Optimization Homework 2

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September 2022

1 Problem 3.15

a)

$$\forall\alpha[0,1], \forall x \in R, u_\alpha(x) = \frac{e^{\alpha log(x)} - e^0}{\alpha - 0}$$

Here we clearly recognize the rate of increase of the function $f_x : \alpha \to e^{\alpha log(x)}$ and thus the limit when $\alpha \to 0$ is going to be the derivative of f_x in zero, and thus:

$$_{\alpha\to 0}u_\alpha(x) = f_x'(0) = log(x)$$

b) We easily compute from question 1: $u_\alpha(1) = 0$. Then u_α is concave because is it the composition of a the concave logarithm function, multiplied by a positive constant (still concave) with a convex increasing function (the exponential), multiplied by a posistive constant (thus still convex non decreasing). In then end, it is the composition of a concave function with a convex non decreasing function and thus u_α is concave.

It is increasing because it is the combination of two increasing function. (Because all the multplicative constant are positive, it keeps the increasing aspects of *log* and *exp*.

2 Problem 3.18

a)

b) We keep the introductory paragraph of the proof in which Z, t, G are introduced and defined, we just change the function by $det(X^{1/n})$. Because the determinant is multiplicative, we can let out the exponent and mutliply the matrices in any order we want:

$$g(t) = det((Z + tG)^{1/N}) = det(Z^{1/2}(1 + tZ^{-1/2}GZ^{-1/2})Z^{1/2})^{1/n}$$
$$= det(Z)^{1/n} \times det(1 + tZ^{-1/2}GZ^{-1/2})^{1/n}$$

And then, we introduce λ_i the eigen value of $Z^{-1/2}GZ^{-1/2}$, which gives us the expression:

$$det((Z+tG)^{1/N}) = det(Z)^{1/n} \times \sum_{i=1}^{n} (1+t\lambda_i)^{1/n}$$

Derivating this function twice, we get:

$$g''(t) = \frac{1}{det(Z)} \frac{n-1}{n^2} \sum_{i=1}^{n} \lambda_i^2 (1+t\lambda_i)^{-1/n} \geq 0$$

It is positive because $\forall i, 1+t\lambda_i \geq 0$, which ends the proof that $det(X^{1/n})$ is a convex function.

3 Problem 3.22

a) The function f f is the composition of the three function $-log$ (convex non increasing), $-log(\sum_{i=1} e^{y_i})$ (concave because opposite of a convex function) and $Ax+b$ where A is a diagonal matrix (linear thus convex). Then, using the slide 3-18 of the lectures, we deduce that f is convex.

b)

c) Using the same argument as b), $-log$ is convex non-increasing and $uv - x^T x$ is concave thus f is convex.

e) Using the same argument as d), $-log$ is convex non-increasing and thus f is convex.

4 Problem 3.24

a) The expectation such as defined in this problem is a linear application of p, and therefore it is convex.

b) Same argument as previously, $P(x > \alpha) : p \rightarrow \sum_{a_i > \alpha a_i p_i}$ is a linear application and therefore convex.

c) This application is also linear in p, and therefore convex.

d)

5 Problem 3.25

First, let us notice that the sum $\sum_{i \in C} p_i - q_i$ is going to be maximal when all its terms are of the same sign. This can happen for only two events $I = \{i \in 1, \ldots, n | p_i \geq q_i\}$ and $J = \{j \in 1, \ldots, n | q_j > p_j\}$. Therefore, we can already simplify the expression of $d_{mp}(p,q)$:

$$d_{mp}(p,q) = max \left\{ \sum_{i \in I} (p_i - q_i), \sum_{j in J} (q_j - p_j) \right\}$$

Then, let us observe that these two sums are actually equal. Indeed, because $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, we get the equality:

$$\begin{aligned}
\sum_{i \in I}(p_i - q_i) &= \sum_{i \in I} p_i - (1 - \sum_{j \in J} q_j) \\
&= \sum_{i \in I} p_i - 1 + \sum_{j \in J} q_j \\
&= \sum_{j in J} (q_j - p_j)
\end{aligned}$$

And then, because $I \cup J = \{1, \ldots, n\}$ we can easily deduce the expression:

$$2d_{mp}(p,q) = \sum_{j in J}(q_j - p_j) + \sum_{i \in I}(p_i - q_i) = \|p - q\|_1$$

Thus

$$d_{mp}(p,q) = \frac{\|p - q\|_1}{2}$$

Then, it comes naturally that since $d_{mp}(p,q)$ is a norm, it is convex (let us remind ourselves that the convexity comes from applying the triangular equality and the homogeneity of the norm).

$$\forall x, y \in dom d_{mp}, \forall \eta \in [0,1], \|\eta x + (1-\eta)y\| \leq \eta\|x\| + (1-\eta)\|y\| = 1$$

The extension of $d_{mp}(p,q)$ on $R^n \times R^n$ is also convex because it is a norm on $R^n \times R^n$.

6 Problem 4.1