Optimization 7

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I apologize for the few numbers of exercice attempted. I had a midterm on sunday.

1 Problem 2.25

First, we need to transform this problem into a problem of finding the zero of an operator. The Langrangian will be:

$$L(x,u) = c^T x + u^T(Ax - b) - \delta_K(x)$$

We can there for introduce the operators $B = N_K$ and $A = \underset{u \in \mathbb{R}^n}{argmin}\{c^T x + u^T(Ax - b)\}$. Then, thanks to the resolvant identity, $\mathbb{J}_{N_K}(x) = \pi_K(x)$, and:

$$x^{k+1/2} = \pi_K(z^k)$$
$$x^{k+1} = J_{\alpha A}(2x^{k+1/2} - z^k) = \{x | x + \underset{u}{inf} L(x,u) = 2x^{k+1/2} - z^k\}$$
$$z^{k+1} = z^k + x^{k+1} - x^{k+1/2}$$

The crux of the problem is therefore to calculate Ax.

2 Problem 2.28

First let us establish the fact that because f and g are CCP function, their subgradient exists an is unique (the gradient) at every point of their interior domain. Furthermore, because they are CCP, their subgradient is also a maximal monotone operator and thus the resolvant of their subgradient is averaged. Thus the composition of their resolvant is also averaged and converges to one fixed point for any starting point thanks to theorem 1, which proves convergence.

Then let us establish the fact that:

$$\mathbb{J}_{\alpha\partial f}(y) = \{t \in \mathbb{R}^n | y = t + \alpha \nabla f(t)\}$$

$$\mathbb{J}_{\alpha\partial g}(x) = \{y \in \mathbb{R}^n | x = y + \alpha \nabla g(y)\}$$

Then, because the functions g and f are convex, we can easily recognize the equivalent problem of finding minimizers, are for a sum of convex function (the norm being convex), the minimizer nullifies the gradient:

$$\mathbb{J}_{\alpha\partial g}(x) = \{y \in \mathbb{R}^n | \frac{1}{\alpha}(x - y) = \nabla g(y)\} = \underset{y \in \mathbb{R}^n}{argmin}\{f(x) + g(y) + \frac{1}{2\alpha}\|x - y\|^2\}$$

Notice that adding the constant $f(x)$ does not change the minimization problem on the right. The same can be said to the problem of the resolvvant of f:

$$\mathbb{J}_{\alpha\partial f}(y) = \{x \in \mathbb{R}^n | \frac{1}{\alpha}(y - x) = \nabla f(x)\} = \underset{x \in \mathbb{R}^n}{argmin}\{f(x) + g(y) + \frac{1}{2\alpha}\|x - y\|^2\}$$

Then, because $\mathbb{J}_{\alpha\partial f}\mathbb{J}_{\alpha\partial g}(x)$ is averaged and we apply the resolvant of ∂g and ∂f succesively, our fixed point iterator will give us the following algorithm:

$$y^{k+1} = \underset{y \in \mathbb{R}^n}{argmin}\{f(x^k) + g(y) + \frac{1}{2\alpha}\|x^k - y\|^2\}$$

$$x^{k+1} = \underset{x \in \mathbb{R}^n}{argmin}\{f(x) + g(y^{k+1}) + \frac{1}{2\alpha}\|x - y^{k+1}\|^2\}$$

And therefore the fixed points will correspond to the minimizers of :

$$\underset{x \in \mathbb{R}^n}{argmin}\{f(x) + g(y) + \frac{1}{2\alpha}\|x - y\|^2\}$$

Let us remind ourselves that we have shown convergence in the first paragraph.

3 Problem 3.1

The function f and the norm being convex, the argmin on the left exists if it is possible to nullify the gradient of this function, ie:

$$\nabla f(x) + A^T(Ax - y) = 0 \equiv$$

$\mathcal{R}(A^T) \cap ri\ dom\ f \neq \emptyset$