# FIRST-ORDER LOGIC

ARTIFICIAL INTELLIGENCE | COMP 131

# TODAY ON AI

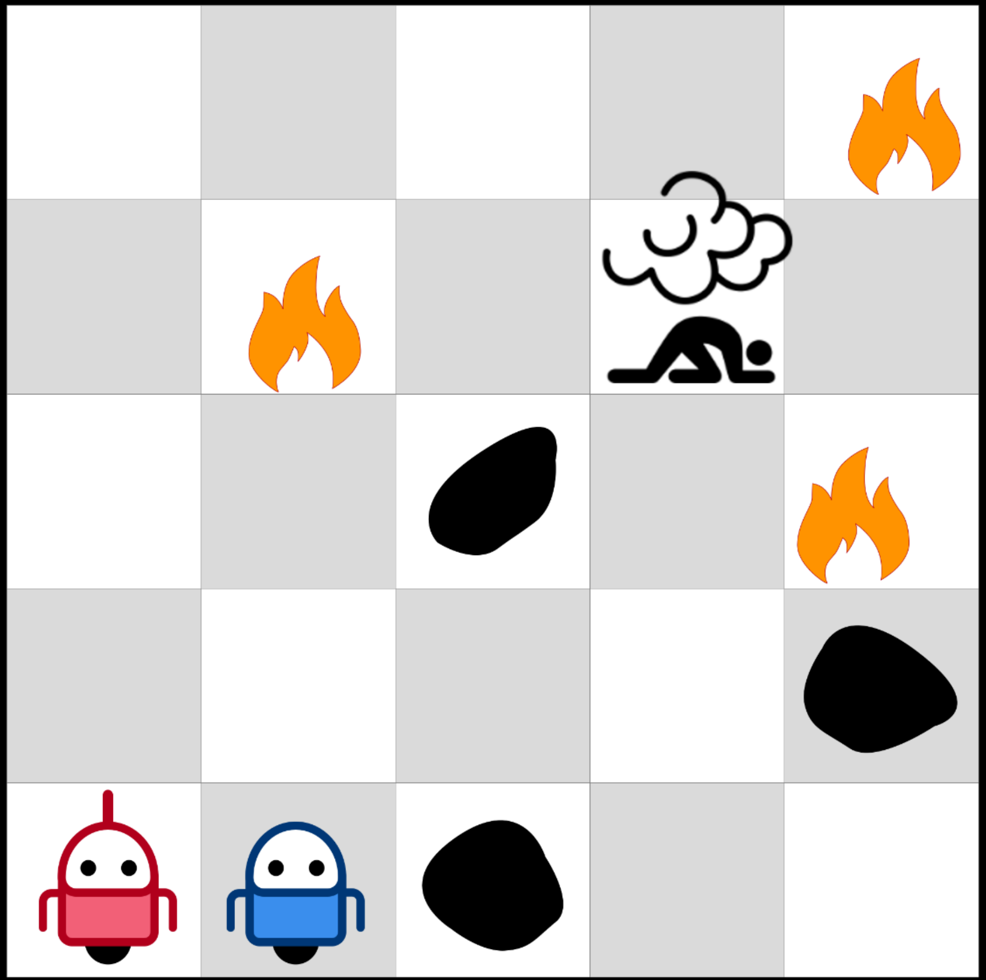
- Disaster zone
- Propositional Logic is a weak language
- First-order Logic
- Questions?

- A building of size **WIDTH** × **LENGTH**

- RR1 starts in square (1, 1)
- BR2 starts in square (2, 1)

- There is a **victim** somewhere

- Robots of type SRR have sensors that can detect **heat** and **drop-offs**
  - They reliably sense drop-offs and victims from exactly one (non-diagonal) block away
  - They are not good enough to say in what direction the victim or hole was



- Propositional Logic is **declarative**
- Propositional Logic is **compositional**: the meaning of $p \land q$ is derived from meaning of $p$ and of $q$
- Propositional Logic is **context-independent** (unlike natural language)
- Propositional Logic assumes that the world contains **only facts**

Unfortunately, Propositional Logic has very **limited** expressive power (unlike natural language):

- Hard to identify **individuals**:
  Every pit causes breeze in adjacent squares
- Can’t directly talk about **properties** and relations between individuals:
  RR1 is red
- **Generalizations**, **patterns**, **regularities** can’t easily be represented:
  All SRR robots have a left arm

SECTION 02

# First-order Logic

**First-order Logic** is a powerful evolution of Propositional Logic. It models the world in terms of:

- **Objects**: things with individual identities
- **Properties**: properties of objects that distinguish them from other objects
- **Relations**: relationships that hold among sets of objects
- **Functions**: a subset of relations where there is only one value for any given input

