leq \int_0^1 \|I - \alpha\nabla^2 f(tx + (1-t)(x-y))dt\|$$

Then, because f is μ-strong $I - \alpha\nabla^2(v) \preceq I - \mu I$, for all v in domf. Thus the norm being the operator norm in that case, we will get $\|I - \alpha\nabla^2(v)\| < |1 - \alpha\mu|\|I\|$. And thus:

$$\|(I - \alpha\nabla f)(x) - (I - \alpha\nabla f)(y)\|^2 \leq \int_0^1 |I - \alpha\mu|dt = |1 - \alpha\mu|$$

Then, we will use

4 Problem 2.20

First let us write $T = (1-\theta)Id + \theta S$ where S is a non-expensive operator. Then, we observe that if x^* is a fixed point of T, it is also a fixed point of S. Thus we have, by triangular inequality of the norm.

$$\|x^{k+1} - x^*\|^2 \leq (1-\theta)\|(x^k - x^*)\|^2 + \theta\|S(x^k) - S(x^*)\|^2$$

And then by non expansivity of T (because T is averaged):

$$\|x^{k+1} - x^*\|^2 \leq \|(x^k - x^*)\|^2$$

Then we notice:

$$\|x^{k+1} - x^K\|^2 = \theta^2(\|x^k\|^2 + \|S(x^k)\|^2 - 2\langle x|S(x^k)\rangle) = \theta^2\|x^k - S(x^k)\|^2$$

Then :

$$V^{k+1} - V^k = \frac{1-\theta}{\theta}\|x^{k+1} - x^k\|^2 + \|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 =$$

I haven't managed this exercice.

Then, because $(V^k)_{k \in N}$ is a decreasing sequence of positive number, it converges toward a limit. Thus the sequence $V^{k+1} - V^k$ converges towards zero, then we have:

$$\frac{1-\theta}{\theta}\|x^{k+1} - x^k\|^2 + \|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 \to 0$$

5 Problem 2.35

a) First let us notice that, because $\forall x \notin K, \delta_K(x) = \infty$

$$(\delta_K)^*(y) = \underset{x \in R^n}{sup}\{y^T x - f(x)\} = \underset{x \in K}{sup}\{y^T x\}$$

Then, if $y \in (-K^*)$, then for all $x \in K$, $y^T x \leq 0$, and because 0 is in K:

$$\forall y \in (-K^*), (\delta_K) * (y) = 0$$

If $y \notin (-K^*)$, then there exist $z \in K$ such that $y^T z = c > 0$. Then by convexity of K, the sequence $(nz)_{n \in N}$ is also in K, and the sequence $(ny^T z)_{n \in N}$ diverges. Thus $(\delta_K)^*(y) = 0$. In the end, we get:

$$(\delta_K)^*(y) = \begin{cases} 0, \text{ if } y \in (-K*) \\ \infty \text{otherwise} \end{cases}$$

And that is exactly δ_{-K*}

b) If $x \in K$, then $N_K(x) = \{y | \forall z \in K, \langle y | z - x \rangle \leq 0\}$. But because K is convex, $x + K \subseteq K$, and thus, all the element of $N_K(x)$ must be in the opposite of the dual of K. Once that is clear, $\{y \in -K^* | \langle u | x \rangle = 0\} \subseteq N_K(x)$. For the other inclusion, suppose $-\langle y | x \rangle > 0$. Let $z \in K$. Then:

$$\forall n > \frac{log(\frac{y^T x}{y^T x})}{log(2)}, \langle y | 2^{-n} z - x \rangle > 0$$

.

So $y^T x = 0$ and thus we have:

$$N_K(x) = \{y \in -K^* | \langle u | x \rangle = 0\}$$

c)