# hw10 Optimization

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## 1 Problem 13.1

Using the fact that the class $\mathcal{L}_R$, $\mathcal{N}_{\frac{1+R}{2}}$, $\mathcal{L}_1$ are all full SRG opetrators (that's the result of exercice 14, but it was written in the chapter so, I am just going to use it as it allows us to make SRG proof), we will proof the inclusion by proving the equivalent problem.

$$\mathcal{G}(\mathcal{L}_R) \subset \mathcal{G}(\mathcal{N}_{\frac{1+R}{2}}) \subset \mathcal{G}(\mathcal{L}_1)$$

Then considering that

$$\mathcal{G}(\mathcal{L}_1) = \{z \in C||z| \leq 1\}$$

ie, the disk of radius equal to 1.

$$\mathcal{G}(\mathcal{L}_R) = \{z \in C||z| \leq R\}$$

ie the disk of radius$R$.

$$\mathcal{G}(\mathcal{N}_{\frac{1+R}{2}}) = \frac{1-R}{2} + \frac{1+R}{2}\mathcal{G}(\mathcal{L}_1)$$

And here the disk of center $\frac{1-R}{2}$ (the real number in the complex plane) and of radius $\frac{1+R}{2}$.

Then, by describing the shapes as we did, it clearly follows that they are all strict subset of one another, as the disk of radius $R$ is included in the disk of center $\frac{1-R}{2}$ and of radius $\frac{1+R}{2}$ (their circle both touch in point $-R$, and because the center of the first disk is included in the sedcond disk and that the radius of the former if smaller than the radius of the latter, the inclusion exist). And then, the disk of center $\frac{1-R}{2}$ and of radius $\frac{1+R}{2}$ is also included in disk of radius 1. (Their circle touch in point 1, the inclusion result from the same argument as above).

## 2 Problem 13.2

Because we use the fact that all the classes of operators used below are all full SRG operators, we will prof the equivalence chain using SRG proof:

If $A \in \mathcal{N}_{\frac{1}{2}}$, then :

$$\mathcal{G}(I-A) = 1 - \mathcal{G}(A)\frac{1}{2} - \frac{1}{2}\mathcal{G}(\mathcal{L}_1)$$

Then, because $\mathcal{G}(\mathcal{L}_1) = -\mathcal{G}(\mathcal{L}_1)$ (as it is simply the disk of radius one), we have:

$$\mathcal{G}(I-A) \subset \mathcal{G}(\mathcal{N}_{\frac{1}{2}})$$

Which is equivalent to:

$$A \in \mathcal{N}_{\frac{1}{2}} \implies I - A \in \mathcal{N}_{\frac{1}{2}}$$

.

If $I - A \in \mathcal{N}_{\frac{1}{2}}$ we also have:

$$\mathcal{G}(2A - I) = \mathcal{G}(-2(I-A) + I) \subset -2\mathcal{G}(\mathcal{N}_{\frac{1}{2}} + 1) = \mathcal{G}(\mathcal{L}_\infty)$$

Which proves: $I - A \in \mathcal{N}_{\frac{1}{2}} \implies 2A - 1 \in \mathcal{L}_1$.

If $2A - 1 \in \mathcal{L}_\infty$, then:

$$A \subset \frac{1}{2} + \frac{1}{2}\mathcal{G}(\mathcal{L}_1) = \mathcal{G}(\mathcal{N}_{\frac{1}{2}})$$

So far, we thus have proven the chain of equivalent relations:

$$A \in \mathcal{N}_{\frac{1}{2}} \iff A \in \mathcal{N}_{\frac{1}{2}} \iff I - A \in \mathcal{N}_{\frac{1}{2}}$$

It is time to prove the last equivalency. First, if $A \in \mathcal{N}_{\frac{1}{2}}$ we have:

$$\mathcal{G}(A) \subset \frac{1}{2} + \frac{1}{2}\mathcal{G}(\mathcal{L}_1) = \mathcal{G}(C_1)$$

Thanks to page 272 of the book where we have $\mathcal{G}(C_\beta$ is the disk of center $\frac{1}{2\beta}$ and radius $\frac{1}{2\beta}$ (here it is case $\beta = 1$. So we finally get with what's above:

$$A \in \mathcal{N}_{\frac{1}{2}} \iff A \in \mathcal{C}_1$$

.

## 3 Problem 13.3

If we use the results of page 272 we get:

$$\mathcal{G}(C_{1/2\theta}) = \theta + \theta\mathcal{G}(\mathcal{L}_1)$$

.

And:

$$\mathcal{G}(\mathcal{N}_\theta) = 1 - \theta + \theta\mathcal{G}(\mathcal{L}_1)$$

.

Thus, because $\mathcal{G}(\mathcal{L}_1)$ is invariant by multiplication by $-1$, we get:

$$1 - \mathcal{G}(C_{1/2\theta}) = \mathcal{G}(\mathcal{N}_\theta)$$

Which proves the equivalence:

$$A \in \mathcal{N}_\theta \iff A \in C_{\frac{1}{2\theta}}$$

# 4 Problem 9.3

First of all, the trivial solution to the primal problem is $x = 0$ as any other value gives $f(x)+g(x) = \infty$. Then, the Fenchell dual problem can not have any solution as:

$$f^*(y) = \underset{x}{sup}\{xy - f(x)\} = \begin{cases} \infty \text{ if } y \geq 0 \\ \frac{1}{4y^2} \text{ if } y < 0 \end{cases}$$

And:

$$g^*(u) = \underset{x}{sup}\{xu - \delta_{\{0\}}(x)\} = 0$$

Thus the problem can be simplified because:

$$\forall u < 0, \quad -f^*(-u) - g^*(u) = -\infty$$

And

$$\forall u > 0, -f^*(-u) - g^*(u) = -\frac{1}{4u^2}$$

The problem doesn't have a solution because the maximum of the problem (ie 0), is attained in $-\infty$.

Furthermore, $argmin(f+g) = \{0\}$ as we have shown earlier and $Zer(\partial f + \partial g) =$ because $g$ only has a subgradient in 0 equal to $R$ while $f$ does not have a subgradient in zero. So they are different.