- Examples:
  - **Objects**: $Victim$, $Robot$, $RR1$, $RB2$, $Sq11$, $Sq12, \dots$
  - **Properties**: $\text{blue}(Robot), \text{red}(Robot)$
  - **Relations**: $\text{color}(x, Red), \text{same\_type}(RR1, RB2)$
  - **Functions**: $\text{location}(Robot, Step0) = Sq11$

- **Term**: Objects, functions, or variables
- **Proposition**: Relations, or property
- **Connectives**:

| | | |
|---|---|---|
| $\wedge$ | AND | Conjunction |
| $\vee$ | OR | Disjunction |
| $\rightarrow$ | IMPLIES | Implication / conditional |
| $\leftrightarrow$ | IS EQUIVALENT | Biconditional |
| $\neg$ | NOT | Negation |

- **Quantifiers**:

| | | |
|---|---|---|
| $\exists$ | EXISTS | Existential quantifier |
| $\forall$ | FOR ALL | Universal quantifier |

- **Operator precedence**: $\neg = \wedge \vee \rightarrow \leftrightarrow$

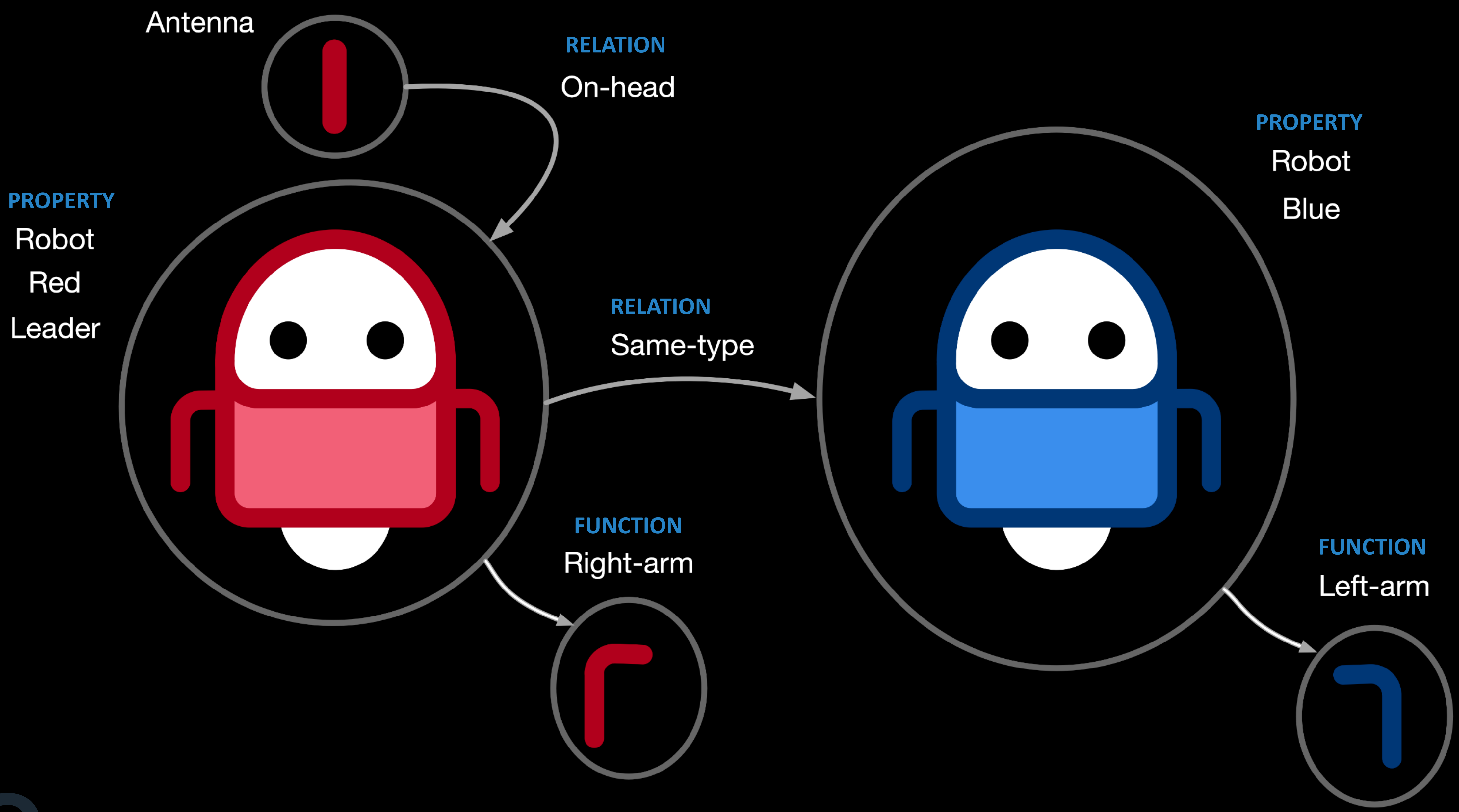
## FIRST-ORDER LOGIC GRAMMAR

| | | |
|---|---|---|
| **Sentence** | $\rightarrow$ | Atomic \| Proposition |
| **Atomic** | $\rightarrow$ | Predicate \| Term \| Term = Term |
| **Proposition** | $\rightarrow$ | (Sentence) |
| | \| | $\neg$ Sentence |
| | \| | Sentence Connective Sentence |
| | \| | Quantifier Variable, … Sentence |
| **Term** | $\rightarrow$ | Function(Term1, Term2, …) |
| | \| | Constant |
| | \| | Variable |
| **Constant** | $\rightarrow$ | $A$ \| $X_1$ \| RR1 … |
| **Variable** | $\rightarrow$ | a \| $x$ \| $s$ … |
| **Predicate** | $\rightarrow$ | **true (T)** \| **false (F)** \| On-head … |
| **Function** | $\rightarrow$ | Left-arm \| Right-arm … |
| **Connective** | $\rightarrow$ | $\wedge, \vee, \rightarrow, \leftrightarrow$ |
| **Quantifier** | $\rightarrow$ | $\exists, \forall$ |

- Sentences are **true** with respect to a **model** and an **interpretation**
- Model contains objects (**domain elements**) and relations among them
- Interpretation specifies referents for:

| | | |
|---|---|---|
| **constant symbols** | → | objects |
| **predicate symbols** | → | relations |
| **function symbols** | → | functional relations |

- A sentence $predicate(term_1, \ldots, term_n)$ is **true** $iff$ the **objects** referred to by $term_1, \ldots, term_n$ are in the **relation** referred to by **predicate**






Example of First-order Logic application

**Existential** quantification:

- $\exists \boldsymbol{x}$: $\boldsymbol{P}(\boldsymbol{x})$ **means** that P holds for some value of x in the domain associated with that variable:

  $\exists x$: $\mathbf{square}(x) \wedge \mathbf{unsafe}(x)$ means **There is an unsafe square**

- It permits one to make a statement about some object without naming it

- The existential quantifier is equivalent to the **disjunction** of **instantiations** of $P(x)$ for all the objects defined in the domain:

$$\begin{aligned}
&\mathbf{square}(x_1) \ \wedge \ \mathbf{unsafe}(x_1) \ \vee \\
&\mathbf{square}(x_2) \ \wedge \ \mathbf{unsafe}(x_2) \ \vee \\
&\mathbf{square}(x_3) \ \wedge \ \mathbf{unsafe}(x_4) \ \vee \\
&\qquad\vdots
\end{aligned}$$

**Existential quantifiers** are normally used with **and** (∧) to specify a list of properties about that object:

$\exists x\colon \mathrm{square}(x) \;\wedge\; \mathrm{free}(x)$ means **There is an object that is square and free**

A common mistake is to use it with an implication (→):

$\exists x\colon \mathrm{square}(x) \rightarrow \mathrm{free}(x)$ means **There is an object for which square implies free**

**Universal** quantification:

- $\forall x$: $P(x)$ means that $P$ holds for all values of $x$ in the domain associated with that variable:

$$\forall x\text{: } \mathbf{adjacent}(x, \mathbf{Sq12}) \wedge \mathbf{unsafe}(x, \mathbf{Sq12}) \rightarrow \mathbf{dropoff\text{-}detected}(x)$$

- The existential quantifier is equivalent to the **conjunction** of instantiations of $P(x)$ for all the objects defined in the domain:

$$\mathbf{adjacent}(x_1, \mathbf{Sq12}) \wedge \mathbf{unsafe}(x_1, \mathbf{Sq12}) \rightarrow \mathbf{dropoff{-}detected}(x_1) \wedge$$
$$\mathbf{adjacent}(x_2, \mathbf{Sq12}) \wedge \mathbf{unsafe}(x_2, \mathbf{Sq12}) \rightarrow \mathbf{dropoff{-}detected}(x_2) \wedge$$
$$\mathbf{adjacent}(x_3, \mathbf{Sq12}) \wedge \mathbf{unsafe}(x_3, \mathbf{Sq12}) \rightarrow \mathbf{dropoff{-}detected}(x_3) \wedge$$
$$\vdots$$

**Universal quantifiers** are often used with "→" to form **rules**.

Universal quantification is rarely used to make blanket statements about every individual in the world:

$\forall x\text{: unsafe}(x)$ means **Every object [in the world] is unsafe**

$\forall x\text{: square}(x) \wedge \text{unsafe}(x)$ means **Every object [in the world] is square and unsafe**

Switching the order of universal quantifiers does not change the meaning:

$$\forall x\ \forall y\!:\ P(x,y) \leftrightarrow \forall y \forall x\ P(x,y)$$

Switching the order of existential quantifiers does not change the meaning:

$$\exists x\ \exists y\!:\ P(x,y) \leftrightarrow \exists y \exists x\ P(x,y)$$

Switching the order of existential and universal quantifiers does change the meaning:

$\forall x\ \exists y\!:\ \mathbf{loves}(\mathrm{x}, \mathrm{y})$ or $\forall x\ (\exists y\!:\ \mathbf{loves}(x, y))$ means **...?**

$\exists \mathrm{y}\ \forall x\!:\ \mathbf{loves}(\mathrm{x}, \mathrm{y})$ or $\exists \mathrm{y}\ (\forall x\!:\ \mathbf{loves}(x, y))$ means **...?**

- We can relate sentences involving ∀ and ∃ using De Morgan's laws:

| SENTENCE | EQUIVALENT |
| --- | --- |
| $\forall x:\ \neg P(x)$ | $\nexists x:\ P(x)$ |
| $\neg\forall x:\ P(x)$ | $\exists x:\ \neg P(x)$ |
| $\forall x:\ P(x)$ | $\nexists x:\ \neg P(x)$ |
| $\exists x:\ P(x)$ | $\neg\forall x:\ \neg P(x)$ |

FOL introduces the concept of equality:

- At timestep 0, the robot was at Square (1, 1).
  $\mathbf{location}(\mathrm{Robot}, \mathrm{T0}) = \mathrm{Sq11}$

- The robot was located at different places in timestep 0 and timestep 1.
  $x = \mathbf{location}(\mathrm{Robot}, \mathrm{T0}) \land y = \mathbf{location}(\mathrm{Robot}, \mathrm{T1}) \land \neg\, (x = y)$
  $x = \mathbf{location}(\mathrm{Robot}, \mathrm{T0}) \land y = \mathbf{location}(\mathrm{Robot}, \mathrm{T1}) \land (x \neq y)$

- The robot is different from the victim.
  $\neg(\mathrm{Robot} = \mathrm{Victim})$
  $\mathrm{Robot} \neq \mathrm{Victim}$

- Every gardener likes the sun.
  $\forall x\colon\ \mathbf{gardener}(x) \rightarrow \mathbf{likes}(x, \mathbf{Sun})$

- You can fool some of the people all the time.
  $\exists x\ \forall t\colon\ (\mathbf{person}(x) \wedge \mathbf{time}(t)) \rightarrow \mathbf{can\text{-}fool}(x, t)$

- You can fool all the people some of the time.
  $\forall x\ \exists t\colon\ (\mathbf{person}(x) \wedge \mathbf{time}(t)) \rightarrow \mathbf{can\text{-}fool}(x, t)$

- All purple mushrooms are poisonous.
  $\forall x\colon\ (\mathbf{mushroom}(x) \wedge \mathbf{purple}(x)) \rightarrow \mathbf{poisonous}(x)$

- No purple mushroom is poisonous.
  $\nexists x\text{: } \textbf{purple}(x) \land \textbf{mushroom}(x) \land \textbf{poisonous}(x)$
  $\forall x\text{: } \textbf{mushroom}(x) \land \textbf{purple}(x)) \to \neg\textbf{poisonous}(x)$

- There are exactly two purple mushrooms.
  $\exists x\ \exists y\text{: } \textbf{mushroom}(x) \land \textbf{purple}(x) \land \textbf{mushroom}(y) \land \textbf{purple}(y) \land \neg(x = y) \land$
  $\quad \forall z\text{: } (\textbf{mushroom}(z) \land \textbf{purple}(z)) \to ((x = z) \lor (y = z))$

- John is not tall.
  $\neg\textbf{tall}(\text{John})$

# PRACTICE

Exercises from the textbook (chapter 8):
8.6, 8.10, 8.11, 8.23, 8.24

# READINGS

Chapters 8

# QUESTIONS ?

# ARTIFICIAL INTELLIGENCE

## COMP 131

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VERSION 3.2